ESSAYS ON UNCERTAINTY
AND ALLOCATION OVER TIME

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C O N T E N T S

CAPITAL RISK, CONSUMPTION and PORTFOLIO CHOICE

This paper is concerned with the interaction of saving and portfolio decisions of a single consumer. Its building blocks are the classical theory of optimal allocation over time, and Arrow's recent formulation of the theory of portfolio selection. The concept of a risk aversion function is extended to a two-period context, and the implications of declining risk aversion are explored. Also discussed are the problems of the effect of changes in the rates of return and in the degree of risk, as well as the question of taxation and risk-taking.

1. Introduction

In the classical analysis of saving, derived from the work of Irving Fisher [5], it is assumed that whatever amount the consumer wishes to save out of current income, is invested at an exogeneously given rate of interest. This rate of interest is most naturally interpreted as a <u>certain</u> rate of return; there is no capital risk in this model. One might, of course, offer the interpretation that the analysis may also be applicable to a world of uncertainty, in which asset yields are not perfectly known, but this escape is not very satisfactory. For in the real world of uncertain asset yields the consumer typically has a choice between several assets when composing a savings portfolio, and casual observation is sufficient to conclude that the resulting portfolio will generally be a diversified one. This line of reasoning suggests that the theory of consumer saving should drop its one-asset assumption, and take account of the insights offered by the modern theory of portfolio selection. The argument works the other way too. Portfolio theory is concerned with the optimal composition of a portfolio of given size. It would seem a promising undertaking to try to work out a theory in which the size and the composition of the portfolio are simultaneously determined.

Capital risk is, of course, not the only kind of risk which is relevant to the consumer's saving-consumption decision, although the present paper concentrates on this type. No attention is paid here to the fact that future income may also be imperfectly known,

nor do we take account of lifetime uncertainty, which
has been discussed in a recent paper by Yaari [12].[1]

Sections 2 and 3 of this paper summarize briefly
the main characteristics of Fisher's theory of saving,
and Arrow's version of the theory of portfolio selection.
In section 4 an integrated model of saving and portfolio
choice is presented, and necessary and sufficient con-
ditions for a local maximum are derived. The concepts
of risk premium and risk aversion function are de-
veloped in section 5. Section 6 is concerned with the
question of whether the risky asset is a normal good.
Section 7 analyzes changes in asset yields, and section
8 is concerned with the effect of a capital gains tax
on saving and risk-taking. The effect of increased
riskiness on present consumption and saving is explored
in section 9. Finally, some concluding remarks are
collected in section 10.

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1) There is not much published work in this field.
Phelps [8] has analyzed consumption allocation over
time with capital risk, but there is no portfolio
choice in his model. Hakansson [6] has, however,
extended Phelps' model to include choice among alterna-
tive investment opportunities. Both of these authors
analyze special forms of additive utility functions
with discounting of an instantaneous utility function.
More in the spirit of the present paper are unpublished
work by Diamond [3] and by Drèze and Modigliani [4].
These authors formulate a two-period model similar to
the one used here, and they do not assume additivity
of the utility function.

2. Fisher's Theory of Saving

The consumer is assumed to have a preference ordering over present consumption, C_1, and "future" consumption, C_2 [2]. This ordering is such that it can be represented by a continuous ordinal utility function

$$U = U(C_1, C_2).$$

Present and future income (Y_1, Y_2) are assumed to be exogeneously given. It is also assumed that the consumer has access to a perfect capital market, in which he can borrow and lend at the same rate of interest, r. The budget constraint is then

$$C_2 = (Y_1 - C_1)(1 + r) + Y_2.$$

The necessary condition for a constrained maximum is

(1) $$U_1 - (1 + r)U_2 = 0$$

or

$$\frac{U_1}{U_2} - 1 = r,$$

which is Fisher's famous rule for optimization over time; equality between the marginal rate of time preference and the rate of interest.

The effect of a change in income (say Y_1) on present consumption can be written as

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2)   See [11] for some comments on the legitimacy of the two-period framework.

(2) $\quad \dfrac{\delta C_1}{\delta Y_1} = (1 + r) \dfrac{(1 + r)U_{22} - U_{12}}{D}$

where

$$D = U_{11} - 2(1 + r)U_{12} + (1 + r)^2 U_{22} < 0$$

as a second-order maximum condition. From this it is easy to see that necessary and sufficient conditions for the marginal propensity to consume to lie between zero and unity (which is equivalent to the requirement that both C_1 and C_2 be superior goods) are[3]

$$(1 + r)U_{22} - U_{12} < 0,$$

$$U_{11} - (1 + r)U_{12} < 0.$$

The interest rate derivative of present consumption is

(3) $\quad \dfrac{\delta C_1}{\delta r} = \dfrac{Y_1 - C_1}{1 + r} \cdot \dfrac{\delta C_1}{\delta Y_1} + \dfrac{U_2}{D}$

with the substitution effect always negative and the income effect positive for a lender, negative for a borrower (assuming that C_1 is not inferior).

3. The Theory of Portfolio Selection

The theory of portfolio selection has recently received a very elegant and general treatment by Arrow [1,2].

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[3] These are local conditions, assuming that $U_1/U_2 = (1+r)$. In general, the condition for absence of inferiority is that U_1/U_2 is decreasing in C_1 and increasing in C_2. This implies

$$U_{11} - \frac{U_1}{U_2} U_{12} < 0 \quad \text{and} \quad \frac{U_1}{U_2} U_{22} - U_{12} < 0.$$

The individual agent ("investor") has a utility-of-wealth function $W(Z)$, where Z refers to final wealth, i.e. wealth at the end of the period for which the investment decision is binding. Marginal utility is everywhere positive and decreasing. Z is defined as

$$Z = a(1 + x) + m(1 + r),$$

where a and m are the amounts invested in the risky and the secure asset, respectively. r is the rate of return on the secure asset, and x is the random rate of return on the risky asset with subjective density function $f(x)$. The budget constraint is

$$A = a + m,$$

A being initial wealth. Final wealth can now be expressed as

$$Z = A(1 + r) + a(x - r).$$

The investor maximizes expected utility

$$E[W(Z)] = \int W(A(1+r) + a(x-r))\ f(x)dx$$

in the von Neumann-Morgenstern sense. The first-order maximum condition can be written as

$$(4) \qquad E[W'(Z)(x-r)] = 0,$$

while the satisfaction of the second-order condition is guaranteed by the assumption of concavity. (4) says, in effect, that expected marginal utility per dollar invested should be equal for the two assets.

Arrow[4] has introduced the concepts of absolute and relative risk aversion. These measures are defined as

$$R_A(Z) = - \frac{W''(Z)}{W'(Z)} \, ,$$

$$R_R(Z) = - \frac{ZW''(Z)}{W'(Z)} \, ,$$

respectively. Note that both measures are positive (under risk aversion) and invariant under positive linear transformations of the utility function.

Arrow advances the hypotheses that $R_A(Z)$ is a decreasing function of Z and that $R_R(Z)$ is an increasing function of Z. Decreasing $R_A(Z)$ implies that "the willingness to engage in small bets of fixed size increases with wealth, in the sense that the odds demanded diminish", while increasing $R_R(Z)$ may be interpreted to mean that "if both wealth and the size of the bet are increased in the same proportion, the willingness to accept the bet (as measured by the odds demanded) should decrease".[5]

The derivative of risky asset holdings with respect to initial wealth is

$$(5) \qquad \frac{\delta a}{\delta A} = - \frac{E[W''(Z)(x-r)]}{E[W''(Z)(x-r)^2]} \, .$$

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4) The exposition in this section leans heavily on that of Arrow [2, pp. 32-44]. The measures of risk aversion used by Arrow were independently developed by J.W.Pratt [10].

5) The quotations are from Arrow [2, pp. 35-36].

The denominator of this expression is clearly negative, so that the sign of the derivative is the same as that of the numerator. It can be shown that decreasing absolute risk aversion implies $E[W''(Z)(x-r)] > 0$, so that the risky asset is a normal good. The proof of this closely resembles those presented in section 5 of the present paper, and will not be given here.

The attractiveness of Arrow's approach - deducing empirically significant conclusions from plausible hypotheses on behaviour in simple risk situations - makes it seem a promising undertaking to reexamine his conclusions within an integrated model of saving and portfolio choice. This is a task to which we will turn in the next section.

One further comment: Arrow assumes that there exists a secure asset, in the sense that its rate of return is known with certainty. The existence of such an asset, either subjectively or in some objective sense, may be questioned on grounds of realism, although in economies characterized by a high degree of price predictability, government bonds or bank deposits might come close to this ideal. But the basic defence of the assumption is an analytical one; we wish to study the choice between assets which are relatively secure and assets which are relatively risky, and this is one way to do it. Another approach is to assume that the probability distribution of the yields can be completely described by means of first and second moments, but that approach is more restrictive.

supposed to be a subset of the interval $[-1, \infty)$.
Combining the last two expressions, we obtain

$$C_2 = Y_2 + (Y_1-C_1)(1+r) + a(x-r).$$

Substituting this into the utility function, we have that expected utility is

(7) $E[U] = \int U(C_1, Y_2 + (Y_1-C_1)(1+r) + a(x-r)) \, f(x)dx,$

where integration is over the range of x.

Maximization of (7) leads to the first-order conditions

(8) $E[U_1 - (1+r) U_2] = 0,$

(9) $E[U_2(x-r)] = 0.$

Equation (8) is a generalization of Fisher's rule, as formulated in (1) above.

Equation (9) is the counterpart of the first-order condition for the pure portfolio model, equation (4), except that the marginal utility of income has been replaced by the marginal utility of future consumption.[7]

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7)   This does not mean that "wealth" or "income" in the pure portfolio model is simply a proxy for future consumption. Traditional portfolio theory can most naturally be interpreted as being concerned with timeless risk prospects, which means that the uncertainty will be removed before the saving-consumption decision is made. In this paper we are concerned with temporal risk prospects. This means that the uncertainty about the yield of the risky asset is not going to be removed until the end of the first period. The distinction between timeless and temporal risks has been stressed by Drèze and Modigliani [4].

4. A General Model of Portfolio Choice and Allocation over Time

We shall study a consumer whose preferences conform to the von Neumann-Morgenstern axioms for rational choice under uncertainty. His preference ordering on consumption profiles can be represented by a continuous cardinal utility function

$$(6) \qquad U = U(C_1, C_2),$$

which is assumed to be at least three times continuously differentiable, and to possess everywhere positive marginal utilities.

The budget constraint is expressed by the equation

$$C_1 + a + m = Y_1,$$

which says that income in the first period can be used to buy consumption goods (for consumption in the same period) or to invest in the risky asset (a) or the secure asset (m). Future consumption is a stochastic variable and is defined as

$$C_2 = Y_2 + a(1+x) + m(1+r),$$

where r and x are to be interpreted as in the previous section. r is taken to be a real number greater than minus one, and the range of the random variable x is

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6) The model has no explicit treatment of prices, because they are of no particular interest for the problems discussed in this paper. But it is clear that all the variables C_1, C_2, a and m could have been written as products of price and quantity components. For general equilibrium analysis this approach is, of course, essential.

The second-order conditions for the local maximum are

$$(10) \quad H = \begin{vmatrix} E[U_{11}-2(1+r)U_{12}+(1+r)^2U_{22}]E[(x-r)U_{12}-(x-r)(1+r)U_{22}] \\ E[(x-r)U_{12}-(x-r)(1+r)U_{22}] \qquad E[(x-r)^2U_{22}] \end{vmatrix} > 0$$

$$(11) \qquad E[U_{11}-2(1+r)U_{12} + (1+r)^2U_{22}] < 0,$$

$$(12) \qquad E[(x-r)^2U_{22}] < 0.$$

This model can now be subjected to comparative statics analysis in the Hicks-Samuelson tradition. Without further assumptions the conclusions that can be drawn are analogous to those of traditional demand analysis in its most general form. Thus, no a priori conclusions can be drawn as to the signs of the income derivatives, except, of course, that their sum must equal unity.[8] As for substitution effects (compensated changes in yield), direct substitution effects are positive, while the signs of the cross substitution effects are indeterminate. An increase in the yield of the secure asset will raise the demand for that asset, while the demand for the risky asset will increase with a shift in the probability distribution of its yield which increases the mean with no change in dispersion. Nothing can be said about the effect on consumption of compensated

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8) One may feel that it would be legitimate to assume that present and future consumption are normal goods. One might then ask whether this has any implications for the income derivatives of asset holdings. The answer is no, apart from the obvious fact that their sum must be positive. It would still be possible for one of the assets to be inferior.

changes in yield, while in the Fisher model this is always negative. The difference is explained by the fact that while Fisher's analysis contains two "goods", ours is a three-good model, where the signs of cross-substitution effects are indeterminate. It is then perfectly possible for yield changes to influence only the composition of the portfolio while leaving consumption unchanged, even when attention is restricted to substitution effects.[9]

5.   The Risk Aversion Function

Pratt [10, pp. 124-125] shows that the function $-W''(Z)/W'(Z)$ (in the symbols of section 3) may be taken as a measure of local risk aversion for timeless risks. For infinitesimal risks he shows that the risk premium, which is defined as the actuarial value of a gamble minus its cash equivalent, will be proportional to this function, which Arrow [2] calls absolute risk aversion. The appealing hypothesis that the risk premium is less, the greater is the wealth of the investor, is then seen to imply that absolute risk aversion is a decreasing function of wealth. We shall now develop the concepts of risk premium and risk aversion in a temporal context. Underlying this development is the basic viewpoint of this paper that saving involves giving up the certainty of present consumption for the uncertainty of future consumption.

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9)   We shall generally assume that assets are held in non-zero (although not necessarily non-negative) quantities. From (9) it is easy to see that the condition for $a=0$ is $E[x] = r$, or, if negative holdings of the risky asset are not allowed, $E[x] \leq r$.

Consider a consumer who, for a given level of C_1, is offered the opportunity of entering into a fair gamble , where the outcomes are C_2+h and C_2-h with equal probability. h is taken to be a very small number, so that this gamble is, in Pratt's words, an infinitesimal risk. The expected utility of this gamble is

$$\tfrac{1}{2}U(C_1,C_2+h) + \tfrac{1}{2}U(C_1,C_2-h).$$

The utility of the expected outcome of the gamble is, of course,

$$U(C_1,C_2),$$

which, under risk aversion (in the sense that $U_{22} < 0$), is greater than the expected utility of the gamble itself. Let the positive risk premium, p, be defined by the equation

$$U(C_1,C_2-p) = \tfrac{1}{2}U(C_1,C_2+h) + \tfrac{1}{2}U(C_1,C_2-h).$$

Multiplying by 2 and subtracting $2U(C_1,C_2)$ on both sides, we get

$$2\{U(C_1,C_2-p) - U(C_1,C_2)\} = U(C_1,C_2+h) -$$

$$U(C_1,C_2) + U(C_1,C_2-h) - U(C_1,C_2).$$

The expression in braces is approximately equal to $-pU_2(C_1,C_2)$. Dividing through by h on both sides, we have, as an approximation,

$$-\frac{2}{h}pU_2(C_1,C_2) = U_2(C_1,C_2) - U_2(C_1,C_2-h).$$

Dividing once more by h, we get, again as an approximation

$$- \frac{2}{h^2} p U_2(C_1, C_2) = U_{22}(C_1, C_2)$$

and

(13)
$$\frac{2}{h^2} p = - \frac{U_{22}(C_1, C_2)}{U_2(C_1, C_2)} \ .$$

The right side is then approximately "twice the risk premium per unit of variance for infinitesimal risks" (Pratt's formulation) for the type of gamble where present consumption is given, and the outcomes are given in terms of quantities of future consumption.

The important thing to notice about the risk aversion function (13) is that, in general, it depends on both C_1 and C_2. If an additive utility function is assumed, the risk aversion function depends on C_2 only. In that case, as shown in an earlier paper [11], the generalization of Arrow's results becomes very simple and direct. However, there does not seem to be any compelling reason for assuming additivity. Indeed, recent work by Pollak [9] seems to show that this implies quite strong restrictions on the preference ordering of the consumer.

We shall assume that the risk premium, and therefore the risk aversion function, is a decreasing function of C_2. This seems an intuitively reasonable assumption, and one which suggests itself naturally from consideration of the additive case. The question of how the risk premium depends on C_1 seems far more complicated, and particular hypotheses do not suggest themselves so easily. However, we shall assume that the risk premium is increasing in C_1. This means that the higher is present consumption, the higher is the consumer's risk premium for gambles on future consumption. It is tempting to call this risk complementarity,

and its opposite (risk premium decreasing in C_1)
risk substitutability. But it should be noted that this
involves no assumption on the sign of U_{12}, which is the
classical measure of complementarity.[10])[12]

We shall now prove two mathematical lemmas, which
will be of importance for comparative statics analysis.

Lemma 1:

$$\frac{\delta}{\delta C_2} \left\{ - \frac{U_{22}}{U_2} \right\} \leqq 0 \quad \text{implies} \quad E[(x-r)U_{22}] \geqq 0$$

if $a \geqq 0$, and

$$E[(x-r)U_{22}] \leqq 0 \quad \text{if} \quad a \leqq 0.$$

Proof: From section 4 above we have that

$$C_2 = Y_2 + (Y_1 - C_1)(1+r) + a(x-r).$$

Define

$$C_2^{\;o} = Y_2 + (Y_1 - C_1)(1+r).$$

Since $C_2 = C_2^{\;o} + a(x-r)$ and $-U_{22}/U_2$ is decreasing
in C_2, we have

$$(14) \qquad \cdot \frac{-U_{22}}{U_2} \leqq \left(- \frac{U_{22}}{U_2} \right)^o \quad \text{if} \quad x \geqq r \quad \text{and} \quad a \geqq 0,$$

where the right-hand side is the risk aversion function
evaluated at $C_2^{\;o}$.

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10) Risk complementarity, as defined here, may be seen as
implying, roughly, decreasing risk aversion at the beginning
of the second period. The higher is consumption today, the
lower are the resources at disposal at time 2, and the higher
is risk aversion.

Trivially

(15) $-U_2(x-r) \leqq 0$ if $x \geqq r$.

Multiply through in (14) by $-U_2(x-r)$.
The inequality is then reversed.

(16) $U_{22}(x-r) \geqq (\frac{U_{22}}{U_2})^0 U_2(x-r)$ if $x \geqq r$ and $a \geqq 0$.

Suppose now that $x \leqq r$. Then inequalities (14) and
(15) are both reversed, and so (16) holds for all x.
Since $(U_{22}/U_2)^0$ is not a random variable, we can take
expectations on both sides of (16) to get

$$E[(x-r)U_{22}] \geqq (\frac{U_{22}}{U_2})^0 E[U_2(x-r)]   \text{ if } a \geqq 0.$$

But the right side is zero because of (9). Hence the
lemma is proved for $a \geqq 0$.

Suppose now that $a \leqq 0$ (short sales of the risky
asset). This will reverse inequality (14) and therefore
(16) as well. Again taking expectations, we have that

$$E[(x-r)U_{22}] \leqq (\frac{U_{22}}{U_2})^0 E[U_2(x-r)]   \text{ if } a \leqq 0,$$

where the right-hand side is zero. This proves the
lemma for $a \leqq 0$.

Lemma 2:

$$\frac{\delta}{\delta C_1} \{- \frac{U_{22}}{U_2}\} \geqq 0   \text{ implies }   E[(x-r)U_{12}] \leqq 0$$

if $a \geqq 0$ and $E[(x-r)U_{12}] \geqq 0$ if $a \leqq 0$.

Proof: Writing out the derivative in full, we obtain

$$\frac{\delta}{\delta C_1} \left\{- \frac{U_{22}}{U_2}\right\} = - \frac{U_{122}U_2 - U_{22}U_{12}}{U_2^2} .$$

But we have also that

$$\frac{\delta}{\delta C_2} \left\{- \frac{U_{12}}{U_2}\right\} = - \frac{U_{122} U_2 - U_{22}U_{12}}{U_2^2} ,$$

so that we may as well base our proof on $-U_{12}/U_2$ being increasing in C_2.

Adopting the notation of the previous proof

$$(17) \qquad \frac{-U_{12}}{U_2} \geqq \left(- \frac{U_{12}}{U_2}\right)^0 \qquad \text{if } x \geqq r \text{ and } a \geqq 0.$$

Multiplying through by $-U_2(x-r)$ we have from (15) and (17)

$$(x-r)U_{12} \leqq \left(\frac{U_{12}}{U_2}\right)^0 U_2(x-r) \qquad \text{if } a \geqq 0.$$

This actually holds for all x, since inequalities (15) and (17) are both reversed if $x \leqq r$.   Taking expectations, it follows that

$$E[(x-r)U_{12}] \leqq 0 \qquad \text{if } a \geqq 0,$$

because $E[U_2(x-r)] = 0$.

If $a \leqq 0$, inequality (17) is reversed. It is then easy to see that

$$E[(x-r)U_{12}] \geqq 0 \qquad \text{if } a \leqq 0.$$

This completes the proof of the lemma.

Corollary:

From lemmas 1 og 2 it follows immediately that

$$E[(x-r)U_{12} - (1+r)(x-r)U_{22}] \leqq 0 \qquad \text{if} \quad a \geqq 0$$

and that

$$E[(x-r)U_{12} - (1+r)(x-r)U_{22}] \geqq 0 \qquad \text{if} \quad a \leqq 0.$$

6. The Non-Inferiority of Risky Assets

Arrow [2] proves that decreasing absolute risk aversion implies that the risky asset is not an inferior good. We are now in a position to prove a similar theorem.

We shall first make the assumption that both present consumption and saving are normal goods, i.e that the marginal propensity to consume lies between zero and one. Since we have that

$$\frac{\delta C_1}{\delta Y_1} = \frac{1}{H} \{E[(1+r)^2 U_{22} - (1+r)U_{12}] \ E[(x-r)^2 U_{22}]$$

$$+ E[(x-r)U_{12} - (1+r)(x-r)U_{22}] \ E[(1+r)(x-r)U_{22}]\}$$

it is easy to see that a necessary condition for $\delta C_1/\delta Y_1$ to be positive is that [11]

$$(18) \qquad E[(1+r)U_{22} - U_{12}] < 0.$$

Moreover, since the marginal propensity to save, $1-\delta C_1/\delta Y_1$, can be written as

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11) From lemmas 1 and 2 the last term in braces is negative. Since $E[(x-r)^2 U_{22}]$ is negative, (18) follows.

$$\frac{\delta S}{\delta Y_1} = \frac{1}{H} \{ E[U_{11} - (1+r)U_{12}] \; E[(x-r)^2 U_{22}]$$

$$- E[(x-r)U_{12} - (1+r)(x-r)U_{22}] \; E[(x-r)U_{12}] \}$$

it follows that a necessary condition for $\delta S/\delta Y_1 > 0$ is that[12]

(19) $E[U_{11} - (1+r)U_{12}] < 0.$

The income derivative of risky assets is

(20) $$\frac{\delta a}{\delta Y_1} = - \frac{1}{H} \{ E[U_{11} - (1+r)U_{12}] \; E[(1+r)(x-r)U_{22}]$$

$$+ E[(1+r)U_{22} - U_{12}] \; E[(1+r)(x-r)U_{12}] \}.$$

The sign of this derivative is ambiguous both for $a \geq 0$ and $a \leq 0$. The reason is that when higher income increases both present consumption and planned future consumption (saving), the increase in future consumption decreases risk aversion while the increase in present consumption increases risk aversion. The ambiguity of this result as compared with Arrow's is not to be deplored. It is true that as a general proposition the hypothesis that the risky asset is a normal good seems preferable to its opposite. But an inter-temporal analysis should keep open the possibility that a consumer experiencing an increase in income should thereby become less willing to gamble on the level of future consumption.

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12) From the two lemmas the last term in braces is positive. So (19) is necessary for the whole expression in braces to be positive.

If one makes the strong assumption of additivity, the expression (20) is very much simplified. With decreasing marginal utility of present consumption, decreasing absolute aversion is a sufficient condition for the risky asset to be a normal good.

We shall not go into the question of the effect of increases in income on relative portfolio shares. This has been discussed elsewhere [11] for the case of an additive utility function. In that case Arrow's conclusion that the income elasticity of the secure asset is at least one must be weakened to the effect that the income elasticity of the secure asset is at least as great as that of the risky asset. This is all on the assumption of increasing relative risk aversion, independent of present consumption. With the present approach, even this result would be hard to uphold without additional assumptions.

7.   Changes in Yield

We first examine the effect of an additive shift in the distribution of the random variable x. Thus, let the yield on the risky asset be $x+\theta$, where θ is the shift parameter, and differentiate with respect to θ. This may be interpreted to mean an increase in the expected value of the yield with all other moments constant. The result is (when the derivative is evaluated at $\theta=0$)

$$(21) \quad \frac{\delta a}{\delta \theta} = \frac{a}{1+r} \frac{\delta a}{\delta Y_1} - \frac{1}{H} E[U_2]E[U_{11}-2(1+r)U_{12}+(1+r)^2 U_{22}],$$

which is a Slutsky equation. The second term on the right
is the substitution effect, which is positive. Let us
assume that $\partial a/\partial Y_1 > 0$ for $a \geqq 0$ and $\partial a/\partial Y_1 < 0$ for
$a \leqq 0$, which might perhaps be taken to be the normal
case[13]. Then the income and substitution effects work
in the same direction. If $a \geqq 0$, an increase in the
expected yield will always increase investment in the
risky asset. If $a \leqq 0$, the interpretation is that an
increase in the expected yield will always reduce the
debt held in units of the risky asset. An example may
perhaps make this clearer: With an uncertain future
price level an increase in the expected rate of price de-
flation will increase investments held in constant nomi-
nal value and decrease debt issued in constant nominal value.

The effect on consumption is

$$(22) \qquad \frac{\delta C_1}{\delta \theta} = \frac{a}{1+r} \frac{\delta C_1}{\delta Y_1} + \frac{1}{H} E[U_2]E[(x-r)U_{12}-(1+r)(x-r)U_{22}].$$

The income effect is positive for $a \geqq 0$ and
negative for $a \leqq 0$. From the corollary in section 5
it follows directly that the substitution effect is nega-
tive for $a \geqq 0$ and positive for $a \leqq 0$. It follows
that the sign of the total effect is indeterminate in both
cases.

It may be of interest to ask what would be the effect
of a general rise in yields, i.e. of an increase in the
rate of interest on the secure asset together with an

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13) That $\partial a/\partial Y_1 < 0$ for $a \leqq 0$ means that the con-
sumer will increase the amount of debt held in the
risky asset.

additive shift in the probability distribution of x.
This can be answered by differentiating with respect to
r and setting $\delta\theta/\delta r = 1$.

$$(23) \quad \left(\frac{\delta C_1}{\delta r}\right)_{\delta\theta/\delta r=1} = \frac{Y_1-C_1}{1+r}\frac{\delta C_1}{\delta Y_1}+\frac{1}{H} E[U_2]E[(x-r)^2 U_{22}].$$

Here the sign of the income effect is positive or
negative, according as $Y_1 - C_1 \lessgtr 0$ and the substitution
effect is negative, independent of the behaviour of the
risk aversion function. It is natural to interpret this
as a direct generalization of the analysis of interest
rate changes under certainty (compare equation (8) above).

What would be the effect on asset holdings of such
a general rise in yields? There will, of course, be income
effects, but of more interest are the substitution effects.
The effect on risky asset holdings is

$$\left(\frac{\delta a}{\delta r}\right)_{\delta\theta/\delta r=1} = \frac{Y_1-C_1}{1+r}\frac{\delta a}{\delta Y_1} - \frac{1}{H} E[U_2]E[(x-r)U_{12}-(1+r)(x-r)U_{22}],$$

with the substitution effect positive for $a \geqq 0$, and
negative for $a \leqq 0$. Thus, with a general rise in yields,
the substitution effect indicates that the risky asset will
generally be substituted for the secure one. In the case
where the consumer takes a short position in the risky
asset, the general rise in expected yields will cause him
to decrease his borrowing in that asset[14].

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14) To arrive at the total effect, account must, of course,
be taken of the income effect. To work out all possible
cases would be very tedious and is left to the inter-
ested reader. The most interesting case may be

$a \geqq 0$, $Y_1-C_1 \geqq 0$, $\partial a/\partial Y_1 > 0$, in which $\partial a/\partial r$ is
positive.

8. Capital Gains Taxation and Risk-Taking

A problem which has been studied by several authors is the following one: Suppose that an individual can invest in two assets, one bearing a secure rate of return of zero, and one risky asset with random yield x. Suppose a proportional tax is levied on investment income with full loss offset provisions. How does this affect the composition of the portfolio? The most modern and general treatment of this problem is that of Mossin [7], who shows that the tax rate derivative of risky asset holdings, $\delta a/\delta t$, is simply equal to $a/(1-t)$.

A question raised by Mossin's analysis is whether his conclusion depends in any essential way on the assumption of a fixed portfolio. In other words, one may ask whether reactions of _saving_ to tax rate changes might not come to dominate the simple reaction pattern implied by his model.

We shall try to answer this question in terms of the analysis of this paper. Future consumption is

$$C_2 = Y_1 + Y_2 - C_1 + ax(1-t).$$

The first-order condition for a maximum of $E[U(C_1,C_2)]$ are

$$E[U_1-U_2] = 0,$$

$$E[U_2x] = 0.$$

Differentiating with respect to t, we obtain

(24) $\quad \dfrac{\delta a}{\delta t} = \dfrac{a}{1-t}$,

(25) $\quad \dfrac{\delta C_1}{\delta t} = 0$.

Consumption is unchanged when the tax rate increases, while the reaction of the demand for the risky asset is exactly the one predicted by Mossin's model. The rationale of this result is simply that it implies that expected utility, $E[U(C_1,Y_1+Y_2-C_1+ax(1-t))]$ remains constant when the tax rate is increased. Constancy of expected utility is clearly the best that the consumer can hope for. When given the opportunity, he should behave so as to achieve just that.

The somewhat surprising simplicity of these results does not carry over to the case where r is not zero. However, as long as the tax is levied on the differential yield $(x-r)$, the results are exactly as before. This case may not be entirely unrealistic. It could be taken to represent a tax on "excess profits". Or, if m is taken to be a debt instrument issued to finance the holding of the risky asset, it would simply represent deductible interest payments on debt.

9.   Variations in the Degree of Risk

The "degree of risk" is an elusive concept when not measured by one statistic as e.g. the variance. Following a suggestion by Arrow [1], we shall analyze the problem by means of shift parameters, paying special attention to

the effect on present consumption and saving.

We note first that the probability distribution
on which expected utility depends, is that of the differ-
ential yield, x-r[15]. A pure increase in dispersion can
now be studied by means of (1) - a multiplicative shift
around zero, and (2) - an additive shift to restore the
mean to its initial value. In combination, this means a
multiplicative shift around the mean. The effect of a
multiplicative shift around zero was examined in section 8
as a change in t, and was shown to have no effect on con-
sumption. The effect of an additive shift was studied in
section 7 and expressed in equation (22). Two cases need
to be considered.

(1) $a \geq 0$. In this case we have $E[x-r] \geq 0$[16], and
     therefore a multiplicative shift around zero will
     increase the mean. It will, therefore, have to
     be followed by an additive shift in the negative
     direction for the mean to be restored. For lenders,
     therefore, an increase in dispersion has the same
     effect on consumption as a decrease in the expected
     yield on the risky asset.

(2) $a \leq 0$. Now we know that $E[x-r] \leq 0$, and it follows
     that a multiplicative shift around zero will decrease
     the mean. The mean has, therefore, to be restored by
     means of an additive shift in the positive direction.
     We conclude that for borrowers, an increase in dis-

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15) Developing the expression for expected utility in a
     Taylor series, it is easy to see that it depends on
     the successive moments of x-r.

16) Compare footnote 10 above.

persion has the same effect on consumption as an in-
.crease in the expected yield on the risky asset,
which in this case serves as a debt instrument.

This connection between the effects of changes in
expected yield and in its riskiness ties in nicely with
the more intuitive view that the effect of uncertainty is
to make the "true" interest rates higher than their expected
values[17].

Equation (22) implies that the effect on consumption
of an increase in expected yield on the risky asset is
indeterminate for all a ;   there are always the con -
flicting tendencies of the substitution and income effects.
A fortiori, this will also be the case for increases in
risk.

10. Concluding Remarks

While the model of this paper may be seen as a
generalization and extension of Fisher's theory of saving,
it would be somewhat unfair to its founder not to note
that it is very much in the spirit of his analysis. It
was Fisher who first stressed the need for simultaneous
analysis of saving and investment decisions, and he was
well aware of the problems raised by uncertainty, although
he did not attempt any formal study of these problems.

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17) Drèze and Modigliani [4] have arrived at the same
     result by a different sort of analysis.

We conclude with a few observations on the generality of the results. Increased generality can be achieved in a number of ways, and we shall comment upon two of them only. The first is an extension to more than two periods; the second is to allow for an arbitrary number of risky assets.

To pass judgement on the restrictiveness of the two-period model it is necessary to be quite clear about the kinds of questions one wants to ask. If one's sole interest is in deriving the implications <u>for present decisions</u> of intertemporal allocation under uncertainty, then the two-period assumption would seem to be adequate. On the other hand, if one is interested in the sequential nature of decision-making, a multiperiod approach becomes essential. The concern of the present paper is, of course, with the former type of problem.

Several of the results of this paper hold for the case of an arbitrary number of risky assets. However, this does not apply without reservations to the results based on the particular hypotheses about the risk aversion function, since these do not seem to carry over to the case with more than one risky asset, or, equivalently, to the case where risky assets are not held in constant proportions independent of income. Thus, for instance, without any specific assumptions about the risk aversion function it would still be true that the effect of increased uncertainty is the same as that of a decrease in expected yield on the risky asset. But it would no longer be possible to determine the signs of the substitution and income effects involved.

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THE EFFECT of UNCERTAINTY on SAVING DECISIONS

Two types of uncertainty concerning the future
are examined in this paper; uncertainty with re-
gard to future income and uncertainty as to the
rate of return on capital investment. Assuming
the existence of risk aversion and decreasing
temporal risk aversion (a concept which is defined
in the paper) it is proved that increased riskiness
of future income will increase saving, while in the
case of capital risk the substitution effect calls
for less saving and the income effect for more.
The analysis is briefly related to empirical studies
of the consumption function.

1. Introduction

How does increased uncertainty about the future affect the consumer's choice between saving and immediate consumption? This question has received considerable attention in the literature, although not often of a formal character. Thus, Alfred Marshall [11, p. 226] wrote:

> "The thriftlessness of early times was in great measure due to the want of security that those who made provision for the future would enjoy it: only those who were already wealthy were strong enough to hold what they had saved; the laborious and self-denying peasant who had heaped up a little store of wealth only to see it taken from him by a stronger hand, was a constant warning to his neighbours to enjoy their pleasure and their rest when they could".

In a more recent discussion of the problem Boulding [2, p. 535] writes:

> "Other things being equal, we should expect a man with a safe job to save less than a man with an uncertain job".

At first glance these two statements may seem inconsistent. But closer inspection reveals that Marshall and Boulding do not really discuss the same kind of uncertainty. While Boulding is concerned with uncertainty concerning future non-capital income, Marshall analyzes the effect of an uncertain yield on capital investment. The role of saving in the two cases is fundamentally different. In Boulding's case of income risk, the role of accumulated savings is that of a buffer providing a guarantee that future consumption will not fall below some minimum level. In other words, accumulated savings is the certain component of total resources available for future consumption.

In the Marshallian case of <u>capital risk</u>, however, the
more one saves, the more one stands to lose. Giving up
a dollar's worth of certain present consumption does not
result in a certain increase in future consumption. It
is by no means obvious that these two types of uncertainty
affect saving decisions in the same manner so that it may
still be possible to reconcile the statements of Marshall
and Boulding, both of which appear to have considerable
intuitive appeal.

There are not many examples of formal treatments of
saving decisions under uncertainty. The approach adopted
in the present paper is similar to that of Drèze and
Modigliani [4], Diamond [3] and Leland [10], all of whom
work within a two-period framework without assuming addi-
tivity of the utility function. Additive utility functions
are assumed by Phelps [13], Hakansson [9] and Mirrlees [12],
who work with n-period or infinite-horizon models. As long
as one is not interested in analyzing sequential decisions,
the two-period model would seem to be adequate, while it
has also the advantage of not requiring the assumption of
additivity[1].

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1)   The present paper may be seen as a companion piece
      to [15], which is chiefly concerned with the inte-
      gration of models of saving and of portfolio choice.

2.   The Risk Aversion Function

Important contributions to the theory of choice
under uncertainty have recently been made by Arrow [1]
and Pratt [14], who have introduced the concept of a
risk aversion function. Arrow and Pratt are concerned
with preferences over probability distributions of final
wealth only, expressed in terms of a concave utility
function $W(Z)$, where Z is final wealth. If the risk
premium is defined as the actuarial value of an uncertain
prospect minus its certainty equivalent, it can be shown
that this risk premium is proportional to the function
$-W''(Z)/W'(Z)$, which Arrow [1] calls absolute risk aversion[2].
It seems reasonable to assume that the risk premium should
be decreasing with wealth, because "it seems likely that
many decision makers would feel they ought to pay less for
insurance against a given risk the greater their assets"
[14, p. 123]. We shall now develop a risk aversion function
for temporal risks, i.e. for prospects the outcomes of
which will not be known until after the saving-consumption
decision has been made, and present a temporal version of
the hypothesis of decreasing risk aversion[3].

The consumer is assumed to have a preference ordering
over present and future consumption (C_1, C_2) which can be
represented by a continuous, cardinal utility function,

(1) $U = U(C_1, C_2)$, $C_1, C_2 \geqq 0$,

---

2)   We shall not here be concerned with the relative risk
      aversion function, which is defined as $-W''(Z)Z/W'(Z)$.

3)   The first part of the following analysis parallels
      that of [15].

which is further assumed to possess continuous derivatives
of first, second and third order with first-order deriva-
tives everywhere positive[4].

Suppose now that a consumer is offered a gamble with
vectors of present and future consumption as outcomes.
Let there be two possible outcomes, (C_1, C_2-h) and (C_1, C_2+h)
occuring with equal probability[5]. The expected utility
of the gamble is then

$$\tfrac{1}{2}U(C_1, C_2+h) + \tfrac{1}{2}U(C_1, C_2-h),$$

while the utility of the expected outcome is

$$U(C_1, C_2).$$

Let the risk premium, p , be defined by the equation

$$U(C_1, C_2-p) = \tfrac{1}{2}U(C_1, C_2+h) + \tfrac{1}{2}U(C_1, C_2-h).$$

Multiplying by 2 and subtracting $2U(C_1, C_2)$ on both
sides of this expression, we obtain

$$2\{U(C_1, C_2-p)-U(C_1, C_2)\} = U(C_1, C_2+h) - U(C_1, C_2) + U(C_1, C_2-h)$$
$$- U(C_1, C_2).$$

Here the expression in braces is approximately equal to
$-pU_2(C_1, C_2)$. We now divide by h on both sides to obtain
as an approximation,

---

4)   Derivatives of U will be denoted by subscripts;
      thus $\delta U/\delta C_1 = U_1$, $\delta^2 U/\delta C_1 \delta C_2 = U_{12}$ etc.

5)   h is taken to be a small number, so that this gamble
      conforms to Pratt's definition of an infinitesimal risk.

$$- \frac{2}{h} p U_2(c_1, c_2) = U_2(c_1, c_2) - U_2(c_1, c_2 - h).$$

Once more dividing by h, we get, again as an approximation,

$$- \frac{2}{h^2} p U_2(c_1, c_2) = U_{22}(c_1, c_2),$$

and

(2) $$\frac{2}{h^2} p = - \frac{U_{22}(c_1, c_2)}{U_2(c_1, c_2)} .$$

The right side is twice the risk premium per unit of variance for infinitesimal risks. In order to have risk aversion $(p > 0)$ we must require that $U_{22} < 0$.

The chief complexity introduced by the risk aversion function (2) as compared with that of Arrow and Pratt, is that it is a function of two variables, so that there is no obvious candidate for <u>the</u> concept of decreasing risk aversion. In [15] it has been suggested that the risk aversion function is decreasing in c_2 and increasing in c_1; this hypothesis was shown to lead to sensible results. We now observe that this implies knowledge of the behaviour of the risk aversion function for opposite movements in c_1 and c_2. With reference to fig. 1 it means that, starting from any point c in the indifference map, the risk aversion function decreases with movements in the NW direction and increases with movements in the SE direction. We shall refer to this assumption as the hypothesis of <u>decreasing temporal risk aversion</u>.

It should be stressed that this hypothesis about the risk aversion function is a restriction on the utility function and should be interpreted solely in terms of
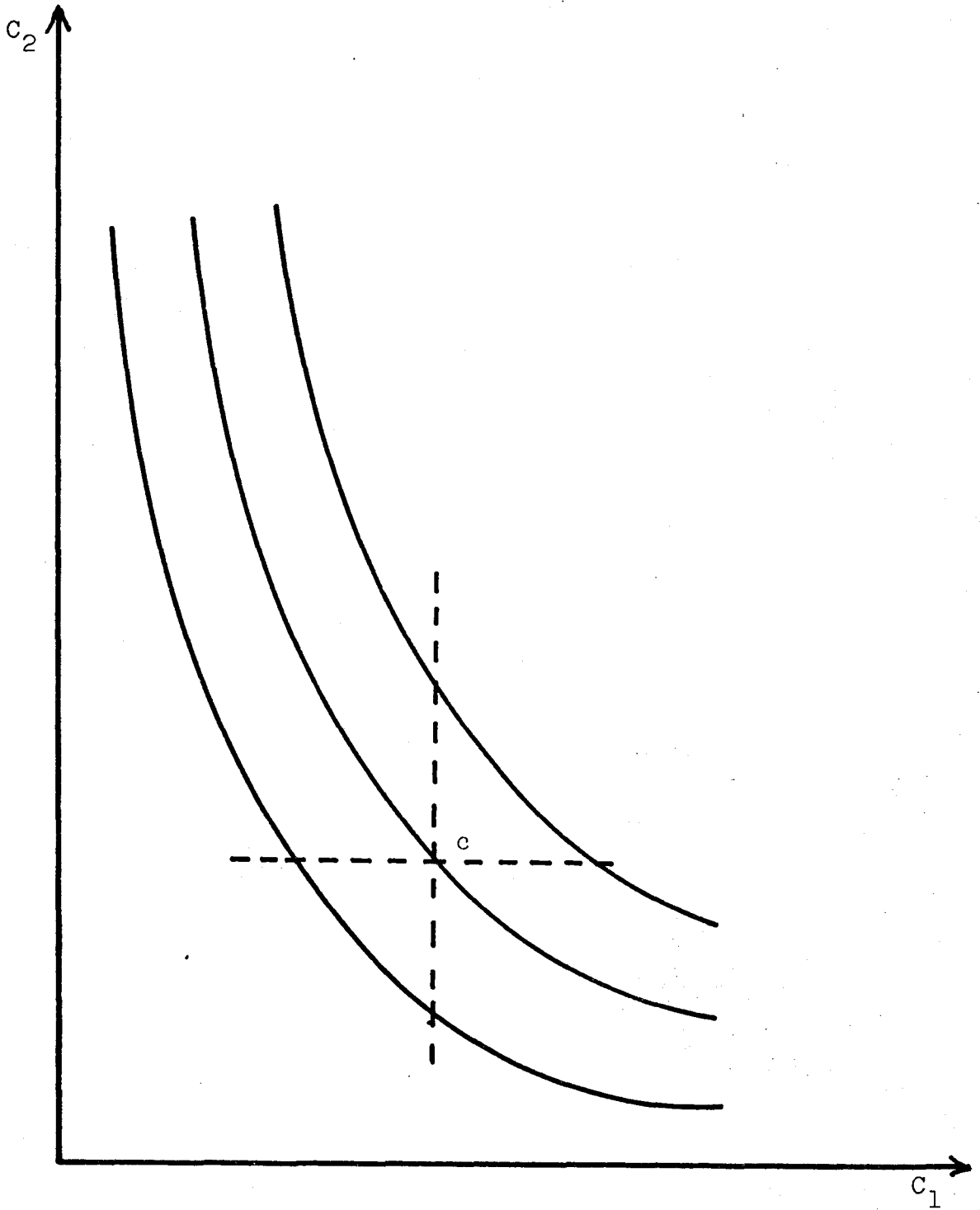
Fig. 1

properties of the preference ordering, independently of
the budget constraint of the particular problem discussed
here. The interpretation is as follows: Suppose a con-
sumer "owns" a consumption vector $c = \{c_1, c_2\}$ and is
offered a gamble where the two possible outcomes are
-h and h of future consumption. He is asked to give
the odds on which he will accept the gamble. Under risk
aversion we know that the odds will be "better than fair";
thus, if $\pi(h)$ is the probability of a gain of h, we
know that the consumer will demand $\pi(h) > \frac{1}{2}$ in order to
accept the gamble. It is reasonable to assume that $\pi(h)$
will be lower, the higher is c_2; this suggests itself
as a natural extension of the Arrow-Pratt assumption.
Likewise, it seems attractive to assume that $\pi(h)$ will
be higher, the higher is c_1; a higher level of present
consumption makes the consumer less inclined to gamble on
the value of future consumption. A fortiori it follows
that $\pi(h)$ will fall with a simultaneous increase in c_2
and decrease in c_1, and that it will rise with a simul-
taneous decrease in c_2 and increase in c_1.[6]

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6) An alternative interpretation of the hypothesis, which
will emerge from the discussion below, is the following:
For any consumption vector $\{c_1, c_2\}$ we may compute its
expected present value as
$$c_1 + (1+r)^{-1} c_2.$$

If c_2 is increased and c_1 is decreased so as to hold
the present value constant, the risk aversion function
will decrease. The hypothesis of decreasing temporal
risk aversion implies that this will be true for all
values of r. Following this interpretation, the hypothe-
sis might alternatively have been denoted "decreasing
risk aversion along a budget line".

Leland's hypothesis [10] is that the risk aversion
function decreases with movements to the NW along an
indifference curve. In the neighbourhood of the opti-
mum these measures will be approximately the same.
Indeed, Leland relies on a Taylor expansion to establish
his result.

So much for interpretation. The hypothesis of decreasing temporal risk aversion can now be written as

$$(3) \quad d \{- \frac{U_{22}}{U_2}\} = \frac{\delta}{\delta C_1} \{- \frac{U_{22}}{U_2}\} (-dC_1) + \frac{\delta}{\delta C_2} \{- \frac{U_{22}}{U_2}\} dC_2 < 0.$$

Without loss of generality we may write

$$dC_2 = (1+r) dC_1,$$

where $(1+r)$ is some nonnegative real number. (3) can then be written as

$$(4) \quad \frac{d}{dC_1} \{- \frac{U_{22}}{U_2}\} = - \frac{\delta}{\delta C_1} \{- \frac{U_{22}}{U_2}\} + (1+r) \frac{\delta}{\delta C_2} \{- \frac{U_{22}}{U_2}\} < 0$$

for all values of $(1+r) \geq 0$.

We now observe that under our continuity assumption the following holds as an identity.

$$\frac{\delta}{\delta C_1} \{- \frac{U_{22}}{U_2}\} = \frac{\delta}{\delta C_2} \{- \frac{U_{12}}{U_2}\}.$$

The inequality in (4) can now be written as

$$(5) \quad \frac{\delta}{\delta C_2} \{ \frac{U_{12} - (1+r) U_{22}}{U_2} \} < 0.$$

This result will prove helpful in the following discussion of the effects of uncertainty.

It is easy to see that if the utility function is additive, the risk aversion function will depend on C_2

only, and the assumption that risk aversion is a decreasing function of C_2 is then sufficient to establish the results derived in the following sections.

3. Income Risk

In this section we shall discuss the effects of increased riskiness of future income on present consumption. The first-period budget constraint facing the consumer is

$$(6) \qquad Y_1 = C_1 + S_1,$$

where Y_1 is income in the first period, assumed to be known with certainty, and S_1 is saving. Future consumption is given by

$$(7) \qquad C_2 = Y_2 + S_1(1+r),$$

where r is the rate of interest, which is assumed to be known in this case of pure income risk, and Y_2 is future income, which is not known in period 1. The consumer's beliefs about the value of future income can be summarized in a subjective probability density function $f(Y_2)$ with mean ξ ; on the basis of this the consumer maximizes expected utility in the von Neumann-Morgenstern sense.

Combining (6) and (7) we can write

$$(8) \qquad C_2 = Y_2 + (Y_1 - C_1)(1+r).$$

Expected utility can the be written as

$$E[U] = \int U(C_1, Y_2 + (Y_1 - C_1)(1+r)) \, f(Y_2) \, dY_2,$$

where integration is over the range of Y_2. Maximizing with respect to C_1, we obtain the first-order condition

(9) $\qquad E[U_1 - (1+r) U_2] = 0,$

and the second-order condition

$$D = E[U_{11} - 2(1+r) U_{12} + (1+r)^2 U_{22}] < 0.$$

The effect of an increase in income (Y_1) can be found by implicit differentiation in (9):

$$\frac{\delta C_1}{\delta Y_1} = -(1+r) \frac{E[U_{12} - (1+r) U_{22}]}{D}.$$

The sign of this derivative cannot be determined a priori, but in the following we shall assume that it is always positive, both under certainty and uncertainty, which implies that

(10) $\qquad U_{12} - (1+r) U_{22} > 0, \qquad E[U_{12} - (1+r) U_{22}] > 0.$

We now wish to examine the effect on present consumption of an increase in the degree of risk concerning future income. This raises the problem of how to measure the "degree of risk" without adopting the rather restrictive mean-variance approach. One solution to this problem, used by Leland [10], is to expand (9) around (Y_1, ξ) ; one then obtains an expression containing the variance of Y_2. Here we shall take a more direct approach.

One can examine two kinds of shift in the probability distribution of Y_2. One is an additive shift, which is equivalent to an increase in the mean with all other moments constant. The other is a multiplicative shift, by which the distribution is "stretched" around zero.[7] A pure increase in dispersion can be defined as a stretching of the distribution around a constant mean. This is equivalent to a combination of additive and multiplicative parameter changes.

Let us write future income as

(11) $\qquad \gamma Y_2 + \theta$,

the expected value of which is

$$E[\gamma Y_2 + \theta].$$

Here γ is the multiplicative shift parameter, and θ is the additive one. Because of the nonnegativity of Y_2 a multiplicative shift around zero will increase the mean. It must, therefore, be counteracted by an additive shift in the negative direction, so that the expected value is held constant. Taking the differential, the requirement is that

$$dE[\gamma Y_2 + \theta] = E[Y_2 d\gamma + d\theta] = 0,$$

which implies that

---

7) Since Y_2 is most naturally interpreted as a nonnegative number, the distribution will really be stretched only on the right side of zero.

$$\frac{d\theta}{d\gamma} = -E[Y_2] = -\xi. \quad [8)$$

We can now substitute (11) into the first-order condition (9) and differentiate with respect to γ. We then obtain

$$(12) \qquad \left(\frac{\delta C_1}{\delta \gamma}\right)_{\frac{\delta\theta}{\delta\gamma}=-\xi} = -\frac{1}{D} E[(U_{12}-(1+r)U_{22})(Y_2-\xi)].$$

It can be shown that decreasing temporal risk aversion is a sufficient condition for this derivative to be negative, so that <u>increased uncertainty about future income decreases consumption (increases saving)</u>.

<u>Proof</u>:   We first define

$$\bar{C}_2 = (Y_1-C_1)(1+r) + \xi.$$

---

8)   A numerical illustration is perhaps in order at this point.   Let there be

Y_2^1	Y_2^2	Mean	Variance
10	20	15	25
12	24	18	36
9	21	15	36

two possible values of future income, Y_2^1 and Y_2^2, occuring with equal probability.   Initially we have $Y_2^1 = 10$ and $Y_2^2 = 20$ with mean and variance as given in the first line of the table.   Multiplying Y_2^1 by 1.2 increases the variance, but it also increases the mean, as shown in the second line.   We can now restore the mean to its original value by subtracting 3 from each Y_2^1 in the second line.   By a combination of a positive multiplicative shift and a negative additive shift, we have obtained an increase in the variance with the mean constant.

From (8) we have that

$$C_2 = \bar{C}_2 + Y_2 - \xi.$$

Because $(U_{12}-(1+r)U_{22})/U_2$ is decreasing in C_2, we must have that

$$(13) \qquad \frac{U_{12}-(1+r)U_{22}}{U_2} \leqq \left\{\frac{U_{12}-(1+r)U_{22}}{U_2}\right\}_\xi \qquad \text{if } Y_2 \geqq \xi.$$

The right side of this inequality is evaluated at $C_2 = \bar{C}_2$ and is not a random variable.

Obviously

$$(14) \qquad U_2(Y_2-\xi) \geqq 0 \qquad\qquad \text{if } Y_2 \geqq \xi.$$

We now multiply on both sides of (13) by $U_2(Y_2-\xi)$. We then obtain

$$(U_{12}-(1+r)U_{22})(Y_2-\xi) \leqq \left\{\frac{U_{12}-(1+r)U_{22}}{U_2}\right\}_\xi \cdot U_2(Y_2-\xi)$$

$$\text{if } Y_2 \geqq \xi.$$

Taking expected values on both sides we have that

$$(15) \quad E[(U_{12}-(1+r)U_{22})(Y_2-\xi)] \leqq \left\{\frac{U_{12}-(1+r)U_{22}}{U_2}\right\}_\xi E[U_2(Y_2-\xi)].$$

We now observe that if $Y_2 \leqq \xi$, inequalities (13) and (14) will both be reversed, so that (15) holds for all Y_2.

To prove that the left side of (15) is negative, it is sufficient to show that the right side is negative.[9] From (10) the expression in braces is positive, so that we have to show that $E[U_2(Y_2-\xi)] \leqq 0$. Since $U_{22} < 0$, we must have

$$(16) \qquad U_2 \leqq (U_2)_\xi \qquad\qquad \text{if } Y_2 \geqq \xi.$$

Trivially,

$$(17) \qquad Y_2 - \xi \geqq 0 \qquad\qquad \text{if } Y_2 \geqq \xi.$$

Multiplying in (16) by $(Y_2-\xi)$ we can write

$$U_2(Y_2-\xi) \leqq (U_2)_\xi(Y_2-\xi).$$

This holds for all Y_2, since inequalities (16) and (17) are both reversed if $Y_2 \leqq \xi$. Taking expectations, we obtain

$$E[U_2(Y_2-\xi)] \leqq (U_2)_\xi E[Y_2-\xi] = 0,$$

which implies

$$E[(U_{12}-(1+r)U_{22})(Y_2-\xi)] \leqq 0.$$

Therefore, since $D < 0$, it follows that the derivative (12) is negative. Q.E.D.[10]

---

9) At this point it is clear that the proof is one of sufficiency, not necessity.

10) It may be of interest to record that in the case of the quadratic utility function $k_1C_1+k_2C_2+k_{12}C_1C_2+k_{11}C_1^2+k_{22}C_2^2$ present consumption is independent of the variance of future income. This function can easily be shown to display _increasing_ temporal risk aversion.

Our analysis thus confirms Boulding's conjecture that increased uncertainty about future income leads to more saving. It is tempting at this point to relate the result to empirical studies of saving behaviour, but this will be reserved for the final section of the paper.

4.   Capital Risk

We now turn to a stylized version of Marshall's "laborious and self-denying peasant". In the first period he can allocate his resources (Y_1) between present consumption (C_1) and capital investment (K):

$$Y_1 = C_1 + K.$$

In general, capital investment is transformed into resources available for future consumption by means of a transformation function $F(K,x)$, where x is a stochastic parameter. We shall assume that the transformation function is of the following simple form:

$$C_2 = K(1+x), \qquad\qquad 1+x \geqq 0,$$

with x as the random rate of return on capital. x may conceivably take on the value -1, in which case one may suppose that the peasant's wealth is "taken from him by a stronger hand"; this represents the lower bound on the range of x.

Combining these two equations, we have that

$$C_2 = (Y_1 - C_1)(1+x).$$

Expected utility is then

$$E[U] = \int U(C_1, (Y_1 - C_1)(1+x))g(x)dx,$$

where $g(x)$ is the subjective density function of x and integration is over the range of x.

Necessary and sufficient conditions for a maximum of $E[U]$ are

$$(18) \qquad E[U_1 - (1+x)U_2] = 0,$$

$$(19) \qquad H = E[U_{11} - 2(1+x)U_{12} + (1+x)^2 U_{22}] < 0.$$

To examine the effect of a pure increase in risk, we proceed exactly as in the preceding section. Writing the yield on capital as $\gamma x + \theta$, we find that for a multiplicative shift around zero to keep the mean constant, we must have

$$dE[\gamma x + \theta] = 0,$$

i.e.

$$\frac{d\theta}{d\gamma} = -\mu,$$

where $\mu = E[x]$.

Differentiating (18) with respect to γ and evaluating the derivative at $(\gamma = 1, \theta = 0)$ we obtain

$$(20) \qquad \left(\frac{\delta C_1}{\delta \gamma}\right)_{\frac{\delta\theta}{\delta\gamma} = -\mu} = -\frac{1}{H} KE[(U_{12} - (1+x)U_{22})(x-\mu)] + \frac{1}{H}E[U_2(x-\mu)].$$

Here the first term is the income effect and the second term is the substitution effect, i.e. the second term is the value of the derivative with E[U] constant.

It can be shown that the <u>existence of risk aversion is a necessary and sufficient condition for the substitution effect to be positive</u>. <u>The additional assumption of decreasing temporal risk aversion is sufficient for the income effect to be negative</u>.[11] The total effect cannot be determined without additional assumptions.

<u>Proof</u>: The proofs of these assertions are very similar to the proof that the derivative (12) is negative, as presented in section 3. Consequently, we shall only sketch the proofs, the details of which will be evident from the previous one.

Define

$$c_2^{\ O} = K(1+\mu)$$

so that

$$c_2 = c_2^{\ O} + K(x-\mu).$$

It is now straightforward to prove that $E[U_2(x-\mu)] \leq 0$. This is done by writing down inequalities similar to (16) and (17) and taking into account that $U_{22} < 0$. The substitution effect is, therefore, positive. Under decreasing temporal risk aversion we must have that

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11) It is also assumed, for the latter result, that present consumption is a normal good.

$$\frac{U_{12}-(1+x)U_{22}}{U_2} \leqq \{\frac{U_{12}-(1+x)U_{22}}{U_2}\}_\mu \qquad \text{if } x \geqq \mu.$$

Multiplying by $U_2(x-\mu)$ on both sides, and taking expectations, it is easy to see that, for all x,

$$E[(U_{12}-(1+x)U_{22})(x-\mu)] \leqq \{\frac{U_{12}-(1+x)U_{22}}{U_2}\}_\mu E[U_2(x-\mu)].$$

The factor in braces is a constant, because it is evaluated at μ, and it is positive because of the non-inferiority of present consumption. But $E[U_2(x-\mu)]$ has been shown to be negative. Hence

$$E[(U_{12}-(1+x)U_{22})(x-\mu)] \leqq 0,$$

and the income effect is accordingly positive.

The intuitive interpretation of the result is fairly simple. An increase in the degree of risk makes the consumer less inclined to expose his resources to the possibility of loss; hence the positive substitution effect on consumption. On the other hand, higher riskiness makes it necessary to save more in order to protect oneself against very low levels of future consumption. This explains the negative income effect on consumption.

How does all this tie in with Marshall's hypothesis? Presumably we should judge him to be correct on his own terms. His statement that increased capital risk will increase present consumption may be seen as amounting to neglect of the income effect, which is what Marshall gener-

ally practiced in his analysis of demand. (See e.g. Friedman's interpretation in [7].) However, a more complete analysis must take account of the income effect, so that it is no longer possible to arrive at any clear conclusion in the general case.

Some readers may feel that the point of Marshall's simple story has been pressed too far. The analysis of the present section is a study of technological uncertainty, in which the rate of return on capital is a continuously distributed random variable. In contrast, Marshall may be concerned with the case where the rate of return on capital is either some positive number $\bar{x}$ with probability p and -1 with probability 1-p. In that case our interpretation of the income effect may not make much sense. Whatever the correct interpretation - and this paper is not chiefly concerned with what Marshall really meant - there is sufficient similarity between his case and ours to consider his comments as being concerned with the effect of capital risk in general.

In analyzing the effect of capital risk it is sometimes desirable to allow for asset choice, so that the consumer may react to change in riskiness by a reallocation among assets. A model along these lines has been studied in [15]. However, the present analysis is not necessarily a step backward. The one-asset model may be of considerable relevance for many real-world problems, since many types of increases in riskiness will apply to the yield on all assets, so that the possibility of hedging against risk by portfolio rearrangements are limited. Moreover, for society as a whole, real capital constitutes the only form that saving can take (at least in a closed economy); the present model may, therefore, be seen as a simplified analysis of optimal growth under uncertainty.

5. Empirical Aspects of the Analysis

In this section we shall comment on some broader implications of the theory presented above. In recent years there has been a great deal of work done on reconciling observed saving behaviour with theories based on utility maximization over time. However, most of the theoretical work[12] has been incomplete in taking no account of the uncertainty of future income and/or the rate of return on savings.

An interesting component of Friedman's permanent income hypothesis [8] is his distinction between permanent and transitory income changes and his hypothesis that the propensity to consume is lower for transitory than for permanent changes in income. This hypothesis can in fact be derived from utility analysis.[13] Assuming that the consumer maximizes $U(C_1, C_2)$ subject to $C_2 = Y_2 + (Y_1 - C_1) \cdot (1+r)$, and provided that there is no uncertainty, the consumption demand function can be written as

$$C_1 = f(V, r), \qquad 0 < f_V < 1,$$

where

$$V = Y_1 + \frac{1}{1+r} Y_2.$$

---

12) Leland [10] has an interesting discussion of the empirical implications of his analysis.

13) The interpretation of this hypothesis has been the subject of some disagreement in the literature. Rather than follow the "extreme" view of Friedman, I chose to adopt the position of Eisner [5].

V is the lifetime income of the consumer. The main point
of Friedman's argument is that changes in current income
influence consumption only through their effect on life-
time income.

The response of consumption to a permanent change in
income is

$$(dC_1)_p = f_v(dY_1 + \frac{1}{1+r} dY_2),$$

and to a transitory change it is

$$(dC_1)_t = f_v dY_1.$$

Hence the marginal propensities to consume are, respectively,

$$(\frac{dC_1}{dY_1})_p = f_v(1 + \frac{1}{1+r} \cdot \frac{dY_2}{dY_1}),$$

$$(\frac{dC_1}{dY_1})_t = f_v,$$

and it is obvious that $(dC_1/dY_1)_p > (dC_1/dY_1)_t$.

If we assume $dY_2/dY_1 = 1$, the former propensity would
be approximately twice the latter for small r; for a
three-period model, which Friedman considers to be reason-
able, it would be approximately three times as large.[14]
If now data from time series show the marginal propensity
to consume to be of the order of 0.9, it seems natural
to see this as an estimate of $(dC_1/dY_1)_p$, and to adopt
$\frac{1}{n} \cdot 0.9$ as the theoretical prediction of $(dC_1/dY_1)_t$
when r is small and the horizon is not too long.

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14) See Eisner's paper ([5], p. 974n.) for an elaboration
of this interpretation.

What is left out here is, of course, that while a change in present income is a hard fact to the individual consumer, the associated change in future consumption is only a belief not held with certainty. If higher present income is associated not only with an increase in expected future income but also in its riskiness, this will tend to bring $(dC_1/dY_1)_p$ and $(dC_1/dY_1)_t$ closer to each other than predicted by the certainty model, because increased riskiness in itself will tend to decrease consumption. While it may be hard to say, in general, whether high incomes are more or less risky than low ones, this consideration should be taken into account when testing the Friedman-Eisner hypothesis against empirical data.[15]

If one does accept the hypothesis that riskiness of income is greatest for high incomes, and if "high" refers to a comparison with the average level of income in any time period, then we have an alternative explanation of the discrepancy between time series and cross-section consumption functions.[16] As all incomes increase over

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15)   There is, however, some evidence for the hypothesis that riskiness of income increases with income in a cross-section material. Eisner ([5],p. 976) reports that the variance of income for "salaried professionals, officials etc." was about twice that of "clerical and sales workers" and 4 to 5 times that of "wage earners" in American cross-section data for 1950. The evidence is not conclusive, since variability of cross-section incomes does not per se imply uncertain income expectations for the individual consumer. However, one might conjecture that individual beliefs about future income are to some extent conditioned by cross-section income patterns.

16)   This has also been pointed out by Leland [10].

time, with relative income positions approximately constant,
any given level of income will fall relatively to the aver-
age level of income and will, therefore, presumably become
less risky. Because of this we would expect more con-
sumption with a given absolute income the higher is the
general income level.

It has often been observed that there is a significant
difference in saving behaviour between wage and salary
earners on the one hand. and self-employed persons on the
other.[17)] Moreover, it is generally accepted that the
latter group, farmers and businessmen, have more variable
incomes than the former. On the reasonable assumption
that ex post variability goes together with ex ante
uncertainty, theoretical considerations should lead us to
expect the self-employed group to save more, and this con-
clusion appears in fact to be supported by empirical research.

However, some care should be taken in identifying empiri-
cal and theoretical results at this point. As far as
reactions to income uncertainty is concerned, comparison
should be restricted to consumers with incomes that are
exogeneous, i.e. independent of their own saving behaviour.
As regards self-employed persons, however, their future
income may depend in an essential way on how much they
save in the present, so that a comparison between these
two groups would rather constitute a test of the effect of
capital uncertainty. But as regards that effect, theory
does not offer any clearcut hypothesis. This is no less

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17)   For a survey and references see Farrell [6].

true if the effect of capital risk is studied in a
two-asset model. In that case it has been shown [15]
that the effect of an increase in risk is the same as
that of a fall in the expected rate of return on the
risky asset[18], the sign of which cannot in general be
determined.

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18)   This conclusion can easily be extended to the
      case of an arbitrary number of risky assets.

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EQUILIBRIUM and EFFICIENCY
in LOAN MARKETS

This paper explores the connection between
competitive equilibrium and Pareto optimality in a
two-period consumption-loans model. It is shown that
an ordinary loan market achieves only a constrained
Pareto optimum, and the nature of the constraint is
identified. An unconstrained Pareto optimum is ob-
tained in a regime of state contingent claims. A
third alternative regime of state contingent rates
of return is also considered.

1.   Introduction

One of the main results of the modern theory of
general equilibrium and welfare economics is that a com-
petitive equilibrium results in a Pareto optimal allocation
of resources.   In a pure exchange system this means that
it is not possible to effect a redistribution of commodi-
ties, which is such that every individual in the economy
prefers the new allocation to the initial competitive one.

The purpose of this paper is to examine the con-
nection between market equilibrium and Pareto optimality
in a two-period consumption loans model, where the future
incomes of consumers are not known with certainty.   It is
shown that an ordinary loan market is generally inefficient,
in the sense that the competitive equilibrium is a
constrained Pareto optimum.   A full Pareto optimum can
be realized by a system of state contingent claims,
similar to that discussed by Arrow [1].   It is further
shown that an alternative system of state contingent
rates of return lacks a unique equilibrium;   thus, it may
but need not realize the full conditions for a Pareto
optimum.

The conclusion that state-contingent claims or
commodities are required to reach an unconstrained Pareto
optimum under uncertainty is implicit in the work of Arrow
[1] and Debreu [3], and it has since been developed in
different contexts by Borch [2, Chap. VIII] and Diamond [4].
In the present paper an attempt is made to show explicitly
why it is that a regime which is known to lead to a Pareto
optimum under certainty, fails to do so when uncertainty
is introduced.   It also seems to be an advantage to frame

the discussion in an intertemporal context[1].

In section 2-5 the main results of the paper will be presented in the form of an example. Generalizations are provided in section 6, while the final section contains some more general remarks suggested by the analysis.

2. The Inefficiency of Loan Markets

We consider an economy with two persons. Each of them owns an income profile (y_1, y_2), and acts so as to maximize a cardinal utility function with consumption in each of the two periods (c_1, c_2) as its arguments. Their income prospects are as follows:

Person 1 has an income of 2 in period 1 and is assured of receiving the same amount in period 2.

Person 2 will receive an income of 1 in period 1, while his income in period 2 will be either 1 or 4, each with probability $\frac{1}{2}$.

We may think of each value of person 2's future income as being derived from a certain "state of the world"; thus, he will get 1 if state θ_1 occurs, and 4 if θ_2 occurs. We may summarize this in the following table:

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1) The basic model bears some resemblance to Samuelson's [5], but it ignores the intergeneration aspects which are central in his analysis.

Person	y_1	θ_1	y_2	θ_2
1	2	2		2
2	1	1		4

Economic theory tells us that by borrowing and lending, our two persons would become able to realize more preferred consumption patterns, and it also tells us that it would be wise to institutionalize this in the form of a loan market with a rate of return r, which is competitively established[2]. Let x denote the amount lent by person 1 (and borrowed by person 2). The consumption pattern established in the market will then be

Person	c_1	θ_1	c_2	θ_2
1	2-x	2+rx		2+rx
2	1+x	1-rx		4-rx

Note that r is interpreted as the gross rate of return, i.e. as one plus the interest rate on loans.

We assume now that both persons have the same cardinal utility function over consumption profiles:

$$(1) \quad u(c_1,c_2) = 72 - (6-c_1)^2 - (6-c_2)^2,$$

$$0 < c_1 < 6, \quad 0 < c_2 < 6.$$

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2) The two-person assumption is not to be taken literally; we may conveniently think of them as groups of identical persons.

For each given r, person 1 will prefer the value of $x = x^1$, which maximizes his expected utility U_1:

$$U_1 = u(2-x, \; 2+rx) = 72 - (4+x)^2 - (4-rx)^2.$$

His supply of funds is determined by the condition

$$\frac{dU_1}{dx} = -2(4+x) + 2(4-rx)r = 0,$$

which gives

$$x^1 = \frac{4r-4}{1+r^2}.$$

It is easy to see that person 1 will be a lender or a borrower according as $r \gtrless 1$.

Person 2 will prefer the value of $x = x^2$, which maximizes

$$U_2 = \tfrac{1}{2}u(1+x, \; 1-rx) + \tfrac{1}{2}u(1+x, \; 4-rx)$$

$$= 72 - (5-x)^2 - \tfrac{1}{2}(5+rx)^2 - \tfrac{1}{2}(2+rx)^2.$$

We must then have

$$\frac{dU_2}{dx} = 2(5-x) - (5+rx)r - (2+rx)r = 0,$$

which yields the solution

$$x^2 = \frac{10 - 7r}{2(1+r^2)}.$$

Whether x^2 will be positive or negative depends on whether $r \lessgtr \frac{10}{7}$.

In equilibrium, supply and demand must be equal, i.e. $x^1 = x^2 = x$; this condition determines r. We find easily that the equilibrium rate of return is

$$r = \frac{6}{5} ,$$

which corresponds to an interest rate of 20 per cent on loans. The corresponding value of x is

$$x = \frac{20}{61} ,$$

which means that in equilibrium person 1 lends an amount of $\frac{20}{61}$ units of income to person 2. The resulting utility levels will be

$$U_1 = 40.26,$$
$$U_2 = 32.76.$$

If there had been no borrowing and lending $(x^1 = x^2 = 0)$, the corresponding utility levels would have been 40 and 32.5, respectively. Thus, not surprisingly, both persons have gained from trade.

Are these gains really as large as they could be ? This amounts to asking whether the achieved solution is really a Pareto optimum; and we shall show that this is not the case. To see this we have to determine the optimal transfers of income between persons 1 and 2 :

Person	c_1	θ_1	c_2	θ_2
1	2-a	2+b_1		2+b_2
2	1+a	1-b_1		4-b_2

We are interested in determining the set of allocations which is such that, given any persons's utility level, the other person's utility is maximized. We can determine this set by maximizing the expression

(2) $V = U_1 + kU_2,$ $k > 0.$

By letting k vary, we obtain all Pareto optimal allocations, conversely, to any Pareto optimal allocation there corresponds a value of k.

Expected utilities are

$$U_1 = \tfrac{1}{2}u(2-a,\ 2+b_1) + \tfrac{1}{2}u(2-a,\ 2+b_2)$$

$$= 72 - (4+a)^2 - \tfrac{1}{2}(4-b_1)^2 - \tfrac{1}{2}(4-b_2)^2,$$

$$U_2 = \tfrac{1}{2}u(1+a,\ 1-b_1) + \tfrac{1}{2}u(1+a,\ 4-b_2)$$

$$= 72 - (5-a)^2 - \tfrac{1}{2}(5+b_1)^2 - \tfrac{1}{2}(2+b_2)^2.$$

We wish to find the maximum of

$$V(a,b_1,b_2) = U_1(a,b_1,b_2) + kU_2(a,b_1,b_2).$$

This is obtained by solving

$$\frac{\delta V}{\delta a} = -2(4+a) + 2k(5-a) = 0,$$

$$\frac{\delta V}{\delta b_1} = 4-b_1 - k(5+b_1) = 0,$$

$$\frac{\delta V}{\delta b_2} = 4-b_2 - k(2+b_2) = 0.$$

The solution is the following:

$$(3) \quad \begin{cases} a &= \dfrac{5k - 4}{k + 1}, \\[2mm] b_1 &= \dfrac{4 - 5k}{k + 1}, \\[2mm] b_2 &= \dfrac{4 - 2k}{k + 1}. \end{cases}$$

Expected utilities can now be expressed in terms of k as follows:

$$U_1 = 72 - 139.5 \left(\frac{k}{1+k} \right)^2,$$

$$U_2 = 72 - 139.5 \left(\frac{1}{1+k} \right)^2.$$

Eliminating k, we obtain an expression for the efficiency frontier, which can be written as

$$(4) \quad U_2 = \sqrt{558(72-U_1)} + U_1 - 139.5.$$

This equation gives the maximum possible utility for person 2 for any given utility of person 1. We now substitute the value for U_1 obtained under the loan market solution, viz. $U_1 = 40.26$. This gives $U_2 = 33.84$ as contrasted with 32.76 under the loan market arrangement; therefore, the market allocation has been shown to be suboptimal. The utility enjoyed by person 2 under the loan market mechanism is not the maximum possible utility, given the utility of person 1.

3.   The Loan Market Allocation as a Constrained
      Pareto Optimum

The set of Pareto optimal allocations was derived
above as an unconstrained maximum of the function V in
(1) for any value of· k.   Or, more precisely, the only
constraint used in deriving the efficient set of allo-
cations, was that the sum of the two persons' consumption
must equal total resources in every period and every state
of the world.   In deriving the efficiency locus, a set of
transfer payments is also determined.· For any value of k,
we can determine the values of the payments (a, b_1, b_2),
which are necessary to achieve the optimum.

We can now show that the market allocation is a
solution to the constrained Pareto optimum defined as
follows:

(5) $\text{Max}_{a,b_1,b_2} V(a,b_1,b_2)$

subject to the condition

(6) $b_1 = b_2.$

Denoting the common value of b_1 and b_2 by b,
we can write our maximum conditions as

$$\frac{\delta V}{\delta a} = -2(4+a) + 2k(5-a) = 0,$$

$$\frac{\delta V}{\delta b} = 2(4-b) - k(5+b) - k(2+b) = 0.$$

This gives the solution

$$a = \frac{5k - 4}{1 + k} \, ,$$

$$b = \frac{8 - 7k}{2(1+k)} \, .$$

The expected utility levels become

$$U_1 = 72 - 137.25 \left(\frac{k}{1+k}\right)^2 \, ,$$

$$U_2 = 69.75 - 137.25 \left(\frac{1}{1+k}\right)^2 .$$

By eliminating k we can again derive the efficiency frontier as

$$(7) \qquad U_2 = \sqrt{549(72-U_1)} + U_1 - 139.5.$$

Setting $U_1 = 40.26$, which was person 1's utility under the loan market regime, we obtain $U_2 = 32.76$, which was the corresponding utility level of person 2.

Visualizing the efficiency frontier as a curve, the efficiency frontier (7) lies below that defined by (4) for unconstrained Pareto optima. This can be seen directly as follows. Let $U_1 = \bar{U}_1$, a constant. Let us denote the value of U_2 obtained under unconstrained maximization as U_2^F and the corresponding utility level under constrained maximization as U_2^C. From (4) and (7) we easily obtain

$$U_2^F - U_2^C = 0.19 \sqrt{72 - \bar{U}_1}.$$

But because $U_1 < 72$, this expression is necessarily positive.

Geometrically, the result may be illustrated as in Fig. 2, where the initial utility level has been chosen
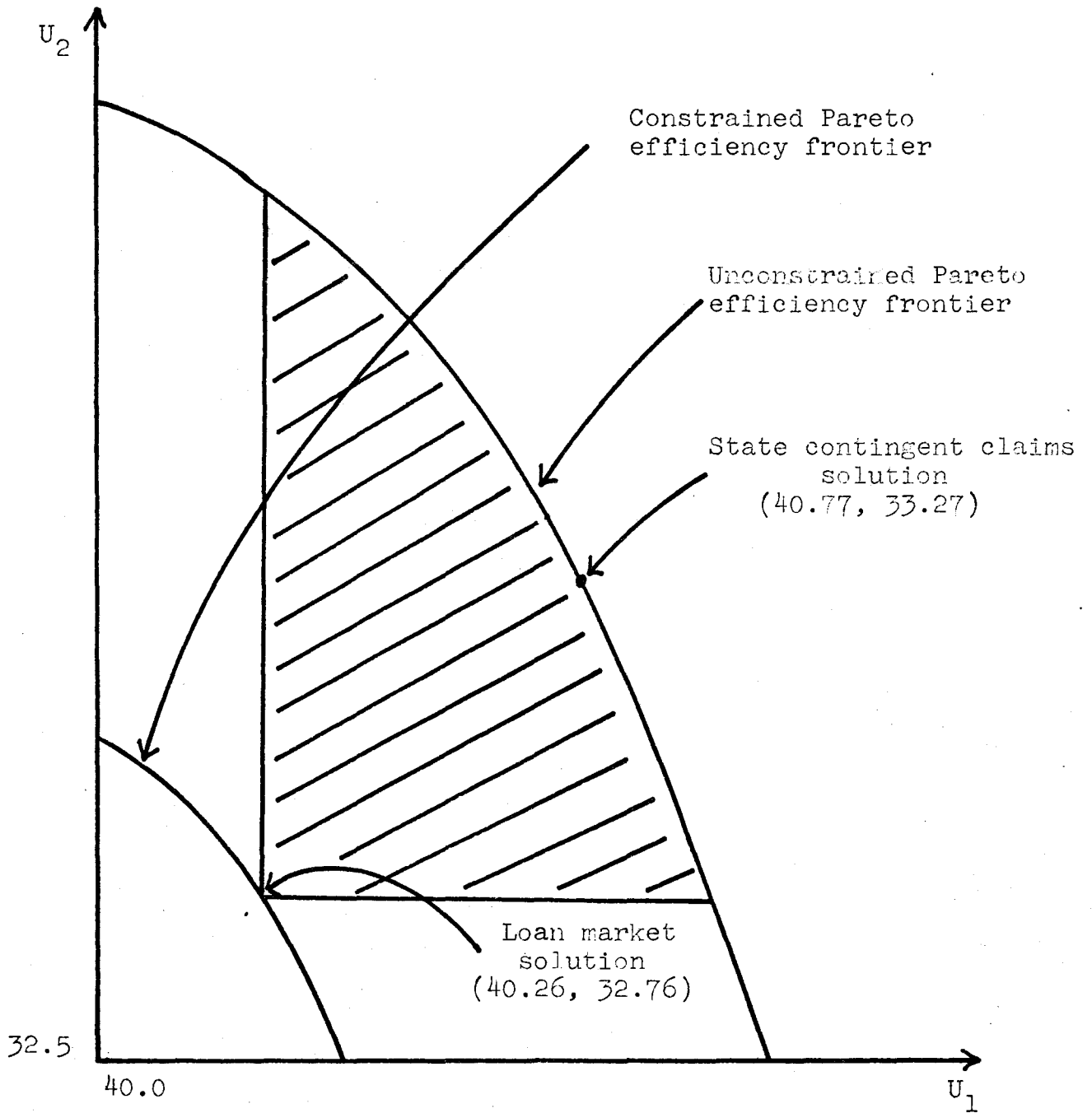
Constrained Pareto
efficiency frontier

Unconstrained Pareto
efficiency frontier

State contingent claims
solution
(40.77, 33.27)

Loan market
solution
(40.26, 32.76)

U_2

U_1

32.5

40.0

Fig. 2

as the origin. The loan market regime results in an allo-
cation on the constrained efficiency frontier, which is
clearly better than the initial allocation. However,
within the constraints set by the income profiles there
is a set of allocations which is preferred by both con-
sumers, and which the loan market mechanism is unable to
achieve.

The fundamental reason why the market fails, is that
it is only able to shift resources between time periods
but not between states of the world within periods. In
our example person 1 bears no risk initially, nor does
the market transfer to him some of the risk born by person 2.
But the Pareto optimal solution shows that it will be to
both persons' advantage to effect such a transfer of risk.
Person 2 will be willing to forego some of his total ex-
pected income in order to reduce the variability in his
second-period consumption if he is sufficiently well compen-
sated in the form of higher expected income. But these
risk preferences cannot be accommodated under the loan
market regime.

4.   State Contingent Claims

Since the loan market mechanism is unable to come
up with a Pareto optimal allocation, it is natural to look
for other forms of market organization and evaluate their
performance. One such alternative organization consists
in establishing markets for state contingent claims ;
certificates paying one unit of income if and only if a
specific state of the world occurs. We shall show that

such a regime is indeed able to achieve an allocation of consumption which is an unconstrained Pareto optimum.

Let z_s be the number of θ_s-certificates $(s=1,2)$ bought by person 1 and sold by person 2, and let p_s be their respective prices. The consumption patterns will then look as follows:

Person	c_1	θ_1	c_2	θ_2
1	$2-p_1z_1-p_2z_2$	$2+z_1$		$2+z_2$
2	$1+p_1z_1+p_2z_2$	$1-z_1$		$4-z_2$

Person 1's expected utility becomes

$$U_1 = \tfrac{1}{2}(2-p_1z_1-p_2z_2,\ 2+z_1) + \tfrac{1}{2}(2-p_1z_1-p_2z_2,\ 2+z_2)$$

$$= 72 - (4+p_1z_1+p_2z_2)^2 - \tfrac{1}{2}(4-z_1)^2 - \tfrac{1}{2}(4-z_2)^2.$$

Person 1's demand for the two types of certificates is determined by the conditions

$$\frac{\delta U_1}{\delta z_1} = -2p_1(4+p_1z_1+p_2z_2) + (4-z_1) = 0,$$

$$\frac{\delta U_1}{\delta z_2} = -2p_2(4+p_1z_1+p_2z_2) + (4-z_2) = 0.$$

For person 2 expected utility is

$$U_2 = \tfrac{1}{2}u(1+p_1z_1+p_2z_2,\ 1-z_1) + \tfrac{1}{2}u(1+p_1z_1+p_2z_2,\ 4-z_2)$$

$$= 72 - (5-p_1z_1-p_2z_2)^2 - \tfrac{1}{2}(5+z_1)^2 - \tfrac{1}{2}(2+z_2)^2,$$

and the first-order conditions for a maximum are

$$\frac{\delta U_2}{\delta z_1} = 2p_1(5-p_1z_1-p_2z_2) - (5+z_1) = 0,$$

$$\frac{\delta U_2}{\delta z_2} = 2p_2(5-p_1z_1-p_2z_2) - (2+z_2) = 0.$$

The two set of first-order conditions gives our two persons'
demand for certificates as functions of their prices.
Setting demand equal to supply for each certificate, we
obtain the equilibrium prices as

$$p_1 = 1/2, \quad p_2 = 1/3,$$

and the equilibrium exchange of certificates as

$$z_1 = -8/31, \quad z_2 = 36/31.$$

The final allocation then becomes

Person	c_1	θ_1	c_2	θ_2
1	54/31	54/31		98/31
2	39/31	39/31		88/31

The resulting utility levels are

$$U_1 = 40.77,$$
$$U_2 = 33.27.$$

This allocation is indeed a solution to the equation (4)
of the unconstrained Pareto efficiency frontier. With
markets for state contingent claims the economy is able
to achieve a full Pareto optimum. Not only is there now
a transfer of resources between periods; there is also
a transfer of risk, whereby person 1 takes over some of
the variability of person 2's income profile.

As the loan market regime results in a constrained
Pareto optimum, it may be useful to consider it as a market
for state contingent claims with the <u>transactions constraint</u>
that certificates can only be bought in pairs. The price
of a pair of certificates is $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$. This gives an
income of 1 in the second period with complete certainty.
Thus, the implied riskless rate of return is $\frac{6}{5}$, precise-
ly as in the loan market. Establishing real markets for
state contingent claims then amounts to removing this
transactions constraint, so that each person can get that
particular coverage that suits his initial income distri-
bution best.

5.   State Contingent Rates of Return

Markets for state contingent claims do not exist in
the real world. The analysis above indicates that such a
regime would have desirable properties, but an objection
to it is that it seems fairly complicated, because, in
general, it would require very many markets. Could not
the same result be achieved by a smaller number of markets?
One possible way out might be a regime of <u>state contingent
rates of return</u>, in which there is only one claim bought

and sold and therefore only one market, but where the rate
of return is contingent upon which state of the world
occurs. A contract will then specify that the rate of
return will be r_1 if θ_1 occurs, and r_2 if θ_2
occurs. Let x as before be the amount lent by person 1
and borrowed by person 2. The market allocation will then
look as follows:

Person	c_1	θ_1	c_2	θ_2
1	$2-x$	$2+r_1 x$		$2+r_2 x$
2	$1+x$	$1-r_1 x$		$4-r_2 x$

Substituting into the utility functions it is easy
to see that the first-order conditions for maximum utility
imply the following demand/supply functions:

$$x^1 = \frac{4(r_1+r_2)-8}{2+r_1^2+r_2^2} \quad ,$$

$$x^2 = \frac{10-5r_1-2r_2}{2+r_1^2+r_2^2} \quad .$$

In equilibrium we must have $x^1 = x^2$. This means
that we must have

$$4(r_1+r_2) - 8 = 10-5r_1-2r_2,$$

i.e.

$$(8) \qquad r_2 = 3 - \frac{3}{2} r_1 .$$

One characteristic of the contingent rate of return
model is clearly that it is unable to determine a unique
equilibrium of rates of return. This is hardly surprising;
we have only one market and as many rates of return as
there are states of the world. The market is only able
to determine the level of rates of return in the form of
a line in the (r_1, r_2) plane; no unique point on the
line can be determined.

For the moment, let us sidestep this problem. One
interesting question is: Does there exist any equilibrium
configuration of rates of return such that the market leads
to a Pareto optimal allocation? We shall show that the
answer to this question is in the affirmative.

To show this, we proceed as follows: From (3) it
follows that a Pareto optimum requires $a = -b_1$, so that
the payment made by person 1 in the first period should
equal his payment in the second period if θ_1 occurs.
We have then that, for person 1 :

$$c_1 = c_2(\theta_1).$$

With state contingent rates of return this implies that

$$2 - x = 2 + r_1 x$$

i.e.

$$r_1 = -1.$$

It then follows from (8) that

$$r_2 = 9/2,$$

and the amount lent by person 1 in the first period is[3]

$$x = 8/31.$$

The final allocation is then easily computed as

Person	c_1	θ_1	c_2	θ_2
1	54/31	54/31		98/31
2	39/31	39/31		88/31

which is exactly the same as the equilibrium solution
under the regime of state contingent claims. This we
know represents a Pareto optimum.

However, this is clearly a special case. Another
equilibrium solution satisfying (8) is $r_1 = r_2 = 6/5$.
This is the equilibrium rate of return under the loan
market regime, and we know that the resulting allocation
is not Pareto optimal.

Since the contingent rate of return regime will
always lead to some shifting of risk, it might be reason-
able to guess that it will always perform better than the
ordinary loan market, except for the case referred to above.
But this conjecture is false. One equilibrium solution is
$r_1 = 2, r_2 = 0$. However, this implies $x = 0$, which is

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3) $r_1 = -1$ means a rate of interest of minus 200 per cent.
Under certainty, such a rate would be meaningless, but
not under uncertainty. The contract would read: "I pay
you $100 this year. If you don't find oil I pay you an-
other $100 next year; if you do, you pay me $450."
There is nothing inherently implausible about this; indeed,
it is the case of the sleeping partner.

the autarky solution, and which is clearly inferior to
the loan market allocation.

Geometrically (Fig.1) each equilibrium solution de-
termines a point in the U_1, U_2 plane, increasing monotoni-
cally from the point (40.0, 32.5) to (40.77, 33.27). As
we have shown, the point (40.26, 32.76) is on this line.
The important point is that we cannot know whether the
contingent rate of return regime will lead to a Pareto
optimal allocation; nor do we know that it will perform
better than the ordinary loan market. However, if the
government or some such external authority were to fix
$r_1 = -1$, then the market could be left to determine
$r_2 = 9/2$, and so achieve an unconstrained Pareto optimum.
But in general the number of states of the world would be
very large, say S, and in that case the government would
have to determine (S-1) rates of return in order to be
sure that the market would realize a Pareto optimal allo-
cation. This appears to reduce the attractiveness of this
regime as an alternative to a system of state contingent
claims.[4]

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4) An alternative way of showing that this regime has one
   equilibrium which is Pareto optimal is the following:
   We know that the state contingent claim regime results
   in a Pareto optimum. Now in that case total saving of
   person 1 is in equilibrium

   $$p_1 z_1 + p_2 z_2 = - \frac{4}{31} + \frac{12}{31} = \frac{8}{31} .$$

   If state of the world θ_1 occurs, he will make a pay-
   ment of $z_1 = 8/31$, i.e. the rate of return is minus one.
   If θ_2 occurs, he will receive 36/31, in which case
   the rate of return is 9/2. Thus, any market solution
   under the state contingent claims regime imply a set of
   state contingent rates of return. If these rates of
   return could be established directly by the market, the
   result would obviously be equivalent.

6. Generalization

All the essential points of this paper have been made in the preceding sections. It is obvious, however, that the framework of the discussion has been very special. We have assumed that there are only two consumers and two possible states of the world in the second period. We have further assumed that the consumers have identical utility functions. And finally, we have assumed that this common utility function is of a special form. All these assumptions will now be relaxed.

Let there be n consumers and S states of the world. Each consumer's preferences over two-period consumption profiles can be summarized by a cardinal, concave utility function

$$u_i = u_i(c_{1i}, c_{2i}), \qquad i = 1, \ldots, n,$$

with first-order derivatives everywhere positive. Following Arrow [1], we also assume that all consumers are characterized by risk aversion. In this context, where uncertainty is connected with events in the second period only, this means that $\delta^2 u_i / \delta c_{2i}^2 < 0$.

Each consumer takes as given an income vector

$$y_i = \{ y_{1i}, y_{2i1}, \ldots, y_{2iS} \}, \quad i = 1, \ldots, n,$$

where y_{1i} is i's income in the first period, which is known with certainty, and where y_{2is} ($s = 1, \ldots, S$) is i's income in period 2 if state of the world s occurs. We also define

$$y_{2s} = \sum_{i=1}^{n} y_{2is}, \qquad s = 1, \ldots, S.$$

as the economy's total second-period income if state of the world s occurs. Let π_{is} be consumer i's subjective probability that state of the world s occurs. The consumer's expected utility is then

$$(9) \quad U_i = \sum_{s=1}^{S} \pi_{is}\, u_i(c_{1i}, c_{2is}), \qquad i = 1, \ldots, n.$$

Under autarky we must have $c_{1i} = y_{1i}$ and $c_{2is} = y_{2is}$ for all i and s.

We shall again define a Pareto optimum by means of the transfer payments required. Let a_{ij} be the transfer paid to i from j in period 1, and b_{ijs} the corresponding transfer payment in period 2 and state of the world s. We then have that

$$(10) \quad c_{1i} = y_{1i} + \sum_{j=1}^{n} a_{ij}, \qquad i = 1, \ldots, n,$$

and

$$(11) \quad c_{2is} = y_{2is} + \sum_{j=1}^{n} b_{ijs}, \qquad \begin{aligned} i &= 1, \ldots, n; \\ s &= 1, \ldots, S. \end{aligned}$$

Summing over i, we see that we must have

$$(12) \quad \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ijs} = 0, \qquad s = 1, \ldots, S,$$

$$(13) \quad \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} = 0.$$

We now define a Pareto optimum as the maximum of the function

$$(14) \qquad V = \sum_{i=1}^{n} k_i U_i,$$

where the k_i's are arbitrary positive constants. Substituting in (9) from (10) and (11) we can write

$$(15) \qquad U_i = \sum_{s=1}^{S} \pi_{is} u_i \left(y_{1i} + \sum_{j=1}^{n} a_{ij}, \; y_{2is} + \sum_{j=1}^{n} b_{ijs} \right).$$

Our problem is now to find the maximum of (14) subject to (12) and (13). The first-order conditions are:

$$(16) \qquad k_i \sum_{s=1}^{S} \pi_{is} \frac{\delta u_i}{\delta c_{1i}} - \lambda = 0, \quad i = 1, \ldots, n,$$

$$(17) \qquad k_i \pi_{is} \frac{\delta u_i}{\delta c_{2is}} - \lambda_s = 0, \quad \begin{array}{l} i = 1, \ldots, n; \\ s = 1, \ldots, S. \end{array}$$

Here λ and λ_s $(s = 1, \ldots, S)$ are Lagrangian multipliers.

Combining (16) and (17) we can write

$$(18) \qquad \frac{\sum_{s=1}^{S} \pi_{is} \dfrac{\delta u_i}{\delta c_{1i}}}{\pi_{is} \dfrac{\delta u_i}{\delta c_{2is}}} = \frac{\sum_{s=1}^{S} \pi_{js} \dfrac{\delta u_j}{\delta c_{1j}}}{\pi_{js} \dfrac{\delta u_j}{\delta c_{2js}}} \qquad \text{for all } i, j, s.$$

In words: In a Pareto optimum the ratio of the expected marginal utility of present consumption to the expected marginal utility of future consumption in state of the world s should be equal for all consumers and all states of the world.

We now proceed to show that a regime of state contingent claims results in a Pareto optimum. Let there be S types of claims, and let z_{is} be the amount bought by i of claims of type s, i.e. claims promising a payment of 1 dollar in period 2, if and only if state of the world s occurs. Let p_s be the market price of this claim. The i'th consumer's budget constraint is then

$$c_{1i} = y_{1i} - \sum_{s=1}^{S} p_s z_{is}.$$

Second-period consumption in state of the world s is

$$c_{2is} = y_{2is} - z_{is}.$$

By substitution we obtain

$$(19) \qquad c_{1i} = y_{1i} - \sum_{s=1}^{S} p_s (c_{2is} - y_{2is}).$$

Each consumer now maximizes (9) subject to (19). This yields

$$\sum_{s=1}^{S} \pi_{is} \frac{\delta u_i}{\delta c_{1i}} - \mu = 0,$$

$$\pi_{is} \frac{\delta u_i}{\delta c_{2is}} - \mu p_s = 0, \qquad s = 1, \ldots, S.$$

where μ is a Lagrangian multiplier. This implies

$$(20) \qquad \frac{\sum_{s=1}^{S} \pi_{is} \frac{\delta u_i}{\delta c_{1i}}}{\pi_{is} \frac{\delta u_i}{\delta c_{2is}}} = \frac{1}{p_s}, \qquad \begin{array}{l} i = 1, \ldots, n, \\ s = 1, \ldots, S. \end{array}$$

But since all consumers take prices as given, we must have that, in equilibrium

$$(21) \qquad \frac{\sum\limits_{s=1}^{S} \pi_{is} \dfrac{\delta u_i}{\delta c_{1i}}}{\pi_{is} \dfrac{\delta u_i}{\delta c_{2is}}} = \frac{\sum\limits_{s=1}^{S} \pi_{js} \dfrac{\delta u_j}{\delta c_{1j}}}{\pi_{js} \dfrac{\delta u_j}{\delta c_{2js}}} \qquad \text{for all } i,j,s,$$

which are identical to conditions (19) for Pareto optimum.

Under the ordinary loan market regime the budget constraint of the i'th consumer is simply

$$(22) \qquad c_{2is} = y_{2is} + (y_{1i} - c_{1i})r.$$

Maximization of (9) subject to this condition yields

$$(23) \qquad \frac{\sum\limits_{s=1}^{S} \pi_{is} \dfrac{\delta u_i}{\delta c_{1i}}}{\sum\limits_{s=1}^{S} \pi_{is} \dfrac{\delta u_i}{\delta c_{2is}}} = r, \qquad i = 1, \ldots, n.$$

This implies that

$$(24) \qquad \frac{\sum\limits_{s=1}^{S} \pi_{is} \dfrac{\delta u_i}{\delta c_{1i}}}{\sum\limits_{s=1}^{S} \pi_{is} \dfrac{\delta u_i}{\delta c_{2is}}} = \frac{\sum\limits_{s=1}^{S} \pi_{js} \dfrac{\delta u_j}{\delta c_{1j}}}{\sum\limits_{s=1}^{S} \pi_{js} \dfrac{\delta u_j}{\delta c_{2js}}}, \qquad i,j = 1, \ldots, n.$$

These conditions do not imply a Pareto optimum. Their interpretation would be as follows: Under the loan market regime the ratio of the expected marginal utility

of present consumption to the expected marginal utility
of future consumption will be equal for all consumers.
It is immediately clear that this is much more restrictive
than the corresponding conditions for the state contingent
claim regime. Indeed, summing over s in conditions (21)
gives (24), while (24) does not imply (21).

As in the example discussed above, we may also see
the loan market allocation as a constrained Pareto optimum.
An unconstrained Pareto optimum is found as the solution
of the following problem:

$$\underset{a_{ij}, b_{ijs}}{\text{Max } V} = \sum_{i=1}^{n} k_i \sum_{s=1}^{S} \pi_{is} v_i \left(y_{1i} + \sum_{j=1}^{n} a_{ij}, y_{2is} + \sum_{j=1}^{n} b_{ijs} \right)$$

subject to

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} = 0,$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} b_{ijs} = 0, \qquad\qquad s = 1, \ldots, S.$$

This is found by maximizing the Lagrangian expression

$$L = V - \lambda \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} - \sum_{s=1}^{S} \lambda_s \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ijs}.$$

A constrained Pareto optimum is defined if we introduce
the additional S constraints

$$b_{ijs} = b_{ij} \qquad \text{for all } s.$$

The Lagrangian then becomes

$$L^* = V^* - \lambda \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} - \lambda^* \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij},$$

and the first-order maximum conditions are

$$k_i \sum_{s=1}^{S} \pi_{is} \frac{\delta u_i}{\delta c_{1i}} - \lambda = 0, \qquad i = 1, \ldots, n,$$

$$k_i \sum_{s=1}^{S} \pi_{is} \frac{\delta u_i}{\delta c_{2is}} - \lambda^* = 0, \qquad i = 1, \ldots, n.$$

It is now easy to see that, in equilibrium, these conditions imply (24), so that the loan market does indeed result in a constrained Pareto optimum.

We shall not elaborate further on the contingent rate of return regime. From the foregoing discussion it should be fairly obvious how the analysis of section 5 can be generalized. The essential point remains true that of the set of possible equilibria there will in general be one which corresponds to a Pareto optimum, but the economy will achieve this only by a stroke of luck.

7. Concluding Remarks

We have shown that, in general, ordinary loan markets will not achieve a Pareto optimal allocation of resources. We have also shown that such a regime will achieve a con-

strained Pareto optimum, in the sense that second-period interpersonal transfers are restricted to be independent of which state of the world occurs.

Will this constraint always be effective? In other words, can it ever happen that the ordinary loan market achieves a full Pareto optimum? The only case I have found appears to be the exception that proves the rule. If all consumers have identical preferences, identical probability beliefs and essentially identical income vectors[5), then the ordinary loan market allocation is a Pareto optimum. But this case is trivial, for with consumers identical in all essential respects, there will be no basis for trade, and the Pareto optimum is simply the autarky solution.

A complete system of state contingent claims seems rather complicated, and if the only practical implication of this paper were that such a system should be established with all possible speed, it could hardly be taken very seriously. But the moral of the story has a wider applicability. Suppose consumer goods could only be bought in identical bundles. This would clearly be suboptimal as compared with a system where consumers themselves determine freely the composition of their consumption. Nevertheless, if consumers were given the choice between two or three different bundles, that would clearly be an improvement,

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5) The word "essentially" is explained as follows. In our
   example where both persons consider the states θ_1 and
   θ_2 to be equally likely, the following income profiles
   will be essentially identical because the probability
   distributions of future income are identical.

y_1	$y_2(\theta_1)$	$y_2(\theta_2)$
2	2	3
2	3	2

although the system would still be far from the optimum.
This is no less true of asset markets. In an ordinary
loan market, there is only one bundle of state contingent
claims. Introducing more types of assets means increasing
the number of different bundles of state contingent claims.
This will be a step in the right direction because it makes
consumers better able to accomodate their time and risk
preferences. In other words, the full optimum is approached
by increasing each consumer's opportunities for risk cover-
age according to his personal preferences, probability beliefs
and income profile.

We have assumed throughout that the utility function
itself is independent of the state of the world. More
formally, we may state this as follows: For all consumers,
given $c_{1i} = \bar{c}_{1i}$, we have that $c_{2is} = c_{2it}$ implies
$u_i(\bar{c}_{1i}, c_{2is}) = u_i(\bar{c}_{1i}, c_{2it})$. The ranking of consumption
profiles is thus assumed to be independent of the state of
the world. This seems reasonable enough for many appli-
cations, but not for all. For instance, if a person's
future income depends on whether he becomes ill or not,
the assumption is unwarranted. What this means, is that in
such a case consumption profiles alone are insufficient as
specifications of events. The utility function of the
consumer would then have to be written as, e.g.,
$u_i(c_{1i}, c_{2i}, \alpha)$ with $\alpha = 1$ if illness does not occur,
and $\alpha = 0$ if it does. Such a formulation would be
relevant in a discussion of loan markets from the viewpoint
of health insurance.

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