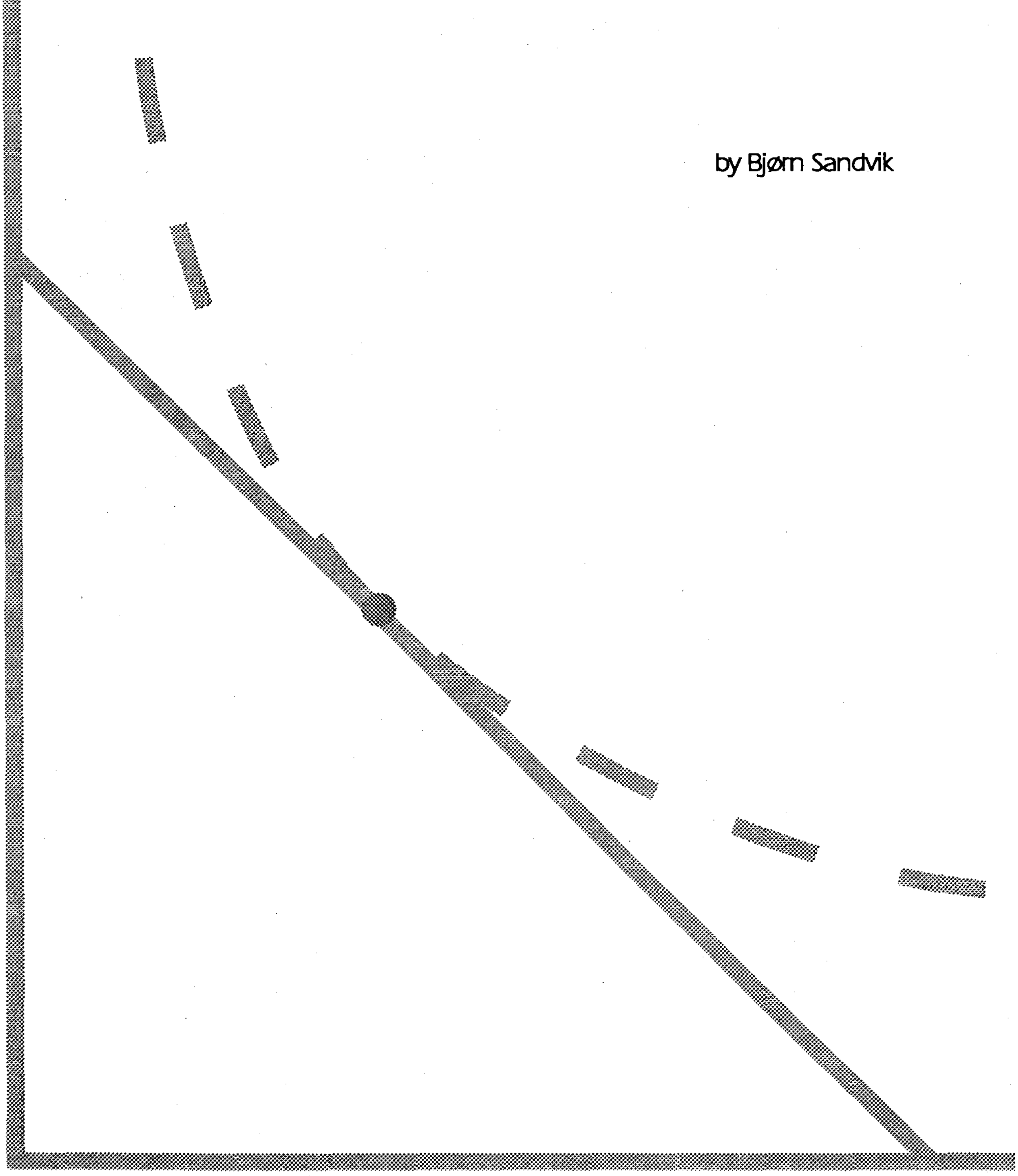
# AN ESSAY ON THE FOUNDATIONS OF THE NEOCLASSICAL THEORY OF THE INDIVIDUAL

by Bjørn Sandvik



Thesis submitted for the degree of Doctor oeconomiae
at the Norwegian School of Economics and Business Administration

Bergen, March 1992

# AN ESSAY ON THE FOUNDATIONS OF THE NEOCLASSICAL THEORY OF THE INDIVIDUAL

by

**Bjørn Sandvik**

Thesis submitted for the degree of Doctor oeconomiae
at the Norwegian School of Economics an Business Administration

Bergen, March 1992

# CONTENTS:

# FOREWORD

The point of departure for this essay was Terje Lensberg's suggestion that I should try to generalize his (1987) characterization of completely separable utility in terms of choice. Chapter 7 contains such a generalization. Trying to work out such a generalization, I came across Uzawa's (1956) early article. This article sparked my interest in trying to reformulate the basic theory of the interrelationship between preferences and choice, which is the central theme of this essay.

I have benefited from discussing the ideas in this essay with many people. The most important ones, which I would thank especially, are Sjur Flåm, Thorsten Hens, Aanund Hylland, Terje Lensberg and my supervisor Lars Thorlund-Petersen.

February 1991
Revised, March 1992

# CHAPTER 1: NEOCLASSICAL DEMAND THEORY[0]

## 1. INTRODUCTION

The main theme of this essay is the neoclassical theory of the individual, or the theory of rational choice as it is also called. To set this theory into into perspective, this chapter presents some ideas underlying the neoclassical approach to demand theory. It also gives a selective picture of the present status of demand theory and serves as an introduction to the more technical work to follow.

Section 2 outlines the core of the neoclassical approach to demand theory as I see it. It consists of two methodological principles, "methodological individualism" and "individual rationality". Section 3 discusses methodological individualism. The rest of this chapter is concerned with the neoclassical theory of the individual, the theory of rational choice in budgetary situations. This theory is also the main theme of the following chapters. In Section 4 we discuss rationality, in Section 5 the relationship between preferences and choice (or action), and in Section 6 the status of rational choice theory. In the three last sections we discuss some additional restrictions on preferences and choice: Section 7 discusses restrictions on income effects based on the law of demand, Section 8 separability, Section 9 expected utility, and Section 10 outlines the rest of the essay.

## 2. THE NEOCLASSICAL RESEARCH PROGRAM

As mentioned, we characterize the neoclassical approach to the explanation of social phenomena by the following two methodological principles:

[0]Thanks are due to Thorsten Hens, Aanund Hylland, Claudia Keser and Hans Larsson for valuable comments to this chapter.

- **Methodological individualism:**

  Social phenomena are to be explained by individual behavior.

- **The rationality principle:**

  Individual behavior is to be explained (intentionally) by rationality notions given the individual's preferences (values) and perceived situational constraints.

The first principle should not totally exclude social and cultural notions in explanations of social phenomena, for this is generally impossible, as argued by Hodgson (1986). Thus, I take methodological individualism to say that one should avoid to refer to social institutions and cultural phenomena as far as possible in the explanations of social phenomena - except of course social phenomena which have already been explained on the neoclassical approach. The second principle is weak, and presumably without empirical content of its own. In the neoclassical approach, this weakness is remedied (and the problem of modeling an individual is made more tractable) by some supplementary assumptions and methodological principles. These assumptions and principles are also devices to reduce the necessary referencing social and cultural phenomena in explaining individual actions:

**Full knowledge:** The individual knows all relevant aspects of his situation.

**Consequentialism:** Preferences are over consequences only.

**Extensionality:** Consequences (or objects) are extensionally given, i.e. independent of their descriptions.

**Preference uniqueness:** The individual has unique preferences.

**Preference exogenity:** Preferences are exogenous.

**Preference invariance:** Preferences are separable over time and situationally in dependent.

We restrict attention to demand theory. **Demand theory** purports to explain behavior in **parametric situations**, i.e. in situations with no strategic interaction between individuals. In that case, it is convenient to identify acts (behavior) with their consequences. Usually , as we will do here, one restricts attention further to **price-generated budgets**, i.e. situations where the alternatives of an individual are constrained by some prices (and an income) in a finite-dimensional, closed, convex, and downward bounded space of goods.[1]

A main task of demand theory is to explain market (or aggregate) demand, i.e. to build a theory with non-trivial restrictions on market demand. Preferably, these restrictions should (together with standard technology assumptions) suffice to justify standard applications of general equilibrium theory, especially comparative statics. Hence one would like to verify uniqueness and some kind of stability (for example of the tâtonnement process) of equilibrium. It is also of interest to generate downward sloping aggregate demand curves or some generalization of this like the law of demand.

Demand theory is also important both for normative problems and the interpretation and explanation of individual action. I do, however, take the neoclassical research program to be mainly concerned with the above mentioned descriptive problem.[2]

One can deviate from any of the supplementary principles if one has a clear idea of some (simple) additional structure. Their function is thus a simplifying one without such ideas.

Full knowledge abstracts from the fallibility of beliefs upon which humans act. It is easily weakened. Instead of knowing the outcomes of actions, for example, it suffices for the theory that one knows the probability distribution of outcomes over different <u>known</u> states.[3]

---

[1]Note that convexity of the space of goods implies perfectly divisible goods.

[2]Criteria for evaluating theories with a normative interpretation are also rather different from the criteria for theories with a descriptive interpretation. This is briefly discussed in Section 6.

[3]As far as I can see, the theory presupposes full knowledge (or common certain beliefs) about <u>some</u> objects. In the above example, one presupposes full knowledge about the state space and the probability distribution.

Consequentialism abstracts from our concern about the actions themselves and their history.[4] For example, preferences are not allowed to depend on a reference level. Experiments by Tversky and Kahnemann (1991), however, indicate such a dependence. Tversky and Kahnemann also outline a theory with reference dependence.

Extensionality implies that individual behavior depends only on the available alternatives, and not on the way these are conceived. Thus it excludes framing effects as discussed in Tversky and Kahnemann (1981). As they make clear, extentionality is a strong assumption, at least in situations involving risk.

Individuals usually do not have unique preferences. For example, an individual's <u>moral</u> preferences are usually different from his <u>egoistic</u> ones. It depends on the situation which preferences influence action. More generally, an individual in a social role activates preferences (values) relevant to that role.[5] This is of little importance for the analysis of traditional commodity demand. It gains importance, however, when one extends the scope of economic analysis.

The assumption of exogenous preferences means that one does not analyze the process of value formation.[6] This is a main shortcoming of neoclassical economics as a comprehensive social theory.[7]

Individual actions are usually observable only one at a time, but the theory presupposes simultaneous choices. Therefore to get empirical content in the theory, one need to specify how preferences develop over time. The simplest alternative is to assume separability over time and say that their intraperiod parts are time invariant (i.e. stationary), possibly with a constant rate of time preference.

---

[4]In some cases, concern about actions can be incorporated into the theory by extending the choice space. Then, however, the choice space soon looses its simple structure.

[5]In some situations one might ascribe individual unique preferences and model their "role-induced part" as situational constraints. This, however, often misrepresents the thinking of individuals, see Sen (1973).

[6]Values could be formed for example through agumentation. In contast to the situation concerning values, one has an interesting (Bayesian) theory of the evolution of <u>beliefs</u>.

[7]Some work, however, has been done on the influence of advertisement on preferences and behavior.

## 3. METHODOLOGICAL INDIVIDUALISM

This section discusses the principle of methodological individualism. This principle is closely related to the **aggregation problem**. The question is to what extent it is possible to get nontrivial restrictions on market (or aggregate) demand functions from restrictions on the individual demand functions generated by a theory of rational individuals. In a sense, this is the crucial step. The reason is that if one can build a nontrivial theory of market demand on the given individual constructs, then objections at the individual level, saying that the theory abstracts too much from reality, do not matter much. This is the case at least until one gets a viable alternative to the neoclassical approach. If something is fundamentally wrong at the individual level, the situation might, however, be different.

Many economists seem to believe that the aggregation step is unproblematic or at least possible. So far, however, this step has only been verified under extremely restrictive assumptions. The classical result is that of Antonielli (1886). He showed that if individual preferences are identical and **homothetic** (i.e. demand is linear in income), then there is a representative consumer[8] with the same preferences. Conversely, he showed that if there is always a representative consumer, then the individuals have identical and homothetic demand functions. Thus one needs some restrictions on the income distribution (or more fundamentally the distribution of endowments) to establish a representative consumer under more plausible preference assumptions. The first such result was by Eisenberg (1961). He showed that with a price independent[9] relative income distribution and homothetic preferences, there is a representative homothetic consumer. Next, Shafer (1977) showed that with a price independent income distribution and demand functions satisfying the **law of demand** (i.e. a negative correlation between changes in prices and quantities demanded - for fixed income), there is a representative consumer satisfying the law of demand. Thus the

[8]An economy admits a **representative consumer** if we can construct an individual generating the aggregate excess demand function of the economy. The existence of a representative consumer clearly depends on the more specific requirements of individuals in the theory.

[9]The typical case is when the individuals' endowments are on a ray through 0 and there are no profit.

aggregation problem is mainly rooted in the influence of prices on the relative income distribution via the resources and profits of the individuals.[10]

These results give conditions under which properties of individual demand are inherited by aggregate demand. Generally, properties of aggregate demand might be different from those of individual demand. An analogy might clarify this: In thermodynamics one has a smooth and stable theory at the macro level involving concepts like temperature, volume and pressure. The movement of the micro units, the particles, however, are stochastic and unpredictable individually. One result giving more structure on aggregate demand than on individual demand is that by Hildenbrand (1983). He showed that the law of demand holds for aggregate excess demand in a large economy, without assuming it to hold for individuals. This is, however, only true under a somewhat implausible assumption on the income distribution,

Furthermore, Samuelson (1956) showed that a representative consumer exists if income is optimally distributed according to some welfare function.

In the negative direction, there are several negative results of increasing strength, originating in Sonnenschein (1973). These results show that only the most trivial properties of market (excess) demand follow from methodological individualism. These are the following: First, market excess demand is homogeneous of degree zero in prices, i.e. only relative prices matter. Secondly, it satisfies **Walras law**, saying that the value of market excess demand is 0. Finally, a certain behavior at the boundary of the choice space is implied.[11] This is so even under very restrictive conditions both on preferences and the distribution of endowments if the number of goods is not larger than the number of individuals.[12] A strong negative result is given by Kirman and Koch (1987). They show that any excess demand function with the above properties can be generated from a pure exchange economy with as many individuals as there are goods - even if the individuals have identical

[10]If individuals satisfy the law of demand, this result justifies class models where all individuals in each class hold endowments of one and the same good, e.g. labour and capital.

[11]Namely that if the prices of some goods goes to zero, then aggregate demand gets unbounded. This follows from monotone preferences.

[12]Otherwise a weakened version of the Slutsky condition holds, see Diewert (1977).

preferences and collinear endowments.[13] A result by Hildenbrand (1989a) points in the same direction. It shows that under very mild preference assumptions, if individual endowments vary, the revealed preference axiom for mean demand holds only on a space of measure zero.

These described negative results seem devastating for the neoclassical research program. They are, however, far from conclusive. As for the result by Hildenbrand (1989a), the conclusion is weak. The revealed preference axiom might for example still hold on appropriate subspaces.

A positive result is by Hens (1990). Instead of investigating properties of equilibria in simultaneous contingent commodities, he looks at equilibria with incomplete markets and sequential trade in reopening spot markets. Then risk averse individuals trade in futures markets to avoid spot market risk. In his model, the result of the future markets trade in previous periods is that spot market endowments get collinear. This then ensures uniqueness and stability of (spot market) equilibria. The result is wrong if markets are organized as contingent contracts as in the Arrow-Debreu model.

The empirical evidence seems to support a more optimistic view of the aggregation problem. Lewbel (1991) shows that a representative consumer model represents aggregate demand reasonably well - especially if one drops the individuals in the tails of the income distribution. The representative consumer corresponds to a cost function with two price indices. Lewbel shows that by allowing one more price index in the cost function, one gets a very good fit. How is this to be reconciled with the negative results presented above? First, there is presumably little price variability in the data. Secondly, by taking the income distribution to be price independent, he sidesteps what is presumably the most important theoretical problem connected to the aggregation problem. This is indicated by the law of demand aggregation result mentioned above. For predictive purposes, assuming that price variability and income price dependence remains small, these objections do not matter much. But we must consider them if we want to explain or understand aggregate demand.

---

[13]Endowments are collinear if they lie on a ray through 0, i.e. spans a one dimensional subspace.

So the neoclassical research program, at least when modified in this direction, is still open. So Kirman's (1989) metaphor of the status of the neoclassical research program: "The Emperor has no clothes" does not seem fitting. On the other hand, it looks as if the popularity of methodological individualism stems more from the lack of viable alternatives than its own progress. The reductionist attitude of methodological individualism surely also has its appeals. Up to now, much of the use of a representative consumer in economics is not much better founded than explanations by the Hegelian "Weltgeist", though we believe it to make more sense.[14] More generally, criticism that explanations in social sciences do not satisfy methodological individualism is unjustified as long as neoclassical economists does not get forward on the aggregation problem.

The rest of this chapter is mainly concerned with the neoclassical theory of the individual, also called the **theory** of **rational choice**.

## 4. RATIONALITY

The theory of rational choice explains individual behavior **intentionally**. Thus individual actions are explained by its values and perceived situational constraints. Causal laws only determine the consequences of actions. Intentional explanations are attractive as they treat individuals as the <u>subjects</u> of history. Hence they fit our way of seeing ourselves - at least in our better moments.

Assume well-defined goals in the form of preferences over a space of goods, and situational constraints in the form of budgets. Then the approach is, as mentioned, to explain individual behavior by assuming rationality. There are, however, many (and usually non-exclusive) ways of explaining rationality. We will say that a preference or choice

---

[14]For example, much of the so called "microfoundations of macroeconomics," consisting essentially in using representative consumers for large aggregates, without aggregation results, have no better microeconomic foundation than traditional "ad hoc" macroeconomics. This does not mean that these representative consumer models are uninteresting, on the contrary, such models are often quite interesting.

property is **rational** if it follows directly from the meaning of the basic terms of the theory.[15]

The above criterion is vague, but has some clear consequences, as it justifies preference **asymmetry** (i.e. that one bundle cannot be better than another if the second is at least as good as the first), **reflexivity** (i.e. that a bundle is at least as good as itself), and **transitivity** (i.e. that if one bundle is preferred to another and this again is at least as good as a third, then the first is also preferred to the third).[16] These notions are direct consequences of the meaning of the terms "better" and "at least as good as". These rationality criteria also justify the **maximal element** (definition of rational) **choice**. This says that a bundle is a rational choice at a budget if it belongs to the budget and all preferred bundles are outside the budget). The idea is simply that one would not choose an alternative if one knew that one had a better one available.[17]

Often, **completeness** (i.e. that of any two bundles, either the first is preferred to the second or the second is at least as good as the first) is also seen as a rationality property of preferences. Under the above conceptions of rationality, this is unwarranted.[18] Is completeness necessary for a theory of choice? No, one can verify existence of individual choice and general equilibrium without it, but at some cost. If not a rationality property, completeness is often seen as a simplifying assumption in the theory of rational choice. This is not always the case, however. Indeed in Chapters 2 to 5 we show that insisting on completeness complicates the task of characterizing preferences in terms of choice. We do this by essentially characterizing the properties of the revealed preference relations - which are generally incomplete. With this change of perspective, a full characterization is fairly straightforward, as should not be very surprising. The approach necessitates some care in

---

[15]Another, more in the spirit of Rawls' (1970) reflective equilibrium, is to say that goals or behaviour is rational if they would not change upon more information of certain kinds. A further strengthening of this is that goals or behavior should not change even if they where made publicly known. This is more in the spirit of Kant's (1785) Categorical Imperative. It is mentioned as a reminder that the neoclassical conception does not exhaust rationality. See also Føllesdal (1982a,b).

[16]Indeed, transitivity is slightly stronger than this, see the definition in Chapter 2.
Trivially, if a bundle is better than another, then it is also at least as good.

[17]It is not necessary to assume selfish preferences, though this simplifies the treatment of general equilibrium.

[18]This is clear under the first interpretation if one thinks for example of the Paretian partial orders.

choosing concepts, however, as concepts which are equivalent with completeness are often different without completeness.

With the above assumptions, the theory might, however, still be vacuous in many situations, as there need not exist any rational choices. To avoid this, one makes some additional assumptions, i.e. that preferences are continuous, convex, and locally nonsatiated. Continuity can be seen as a regularity assumption, as it is not possible to test for it on a finite data set. **Convexity** (saying that any convex combination of a set of points is at least as good as one of these points) is in contrast a strong assumption. Thinking of aquavit and antabus, preferences are not always convex. It cannot be falsified in a finite data set based on observations, however. Convexity is not needed for a non-vacuous theory of the consumer with transitive preferences. But it is needed for the existence of a general equilibrium with a finite number of individuals, and is therefore usually assumed. The final assumption is **local nonsatiation**, saying that arbitrary close to any given bundle, there is a preferred one. This also seems unproblematic. Below I shall strengthen local nonsatiation to **monotonicity**, saying that a bundle is preferred to another if one has more of every good. This is to avoid negative prices. It can be replaced by a similar assumption on the technology. Under these additional assumptions, choice is non-vacuous, at least at budgets with positive prices.

Full knowledge (or certain belief) is, however, needed for some kind of objects - e.g. the state space and a joint probability distribution. Without it, the maximal element definition of rational choice is generally empty. Then one has to be satisfied with a weaker concept of rationality. One such is Simon's (1972) notion of **satisficing**, saying that the individual sets some aspiration level, and ends search when this is obtained.[19] Another is the finite automat explication, introducing bounded rationality through a finite memory, as in Rubinstein (1986).[20]

---

[19]This theory is partial as it does not explain how the aspiration level is set or changed. This is similar to the tratment of preferences in the neoclassical theory. Preferences, however, seem more stable than aspiration levels, and hence easier to access.

[20]As indicated, bounded rationality is of lesser interest for normative purposes. For descriptive purposes, this is not so, as it is mainly an empirical question which theories are to be preferred. Then also more evolutionary oriented theories like the one by Heiner (1983) are of interest.

Rationality is not needed for well-behaved aggregate demand. As shown by Becker (1962), aggregate demand behaves nicely in a large economy when the individuals "choose" randomly from a common distribution over a common budget constraint. Indeed, then even the law of demand holds for the aggregate. This follows from the law of large numbers.

Above, rationality was discussed in connection with the explanation of action. As argued by Føllesdal (1982a,b), however, at a more fundamental level rationality is a prerequisite for understanding other people. The reason is that rationality is presupposed in interpreting or understanding behavior and identifying actions.[21] This builds on a distinction between **behavior** and actions. While behavior can be described without reference to intentions or goals, actions cannot. Thus to say that an individual performs a certain action, implies saying something about his goals. The claim is also that actions are important in human relationship. This (more hermeneutic) view of economics is also advocated by Andreassen (1989). It is at cross with the narrow behaviorism of some economists, like the early Samuelson (i.e. Samuelson (1938)).

Rationality as a prerequisite for identifying actions is an idea underlying many humanistic approaches to psychotherapy, like Greenwald's (1974) "decision therapy," Ellis' (1973) "rational emotive therapy," and Perls, Hefferline, and Goodman's (1951) "gestalt therapy." These humanistic approaches assume that individuals' actions, however bizarre, are rational. The assumption of unity of the individual are, however, often dispensed with. The point is that the rationality assumption is used to get the patient's underlying values and beliefs into daylight. Then the patient can work consciously with these values and beliefs, and the different parts of the individual can be reconciled with each other.

[21]This idea goes back at least to Husserl (1913).

## 5. PREFERENCES AND CHOICE

There are two simple ways of looking at the interrelationship between preferences and choice. The classical one is the **preference point** of **view**. Here, as above, one takes preferences as given and relatively unproblematic, and one asks: What are the choice consequences of given preference assumptions? This is the **first** basic **question**. The **choice point of view** has a narrower conception of evidence. It assumes that the only evidence one can get about preferences comes from individual choices.[22] Here a natural question to ask is whether a pattern of behavior (choice correspondence) can be seen as generated by preferences of certain kinds. Or in other words, to ask which kinds of preferences, if any, are consistent with a certain pattern of behavior. This is the **second basic question**. This point of view originates in Samuelson (1938), who under strong operationalist influence set out to eliminate theoretical concepts like preferences from demand theory.

In reality, the relationship between preferences and choice regarding evidence is more complex than expressed in either point of view. On the one hand, it does make sense to ask people about their preferences. On the other hand this is not unproblematic. The basic questions do, however, make sense independently of these two points of view.

A **complete** characterization gives both necessary and sufficient conditions. Thus a complete characterization answers both basic questions, since the necessary conditions for one basic question are sufficient ones for the other. Hence a complete characterization shows the contents of the preference and the choice formulations of the theory to be essentially the same. Most of the characterization results in the literature are not complete in this sense, however.

Historically, one started with a utility function and derived demand properties from it. This culminated in the work of Johnson (1913) and Slutsky (1915) who introduced the Slutsky equation, relating the demand and the compensated demand function. The derivative of the latter is the Slutsky matrix. Johnson and Slutsky answered the first basic question by showing that the Slutsky matrix was symmetric and negative semidefinite.

---

22Usually, one restricts attention further to price-generated budgets.

As it was hard to derive additional nontrivial properties of choice from the standard preference assumptions, one started to ask whether these properties were all one could get. This led to the **integration approach** to deal with the second basic question, summarized in Hurwicz (1971). The integration approach starts with the Slutsky conditions for a given demand function. The compensated demand given by the Slutsky equation is integrated with respect to prices. This yields an income compensation (or expenditure) function, with some prices and an income as parameters (initial conditions). Looked upon as a function of these parameters, the income compensation function can be shown to be an indirect utility function of the desired kind. From this one obtains the direct utility function by duality. The integration approach, however, requires some extra differentiability assumptions (or at least Lipschitz continuity), which do not follow from the standard preference assumptions.

Another approach to the second basic question, is the **revealed preference approach.** This approach originates in the work by Samuelson (1938). He introduced the **single-valued axiom of revealed preference** to purge theoretical constructs like preferences from demand theory.[23] This axiom says that if a bundle is chosen where another is accessible, then the latter cannot be chosen where the first is accessible. Thus the single-valued axiom excludes situations like the one in Figure 1.[24] Samuelson also showed that the axiom implies

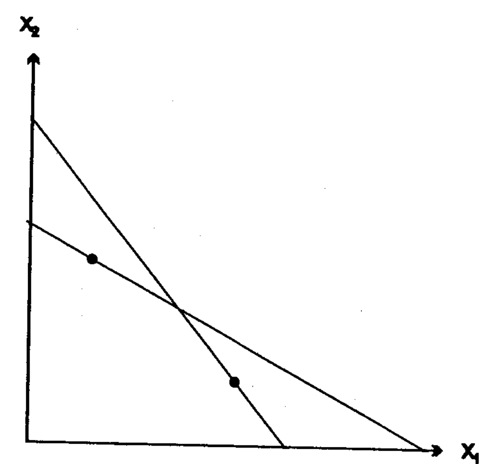


Figure 1

[23]For aggregate excess demand, this was essentially already formulated by Wald (1936) as a basis for his proof of the existence of general equilibrium.

[24]This motivation is misconceived, but fruitful. The axiom of revealed preference is stated solely in the language of choice. Thus it might seem to make preferences superfluous. The axiom of revealed preference is, however, only plausible if one thinks of choice as generated by some underlying preferences. Additionally, the characterization results which grew out of Samuelson's work has shown that the content of the preference and the choice based theories are the same.

negative semidefiniteness of the Slutsky matrix.[25] The next important step within the revealed preference approach was taken by Houtakker (1950), building on Little (1949) and Samuelson (1948). He formulated the (single-valued) transitive axiom of revealed preference (often called the strong axiom). This extends the (single-valued) axiom in a transitive manner. Houtakker proved that under some assumptions, the transitive axiom guarantees the existence of a standard utility function generating the given choice. He thereby gave the first rough characterization of the above kind. The argument was later on completed by Uzawa (1959), Hurwicz and Uzawa (1971), and Stigum (1973).[26]

A simplified (sometimes called set-theoretic) version of the revealed preference approach answers the second basic question by showing that the preferences naturally generated by a choice have the appropriate properties. This approach originates in Uzawa (1956), which itself is based on some lectures by Houtakker in Tokyo. Uzawas article, however, has been virtually unknown, or at least not well understood, so this approach is usually attributed to Richter (1966,1971). Chapter 2 works out the ideas of Uzawa (1956) in more detail. We claim that with some minor modifications, Uzawas original approach is the most appropriate way to analyze the characterization problem.

Both approaches have problems in establishing full continuity of the generated preferences without additional assumptions. Within the revealed preference approach, the strongest result so far without extra assumptions is by Hurwicz and Richter (1971) giving only upper semicontinuous preferences. This is due to the insistence on complete preferences. Weakening this not especially plausible requirement, one can establish a characterization of the desired kind rather straightforwardly, as shown in Chapter 3.

In the result of Chapter 3, the generated preferences are not unique. For descriptive

[25]A converse result also holds. Indeed as shown by Kihlstrom, Mas-Colell and Sonnenschein (1976) for a differentiable choice function, the axiom is intermediate between negative definiteness and negative semidefiniteness of the Slutsky matrix (a simpler proof of this is in Hildenbrand and Jerison (1988)). The symmetry of the Slutsky matrix similarly corresponds to going from the axiom to the transitive axiom of revealed preference.

[26]The distinction between the axiom and the strong axiom is mainly of interest when attention is restricted to the price-generated budgets. Allowing all three-element sets as 'budgets', these axioms are equivalent, as shown by Arrow (1957).

purposes there is indeed no need for such uniqueness.[27] This (ordinal) uniqueness question was, however, answered by Mas-Colell (1978a,b) and shown to be related to an income Lipschitz condition on choice. Such conditions have had a central place in most work on the second basic question.

In the thirties, the question of the status of transitivity (or as stated then, Slutsky symmetry) was much discussed. More recently, after Sonnenschein's (1971) discovery that transitivity of preferences is unnecessary for the existence of choice, and Mas-Colell's (1974) discovery that the same is true for the existence of general equilibrium, interest in demand theory without transitivity has reappeared. A central point in the discussion has been the Kihlstrom, Mas-Colell, Sonnenschein, and Shafer (1976) conjecture. This says that the choice consequences of the standard preferences assumptions except transitivity are identical to the standard ones when the (single-valued) transitive axiom is replaced by the (single-valued) axiom of revealed preference. Kim and Richter (1986) provided a counter-example to this conjecture. In Chapter 3, we show that the counterexample stems from an artificial restriction in the definition of choice continuity. Thus, with a more natural definition of continuity, the conjecture is true.

As transitivity is a rationality property, why should one be interested in a theory without it? As with other rationality notions, it might be invalid empirically - i.e. the actual behavior of human beings need not be rational in this sense. Having less empirical content, a theory without transitivity obviously stands better up to empirical tests. The loss of empirical content is, however, undesirable. For descriptive purposes, one might instead consider adding the law of demand to the theory without transitivity, since the law of demand is a strong and usually empirically valid property fitting nicely into a theory without transitivity.

Another important development is Afriat's (1967a,1967b,1973,1976) **finitary approach**. It is a natural development of the choice point of view. Afriat's starting point is that one only has a <u>finite number</u> of <u>observations</u> of choices in different price-generated situations, i.e. that one knows only a finite part of the whole demand correspondence. This simplifies

---

[27]As a basis for a normative interpersonal theory, however, preferences should ideally have strong cardinal uniqueness properties, and be interpersonally comparable.

considerably answering the second basic question, by making continuity considerations essentially trivial. It throws less light on the first basic question, however. The basic result of Afriat (1967a) says that if a finite (as a set) demand correspondence satisfies the transitive (or strong) axiom of revealed preference, then there exists a concave, continuous, and monotone utility function that generates (an extension of) the demand correspondence.[28] This might seem surprising at first, since it says that neither continuity, nor concavity has any empirical content in this context. Continuity, however, is trivial by the assumed finiteness. Neither can violations of quasiconcavity be detected, since upper level sets cannot be discriminated from their convex hulls in this context.[29] Thus, the only surprising result is that full concavity is also without empirical content in this context.

Whereas concavity puts no additional restrictions on a finite demand correspondence, it does for an infinite demand correspondence. Chapter 6 will give a full characterization of the demand correspondences which can be represented by concave utility functions. This builds on the characterization of the subdifferential of a convex function in Rockafellar (1970), and the standard first-order conditions. The general idea in Chapter 6 is contained in Afriat's work, but the extension from finite choices is new. In contrast, Kannai (1977) have given three different characterizations of <u>preferences</u> which admits concave utility representations, but these are more complicated than ours in terms of choice.

The finitary approach is essentially the revealed preference approach restricted to finite choices. Of course, the finiteness restriction makes new results and types of arguments available. These are of independent interest.

In both the finitary and the revealed preference approach one can easily introduce and characterize more specific structure like homotheticity and separability. In the finitary approach this is done by Afriat (1967b,1977,1981) in a series of articles. Varian (1983) and Afriat (1987) overview these results.

Empirical work in demand theory usually assumes specific parametric functional forms. With these functional forms, however, it is difficult to test for functional structure, like

---

[28]Indeed a slightly weaker form of the transitive axiom is sufficient.

[29]With more general budget sets - or if <u>tests</u> are allowed, one might violate quasiconcavity.

separability, see Blackorby, Primont, and Russell (1978), Chapter 8.3. But the nonparametric revealed preference conditions we introduce are easily testable. It thus seems appropriate to first test for structure nonparametrically. Then as a second step one can choose parametric forms, according to the results of the structural tests. A problem with these nonparametric tests has been that they are does not say if the conditions are approximately satisfied. In Chapter 4 and 5 I show how one also can easily get approximate nonparametric measures of the satisfaction of some of the interesting preference restrictions.[30] These measures are nonstochastic, however. Stochastic nonparametric tests have been developed by Varian (1985) and Epstein and Yatchew (1985).

## 6. THE STATUS OF THE NEOCLASSICAL THEORY OF THE INDIVIDUAL

Does rational choice theory stand up to the available evidence? It is easy to construct social situations where each of the assumptions in Section 2 are systematically violated. Cognitive psychologists like Tversky (1969) and Tversky and Kahnemann (1981) have given empirical evidence of this. This was to be expected. An analogy from mathematics is that standard arithmetics does not always give the best predictions of the results of our calculations. In fact, even systematic violations do not matter much for a theory of rational behavior, as long as they are unreflected. If, however, individuals persist in their habits after having understood the theory and its implications, then such a normative theory is in trouble.[31]

For a descriptive theory of individual behavior the evidence is less comforting.[32] As the main goal of the neoclassical research program was a descriptive theory of market behavior

[30]The problem remains, that these results concerns individual demand, while one usually is more interested in aggregate demand.

[31]Such is the status of the expected utility hypothesis, as discussed below in Section 9. It is also not clear weather the reference dependence investigated in Tversky and Kahnemann (1991) is irrational. If it is not, this dependence is also interesting normatively.

[32]Schick (1987) proposes to save rationality by loosening the extensionality assumption. Then, however, the theory ends up as essentially without empirical content.

based on rational choice theory, this might look devastating to this research program. This is not the case, however. The point is simple. It is the theory of market demand as a whole which should be tested empirically. Thus, as mentioned, if one can build a nontrivial theory of market demand on the given individual constructs, objections on the individual level do not matter much before one gets an alternative to the neoclassical approach.

Indeed, individual behavior might not be sufficiently regular to be suitable for explanation. One might instead base a theory of market behavior on the behavior of larger groups of individuals with the same relevant observable characteristics. The law of large numbers makes regularity more plausible for such aggregates than for the individuals. This is the basis for an alternative research program proposed by Hildenbrand (1989b).

The theory of rational choice outlined is essentially static, but individual choices usually take place sequentially over time. This leads to problems of periodization and assigning the appropriate period budget to the individual.

The general theory of rational choice in price-generated situations does not excel in empirical content. Therefore one looks for additional structure ot be imposed, depending on the particular context in question. In the next three sections, different such structures are shortly discussed. The first section treats a hierarchy of restrictions on income effects, based on the law of demand, going from homotheticity to the standard general theory. The second treats separability notions, being presumably the most commonly imposed type of restriction in demand theory. Finally, the expected utility hypothesis is discussed.[33]

## 7. THE LAW OF DEMAND

The **law of demand** says that there is a negative correlation between price and quantity changes for fixed income. Thus it generalizes a downward sloping demand curve for fixed income. Downward sloping demand curves was taken as intuitively evident by Walras and Edgeworth. In a sense, the law of demand is a property one always wanted to prove in

[33]In Chapter 5, it is also shown how these types of restrictions interact.

demand theory. Though usually assumed in applied work, it was not much discussed theoretically until Shafer (1977) and Hildenbrand (1983). The reason was presumably that it did not hold in standard theory as sketched above.[34] One also had difficulties in finding preference restrictions corresponding to the law of demand in the standard theory. The latter problem was solved by Kannai (1989). His characterization is fairly complex. Chapter 4 below, however, gives an essentially trivial characterization of the law of demand in a theory without transitivity.

The law of demand is slightly stronger than negative definiteness of the derivative of demand - as the axiom of revealed preference is slightly stronger than negative definiteness on the appropriate tangent plane. The latter is again equivalent to negative semidefiniteness of the derivative of compensated demand, i.e. the Slutsky matrix.

A property related to the law of demand in the same way as the transitive (strong) axiom relates to the axiom of revealed preference is called **cyclical monotonicity**. Cyclical monotonicity corresponds to existence of a concave utility function of which demand is simply the derivative. In that case, (Marshallian) consumer surplus is a concave utility function. Cyclical monotonicity is equivalent to homotheticity and the transitive axiom. It is easily characterized in terms of (transitive) preferences.[35]

One can extend the law of demand (and cyclical monotonicity) to a hierarchy going from homotheticity to the axiom (transitive axiom) of revealed preference, as is done in Chapter 4. All these concepts are easily testable on finite data sets.[36] Also the parameters of these hierarchies give rise to natural measures of the perversity of income effects allowed by a choice, or in other words, measures of the degree of homotheticity and the degree of satisfaction of the law of demand.

[34]The classical counterexample to the law of demand is the Giffen paradox.

[35]This was shown by Shafer (1977) who also introduced these notions. By the stated equivalence, these results coincide with well-known ones.

[36]Provided of course that individual demand is made testable by the additional hypotheses discussed earlier.

## 8. SEPARABILITY

Separability is presumably the most important type of additional restrictions in demand (and production) theory. Indeed even when setting up an applied demand system, one usually more or less explicitly assumes both that demand in the period studied is independent both of demand in earlier and later periods, as well as from the demand for goods not captured by the model in the period studied. This follows directly from a definition of separable choice first formulated by Lau (1969) and Pollak (1970): A demand function (choice) is (weakly) **decentralizable** with respect to a subgroup of goods if the subgroup demand only depends on the subgroup budget. Decentralization ensures that the subgroup revealed preference relations are well-defined relations. A slightly stronger assumption is that they constitute preferences, i.e. are asymmetric. I call this the **subgroup axiom** as it is a subgroup variant of the axiom of revealed preference. Chapter 5 below shows that the subgroup axiom characterizes preference separability. As the standard revealed preference axiom, the subgroup axiom is easily testable on finite data sets. A non-parametric type of testing for separability is better than the use of flexible functional forms. Blackorby, Primont, and Russell (1978, Chapter 8.2) have shown that the use of flexible functional forms provide problems when one wants to test for general separability.

The basic concept of separable preferences was introduced by Stigum (1967) and Gorman (1968).[37] It says that a group of goods is **separable** from the rest if the preferences between the goods in the group is independent of the amount of goods held outside the group. This was later slightly generalized by Bliss (1975).

An interesting result is presented by Gorman (1968). He shows that separability is inherited under intersections, unions, differences, and symmetric differences of subgroups of goods.

---

[37]The basic notion of separability goes further back to Sono (1945) and Leontief (1947a,b). Assuming differentiability, they showed that separable utility corresponds to an independence property of the marginal rate of substitution.

When all subgroups in a partition are separable from their complement (in the partition) we have **complete separability** (with respect to the partition). It gives the existence of an additive utility representation, (shown by Debreu (1959b)). The above result by Gorman reduces the task of checking for additive utility representations considerably.

Lensberg (1987) gives a full characterization of complete separability (with one dimensional factor spaces) in terms of demand functions. His characterization incorporates a solution to the integrability problem. In Chapter 7, Lensberg's characterization is generalized to choice correspondences. The argument is also simplified. Furthermore it is shown that additionally assuming concave utility corresponds to supposing that all goods are normal.

A stronger notion of separability is homothetic separability, which in addition to separability requires that the subgroup choice is homothetic. In contrast to the standard notion of separability, this notion is **self-dual**, i.e. preserved when the roles of goods and (income normalized) prices are interchanged. It is shown by Blackorby, Primont, and Russell (1978, Chapter 5) that homothetic separability in a partition is necessary and sufficient for **additive price aggregation**, saying that there are well-behaved price and quantity indexes for the groups involved.

Chapter 5 below introduces a notion of separability of choice which is simply a subgroup version of the revealed preference axiom. This is simpler both intuitively and computationally than previous nonparametric measures of separability, like the ones in Varian (1983). Based on the I-axiom, a testable hierarchy going from separability to homothetic separability is introduced. This hierarchy gives a nonparametric measure of homothetic separability, given separability. When separability does not hold, this hierarchy is of no use. Then subgroup versions of the measures in Jerison and Jerison (1989) are presumably the way to go.

## 9. CHOICE UNDER RISK AND EXPECTED UTILITY

A risky situation is one where one does not know which state of affairs will obtain at the time of choosing how to act.[38] The standard theory of choice under risk views individuals as acting to maximize expected utility, where the **expected utility** of an act is the sum of the utility of its consequences in each state,[39] weighted by the probabilities of each state. This idea goes back to Bernoulli, who proposed it in 1738 as a resolution to the so called "St Petersburg paradox". The plausibility of the expected utility idea is hard to ascertain directly. Hence it is interesting to characterize the expected utility hypothesis in more accessible terms.

The expected utility hypothesis has been characterized in two different ways. The most common one is in terms of preferences over probability distribution over consequences. The first such characterization was done by von Neumann and Morgenstern (1947). The second approach was initiated by Ramsey (1926) and developed by Savage (1954). Here, expected utility is characterized in terms of preferences over acts, thus knowledge by the individual of the probability distribution is not presupposed. Instead, conditions are given under which (subjective) probabilities can be extracted from the preferences over acts. When these conditions are satisfied, one can look upon an individual as if it has a utility function over consequences and a probability distribution over states, with the expected utility property. This is the more interesting approach, and is much more in the spirit of standard general equilibrium framework of Debreu (1959). Savage's characterization does, however, deviate from this framework by presupposing a convex (and hence infinite) set of states. The first such characterization of expected utility in a finite state framework was by Stigum (1972).

Lensberg (1985) characterized an expected utility function with a strictly concave state utility function in terms of demand functions associating acts to certain lotteries.[40] Chapter 7

---

[38]We assume that the lacking knowledge is stochastic, i.e. not caused by other rational agents, and that the probability distribution is known.

[39]Acts are usually identified with functions from states to consequences.

[40]To avoid the integrability problem which one faces when going from choice to utility he assumed only one basic good.

simplifies (and slightly generalizes) Lensberg's argument by using a slightly different class of lotteries. It shows that an expected utility function is characterized by separability and **diagonal invariance**, saying that sure outcomes are always chosen at the same relative odds.

With a finite set of states, the state utility function is no longer cardinal, in contrast to Savages case with a convex set of states. Also, the probabilities are generally no longer unique, see Fishburn and Odlyzko (1989, Lemma 1).

Diagonal invariance is generalized by Hens (1989), who proceeded to give a characterization of expected utility in terms of preferences (à la Savage) with many goods, and without my concavity assumption. In addition to the generalized diagonal invariance, and separability (i.e. the sure things principle) the characterization involves state independence, which is trivial in the case with only one good. The resulting characterization is simpler than the one by Stigum (1972), mentioned above.

So much for characterizations of expected utility in terms of preferences or choice, which outlines more clearly the implications of the expected utility hypothesis. How does the hypothesis stand up to empirical tests? As discussed in Machina (1989b), the answer is rather negative. Many alternatives to the expected utility hypothesis have therefore been proposed. Machina outlines some of these and shows that especially the so called regret theory accords fairly well with the available evidence.

What about expected utility as a hypothesis about rationality? Diagonal invariance does not look very much like a rationality property: Why should one always choose sure outcomes at the same odds, independent of the size of the sure outcome? Separability looks more plausible when it comes to choice under risk. But Machina's (1989a) parental inheritance example shows that also separability is counter intuitive in certain situations. Hence it can hardly be seen as a rationality property under the choice under risk interpretation either - at least not without narrowing down the interpretation.

## 10. AN OUTLINE OF THE REST OF THIS ESSAY

Chapter 2 deals with the basic relationship between preferences and choice without requiring any special structure of the choice and budget spaces. A quite general characterization is possible in this case. The following three chapters build on Chapter 2, but restrict attention to the classical case where the choice space is the nonnegative orthant of consumer space and budget sets are price-generated.

Chapter 3 gives a full characterization except that completeness is slightly weakened. We also characterize similar preferences without transitivity.

Chapter 4 introduces two hierarchies. One goes from homotheticity via the law of demand to the axiom of revealed preference. The other one goes from homotheticity to the transitive axiom of revealed preference. These two hierarchies are easily characterized in terms of preferences without and with transitivity respectively.

Chapter 5 introduces a subgroup version of the axiom of revealed preference and a hierarchy based on it going from separability to homothetic separability. This hierarchy is also are characterized in terms of preferences.

In Chapter 6 we characterize the demand correspondences stemming from concave utility functions. We also give a simple proof of the existence of least concave utility functions generating a given choice.

In Chapter 7 we characterize the demand correspondences generated by additively separable utility functions. It also extends the characterization to the cases when the component utility functions are concave and of the expected utility kind, respectively.

Finally, in Chapter 8 we characterize the demand correspondences generated by leximin preferences.

May 1986
Revised, May 1990

# CHAPTER 2: THE BASIC THEORY[0]

## 1. INTRODUCTION

Uzawa (1956)[1] gives the first "modern" (i.e. non-analytic or set-theoretic) treatment of the problem of characterizing corresponding classes of preferences, $\mathfrak{P}$, and (rational) choices (demand correspondences), $\mathfrak{C}$. For such a characterization one needs to show, first that the rational choices of any preferences in $\mathfrak{P}$ belongs to $\mathfrak{C}$, and secondly, that any choice in $\mathfrak{C}$ is the rational choice of some preferences in $\mathfrak{P}$. Such a characterization answers the following two questions: First, given some preference structure and assuming rationality, what structure of behavior follows? Secondly, given some structure of behavior, what preference structure (if any) can be attributed to an individual, presupposing again that he is rational? These are central questions in the theory of individual demand.

Despite some obscurities in Uzawas paper, I claim that his approach, slightly modified, is the most direct and suitable for this problem. This seems to have escaped notice by later writers. This paper justifies the claim by extending and clarifying Uzawas approach.

The approach gives a unified treatment of the preference counterparts of three revealed preference axioms: Arrow's (1957) basic axiom, the weakly transitive axiom, and the transitive axiom. The two latter are variants of the rationality condition in Uzawa (1956) and the congruence axiom in Richter (1966). The main result is that on the domain of the choice, these revealed preference axioms characterize classes of preferences which are (partially) recoverable from their rational choice correspondences in natural ways. The basic axiom result is new, the weakly transitive axiom result is essentially the main result in Uzawa (1956), and the transitive one corresponds to the central part of Richter (1971), Theorem 8.

---

[0]Thanks are due to Aanund Hylland, Terje Lensberg, and Lars Thorlund-Petersen for valuable comments to this chapter.

[1]Though his ideas are largely based on Houtakker's lectures at Tokio University in 1955, see Houtakker (1965).

Arrow's basic axiom has some nice properties not shared by Richter's (1971) V-axiom. First, the maximal and the best element definitions of rational choice are equivalent in any theory where the basic axiom holds. This is a result in the theory in contrast to Kim and Richter's (1986, Section 6) meta-result, based on the V-axiom. Secondly it has a nice extension property, as any preferences extending the preferences corresponding to the basic axiom, generates the same choice.[2] Thirdly, it is equivalent to what I call partial recoverability of choice. Fourthly, it is self-dual. And finally, it is, under weak conditions, equivalent to a useful notion (also from Arrow (1957)), called inclusion invariance.

The transitive and the weakly transitive axioms are equivalent if choice is weakly single-sectioned. This notion corresponds to indifference curves with no adjacent kinks and flats. Thus, given this condition, nothing is gained by adding transitive indifference. This strengthens the main result in Kim (1987), who shows a similar result for single-valued choice.

I also argue that the revealed preference axioms are not rationality properties of choice.

The main differences from the standard approach by Richter (1966,1971) are: First, preferences are not required to be complete. Secondly, both a strong and a weak preference relation are taken as basic concepts. Thirdly, a generalized notion of transitive closure is used instead of the traditional one. Fourthly, rationalizability concepts are replaced by slightly stronger notions of (partial) recoverable choice. Finally, the maximal element definition of rational choice is used instead of the best element one. Of these modifications, only the fourth is fully realized in Uzawa's article. The first change allows the naturally generated (revealed) preferences to be preferences, which they are not generally if completeness is required. The second makes the framework more appropriate for studying incomplete preferences. Thereby a strengthened notion of preference asymmetry is intuitively evident. The revealed preference axioms then simply express the required asymmetry of the appropriate generated (revealed) preferences. The third change allows us to get the desired transitivity properties of the generated (revealed) preferences. The last modification ensures that the revealed preference axioms are equivalent to partial recoverability of choice (from naturally generated preferences,

---

[2]Appendix 1 discusses (and simplifies) a similar result due to Clark (1988) using the best element definition of rational choice.

Lemma 2), and to full recoverability on the domain of choice The latter result was derived by Clark (1985, Theorem 3) in the weak and weakly transitive axiom cases. I found it, and the nice underlying Lemma 2, however, independently by analyzing Uzawa's article - which contains the "only if" part of Lemma 2 in the weakly transitive case treated there.

So far our main argument for the incomplete preference framework is that it is the natural one for the revealed preference assumptions. In the next chapter it is furthermore shown that the framework also gives simple and natural characterizations of demand theory.

The rest of the paper is organized as follows: Section 2 introduces some material on relations up to a natural weakened preference concept. Section 3 introduces the basic concepts of revealed preference theory: The natural maps between preferences and choice, the revealed preference axioms, and the partial recoverability notions. With the aid of these concepts the main results are proven quite simply. Section 4 gives a short treatment of indirect preferences and duality, and the conclusion discusses briefly the relevance of the results.

There are three appendices. The first shortly discusses Richter's (1971) V-axiom, based on the best element definition of rational choice and the extension property in this case. The second introduces the notion of inclusion invariance (from Arrow (1957)), and shows it to be equivalent to the basic axiom under weak conditions. It also gives conditions under which the V-axiom implies the basic axiom. The third treats completeness.

## 2. RELATIONS

Relations (correspondences) are identified with their graphs. Hence binary relations are sets of ordered pairs. Let $\mathcal{X}$ and $\mathcal{P}$ be nonempty sets, called the **goods** and **budget space**, respectively, with typical elements **x** and p. Given a relation $B \subseteq \mathcal{X}\times\mathcal{P}$, the **inverse** of B, $\mathbf{B^{-1}} = \{(p,x)\,|\,(x,p) \in B\}$. Also $\mathbf{B(p)} = \{x\,|\,(x,p) \in B\}$ is the **upper section** or **value** of B at p, $\mathbf{B^{-1}(x)} = \{p\,|\,(x,p) \in B\}$ is the **lower section** or **inverse value** of B at x, $\mathbf{DB} = \{p\,|\,B(p) \neq \emptyset\}$ is the (effective) **domain** of B, and $DB^{-1}$ is the **inverse domain** or **range** of B.

A relation B is a **budget correspondence** over $\mathcal{X}\times\mathcal{P}$ if $B \subseteq \mathcal{X}\times\mathcal{P}$, $\mathcal{P} = DB$, and $\mathcal{X} = DB^{-1}$; and a relation c is a **choice** (correspondence relative to B) if $c \subseteq B$. In what follows, **B** is a fixed budget correspondence over $\mathcal{X}\times\mathcal{P}$ and **c** a choice. Lower case p's should be thought of as situational parameters, and B(p) as the subset of $\mathcal{X}$ from which it is possible to choose in situation p, read: the **budget** (at) **p.**[3] Similarly, $x \in c(p)$ is read: x is a **choice at p**. A choice c is **single-valued** if for all p and $x,x' \in c(p)$, $x = x'$.

Let P,Q, and R be relations on $\mathcal{X}$ (i.e. $P,Q,R \subseteq \mathcal{X}^2 = \mathcal{X}\times\mathcal{X}$). The **composition** of P and R, **P∘R**, is defined by $x' \in P{\circ}R(x)$ if there is $x'' \in R(x)$ such that $x' \in P(x'')$. The following slight generalization of the transitive closure of a relation is essential: The **P-closure** of R (with respect to ∘), ${}^{\mathbf{P}}\mathrm{R}$, is defined inductively by $R \subseteq {}^{P}R$ and if $Q \subseteq {}^{P}R$, then $P{\circ}Q, Q{\circ}P \subseteq {}^{P}R$. Hence ${}^{P}R$ is the smallest relation extending R and closed under composition with P. This follows as it is easy to show by induction, first that the P-closure is P-closed (i.e. ${}^{P}({}^{P}R) \subseteq {}^{P}R$), and secondly that inclusion is preserved by P-closures (i.e. if $R \subseteq R'$, then ${}^{P}R \subseteq {}^{P}R'$).

A relation Q over $\mathcal{X}$ is **reflexive** if for all x, $x \in Q(x)$. A pair of relations (P,R) over $\mathcal{X}$ is **asymmetric** if for all x and x', not both $x' \in P(x)$ and $x \in R(x')$; and **preferences** over $\mathcal{X}$ if $P \subseteq R \subseteq \mathcal{X}$, R is reflexive, and (P,R) is asymmetric.[4] In the sequel **(P,R)** are preferences over $\mathcal{X}$. Thus P is a strict and R a weak preference relation.

---

[3]The terminology stems from the standard demand theory interpretation. There $\mathcal{X} = \mathcal{P} = \mathbb{R}^{I^0}$ for a finite set $I^0$ and B is defined by $x \in B(p)$ if $px \leq 1$. Here $I^0$ is a interpreted as a set of goods and x and p are their quantities and **budgets** (i.e. prices divided through income), respectively. (The budgets are intuitively the values of B, but are identified with the corresponding arguments as B is bijective.) Nothing in this article depends on these choices and interpretation, however.

[4]Thus preferences (P,R) only requires that $P \subseteq R\text{-}R^{-1}$ in addition to the reflexivity of R.

Preferences (P,R) is **transitive** if $R \circ R \subseteq R$, $R \circ P \subseteq P$, and $P \circ R \subseteq P$; **weakly transitive** if $P \circ P \subseteq P$, $R \circ P \subseteq R$, and $P \circ R \subseteq R$; and **complete** if for all x and x', $x' \in P(x)$ or $x \in R(x')$.[5] Finally say that (P',R') **extends** (P,R) (or (P,R) is **subpreferences** of (P',R')) if $P \subseteq P'$ and $R \subseteq R'$ (Denoted **(P,R) ⊆ (P',R')**).

**Remark 1:** With both a strong and a weak preference relation as basic concepts, the above (strengthened) asymmetry is a natural defining property of preferences, as it is a direct consequence of the intended meaning of the preferences (P,R). For to say that a person both prefers one situation to another, and at the same time finds the latter at least as good (in the same sense - and seriously), indicates, I think, that one does not understand the language used. "Properties following directly from the meaning of the terms involved", is one interpretation of "rationality assumptions" in economics. Under this interpretation, completeness is hardly a rationality assumption, in contrast to different transitivity notions (together with optimization - if this is well-defined). Another way to justify the rationality label is through "money pump" arguments. Again completeness is hardly justifiable.

**Remark 2:** It is easily verified by induction that $({}^{P}P, {}^{P}R)$ and $({}^{R}P, {}^{R}R)$ are the weakly transitive and the transitive closure of (P,R), respectively. In the same manner one verifies that the condition for these to be preferences can be simplified:

$({}^{P}P, R)$ is asymmetric if and only if $({}^{P}P, {}^{P}R)$ is asymmetric.

$(P, {}^{R}R)$ is asymmetric if and only if $({}^{R}P, {}^{R}R)$ is asymmetric.

In the first case both sides say that there are no preference cycle with at most one weak relation, whereas in the latter case they say that there are no such cycle with at least one strict one.

---

[5] This notion of completeness coincides with the traditional one, as it is easy to verify that (P,R) is complete if and only if $P = R - R^{-1}$ and for all x and x', $x' \in R(x)$ or $x \in R(x')$. If preferences are complete, transitivity and weak transitivity coincides with transitivity of R and P, respectively. Without completeness, however, our notions are generally stronger.
A little aside: The old problem (Richter (1971)p. 36) of characterizing complete but not necessarily reflexive preferences, seems to be a "Scheinproblem". The point is that without a reflexive relation (other than the identity) it is not even clear how to <u>define</u> completeness.

## 3. THE GENERAL THEORY

As mentioned, the problem is to characterize corresponding classes of preferences $\mathfrak{P}$ and rational choices $\mathfrak{C}$. This is done by means of natural maps between the two types of objects. The first task is to introduce these maps, namely the rational choice map $\gamma$ generating a rational choice (correspondence) from preferences (P,R) and natural revealed preference maps $\pi$ generating preferences from a choice (correspondence) c.

Given preferences, the natural way to express rational choice is to require preferences to be maximized in any situation - if this is well-defined. This can be done in two ways: The **maximal element choice, $c^P$**, is defined by $x \in c^P(p)$ if $x \in B(p)$, and for all $x' \in P(x)$, $x' \notin B(p)$; and the **best element choice, $c^R$**, is defined by $x \in c^R(p)$ if $x \in B(p)$, and for all $x' \in B(p)$, $x \in R(x')$. Corresponding to these two definitions, there are two natural maps from preferences to choice. Only the first is used here, however, namely the **rational choice** map, $\gamma$, defined by $\gamma(P,R) = c^P$. Also say that $\gamma(P,R)$ is the **rational choice**, given (P,R).

To obtain the revealed preference maps, first define the **preferences generated by c**, denoted **$(P^c,R^c)$**, by $x \in P^c(x')$ if there is p such that $x \in c(p)$ and $x' \in B(p)\setminus c(p)$; and $x \in R^c(x')$ if $x = x'$ or there is p such that $x \in c(p)$ and $x' \in B(p)$. Define the **direct**, **weakly transitive**, and **transitive revealed preference maps**, ${}^0\pi$, ${}^P\pi$, and ${}^R\pi$, respectively, by ${}^0\pi(c) = (P^c,R^c)$, ${}^P\pi(c) = ({}^{P^c}P^c, {}^{R^c}R^c)$, and ${}^R\pi(c) = ({}^{R^c}P^c, {}^{R^c}R^c)$. The latter terms are chosen as, by Remark 2, ${}^P\pi(c)$ is the smallest weakly transitive and ${}^R\pi(c)$ the smallest transitive extension of the preferences generated by c, ${}^0\pi(c) = (P^c,R^c)$.

The "preference" terminology is so far misleading, as preference asymmetry is not ensured. The following **revealed preference axioms** justifies this in the respective cases: A choice c satisfies the **basic axiom** if $(P^c,R^c)$ is asymmetric, the **weakly transitive axiom** if $({}^{P^c}P^c,R^c)$ is asymmetric, and the **transitive axiom** if $(P^c,{}^{R^c}R^c)$ is asymmetric.[6] By Remark 2,

---

[6]There are many variants of the revealed preference axioms, and terminology is not fixed. The most well-known are the single-valued ones: A choice c satisfies the **single-valued axiom** if $R^c$ is antisymmetric and **the single-valued transitive axiom** if ${}^{R^c}R^c$ is antisymmetric, where a relation Q is **antisymmetric** if for all x and x', $x \in Q(x')$ and $x' \in Q(x)$ implies that $x = x'$. The single-valued (transitive) axiom is equivalent to single-valuedness and the (transitive) axiom. The first of these is Samuelson's (1938) weak axiom, and the second Houtakker's (1950) strong axiom.

these axioms are equivalent to the asymmetry of ${}^{0}\pi(c)$, ${}^{P}\pi(c)$, and ${}^{R}\pi(c)$, respectively.

To obtain the desired characterization of classes of preferences $\mathfrak{P}$ and choices $\mathfrak{C}$ by means of the natural maps, a necessary requirement is that any choice is recoverable as the rational choice from its generated preferences. Hence, presupposing the rational choice map, say that a choice c is **$\pi$-recoverable** if $c = \gamma\circ\pi(c)$. More specifically one might say that c is **recoverable** if $\pi = {}^{0}\pi$, **weakly transitive recoverable** if $\pi = {}^{P}\pi$, and transitively recoverable if $\pi = {}^{R}\pi$. Analogously, preferences (P,R) is **$\pi$-recoverable** if $\pi\circ\gamma(P,R) = (P,R)$.[7] Choices and preferences are **partially $\pi$-recoverable** if = is replaced by $\subseteq$ in the above definitions.

The characterization theorem can now be stated. The first part expresses that, on its domain, $\pi$-recoverability of choice is equivalent to the corresponding revealed preference axiom. Thus the revealed preference axioms have a double role: They ensure that the appropriate revealed preferences are preferences, and at the same time that, on its domain, choice is recoverable from these preferences.

The second part says, first, that $\gamma$ and $\pi$ are bijections between the classes $\mathfrak{P}^{\pi}$ of $\pi$-recoverable preferences (relative to the budget space $\mathcal{P}$) and $\mathfrak{C}^{\pi}$ of $\pi$-recoverable choices c restricted to Dc. Thus these classes are in a sense identical. Next, let $\mathfrak{P}^{\pi}_{PR}$ be the class of partially $\pi$-recoverable preferences (relative to $\mathcal{P}$). Clearly preferences in $\mathfrak{P}^{\pi}_{PR}$ extend the ones in $\mathfrak{P}^{\pi}$. More interestingly, any extension of some preferences in $\mathfrak{P}^{\pi}$ generate the same choice as these, at least on the domain of choice. Hence, given some preferences (P,R) in $\mathfrak{P}^{\pi}_{PR}$, $\pi\circ\gamma(P,R)$ is the observationally relevant subpreferences of (P,R), relative to $\mathcal{P}$.

A diagram of classes and maps (arrows) is **commutative** if all maps from one class into another (or the same) class obtained by the composition of maps in the diagram are equal. More formally the result is:

[7]More abstractly, if they hold universally, these definitions require the revealed preference maps to be right and left inverses to the rational choice map, respectively.

**CHARACTERIZATION THEOREM:** For each $\pi$:

1) On Dc, $\pi(c)$ is asymmetric if and only if $c = \pi \circ \gamma(c)$.

2) The following diagram is commutative:

$$\begin{tikzcd}
\mathfrak{P}^{\pi}_{PR} \arrow[d, shift left] \arrow[dr] & \\
\mathfrak{P}^{\pi} \arrow[u, shift left] \arrow[r, shift left] \arrow[loop left, "\iota"] & \mathfrak{C}^{\pi} \arrow[l, shift left] \arrow[loop right, "\iota"]
\end{tikzcd}$$

Here $\uparrow$ is an arbitrary extension, $\downarrow$ the restriction map, and $\iota$ the identity map on a class.

Roughly, the result says that the basic axiom characterizes (partially) $^{0}\pi$-recoverable preferences, the weakly transitive axiom characterizes (partially) $^{P}\pi$-recoverable preferences, and the transitive axiom characterizes (partially) $^{R}\pi$-recoverable preferences.

Instead of "shrinking" preferences (P,R) down to their observational part, $\pi \circ \gamma(P,R)$, they have usually been extended to obtain completeness. Then observationally irrelevant information (relative to the budget space, $\mathcal{P}$) is generally built into the preferences. In this abstract case, however, completeness imposes no extra restrictions on choice. In the transitive case, the conceptually somewhat complex partial recoverability notion can be replaced by the simpler completeness in the characterization. (The last two statements are justified in Appendix 3).

The theorem will now be verified. First, $\pi$-recoverability is trivially preserved by both the rational choice and the revealed preference maps (i.e. if (P,R) is $\pi$-recoverable, then $\gamma(P,R)$ is $\pi$-recoverable; and for all $\pi$, if c is $\pi$-recoverable, then $\pi(c)$ is $\pi$-recoverable). This verifies the lower part of the diagram.

It is well-known that the rational choice map reverses inclusions on the P-component, which immediately gives that it also preserves partial $\pi$-recoverability:[8]

**LEMMA 1:** If $P' \subseteq P$, then $c^{P} \subseteq c^{P'}$.

---

[8]In contrast, inclusions are preserved on the R-component.

Proof: Let $P' \subseteq P$ and $x \in c^{P}(p)$. Then $x \in B(p)$. Let $x' \in P'(x)$. Then by assumption, $x' \in P(x)$. Hence, since $x \in c^{P}(p)$, $x' \notin B(p)$, so $x \in c^{P'}(p)$. □

Partial $\pi$-recoverability of choice is equivalent to the revealed preference axioms. E.g., in the basic axiom case, $c \subseteq c^{P^c}$ is equivalent to the asymmetry of $(P^c,R^c)$. Or in terms of the revealed preference and rational choice mappings:

**LEMMA 2:** For all $\pi$: $c \subseteq \gamma \circ \pi(c)$ if and only if $\pi(c)$ is asymmetric.

Proof: **Basic axiom:** For the "only if" part, let $x' \in P^c(x)$ and $x \in R^c(x')$. Then there is p such that $x \in c(p)$ and $x' \in B(p)$. Hence by assumption $x \in c^{P^c}(p)$. Clearly $x \in B(p)$, and since $x' \in P^c(x)$, $x' \notin B(p)$, contradiction. Conversely, assume $x \in c(p) \backslash c^{P^c}(p)$. Then there is $x' \in P^c(x)$ such that $x' \in B(p)$. But then $x \in R^c(x')$, contradicting asymmetry.
**Weakly transitive axiom:** Substitute ${}^{P^c}P^c$ for $P^c$ in the basic axiom case proof.
**Transitive axiom:** Let $x' \in P^c(x'')$ and $x'' \in {}^{R^c}R^c(x')$. Then there is $x \in R^c(x')$ such that $x' \in {}^{R^c}P^c(x)$. Next, substitute ${}^{R^c}P^c$ for $P^c$ in the basic axiom case proof. □

We now verify the converse of partial recoverability on the domain of choice. Together with Lemma 2, this establishes the first part of the theorem:

**LEMMA 3:** On Dc, for all $\pi$: $\gamma \circ \pi(c) \subseteq c$.

Proof: As $\gamma$ reverses inclusions, the result follows from the fact that ${}^{0}\pi(c) \subseteq {}^{P}\pi(c) \subseteq {}^{R}\pi(c)$ if one can show that $\gamma \circ {}^{0}\pi(c) \subseteq c$. Assume not, and let $x \in c^{P^c}(p) \backslash c(p)$, where $p \in Dc$. Let $x' \in c(p)$. Then since $x \in B(p)$, $x' \in P^c(x)$, contradicting $x \in c^{P^c}(p)$. □

Finally we have an extension property, saying that any preferences extending the preferences generated by a choice rationalize it - at least on its domain:[9]

---

[9]Appendix 1 states a variant of this result using the best element definition of rational choice.

**LEMMA 4:** On Dc, if $(P^c,R^c) \subseteq (P,R)$, then $c = c^P$.

**Proof:** On Dc, it follows directly from Lemmas 1 and 3 that $c^P \subseteq c$. For the converse, assume that $x \in c(p)\backslash c^P(p)$. Then there is $x' \in P(x) \cap B(p)$. Hence $x \in R^c(x')$, and therefore since $R^c \subseteq R$, also $x \in R(x')$, contradicting preference asymmetry. □

Together with Lemma 3 and the corollary to Lemma 1, this verifies the upper part of the diagram, and the theorem is proved.

So far, the R-part of partial $^0\pi$-recoverability has not been used. I show that it implies that the maximal and best element choices are equal, by first establishing that preference asymmetry implies $c^R \subseteq c^P$, and next that the R-part of partial $^0\pi$-recoverability is equivalent to $c^P \subseteq c^R$:

**PROPOSITION 1:** 1) $c^R \subseteq c^P$

2) $c^P \subseteq c^R$ if and only if $R^{c^P} \subseteq R$

**Proof: 1):** Let $x \in c^R(p)$. Then $x \in B(p)$. Let $x' \in P(x)$ and assume $x' \in B(p)$. Since $x \in c^R(p)$, then $x \in R(x')$, contradicting asymmetry. Hence $x' \notin B(p)$, so $x \in c^P(p)$.

**2):** For the "if" part, let $x \in c^P(p)$ and $x' \in B(p)$. Then $x \in B(p)$ and $x \in R^{c^P}(x')$. Hence by assumption $x \in R(x')$, thus $x \in c^R(p)$. Conversely, let $x' \in R^{c^P}(x)$. Then there is p such that $x' \in c^P(p)$ and $x \in B(p)$. Hence by assumption $x' \in c^R(p)$, so by the definition of $c^R$, $x' \in R(x)$.□

It follows from the characterization theorem that the maximal and the best element definitions of rational choice are equal in any theory where the basic axiom holds.[10] Next, the

---

[10]Kim and Richter (1986) use the weaker V-axiom instead of the axiom. Then the equality between the maximal and the best element definitions of rational choice is no longer generally valid. They get a meta-result corresponding to Proposition 1 in their Section 6, however, by exhibiting a correspondence between proofs in the frameworks based on the maximal and the best element definition, respectively, by mapping R on $\mathbf{P^*} = \mathcal{X}\text{-}R^{-1}$ and P on $R^* = \mathcal{X}\text{-}P^{-1}$. The core of their result is two convexity concepts of preferences which are carried into each other by these maps, together with the fact that completeness is sufficient to guarantee the equivalence between the two definitions of rational choice.

weakly transitive and the transitive axioms are equivalent if a choice c is not **many-to-many** (i.e. if for all p, p', $x \in c(p)$, and $x' \in c(p')\cap c(p)$, $p = p'$ or $x = x'$), expressing that indifference curves have no adjacent kinks and flats:

**PROPOSITION** 2: If a choice is satisfies the weakly transitive axiom and is not many-to-many, then it satisfies the transitive axiom.

**Proof:** To show that $({}^{R^c}P^c, {}^{R^c}R^c)$ is asymmetric, i.e. that there are no revealed preference cycles with at least one strict relation, $P^c$, by induction on the number of weak relations, $R^c$.

**Base case:** Here there is one weak relation only. This is the weakly transitive axiom.

**Induction case:** Assume inductively that there are no such cycle with less than n weak relations. Such a cycle with n weak relations can then be decomposed as: $x \in P^c(x')$, $x' \in Q(x'')$ and $x'' \in R^c(x)$, where Q contains n-1 weak relations. Then there is p" such that $x'' \in c(p'')$, and $x \in B(p'')$. If $x \notin c(p'')$, then $x'' \in P^c(x)$, and the result follows by the induction assumption. Otherwise, $x \in c(p'')$. Then since c is not many-to-many, $x = x''$ or $p = p''$. In the first case, $x \in P^c(x')$ and $x' \in Q(x)$. In the second case, $x'' \in P^c(x')$ and $x' \in Q(x'')$. In both cases the result follows by the induction assumption. □

Trivially, the same result follows if choice is single-valued (or inversely single-valued). Hence an analogy of the main result in Kim (1987), namely that in the single-valued case, the choice consequences of partially ${}^{P}\pi$- and ${}^{R}\pi$-recoverable preferences are identical, is an obvious corollary to the characterization theorem.[11]

The revealed preference axioms are often looked upon as rationality properties. If rationality is explained as in Remark 1, this is unwarranted, as preference completeness seems

---

[11]Even in the single-valued case, this result improves the one in Kim (1987) mentioned in the introduction. He uses two other transitivity notions which are less intuitive and intuitively somewhat stronger than weak transitivity: Preferences (P,R) is **semi-transitive** if $R \circ P \circ P \subseteq P$ and $P \circ P \circ R \subseteq P$, and **pseudo-transitive** if $P \circ R \circ P \subseteq P$. He also assumes that $P = R - R^{-1}$. To show that weak transitivity follows from these notions, however, needs some extra assumptions (to establish $P \circ R \subseteq R$ and $R \circ P \subseteq R$). In both cases, completeness clearly suffices. In the semi-transitive case, if $\mathcal{X}$ is a topological space, open lower P-sections and **local nonsatiation** (i.e. for all x, $x \in \mathrm{cl}P(x)$ where **cl** denotes topological closure) suffices, since then $P \subseteq P \circ P$.

necessary to justify them. For the three revealed preference axioms in the text, this follows from the discussion in Appendix 3. On the surface, the V-axiom is different, as it follows directly from the definitions involved. One of these, however, is the best element definition of rational choice, which roughly only makes sense if preferences are complete. Thus completeness is again presupposed.

## 4. DUALITY AND INDIRECT PREFERENCES

**Indirect preferences** are preferences over the space of budgets $\mathcal{P}$. The notation chosen makes the duality between direct and indirect preferences transparent. Definitions concerning indirect preferences are structurally identical to the direct ones, but interpretations differ. The basic idea is from Richter (1979). The theory presented is, however, stated in terms of preferences instead of utility. By Lemma 2, our theory gets more content as our revealed preference axioms are exact conditions for analogies of his notions of partial rationalization.

In the following, **(W,V)** is indirect preferences. The expression $p' \in W(p)$ is interpreted as: The budget p' is worse than p. Similarly $p' \in V(p)$ is interpreted as: p' is at least as bad as p. The **minimal element choice,** $c^W$, is defined by $p \in c^W(x)$ if $p \in B^{-1}(x)$ and for all $p' \in W(p)$ $p' \notin B^{-1}(x)$, the **worst element choice,** $c^V$, is defined by $p \in c^V(x)$ if $p \in B^{-1}(x)$ and for all $p' \in B^{-1}(x)$, $p \in V(p')$, and the **indirect rational choice map,** $\gamma^*$, from indirect preferences to inverse choices, is defined by $\gamma^*(W,V) = c^W$.

The **indirect preferences generated by** $\bar{c} = c^{-1}$, $(\mathbf{W^{\bar{c}},V^{\bar{c}}})$, is defined by $p \in W^{\bar{c}}(p')$ if there is x such that $p \in c^{-1}(x)$ and $p' \in B^{-1}(x) \backslash c^{-1}(x)$, and $p \in V^{\bar{c}}(p')$ if $p = p'$ or there is x such that $p \in c^{-1}(x)$ and $p' \in B^{-1}(x)$. The revealed indirect preference maps ${}^0\pi^*$, ${}^W\pi^*$, and ${}^V\pi^*$ are defined in the obvious way.

The setup is completely dual to the one of usual revealed preference theory. Thus proofs and results immediately carry over back and forth by replacing signs in the one line below with the corresponding ones in the other (also in composites like $c^{Pc}$ and ${}^R\pi$):

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $\mathcal{X}$ | x | B | c | P | R | $\gamma$ | $\pi$ |
| $\mathcal{P}$ | p | $B^{-1}$ | $c^{-1}$ | W | V | $\gamma^*$ | $\pi^*$ |

Many choice properties are self-dual, e.g. weakly single-sectioned. The next lemma shows that our revealed preference axioms are self-dual:

**LEMMA 5:** For all $\pi$, $\pi(c)$ is asymmetric if and only if $\pi^*(c^{-1})$ is asymmetric.

**Proof:** I only prove the basic axiom case, the other cases are similar. Also I only prove the "only if" part, as the converse is dual. Let $p' \in W^{\bar{c}}(p)$ and $p \in V^{\bar{c}}(p')$. Then there is x' such that $p' \in c^{-1}(x')$ and $p \in B^{-1}(x')\backslash c^{-1}(x')$ and there is x such that $p \in c^{-1}(x)$ and $p' \in B^{-1}(x)$. Hence $x' \in c(p')$, $x' \in B(p)\backslash c(p)$, $x \in c(p)$, and $x \in B(p')$. But then $x' \in R^c(x)$ and $x \in P^c(x')$, contradicting the asymmetry of $\pi(c)$. □

The single-valued axioms, on the other hand, are not self-dual. Their duals have been treated by Sakai (1977). Note, however, that by the present duality mappings, there are no need for a separate theory characterizing indirect preferences, as the structure of this dual theory is the same as that of the primal one.

## 5. CONCLUDING REMARKS

The abstract nature of the theory presented makes it quite generally applicable, and at the same time quite "airy". What it does is to provides a conceptual framework for more specific theories, i.e. ones applicable only in narrower contexts, where more structure is introduced into the theory. A nice feature of the theory is then that extensions of the characterization theorem are obtained simply by showing that the additional corresponding notions of preferences and choice are carried into each other under the rational choice and the appropriate revealed preference map.

Which additional properties hold in the theory are of course dependent upon the particular interpretation one has in mind which implies a choice of the choice space $\mathcal{X}$, the budget space

$\mathcal{P}$, and the budget correspondence B. The following chapters impose additional restrictions within the standard demand theory framework of Footnote 2.

## APPENDIX 1:
## THE V-AXIOM AND THE BEST ELEMENT DEFINITION OF RATIONAL CHOICE

The characterization theorem needs the maximal element definition of rational choice, except in the transitive case. In the nontransitive case, the analog recoverability property using the best element definition of rational choice is Richters (1971) **V-axiom:** $c^{R^c} \subseteq c$. (The converse $c \subseteq c^{R^c}$ is trivial). The V-axiom is weaker than the basic axiom, even for single-valued choice. If, however, the values of the budget correspondence are additionally closed under intersections, the V-axiom implies the basic axiom, as shown in Appendix 2. In contrast to the basic axiom, the V-axiom is not self-dual, as shown in Example 1.

In the weakly transitive case, the analogy of the V-axiom is (*): $c^{R'} \subseteq c$, where $R' = {}^{P^c}R^c$. In general, (*) does not imply the basic axiom. This follows as (*) is satisfied in Example 1 below. If choice is single-valued, however, (*) implies the weakly transitive axiom, as is easily shown. One might believe that (*) characterizes complete, weakly transitive preferences, in analogy with Richter's (1971) result, that the V-axiom characterizes complete preferences. This is not generally the case, however, as is verified in Example 2 below.

With the best element definition of rational choice, the extension property is the following, saying that preferences extending the revealed preferences generate the same choice, at least on the domain of choice:[12]

**LEMMA 4'**: If (P,R) is preferences extending $(P^c,R^c)$, then on Dc, $c = c^R$.

[12]This is a strenghtening of the interesting part of the main result in Clark (1988). Clark's article is quite similar to parts of the present work, except that he uses the best element definition of rational choice instead of the maximal element one. The strengthening is that neither is the axiom needed, nor need the strict preference relation be the asymmetric part of the weak one.

Proof: Clearly $c \subseteq c^{R^c}$ and since the best element definition preserves inclusions, $c^{R^c} \subseteq c^{R}$. Conversely, assume that $x \in c^R(p) \backslash c(p)$, and let $x' \in c(p)$. Then $x' \in P^c(x) \subseteq P(x)$. But since $x \in c^R(p)$, $x \in R(x')$, contradicting preference asymmetry. □

As this result does not use the basic axiom, one might believe that it can be applied in a theory based on the V-axiom also. There, however, one use $R^c \backslash (R^c)^{-1}$ instead of $P^c$ as a strict preference relation. Without the basic axiom generally only $R^c \backslash (R^c)^{-1} \subseteq P^c$, which does not ensure that extensions of $(R^c \backslash (R^c)^{-1}, R^c)$ extend $(P^c, R^c)$. Hence one really need the basic axiom for the extension property also in this case.

**Example 1:** The V-axiom is not self-dual: Let $\mathcal{X} = \{0,1,2\}$ and $\mathcal{P} = \{0,1\}$, and let B and c be given by: $B(0) = \{0,1,2\}$, $c(0) = \{1,2\}$, and $B(1) = c(1) = \{0,1\}$. One easily verifies that the V-basic axiom holds at the six possible combinations in $\mathcal{X} \times \mathcal{P}$. Also, $0 \subseteq \bar{c}^{V'}(0) \backslash c(0)$, where $V' = V^{\subset}$, hence the dual of the V-axiom is false. Clearly also the basic axiom is violated here.

**Example 2:** Counterexample to: If (P,R) is weakly (semi- or pseudo-) transitive and complete with single-valued rational choice, then (*) holds for $c^R$: Let $\mathcal{X} = \{0,1,2\}$ and $\mathcal{P} = \{0',1',2'\}$. Let R be reflexive, $0 \in R(2)$, $2 \in R(1)$, and $1 \in R(0)$. Furthermore let $B(0') = \{0,1\}$, $B(1') = \{1,2\}$, and $B(2') = \{0,2\}$. Then $0 \notin c^R(0')$, $2 \in c^R(1')$, $0 \in c^R(2')$, and $2 \notin c^R(2')$. Hence $0 \in P^{c^R}(2)$ and $2 \in R^{c^R}(1)$. Thus (0,0') violates (*) for $c^R$, but all the assumptions are satisfied.

## APPENDIX 2: INCLUSION INVARIANCE

Under weak assumptions, the basic axiom is characterized by another interesting choice notion (due to Arrow (1957)), called inclusion invariance: A choice c is **downward inclusion invariant** ($\mathrm{II}\downarrow$) if $B(p') \subseteq B(p)$ implies $c(p) \cap B(p') \subseteq c(p')$, it is **upward inclusion invariant** ($\mathrm{II}\uparrow$) if $B(p') \subseteq B(p)$ and $c(p) \cap B(p') \neq \emptyset$ implies $c(p') \subseteq c(p)$, and it is **inclusion invariant** (II) if it is both downward and upward inclusion invariant.[13] [14] This says that if the budget at p' is contained in the budget at p, then the choices at p' are exactly the choices at p contained in the budget at p', provided any such exists.

To verify that inclusion invariance is equivalent to the basic axiom, I first show it to follow from ${}^0\pi$-recoverability (and hence of the basic axiom on the domain, by the first part of the characterization theorem). Next, with sufficiently many budgets, the basic axiom follows from inclusion invariance (The proofs are slight generalizations of ones by Arrow):

**LEMMA 6:** A ${}^0\pi$-recoverable choice is inclusion invariant.

**Proof:** Let $B(p') \subseteq B(p)$ and $x \in c(p) \cap B(p')$.

We first show that $x \in c(p')$. By recoverability, first $x \in c^{Pc}(p)$, and secondly, it is sufficient to show that $x \in c^{Pc}(p')$. If not, since $x \in B(p')$, there is $x' \in P^c(x) \cap B(p')$. But then $x' \in B(p)$, contradicting $x \in c^{Pc}(p)$. Secondly, let $x' \in c(p')$. To show that $x' \in c(p)$. If not, then $x \in P^c(x')$. Then, since by recoverability $x' \in c^{Pc}(p')$, it follows that $x \notin B(p')$, contradiction. □

---

[13]The downward notion is due to Nash (1950). It is also called independence of irrelevant alternatives, condition $\alpha$, or Chernoffs axiom. If choice is at most single-valued, it implies full inclusion invariance. In contrast to the revealed preference axioms above (including the V-axiom), downward inclusion invariance is a rationality property of choice, under the interpretation of Remark 1, as it easily follows from the maximal element definition of rational choice alone (given preferences). The upward notion is also called condition $\beta$+. It is easily verified by a counterexample that this is not a rationality property in the above sense.

[14]If B is defined over $\mathbb{R}^n_+ \times \mathbb{R}^n_+$ by $x \in B(p)$ if $px \leq 1$, then $B(p') \subseteq B(p)$ is equivalent to $p' \geq p$, thus on "vertical" budgets the duál of downward inclusion invariance (and hence by the following lemma, the axiom) implies that choice is not single-valued.

**LEMMA** 7: Let the values of B be closed under intersections (with at least two elements). Then an inclusion invariant choice satisfies the basic axiom.[15]

**Proof:** Assume $x' \in P^c(x)$ and $x \in R^c(x')$. Then there is p' such that $x' \in c(p')$ and $x \in B(p')\setminus c(p')$ and there is p such that $x \in c(p)$ and $x' \in B(p)$. Let $B(p'') = B(p) \cap B(p')$. Then $x,x' \in B(p'')$, so by downward inclusion invariance, $x,x' \in c(p'')$. But as $x' \in c(p') \cap B(p'')$, by upward inclusion invariance, $c(p'') \subseteq c(p')$. Hence $x \in c(p')$, contradiction. □

As it is well-known that the V-axiom implies downward inclusion invariance, the relationship between the different primal "variants" of the basic axiom are in general as follows:

V ⟵ A ⇄ PR<br>
↑↓ ↑ ↙<br>
II↓ ⇄ II

Here **A** stands for the basic axiom, **V** for the V-axiom, **PR** for partial recoverability of choice, and the different arrows are implications with the following meaning:

⟶: holds in general.

⇀: holds on the domain of c.

⟶: holds if the values of B are closed under intersections.

⟹: holds if c is single-valued.

Hence if the values of B are closed under intersections and choice is single-valued, the V-axiom is equivalent to the basic axiom.

---

[15]Despite Arrows argument, Lemma 7 does not seem to be well-known. E.g. Suzumura (1983) denies it, and provides a counterexample (Example 5) violating the closure property. This closure property, however, seems quite natural in his social choice framework.

## APPENDIX 3: COMPLETENESS

That completeness imposes no extra restrictions on choice is, in our abstract cases, a consequence of the characterization theorem and the following essentially well-known result:[16]

**PREFERENCE EXTENSION THEOREM:**

1) Any preferences have a complete extension.

2) Any weakly transitive preferences have a complete weakly transitive extension.

3) Any transitive preferences have a complete transitive extension.

**Proof: 1):** Trivial by adding $x' \in R(x)$ at "undecided" pairs.

**2):** Let (P,R) be weakly transitive preferences and $\hat{R} = \mathcal{X}^2 \backslash P^{-1}$. Then $(P,\hat{R})$ is a complete extension of (P,R). It remains to show that $(P,\hat{R})$ is weakly transitive. It suffices to show that $P \circ \hat{R} \subseteq \hat{R}$, as $\hat{R} \circ P \subseteq \hat{R}$ is similar. Let $x'' \in P(x')$ and $x' \in \hat{R}(x)$. Assume $x'' \notin \hat{R}(x)$. Then $x \in P(x'')$. Hence since (P,R) is weakly transitive, $x \in P(x')$, contradiction.

**3):** This is a variant of the order extension theorem. It is proved by applying Zorn's lemma: Let (P,R) be transitive preferences, $\hat{\mathfrak{P}}$ the set of transitive preferences (over $\mathcal{X}$), extending (P,R) and ordered by inclusion. Let $\{(P_i,R_i)\}_I$ a linearily ordered subset of $\hat{\mathfrak{P}}$. Obviously, $(\cup_I P_i, \cup_I R_i)$ is an upper bound for $\{(P_i,R_i)\}_I$. It is also straightforward to check that $(\cup_I P_i, \cup_I R_i)$ is transitive preferences, so $(\cup_I P_i, \cup_I R_i) \in \hat{\mathfrak{P}}$. Hence by Zorn's lemma, $\hat{\mathfrak{P}}$ has a maximal element, $(P^\bullet,R^\bullet)$. It remains to show that $(P^\bullet,R^\bullet)$ is complete. If not, there are x' and x" such that no $R^\bullet$ relation holds between them. Extend $(P^\bullet,R^\bullet)$ to $(P^\circ,R^\circ)$ by (I want $x'' \in R^\circ(x')$, the rest ensures transitivity):

If $x''' \in R^\bullet(x'')$ and $x' \in R^\bullet(x)$, then $x''' \in R^\circ(x)$.

If $x''' \in P^\bullet(x'')$ and $x' \in R^\bullet(x)$, then $x''' \in R^\circ(x)$.

If $x''' \in R^\bullet(x'')$ and $x' \in P^\bullet(x)$, then $x''' \in R^\circ(x)$.

By construction $(P^\circ,R^\circ)$ is transitive, and it is also easy to check that it is asymmetric. Hence $(P^\circ,R^\circ) \in \hat{\mathfrak{P}}$, contradicting the maximality of $(P^\bullet,R^\bullet)$. □

---

[16]The next chapter shows that (iii) is wrong if continuity is added.

It follows that one can add completeness to the preference properties on the top left of the diagram in the characterization theorem. In the transitive case, partial recoverability can be dropped, giving a variant of Richter (1971), Theorem 8. This is a consequence of the following trivial result:

**LEMMA 8:** 1) If (P,R) is complete, then $R_{c^P} \subseteq R$.
2) If (P,R) is complete and transitive, then $P_{c^P} \subseteq P$.

**Proof:** 1): Assume $x \in R_{c^P}(x')\backslash R(x')$. Then there is p such that $x \in c^P(p)$ and $x' \in B(p)$, and by completeness, $x' \in P(x)$, contradicting the definition of $c^P$.
2): Assume $x \in P_{c^P}(x')\backslash P(x')$. Then there is p such that $x \in c^P(p)$ and $x' \in B(p)\backslash c^P(p)$. Hence there is $x'' \in P(x') \cap B(p)$. By completeness, $x' \in R(x)$, so by transitivity, $x'' \in P(x)$, contradicting the definition of $c^P$. □

Together with Proposition 1, (i) shows that the two definitions of rational choice are also identical if preferences are complete.

**Remark 3:** Having thus, in the transitive axiom case, obtained complete, transitive preferences rationalizing a given choice, there might still be no utility function doing the same. But the reason for this is a rather trivial one, namely that there are too few reals (i.e. that there are no injective order preserving mapping from the preferences to the reals with the standard orderings).

**Remark 4:** As is well-known, completeness is not needed for equilibrium existence results. For the general possibility of inferring preferences from choices the situation is slightly different. The reason is that the basic axiom corresponds to partial recoverability of preferences, and as seen, some measure of completeness (and transitivity) seems involved in this.

August 1989
Revised, February 1992

# CHAPTER 3: PREFERENCES, CHOICE AND CONTINUITY[0]

## 1. INTRODUCTION

A classical problem in individual demand theory is the full characterization of the choice consequences of **standard preferences** (i.e. preferences which are complete, transitive, convex, continuous, and monotone) on price generated budgets. This problem can naturally be divided into two subproblems: First, to derive the appropriate properties of choice from the standard preferences, presupposing rationality. Secondly, to construct standard preferences generating the choice from a choice with these properties. This problem is so far unsolved.

Gorman (1971) shows that standard preferences are underdetermined by the choices they generate, by showing that different preferences might generate the same choice. He then proceeded to give conditions under which they are fully determined. We instead characterize the class of preferences where differences have behavioral content. Thus whereas Gorman in a sense starts with the standard preference properties, we start with the "standard" choice properties, and determines the preference properties of the revealed preference relations. As should not be very surprising, this allows a full characterization. The weakening involved on the preference side is simply to replace completeness with what I call partial recoverability. As completeness is not too plausible intuitively, this should not worry much. Weather these "behavioral" preferences also have a complete, transitive, and continuous extension, seems to be an open question. If so, one would have a solution to the main problem, although without preference uniqueness.

Compared to Hurwicz and Richter (1971), the characterization involves full preference continuity. We define preference continuity in terms of the strict preference relation (as two arguments by Uzawa (1959) and Stigum (1973) establish that the transitive strict revealed

[0]Thanks are due to Michael Jerison, Terje Lensberg, Bernt Stigum, and Lars Thorlund-Petersen for valuable comments to this chapter.

preference relation is open), and using the maximal element definition of rational choice. The equivalence between different versions of preference concepts generally fails in the absence of completeness. Therefore care is needed to choose appropriate concepts to get the result.

However, the characterization additionally involves preference , which corresponds to inverse singlevaluedness (differentiability) of the choice function.

There is no need for any Lipschitz condition. One reason for this is the absence of any (complete) preference uniqueness requirements in the above problem. That the Lipschitz condition is related to the uniqueness question was noted by Mas-Colell (1977a,b). He proceeded to give a characterization with such a uniqueness requirement. This characterization also involves a Lipschitz condition on preferences. Our characterization without this uniqueness requirement is much simpler, however. This is nice, as uniqueness does not seem crucial, at least for descriptive purposes.

An even simpler characterization is possible if transitivity is dropped. Then we define preference continuity in terms of the weak preference relation, as it is straightforward to prove that the weak (direct) revealed preference relation is closed.[1] Here, strict monotonicity and smoothness are superfluous in the characterization. The characterization, together with an extension result providing completeness, is essentially a simplified variant of some results in Kim and Richter (1986). The main simplification is that the desired preference properties are established directly for the naturally generated revealed preferences.

A corollary treating single-valued choice shows that the Kihlstrom, Mas-Colell, Sonnenschein, and Schafer (1976) conjecture (saying that replacing the single-valued transitive axiom of revealed preference by the single-valued axiom corresponds to removing transitivity of preference) is correct.[2] But what about Kim and Richter's (1986) counterexample to this conjecture? They somewhat unnaturally restrict continuity of the choice function to the interior of the budget space, i.e. excludes zero prices. When this restriction is lifted, the counterexample vanishes. Indeed without this restriction continuity and the single-valued axiom imply the C-axiom they use to prove a similar result, as shown in an appendix.

[1]Hence, it is natural to use the best element definition of rational choice.

[2]Their conjecture omits an obvious "convexity" property of choice, i.e. that each interior bundle is chosen at some budget. R. John's (unpublished) counterexample is due to this omission.

Some additional restricting on the behavior at the boundary of the choice space is also needed. We assume that indifference surfaces are tangent to the boundary. It corresponds to choices always being in the interior of the choice space. This assumption, however, can be weakened considerably, as will be indicated.

## 2. CONCEPTS AND NOTATION

In the following, let $\mathcal{X} = \mathcal{P} = \mathbb{R}_+^{I^0}$, where $I^0$ is finite. Also, let **x** and **p** be elements of $\mathcal{X}$ and $\mathcal{P}$, respectively. Under the standard interpretation, $I^0$, $\mathcal{X}$, and $\mathcal{P}$ are the sets of goods, quantities and budgets (i.e. income normalized prizes), respectively. Define the **budget**, **interior budget**, and **hyperplane budget correspondences** on $\mathcal{X}\times\mathcal{P}$, B, **b**, and **H**, respectively, by $x \in B(p)$ if $px \leq 1$, $x \in b(p)$ if $px < 1$, and $x \in H(p)$ if $px = 1$. The budget correspondence has convex and nonempty sections, is closed, and lower hemicontinuous.

We introduce some preference concepts. Preferences (P,R) are **monotone** if for all x and $x' \gg x$, $x' \in P(x)$, **strictly monotone** if for all x and $x' > x$, $x' \in P(x)$, **convex** if for all finite sets S, $\{x_s\}_S$, and $x \in [x_s]_S$, there is $s \in S$ such that $x \in R(x_s)$, **strictly convex** if for all finite sets S, $\{x_s\}_S$, and $x \in [x_s]_S \backslash \{x_s\}_S$, there is $s \in S$ such that $x \in P(x_s)$, **P-open** if P is open in $\mathcal{X}$, **R-closed** if R is closed, **smooth** (in $\mathcal{X}$) if for all $x \in \text{int}\mathcal{X}$, x', and x", $\langle x,x'] \cap R(x) = \langle x,x''] \cap R(x) = \emptyset$ implies that $\langle x',x''\rangle \subseteq R^{-1}(x))$, and satisfies **boundary tangency** (of indifference surfaces) if for all $x \notin \text{int}\mathcal{X} \cup \{0\}$ and $p \in B^{-1}(x) \cap \text{int}\mathcal{P}$, $P(x) \cap B(p)$ is nonempty.[3] The convexity concept is from Kim and Richter (1986) and says that a point on a line is at least as good as one end point, see Figure 1.

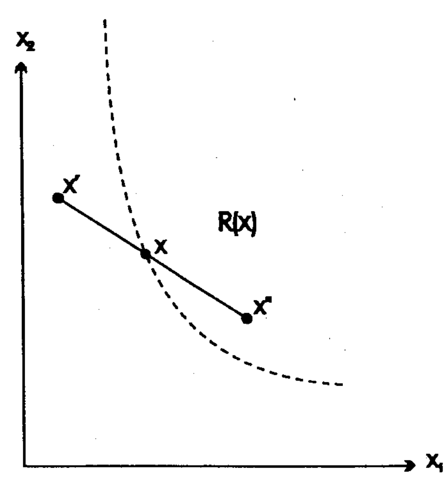



Figure 1

[3]$[x_s]_S$ is the closed polyhedron generated by the points $\{x_s\}_S$. Also $\gg$ and $>$ are the standard strict Euclidian orders defined by a strict inequality in all and at least one component.

The strict convexity concept in the same spirit is new. Both convexity concepts imply the standard ones if preferences are transitive and are implied by these if preferences are complete. Smoothness says that a given (interior) point are at least as good as any line between points on lines through the given point which are not as good as the given point. A typical violation is shown in Figure 2. Boundary tangency is illustrated in Figure 3. Convexity, strict convexity, smoothness, and boundary tangency are preserved under preference extensions.

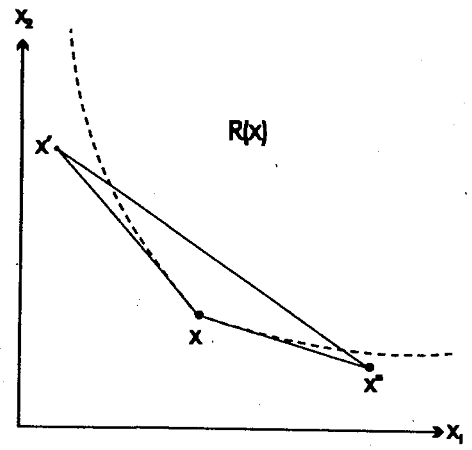



Figure 2

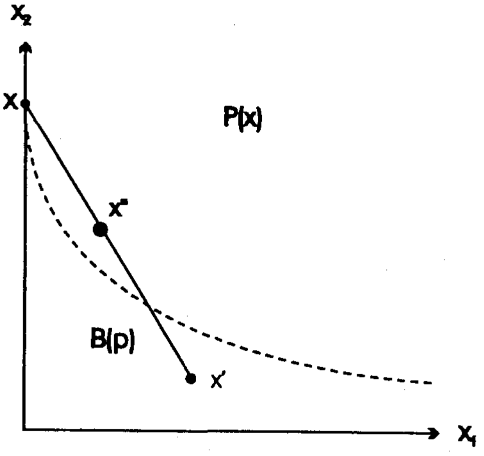



Figure 3

Next comes the corresponding choice concepts. Say that a choice c satisfies the **budget identity** if $c \subseteq H$, has a **large domain** if $\text{int}\mathcal{P} \subseteq Dc$, **large range** if $\text{int}\mathcal{X} \subseteq Dc^{-1}$, **large domains** if it has a large domain and a large range, **interior domain** if $Dc \subseteq \text{int}\mathcal{P}$, **interior range** if $Dc^{-1} \subseteq \text{int}\mathcal{X}$,[4] **interior domains** if it has interior domain and interior range, is **single-valued** if for all p and $x,x' \in c(p)$, $x = x'$, **inversely single-valued** (i.e. differentiable) if $c^{-1}$ is single-valued, and **one-to-one** (or single-sectioned) if it is both single-valued and inversely single-valued.

Departing slightly from the terminology of the preceding chapter, a choice c is **recoverable** if $c^{R^c} = c$ and **transitively recoverable** if $c^{P'} = c$, where $P' = {}^{R^c}P^c$.

The characterizations build on the characterization in Chapter 2, relating the partial recoverability notions to each other and axioms of revealed preference over <u>abstract</u> goods and budget spaces. There the characterizations are restricted to the domain of choice. The following result supplements it:

---

[4]This corresponds to essensiality. Thus essensiality is dual to the strengthening of monotonicity to strict monotonicity.

**LEMMA** 1: If the choice c is closed, has a large domain, and satisfies the budget identity and the basic axiom, then $Dc^{R^c}$, $Dc^{R^c_{P^c}} \subseteq Dc^{P^c} \subseteq Dc$.

Proof: (Figure 4) I first show that $Dc^{P^c} \subseteq Dc$. Let c(p) be empty and $x \in B(p)$. As c has a large domain, $p \notin \text{int}\mathcal{P}$. Let I be the positive components of p (in Figure 4, I = {2,3}), $x' = x+e_{-I}$, and $p^t = tp+ (1-t)/(e_{-I}x'_{-I})e_{-I}$. Then for $t \in \langle\frac{1}{2},1\rangle$, $p^t \in \text{int}\mathcal{P}$ and $x \in b(p^t)$. Let $x^t \in c(p^t)$. Then by the budget identity, $x^t \in P^c(x)$. Assume that $\{x^t\}_{\langle\frac{1}{2},1\rangle}$ is bounded and let $x^t \to \bar{x}$, subsequentially as $t \to 1$. Then, since c is closed, $\bar{x} \in c(p)$, contradiction. Hence $\{x^t\}_{\langle\frac{1}{2},1\rangle}$ is unbounded, so there is t sufficiently near 1 such that $x^t \in B(p)$. Hence $c^{P^c}(p)$ is empty.

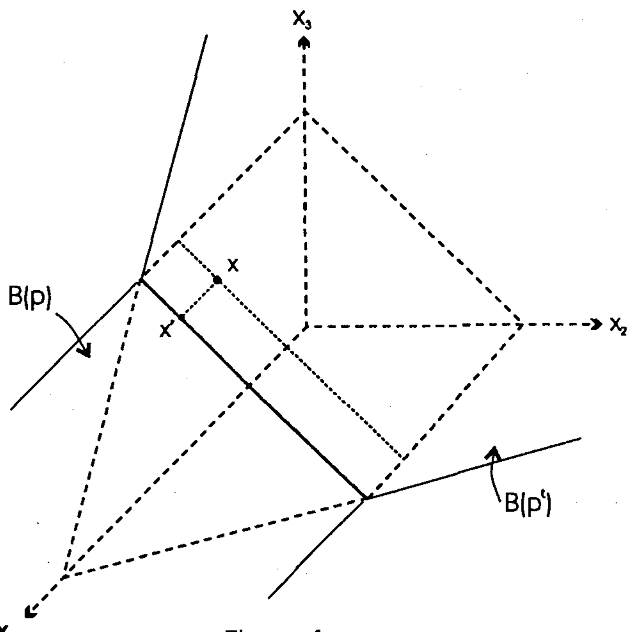


Figure 4

The rest is direct, using the basic axiom in the $Dc^{R^c}$ case. □

## 3. RESULTS

The main result without transitivity is simplest, and stated first:

**THEOREM** 1: 1) If preferences (P,R) is monotone, R-closed, convex, partially recoverable, and satisfies boundary tangency, then the best element choice, $c^R$, is closed, has large domains, interior range, and satisfies the budget identity and the basic axiom.

2) If a choice c is closed, has large domains, interior range, and satisfies the budget identity and the basic axiom, then the generated preferences, $(P^c,R^c)$, is monotone, R-closed, convex, partially recoverable, satisfies boundary tangency, and generates c.

**Proof:** 1): **Closed:** Let $x_n \in c^R(p_n)$ and $(x_n,p_n)\to(x,p)$. Then $x_n \in B(p_n)$, so since B is closed, $x \in B(p)$. Let $x' \in B(p)$. Then, since B is lower hemicontinuous, there are $x_n'\to x'$ such that $x_n' \in B(p_n)$. Hence, since $x_n \in c^R(p_n)$, $x_n \in R(x_n')$. Since R is closed, $x \in R(x')$, so $x \in c^R(p)$.

**Budget identity:** Assume $x \in c^R(p)\cap b(p)$. Then there is $x' \in B(p)$ such that $x' \gg x$. Then $x \in R(x')$. But by monotonicity, $x' \in P(x)$, contradiction.

**Large domain:** Let $p \in \text{int}\mathcal{P}$. Then B(p) is compact. Let $R_p(x) = R(x)\cap B(p)$. Then, since R(x) is closed, $R_p(x)$ is compact. Let $x \in [x_s]_S$, where $x_s \in B(p)$ for all $s \in S$ and S is finite. Then $x \in B(p)$ and by convexity there is $s \in S$ such that $x \in R(x_s)$. Hence $x \in R_p(x_s)$, so by Fan (1961, Lemma 1), $\cap_{B(p)}R_p(x)$ is nonempty, i.e. $c^R(p)$ is nonempty.

**Large Range:** Let $x \in \text{int}\mathcal{X}$. Assume $R^{-1}(x) \neq \mathcal{X}$, as the result is trivial otherwise. Define $\hat{P}$ by $x' \in \hat{P}(x)$ if $x \notin R(x')$. Then $\hat{P}(x)$ is nonempty, and since $R^{-1}(x)$ is closed, $\hat{P}(x)$ is open in $\mathcal{X}$. Assume $x \in \text{co}\hat{P}(x)$. Then $x \in [x_s]_S$, where for all $s \in S$, $x_s \in \hat{P}(x)$, i.e. $x \notin R(x_s)$. But by convexity for some $s \in S$, $x \in R(x_s)$, contradiction. Hence $x \notin \text{co}\hat{P}(x)$. Since $\hat{P}(x)$ is open in $\mathcal{X}$, so is $\text{co}\hat{P}(x)$. Hence by separation (and monotonicity) there is p such that $x \in B(p)$ and $B(p)\cap\text{co}\hat{P}(x)$ is empty. Let $x'' \in B(p)$ and assume $x \notin R(x'')$. Then $x'' \in \hat{P}(x) \subseteq \text{co}\hat{P}(x)$, contradiction. Hence $x \in c^R(p)$.

**Interior range:** Let $x \in c^R(p)\setminus\text{int}\mathcal{X}\cup\{0\}$. Then $x \in c^P(p)$, contradicting boundary tangency.

**Basic axiom** and **recoverable choice:** Proposition 1 in Chapter 2 shows that R-partial recoverability implies the equality of the two definitions of rational choice. The basic axiom and recoverability then follows from the characterization theorem given there and Lemma 1.

2): **Monotone:** Let $x,x' \in Dc^{-1}$, $x \gg x'$, and $x \in c(p)$. Then $x' \in b(p)$, so by the budget identity, $x' \notin c(p)$. Hence $x \in P^c(x')$.

**R-closed:** Let $x,x' \in Dc^{-1}$, $x_n \in R^c(x_n')$, and $(x_n,x_n')\to(x,x')$. Then there is $p_n$ such that $x_n \in c(p_n)$ and $x_n' \in B(p_n)$. By interior range, $\{p_n\}$ is bounded, so let $p_n\to p$ subsequentially. Then since c and B are closed, $x \in c(p)$ and $x' \in B(p)$, so $x \in R^c(x')$.

**Convex:** Let $x \in [x_s]_S$ and $x \in c(p)$. Then for some $s \in S$, $x_s \in B(p)$. Hence $x \in R^c(x_s)$.

**Boundary tangency:** Let $x \notin \text{int}\mathcal{X}\cup\{0\}$ and $p \in B^{-1}(x)\cap\text{int}\mathcal{P}$. Since $p \in \text{int}\mathcal{P}$, large domain, there is $x' \in c(p)$. By interior range, $x \notin c(p)$. Hence $x' \in P^c(x)\cap B(p)$. □

Substituting the V-axiom for the basic axiom (which is not essentially used here) and completeness for partial recoverability, the second part of this result together with Proposition 1 of Appendix 1 is a simplified version of Kim and Richter's (1986) Theorem 8, except that they use a compact choice set $\mathcal{X}$ instead of our assumptions of interior range. By strengthening convexity to strict convexity, partial recoverability can be avoided in the above theorem[5]:

**COROLLARY:** 1) If preferences (P,R) is monotone, R-closed, strictly convex, and satisfies boundary tangency, then the best element choice, $c^R$, is closed, single-valued, has large domains, interior range, and satisfies the basic axiom.

2) If the choice c is is closed, single-valued, has large domains, interior range, and satisfies the basic axiom, then the generated preferences, $(P^c,R^c)$, is monotone, R-closed, strictly convex, satisfies boundary tangency, and generates c.

**Proof:** One only needs to verify the single-valued axiom and strict convexity:

**Single-valued axiom:** Assume $x \in R^{c^R}(x')$, $x' \in R^{c^R}(x)$ and assume $x \neq x'$. Then there is p such that $x \in c^R(p)$ and $x' \in B(p)$, and p' such that $x' \in c^R(p')$ and $x \in B(p')$. Let $x'' \in \langle x,x' \rangle$. Then $x'' \in B(p) \cap B(p')$. Hence by the definition of $c^R$, $x,x' \in R(x'')$. But by strict convexity, $x'' \in P(x) \cup P(x')$, contradiction.

**Strictly convex:** Let $x \in [x_s]_S \backslash \{x_s\}_S'$ and $x \in c(p)$. Then for some $s \in S$, $x_s \in B(p)$. If $x_s \in c(p)$, then by single-valuedness, $x = x_s$, contradiction. Hence $x_s \notin c(p)$, so $x \in P^c(x_s)$. □

This corollary, together with the extension theorem below and the extension property of Chapter 2, verifies the Kihlstrom, Mas-Colell, Sonnenschein, and Shafer (1976) conjecture. The direct use of the revealed preferences in the second part of this theorem is made possible by the strict convexity concept, which in this context is weaker than the standard one, but still sufficient.

---

[5]Partial recoverability is somewhat less desirable in a characterization without transitivity, as transitivity seems necessary (in addition to completeness) to derive it from more traditional preference assumptions.

The result with transitivity has essentially the same structure as Theorem 1. It is, however, based on the maximal element definition of rational choice. Also, it uses some extra assumptions, namely smoothness and strict monotonicity:

**THEOREM 2:** 1) If preferences (P,R) is transitive, strictly monotone, partially recoverable, P-open, convex, smooth, and satisfies boundary tangency, then the maximal element choice, $c^P$, is closed, inversely single-valued, has large interior domains, and satisfies the budget identity and the transitive axiom.

2) If the choice c is is closed, inversely single-valued, has large interior domains, and satisfies the budget identity and the transitive axiom, then the generated transitive preferences, $({}^{Rc}P^c, {}^{Rc}R^c)$, is strictly monotone, partially recoverable, P-open, convex, smooth, satisfies boundary tangency, and generates c.

Proof: 1): **Closed:** Let $x_n \in c^P(p_n)$ and $(x_n,p_n) \to (x,p)$. Then $x_n \in B(p_n)$, so since B is closed, $x \in B(p)$. Let $x' \in P(x)$, and assume $x' \in B(p)$. Then, since B is lower hemicontinuous, there are $x_n' \to x'$ such that $x_n' \in B(p_n)$. Since P is open (in $\mathcal{X}$), for sufficiently large n, $x_n' \in P(x_n)$, contradicting $x_n \in c^P(p_n)$. Hence $x' \notin B(p)$, so $x \in c^P(p)$.

**Budget identity:** Assume $x \in c^P(p) \cap b(p)$. Then there is $x' \gg x$ such that $x' \in B(p)$. By monotonicity, $x' \in P(x)$, contradiction.

**Interior domain:** Let $x \in c^P(p)$, where $p \notin \mathrm{int}\mathcal{P}$. Then there is $x' \in B(p)$ such that $x' > x$, so by strict monotonicity, $x' \in P(x)$, contradiction.

**Interior range:** Essentially as in the previous proof.

**Large domain:** Let $p \in \mathrm{int}\mathcal{P}$ and assume that for all $x \in B(p)$ there is $x' \in P(x) \cap B(p)$. Clearly $\{P^{-1}(x) | x \in B(p)\}$ is a covering of B(p), and since P has open lower sections and B(p) is compact, it has a finite subcovering $\{P^{-1}(x_s)\}_S$. But then by assumption, for all $s \in S$ there is $t \in S$ such that $x_t \in P(x_s)$. Hence since S is finite and P is transitive, there is $s \in S$ such that $x_s \in P(x_s)$, contradiction.

**Large range:** Let $x \in \text{int}\,\mathcal{X}$. Assume that $P(x)$ is nonempty, as the result is trivial otherwise. By open upper sections, $P(x)$ is open in $\mathcal{X}$, hence so is $coP(x)$. Assume $x \in coP(x)$. Then there is a finite set $S$ such that $x'' \in [x_s]_S$ and $x_s \in P(x)$ for all $s \in S$. But by convexity, for some $s \in S$, $x \in R(x_s)$, contradiction. Hence $x \notin coP(x)$. By separation (and monotonicity) there is $p$ such that $x \in B(p)$ and $B(p) \cap coP(x)$ is empty. But then $x \in c^P(p)$.

**Transitive axiom and transitive recoverability:** This follows from the characterization theorem in Chapter 2, together with Lemma 1.

**Inversely single-valued:** (Figure 5) Let $x \in c^P(p') \cap c^P(p'')$, where $p' \neq p''$. Then there is $x' \in b(p') \cap \text{int}\mathcal{X}$, $x'' \in b(p'') \cap \text{int}\mathcal{X}$, and $\bar{x} \in \langle x',x'' \rangle$ such that $\bar{x} \gg x$. By monotonicity, $\bar{x} \in P(x)$. Let $\bar{x}' \in \langle x,x' ]$. Assume that there is $\bar{x}' \in \langle x,x' ] \cap R(x)$. Then by transitivity, $\bar{x}' \in c(p')$. But $\bar{x}' \in b(p')$, contradicting monotonicity. Hence $\langle x,x' ] \cap R(x) = \emptyset$. Similarly $\langle x,x'' ] \cap R(x'') = \emptyset$, so by smoothness, $\langle x',x'' \rangle \subseteq R^{-1}(x)$. Hence $x \in R(\bar{x})$, contradiction.

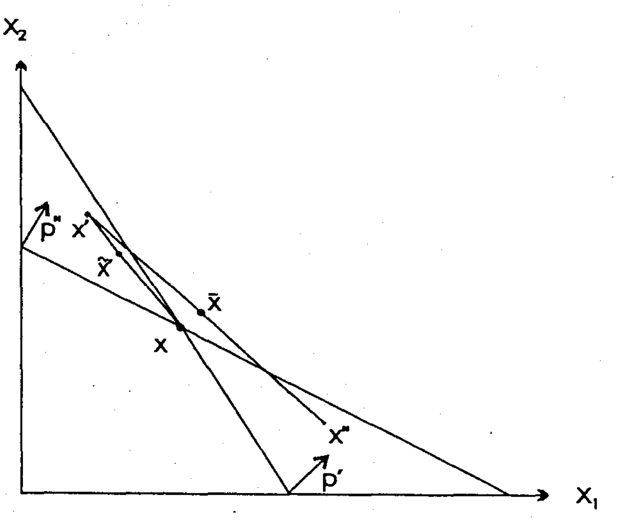



Figure 5

2): **Transitive:** $({}^{R^c}P^c, {}^{R^c}R^c)$ is transitive by definition.

**Strictly monotone:** As monotonicity is preserved under extensions, it is sufficient to prove this for the generated preferences $(P^c,R^c)$. Let $x > x'$, $x,x' \in Dc^{-1}$, and $x \in c(p)$. By interior domain, $p \gg 0$. Hence $x' \in b(p)$, so by the budget identity, $x' \notin c(p)$. But then $x \in P^c(x')$.

**Convex:** Follows from the previous proof as convexity is preserved under extensions.

**Smooth:** Since smoothness is preserved under extensions of preferences, it will suffice to prove this for the generated preferences $(P^c,R^c)$. Let $\langle x,x' ] \cap R^c(x) = \langle x,x'' ] \cap R^c(x) = \emptyset$, where $x,x',x'' \in \text{int}\mathcal{X}$. Then by Lemma 2, $x \in R^c(x') \cap R^c(x'')$. Hence there is $p'$ and $p''$ such that $x \in c(p') \cap c(p'')$, $x' \in B(p')$ and $x'' \in B(p'')$. By inverse single-valuedness, $p' = p''$. Hence $x',x'' \in B(p')$, so $\langle x',x'' \rangle \subseteq (R^c)^{-1}(x)$.

P-open: I first verify that ${}^{P^c}P^c$ has open lower sections (Figure 6). Let $x \in P^c(x')$, $x,x' \in Dc^{-1}$ and $x_n \to x$. Then there is p such that $x \in c(p)$ and $x' \in B(p)-c(p)$. Since c(p) is closed, there is $x'' \in \langle x,x'\rangle \backslash c(p)$. By large range, let $x'' \in c(p'')$, so by the basic axiom, $x \notin B(p'')$. Hence $x' \in b(p)$. But then by the budget identity, $x'' \in P^c(x_n')$, so $x \in {}^{P^c}P^c(x_n')$. The general case follows by induction. Note that the same argument also verifies that $P^c$ and hence ${}^{P^c}P^c$ is order dense.

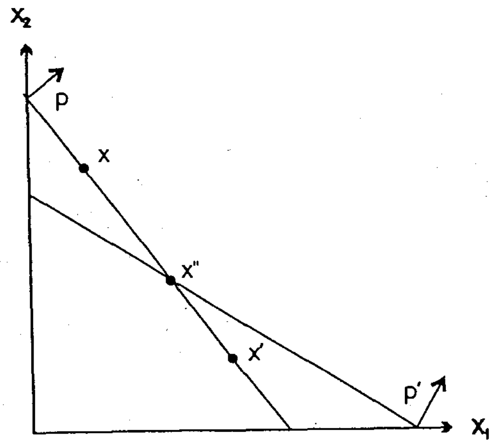


Figure 6

Next, I verify that ${}^{P^c}P^c$ has open upper sections (Figure 7). Let $x \in P^c(x')$ and $x,x' \in Dc^{-1}$. Then there is p such that $x \in c(p)$ and $x' \in B(p)-c(p)$. Let $x' \in c(p')$. Assume that for all $p'' \in \langle p,p'\rangle$ there is $x'' \in c(p'') \cap B(p')$. Let $x'' \to \tilde{x}$, subsequentially as $p'' \to p$ ($\{x''\}$ is bounded as p is interior). Then since c and B are closed, $\tilde{x} \in c(p) \cap B(p')$. But then $\tilde{x} \in P^c(x')$ and $x' \in R^c(\tilde{x})$, contradicting the basic axiom. Hence there is $p'' \in \langle p,p'\rangle$ such that $c(p'') \cap B(p')$ is empty. Let $x'' \in c(p'')$. Clearly $x' \in B(p'')$ and since $c(p'') \cap B(p') = \emptyset$, $x' \notin c(p'')$. Hence $x'' \in P^c(x') \cap b(p)$. Let $x_n \to x$, $x_n \in c(p_n)$. By interior range, $x \in \text{int}\mathcal{X}$, so $\{p_n\}$ is bounded for n sufficiently large. Let $p_n \to \bar{p}$ subsequentially. By inverse single-valuedness, $\bar{p} = p$. But then for n sufficiently large, $x'' \in b(p_n)$, so $x_n \in P^c(x'')$, hence $x_n \in {}^{P^c}P^c(x')$.

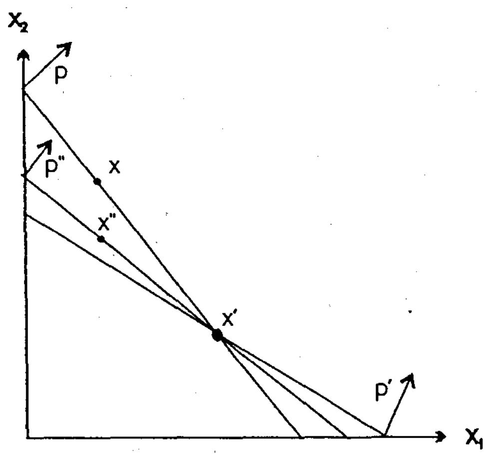


Figure 7

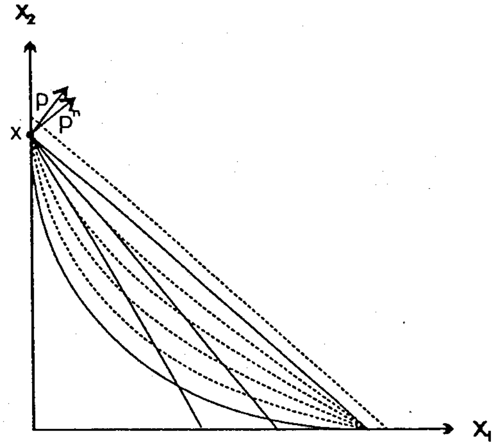
That ${}^{P^c}P^c$ is open in $\mathcal{X}$ now follows from Bergstrom, Parks, and Rader (1976, Theorem 1), as it is transitive, order dense, and has open sections. By Proposition 2.2, it follows from inverse single-valuedness that ${}^{R^c}P^c = {}^{P^c}P^c$, so ${}^{R^c}P^c$ is open in $\mathcal{X}$.

Boundary tangency: Follows from the previous proof as it is preserved under preference extensions. □

**Remark** 1: Inverse single-valuedness (i.e. smoothness) is used twice above: To show that ${}^{Pc}Pc$ has open upper sections and ${}^{Rc}Pc = {}^{Pc}Pc$.

**Remark** 2: It is clear from the proof of Theorem 2 that it is also valid when transitivity is replaced by weak transitivity and the transitive axiom by the weakly transitive one.

**Remark** 3: One would like similar results under alternative boundary assumptions. A simple alternative is the polar one, saying that indifference surfaces cross the boundary of the choice space. A simple way to express this is: For all $x \notin \text{int}\mathcal{X}\cup\{0\}$, there is $p \in H^{-1}(x)\cap\text{int}\mathcal{P}$ such that $P(x)\cap B(p)$ is empty. The corresponding choice notion is that the range of c, $Dc^{-1} = \mathcal{X}\setminus\{0\}$. The problem with this concept is that it does not suffice to show continuity of the revealed preferences. E.g., in the proof of R-closedness in the first theorem, the sequence $\{p_n\}$ might be unbounded, as illustrated in Figure 8. The problem is eliminated by the following strengthening of the above concepts: Preferences (P,R) satisfies **boundary crossing** if for each compact $\mathcal{X}' \subseteq \mathcal{X}$ such that $0 \notin \mathcal{X}'$, there is a compact $\mathcal{P}' \subseteq \mathcal{P}$ such that for all $x \in \mathcal{X}'\setminus\text{int}\mathcal{X}$ there is $p \in \mathcal{P}'$ and such that $P(x)\cap B(p) = \emptyset$. The corresponding choice notion is similar: A choice has (strict) **full range** if for each compact $\mathcal{X}' \subseteq \mathcal{X}$ such that $0 \notin \mathcal{X}'$ there is a compact $\mathcal{P}' \subseteq \mathcal{P}$ such that for all $x \in \mathcal{X}'\setminus\text{int}\mathcal{X}$ there is $p \in \mathcal{P}'$ such that $x \in c(p)$. Thus all points on a bounded part of the boundary is chosen at some uniformly bounded relative prices. With these concepts replacing boundary tangency and interior range assumption, we again get full characterizations, as is easily verified.



Figur 8

Both boundary assumption are restrictive. The reason is that they presuppose the same type of boundary behavior everywhere. There are, however, no problems in extending the results to cases which are mixtures of our two boundary assumptions. I.e., where the set of goods can be partitioned into one group satisfying boundary tangency, and another satisfying boundary crossing. The resulting characterization is sufficient for most applications.

Remark 4: As interior domain and interior range are dual concepts, so are also their preference counterparts, strict monotonicity and boundary tangency.

Remark 5: Interior domain (and hence the strict monotonicity) is used in Theorem 2 only to ensure that some goods sequences are bounded in the proof of preference continuity. Hence, in analogy with the discussion in Remark 3, it can be replaced with the dual of full range, i.e. full domain. On the preference side the counterpart of this is the following: Preferences (P,R) is **locally satiated** (with respect to all goods) if for each each compact $\mathcal{P}' \subseteq \mathcal{P}$ such that $0 \notin \mathcal{P}'$, there is a compact $\mathcal{X}' \subseteq \mathcal{X}$ such that for all $p \in \mathcal{P}' \setminus \mathrm{int}\mathcal{P}$ there is $x \in \mathcal{X}'$ such that $P(x) \cap B(p) = \emptyset$. A counterexample to the uniformity property in this case is given in Figure 9.

Again, mixtures of the two dual boundary assumptions are possible. But intuitively, the above treatment of boundary of the choice space is more interesting.

Characterizations of more specific types of preferences follows easily, as is verified in Chapter 4 and 6 for the law of demand and separability type of restrictions, respectively.

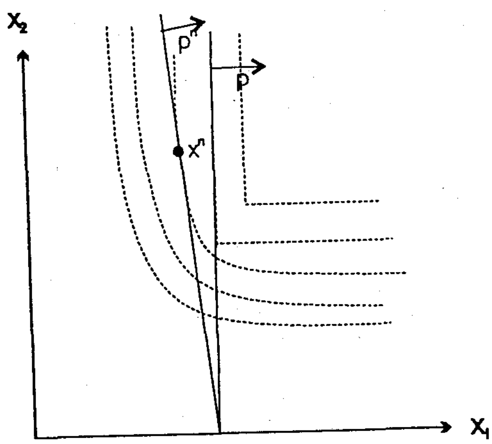


Figure 9

Theorem 2 assumes that inverse choice is single-valued, i.e. that preferences are smooth. Thorlund-Petersen (1985) shows that smoothness can be dropped if one instead assumes finite transitivity. A choice, c, is **finitely transitive** if for all x', $x \in R^c(x')$, and neighborhoods $U_x$ and $U_{x'}$, there is a compact set K and a natural number d such that for all $\hat{x} \in U_x$ and $\hat{x}' \in U_{x'}$, there is $x_1,\dots,x_k$ with $k \leq d$ such that $\hat{x} \in R^c(x_1)$, $x_1 \in R^c(x_2),\dots,x_k \in R^c(\hat{x}')$. This says that if a point is (indirectly) weakly revealed preferred to another, then this can be done locally in a finite number of steps. Under these conditions one also gets an utility function.

## 4. COMPLETENESS

In Theorem 1 and its corollary (i.e. without transitivity) completeness can be added to the preference properties in Part 2. The argument modifies one by Kim and Richter (1986):[6]

**EXTENSION THEOREM:** Any monotone, convex, and R-closed preferences have a complete R-closed extension.

**Proof:** Let preferences (P,R) be monotone, convex, and R-closed. Define $R^*$ by $R^*(x,x') = R(x) \cap R(x') \cap [x,x']$. Then since R is closed, so is $R^*$ and $R(x) \cup R(x')$. By convexity, $[x,x'] \subseteq R(x) \cup R(x')$, and by monotonicity, $R(x) \cup R(x')$ is connected. Hence as $R(x) \cap [x,x']$ and $R(x') \cap [x,x']$ are closed and nonempty, $R^*(x,x')$ is nonempty (since a closed connected set cannot be partitioned into two nonempty closed sets). Extend R to R' by adding $x \in R'(x')$ if $d(x,R^*(x,x')) \leq d(x',R^*(x,x'))$ at pairs undecided by (P,R), where d is the standard metric. Then (P,R') is complete by definition. It remains to show that R' is closed. Let $x_n \in R'(x_n')$ and $(x_n,x_n') \to (x,x')$. If $x_n \in R(x_n')$ for arbitrary large n, then since R is closed, $x \in R(x') \subseteq R'(x')$. Otherwise, $d(x_n,R^*(x_n,x_n')) \leq d(x_n',R^*(x_n,x_n'))$ for sufficiently large n. Hence since $R^*$ is closed and d is continuous, $d(x,R^*(x,x')) \leq d(x',R^*(x,x'))$, i.e. $x \in R'(x')$. □

This result is sufficient as the convexity notions are preserved under extensions and any extension of the generated preferences gives rise to the same rational choice by the characterization theorem in Chapter 2.

**Remark:** A similar result is trivial for weakly transitive preferences, as continuity is preserved by the construction in the extension theorem in Chapter 2. I.e. the following holds: Any P-open and weakly transitive preferences has a complete, P-open and transitive extension. The extension property with full transitivity, on the other hand, seems still to be an open question, as mentioned in the introduction.

---

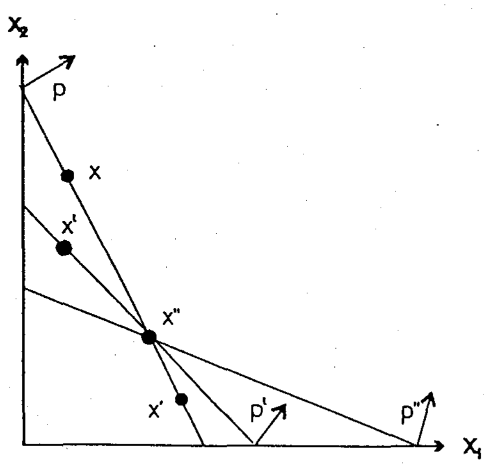
[6]The modification is that I prove the constructed extension, R', to be continuous, whereas they look at its closure and verifies convexity. In my framework, however, taking the closure might conflict with preference asymmetry.

## APPENDIX 1: WEAK AXIOMS

There are some weaker variants of the revealed preference axioms. In contrast to the basic revealed preference axioms, they are easily decidable on finite choices and also preserved under subchoices. First, define the **strengthened strict revealed preference relation**, $^+P^c$, by $x \in {}^+P^c(x')$ if there is p such that $x \in c(p)$ and $x' \in b(p)$. If choice satisfies the budget identity, $^+P^c \subseteq P^c$, and the terminology is justified. Adding continuity and transitivity makes the two concepts equivalent:[7]

**PROPOSITION 1**: If a choice c is closed, convex-valued, with large interior domains, and satisfies the budget identity, then $P^c \subseteq {}^+P^c \circ {}^+P^c$.

**Proof**: (Figure 10) Let $x \in P^c(x')$. Then there is p such that $x \in c(p)$ and $x' \in B(p) \backslash c(p)$. Let $x' \in H(p)$ as the result is trivial otherwise. Since $c(p)$ is closed, there is $x'' \in \langle x,x' \rangle \backslash c(p)$. Since c is convex-valued and satisfies the budget identity, let $p'' \in H^{-1}(x'')$ be such that $c(p) \cap B(p'')$ is empty. Let $p^t = (1-t)p+tp''$ for $t \in [0,1]$, and let $x^t \in c(p^t)$. Then for all $t > 0$, $x' \in b(p^t)$, so $x^t \in {}^+P^c(x')$. Since c is closed, $x^0 \in c(p)$, so $x^0 \notin B(p'')$. But then since B is closed, for t sufficiently small, $x^t \notin B(p'')$. Since $(1-t)px^t+tp''x^t = p^t x^t = 1$, $x^t \in b(p)$, so $x \in {}^+P^c(x^t)$. □




Figur 10

A choice c satisfies the **weak axiom** if $({}^+P^c, R^c)$ is asymmetric, and the **transitive weak axiom** if $({}^+P^c, {}^{R^c}R^c)$ is asymmetric. Thus these axioms are equal to the basic axiom and the transitive axiom, except for the stronger strict revealed preference relation. In contrast to the

---

[7]This result is due to McFadden (1979). His additional assumtion of single-valued choice is, however, superfluous.

basic axiom and the transitive axioms, the weak and the weak transitive axiom are decidable on finite (as sets) choices and preserved under subchoices. By Proposition 1, the transitive weak axiom is equivalent to the transitive axiom under the stated assumptions.

The weak axiom is from Hicks (1956), and often called the weak weak axiom. For a one-to-one choice (i.e. a differentiable choice function), the weak axiom is equivalent to negative semidefiniteness, and the transitive weak axiom is equivalent to symmetry and negative semidefiniteness of the Slutsky matrix. The first of these results was proved by Kihlstrom, Mas-Colell, and Sonnenshein (1976), and simplified in Hildenbrand and Jerison (1988). The weak axioms are self-dual.

The weak axiom does not imply the basic axiom. Hence, the extension property (Lemma 2.4) and the equality of the two definitions of preference generated choice (Proposition 2.1) does not generally hold for the weak axiom.

The transitive weak axiom Varian's (1982) "generalized axiom" and goes back to Afriat (1967). Afriat showed that on a finite choice, it implies the existence of a concave utility function rationalizing the choice. A variant of this result will be proved in Chapter 6, using a more general characterization of the choices generated by a concave utility function.

Varian (1982, 449-450) claims that Afriat's result shows that monotonicity is unobservable on finite choices (data sets). This interpretation cannot be correct, as monotonicity implies the observable budget identity. The point is that the budget identity is "baked" into the weak, and hence also the weak transitive axiom. Hence violations of the budget identity will be taken as violations the weak transitive axiom in Afriat's case. This is supported by the following trivial result:

**PROPOSITION** 2: A choice which has large range and satisfies the weak axiom also satisfies the budget identity.

Proof: Assume not, and let $x \in c(p) \cap b(p)$. Let $x' \in B(p)$ and $x' \gg x$.By large range, let $x' \in c(p')$. Then $x \in b(p)$, so $x \in R^c(x')$ and $x' \in {}^{+}P^c(x)$, contradicting the weak axiom. □

## APPENDIX 2: THE C-AXIOM

Now for a strengthening of the basic axiom. A choice c satisfies the **C-axiom** if $x_n \in R^c(x_n')$, $(x_n,x_n') \to (x,x')$, and $x' \in R^c(x)$ implies that $x = x'$. Kim and Richter (1986) use this continuous version of the single-valued axiom to prove the Kihlstrom, Mas-Colell, Sonnenschein, and Shafer (1976) conjecture. As we do not require strictly positive prices, it is a consequence of the single-valued axiom:

**PROPOSITION 3:** If a choice is closed, has interior range, and satisfies the single-valued axiom, then it satisfies the C-axiom.

**Proof:** Let $x_n \in c(p_n)$, $x_n' \in B(p_n)$, $(x_n,x_n') \to (x,x')$, and $x' \in R^c(x')$. Then $\{p_n\}$ is bounded, since otherwise $x \notin \mathrm{int}\mathcal{X}$, contradicting interior range. Let $p_n \to p$, subsequentially. Then since c is closed, $x \in c(p)$ and $x' \in B(p)$, i.e. $x \in R^c(x')$. But then by the basic axiom, $x = x'$. □

August 1989
Revised, March 1992

# CHAPTER 4: THE LAW OF DEMAND AND RELATED NOTIONS[0]

## 1. INTRODUCTION

The **law of demand** expresses a negative relationship between price and quantity changes for a given income. It (or a partial version of it) was taken as intuitively obvious by Walras (1871, 12. Leçon) and other early neoclassics. Here the law of demand is explicated by a monotonicity notion taken from Rockafellar (1970, Chapter 24).[1]

Monotonicity is a strong concept, but often valid empirically. It has three nice properties. First, it is easily characterized in preference terms, at least if preferences are not required to be transitive. Secondly, it is preserved under aggregation over individuals if their endowments are **collinear** (i.e. on a ray through 0). And finally, (some variants of) it is easily testable on finite data sets.

Monotonicity is extended in two directions. First, a hierarchy is introduced, differing only in the order of the mean used in the definition of monotonicity. The first order (or arithmetic) mean is the law of demand, the infinite order mean is the basic axiom of revealed preference, and the zero-order (or geometric) mean (slightly weakened) corresponds to homotheticity. This hierarchy gives a natural measure of the "perversity" of income effects, and thereby a natural nonparametric way of imposing and testing stronger restrictions than the Slutsky ones on income effects. Thus it suggests a nonparametric alternative to the Slutsky equation approach to empirical demand analysis. The characterization in terms of preferences easily carries over to this hierarchy.

---

[0]An earlier version of this chapter was presented at the Econometric Society meeting in Munich in September 1989. Thanks are due to Tore Ellingsen, Thorsten Hens, Michael Jerison, Terje Lensberg, Kjell-Erik Lommerud, Rolf Schmachtenberg, and Lars Thorlund-Petersen for valuable comments and discussion.

[1]With a slight change: Rockafellars (cyclical) monotonicity corresponds to my weak (cyclical) monotonicity, and his maximal (cyclical) monotonicity to my (cyclical) monotonicity. This explication of the law of demand is not universal. I.e. Mas-Colell (1985) uses the law of demand for the similar property of <u>compensated</u> demand, which holds generally in the standard model.

Secondly, a stronger notion called cyclical monotonicity is introduced. This is related to monotonicity as the transitive axiom is related to the basic axiom. It is shown that cyclical monotonicity is another characterization of homotheticity in the presence of transitivity. By some results of Rockafellar (1970), cyclical monotonicity also characterizes concave consumer surplus functions which are also utility functions, see Chapter 6. As monotonicity, cyclical monotonicity is preserved under aggregation if endowments are collinear. A hierarchy based on cyclical monotonicity is also introduced, as well as some variants of the concepts. The characterization in terms of preferences (this time transitive) easily carries over to the hierarchies. So far, however, neither the aggregation result nor the characterization in terms of utility is extended to the hierarchies. This hierarchy gives a measure of the degree of transitive homotheticity (i.e. cyclical monotonicity), and thereby also a measure of the "correctness" of (Marshallian) consumer surplus.

Monotonicity and cyclical monotonicity were used by Shafer (1977), calling them strong acyclicity of degree 2 and $\infty$, respectively. He also proved the aggregation result.

The rest of this paper is organized as follows: Section 2 introduces the basic monotonicity notion explicating the law of demand, as well the hierarchy based on it. Section 3 introduces cyclical monotonicity, as well as some variants of the concepts. Section 4 characterizes the monotonicity notions in terms of preference terms in an elementary way. Section 5 gives the aggregation results.

## 2. MONOTONICITY

We continue to use the notation of the previous chapters. The notion explicating the law of demand is the following: A choice c is **monotone** if for all $p_1$, $p_2$, $x_1 \in c(p_1)$, and $x_2 \in c(p_2) \backslash c(p_1)$, $(p_2-p_1)(x_2-x_1) < 0$. Thus monotonicity expresses a negative correlation between price and quantity changes for a given income. The above inequality is equivalent both to $(x_1+x_2)/2 \notin B((p_1+p_2)/2)$ and to $(p_1x_2+p_2x_1)/2 > 1$. The first of these says that the mean bundle is outside the mean budget, and allows a nice geometrical interpretation (see Figure 1), whereas the second generalizes in different directions. Note that monotonicity is **self-dual** (i.e. invariant when x's and p's are interchanged).

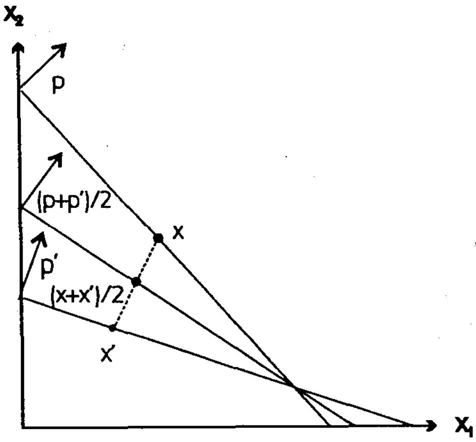



Figure 1

For the hierarchy we need the generalized means: Define the **$\alpha$-th order mean** of the positive reals $y_1$ and $y_2$, **$m^\alpha(y_1,y_2)$**, by $m^\alpha(y_1,y_2) = (((y_1)^\alpha+(y_2)^\alpha)/2)^{1/\alpha}$ for $\alpha$ positive, $m^0(y_1,y_2) = (y_1y_2)^{\frac{1}{2}}$, and $m^\infty(y_1,y_2) = \sup\{y_1,y_2\}$. Then $m^\alpha(y_1,y_2)$ is strictly increasing in $\alpha$ if $y_1 \neq y_2$, and constant if $y_1 = y_2$.[2]

The monotonicity hierarchy can now be defined by simply replacing the arithmetic mean by the $\alpha$-order mean in the definition of monotonicity: A choice c is **$\alpha$-monotone** if for all $p_1$, $p_2$, $x_1 \in c(p_1)$, and $x_2 \in c(p_2)\backslash c(p_1)$, $m^\alpha(p_1x_2,p_2x_1) > 1$. As the $\alpha$-th order mean is increasing in $\alpha$, $\alpha$-monotonicity is of decreasing strength in $\alpha$. Here $\infty$-monotonicity is the basic axiom (of revealed preference) from Arrow (1957),[3] 1-monotonicity is monotonicity, whereas 0-monotonicity is inconsistent, but a slightly weaker variant corresponds to homotheticity, as explained in the next section. By the result mentioned in Footnote 3, $\infty$-

[2]For a proof, see Leichtweiss (1980), Hilfsatz 21.1.

[3]For single-sectioned choice, this is between negative semidefiniteness and negative definiteness of the Slutsky matrix in strength, see Kihlstrom, Mas-Colell, and Sonnenschein (1976).

monotonicity corresponds to Slutsky negative definiteness. Thus the monotonicity hierarchy is a simple nonparametric way to introduce more restrictive conditions on income effects than Slutsky negative definiteness.

The **monotonicity measure** of a choice c, $\mu(c) = \inf_\alpha\{\alpha | c \text{ is } \alpha\text{-monotone}\}$. Thus, $\mu(c)$ measures the "perversity" of income effects of the choice c. For each $\alpha$, $\alpha$-monotonicity is easily testable on finite data sets.[4] Hence it is also easy to calculate the monotonicity measure, which should therefore be of great interest empirically. In contrast to standard parametric estimation and test techniques, nonparametric tests are exact. One can, however, incorporate errors by accepting monotonicity even if the monotonicity measure is slightly greater than one.[5]

Here, choice is a function of the budget. But one is often more interested in demand as a function of relative prices. These influence the budget directly, but also indirectly through income. Assuming endowments, $\tilde{x}$, and no profit income, choice as a function of relative prices, $c^{\tilde{x}}$, is defined by $x \in c^{\tilde{x}}(p)$ if $x \in c(p/p\tilde{x})$. The monotonicity hierarchy obviously carries over to this concept if prices are normalized such that income always equals one, i.e. belongs to $H^{-1}(\tilde{x})$. Thus it follows that for all $p,p' \in H^{-1}(\tilde{x})$, $x \in c^{\tilde{x}}(p)$, and $x' \in c^{\tilde{x}}(p')$, $m^\alpha(px',p'x) \geq 1$.

What about **excess** choice, $\mathbf{e}^{\tilde{x}}$, defined by $z \in e^{\tilde{x}}(p)$ if $z+\tilde{x} \in c^{\tilde{x}}(p)$? If the choice c is monotone, the corresponding excess choice, $\mathbf{e}^{\tilde{x}}$, is **excess monotone** (i.e. for all $p,p' \in H^{-1}(\tilde{x})$, $z \in e^{\tilde{x}}(p)$, and $z' \in e^{\tilde{x}}(p')$, $(p'-p)(z'-z) \leq 0$). Thus again the law of demand follows as above when prices are normalized so that the income always equals one.[6] Excess monotonicity has a nice geometrical interpretation in $\mathbb{R}^2$: The endowment vector $\tilde{x}$ is the minimum (maximum) allowed steepness of the graph of excess demand in the second (fourth) quadrant. Thus with endowments of only one good, the graph of an typical individual excess demand function satisfying excess monotonicity looks like the one in

[4]Not quite, as $x' \notin c(p)$ is unobservable. The next section introduces related observable concepts. On a finite data set, the obtained monotonicity measure is clearly only a lower limit of the monotonicity measure of the "full" choice.

[5]It is obviously of interest to find criteria delineating "slight" here.

[6]This was noted by Mas-Colell (1988).

Figure 2. It is thus easy to construct examples where excess monotonicity does not aggregate, by giving two individuals each endowments of only one good. Thus to obtain aggregation, one needs narrow restrictions on both endowments and preferences.[7] The monotonicity hierarchy, however, has no simple counterpart for excess choice. Thus it seems better to investigate such preference restrictions in the choice framework rather than in the excess choice one.

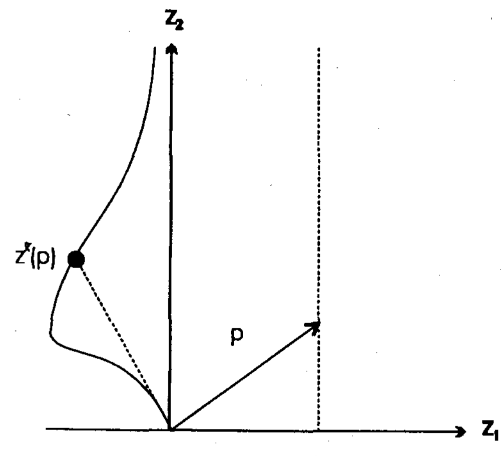


Figure 2

### 3. CYCLICAL MONOTONICITY

Cyclical monotonicity has the same relationship to monotonicity as the transitive axiom of revealed preference has to the basic axiom. To introduce it (and some variants of monotonicity) requires more concepts. Let S stand for finite <u>ordered</u> sets with $|S|$ elements. Elements **s** of S are usually interpreted as time periods, but other interpretations (e.g. as states in a theory of choice under risk) are also of interest. Let $\mathcal{Y} = \mathbb{R}_+$, $y_S \in \mathcal{Y}^S$, and $\alpha \in \mathbb{R}_+ \cup \{\infty\}$. The **$\alpha$-th order mean** of $y_S$, $m^\alpha(y_S)$, is defined in the obvious way by $m^\alpha(y_S) = ((\Sigma_S y_s^\alpha)/|S|)^{1/\alpha}$ for $\alpha$ positive, and let $m^0(y_S) = (\Pi_S y_s)^{1/|S|}$ and $m^\infty(y_S) = \sup y_S$.[8] This hierarchy is also increasing in $\alpha$:[9]

[7] In fact even homotheticity of individual choice is not enough for the axiom of aggregate demand without additional assumptions on the distribution of resources.

[8] One could also use weighted means here, and the following results would still go through. Having found no natural interpretations for these, I stick to the symmetric means.

[9] For a proof, see Leichtweiss (1980), Hilfsatz 21.1.

**LEMMA** 1: If $\alpha < \alpha'$, then $m^{\alpha}(y_S) \leq m^{\alpha'}(y_S)$, and if not all components are equal, then $m^{\alpha}(y_S) < m^{\alpha'}(y_S)$.

For $s \in S$, let $s+$ be the next element in $S$, with the convention that the first element is next to the last, and let $S+$ be equal to $S$, except that each element is replaced by the next. Denote elements of $\mathcal{X}^S$ and $\mathcal{P}^S$ by $x_S$ and $\mathbf{p}_S$, respectively. Also let $p_S x_S \in \mathbb{R}^S_+$, where scalar products are componentwise in the obvious way. Furthermore, if $Q \subseteq A \times B$, write $\mathbf{a}_S \in Q(\mathbf{b}_S)$ if for all $s \in S$, $a_s \in Q(b_s)$, and $a_S \notin \mathbf{Q(b_S)}$ if for some $s \in S$, $a_s \notin Q(b_s)$. The generalized monotonicity notions can now be introduced, where $k \in \mathbb{N}$:

A choice c is **generalized** $(\alpha,k)$**-monotone** if for all $S$, $p_S$, and $x_S \in c(p_S)$, $|S| \leq k$ implies $m^{\alpha}(p_S x_{S+}) \geq 1$, and it is **weakly** $(\alpha,k)$-monotone, $(\alpha,k)$**-semimonotone**, $(\alpha,k)$**-monotone**, and **single-valued** $(\alpha,k)$**-monotone** if additionally a strict inequality holds if for some $s \in S, x_{s+} \in b(p_s)$, $x_{s+}/p_s x_{s+} \notin c(p_s)$, $x_{s+} \notin c(p_s)$, and $x_{s+} \neq x_s$, respectively. Monotonicity is the same as (1,2)-monotonicity. Generally the (1,2)-prefix is omitted and $(\alpha,2)$- is written $\alpha$-. Thus e.g. generalized $\alpha$-monotonicity says that the mean of order $\alpha$ of the "values" of the chosen bundles, evaluated at the previous budgets along cycles of length <u>two</u> is at least one. A choice c is **cyclically** $\alpha$**-monotone** if it is $(\alpha,k)$-monotone for all $k \in \mathbb{N}$, and similarly for the other notions. As the other notions, $(\alpha,k)$-monotonicity is of decreasing strength in $\alpha$ for finite k. Generalized, weak, and standard $(\alpha,k)$-monotonicity are self-dual.[10] The weak notions are decidable on finite choices, in contrast to the standard ones. They are also generally equivalent to the basic notions when attention is restricted to finite (as sets) choices. We have introduced the semi-notions is that for $\alpha = 0$, the standard notions are inconsistent, whereas the seminotions characterize homotheticity. The generalized and weak notions are preserved by subchoices. If choice satisfies the budget identity, the relationship between the different monotonicity notions is illustrated in the diagram for given $(\alpha,k)$, where arrows indicate decreasing strength:

$$\begin{tikzcd}
& \text{single-valued} \arrow[d] & \\
& \text{basic} \arrow[dl] \arrow[dr] & \\
\text{semi-} \arrow[dr] & & \text{weak} \arrow[dl] \\
& \text{generalized} &
\end{tikzcd}$$

[10]Similarly, the largest k such that c satisfies the k-axiom is a (discrete) measure of the degree of transitivity of the (preferences corresponding to the) choice. Other such transitivity measures are discussed in Jerison and Jerison (1989).

The three variants of the concepts in the middle are rather close:, as shown by the following result:

**PROPOSITION 1**: 1) If $\alpha < \alpha'$ then an $\alpha$-semimonotone choice is $\alpha'$-monotone.

2) An $\alpha$-semimonotone and single-valued choice is single-valued $\alpha$-monotone for $\alpha > 0$.

3) If a closed, convex-valued choice which satisfies the budget identity with large interior domain is weakly cyclically $\alpha$-monotone, then it is cyclically $\alpha$-monotone.

**Proof: 1)**: Let $x \in c(p)$, $x' \in c(p')\backslash c(p)$. By weak $\alpha$-monotonicity, $m^{\alpha}(px',p'x) \geq 1$. Furthermore $m^{\alpha'}(px',p'x) \geq m^{\alpha}(px',p'x)$, with a strict inequality if the terms are unequal. That case is trivial. So assume that $px' = p'x = t$. Clearly $t \geq 1$. If $t > 1$, $m^{\alpha}(px',p'x) > 1$, and if $t = 1$, then $x'/px' = x' \notin c(p)$, so the result follows by $\alpha$-semimonotonicity.

**2)**: Let $x \in c(p)$, $x' \in c(p')$, and $x \neq x'$. If $x'/px' \notin c(p)$, the result follows by $\alpha$-semimonotonicity, hence assume $x'/px' \in c(p)$. By single-valuedness, $x = x'/px'$, so $p'x = 1/px'$. By weak $\alpha$-monotonicity, $m^{\alpha}(px',p'x) = m^{\alpha}(px',1/px') \geq 1$. But this inequality is strict if $px' \neq 1$ for $\alpha > 0$, and if $px' = 1$, $x \notin c(p)$, contradiction.

**3)**: Let $x_S \in c(p_S)$ and $x_s \notin c(p_{s+})$ for some $s \in S$. Then by generaqlized $\alpha$-monotonicity, $m^{\alpha}(p_S x_{S+}) \geq 1$. Assume $m^{\alpha}(p_S x_{S+}) = 1$, as the result is trivial otherwise. Assume furthermore that for all s, $p_s x_{s+} = 1$. Then $x_{s+} \in B(p_s)$, contradicting the transitive axiom, which follows from the assumption by Lemma 3.2. Hence for some s, $p_s x_{s+} < 1$, so $x_{s+} \in b(p_s)$. But then by the assumption, $m^{\alpha}(p_S x_{S+}) > 1$. □

The basic notions are essentially the maximally generalized ones, where a choice is **maximally generalized monotone** if it is monotone and it has no maximally generalized monotone extension. At least we have:

**PROPOSITION** 2: 1) If a choice is maximally generalized monotone and satisfies the budget identity, then it is monotone.

2) If a choice c is closed, monotone, and there is a closed set A such that int$A \subseteq D^{-1}c \subseteq A$, then it is maximally generalized monotone.

**Proof: 1):** Assume that c is not monotone. Then there is $x \in c(p)$ and $x' \in c(p')\backslash c(p)$ such that $px'+p'x = 2$. Let $x' \in B(p)$, since otherwise $p'x < 1$, so by the budget identity, $x \notin c(p')$, so we could replace $(x',p)$ by $(x,p')$. Let $c' = c\cup\{(x'/px')\}$. Then $x'/px' \notin c(p)$. If $px' < 1$, this follows from the budget identity, and if $px' = 1$, from the assumption. Hence c' extends c. It follows that c' is generalized monotone by looking at the three permutations of the elements of c', violating the maximality of c.

2: This is essentially the proof of Proposition 6.1, Part 2). □

Proposition 6.1 shows that a similar result holds for cyclical monotonicity.

Single-valued monotonicity is Koch's (1987) "law of demand". For $\alpha = \infty$, one obtains different revealed preference axioms. More specifically, weak $\infty$-monotonicity is equivalent to the **weak axiom**, $\infty$-monotonicity to the basic axiom (from Arrow (1957)), and single-valued $\infty$-monotonicity to the single-valued axiom (usually called Samuelson's (1938) weak axiom). Similarly, cyclical $\infty$-monotonicity is equivalent to the transitive axiom (i.e. Richters (1966) congruence axiom), etc.

At the other end of the hierarchies, 0-monotonicity is inconsistent, whereas 0-semimonotonicity and cyclical 0-semimonotonicity corresponds to homotheticity in theories with and without transitivity. Verifying this needs some more terminology: A choice c is **homothetic** if for all p, $x \in c(p)$, and $t > 0$, if $tx \in Dc^{-1}$, then $tx \in c(p/t)$. For a relation Q, its **homothetic extension**, $\hat{Q}$, is defined by $x \in \hat{Q}(x')$ if there is $t > 0$ such that $tx \in Q(tx')$. Then a choice c satisfies the **homothetic axiom** if $(\hat{P}^c,\hat{R}^c)$ is asymmetric, and the **transitive homothetic axiom** if $(\hat{P}^c,{}^{\hat{R}^c}\hat{R}^c)$ is asymmetric. We are then ready to prove an analogy of Varian (1983), Theorem 2 for not necessarily finite choices:

**PROPOSITION** 3: The choice properties in the following two lists are equivalent:

| | |
|---|---|
| 1) Homotheticity and basic axiom. | Homotheticity and transitive axiom. |
| 2) The homothetic axiom. | The transitive homothetic axiom. |
| 3) 0-semimonotonicity. | Cyclical 0-semimonotonicity. |

Proof: I verify this in the nontransitive case. The transitive case is similar.

**1)⇒2):** Assume $x \in c(p)$, $x' \in B(p)\backslash c(p)$, $tx' \in c(p'/t)$, and $tx \in B(p'/t)$. By homotheticity, $x' \in c(p')$ and $x \in B(p')$, contradicting the basic axiom.

**2)⇒3):** Let $x \in c(p)$, $x' \in c(p')$, and $x'/px' \notin c(p)$. Then $x \in P^c(x'/px')$. If $(px')x \in B(p')$, then $x' \in R^c((px')x)$, contradicting the homothetic axiom. Hence $(px')x \notin B(p')$, i.e. $(px')(p'x) > 1$.

**3)⇒1): Homothetic:** Let $x \in c(p)$, $tx \in Dc^{-1}$, and assume $tx \notin c(p/t)$. Let $tx' \in c(p/t)$. Then by 0-semimonotonicity, $1 \geq ((p/t)x)(ptx') > 1$, contradiction. □

Hence the (transitive) homothetic axiom is self-dual, at least on the interior of the domain. The weak homothetic axiom is also equivalent to weak 0-monotonicity, but these does not quite imply homotheticity. Weak cyclical 0-monotonicity is Afriat's (1981) homogeneous consistency.[11] For negative $\alpha$, the monotonicity notions are inconsistent.

Parallel to the monotonicity measure, the **k-monotonicity measure**, $\mu^k$, is defined by $\mu^k(c) = \inf_\alpha\{\alpha | c \text{ is } (\alpha,k)\text{-monotone}\}$, and the **cyclical monotonicity measure**, $\hat{\mu}$, by $\hat{\mu}(c) = \inf_\alpha\{\alpha | c \text{ is cyclically } \alpha\text{-monotone}\}$. Clearly, $k \leq k'$ implies $\mu^k(c) \leq \mu^{k'}(c)$. When the cyclical monotonicity measure, $\hat{\mu}(c) \leq 1$, the Samuelson' result mentioned in a footnote to Chapter 6.2, shows that preferences are homothetic (and transitive). Hence in the transitive case, the cyclical monotonicity measure, as well as the monotonicity measure, measures a degree of homotheticity. Proposition 6.2 also shows that the cyclical monotonicity measure measures the deviation between the money metric and consumer surplus. They thus give a non-

[11]He also showed that weak cyclical 0-semimonotonicity characterizes homothetic and transitive preferences on finite choices. Indeed, he showed this condition to give more in that context, namely a utility function which is additionally concave and continuous. He has also shown a similar result for the weak transitive axiom. Thus Afriat has characterized limiting cases of the weak cyclically monotone hierarchy, but did not notice the hierarcy.

parametric alternative to Vives (1987) approach to small income effects.[12]

These hierarchies give a natural way to impose and test restrictions on income effects. The monotonicity measures also allow one to non-parametrically express that the law of demand or homotheticity is approximately satisfied in a finite data set. These hierarchies are more informative alternatives to the Slutsky restrictions when it comes to empirical testing.[13]

Monotonicity stands up to empirical tests in most situations, or so I believe, whereas cyclical monotonicity (i.e. homotheticity and transitivity) is often rejected.[14] This suggest that it is reasonable to build empirical theories of choice on monotonicity only. In terms of preferences, this means that one drops transitivity. The resulting theory is simpler and in many ways stronger than the standard theory, and presumably fares better empirically.

The relationships between the different monotonicity concepts is illustrated by the following diagram, where arrows are in the direction of decreasing strength, $\alpha$ increases downwards in the diagram, and k to the left:

$$
\begin{tikzcd}
\begin{array}{l}\text{transitive homothetic axiom}\\ (\text{cyclical } 0\text{–semimonotonicity})\end{array} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \begin{array}{l}\text{homothetic axiom}\\ \text{and transitive axiom}\end{array} \arrow[r] \arrow[d] & \begin{array}{l}\text{homothetic axiom}\\ (0\text{–semimonotonicity})\end{array} \arrow[d] \\
\text{cyclical monotonicity} \arrow[r] \arrow[d] & \begin{array}{l}\text{monotonicity and}\\ \text{transitive axiom}\end{array} \arrow[r] \arrow[dl] & \text{monotonicity} \arrow[d] \\
\begin{array}{l}\text{transitive axiom}\\ (\text{cyclical } \infty\text{–monotonicity})\end{array} \arrow[rr] & & \begin{array}{l}\text{basic axiom}\\ (\infty\text{–monotonicity})\end{array}
\end{tikzcd}
$$

[12]For a sample of size k, the cyclical monotonicity coincides with the k-monotonicity measure. It would be desirable with bounds on the deviation as functions of the monotonicity measures, as well as a treatment of the relationship to Vives analysis.

[13]For a differentiable choice function, it follows from Proposition 2 of Chapter 6 that cyclical monotonicity corresponds to symmetry of the derivative matrix of demand in addition to monotonicity. The latter is slightly weaker than negative definiteness of the derivative of demand. Thus monotonicity and cyclical monotonicity impose the same restrictions on the derivative of demand as the basic axiom and the strong axiom on the Slutsky matrix.

[14]Well - with aggregate data even a representative homothetic consumer is usually not rejected if one has at least five goods aggregates and less than thirty years, see e.g. Houtman and Maks (1983). The explanation is presumably that data are too sparce and have too little price variability in the space considered.

Whereas the middle cases coincide with the left ones in the two limiting cases (with respect to $\alpha$), this is not the case generally. That cyclical monotonicity implies homotheticity follows from Proposition 2 in Chapter 6, as noted in a footnote there.

For single consumers, these classification schemes seem quite satisfactory. But economists are usually more interested in aggregate demand. What about that? Recently Hardle, Hildenbrand, and Jerison (1988), have used average derivative techniques to test the analog of the homothetic axiom in the diagram on panel data, with a somewhat strong maintained hypothesis called metonymy. Given this hypothesis, the analog fares quite well. But as in the last footnote, the reason for this might be little price variability in the data, rather than homothetic preferences.

## 4. PREFERENCE CHARACTERIZATIONS

In this section we introduce preference restrictions corresponding to the different monotonicity notions of choice, and verifies the characterizations in case choice is single-valued. More specifically, $\alpha$-monotonicity is characterized in a theory without transitivity and cyclical $\alpha$-monotonicity in a theory with transitivity. These characterizations are simple, but the preference conditions involved are not very intuitive.

Recently, Kannai (1989), building on Mitjuschin and Polterovich (1978), gave another more complicated characterization of single-valued monotonicity in terms of ordinal properties of utility functions (i.e. with transitivity), assuming that these are twice continuously differentiable. Kannai's characterization of the middle case in the above diagram, whereas I characterize the bordering hierarchies.

Let $\mathcal{U} = \mathbb{R}_+$ and $u_S$ denote elements of $\mathcal{U}^S$. The preference counterparts of the choice monotonicity notions are the following: Preferences (P,R) is **generalized ($\alpha$,k)-harmonic** if for all S with $|S| \leq k$, $x_S$, and $u_S$, $x_S \in R(x_{S+}/u_{S+})$ implies $m^\alpha(u_S) \geq 1$, and **($\alpha$,k)semi-harmonic**, **($\alpha$,k)-harmonic**, and **strictly ($\alpha$,k)-harmonic** if one additionally has a strict inequality if for some $s \in S$, $x_s \in P(x_{s+}/u_{s+})$, $x_s \in P(x_{s+})$, and $x_s \neq x_{s+}$, respectively.[15] Say that preferences are **$\alpha$-harmonic** if they are $(\alpha,2)$-harmonic and **cyclically $\alpha$-harmonic** if they are $(\alpha,k)$-harmonic for all k, etc. Again, drop $\alpha$ if it is one.

The characterizations build on the ones in Chapter 3, which essentially show that complete characterizations of preferences in terms of choice under standard assumptions are fairly straightforward if completeness is weakened.

It follows directly from the definitions that each monotonicity notion of choice is equivalent to the corresponding harmony notion of the generated preferences. This is verified next for the standard ones only:

**LEMMA 2:** The choice c is $(\alpha,k)$-monotone if and only if $(P^c,R^c)$ is $(\alpha,k)$-harmonic.

**Proof:**$\Rightarrow$: Let $x_S \in R^c(x_{S+}/u_{S+})$ and $x_s \in P^c(x_{s+})$ for some $s \in S$. Then there is $p_S$ such that $x_S \in c(p_S)$ and $x_{S+}/u_{S+} \in B(p_S)$. Hence by $(\alpha,k)$-monotonicity, $m^\alpha(p_S x_{S+}) > 1$. But by the definition of B, $u_{S+} \geq p_S x_{S+}$, hence $m^\alpha(u_{S+}) \geq m^\alpha(p_S x_{S+}) > 1$.
$\Leftarrow$: Let $x_S \in c(p_S)$, $x_{S+} \notin c(p_S)$, and $x_{S+}/u_{S+} \in H(p_S)$. Then $p_S x_{S+} = u_{S+}$, so $m^\alpha(p_S x_{S+}) = m^\alpha(u_{S+})$. Also $x_S \in R^c(x_{S+}/u_{S+})$, and for some $s \in S$, $x_s \in P^c(x_{s+})$, so by $(\alpha,k)$-harmony, $m^\alpha(u_{S+}) > 1$, hence $m^\alpha(p_S x_{S+}) > 1$. □

Since trivially $R^{c^R} \subseteq R$, it follows from a variant of Lemma 2 that strict $\alpha$-harmony implies single-valued $\alpha$-monotonicity. Together with the corollary to Theorem 3.1, noting that single-valued monotonicity implies the single-valued axiom, Lemma 2 gives the following characterization of single-valued monotonicity in terms of preferences:

[15]The characterization of the weak monotonicity notions on thew other hand is similar, but in terms of the strenghtened strict revealed preference relation, $^+P^c$.

**THEOREM** 1: 1) If preferences (P,R) is R-closed, convex, strictly $\alpha$-harmonic, and satisfies boundary tangency, then the generated best element choice, $c^R$, is closed, single-valued $\alpha$--monotone, with large domains, and interior range.

2) If the choice c is closed, single-valued, $\alpha$-monotone, with large domains, and interior range, then the generated preferences, $(P^c,R^c)$, is R-closed, convex, strictly $\alpha$-harmonic, satisfies boundary tangency, and generates c.

To get a similar characterization of cyclical monotonicity one need the following result for the cyclical notions, whose proof is straightforward by induction:

**LEMMA** 3: If the generated preferences $(P^c,R^c)$ is strictly cyclically $\alpha$-harmonic, then so is the generated transitive preferences $({}^{R^c}P^c, {}^{R^c}R^c)$.[16]

By partial recoverability, cyclical $\alpha$-harmony implies cyclical $\alpha$-monotonicity. Also cyclical monotonicity implies the strong axiom. These facts, Lemmas 2 and 3, and Theorem 3.2 prove the following characterization of cyclical $\alpha$-monotonicity:

**THEOREM** 2: 1) If the preferences (P,R) is partially recoverable, strictly monotone, transitive, P-open, convex, cyclically $\alpha$-harmonic, and satisfies boundary tangency, then the generated maximal element choice, $c^P$, is closed, cyclically $\alpha$-monotone, satisfies the budget identity and has large interior domains.

2) If a choice c is closed, cyclically $\alpha$-monotone, satisfies the budget identity, and has large interior domains, then the generated transitive preferences, $({}^{R^c}P^c, {}^{R^c}R^c)$, is partially recoverable, strictly monotone, transitive, P-open, convex, cyclically $\alpha$-harmonic, and generates c.

**Remark**: As in Chapter 3, boundary tangency can be weakened in the characterization by making corresponding changes on the choice side.

---

[16]The same holds for the intermediate degrees of transitivity if one has $(\alpha,k)$-harmony with k at least as great as the desired degree of transitivity.

## 5. AGGREGATION

This section shows that if the endowments of the individuals are collinear, then monotonicity and cyclical monotonicity are preserved under aggregation.[17] In other words, if resources are collinear and all individuals satisfy one of these properties, then there exists a representative consumer with the same property.

Let I be the set of individuals, $m_I$ and $m_S$ the (arithmetic) means with respect to I and S, and $z_s^I$ the relative income distribution in situation s (i.e. with $m_I z_s^I = 1$).[18] Let $c^i$ be the choice of individual i, and define **mean choice**, $\bar{c}$, by $m_I x^I \in \bar{c}(p)$ if for all $i \in I$, $x^i \in c^i(p/z^i)$, where the mean is taken componentwise. This says that mean demand only depends on the income distribution through mean income, and is not well-defined generally. But in our case, things works out nicely:

**THEOREM 3:** If each individual $i \in I$ has a closed-valued, and k-monotone choice $c^i$ satisfying the budget identity, and the relative income distribution $z_s^I$ is independent of s, then the mean choice of order , $\bar{c}$, is well-defined and k-monotone.

**Proof:** Let $m_I x_S^I \in \bar{c}(p_S)$ and $m_I x_{S+}^I \notin \bar{c}(p_S)$. Then for all i, $x_S^i \in c(p_S/z_S^i)$, and for some i, $x_{S+}^i \notin c(p_S/z_S^i)$. Let $v_s^i = \inf_x\{p_s x/z_s^i \mid x \in c^i(p_{s+}/z_{s+}^i)\}$. Then since $c^i$ is closed-valued and cyclically monotone, $m_S v_S^i > 1$. Hence $m_S(p_S m_I x_{S+}^I) \geq m_S m_I(v_S^I z_S^I) = m_I m_S(v_S^I z_S^I) = m_I((m_S v_S^I)(m_S z_S^I)\text{-}cov_S(v_S^I, z_S^I)) > 1\text{-}m_I cov_S(v_S^I, z_S^I) \geq 1$. The last inequality follows as for each $i \in I$, $cov_S(v_S^i, z_S^i) = 0$, by the assumption on the income distribution. □

As seen from the proof, the essential condition for the result is the mean covariance one, $m_I cov_S(v_S^I, z_S^I) \leq 0$. This condition is also necessary in the sense that if it is violated, there are mean choices violating k-monotonicity. Essentially the same result holds for the

[17]This should be no surprise, as monotonicity and cyclical monotonicity corresponds to negative definiteness and symmetry and negative definiteness of the derivative of demand, respectively, and these notions are preserved under addition.

[18]If the resources are collinear, then the relative income distribution is independent of the situations (relative prices), at least if the resources are the only source of income.

other monotonicity notions. This result was established by Shafer (1977). The only novelty here is the extraction of the mean covariance condition.[19]

This simple result is important as it sometimes justifies reasoning in terms of a representative consumers. E.g., in class models where individuals in each class hold endowments of only one good and satisfies the law of demand, a representative consumer for each class satisfying the law of demand is justified. Secondly, monotonicity implies the basic axiom for mean choice - which together with differentiability straightforwardly implies uniqueness and stability (with respect to the tâtonnement process) of Walrasian equilibrium.

Theorem 3 rather trivially generalizes to the case where individuals' endowments are still on a line, but not necessarily through 0, as we shall see.[20] The idea is to translate the monotonicity concept in the same manner as the origin. I only write out the standard case: A choice c is **$x^0$-monotone** if for all p, p', $x \in c(p)$, and $x' \in c(p')\backslash c(p)$, $m(\frac{p(x'-x^0)}{p(x-x^0)}, \frac{p'(x-x^0)}{p'(x'-x^0)}) > 1$.[21] Here again a hierarchy can be defined by varying the mean as before. What is the relationship between the translated and the standard hierarchy? In general the $x^0$-homothetic axiom does not even imply monotonicity, as is seen by choosing $x^0 = (-1,-1)$, $p = (0.1,0.65)$, $p' = (3.5,0.1)$, $x = (0.1,1.5)$, $x' = (0.1,1.8)$. Then $(px'+p'x)/2 = 0.86$, hence monotonicity is violated, whereas the budget conditions and the translated homotheticity condition are roughly satisfied. For $\alpha = \infty$, i.e. in the basic axiom case, the translated hierarchies coincides with the standard ones, thus $x^0$-homotheticity and $x^0$-monotonicity implies the basic axiom.

The aggregation result for monotonicity (Theorem 3) easily generalizes to translated monotonicity in the following form: If the individuals choices are $x^0$-monotone and the individuals endowments are on a line through $x^0$, then mean demand is also $x^0$-monotone. Thus if endowments are on a line, to ensure aggregation, it is sufficient to find that the individual choices are $x^0$-monotone for some $x^0$ on the line. This might help a little for aggregation within endowment characterized groups, but does not help much generally.

---

[19]Cyclical monotonicity is equivalent to homotheticity and transitivity, as indicated in Footnote 6.9. Hence the cyclical monotonicity version of Theorem 4 is essentially Eisenberg's (1961) aggregation result.

[20]But what one would like is results with endowments distributed in a some larger subspace.

[21]Note that one does not require $x^0 \in \mathcal{X}$. To avoid some complications, one would like $px^0 < 1$, for all p, which in our case is ensured if $x^0 \ll 0$.

April 1990

# CHAPTER 5: SEPARABILITY

## 1. INTRODUCTION

In this chapter we will investigate what is an appropriate notion of separable choice. The basic concept of separability is with respect to a group of goods, and this is the separability notion we will treat here. In terms of preferences it says preferences between goods in the group is independent of the amount held of goods outside the group. This concept was made precise by Stigum (1967) and Gorman (1968) and slightly generalized by Bliss (1975), though the general idea has been known for a long time.

A corresponding choice property is that the choice with respect to a group of goods only depends on the group budget. This is Blackorby, Primont, and Russell's (1978) (weak) decentralization, going back to Lau (1969) and Pollak (1970). It essentially says that the subgroup revealed preference relations are well-defined relations. A slight strengthening of this concept I call independence. It is a choice counterpart of Sono (1945) and Leontief's (1947a,b) independence property of the marginal rate of substitution. As one might expect from Sono and Leontief's results, independence is a fundamental notion of separable choice. We show that independence is equivalent to a subgroup version of the basic revealed preference axiom. In addition to saying that the subgroup revealed preference relations are well-defined relations, as required by decentralization, the subgroup axiom demands that these relations constitute preferences, i.e. are asymmetric. The subgroup axiom is easily testable on finite data sets. It expresses the separability idea more directly, and are more computationally efficient than previous nonparametric notions like the ones in Varian (1983). It is easy to show that the subgroup axiom characterizes separable preferences. Also a hierarchy of concepts between separability and homothetic separability is introduced and similarly characterized. This is analogous to the hierarchies in the previous chapter.

## 2. SEPARABLE PREFERENCES

The **section** of a relation $Q \subseteq \mathcal{X}^2$ at $x_{-I}$, $Q_{x_{-I}}$, is defined by $x_I \in Q_{x_{-I}}(x'_I)$ if $x \in Q(x'_I+x_{-I})$. Preferences (P,R) is **I-separable** if for all $x_{-I}$ and $x'_{-I,}$ $R_{x_{-I}} \subseteq R_{x'_{-I}}$, and **generalized I-separable** if for all x and x', $R_{x_{-I}}(x_I) \subseteq R_{x'_{-I}}(x'_I)$ or $R_{x'_{-I}}(x'_I) \subseteq R_{x_{-I}}(x_I)$. Separability says that the sections at different points (in $\mathcal{X}_{-I}$) are equal, and generalized separability that the upper sections are nested. These definitions are due to Stigum (1967) and Bliss (1975), respectively.[1]

Since we do not assume complete preferences, it is convenient to work with slightly weaker separability concepts. These are structurally similar to our general definition of preferences. For this, define the **I-sections** of a relation Q, $Q_I$, by $x_I \in Q_I(x'_I)$ if there is $x_{-I}$ such that $x_I \in Q_{x_{-I}}(x'_I)$, and say that preferences (P,R) is **I-asymmetric** if their I-sections, $(P_I,R_I)$, is asymmetric, i.e. if the sections are preferences in their own right, and **generalized I-asymmetric** if $P_I$ is asymmetric.[2] If preferences are complete (and additionally transitive in the generalized case), these notions imply the standard ones, as is easily verified.

The generalized notions are of interest only when preferences violates **I-local nonsatiation**, (i.e. for all x there are $x^n_I \to x_I$ such that $x^n_I \in P_{x_{-I}}(x_I)$).[3] For I-asymmetry this is is a consequence of the following result, which is analogous to the similar result for I-separability in Blackorby, Primont, and Russell (1978, Lemma 3.1):

**LEMMA 1:** Let the preferences (P,R) be transitive, P-open and I-locally nonsatiated. Then generalized I-asymmetry implies I-asymmetry.

Proof: Assume $x \in P(x'_I+x_{-I})$ and $x' \in R(x_I+x'_{-I})$. By I-local nonsatiation, there are $x^n_I \to x'_I$ such that $x^n_I+x'_I \in P(x')$. Hence by transitivity, $x^n_I+x'_I \in P(x_I+x'_{-I})$. But since P is open, for sufficiently large n, $x \in P(x^n_I+x'_{-I})$, contradicting generalized I-asymmetry. □

---

[1]Blackorby, Primont, and Russell (1978) call these notions strict separability and separability, respectively.

[2]That generalized I-separability implies generalized I-asymmetry presupposes transitivity.

[3]The typical example is Leontief preferences.

We introduce a similar notion of homothetic separability. First, the **homothetic extension** of the relation Q, $\hat{Q}$, is defined by $x \in \hat{Q}(x')$ if there is $t > 0$ such that $tx \in Q(tx')$. Then preferences (P,R) is **I-homothetic separable** if $\hat{R}_I \subseteq R_I$, and **I-homothetically asymmetric** if $(\hat{P}_I,\hat{R}_I)$ asymmetric. The latter notion is generally weaker, but implies the former if preferences are complete.

Finally, hierarchies between homothetic I-asymmetry and I-asymmetry are introduced. Preferences (P,R) is **generalized $(\alpha$,I)-harmonic** if for all x and x', if $x_I \in R_I(x_I'/u')$ and $x_I' \in R_I(x_I/u)$, then $m^\alpha(u,u') \geq 1$, where $\mathbf{m}^\alpha$ is the $\alpha$-order mean. Also, the preferences are **$(\alpha$,I)-semiharmonic**, **$(\alpha$,I)-harmonic**, and **single-valued $(\alpha$,I)-harmonic** if a strict inequality holds if additionally $x_I \in P_I(x_I'/u')$, $x_I \in P_I(x_I')$, and $x_I \neq x_I'$, respectively. These notions are of decreasing strength in $\alpha$ and $(\infty$,I)-harmony is equivalent to I-asymmetry.

## 3. SEPARABLE CHOICE

We then introduce our concepts of separable choice: A choice c is **I-independent** if for all p, $x \in c(p) \cap \text{int}\mathcal{X}$, and $x'_{-I} \in \text{int}\mathcal{X}_{-I}$, there is p' such that $x_I + x'_{-I} \in c(p')$ and $p_I'/p_I'x_I = p_I/p_Ix_I$ (see Figure 1). Here $p_I/p_Ix_I$ is the **I-section** of p at $x_{-I}$. I-independence says that an interior point with the same I-projection as some chosen point is chosen at some budget with the same I-section. It is implicit in Lensberg (1987) and closely related to Sono (1945) and Leontief's (1947a,b) independence property of the marginal rate of substitution. A weaker notion is Lau (1969) and Pollak's (1970) (weak) **I-decentralization**, defined by for all p, p', $x \in c(p) \cap \text{int}\mathcal{X}$, $x' \in c(p') \cap \text{int}\mathcal{X}$, $p_I/p_Ix_I = p_I'/p_I'x_I'$ implies that $x_I' + x_{-I} \in c(p)$.[4] This expresses that if the I-sections at chosen

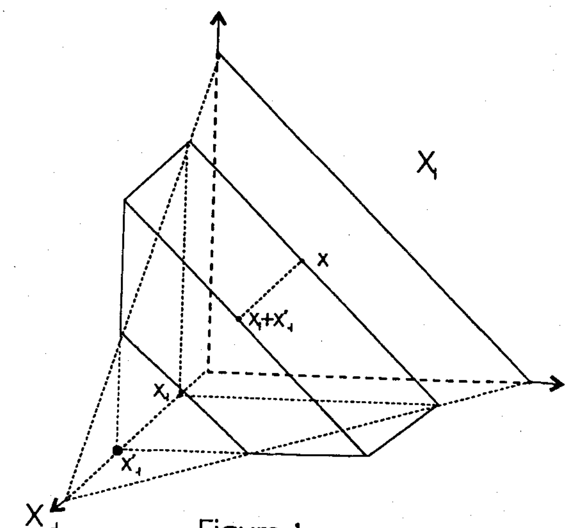


Figure 1

[4]That *separability implies decentralizability follows from the axiom.

(interior) points are equal, then the sectional choices are also equal,[5] or in other words that the **I-sectional choice**, $c_I$, (defined by $x_I \in c_I(p_I)$ if there is $x_{-I}$ and p' such that $x \in c(p')$ and $p_I = p'_I/p'_I x_I$) is well-defined.

Subgroup analogies of the revealed preference axioms are introduced next. Write $Q^c_I$ for $(Q^c)_I$ and say that a choice c satisfies the **generalized I-axiom** if $^+P^c_I$ is asymmetric, the **weak I-axiom** if $(^+P^c_I,R^c_I)$ is asymmetric, the **I-axiom** if $(P^c_I,R^c_I)$ is asymmetric, and the **single-valued I-axiom** if $R^c_I$ is antisymmetric.[6] As the corresponding revealed preference axioms these axioms are generally of increasing strength and coincide if choice is single-valued. Whereas decentralization expresses that the revealed preference sections, $(P^c_I,R^c_I)$, are well-defined relations on $\mathcal{X}_I$, the I-axiom says additionally that they constitute preferences, i.e. are asymmetric.[7] The I-axiom is essentially equivalent to I-independence:

**PROPOSITION 1:** 1): An I-independent choice which satisfies the basic axiom and the budget identity also satisfies the I-axiom on the interior of its range.

2): If a choice is closed, inversely convex-valued, has large inverse domain, and satisfies the I-axiom, then it is I-independent.

**Proof:** 1): Assume $x \in c(p)$, $x'_I+x_{-I} \in B(p)\backslash c(p)$, $x' \in c(p')$, and $x_I+x'_{-I} \in B(p')$. By I-independence, there is p" such that $x'_I+x_{-I} \in c(p'')$ and $p''_I/p''_I x'_I = p'_I/p'_I x'_I$. But then $p''_I x_I/p''_I x'_I = p'_I x_I/p'_I x'_I \leq 1$, so $x \in B(p'')$,[8] contradicting the basic axiom.

[5]In Chapter 6, it is shown that complete decentralizability (with respect to the finest partition) is sufficient to characterize an additively separable utility function.

[6]These axioms express the idea of separable choice more directely than previous nonparametric notions, like the ones used in Varian's (1983) Theorem 3. Also as these axioms avoid existential quantifiers, tests based on them should be computationally more efficient.

[7]With only two goods, all the separability notions defined here are essentially powerless, e.g. the I-axiom follows from strict monotonicity.

[8]Note that if $x \in H(p)$, then $x'_I+x_{-I} \in B(p)$ if and only if $p_I x'_I/p_I x_I \leq 1$, i.e. $x'_I \in B_I(p_I/p_I x_I)$.

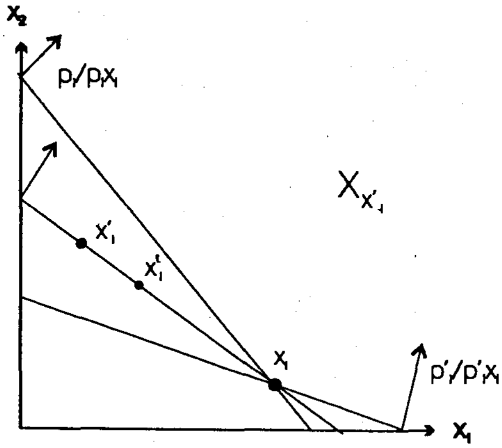



Figure 2

2): (Figure 2) If not, there is p, $x \in c(p)$, and $x_{-I}$ such that for all $p' \in c^{-1}(x_I+x'_{-I})$, $p_I/p_Ix_I = p_I'/p_I'x_I$. Then there is $x_I'$ such that $x' \in b(p/p(x_I+x'_{-I}))\backslash B(p')$. Let $x^t = tx'+(1-t)(x_I+x'_{-I})$ and $x^t \in c(p^t)$. If $x_I+x'_{-I} \in B(p^t)$, then $x^t \in R(x_I+x'_{-I})$. But by the budget identity, $x \in P^c(x_I^t+x_{-I})$, contradicting the I-axiom. Hence $x_I+x'_{-I} \notin B(p^t)$, so $x' \in B(p^t)$ for all $t \in <0,1]$. Hence since c is closed, $x_I+x'_{-I} \in c(p^0)$ and $x' \in B(p^0)$, contradicting the inverse convex-valuedness of c. □

Lemma 2 is analogous to Lemma 4 in Lensberg (1987) and Lemma 1 in Chapter 6. These, however, essentially presuppose complete separability (and one-dimensional factor spaces) due to the weaker separability concepts used (e.g. decentralization in the latter case), whereas the I-axiom also works for separability more generally.[9]

The choice notions corresponding to homothetic separability are: A choice c is I-**homothetic** if for all p, $x \in c(p)$, $x'_{-I}$, and $t > 0$, there is p' such that $t(x_I+x'_{-I}) \in c(p'/t)$, and $p_I'/p_I'x_I = p_I/p_Ix_I$,[10] satisfies the **generalized homothetic I-axiom** if ${}^{+}\hat{P}_I^c$ is acyclic, the **homothetic I-axiom** if $(\hat{P}_I^c,\hat{R}_I^c)$ is asymmetric, and the **single-valued homothetic I-axiom** if $\hat{R}_I^c$ is anti-symmetric.

Next, I introduce hierarchies between the above I-axioms and the I-homothetic axioms, which are subgroup versions of the monotonicity hierarchies in Chapter 4. Then a choice c is **generalized ($\alpha$,I)-monotone** if for all p, p', $x \in c(p)$, and $x' \in c(p')$, $m^{\alpha}(p_Ix_I'/p_Ix_I,p_I'x_I/p_I'x_I') \geq 1$, it

[9]Assuming single-sectioned choice, it is straightforward to extend the result by Kihlstrom, Mas-Colell, and Sonnenschein (1976) to show that the weak I-axiom is equivalent to I-separability and negative semidefiniteness of the I-submatrix of the Jacobean of inverse demand on the appropriate hyperplane. As negative definiteness is inherited by submatrices and the axiom holds in the following, this is of less interest, however.

[10]I-homotheticity is a homothetic version of I-*separability. Indeed it is equivalent to I-*separability and homotheticity of the I-sectional choice, $c_I$, as is easily verified.

is **($\alpha$,I)-semimonotone**, **($\alpha$,I)-monotone**, and **single-valued ($\alpha$,I)-monotone** if additionally a strict inequality holds if $(p_Ix_I/p_Ix_I')x_I'+x_{-I} \notin c(p)$, $x_I'+x_{-I} \notin c(p)$, and $x_I' \neq x_I$, respectively. Thus ($\alpha$,I)-monotonicity says that the I-sectional choice is $\alpha$-monotone. When $\alpha = \infty$ these notions are trivially equivalent to the above I-axioms. At the other end of the hierarchy, (0,I)-monotonicity is inconsistent, but (0,I)-semimonotonicity is equivalent to the homothetic I-axiom,[11] or more generally:

**PROPOSITION 2:** For a closed, inversely convex-valued choice with large inverse domain satisfying the budget identity, the following properties are equivalent:

1): Homotheticity of the I-sectional choice and the I-axiom.

2): I-homotheticity and the I-axiom.

3): The homothetic I-axiom.

4): (0,I)-semimonotonicity.

**Proof: 1)⇒2):** Let $x \in c(p)$. Then $x_I \in c(p_I/p_Ix_I)$, so by homotheticity of $c_I$, $tx_I \in c_I(p_I/tp_Ix_I)$. Hence $tx \in c(p'/t)$, where $p'/p_I'x_I = p_I/p_Ix_I$. By the second part of Proposition 1, the I-axiom implies I-independence, hence there is p" such that $t(x_I+x'_{-I}) \in c(p''/t)$, where $p_I''/p_I''x_I = p_I'/p_I'x_I$.
**2)⇒3):** Assume $x \in c(p)$, $x_I+x'_{-I} \in B(p)\backslash c(p)$, $tx' \in c(p'/t)$, and $t(x_I+x'_{-I}) \in B(p'/t)$. By I-homotheticity, there is p" such that $x_I'+x_{-I} \in c(p'')$, where $p_I''/p_I''x_I' = p_I'/p_I'x_I'$. But then as $x_I+x'_{-I} \in B(p')$, $p_I''x_I/p_I''x_I' = p_I'x_I/p_I'x_I' \leq 1$. Hence $x \in B(p'')$, contradicting the I-axiom.
**3)⇒4):** Let $x \in c(p)$, $x' \in c(p')$, and $(p_Ix_I/p_Ix_I')x_I'+x_{-I} \notin c(p)$. Then $x \in P^c((p_Ix_I/p_Ix_I')x_I' +x_{-I})$, i.e. $x_I \in P_I^c((p_Ix_I/p_Ix_I')x_I')$. Assume that $(p_Ix_I'/p_Ix_I)x_I+x'_{-I} \in B(p')$, then $x' \in R^c((p_Ix_I'/p_Ix_I)x_I+x'_{-I}))$, i.e. $x_I' \in \hat{R}_I^c((p_Ix_I'/p_Ix_I)x_I)$, contradicting the I-homothetic axiom. Hence $(p_Ix_I'/p_Ix_I)x_I+x'_{-I} \notin B(p')$, i.e. $(p_Ix_I'/p_Ix_I)(p_I'x_I/p_I'x_I') > 1$.
**4)⇒1):** Let $x_I \in c_I(p_I^0)$, and assume that $tx_I \notin c_I(p_I^0/t)$. Let $tx_I' \in c_I(p_I/t)$. Then there is $x_{-I}$, p, $x'_{-I}$, and p' such that $x \in c(p)$, $x' \in c(p')$, and $p_I/p_Ix_I = p_I^0 = p_I'/p_I'x_I'$. Clearly $((p_I'/t)tx_I'/(p_I'/t)x_I)x_I+x'_{-I} = t(x_I+x'_{-I}) \notin c(p/t)$. Hence by (0,I)-semimonotonicity, $1 = p_I^0(tx_I')(p_I^0/t)x_I > 1$. □

[11]Again, the (0,I)-semimonotonicity notions are simpler and computationally more efficient than previous nonparametric conditions for homothetic separability, like the ones in Varian (1983), Theorem 5.

The homothetic separability notions are **self-dual** (i.e. preserved when x's and p's are interchanged),[12] since the definitions of (0,I)-monotonicity are. (1,I)-monotonicity is called **I-monotonicity** etc. As monotonicity (i.e. the law of demand), I-monotonicity is presumably an important concept, though I have so far found no uses for it. All monotonicity concepts are easily testable on finite data sets.[13]

## 4. CHARACTERIZATION RESULTS

This section characterizes some of the above preference separability concepts in terms of the corresponding choice ones. Building on the characterization results in Chapter 3 and 4, this is straightforward. With transitivity one has the following result:

**THEOREM 1:** 1) If preferences (P,R) is transitive, strictly monotone, convex, P-open, smooth, strictly $(\alpha,I)$-harmonic, and satisfies boundary tangency, then the maximal element choice, $c^P$, is closed, single-valued $(\alpha,I)$-monotone, has large interior domains, and satisfies the transitive axiom.

2) If the choice c is closed, single-valued $(\alpha,I)$-monotone, has large interior domains, and satisfies the transitive axiom and the budget identity, then the generated transitive preferences, $({}^{R^c}P^c, {}^{R^c}R^c)$, is strictly monotone, partially recoverable, convex, P-open, smooth, strictly $(\alpha,I)$-harmonic, satisfies boundary tangency, and generates c.

---

[12]This is not the case for the weaker notions despite the fact that monotonicity is self-dual. This is due to the role of the group income shares.

[13]Group variants of cyclical monotonicity from Chapter 4 can also be defined: A choice c is **weakly cyclically $(\alpha,I)$-monotone** if for all S, $p_S$, and $x_S \in c(p_S)$, $m^\alpha_S(p_S(x_{IS+}+x_{-IS})) \geq 1$. The stronger cyclical notions are defined in the obvious way. Again, strength is decreasing in $\alpha$, cyclical $(\infty,I)$-monotonicity is equivalent to the I-transitive axiom and cyclical (0,I)-monotonicity the homothetic I-transitive axiom. The latter corresponds to homothetic separability (of group I) in theories with transitivity. Clearly the I-monotonicity notions imply the corresponding I-axioms which are equal to $(\infty,I)$-monotonicity.

Proof: By Theorem 2 of Chapter 3, only $(\alpha,I)$-monotonicity and $(\alpha,I)$-harmony need to be verified:

**Single-valued $(\alpha,I)$-monotonicity:** Let $x \in c^P(p)$, $x' \in c^P(p')$, and $x_I' \neq x_I$. Since $(p_I x_I/p_I x_I')x_I'+x_{-I} \in B(p)$, $x \in R^{c^P}((p_I x_I/p_I x_I')x_I'+x_{-I})$. Then by partial recoverability, $x \in R((p_I x_I/p_I x_I')x_I'+x_{-I})$, i.e. $x_I \in R_I((p_I x_I/p_I x_I')x_I'$. Similarly, $x_I' \in R_I((p_I' x_I'/p_I' x_I)x_I)$, and the result follows by $(\alpha,I)$-harmony.

**Strict $(\alpha,I)$-harmony:** Assume $x \in R^c((x_I'/u'+x_{-I})$, $x' \in R^c((x_I/u+x'_{-I})$ and $x_I \neq x_I'$. Then there are p and p' such that $x \in c(p)$, $x_I'/u'+x_{-I} \in B(p)$, $x' \in c(p')$, and $x_I/u+x'_{-I} \in B(p')$. Hence by single-valued $(\alpha,I)$-monotonicity, $m^{\alpha}(p_I x_I'/p_I x_I, p_I' x_I/p_I' x_I')) > 1$. But then since $u' \geq p_I x_I'/p_I x_I$ and $u \geq p_I' x_I/p_I' x_I'$, so $m^{\alpha}(u,u') > 1$. □

Building on Theorem 1 in Chapter 3, one gets a similar characterization in the absence of transitivity by essentially the same argument:[14]

**THEOREM 2:** 1) If preferences (P,R) is R-closed, convex, strictly $(\alpha,I)$-harmonic, and satisfies boundary tangency, then the generated best element choice, $c^R$, is closed, single-valued, $(\alpha,I)$--monotone, with large domains and interior range.

2) If the choice c is closed, single-valued $(\alpha,I)$-monotone, with large domains and interior range, then the generated preferences, $(P^c,R^c)$, is R-closed, convex, strictly $(\alpha,I)$-harmonic, satisfies boundary tangency, and generates c.

Let $\mathcal{I}$ be a set of subsets of $I^0$, and say that preferences (P,R) is **$\mathcal{I}$-separable** if it is I-separable for each $I \in \mathcal{I}$, and similarly for the other separability notions. It is obvious that the above characterizations carries over to these many-group notions. If the set $\mathcal{I}$ is large, one would expect some redundancies in the characterization.

[14]Note that smoothness and strict convexity are unnecessary in this case.

July 1991
Revised December 1991

# CHAPTER 6: CONCAVE UTILITY[0]

## 1. INTRODUCTION

Concave utility functions are important in economics. This article characterize their demand (behavioral) implications by giving necessary and sufficient conditions on a choice (demand correspondences) for the existence of concave utility functions which generates the choice. Our construction is a based on two ideas: The first is Rockafellar's (1970) characterization of the superdifferential of a concave function. The second is that the superdifferential of a concave function is equal to the choice it generates, up to a multiplier - the marginal utility of income. This is a direct consequence of the first-order conditions for the utility maximizing problem, which in the concave case are both necessary and sufficient.

Rockafellar's characterization of the superdifferential of a concave function is in terms of a notion called "cyclical monotonicity." For a differentiable function of one argument, this simply says that the superdifferential function is decreasing. Our characterization of the choices generated by concave utility functions uses a weaker notion called "cyclical quasimonotonicity", differing by the fact that it admits the marginal utilities of income as multipliers. For finite choices,[1] cyclical quasimonotonicity is a variant of Afriat's (1976) "system of multipliers". Thus the finite choice version of this result has been known for a long time. What we do here is to extend Afriat's results from finite to more general choices.

As for related work, Kannai (1977, 1986) characterized concavifiable preferences in three different ways. Our main result appears simpler than the ones offered by Kannai. This is because the condition for concavifiability intuitively involves both quantities and prices. For choices both are primitives, but for preferences the conditions involving prices must be expressed in terms of quantities only. The result also show that Kannai's (1977) remark: "In

---

[0]I am grateful to Lars Thorlund-Petersen and especially to Sjur Flåm for valuable comments to this chapter.

[1]A finite choice is a choice which is finite as a set, i.e. one based on a finite set of observations.

our context [Rockafellar's characterization] does not appear to be so useful" does not substantiate in our slightly different context.

As noted by Debreu (1976) and Kannai (1977), cardinal utility makes sense with concave utility. The reason being that attention can - without loss of generality - be restricted to the least concave utility functions, and these are cardinally determined. This is shown by a simple proof of the existence of least concave utility functions for a choice. In contrast, the starting point for Debreu's (1976) classic proof is some preferences.

The plan is as follows: For background and perspective, we first review Rockafellar's results. Thereafter we give our main result, and finally treat least concavity.

## 2. SUPERDIFFERENTIALS OF CONCAVE FUNCTIONS

This section restates Rockafellar's (1970, Theorem 24.8 and 24.9) characterization of the superdifferential of a concave function. In our jargon, they say that the superdifferentials of a concave function are the maximally generalized cyclically monotone choices.[2] For our purposes it is convenient to add two minor new results: First, maximally generalized cyclical monotonicity is equivalent to the somewhat simpler concept of cyclical monotonicity, which avoids the maximality construction. Secondly, the concave integral of a cyclically monotone choice is a utility function - i.e. generates the choice.

Consider the class of functions $\mathcal{U} = \{u:\mathbb{R}^l \to \mathbb{R} \cup \{-\infty\} \mid u \text{ is closed, concave, and } u(x_0) = 0\}$.[3] Let $u \in \mathcal{U}$. The **superdifferential of u, $\partial u \subseteq \mathbb{R}^l \times \mathbb{R}^l$**, is defined by $p \in \partial u(x)$ if for all x', $u(x') \le u(x) + p(x'-x)$. Trivially, if $u \in \mathcal{U}$, then $\partial u$ has convex values, and if u is closed, then $\partial u$ is closed. A correspondence $d \subseteq \mathbb{R}^l \times \mathbb{R}^l$ is said to be (decreasingly) **generalized cyclically monotone** if for all finite sequences $\{(p_i,x_i)\} \subseteq d$, $\Sigma_i p_i(x_{i+}-x_i) \ge 0$, where **i+** denotes the successor index of i in the sequence, assuming that the successor of the last element is the

[2]Rockafellar's "cyclical monotonicity" is our "generalized cyclical monotonicity." This is to keep terminology in line with the revealed preference one in the preceeding chapters.

[3]The assumption $u(x^0) = 0$ is a normalization to get rid of the indetermined integration constant.

very first one.[4] We single out the class $\mathfrak{D} = \{d \subseteq \mathbb{R}^l\times\mathbb{R}^l \mid d$ is generalized cyclically monotone$\}$, and focus on the subclass $\mathfrak{D}^* = \{d \in \mathfrak{D} \mid d \subset d'$ implies $d' \notin \mathfrak{D}\}$, i.e. the maximally generalized cyclically monotone correspondences. In the one-dimensional case ($l = 1$), $\mathfrak{D}$ is simply the set of nonincreasing curves in $\mathbb{R}^2$, and $\mathfrak{D}^*$ is the set of connected nonincreasing curves. Theorem 1 below says that $\mathfrak{D}^*$ is the set of superdifferentials of concave functions. To show this, one need to define an inverse operation to the superdifferential, $\partial$, i.e. an integral. Let $d \in \mathfrak{D}$. Define the (concave) **integral of d**, given $p_0$, $\mathbf{u^d} = \mathbf{u^d_{p0}}$,[5] by $u^d_{p0}(x) = \inf\{p_0(x_1\text{-}x_0)+...+p_{n-1}(x_n\text{-}x_{n-1})+p_n(x\text{-}x_n) \mid \{(p_i,x_i)\} \subseteq d$ is a finite sequence$\}$. The terminology is justified by Rockafellar's (1970) Theorem 24.8 and 24.9 which can be restated compactly in the form:

**THEOREM 1:** 1) If $u \in \mathfrak{U}$, then $\partial u \in \mathfrak{D}^*$ and $u^{\partial u} = u$.
2) If $d \in \mathfrak{D}$, then $u^d \in \mathfrak{U}$ and $d \subseteq \partial u^d$. If $d \in \mathfrak{D}^*$, then $d = \partial u^d$.

The **domain** of a correspondence d, is defined as $\mathbf{Dd} = \{x \mid d(x) \neq \emptyset\}$. Superdifferentials of concave functions can, without the maximality construction, be characterized as the closed and cyclically monotone correspondences with essentially convex domains:

**PROPOSITION 1:** $d \in \mathfrak{D}^*$ if and only if it is closed, cyclically monotone, and there is a closed convex set A such that $\text{int}A \subseteq Dd \subseteq A$.

Proof: $\Rightarrow$: Let $d \in \mathfrak{D}^*$. Then by Theorem 1, $d = \partial u$ for some closed concave function u. Clearly, such a superdifferential is closed, and by Rockafellar's (1970) Theorem 23.4, $\text{int}A \subseteq Dd \subseteq A$, where $A = \text{cl}Du$. Hence it suffices to show that $\partial u$ is cyclically monotone. Let $p_i \in \partial u(x_i)$ for $i = 1,...,n$ and $p_n \notin \partial u(x_1)$. Since u is concave, by Theorem 1 again, $\partial u$ is generalized cyclically monotone, i.e. $\sum_1^n p_i(x_{i+}\text{-}x_i) \geq 0$. Assume that $\sum_1^n p_i(x_{i+}\text{-}x_i) = 0$. Since $p_n \notin \partial u(x_1)$, there is $x_{n+1}$ such that $u(x_{n+1})\text{-}u(x_1) > p_n(x_{n+1}\text{-}x_1)$. But $u(x_{n+1})\text{-}u(x_1) =$

---

[4]It is **cyclically monotone** if a strict inequality holds if for some i, $(p_i,x_{i+}) \notin d$.

[5]In the following the index $p^0$ is dropped from the notation.

$\sum_1^n(u(x_{i+1})-u(x_i)) \leq \sum_1^n p_i(x_{i+1}-x_i) = p_nx_{n+1}-p_nx_1$, contradiction. The last equality follows from $\sum_1^n p_i(x_{i+}-x_i) = 0$.

$\Leftarrow$: It suffices to show that if d is closed and monotone, then it is maximally generalized monotone.[6] The conclusion then follows since generalized cyclical monotonicity implies generalized monotonicity and a generalized cyclically monotone extension is also a generalized monotone extension. Let d be closed and monotone, assume that d' is a generalized monotone extension of d, and let $p' \in d'(x) \setminus d(x)$, where $x \in \text{int}Dd$ and $p \in Dd^{-1}$. Let $x' \in d^{-1}(p')$, $x^t = tx'+(1-t)x$, and since $x^t \in Dd$, $p^t \in c(x^t)$ for $t \in \langle 0,1\rangle$. Then since d' is generalized monotone, $(p^t-p')(x'-x) = (p^t-p')(x^t-x)/t \leq 0$ for $t \in \langle 0,1\rangle$. Since $x \in \text{int}Dd$, by Rockafellar's (1970) Theorem 23.3, $\partial f(x)$ is bounded. Hence $\{p^t\}$ is bounded as $t\to 0$. Let $p^t\to p$, subsequentially as $t\to 0$. Then $(p-p')(x-x') \leq 0$. But since d is closed, $p \in d(x)$. Hence since d is monotone, $(p'-p)(x'-x) < 0$, contradiction. □

For the rest of this chapter, we restrict ourselves to a demand theory framework.

Recall some notation: The **budget correspondence**, $B \subseteq \mathcal{X}\times\mathcal{P}$, is defined by $x \in B(p)$ if $px \leq 1$, $x \geq 0$, and $p \geq 0$.[7] If $c \subseteq B$, then c is a **choice** (demand correspondence). A choice c satisfies **budget identity** if $x \in c(p)$ implies $px = 1$. The **choice generated by a utility function u, $c^u$**, is defined by $x \in c^u(p)$ if $x \in B(p)$ and for all x', if $u(x') > u(x)$, then $x' \notin B(p)$.

We show that for a generalized cyclically monotone choice, the integral is indeed a utility function:

**PROPOSITION** 2: Let the choice c satisfy the budget identity. Then:

1) If c is generalized cyclically monotone, then $u^c$ generates an extension of c.

2) If c is maximally generalized cyclically monotone, then on Dc, $u^c$ generates c.

---

[6]A choice is (generalized) **monotone** if one restrict attention to pairs in the definition of (generalized) cyclical monotonicity.

[7]Thus we have normalized prices by setting income equal to 1.

Proof: This follows from Theorem 1 if we can show: i) If $c^{-1} \subseteq \partial u^c$, then $c \subseteq c^{u^c}$ and ii) that restricted to Dc, if $c^{-1} = \partial u^c$, then $c = c^{u^c}$.

1): Assume $x \in c(p)\backslash c^{u^c}(p)$. Then $p \in \partial u^c(x)$ and there is $x' \in B(p)$ such that $u^c(x') > u^c(x)$. Then by the definition of $u^c(x)$, there is a finite sequence $\{(x_i,p_i)\} \subseteq c$ such that $u^c(x') > p_0(x_1-x_0)+...+p_n(x-x_n)$. Hence by the definition of $u^c(x')$, $u^c(x')+p(x'-x) > p_0(x_1-x_0)+...+p_n(x-x_n)+p(x'-x) \geq u^c(x')$. Hence $p(x'-x) > 0$, contradicting $x' \in B(p)$ and the budget identity.

2): Assume $x \in c^{u^c}(p)\backslash c(p)$. Then by assumption, $p \notin \partial u^c(x)$. Hence there is x' such that $u^c(x')-u^c(x) > p(x'-x)$. Let $x^* \in c(p)$. Then by 1), $x^* \in c^{u^c}(p)$, so $u^c(x^*) = u^c(x)$. By assumption, $p \in \partial u^c(x^*)$. Hence $u^c(x')-u^c(x) = u^c(x')-u^c(x^*) \leq p(x'-x^*) = p(x'-x)$ by the budget identity, contradiction. □

The integral of a generalized cyclically monotone choice is simply the (Marshallian) consumer surplus - the minimum willingness to pay to get to x from $x^0$ along some path.[8]

## 3. CONCAVIFIABLE CHOICE

Theorem 1 gave a one-to-one correspondence between closed concave functions and superdifferentials, i.e. (maximally generalized) cyclically monotone correspondences. We use this result to verify a similar one-to-one correspondence between the class of closed concave utility functions $\mathfrak{U}$ and a class of pairs of choices and associated multiplier maps.[9] This correspondence characterizes closed concave utility functions in terms of choice. In view of Theorem 1, it suffices to show that there is a one-to-one correspondence between the class of superdifferentials, $\mathfrak{D}^*$, and the class of pairs of choices and the associated multiplier maps. As mentioned, this correspondence is essentially a straightforward

---

[8]Since a cyclically monotone choice is the superdifferential of a utility function, the marginal utility of income is one, i.e. independent of prices. But as shown by Samuelson (1942), this implies that choice is homothetic. Hence cyclical monotonicity is yet another characterization of homotheticity.

[9]A multiplier map for a choice intuitively associate (Lagrangean) multipliers to each pair of budget and choice at that budget.

consequence of the first-order conditions for a concave utility maximization problem. We need, however, some terminology to express this simple idea in an appropriate way.

A correspondence $\tau \subseteq \mathbb{R}_+ \times (\mathcal{X} \times \mathcal{P})$ is a **multiplier map** if $(t,x,p) \in \tau$ implies $t = 0$ or $px = 1$.[10] Let $\mathfrak{T}$ be the set of multiplier maps. A multiplier map $\tau$ is an (integrating) **multiplier map for a choice c** if $c \subseteq D\tau$ and for all finite sequences $\{(x_i,p_i)\} \subseteq c$, and $t_i \in \tau(x_i,p_i)$, $\Sigma_i t_i p_i(x_{i+}-x_i) \geq 0$, and it is a **binary multiplier map for c** if the same holds for all pairs in c (i.e., if $x \in c(p)$ and $x' \in c(p')$, $t \in \tau(x,p)$, and $t' \in \tau(x',p')$, then $t(px'-1)+ t'(p'x-1) \geq 0$). Let $\mathfrak{T}^c$ be the set of multiplier maps for c. A multiplier map for a choice c associates multipliers to points in c such that 1) the complementary slackness condition holds and 2) the result of scaling budgets by associated multipliers is generalized cyclically monotone - and hence has an integrable extension. Different multiplier maps for a choice corresponds to different concave utility representations. The values of the multiplier maps are essentially Lagrangian multipliers, i.e. marginal utilities of income. Multiplier maps are preserved under sub-choices, i.e. if $c' \subseteq c$ and $\tau \in \mathfrak{T}^c$, then $\tau \in \mathfrak{T}^{c'}$.

Let $\mathfrak{C} = \{c \mid c \subseteq B \text{ and } \mathfrak{T}^c \neq \emptyset\}$ be the set of choices with associated multiplier maps. Such choices we will also call **generalized cyclically quasimonotone.** Let $\mathfrak{C}^* = \{c \in \mathfrak{C} \mid c' \text{ is a choice and } c \subset c' \text{ implies } c' \notin \mathfrak{C}\}$ be the set of **maximally cyclically generalized monotone** choices, and $^*\mathfrak{T}^c = \{\tau \in \mathfrak{T}^c \mid \text{for all } \tau' \supset \tau,\ \tau' \notin \mathfrak{T}^c\}$ the set of **maximal multiplier maps for c.** The main result below gives a one-to-one correspondence between elements $u \in \mathfrak{U}$ and pairs of elements $c \in \mathfrak{C}^*$ and $\tau \in {}^*\mathfrak{T}^c$. To show this we again need some more notation:

Given $d \in \mathfrak{D}$, define the **choice generated by d, $c^d$**, and the **multiplier map generated by d, $\tau^d \in \mathfrak{T}^{c^d}$**, by $x \in c^d(p)$ and $t \in \tau^d(x,p)$ if $tp \in d(x)$, $t \geq 0$, $x \in B(p)$ and $t = 0$ or $px = 1$. Conversely, given $c \in \mathfrak{C}$ and $\tau \in \mathfrak{T}^c$, define the **superdifferential generated by c and $\tau$, $d^{\tau c}$**, by $p \in d^{\tau c}(x)$ if there is $p' \in c^{-1}(x)$ and $t \in \tau(x,p')$ such that $p = tp'$. By the Kuhn-Tucker theorem, the superdifferential of a concave function has value zero or is equal to its generated choice up to a multiplier - the marginal utility of income. We use this to show

---

[10]With our price normalization, this is essentially the complementary slackness condition.

that $c^d$ is appropriately defined, i.e. that the choice generated by the superdifferential of a concave utility function is equal to the choice generated by the utility function:[11]

**LEMMA 1:** If u is concave, then $c^u = c^{\partial u}$.

Proof: $\mathbf{c^u \subseteq c^{\partial u}}$: Let $x \in c^u(p)$. By the Kuhn-Tucker theorem (e.g. Rockafellar (1970, Theorem 28.3)), there is $t \geq 0$ such that $tp \in \partial u(x)$, $x \in B(p)$, and $t(1-px) = 0$. Hence by the definition of $c^d$, $x \in c^{\partial u}(p)$.

$\mathbf{c^{\partial u} \subseteq c^u}$: Assume $x \in c^{\partial u}(p) \backslash c^u(p)$. Since $x \in c^{\partial u}(p)$, $x \in B(p)$. Hence, since $x \notin c^u(p)$, there is $x' \in B(p)$ such that $u(x') > u(x)$. But since $x \in c^{\partial u}(p)$, there is $t \geq 0$ such that $tp \in \partial u(x)$ and $t = 0$ or $px = 1$. But by the definition of $\partial u$, $u(x')-u(x) \leq tp(x'-x) \leq 0$, contradiction. □

The announced correspondence between superdifferentials and pairs of choices and associated multiplier maps is:

**PROPOSITION** 3: 1) If $d \in \mathfrak{D}$, then $c^d \in \mathfrak{C}$, $\tau^d \in \mathfrak{T}^c$, and $d \subseteq d^{\tau^d c^d}$.

2) If $c \in \mathfrak{C}$ and $\tau \in \mathfrak{T}^c$, then $d^{\tau c} \in \mathfrak{D}$, $c \subseteq c^d$ and $\tau \subseteq \tau^d$, where $d = d^{\tau c}$.

1*) If $d \in \mathfrak{D}^*$, then $c^d \in \mathfrak{C}^*$, $\tau^d \in {}^*\mathfrak{T}^c$, and $d = d^{\tau^d c^d}$.

2*) If $c \in \mathfrak{C}^*$ and $\tau \in {}^*\mathfrak{T}^c$, then $d^{\tau c} \in \mathfrak{D}^*$, $c = c^d$ and $\tau = \tau^d$, where $d = d^{\tau c}$.

Proof: **1)** Let $d \in \mathfrak{D}$. By the definitions, $c^d$ is a choice and $\tau^d$ a multiplier map. To show that $\tau^d$ is a multiplier map for $c^d$. Let $\{(x_i,p_i)\} \subseteq c^d$ be a finite sequence, and $t_i \in \tau^d(x_i,p_i)$. Then by the definition of $\tau^d$ and $c^d$, there is $t_i \geq 0$ such that $t_ip_i \in d(x_i)$. Hence $\Sigma_i t_i p_i(x_{i+}-x_i) \geq 0$, since d is generalized cyclically monotone. Hence $c^d \in \mathfrak{C}$ and $\tau^d \in \mathfrak{T}^c$.

---

[11]To ensure that the nonnegativity constraints are never binding, we impose a constraint on $\mathfrak{U}$: If $x \in \mathcal{X} \backslash \mathrm{int}\mathcal{X}$, then there is a sequence $\{x_n\} \subseteq \mathrm{int}\mathcal{X}$ such that $u(x_n) \geq u(x)$ and $x_n \to x$.

$d \subseteq d^{\tau^d c^d}$: Let $p \in d(x)$. There are two cases: i) Assume $p = 0$. Let $x \in B(p')$. Then by the definitions of $c^d$ and $\tau^d$, $x \in c^d(p')$ and $0 \in \tau^d(x,p')$. Hence by the definition of $d^{\tau c}$, $0 \in d^{\tau^d c^d}(x)$. ii) Assume $p \neq 0$. Then for some $t > 0$, $(p/t)x = 1$. Then by the definitions of $c^d$ and $\tau^d$, $x \in c^d(p/t)$ and $1/t \in \tau^d(x,p/t)$. Hence by the definition of $d^{\tau c}$, $p \in d^{\tau^d c^d}(x)$.

**2)** Let $c \in \mathfrak{C}$ and $\tau \in \mathfrak{T}^c$. Let $\{(p_i,x_i)\} \subseteq d^{\tau c}$ be a finite sequence. Then by the definition of $d^{\tau c}$, there is $p_i' \in c^{-1}(x_i)$ and $t_i \in \tau(x_i,p_i')$ such that $p_i = t_i p_i'$. But then, since $\tau$ is a multiplier map for c, $\Sigma_i p_i(x_{i+}-x_i) = \Sigma_i t_i p_i'(x_{i+}-x_i) \geq 0$. Hence $d^{\tau c} \in \mathfrak{D}$.

$c \subseteq c^{d^{\tau c}}$ and $\tau \subseteq \tau^{d^{\tau c}}$: Let $x \in c(p)$ and $t \in \tau(x,p)$. Then $x \in B(p)$, $t = 0$ or $px = 1$, and by the definition of $d^{\tau c}$, $tp \in d^{\tau c}(x)$. Then by the definitions of $c^d$ and $\tau^d$, $x \in c^{d^{\tau c}}(p)$ and $t \in \tau^{d^{\tau c}}(x,p)$.

**1*)** Let $d \in \mathfrak{D}^*$. Then by 1), $c^d \in \mathfrak{C}$ and $\tau^d \in \mathfrak{T}^c$. Assume $\tau^d \notin {}^*\mathfrak{T}^c$. Then there is $\tau \in \mathfrak{T}^c$ such that $\tau^d \subset \tau$. Let $(t,x,p) \in \tau \backslash \tau^d$. Since $\tau \in \mathfrak{T}^c$, $(x,p) \in c$. Let $d' = d \cup \{(tp,x)\}$. Then since $d \in \mathfrak{D}^*$, $d' \notin \mathfrak{D}$, so there is a finite sequence $\{(p_i,x_i)\} \subseteq d$ such that $p_1(x_2-x_1)+...+p_n(x-x_n)+tp(x_1-x) < 0$. By the definitions of $c^d$ and $\tau^d$, $(x_i,p_i/p_ix_i) \in c^d$ and $(p_ix_i,x_i,p_i/p_ix_i) \in \tau^d \subseteq \tau$. But this contradicts $\tau \in \mathfrak{T}^c$. Hence $\tau^d \in {}^*\mathfrak{T}^c$.

Assume $c^d \notin \mathfrak{C}^*$. Then there is $c \in \mathfrak{C}$ such that $c^d \subset c$. Let $(x,p) \in c \backslash c^d$. Then for all $t \geq 0$, $(tp,x) \notin d$. Let $d' = d \cup \{(tp,x)\}$. Then since $d \in \mathfrak{D}^*$, $d' \notin \mathfrak{D}$, so there is a finite sequence $\{(p_i,x_i)\} \subseteq d$ such that $p_1(x_2-x_1)+...+p_n(x-x_n)+tp(x_1-x) < 0$. By the definitions of $c^d$ and $\tau^d$, $(x_i,p_i/p_ix_i) \in c^d \subseteq c$ and $(p_ix_i,x_i,p_i/p_ix_i) \in \tau^d \in \mathfrak{T}^c$. But this contradicts $c \in \mathfrak{C}$. Hence $c^d \in \mathfrak{C}^*$.

$d = d^{\tau^d c^d}$: Since $d \in \mathfrak{D}^*$, this follows from $d \subseteq d^{\tau^d c^d}$.

**2*)** Let $c \in \mathfrak{C}^*$ and $\tau \in \mathfrak{T}^c$. Then by 2), $d^{\tau c} \in \mathfrak{D}$. Assume $d^{\tau c} \notin \mathfrak{D}^*$. Then there is $d \in \mathfrak{D}$ such that $d^{\tau c} \subseteq d$. Let $p \in d(x) \backslash d^{\tau c}(x)$. There are two cases: i) Assume $x \notin c(p/px)$. Let $c' = c \cup \{(x,p/px)\}$. Since $c \in \mathfrak{C}^*$, $c' \notin \mathfrak{C}$, so there is a finite sequence $\{(x_i,p_i)\} \subseteq c$ and $t_i \in \tau(x_i,p_i)$ such that $t_1p_1(x_2-x_1)+...+t_np_n(x-x_n)+p(x_1-x) < 0$. But by the definition of $d^{\tau c}$, $t_ip_i \in d^{\tau c}(x_i) \subset d(x_i)$, contradicting $d \in \mathfrak{D}$. ii) Assume $px \notin \tau(x,p/px)$ and let $\tau' = \tau \cup \{(px,x,p/px)\}$. Then since $\tau \in {}^*\mathfrak{T}^c$, $\tau' \notin \mathfrak{T}^c$, hence there is a finite sequence $\{(x_i,p_i)\} \subseteq c$ such that $t_1p_1(x_2-x_1)+...+t_np_n(x-x_1)+p(x_1-x) < 0$. By the definition of $d^{\tau c}$, $(t_ip_i,x_i) \in d^{\tau c} \subseteq d$. Since $(p,x) \in d$, this contradicts $d \in \mathfrak{D}$. Hence in any case, $d^{\tau c} \in \mathfrak{D}^*$.

$c = c^{d^{\tau c}}$: Since $c \in \mathfrak{C}^*$, this follows from $c \subseteq c^{d^{\tau c}}$. □

A choice is said to be **generalized cyclically quasimonotone** if it has a multiplier map. For finite choices, this amounts to Afriat's (1976) "system of multipliers". This notion is clearly **self-dual**, i.e. preserved by interchanging elements of $\mathcal{X}$ and $\mathcal{P}$. By Theorem 2, a generalized cyclically quasimonotone choice has a maximal extension (with respect to $\subseteq$) with the same properties. This is so because $c \in \mathfrak{C}$ and $\tau \in \mathfrak{T}^c$ implies $c^{d^{\tau c}} \in \mathfrak{C}^*$.

The characterization of closed concave utility functions is a direct consequence of Theorem 1, Proposition 3, and Lemma 1. Specifically, for $c \in \mathfrak{C}$ and $\tau \in \mathfrak{T}$, the **utility function generated by c and** $\tau$, $u^{\tau c} = u^d$, where $d = d^{\tau c}$. Conversely, for $u \in \mathfrak{U}$, let the multiplier map **generated by u,** $\tau^u = \tau^{\partial u}$. The main result is:

**THEOREM 2:** 1) If $u \in \mathfrak{U}$, then $c^u \in \mathfrak{C}^*$, $\tau^u \in {}^*\mathfrak{T}^c$ and $u = u^{\tau^u c^u}$.
2) If $c \in \mathfrak{C}$ and $\tau \in \mathfrak{T}^c$, then $u^{\tau c} \in \mathfrak{U}$, $c \subseteq c^u$ and $\tau \subseteq \tau^u$, where $u = u^{\tau c}$.
2*) If $c \in \mathfrak{C}^*$ and $\tau \in {}^*\mathfrak{T}^c$, then $c = c^u$ and $\tau = \tau^u$, where $u = u^{\tau c}$.

**Proof of Theorem 2:** 1) Let $u \in \mathfrak{U}$. Then by Theorem 1, $\partial u \in \mathfrak{D}^*$. Hence by Lemma 1 and Proposition 3, $c^u = c^{\partial u} \in \mathfrak{C}^*$ and $\tau^u = \tau^{\partial u} \in {}^*\mathfrak{T}^{c^u}$. By Theorem 1 also $u = u^{\partial u}$, and by Lemma 1 and Proposition 3, $\partial u = d^{\tau^{\partial u} c^{\partial u}} = d^{\tau^u c^u}$. Hence by definition, $u = u^{\tau^u c^u}$.
2) Let $c \in \mathfrak{C}$ and $\tau \in \mathfrak{T}^c$. Then by Proposition 3, $d^{\tau c} \in \mathfrak{D}$. Hence by the definition and Theorem 1, $u^{\tau c} \in \mathfrak{U}$. Also by Proposition 3, $c \subseteq c^d$ and $\tau \subseteq \tau^d$, where $d = d^{\tau c}$. By Theorem 1, $d \subseteq \partial u^d$. Hence, since $c^d$ preserves $\subseteq$, $c \subseteq c^{\partial u^d} = c^{u^d} = c^{u^{\tau c}}$, and similarly, $\tau \subseteq \tau^{u^{\tau c}}$.
2*) Similar to 2).

Theorem 2 says that maximally generalized cyclical quasimonotonicity characterizes choices generated by closed concave utility functions. Furthermore, for such choices, there is a one-to-one correspondence between concave utility functions and maximal multiplier maps.

Next, we introduce a variant of maximally generalized cyclical quasimonotonicity which avoids the maximality construction: A choice c is **cyclically quasimonotone** if there is a closed $\tau \in \mathfrak{T}^c$ such that for all finite sequences $\{(x_i,p_i)\} \subseteq c$ and $t_i \in \tau(x_i,p_i)$, if for some i, $x_{i+} \notin c(p_i)$ or $t_i \notin \tau(x_{i+},p_i)$, then $\Sigma_i t_i p_i(x_{i+}-x_i) > 0$. In analogy with Proposition 1, closedness and cyclical quasimonotonicity on a convex set characterize the choices generated by concave utility functions:

**PROPOSITION 4**: $c \in \mathfrak{C}^*$ if and only if it is closed and cyclically quasimonotone, where $\text{int}A \subseteq Dc^{-1} \subseteq A$ for some closed convex set A.

**Proof**: $\Rightarrow$: Let $c \in \mathfrak{C}^*$ and pick $\tau \in {}^*\mathfrak{T}^c$. Then by Proposition 3, $d = d^{\tau c} \in \mathfrak{D}^*$. Hence by Proposition 1, d is closed and cyclically monotone and there is a closed convex set A such that $\text{int}A \subseteq Dd \subseteq A$. But then by Proposition 3 again, $c = c^d$. Hence by definition, c is cyclically quasimonotone. Also, since d is closed, so is c and $\tau$. Finally $Dc^{-1} = Dd$.
$\Rightarrow$: Let c be closed and cyclically quasimonotone with $\text{int}A \subseteq Dc^{-1} \subseteq A$ for a closed convex set A. Then there is a closed $\tau \in \mathfrak{T}^c$ such that if for some i, $x_{i+} \notin c(p_i)$ or $t_i \notin \tau(x_{i+},p_i)$, then $\Sigma_i t_i p_i(x_{i+}-x_i) > 0$. Clearly $Dd^{\tau c} = Dc^{-1}$, and $\tau$ and c closed implies that $d = d^{\tau c}$ is closed. Hence it remains to show that d is cyclically monotone, since then by Proposition 1, d $\mathfrak{D}^*$, so by Proposition 3, $c = c^d \in \mathfrak{C}^*$ and $\tau = \tau^d \in {}^*\mathfrak{T}^c$. Let $\{(p_i,x_i)\} \subseteq d$ be a finite sequence with $p_i \notin d(x_{i+})$ for some i. Then by the definition of $d^{\tau c}$, for each i there is $p_i' \in c^{-1}(x^i)$ and $t_i \in \tau(x_i,p_i')$ such that $p_i = t_i p_i'$, and for some i, $x_{i+} \notin c(p_i')$ or $t_i \notin \tau(x_i,p_i')$. Hence $\Sigma_i p_i(x_{i+}-x_i) = \Sigma_i t_i p_i'(x_{i+}-x_i) > 0$ by the above. Hence d is cyclically monotone. □

For finite choices, generalized cyclical quasimonotonicity is implied by the generally much weaker transitive weak axiom:

**PROPOSITION 5**: A finite choice which satisfies the transitive weak axiom is generalized cyclically quasimonotone.

**Proof:** Let X be finite $(x_i,p_i) \in c$, and $M^c(X) = \{x \in X | \text{for all } x' \in P^c(x)\ x' \notin X\}$. We make an inductive definition: Let $X_0 = X$, $\hat{X}_0 = \emptyset$, $x_0 \in M^c(X_0)$, and $t_0 = 1$. Inductively, let $X_{i+1} = X_i - R^c(x_i)$, $\hat{X}_{i+1} = \hat{X}_i \cup R^c(x_i)$, $x_{i+1} \in M^c(X_{i+1})$, and $t_{i+1} = \max\{\max_{\{x_j\} \subseteq \hat{X}_i, \{x_k\} \subseteq R^c(x_i+1)} \Sigma_j t_j(p_j x_{j+1}-1)/\Sigma_k(p_k x_{k+1}-1), 1\}$. We show by induction on i that $\{x_j\} \subseteq \hat{X}_{i+1}$, $\Sigma_j t_j(p_j x_{j+1}-1) \geq 0$:

**Base case:** $\hat{X}_1 = R^c(x_0)$: Let $x_i, x_j \in R^c(x_0)$. Then since $x_0$ is maximal, $x_0 \in R^c(x_i)$. Hence $p_i x_j \geq 1$, since otherwise $x_i \in {}^+P^c(x_j)$, contradicting the weak transitive axiom.

**Induction case:** Let $x_k \in R^c(x_{i+1})$. If $x_{k+1} \in R^c(x_{i+1})$, then $p_k x_{k+1} \geq 1$ as in the base case. If $x_{k+1} \in \hat{X}_{i+1}$, then $p_k x_{k+1} > 1$, since otherwise $x_k \in R^c(x_{k+1})$, so $x_k \in \hat{X}_{i+1}$, contradiction. Hence the denominator in the definition of $t_{i+1}$ is positive. But then for all $x_k \in \hat{X}_{i+1}$, $t_{i+1}\Sigma_k(p_k x_{k+1}-1) \geq \Sigma_j t_j(p_j x_{j+1}-1)$, and the result follows. □

This proof corresponds to Algorithm 3 in Varian (1982), but verifies Afriat's (1976) "system of multipliers" instead of his "system of multipliers and levels".

Together with Theorem 2, this proposition shows that the existence of a concave utility function has no testable implications on finite data sets when attention is restricted to price-generated budgets. This is the classical result by Afriat (1967).

## 3. LEAST CONCAVITY

The class of utility function for a given choice are generally only ordinally determined (i.e. up to a positive monotone transformation). Debreu (1976) has shown for concave utility functions, one may without loss of generality restrict attention to the subclass of least concave utility functions and the latter is cardinally determined (i.e. invariant up to a positively affine transformation). Here we give a simple proof of a variant of Debreu's result ensuring the existence of least concave utility functions. In contrast to Debreu who defines least concavity with respect to preferences, we define it with respect to a choice, which seems to simplify matters.

A function u is **least concave** for a choice c if $u \in \mathcal{U}^c$ and for all $u' \in \mathcal{U}^c$, there is an increasing concave function $f: u(\mathcal{X}) \to \mathbb{R}$ such that $u' = f \circ u$. Clearly, any two least concave functions for a choice are increasing affine transformations of each other. We will prove the existence of least concave functions.

On $\mathcal{U}$, define the (weak) **less concave** relation, $\precsim$, by $u \precsim u'$ if there is an increasing concave function $f: u(\mathcal{X}) \to \mathbb{R}$ such that $u' = f \circ u$. Clearly, $\precsim$ is a partial order.[12] Let $\mathcal{U}_0 = \{u \in \mathcal{U} \mid (x^0,p^0) \in c^u \text{ and } \sup \tau^u(x^0,p^0) = t^0\}$, i.e. the concave functions (with value 0 at $x^0$) where the associated multipliers attain the same supremum (in the direction $p^0$).[13] In the following restrict $\precsim$ to $\mathcal{U}_0$. Here the less concave relation, $\precsim$, reverses the standard order on $\mathbb{R}$, and an infimum with respect to $\precsim$ is a pointwise supremum of values on $\mathbb{R}$:

**LEMMA 3:** 1) On $\mathcal{U}_0$, if $u \precsim u'$, then for all x, $u(x) \geq u'(x)$.

2) On $\mathcal{U}_0$, if $\underline{u} = \inf_{\precsim}\{u_i\}_{i \in I}$, then for all x, $\underline{u}(x) = \sup\{u_i(x)\}_{i \in I}$.

**Proof: 1):** Let $u \precsim u'$. Then there is an increasing concave function f such that $u' = f \circ u$. Since f is concave, for all x, $u'(x)-u'(x^0) \leq \partial f(u(x^0))(u(x)-u(x^0))$. By the supremum condition on the associated multipliers in $\mathcal{U}_0$, $\partial f(u(x^0)) \leq 1$. Hence $u'(x) \leq u(x)$.

**2):** Let $\underline{u} = \inf_{\precsim}\{u_i\}_I$. By the definition of $\underline{u}$, for all $i \in I$, $\underline{u} \precsim u_i$. Hence by 1), for all x, $\underline{u}(x) \geq u_i(x)$. Next, let $\alpha < \underline{u}(x)$. It suffices to show that $\alpha < u_i(x)$ for some $i \in I$. Define the strictly concave function $f: u^*(\mathcal{X}) \to \mathbb{R}$ by $f(t) = at/(a+t)$. Then $f(0) = 0$ and $\partial f(0) = \{1\}$. Let $a = \alpha'\underline{u}(x)/(\underline{u}(x)-\alpha')$. Then $f(\underline{u}(x)) = \alpha' = (\alpha+\underline{u}(x))/2$. Let $u' = f \circ \underline{u}$. Then $u' \in \mathcal{U}_0$, $\underline{u} \prec u'$ and $\alpha < u'(x)$. Hence by the definition of $\underline{u}$, there is $i \in I$ such that $u_i \precsim u'$. But then by 1), $\alpha < u'(x) \leq u_i(x)$. □

From Lemma 3, it follows that $\precsim$ is antisymmetric (on $\mathcal{U}_0$). We are ready to prove a variant of the main result in Debreu (1976):

---

12The associated strict relation, $\prec$, is defined in the standard way: $u \prec u'$ if $u \precsim u'$ and not $u' \precsim u$.

13Since superdifferentials are closed-valued, the supremum is obtained. Though supressed in the notation, the class depends parametrically on both $x^0$, $p^0$, and $t^0$. The point of fixing these values is to normalize away the cardinal indeterminacy of the least concave utility functions.

**PROPOSITION 6:** 1) A concave function generating a choice c has a less concave function which is least concave for c.

**Proof:** 1: Let $u \in \mathcal{U}_0^c$, the set of concave functions generating c, with value 0 in $x^0$. Let $\mathcal{U}_0^u = \{u' \in \mathcal{U}_0^c \mid u' \preceq u\}$, and let $\{u_i\}_I$ be an ordered subset of $\mathcal{U}_0^u$. Clearly, $\{u_i\}_I$ is bounded below (in $\mathcal{U}_0$) by the appropriate affine function. Let $\underline{u} = \inf_{\preceq}\{u_i\}_I$. By Zorn's lemma, it remains to show that $\underline{u} \in \mathcal{U}_0^c$. By the definition of $\underline{u}$, $\underline{u} \preceq u_i \preceq u$. Hence $\underline{u}$ is concave. It remains to show that $c \subseteq c^{\underline{u}}$. Assume $x \in c(p)-c^{\underline{u}}(p)$. Then for all i, $x \in c^{u_i}(p)$ and there is $x' \in B(p)$ such that $\underline{u}(x') > \underline{u}(x)$. Hence by Lemma 3, for some i, $u_i(x') > \underline{u}(x)$. But then by Lemma 3 again, $u_i(x') > u_i(x)$, contradicting $x \in c^{u_i}(p)$. □

To fill out the picture of Theorem 2, corresponding to less concave among concave utility functions define the (weak) **less decreasing** relation, $\preceq$, on $\mathfrak{D}$ (i.e. subsets of super-differentials by $d \preceq \hat{d}$ if for all x, x', $p \in d(x)$, $p' \in d(x')$, $\hat{p} \in \hat{d}(x)$ and $\hat{p}' \in \hat{d}(x)$, $((\hat{p}'-\hat{p})-(p'/p))$ $(u^0(x')-u^0(x)) \leq 0$ for some $u^0 \in \mathcal{U}$. The corresponding **relative decreasing** relation between multiplier maps for a choice, $\preceq$, on $\mathfrak{T}^c$ is defined by $\tau \preceq \hat{\tau}$ if for all $(x,p),(x',p') \in c$, $t \in \tau(x,p)$, $t' \in \tau(x',p')$, $\hat{t} \in \hat{\tau}(x,p)$, and $\hat{t}' \in \hat{\tau}(x',p')$, $(\hat{t}'/t'-\hat{t}/t)(u^0(x')-u^0(x)) \leq 0$ for $u^0(x) \neq 0$, and $\tau(x,p) \subseteq \hat{\tau}(x,p)$ otherwise. To verify this is straightforward, so we omit it.

May 1988
Revised, February 1990

# CHAPTER 7: COMPLETELY SEPARABLE UTILITY[0]

## 1. INTRODUCTION

As mentioned in Chapter 3, so far, there is no full characterization of the choice (demand) consequences of a standard utility function. Lensberg (1987), however, gave such a characterization in the special case when the utility function is additionally completely (additive) separable and strictly quasiconcave. In this Chapter we weaken Lensberg's strict quasiconcavity assumption to quasiconcavity. The argument is also simplified, and its structure made clearer. Furthermore, we show that similar characterizations easily follow when the utility function additionally has concave components and is of the expected utility form, respectively.

The first results in this direction, by Sono (1945) and Leontief (1947a,b), characterized different kinds of separable utility in terms of an independence property of the marginal rate of substitution. The present argument also proceeds via this independence property, and can thus be seen as extending their work. With hindsight, the step from marginal rates of substitution to choice seems fairly easy, especially if one assumes differentiability. Lau (1969) and Pollak (1970) introduced a corresponding choice concept, called decentralization by Blackorby, Primont, and Russell (1978, Chapter 5.3). Blackorby, Primont and Russell also slightly generalized Pollak's characterization of separable utility by means of what they called "strong decentralizability." Here the slightly weaker notion of (weak) decentralization is used. In addition to avoiding differentiability assumptions, the present work (as the one by Lensberg) puts the result into an integrability framework, i.e. it does not presuppose any utility notions when showing that separable choice implies separable utility. The cost of this is that the result is restricted to one-dimensional factor spaces.

[0]I am grateful to Sjur Flåm, Bernt Stigum, William Thomson, Lars Thorlund-Petersen, and especially Terje Lensberg for valuable comments to this chapter.

Chapter 5 above established a similar characterization of separable preferences in terms of choice which does not presuppose complete separability. It uses the stronger I-axiom instead of decentralizability as the separability notion of choice. The I-axiom makes the restriction to one-dimensional factor spaces unnecessary. On the other hand, in view of the lack of an extension theorem in Chapter 3, this does not quite imply the existence of (separable) utility function.

The main differences compared to Lensberg (1987) are: First, his separability concept, called multilateral stability, and going back to Harsanyi (1959), is replaced by Blackorby, Primont, and Russell's (1978) (weak) decentralization. The reason for this is that multilateral stability does not easily generalize to correspondences. Secondly, the argument is formulated in a demand theory framework, admitting only price-generated budgets in a fixed finite Euclidean space, whereas Lensberg admits more general convex budgets and an infinite sequence of such spaces. This allows the continuity concept to be simplified by using the standard topology of Euclidean space instead of the Hausdorff one on the set of closed subsets. Finally, a new argument is needed to show that inverse choice is single-valued almost everywhere, as Lensberg's proof at this point relies heavily on the use of a choice function. With these modifications, the argument roughly follows the one in Lensberg (1987).

The rest of the paper is organized as follows: After some preliminary results in Section 2, the main result is given in Section 3. Sections 4 and 5 show similar characterizations when the utility function additionally has concave components, and is of the expected utility kind, respectively. Finally, some interpretations is shortly discussed in Section 6.

## 2. CONCEPTS AND NOTATION

We stick to the previous notation, but need some additions. Let $\mathbf{e} \in \mathbb{R}_+^{I0}$ be the vector with unit components. For simplicity, we usually write $x_i$ for $x_{\{i\}}$. A correspondence c is **unbounded** (outside Dc) if for all $p \notin Dc$, $p^n \to p$, and $x^n \in c(p^n)$, $\{x^n\}$ is unbounded.

A function u is **completely separable** (with respect to the finest partition of $I^0$) if there is $\{u_i\}_{I^0}$ such that $u(x) = \Sigma_i u_i(x_i)$. Our separability notion of choice is the fairly weak decentralization concept. Recall that for $x \in B(p)$ and $p_I x_I \neq 0$, the **section** of the budget p at $x_{-I}$, $\mathbf{p|x_{-I}} = p_I/p_I x_I$ (See Figure 1). Now, a choice c is **I-decentralizable** if for all p, p', $x \in c(p)\cap \mathrm{int}\mathcal{X}$, $x' \in c(p')\cap \mathrm{int}\mathcal{X}$, $p|x_{-I} = p'|x'_{-I}$ implies $x_I + x'_{-I} \in c(p')$ (see Figure 2). This expresses that if the sections at chosen (interior) points are equal, then the sectional choices are also equal, or in other words, that the choice from group I is only dependent on the group budget.[1] Also, a choice c is **completely decentralizable** (with respect to the finest partition of $I^0$) if it is I-decentralizable for all I.

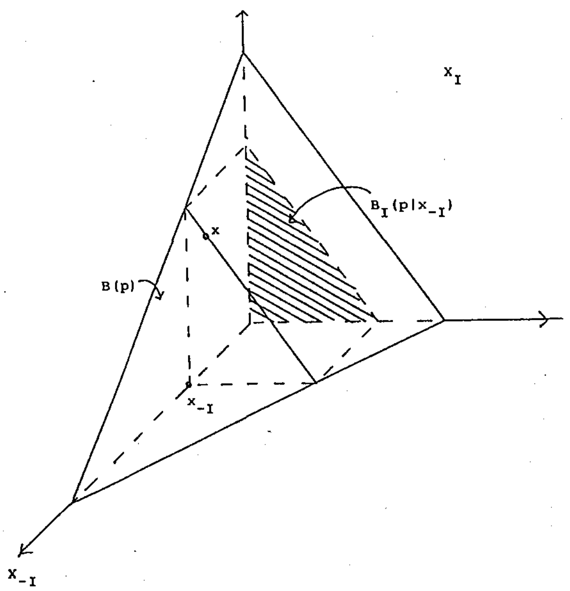



Figure 1

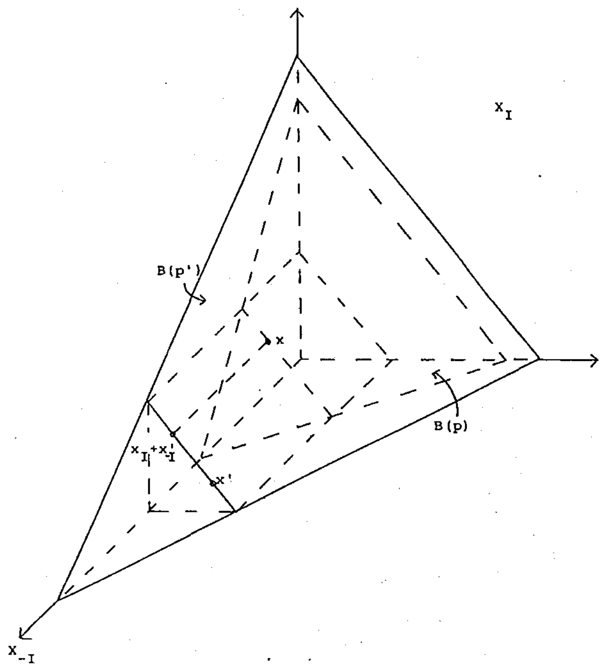



Figure 2

## 3. THE CHARACTERIZATION OF COMPLETELY SEPARABLE UTILITY

The characterization of completely separable utility for spaces of dimension three or more is given by the following two theorems:

**THEOREM 1:** If u is a completely separable, continuous, quasiconcave, and strictly increasing utility function, then its generated choice, $c^u$, is completely decentralizable, closed, has large interior domain, large range, and satisfies the budget identity and the basic axiom.

---

[1]Thus I-sectional budgets are well-defined objects (with equality) in the theory.

Proof: We only verify decentralizability, by showing that if for all x and I, $u(x) = u_I(x_I) + u_{-I}(x_{-I})$, then $c^u$ is I-decentralizable. Let $x \in c^u(p)$, $p|x_{-I} = p'|x'_{-I}$ and $x' \in c^u(p')$. To show $x'_I + x_{-I} \in c^u(p)$. Since $p'|x'_I = p|x_{-I}$, $x_I + x'_{-I} \in B(p')$. Hence since $x' \in c^u(p')$, $u(x'_I) \geq u(x_I + x'_{-I})$, so by the assumption on u, $u(x'_I + x_{-I}) \geq u(x)$. But then since $x \in c^u(p)$ and $x'_I + x_{-I} \in B(p)$, $x' + x_{-I} \in c^u(p)$. □

**THEOREM 2:** If $|I^0| \geq 3$ and the choice c is completely decentralizable, closed, has large interior domain, large range, and satisfies the budget identity and the basic axiom, then there is a completely separable, continuous, quasiconcave, and strictly increasing utility function, $u^c$, which generates c.

**Remark 1:** Debreu and Koopmans (1982, Theorem 9) have shown that the continuity assumption can be dropped in Theorem 1.

**Remark 2:** A stronger separability concept, namely the I-axiom also follows in Theorem 1, with essentially the same proof.[2] As noted there, I-decentralizability is too weak to characterize I-separability of preferences, this works nicely for the I-axiom, in contrast to the I-axiom.

Does one get a characterization without the restriction to one-dimensional factor spaces by replacing I-decentralizability by the I-axiom? No, since that would presuppose a solution to the general integrability problem on the subspaces. On the other hand if one, as is often done, presupposes the existence of a utility function generating the choice such a result follows.

**Remark 3:** No strong axiom is needed in Theorem 2, though one gets transitivity out. The reason for this is that complete decentralizability and one-dimensional factor spaces allows one to reduce the problem to a set of two-dimensional ones. For such problems, the basic axiom implies the strong one, as the result by Rose (1958) and Afriat (1965) easily generalizes to correspondences. That nothing is gained by adding transitivity is also noted by Epstein (1987).

---

[2]Note that using the I-axiom would slightly simplify the proof of Lemma 1.

**Remark 4:** Theorem 2 is stated in terms of complete separability. But looking more closely at the proof, especially Lemma 1 and the definition of the generated utility function, one sees that what is actually needed is separability with respect to a **binary chain** in $I^0$, i.e. a collection of two-element sets $\{I_n\}$ which exhaust $I^0$ and satisfies $I_j \cap I_{j+1}$ nonempty. This accords to Gorman's (1968) result on the identification of separable sections.

**Remark 5:** It is fairly straightforward to extend the characterization to larger classes of "budgets", as long as these are convex, **downward monotone** (i.e. if $x \in B(q)$ and $x' \leq x$, then $x' \in B(q)$), and have a nonempty interior. Lensberg (1987) works with the class of all such budgets.

**Remark 6:** As by Lensberg, using this larger class of budgets and Hausdorff continuity make the large range assumption superfluous in Theorem 2. In contrast to by Lensberg, however, it seems that one cannot weaken the separability notion to one involving only two-element groups. Also one cannot drop the basic axiom. The main reason for this is that the proof of Lensberg's Lemma 1 relies heavily on the lower hemicontinuity of the choice function. Furthermore, the conclusion of this lemma does not in general imply the basic axiom if choice is not single-valued.

**Remark 7:** The dual of the above result characterizes completely separable indirect utility.

The proof of Theorem 2 proceeds via the stronger I-independence notion of separable choice. This is closely related to Sono (1945) and Leontief's (1947a,b) independence property of the marginal rate of substitution. Recall from Chapter 5 that a choice c is **I-independent** if for all p, $x \in c(p) \cap \mathrm{int}\mathcal{X}$, and $x' \in \mathrm{int}\mathcal{X}$, there is p' such that $x_I + x'_{-I} \in c(p')$ and $p'|x'_{-I} = p|x_{-I}$.[3] This says that an interior point with the same I-projection as some chosen point are chosen at some budget with the same I-section. Also a choice c is **completely independent** if it is I-independent for all I. Under the conditions of Theorem 2, complete decentralizability implies complete independence:

[3] That independence implies decentralizability follows from the axiom.

**LEMMA** 1: If the choice c is completely decentralizable, closed, has a large interior domain, and satisfies the basic axiom, then it is completely independent.

**Proof:** Let $x \in c(p) \cap \mathrm{int}\mathcal{X}$, and $x'_{-I} \in \mathrm{int}\mathcal{X}_{-I}$. We verify that c is I-independent (i.e. that there is p' such that $x_I + x'_{-I} \in c(p)$ and $p'|x'_{-I} = p|x_{-I}$) by induction on the size of -I.

**Base case:** $-I = \{j\}$. (See Figure 3). Let $p^t = tp_I + ((1-t)/x'_j)e_j$. Then $p^t|x'_j = p|x_j$ for all $t \in <0,1>$. It is sufficient to find $t \in <0,1>$ and $x''_I + x'_j \in c(p^t)$, since then by decentralizability, $x_I + x'_j \in c(p^t)$. Let $\tilde{c} \subseteq \mathcal{X} \times <0,1>$ be defined by $\tilde{c}(t) = c(p^t)_j$. As c has large domain and satisfies the basic axiom $\tilde{c}$ is convex-valued. Furthermore by the definition of $p^t$ and the lemma of the appendix, $\tilde{c}$ is upper hemicontinuous. Hence by the theorem of the appendix, its range $\tilde{c}(<0,1>)$ is connected. Let $x^t \in c(p^t)$. It is sufficient to show, first that there is $t \in <0,1>$ such that $x^t_j \geq x'_j$, and secondly that there is $t \in <0,1>$ such that $x^t_j \leq x'_j$.

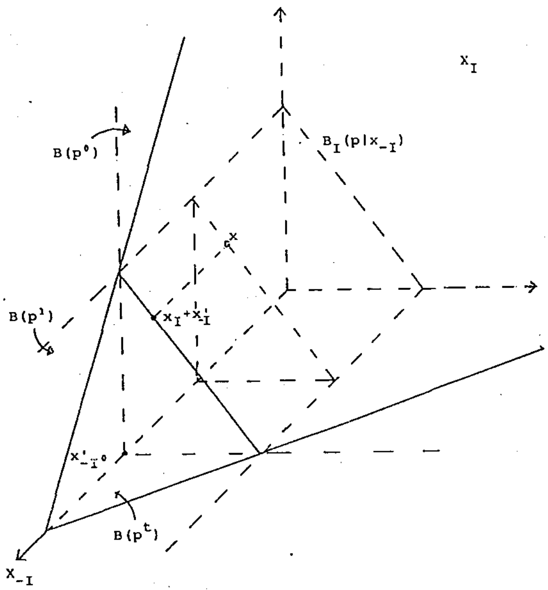


Figure 3

First, let $t \to 1$. Since c has interior domain, $p^1 \notin Dc$, hence since c is unbounded, $\{x^t\}$ is unbounded. Hence since $p^1_I \in \mathrm{int}\mathcal{P}_I$, $\{x^t_I\}$ is unbounded, which gives $x^t_j \geq x'_j$ for t sufficiently large.
Secondly, let $t \to 0$. Since c has interior domain, $p^0 \notin Dc$, hence since c is unbounded, $\{x^t\}$ is unbounded, and since $p^0_I \in \mathrm{int}\mathcal{P}_I$, $\{x^t_I\}$ is unbounded. Then for sufficiently small t, $p^t_I(x^t_I - x'_I) > 0$, hence $p^t_j(x^t_j - x'_j) \leq 0$, so $x^t_j - x'_j \leq 0$.

**Induction case**: Let $(\{k\},L)$ be a proper partition of -I. By the induction assumption, c is $I \cup \{k\}$-independent, so there is a p' such that $x_I + x_k + x'_L \in c(p)$ and $p'|x'_L = p|x_L$. Next, by the base case, c is also $I \cup L$-independent, hence there is p'' such that $x_I + x'_{-I} = x_I + x'_k + x'_L \in c(p'')$, where $p''|x'_k = p'|x_k$. Finally, as the order of sectioning does not matter, $p''|x'_{-I} = p''|x'_k|x'_L = p'|x_k|x'_L = p'|x'_L|x_k = p|x_L|x_k = p|x_{-I}$. □

Define the **single-valued domain** of c, **Sc** = $\{p|c(p)$ is a singleton$\}$. Next, on the single-valued inverse domain, $Sc^{-1}$, let $\mathbf{p^c}(\cdot)$ be the **inverse choice function** determined by c, $\mathbf{p^c_I}(x) = p^c(x)|x_{-I}$ the corresponding I-section, and (for $i \neq j$) $\mathbf{M^c_{ij}}(x) = p^c_{ij}(x)_i/p^c_{ij}(x)_j$ the **marginal rate of substitution** between i and j at x. Clearly, if c is I-independent, then $p^c_I(x)$ is independent of $x_{-I}$. Hence if c is $\{i,j\}$-independent, then $M^c_{ij}(x)$ is independent of $x_{-ij}$, which is Sono and Leontief's independence property. Clearly also $M^c_{ij}(x) = M^c_{ji}(x)^{-1}$ and $M^c_{ij}(x) = M^c_{ik}(x)M^c_{kj}(x)$. If inverse choice were single-valued, the construction of Sono and Leontief gives a completely separable utility function by integration of the marginal rates of substitution. Though inverse choice is not single-valued everywhere, it is single-valued almost everywhere, and this is sufficient for the construction to go through:[4]

**LEMMA** 2: If the choice c is closed, has large domains, and satisfies the basic axiom and the budget identity, then $\mathrm{int}\mathcal{X} \subseteq S(c)$ almost everywhere (with respect to the Lebesgue measure).

**COROLLARY:** There is $x^0 \in Sc^{-1}$ such that for each i, $x_i+x^0_{-i} \in Sc^{-1}$ for almost all $x_i$.

This follows from the lemma by the definition of the product measure. This corollary is what is used of Lemma 2 in the following.

**Remark** 8: By duality, choice is single-valued almost everywhere under the same assumptions.
**Remark** 9: Lensberg (1987, Lemma 7) applies only a two-dimensional analogy of Lemma 2. Similarly, complete decentralizability (separability) makes a two-dimensional version of Lemma 2, with a slightly simpler proof, sufficient here. Lemma 2 is of independent interest, however. E.g., it seems just what is needed to extend Hurwicz (1971) integrability results (without separability) to the general case of demand correspondences.

---

[4]The argument is a translation into choice terms of the main argument in Mas-Colell (1976). The proof is given in Appendix 2.

Next, independence implies that inverse single-valuedness is inherited from the adjacent corners of a rectangle (see Figure 4): [5]

**LEMMA 3**: Let $I\setminus J \neq \emptyset$, $J\setminus I \neq \emptyset$, $I\cap J \neq \emptyset$, $I\cup J = I^0$, and let the choice c be I- and J-independent. Then for all x and x', $x_I+x'_{-I}$, $x_J+x'_{-J} \in Sc^{-1}$ implies that $x \in Sc^{-1}$.

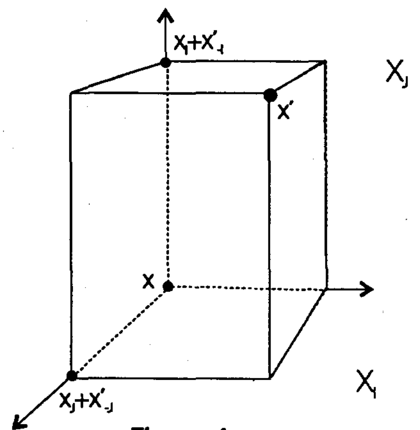


Figure 4

**Proof**: Clearly $x \in \text{int}X$. Let $x \in c(p)\cap c(\bar{p})$. By I-independence there are p' and $\bar{p}'$ such that $x_I+x'_{-I} \in c(p')\cap c(\bar{p}')$, $p'|x'_{-I} = p|x_{-I}$, and $\bar{p}'|x'_{-I} = \bar{p}|x_{-I}$. Since $x_I+x'_{-I} \in Sc^{-1}$, $p' = \bar{p}'$, so $p|x_{-I} = \bar{p}|x_{-I}$. Similarly, J-independence gives $p|x_{-J} = \bar{p}|x_{-J}$. Hence $p = \bar{p}$, as these are uniquely determined by their sectional components, since $I\cap J \neq \emptyset$. □

Together, Lemmas 2 and 3 show that the only affine subspaces without single-valuedness almost everywhere are normal to some axes.

For the rest of this section, let c be a choice satisfying the assumptions of Theorem 2.

On $Sc^{-1}$, define the **marginal rate of substitution** of $x_i$, relative to $x^0$,[6] $m^c_i$, by $m^c_i(x_i) = M^c_{i1}(x_i+x^0_{-i})$ for $i \neq 1$, and $m^c_1(x_1) = M^c_{12}(x_1+x_{-1})M^c_{21}(x^0)$, and the **utility function generated by c, $u^c$**, by $u^c(x) = \sum_i u_i(x_i)$, where $u^c_i(x_i) = \int_{x^0_i}^{x_i} m^c_i(t)dt$ for $x_i > 0$ and $u^c_i(0) = \lim u^c_i(t)$ when $t \to 0$.

As the choice c is closed, $m^c_i$ is continuous and hence measurable on $\{x_i | x_i+x^0_{-i} \in Sc^{-1}\}$, i.e. almost everywhere. Also since c has an interior domain, $m^c_i(x_i)$ is bounded on $[x^0_i,x_i]$ for $x_i > 0$. Thus $u^c_i$ is well-defined and strictly increasing, so by definition $u^c$ is strictly increasing and continuous on $X$. As $u^c$ is trivially completely separable, it only remains to show that $u^c$ is

[5]This lemma presupposes at least three goods - as decentralizability (as most other separability concepts) is powerless with only two goods, being a consequence of strict monotonicity.

[6]The bundle $x^0$ is defined in the corollary to Lemma 2.

quasiconcave and generates c. For this, first note that if well-defined, the inverse choice function is proportional to the derivative (denoted **d**) of the generated utility function:[7]

**LEMMA 4:** If $x \in Sc^{-1}$, then $du^c(x)$ is proportional to $p^c(x)$.

**Proof:** Let $x \in Sc^{-1}$, $m_i = m_i^c$, and $M_{ij} = M_{ij}^c$. By Lemma 3, since $x^0 \in Sc^{-1}$, for arbitrary i and j, $x_i+x^0_{-i}, x_{ij}+x^0_{-ij} \in Sc^{-1}$. Furthermore, on $Sc^{-1}$, $du^c(x) = (m_i(x_i))_{I^0}$. Hence by the definition of $M_{ij}$, it is sufficient to show that for $i \neq j$, $m_i(x_i)/m_j(x_j) = M_{ij}(x)$. There are three cases where the independence property of $M_{ij}$ is used repeatedly:

1) Let $i,j \neq 1$. Then $m_i(x_i)/m_j(x_j) = M_{i1}(x_i+x^0_{-i})M_{1j}(x_j+x^0_{-j}) = M_{i1}(x_{ij}+x^0_{-ij})M_{1j}(x_{ij}+x^0_{-ij}) = M_{ij}(x_{ij}+x^0_{-ij}) = M_{ij}(x)$.

2) Let $j \neq 2$. Then $m_1(x_1)/m_j(x_j) = M_{12}(x_1+x^0_{-1})M_{21}(x^0)\ M_{1j}(x_j+x^0_{-j}) = M_{12}(x_1+x^0_{-1}) \cdot M_{21}(x_j+x^0_{-j})M_{1j}(x_j+x^0_{-j}) = M_{12}(x_1+x^0_{-1})M_{2j}(x_j+x^0_{-j}) = M_{12}(x_{1j}+x^0_{-1j})M_{2j}(x_{1j}+x^0_{-1j}) = M_{1j}(x_{1j}+x^0_{-1j}) = M_{1j}(x)$.

3) $m_1(x_1)/m_2(x_2) = (m_1(x_1)/m_j(x_j))(m_j(x_j)/m_2(x_2)) = M_{1j}(x)M_{j2}(x) = M_{12}(x)$ by the previous cases. □

Finally, it is verified that $u^c$ extends the preferences generated by c, from which the two remaining properties follows. This will finish the proof of Theorem 2:

**LEMMA 5:** For all x,x', if $x' \in P^c(x)$, then $u^c(x') > u^c(x)$, and if $x' \in R^c(x)$, then $u^c(x') \geq u^c(x)$.

**Proof:** This is verified for the strict relation (the weak relation case is simpler). Let $x' \in P^c(x)$. Then there is p such that $x' \in c(p)$ and $x \in B(p) \backslash c(p)$. There are two cases:

1) Assume that $\langle x,x' \rangle \subseteq Sc^{-1}$ almost everywhere. Since c(p) is closed, and $x \notin c(p)$, there is $x'' \in \langle x,x' \rangle$ such that $c(p) \cap \langle x,x'' \rangle = \emptyset$. Hence for $x^1 \in \langle x,x'' \rangle$, $x' \in P^c(x^1)$. Let $x^1 \in c(p^1)$. Then by the basic axiom, $x' \notin B(p^1)$, i.e. $p^1(x'-x^1) > 0$. Hence since $x^1 \in \langle x,x'' \rangle$, $p^1(x''-x) > 0$. Next, let $x^2 \in \langle x',x'' \rangle$ and $x^2 \in c(p^2)$. Then clearly $x' \in R^c(x^2)$. If $x' \in b(p^2)$, then since c satisfies the

[7]Or as usually stated, that the marginal rate of substitution is equal to the price ratio.

budget identity, $x^2 \in P^c(x')$, contradicting the basic axiom. Hence $x' \notin b(p^2)$, so $p^2(x'-x^2) \geq 0$, from which also $p^2(x'-x'') \geq 0$. Then by Lemma 4, $du(x^1)(x''-x) > 0$ for all $x^1 \in \langle x,x''\rangle \cap Sc^{-1}$, and $du(x^2)(x'-x'') \geq 0$ for all $x^2 \in \langle x',x''\rangle \cap Sc^{-1}$. Let $x^t = tx'+(1-t)x$. By the case assumption, $x^t \in Sc^{-1}$ for almost all $t \in [0,1]$, so $du(x^t)(x'-x)$ is integrable on $Sc^{-1}$. Hence: $0 < \int_0^1 du(x^t)(x'-x)dt = \Sigma_i \int_0^1 du(x^t)(x_i'-x_i)dt = \Sigma_i(u_i(x_i^1)-u_i(x_i^0)) = u(x')-u(x)$.

2) Otherwise, by Lemma 3 and the corollary to Lemma 2, x and x' have some common coordinates, -I. By Lemma 2 and the definition of the product measure there are $\tilde{x}$ and $\tilde{x}'$ with $\tilde{x}_I = x_I$, $\tilde{x}_I' = x_I'$, and $\tilde{x}_{-I} = \tilde{x}'_{-I}$ (i.e. parallel translations) such that $\langle \tilde{x},\tilde{x}'\rangle \subseteq Sc^{-1}$ almost everywhere. By I-independence there is $\tilde{p} \in \mathrm{int}\mathcal{P}$ such that $\tilde{x}' \in c(\tilde{p})$ and $\tilde{p}|\tilde{x}_{-I} = p|x_{-I}$. Thus $\tilde{x} \in B(\tilde{p})$. Assume $\tilde{x} \in c(\tilde{p})$. Then, by I-independence again, there are $\hat{p} \in \mathrm{int}\mathcal{P}$ such that $x \in c(\hat{p})$ and $\hat{p}|x_{-I} = \tilde{p}|\tilde{x}_{-I}$. Hence $x' \in B(\hat{p})$, so $x \in R^c(x')$, contradicting the basic axiom. Hence $x \notin c(\tilde{p})$, so $\tilde{x}' \in P^c(\tilde{x})$. Since $\langle \tilde{x},\tilde{x}'\rangle \subseteq Sc^{-1}$ almost everywhere, by case 1) $u(\tilde{x}') > u(\tilde{x})$. Hence, by definition, $u(x') > u(x)$. □

**LEMMA 6:** The generated utility function, $u^c$, is quasiconcave and generates c.

**Proof:** To verify quasiconcavity, let $u(x'),u(x'') \geq \bar{u}$, where $u = u^c$, and let $x \in \langle x',x''\rangle$. To show that $u(x) \geq \bar{u}$. Assume that $\langle x',x''\rangle \subseteq \mathrm{int}\mathcal{X}$. Let $x \in c(p)$. By the definition of B, $x' \in B(p)$ or $x'' \in B(p)$. Assume without loss of generality that $x' \in B(p)$. Then $x \in R^c(x')$, hence by Lemma 5, $u(x) \geq u(x') \geq \bar{u}$. If $\langle x',x''\rangle \not\subseteq \mathcal{X}$, then the result follows by continuity from the previous case. To verify that u generates c, let $x \in c(p)$, and assume $x' \in B(p)$. Then $x \in R^c(x')$, hence by Lemma 5, $u(x) \geq u(x')$, so $x \in c^u(p)$. Thus $c \subseteq c^u$. To show the converse, assume that $x \in c^u(p)\backslash c(p)$. Since u is strictly increasing, $p \in \mathrm{int}\mathcal{P}$. Hence there is $x' \in c(p)$, so $x' \in P^c(x)$. But then by Lemma 5, $u(x')>u(x)$, contradicting $x \in c^u(p)$. □

## 4. CONCAVE UTILITY

As an application of the main result, this section verifies that strengthening quasiconcavity of the utility function to concavity in the main result corresponds to adding normality (of all goods) on the choice side. Results in this direction has been proved by Gahvari (1986), and Thorlund-Petersen (1980, Theorem 6). The former assumes differentiability, which is dispensed with by Thorlund-Petersen. The main novelty here is that the result is stated without reference to preferences (utility) and thus gives an integrability result. A preference characterization of the resulting class of choices (or utility functions) has been given by Yaari (1978), in terms of his principle of diminishing eagerness to trade.

To verify the result, note that the **superdifferential** of a concave function u, $\partial u$, (defined by $p \in \partial u(x)$ if for all x', $u(x')-u(x) \leq p(x'-x)$) is equal to the inverse choice generated by u, except for normalizations, and also that the **indirect utility function, u***, (defined by $u^*(p) = \max_x\{u(x) \mid px \leq 1\}$) is concave in income:[8]

**LEMMA 7:** Let u be concave. Then for all p and x:

1) $p \in \partial u(x)$ implies that $x \in c^u(p/px)$.

2) $x \in c^u(p)$ implies that there is $t \geq 0$ such that $tp \in \partial u(x)$ (if u is strictly increasing, $t > 0$).

3) $u^*(y) = u^*(p/y)$ is concave in y.

**Proof:** 1) Obviously $x \in B(p/px)$. Let $x' \in B(p/px)$, i.e. $p(x'-x) \leq 0$. Then since u is concave and $p \in \partial u(x)$, $u(x')-u(x) \leq p(x'-x) \leq 0$. Hence $x \in c^u(p/px)$.

2) Let $x \in c^u(p)$. Then x solves $u^*(p) = \max_x\{u(x) \mid px \leq 1\}$. Hence by the Kuhn-Tucker theorem, there is $t \geq 0$ such that $0 \in \partial u(x)\setminus\{tp\}$, i.e. $tp \in \partial u(x)$.

3) Let x solve $u^*(y)$, x' solve $u^*(y')$, $y^t = ty'+(1-t)y$, $x^t$ solve $u^*(y^t)$, and $t \in \langle 0,1\rangle$. Assume as well that $y \geq y' > 0$. Then $p(tx'+(1-t)x) \leq y^t$. Hence by concavity, $tu^*(y')+(1-t)u^*(y) = tu(x')+(1-t)u(x) \leq u(tx'+(1-t)x) \leq u(x^t) = u^*(y^t)$. □

---

[8]The first two parts of this lemma is a variant of Lemma 1 in Chapter 5.

If u is completely separable, the superdifferential of u is equal to the cartesian product of the component superdifferentials. Say that a choice c is **normal** (with respect to all goods) if for all $t \geq 1$ and $x \in c(p)$ there is $x' \in c(tp)$ such that $x' \leq x$. The characterization is given by the main theorems and the following two lemmas:

**LEMMA** 8: If u is completely separable, continuous, and strictly increasing utility function with concave components, then its generated choice, $c^u$, is normal.

**Proof:** Let $u(x) = \Sigma_i u_i(x_i)$ and assume that $c^u$ is not normal. Then there is p, $p' = \bar{t}p \geq p$, $x \in c^u(p)$, $x' \in c^u(p')$, and i such that $x_i' > x_i$. By Lemma 7 there is $t > 0$ and $t' > t$ such that $tp \in \partial u(x)$ and $t'p' \in \partial u(x')$. By complete separability, $tp_i \in \partial u_i(x_i)$ and $t'p_i' \in \partial u_i(x_i')$, so by concavity, $(t'p_i'-tp_i)(x_i'-x_i) \leq 0$, contradiction. □

**LEMMA** 9: If $|I^0| \geq 3$ and the choice c is normal, completely decentralizable, closed, has large interior domain, large range, and satisfies the budget identity and the basic axiom, then the generated utility function, $u^c$, is concave.

**Proof:** (see Figure 5) Let $x_i+x_{-i}^0 \in c(p) \cap Sc^{-1}$, $x_i'+x_{-i}^0 \in c(p)$, and $x_i' > x_i$. By the definition of $u^c$, it is sufficient to show that $p_i' \geq p_i$. Let $x_i+x_{-i}^0 \in H(tp')$. Then $t \geq 1$, hence since c is normal, there is $x'' \in c(tp')$ such that $x'' \leq x_i'+x_{-i}^0$. Assume $p_i' > p_i$. Then $x'' \in B(p)$. Hence $x_i+x_{-i}^0 \in R^c(x'')$. Assume that $x_i+x_{-i}' \in c(tp')$. Then since $x_i+x_{-i}^0 \in Sc^{-1}$, $p = tp'$, contradicting $p_i' > p_i$. Hence $x_i+x_{-i}' \notin c(tp')$. But then $x'' \in P^c(x_i+x_{-i}^0)$, contradicting the basic axiom. □

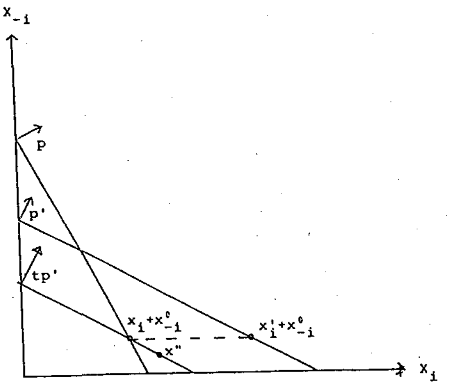



Figure 5

## 5. SUBJECTIVE EXPECTED UTILITY

As a further application of the main result, this section verifies that in a finite state space with one basic good, and a finite number of states, subjective expected utility can be characterized by adding a simple property called diagonal invariance to the choice properties of the main result. Diagonal invariance simplifies Lensberg's (1985) "constant beliefs", which again is a translation into choice terms of Savage's (1954) axiom P4, also called "ordering of events". The argument generalizes the one by Lensberg slightly by admitting risk neutrality, and also simplifies it. Savage's (1954) original result in contrast, has a convex state space, and obtains cardinal uniqueness of the state utility function, which is lost here. The first finite state space characterization of expected utility was by Stigum (1972) (and somewhat strengthened in Stigum (1990, Theorem 19.5)) in terms of preferences. Characterizing expected utility in terms of preferences, he avoids the integrability problem, and can treat the case of more basic goods.

A choice c is **ray invariant** if there are $x^0 \in \mathrm{int}\mathcal{X}$ and $p^0$ such that for all $t > 0$, $tx^0 \in c(p^0/t)$.[9] This expresses that there is one ray on which all points are chosen at the same relative prices, i.e. that there is some linear expansion path. Also an utility function u is an (slightly generalized) **expected utility function** if there is $x \in \mathrm{int}\mathcal{X}$, $y^0 \in \mathbb{R}_+^{I0}$ such that $y^0e = 1$, and a concave strictly increasing function u: $\mathbb{R} \to \mathbb{R} \cup \{\infty\}$ such that $u(x) = \sum_i y_i^0 u(x_i/x_i^0)$. Note that in the standard expected utility case, $x^0 = e$. In this case we term ray invariance **diagonal invariance.** This means that points on the diagonal are always chosen at the same relative prices, or under the choice under uncertainty interpretation, that sure outcomes (constant acts) are always chosen at the same odds. These odds can then be taken as the individuals subjective probabilities of the states. The characterization is given by the main theorems and the following two lemmas:[10]

---

[9]Lensberg (1985) uses a somewhat more complicated notion defined on bilateral lotteries, called "constant beliefs", which is a translation into choice terms of Savages (1954) "ordering of events". The present notion is extracted from his proof.

[10]One-dimensional factor spaces makes Savages (1954) "state independence" redundant here. "Decentralization" and "diagonal invariance" in this subjective framework corresponds to Machina's (1989b) "replacement" and "mixture separability" in an "objective" framework.

**LEMMA 10:** If u is an expected utility function, then its generated choice $c^u$ is ray invariant.

**Proof:** Let $u(x) = \sum_i y_i^0 \bar{u}(x_i/x_i^0)$ and let $p_i^0 = y_i^0/x_i^0$. Clearly $tx^0 \in H(p^0/t)$. Let $tx \in B(p^0/t)$. To show that $u(tx^0) \geq u(tx)$. Since $\bar{u}$ is concave and increasing, there is $t' \geq 0$ such that for each i, $\bar{u}(tx_i/x_i^0)-\bar{u}(t) \leq t't(x_i-x_i^0)/x_i^0$. Hence since $x \in B(p^0)$, $u(tx)-u(tx^0) \leq tt'p^0(x-x^0) \leq 0$. □

**LEMMA 11:** If $|I^0| \geq 3$ and the choice c is ray invariant, completely decentralizable, has large interior domain, large range, and satisfies the budget identity and the basic axiom, then the generated utility function, $u^c$, is an expected utility function.

**Proof:** (See Figure 6). Since c is ray invariant, there is $p^0$ and $x^0 \in \text{int}\mathcal{X}$ such that $tx^0 \in c(p^0/t)$ for all $t > 0$. Then by Lemma 4, for almost all t, $du(tx^0)$ is proportional to $p^0$, where $u = u^c$. Let $\bar{u}(t) = u_i(tx_i^0)/p_i^0$ for some i such that $p_i^0 > 0$. Then $du_j(tx_j^0) = p_j^0 d\bar{u}(t)$ almost everywhere, thus $u_j(tx_j^0) = y_j^0\bar{u}(t)$ for all j and $t > 0$ where $y_i^0 = p_i^0 x_i^0$. Obviously, $\bar{u}$ is strictly increasing, thus it remains only to show that it is concave. For given t,t', let $\bar{t} = y_I^0 e_I t + y_{-I}^0 e_{-I} t'$ and $\bar{x} = tx_I^0 + t'x_{-I}^0$. Then $\bar{x} \in B(p^0/\bar{t})$, so since $\bar{t}x^0 \in c(p^0/\bar{t})$, $\bar{t}x^0 \in R^c(\bar{x})$. Hence by Lemma 5, $u(\bar{t}x^0) \geq u(\bar{x})$. But then $\bar{u}(\bar{t}) = u(\bar{t}x^0) \geq u(\bar{x}) = y_I^0 e_I \bar{u}(t) + y_{-I}^0 e_{-I} \bar{u}(t')$. Thus $\bar{u}$ is concave. □

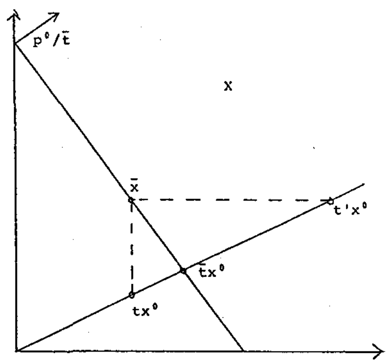


Figure 6

**Remark 10:** By duality this also gives a characterization of expected indirect utility. The latter notion has recently been proposed by Yaari (1987) as an alternative to expected utility as a theory of choice under risk. As ray invariance is self-dual, the question about choosing primal or dual expected utility is essentially the question of in which space separability makes most sense, at least in our context.

**Remark** 11: The present finite state characterization of expected utility does not imply the cardinal uniqueness property of the state utility function which is obtained in Savage's (1954) framework with a convex set of states.

**Remark 12:** Hens (1989) has extended the notion of diagonal invariance to the case with many basic goods, and given a characterization of expected utility in terms of preferences by means of it, which avoids the convexity assumption, i.e. (non-negative) risk aversion. The resulting characterization is simpler than the one by Stigum (1972).

## 6. SOME ADDITIONAL INTERPRETATIONS.

Under the standard demand theory interpretation, complete separability (as well as normality and ray invariance) does not seem generally plausible, though often assumed in applied work. Under some other interpretations, however, the intuitive validity of these notions seems less questionable.

First, interpreting the theory as choice under risk with only one basic good (money), $I^0$ is the (finite) set of (independent) **states**, subsets of $I^0$ are **events**, $\mathcal{X}$ is the set of **actions**, $\mathbf{x}_i$ the **outcome** of action x in state i (measured in the basic good), and $\mathcal{P}$ is the set of **lotteries** where a lottery **p** describes an odds vector, p/pe, and an initial position $\tilde{x}$ such that $p\tilde{x} = 1$. Note that the assumed form of the space $\mathcal{X}$ excludes the possibility of bankruptcy, and that $u(t) = u(tx^0)$ is the utility of the sure outcome t in the standard case where $x^0 = e$. In this case, the subjective probabilities are simply the odds at which sure outcomes are chosen.

Under this choice under risk interpretation, complete decentralization (separability) expresses that the choice given an event is independent of what happens if the event does not occur, normality that by an increase in income one would not choose to get less in any state than before, and diagonal invariance that a sure outcome is chosen at the same relative state contingent prices (or odds) independent of the initial income. Separability (i.e. complete decentralization) and normality seem more plausible than diagonal invariance. Diagonal invariance makes it possible to define subjective probabilities ($y^0$) dependent solely on the underlying space. For local analysis this might be more than necessary, i.e. under some smoothness assumptions complete decentralization and normality might suffice to define probabilities locally. But there are situations where also decentralization is counter intuitive, as in Machina's (1989a) parental inheritance example. Indeed working with objective probabilities, Machina (1982) shows that smoothness alone is sufficient for the local properties of the expected utility hypothesis. Whether one similarly can get rid of separability also in this subjective probability framework is an interesting question.

Secondly, interpreting the theory as normative social choice, $I^0$ is a set of individuals, and elements of $\mathcal{X}$ are fully cardinal, interpersonally comparable measures of the "welfare"[11] of the individuals and elements of $\mathcal{P}$ are (hypothetical) constant "welfare" tradeoffs situations. Under this interpretation, complete decentralizability expresses that the choice for a group of individuals is independent of the allocation to individuals outside the group as long as the situation with respect to the group is not changed, normality that an increase in resources with the same welfare trade off makes one choose no one worse off, and **strong diagonal invariance** (i.e. ray invariance where $x^0 = p^0 = e$) that a symmetric allocation is chosen in a symmetric situation. Normality thus seems rather generally acceptable, whereas complete decentralizability and strong diagonal invariance seems reasonable when one has to do with roughly equal individuals, expressing their independence and equality, respectively.

So far, however, this is only a partial normative theory. To get a full one, one should at the same time justify from more specified normative ideas both the "welfare" measure of the individuals and the social evaluation thereof possibly expressible in the above theory. Finally, note that it is only in normatively relatively simple situations that one should expect it to be possible to express relevant considerations in such a simple framework as the above.

---

[11]"Welfare" may be interpreted subjectively as a measure of satisfaction or what people actually desire, or more objectively as a standard of living, or an index of primary goods in Rawls (1970) sense. For the cardinality and interpersonal comparability requirements of the framework, it seems clear that one should here stick to the latter type of interpretation here. What is intended here is a measure of the normatively relevant characteristics of the individuals, whatever this might be.

## APPENDIX 1: WEIERSTRASS' THEOREM FOR CORRESPONDENCES

Lemma 1 uses a generalization of the fact that a continuous function maps an interval on an interval. Let $\mathcal{X}$ and $\mathcal{P}$ be topological spaces, and let $A \subseteq \mathcal{X}$ and $c \subseteq \mathcal{X}\times\mathcal{P}$. The **strong inverse** (image) of A under c, **$c^+(A)$** $= \{p \mid c(p) \subseteq A\}$ and c is **upper hemicontinuous** if for each open $U \in \mathcal{X}$, $c^+(U)$ is open in P. Furthermore, a pair of open sets $(U_1,U_2)$ is a **separation** of a set $A \subseteq \mathcal{X}$ if $A \subseteq U_1 \cup U_2$, $A \cap U_1 \neq \emptyset$, $A \cap U_2 \neq \emptyset$, and $A \cap U_1 \cap U_2 = \emptyset$. And finally, $A \subseteq \mathcal{X}$ is **connected** if it has no separation. The result is:

**THEOREM:** If $\mathcal{X}$ and $\mathcal{P}$ are topological spaces, $\mathcal{P}$ is connected, and $c \subseteq \mathcal{X}\times\mathcal{P}$ is upper hemcontinuous, nonempty- and connected-valued, then it's range, $Dc^{-1}$, is connected.

**Proof:** Let $(U_1,U_2)$ be a separation of $Dc^{-1}$. It is sufficient to show that $(c^+(U_1),c^+(U_2))$ is a separation of $\mathcal{P}$. First, since c is upper hemicontinuous, $c^+(U_1)$ and $c^+(U_2)$ are open. Next, their intersection is empty. If not, assume that $p \in c^+(U_1) \cap c^+(U_2)$. It follows that $c(p) \subseteq U_1 \cap U_2$, contradicting $c(p) \neq \emptyset$. Third, $\mathcal{P} \subseteq c^+(U_1) \cup c^+(U_2)$. If not, there is $x_1,x_2 \in c(p)$ such that $x_1 \notin U_1$ and $x_2 \notin U_2$. Hence since $U_1$ and $U_2$ covers $Dc^{-1}$, $x_1 \in U_2$ and $x_2 \in U_1$, contradicting the connectedness of c(p). Finally, the two sets are nonempty. E.g. assume that $c^+(U_1) = \emptyset$. Then $Dc^{-1} \subseteq U_2$, contradiction. □

To apply this result in Lemma 1, one needs that the closedness of the choice correspondence implies its upper hemicontinuity, at least on the interior of the budget space:

**LEMMA:** If c is closed, it is also upper hemicontinuous on $\mathrm{int}\mathcal{P}$.

**Proof:** Let $U \subseteq \mathcal{X}$ be open. To show $c^+(U) = \{p \in \mathrm{int}\mathcal{P} \mid c(p) \subseteq U\}$ is open. Let $p \in \mathrm{int}\mathcal{P}$, $c(p) \subseteq U$, and $p_n \to p$. To show $c(p_n) \subseteq U$ for n sufficiently large. If not, then there is $x_n \in c(p_n)$, and since $p \in \mathrm{int}\mathcal{P}$, $\{x_n\}$ is bounded. Let $x_n \to x$, subsequentially. Then since c is closed and U is open, $x \in c(p)\setminus U$, contradiction. □

## APPENDIX 2: PROOF OF LEMMA 2

Let $\bar{S}c^{-1} = \{x \in \mathrm{int}\mathcal{X} \mid c^{-1}(x)$ has at least two elements$\}$. Then $\bar{S}c^{-1} = \mathrm{int}\mathcal{X} \backslash Sc^{-1}$. To show that $\bar{S}c^{-1}$ has measure 0. Let $\tilde{H}(x) = \cap\{H(p) \mid x \in c(p)\}$ be the intersection of the supporting hyperplanes at x. Furthermore for a ray (1-dimensional subspace) L, let $\bar{S}_L c^{-1} = \{x \in \bar{S}c^{-1} \mid$ L is independent of $\tilde{H}(x)$ and $L^{\perp} \cap \mathrm{ri}c^{-1}(x) \neq \emptyset\}$, where riA is the relative interior of A and $L^{\perp}$ the normal space of L.

First, I show that $\bar{S}c^{-1} \subseteq \cup\{\bar{S}_L c^{-1} \mid L \in \mathcal{L}\}$, where $\mathcal{L}$ is the set of rays with a rational basis. By the basic axiom, $c^{-1}$ is convex-valued on $Dc^{-1}$. Hence, as $\mathrm{int}X \subseteq Dc^{-1}$, $\bar{S}c^{-1} = \{x \in \mathrm{int}\mathcal{X} \mid \dim c^{-1}(x) \geq 1\}$, where dim is the affine dimension. Let $x \in \bar{S}c^{-1}$ and $p \in \mathrm{ri}c^{-1}(x)$. Then $\dim\tilde{H}(x) < |I^0|-1$, since $\dim\tilde{H}(x)+\dim c^{-1}(x) = |I^0|-1$. Hence there is a ray L normal to both p and $\tilde{H}(x)$. Let L' be sufficiently near L with a rational basis. Then $L' \in \mathcal{L}$, L' is independent of $\tilde{H}(x)$, and there is $p' \in \mathrm{ri}c^{-1}(x)$ which is normal to L'. Hence $x \in \bar{S}_{L'}c^{-1}$. Thus since $\mathcal{L}$ is countable, it is sufficient to show that $\bar{S}_L c^{-1}$ has measure 0 for $L \in \mathcal{L}$.

$\bar{S}_L c^{-1}$ is measurable. This follows from Theorem III.30 in Castaing and Valadier (1977). First, by the dual of the lemma in the appendix, $c^{-1}$ is upper hemicontinuous on $\mathrm{int}\mathcal{X}$, and hence measurable. But then there is a countable sequence $\{\sigma_n\}$ of measurable selections such that for each $x \in \mathrm{int}\mathcal{X}$, $c^{-1}(x) = \mathrm{cl}\cup\{\sigma_n(x)\}$. Hence $Sc^{-1} = \{x \in \mathrm{int}\mathcal{X} \mid \mathrm{cl}\cup\{\sigma_n(x)\}$ is a singleton$\} = \{x \in \mathrm{int}\mathcal{X} \mid$ for all n,m, $\sigma_n(x) = \sigma_n(x)\}$, so $Sc^{-1}$ is measurable. Next since $c^{-1}$ is measurable, so is $\mathrm{ri}c^{-1}$ where the relative interior is only taken on the left. Hence, since $L^{\perp}$ is closed, its lower inverse $\{x \in \mathrm{int}\mathcal{X} \mid L^{\perp} \cap \mathrm{ri}c^{-1}(x) \neq \emptyset\}$ is measurable. It follows that $\bar{S}_L c^{-1}$ is measurable.

Hence, by a basic property of the product measure, it suffices to show that for all translates L' of L, $\bar{S}_L c^{-1} \cap L'$ is at most singleton. Assume $x \neq x'$, where $x,x' \in \bar{S}_L c^{-1} \cap L'$ for some translate L' of L. Then there is $p \in \mathrm{ri}c^{-1}(x)$ and $p' \in \mathrm{ri}c^{-1}(x')$. Hence since $x,x' \in L'$ and L' is normal to both p and p', $x' \in B(p)$ and $x \in B(p')$. If $x' \notin c(p)$, then $x \in P^c(x')$ and $x \in R^c(x')$, contradicting the basic axiom. Thus $x' \in c(p)$. Since $x \neq x'$, L' is independent of $\tilde{H}(x)$, and $p \in \mathrm{ri}c^{-1}(x)$, there is $p'' \in c^{-1}(x)$ such that $x' \in b(p'')$. But then since c satisfies the budget identity, $x \in P^c(x')$, contradicting the basic axiom again. Hence $\bar{S}_L c^{-1}$ has measure 0. □

January 1987
Revised, March 1992

# CHAPTER 8: LEXIMIN CHOICE[0]

## 1. INTRODUCTION

This chapter characterizes **leximin choice** (i.e. the choice generated by leximin preferences) as the choice which is nonempty-valued, diagonal (egalitarian) and completely decentralizable (separable). The argument is simple and proceeds via an inductive definition of leximin choice.

The result is mainly of interest under a social choice interpretation, as outlined at the end of Chapter 7. There are three reasons for this. First, leximin choice is simply the maximal diagonal (equal component) choice when restricted to price-generated budgets of standard demand theory. Secondly, neither of the two properties seems plausible under the standard interpretation. Hence we extend the class of budgets. Secondly, under the social choice interpretation complete separability is reasonable in many contexts as discussed in Chapter 7, and diagonality expresses that one should only deviate from equality in case of Pareto improvements.

Compared to the earlier characterizations by Imai (1983) and Lensberg (1986), the present one is more direct. Imai (1983) characterizes leximin choice as the choice function which satisfies (in his social choice language) efficiency (Pareto optimality), symmetry, invariance under linear utility transformations, independence under linear utility transformations other than the ideal point, and individual monotonicity. Lensberg's (1986) result is nearer to the present one, as he characterizes leximin choice as the choice function which satisfies efficiency, anonymity, individual monotonicity and complete separability (in a slightly different form from ours). Both also assume that the (generalized) budgets are convex, which is unnecessary here, as one should expect. This is nice, as such convexity is not plausible under the social choice interpretation. A consequence is that leximin choice is no longer single-valued, indeed it is also no longer convex-valued.

---

[0]Due thanks to Terje Lensberg for valuable comments to this chapter.

## 2. THE CHARACTERIZATION

Let $\mathcal{X} = \mathbb{R}^I_+$ where I is a finite set, and let $\mathfrak{B}$ be a collection of nonempty, closed, and downward monotonic subsets of $\mathcal{X}$,[1] which excludes some point in $\mathcal{X}$. Elements of $\mathfrak{B}$ are denoted **B** and called (generalized) **budgets**. The terms **x**, **i**, and **t** denotes elements of $\mathcal{X}$, I, and scalars, respectively. If $c \subseteq \mathcal{X}\times\mathfrak{B}$ ,[2] then c is a **choice** (correspondence).

The **lexicographic order** on $\mathbb{R}^I$, denoted $>_l$, is defined by $x' >_l x$ if there is $i \in I$ such that $x'_i > x_i$, and for all $j < i$, $x'_j = x_j$. An **ordering permutation** is a function $\sigma$ permuting the elements of $\mathcal{X}$, such that for all x and $i \leq j$, $\sigma(x)_i \leq \sigma(x)_j$. The **leximin preferences**, denoted **$(P_l,R_l)$**, is defined by $x' \in P_l(x)$ if $\sigma(x') >_l \sigma(x)$, and $x' \in R_l(x)$ if $\sigma(x') = \sigma(x)$ or $x' \in P_l(x)$, where $\sigma$ is an ordering permutation.

The **diagonal point** of B, $e^B = \max_t\{te \mid te \in B\}$ where **e** has unit components. Let **I(x,B)** = $\{i \in I \mid$ there is $x' \in B$ such that $x' \geq x$ and $x'_i > x_i\}$, the coordinate directions with points in B exceeding x. Say that a choice c is (weakly) **diagonal** if $x \in c(B)$ and $I(e^B,B) = \emptyset$ implies that $x = e^B$. Thus a choice is diagonal if it consists of exactly the diagonal point at each budget which does not contain any points above the diagonal point. This expresses equality. The concepts are illustrated in Figure 1, where $I(e^B,B) = \{2\}$ and $I(x,B) = \emptyset$.

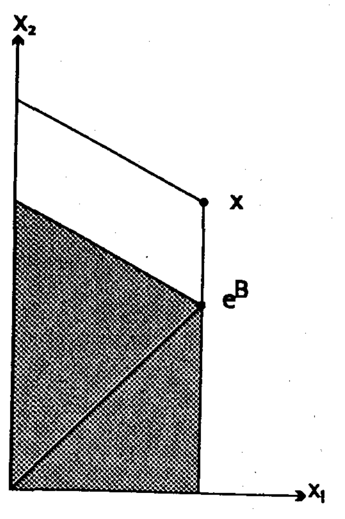


Figure 1

Let $J \subset I$ be nonempty. The **section** of B at $x_{-J}$, denoted $\mathbf{B}|x_{-J}$ is defined by $x_J \in B|x_{-J}$ if $x \in B$. A choice c is **decentralizable** with respect to J if for all x, x', B, B', $x \in c(B)$, $x' \in c(B')$, and $B|x_{-J} = B'|x'_{-J}$ implies that $x_J + x'_{-J} \in c(B')$. This is the earlier decentralization concept, except that the class of price-generated budgets is extended to the class of convex and downward

[1]A set B is **downward monotone** in $\mathcal{X}$ if $x \in B$, and $x' \leq x$ and $x' \in \mathcal{X}$, then $x' \in B$.

[2]Inclusion between sets is denoted by $\subseteq$, and strict inclusion by $\subset$.

monotone budget sets. The **leximin choice**, denoted $c^{P_l}$, is given by $x \in c^{P_l}(B)$ if $x \in B$ and for all $x' \in P_l(x)$, $x' \notin B$. The characterization can then be stated formally as:

**THEOREM:** A choice is nonempty-valued, diagonal, completely decentralizable if and only if it is the leximin choice, $c^{P_l}$.

The only if part of the theorem is straightforward. Hence we set out to prove the converse. This is done in two steps.First, we define inductively a choice d, which we show that is the leximin choice. For $x \in B$, let $\mathbf{t_i(x,B)} = \max_t\{t \mid tx_i + x_{-i} \in B\}$, the (maximal) **multiplier** of x in direction i in B. Define the **level n extended diagonal choice, $\mathbf{d^n}$**, inductively by $d^0(B) = \{e^B\}$, and $d^{n+1}(B) = \{t^n(B)x_{I^n(x,B)} + x_{-I^n(x,B)} \mid x \in d^n(B)\}$, where $\mathbf{t^n(B)} = \min_x \min_i\{t_i(x,B) \mid x \in d^n(B)$ and $i \in I(x,B)\}$, the minimal multipliers of points in $d^n(B)$, and $I^n(x,B) = \{i \in I(x,B) \mid m_i(x,B) = m^n(B)\}$, the directions of the minimal extensions. Here the level 0 extended diagonal of B is simply the diagonal point of B, and the extended diagonal of level n+1 consist of the minimax extensions in B of points in the level n extended diagonal, in the directions of the axes. Define $\mathfrak{B}^n = \{B \in \mathfrak{B} \mid$ for all $x \in d^n(B)$, $I(x,B) = \emptyset\}$, i.e. the points where the extended diagonal construction terminates. Clearly $\mathfrak{B} = \cup_n \mathfrak{B}^n$. If $B \in \mathfrak{B}^n$, write $\mathbf{d(B)}$ for $d^n(B)$ and call d the **extended diagonal choice**. In Lemma 2, we show that the extended diagonal choice equals the leximin choice. For this we need that if a point belongs to the extended diagonal, then so does any other point in B which is equal to it up to a permutation. We also add that the inclusion invariance property of Chapter 2 holds for the extended diagonal. This is used in Lemma 3 below:

**LEMMA 1:** 1) If $\sigma(x) = \sigma(x')$ and $x' \in B$, then for all n, $x \in d^n(B)$ implies $x' \in d^n(B)$.
2) If $B' \subseteq B$ and $d^n(B) \cap B' \neq \emptyset$, then $d^n(B') = d^n(B) \cap B'$.

**Proof:** Obvious by induction on n. □

**LEMMA 2:** $d = c^{P_l}$.

Proof: $\subseteq$: Assume $x \in d(B)\backslash c^{P_1}(B)$. Then there is $x' \in B$ such that $\sigma(x') >_L \sigma(x)$. Hence there is j such that $\sigma(x)_i = \sigma(x')_i$ for $i < j$ and $\sigma(x)_j < \sigma(x')_j$. But then since $\sigma$ is an ordering permutation, $I(x,B) \neq \emptyset$, contradiction.

$\supseteq$: Assume $x \in c^{P_1}(b)\backslash d(B)$ and let $x' \in d(B)$. Then $I(x,B) = \emptyset$, $x \in B$, and $x \in R_l(x')$. Hence by Lemma 1, Part i), $x \in d(B)$, contradiction. □

To prove the theorem it remains to show that if a choice satisfies the stated properties, then it equals the leximin choice. This is shown by induction on $\mathfrak{B}^n$ in Lemma 4, below. To use separability in the induction step, we need for each $B \in \mathfrak{B}^{n+1}$ to define a set $B^* \in \mathfrak{B}^n$ with the "same sections" as B. The idea is to "shrink" B uniformly from the points in $d^{n+1}(B)$ to the points in $d^n(B)$. Let $B \in \mathfrak{B}^{n+1}$. Define **B*** $= \{x_{I^n(x',B)}/t^n(B)+x_{-I^n(x',B)} \mid x \in B$ and $x' \in d^n(B)\}$, thus in Figure 1, B* is the shaded area. Then we have the desired properties:

**LEMMA 3:** Let $B \in \mathfrak{B}^{n+1}$. Then:

1): $B^* \subseteq B$ and $d^n(B) \subseteq B^*$.

2): $d^n(B^*) = d^n(B)$ and $B^* \in \mathfrak{B}^n$.

3): For all $x \in B^*$ and $x' \in d^n(B)$, $B \mid x_{I^n(x',B)} = B^* \mid x_{I^n(x',B)}/t^n(B)$.

Proof: **1):** This is immediate by the definitions.

**2):** The first part follows from i) and Lemma 1, Part 2).

**3):** This is also immediate by the definitions. □

**LEMMA 4:** If a choice c is nonempty-valued, diagonal and completely decentralizable, then c = d, the leximin choice.

Proof: We show that $c(B) = d(B)$ for all $B \in \mathfrak{B}^n$ by induction on n.

**Base case:** $B \in \mathfrak{B}^n$. Then $I(e^B,B) = \emptyset$, so by diagonality, $c(B) = \{e^B\} = d(B)$.

**Induction case**: Let $B \in \mathfrak{B}^{n+1}$.

$\supseteq$: Let $x' \in d(B) = d^{n+1}(B)$. Then $x' = t^n(B)x_{I(x,B)}+x_{-I(x,B)}$ for some $x \in d^n(B)$. By Lemma 3, Part 2), $d^n(B) = d^n(B^*) = d(B^*)$. Hence by induction $x \in c(B^*)$. But by Lemma 3, Part 3), $B|x_{I(x',B)} = B^*|x'_{I(x',B)}$. Hence by decentralization with respect to $-I(x',B)$, $x \in c(B)$.

$\subseteq$: Let $x \in c(B)$ and $x' \in d^n(B)$. Then $x_{I(x',B)}/t^n(B)+x_{-I(x',B)} \in B^*$. By Lemma 3, Part 3), $B|x_{I(x',B)} = B^*|x'_{I(x',B)}$. Hence by decentralization with respect to $-I(x',B)$, $x_{I(x',B)}/t^n(B)+x_{-I(x',B)} \in c(B^*)$. Hence by induction, $x_{I(x',B)}/t^n(B)+x_{-I(x',B)} \in d(B^*) = d^n(B^*) = d^n(B)$. But then $x \in d^{n+1}(B) = d(B)$. □

## REFERENCES


AFRIAT, S. (1965): "The equivalence in two dimensions of the strong and weak axioms of revealed preference," **Metroeconomica** 17.

AFRIAT, S. (1967a): "The construction of utility functions from expenditure data," **International Economic Review 8**, 67-77.

AFRIAT, S. (1967b): "The construction of separable utility functions from expenditure data". Mite, Purdue.

AFRIAT, S. (1973): "On a system of inequalities in demand analysis: An extension of the classical method," **International Economic** Review **14**, 460-472.

AFRIAT, S. (1976): **The Combinatorial Theory** of **Demand.** London: Input-Output Publishing.

AFRIAT, S. (1977): **The Price Index.** London: Cambridge University Press.

AFRIAT, S. (1981): "On the constructability of consistent price indices between several periods simultaneously," in DEATON, A. (ed.): **Essays in** Applied **Demand Analysis.** Cambridge: Cambridge University Press.

AFRIAT, S. (1987): **Logic of Choice and Economic** Theory. Oxford: Clarendon Press.

ANDREASSEN, L. (1989): "Den representative aktør: En Frankenstein i møte med virkeligheten." In BREKKE, K.A. og TORVANGER, A. (eds.): **Vitskapsfilosofi og Økonomisk** Teori. Oslo: Central Bureau of Statistics.

ANTONIELLI, G.B. (1886): **Sulla Teoria Matematica della Economica Politica.** Pisa: Nella Tipographia del Folchetto. Translated in CHIPMAN, J.S., HURWICZ, L., RICHTER, M.K., and SONNENSCHEIN, H.F. (1971): **Preferences, Utility and Demand.** New York: Harcourt Brace Jovanovich.

ARROW, K.J. (1957): "Rational choice functions and orderings," **Economica 26**, 121-27.

BECKER, G. (1962): "Irrational behavior and economic theory," **Journal of Political Economy** 70, 1–13.

BERGSTROM, T.C., PARKS, R.P., and RADER, T. (1976): "Preferences which have open graphs," **Journal of Mathematical Economics** 3, 265-268.

BEWLEY, T. (1977): "The permanent income hypothesis: A theoretical formulation," **Journal of Economic Theory 16**, 252-292.

BLACKORBY C., PRIMONT, D. and RUSSELL, R.R. (1978): **Duality, Separability and Functional Structure: Theory and Economic Applications.** New York: North Holland.

BLISS, C.J. (1975): **Capital Theory and the Distribution of Income.** Amsterdam: North Holland.

BROWNING, M., DEATON, A., and IRISH, M. (1985): "A profitable approach to labor supply and commodity demands over the life-cycle," **Econometrica 53**, 503-543.

CASTAING, C. and VALADIER, M. (1977): **Convex Analysis and Measurable Multi-functions.** Berlin: Springer.

CHIPMAN, J.S., HURWICZ, L., RICHTER, M.K., and SONNENSCHEIN, H.F. (1971): **Preferences, Utility and Demand.** New York: Harcourt Brace Jovanovich.

CLARK, S.A. (1985): "A complementary approach to the strong and weak axioms of revealed preference," **Econometrica 53**, 1459-1464.

CLARK, S.A. (1988): "An extension theorem for rational choice functions," **Review of Economic Studies** 55, 485-492.

DEATON, A. and MUELLBAUER, J. (1980): **Economics and Consumer Behavior**.Cambridge: Cambridge University Press.

DEBREU, G. (1959a): **Theory of Value**. New Haven: Yale University Press.

DEBREU, G. (1959b): "Topological methods in cardinal utility theory," in ARROW, K.J., KARLIN, S., and SUPPES, P. (eds.): **Mathematical Methods in** the **Social Sciences**. Stanford: Stanford University Press.

DEBREU, G. (1976): "Least concave utility functions," **Journal** of **Mathemathical Economics 3**, 121-129.

DEBREU, G. and KOOPMANS, T.C. (1982): "Additively decomposed quasiconvex functions," **Mathematical Programming 24**, 1-38.

DIEWERT, W.E. (1977): "Generalized Slutsky conditions for aggregate consumer demand functions," **Journal of Economic Theory 15**, 353–362.

EDGEWORTH, B. (1881): **Mathematical Psychics**. London: Keegan Paul.

EISENBERG, B. (1961): "Aggregation of utility functions," **Management Science 7**, 337-350.

ELLIS, A. (1973): "A rational-emotive approach to interpretation," in ELLIS, A.: **Humanistic Psychotherapy**. New York: McGraw Hill.

ELSTER, J. (1983): **Sour Grapes**. Cambridge: Cambridge University Press.

EPSTEIN, L.G. (1987): "The unimportance of intransitivity of separable preferences," **International Economic Review 28**, 315–322.

EPSTEIN, L.G. and YATCHEW, A.J. (1985): "Non–parametric hypothesis testing procedures and applications to demand analysis," **Journal** of **Econometrics 30**, 149–169.

FAN, K. (1971): "A generalization of Tychonoffs fixed point theorem," **Mathematische Annalen** 142, 305-310.

FISHBURN, P.C. and ODLYZKO, A.M. (1989): "Unique subjective probability on finite sets," **Journal of Rahmanujan Mathematical** Society 2.

FREIXAS, X. and MAS-COLELL, A. (1987): "Engel curves leading to the weak axiom in the aggregate," **Econometrica 55**, 515–532.

FRIEDMAN, M. (1957): A **Theory of the Consumption Function**. Princeton: Princeton University Press.

FØLLESDAL, D. (1982a): "Intentionality and behaviorism," in COHEN, L.J., LOS, H., **and PODEWSKI, K.-P.** (eds.): **Logic, Methodology,** and **Philosophy** of **Science VI**. Amsterdam: North Holland.

FØLLESDAL, D. (1982b): "The status of rationality assumptions in interpretation and in the explanation of action," **Dialectica 36**, 302-316.

GAHVARI, F. (1986): "A note on additivity and diminishing marginal utility," **Oxford Economic Papers 38**, 185-186.

GORMAN, W.M. (1968): "The structure of utility functions," **Review of Economic Studies 35**, 369-390.

GORMAN. T. (1971): "Preference, revealed preference, and indifference," in CHIPMAN, J.S., HURWICZ, L., RICHTER, M.K., and SONNENSCHEIN, H.F. (eds.) (1971): **Preferences, Utility and Demand.** New York: Harcourt Brace Jovanovich.

GREENWALD, H. (1974): **Decision therapy**. New York: Wyden.

HARDLE, W., HILDENBRAND, W. and JERISON, M. (1988): "Empirical evidence on the law of demand." Universitat Bonn, SFB 303. Discussion Paper A–193.

HARSANYI, J.C. (1959): "A bargaining model for the cooperative n-person game," in TUCKER, A.W. and LUCE, R.D. (eds.), **Contributions to the Theory of Games IV**. Princeton: Princeton University Press.

HECKMAN, J.J. (1974): "Life cycle consumption and labor supply: An exploration of the relationship between income and consumption over the life cycle," **American Economic Review 64**, 188-194.

HEINER, R.A. (1983): "The origin of predictable behavior," **American Economic Review 73**, 560-595.

HENS, T. (1989): "A characterization of subjective expected utility in a model with a continuum of consequences and a finite number of states". Universität Bonn, SFB 303. Discussion Paper A-256.

HENS, T. (1990): "On the structure of market excess demand in an intertemporal general equilibrium model". Universität Bonn, SFB 303. Discussion Paper A-280.

HICKS, J.R. (1956): **A revision of Demand Theory**. Oxford: Clarendon Press.

HICKS, J.R. and ALLEN R.G.D. (1934): "A reconsideration of the theory of value," **Economica 1**, 52-76 and 196-219.

HILDENBRAND, W. (1983): "On the "law" of demand," **Econometrica** 51, 997-1019.

HILDENBRAND, W. (1989a): "The weak axiom of revealed preference for market demand is strong," **Econometrica 57**, 979-985.

HILDENBRAND, W. (1989b): "Facts and ideas in microeconomic theory," **Europeean Economic Review 33**, 251-276.

HILDENBRAND, W. and JERISON, M. (1988): "The demand theory of the weak axioms of revealed preference", **Economic Letters** 29, 209–213.

HILDENBRAND, W. and KIRMAN, A.P. (1988): **Equilibrium Analysis**. Amsterdam: North Holland.

HODGSON, G. (1986): "Behind methodological individualism", **Cambridge Journal of Economics 10**, 211-224.

HOUTMAN, M. and MAKS, J.A.H. (1983): "The existence of homothetic utility functions generating Dutch consumer demand data". University of Groningen, memorandum 158.

HOUTMAN, M. and MAKS, J.A.H. (1985): "The consistency of aggregate random data with the hypothesis of cost minimization and utility maximization". []

HOUTAKKER, H. (1950): "Revealed preferences and the utility function," **Economica 17**, 159-174.

HOUTAKKER, H.S. (1965): "On the logic of preference and choice," in TYMINECKA, A.-T. and PARSONS, C. (eds.): **Contributions to Logic and Methodology in Honour of J.M. Bocheński.**

HURWICZ, L. (1971): "On the problem of integrability of demand functions," in CHIPMAN, J.S., HURWICZ, L., RICHTER, M.K., and SONNENSCHEIN, H. (eds.): **Preferences, Utility and Demand.** New York: Harcourt Brace Jovanovich.

HURWICZ, L. and RICHTER, M.K. (1971): "Revealed preferences without demand continuity assumptions," in CHIPMAN et al., 59-76.

HURWICZ, L. and UZAWA, H. (1971): "On the integrability of demand functions," in CHIPMAN et al.

HUSSERL, E. (1913): **Ideen zu einer reinen Phänomenologie und Phänomenologischen Philosophie.** Jahrbuch fur Philosophie und Phänomenologische Forschung 1.

IMAI, H. (1983): "Individual Monotonicity and Lexicographic Maximin Solution," **Econometrica 51**, 398-401.

JERISON, D. and JERISON, M. (1989): "Approximately rational consumer demand and Ville cycles". Manuscript, University of Bonn.

JOHNSON, W.E. (1913): "The pure theory of utility curves," **Economic Journal 23**, 483-513.

KANNAI, Y. (1977): "Concavifiability and constructions of concave utility functions," **Journal of Mathemathical Economics 4**, 1-56.

KANNAI, Y. (1986): "Engel curves, marginal utility of income, and concavifiable preferences," in Hildenbrand, W. and Mas-Colell, A. (eds.): **Contributions to Mathemathical Economics.** Amsterdam: North Holland.

KANNAI, Y. (1989): "A characterization of monotone individual demand functions," **Journal of Mathemathical economics** 18, 87–94.

KANT, I. (1785): **Grundlegung zur Metaphysik der Sitten.**

KIHLSTROM, R., MAS-COLELL, A., and SONNENSCHEIN, H. (1976): "The demand theory of the weak axiom of revealed preference," **Econometrica 44**, 971-78.

KIM, T. (1987): "Intransitive indifference and revealed preference," **Econometrica 55**, 163-167.

KIM, T. and RICHTER, M.K. (1986): "Nontransitive, nontotal consumer theory," **Journal of Economic Theory 38**, 324-363.

KIRMAN, A. (1989): "The intrinsic limits of modern economic theory: The emperor has no clothes," **The Economic Journal 99**, 126-139.

KOCH, K.J. (1987): **Consumer Demand and Aggregation.** Dissertation, Universität Bonn.

LAU, L.J. (1969): "Budgeting and decentralization of allocation decisions". Memorandum 89, Center for Research in Economic Growth, Stanford University.

LEICHTWEISS, K. (1980): **Konvexe Mengen.** Berlin: Springer.

LENSBERG, T. (1985): "A choice theoretic approach to expected utility". Discussion Paper no. 7, The Norwegian School of Economics and Business Administration.

LENSBERG. T. (1986): "Stability and the Leximin Solution," in: **Stability, Collective Choice and Separable Welfare.** Doctoral Dissertation, the Norwegian School of Economics and Business Administration.

LENSBERG, T. (1987): "Stability and collective rationality," **Econometrica 55**, 935-961.

LEONTIEF, W. (1947a): "A note on the interrelation of subsets of independent variables of a continuous function with continuous first derivatives," **Bulletin of the American Mathematical Society 53**, 343-350.

LEONTIEF, W. (1947b): "Introduction to a theory of internal structure of functional relationships," **Econometrica 15**, 361-373.

LEWBEL, A. (1991): "The rank of demand systems: Theory and nonparametric estimation," **Econometrica 59**, 711–739.

LITTLE, I. (1949): "A reformulation of the theory of consumer behavior," **Oxford Economic Papers NS**1, 90-99.

MACHINA, M.J. (1982): "Expected utility" analysis without the independence axioms, **Econometrica 50**, 277-323.

MACHINA, M.J. (1989a): "Dynamic consistency and non-expected utility models of choice under uncertainty," **Journal** of **Economic Litterature 27**, 1622-1688.

MACHINA, M.J. (1989b): "Choice under uncertainty: Problems solved and unsolved," in HEY, J.D. (ed.): **Current Issues in Microeconomics.** London: Macmillan.

MARSHALL, A. (1895): **Principles** of **Economics**, 3. ed. London: MacMillan.

MAS-COLELL, A. (1974): "An equilibrium existence theorem without complete or transitive preferences," **Journal** of **Mathematical** Economics 1, 237-243.

MAS-COLELL, A. (1976): "A Remark on a smoothness property of convex, complete preorders," **Journal of Mathematical Economics** 3, 103-105.

MAS-COLELL, A. (1977a): "The recoverability of consumers' preferences from market demand behavior," **Econometrica 45**, 1409-1430.

MAS-COLELL, A. (1977b): "On revealed preference analysis," **Review of Economic Studies 45,** 121-131.

MAS-COLELL, A. (1985): **The** Theory of **General Equilibrium. A Differentiable** Approach. Cambridge: Cambridge U.P.

MAS-COLELL, A. (1988): "On the uniqueness of equilibrium once again," Mimeo, Harvard University.

MCFADDEN, D. (1979): "Computability tests for SARP," **Journal** of Mathematical Economics **7**, 17-27.

MITJUSCHIN, L.G. and POLTEROVICH, W. (1978): "Criteria for monotonicity of demand functions," (in Russian) **Economica** i **Mathematicheskie** Metody 14, 122-128.

NASH, J.F. (1950): "The bargaining problem," **Econometrica 18**, 155-162.

PERLS, F.S., HEFFERLINE, R.F., and GOODMAN, P. (1951): **Gestalt therapy.** New York: Dell.

POLLAK, R.A. (1969): "Conditional demand functions and consumption theory," **Quarterly Journal of Economics 83**, 60-78.

POLLAK, R.A. (1970): "Budgeting and decentralization". Discussion paper 157, Department of Economics, University of Pennsylvania.

RAMSEY, F.P. (1926): "Truth and Probability," in: **The Foundation of Mathematics and other Logical Essays**, edited by Braithwaite, R.B.. London: Routledge and Keagan Paul (1954).

RAWLS, J. (1970): **A Theory of Justice.** Oxford: Oxford University Press.

RICHTER, M.K. (1966): "Revealed preference theory," **Econometrica 34**, 635-645.

RICHTER, M.K. (1971): "Rational choice," in CHIPMAN, J.S., HURWICZ, L., RICHTER, M.K, and SONNENSCHEIN, H. (eds.) (1971): **Preferences, Utility and** Demand. New York: Harcourt Brace Jovanovich.

RICHTER, M.K. (1979): "Preferences, choice and duality," **Journal of Economic Theory 20**, 131-181.

ROCKAFELLAR, R.T. (1970): **Convex Analysis.** Princeton: Princeton University Press.

ROSE, H. (1958): "Consistency of preference: The two-commodity case," **Review of Economic Studies 35**, 124-125.

RUBINSTEIN, A. (1986): "Finite automata play the repeated prisoners dilemma," **Journal of Economic Theory 39**, 83-96

SAKAI, Y. (1977): "Revealed favorability, indirect utility, and direct utility," **Journal of Economic Theory 14**, 113-129.

SAMUELSON, P.A. (1938): "A note on the pure theory of consumer's behavior," **Economica 5**, 61-71 and 353-54.

SAMUELSON, P.A. (1942): "On the constancy of the marginal utility of income" in LANGE, O., MCINTYRE, F., and YNTEMA, T.O. (eds.): **Studies in Mathematical Economics and Econometrics.** Chicago: University of Chicago Press.

SAMUELSON, P.A. (1948): "Consumption theory in terms of revealed preference," **Economica 15**, 243-253.

SAMUELSON, P.A. (1956): "Social indifference curves," **Quarterly Journal** of **Economics 60**, 1-22.

SAVAGE, L.J. (1954): **The Foundation** of **Statistics.** New York: Wiley.

SCHICK, F. (1987): "Rationality. A third dimension," **Economics and Philosophy 3**, 49-66.

SEN, A. (1973): "Behavoior and the concept of preference," **Economica 40**, 241-251.

SHAFER, W. (1977): "Revealed preference and aggregation," **Econometrica** 48, 1173-1182.

SHAFER, W. and SONNENSCHEIN, H. (1982): "Market demand and excess demand functions," in ARROW, K.J. and INTRILLIGATOR, M.D.: **Handbook of Mathematical Economics**, vol II. Amsterdam: North Holland.

SIMON, H. (1972): "Theories of bounded rationality," in MCGUIRE, C.D. and RADNER, R. (eds.): **Decision and Organisation**. Amsterdam: North Holland.

SONNENSCHEIN, H. (1971): "Demand theory without transitive preferences, with applications to the theory of competetive equilibrium," in CHIPMAN et al., 215-223.

SONNENSCHEIN, H. (1973): "Does Walras identity and continuity characterize the class of community excess demand functions," **Journal** of **Economic Theory** 6, 345-354.

SONNENSCHEIN, H. (1974): "Market excess demand functions," **Econometrica** 40, 549-563.

SONO, M. (1945): "The effect of price changes on the demand and supply of separable goods," **Kokumin Keisai Zasshi** 74, 1-51. (Translated in **International Economic Review** 2, 239-271, 1961).

SLUTSKY, E. (1915): "On the theory of the budget consumer," **Giomale degli economisti** 6. (Translated in the American Economic Association, **Readings in Price Theory**, Chicago: Irwin, 1952, 27-56).

STIGLER, G. (1950): "The development of utility theory," **Journal of Political** Economy **58**, 307-327.

STIGUM, B.P. (1967): "On certain problems of aggregation," **International Economic Review 8**, 349-367.

STIGUM, B.P. (1972): "Finite state space and expected utility maximization," **Econometrica 40**, 253-259.

STIGUM, B.P. (1973): "Revealed preference - A proof of Houtakker's theorem," **Econometrica 41**, 411-423.

STIGUM, B.P. (1990): **Towards a Formal Science of Economics**. London: MIT press.

SUZUMURA, K. (1983): **Rational Choice, Collective Decisions and Social Welfare**. London: Cambridge University Press.

THORLUND-PETERSEN, L. (1980): "Concave, additive utility representations and fair division of a random income". Mimeo, University of Copenhagen.

THORLUND-PETERSEN, L. (1985): "Revealed preferences by topological methods," **Economic Letters 18**, 317-319.

TVERSKY, A. (1969): "Intransitivity of preferences," **Psychological Review** 76, 31-45.

TVERSKY, A. (1975): "A critique of expected utility theory: Descriptive and normative considerations," **Erkenntnis** 9, 163-173.

TVERSKY, A. and KAHNEMANN, D. (1981): "The framing of decisions and the psychology of choice," **Science 211**, 453-458.

TVERSKY, A. and KAHNEMANN, D. (1991): "Loss aversion in riskless choice: A reference-dependent model," **Quarterly Journal of Economics**, 1039–1061.

UZAWA, H. (1956): "Note on preference and axioms of choice," **Annals of the Institute for Statistical Mathematics 8**, 35-40.

UZAWA, H. (1959): "Preferences and rational choice in the theory of consumption," in ARROW, K.J., KARLIN, S., and SUPPES, P. (eds.): **Mathematical Methods in the Social Sciences**. Stanford: Stanford University Press.

VARIAN, H. (1982): "The nonparametric approach to demand analysis," **Econometrica 50**, 945-973.

VARIAN, H. (1983): "Non-parametric tests of consumer behaviour," **Review of Economic Studies 50**, 99-110.

VARIAN, H. (1985): "Non-parametric analysis of optimizing behavior with measurement error," **Journal of Econometrics** 30, 445–458.

VIVES, X. (1987): "Small income effects: A Marshallian theory of consumer surplus and downward sloping demand," **Review of Economic Studies** 54, 87-103.

VON NEUMANN, J. and MORGENSTERN, O. (1947): **Theory of games and economic behavior** (2.ed.). Princeton: Princeton University Press.

WALD, A. (1936): "Über einige Gleichungssysteme der mathematischen Ökonomie," **Zeitschrift fur Nathionalökonomie 7.**

WALRAS, L. (1871): **Element d'economie politique pure, ou theorie de la richesse sociale.** References are to the 4. revised edition, Paris (1952): Libraire General de Droit et Jurisprudence.

YAARI, M.E. (1978): "Separable concave utility and the principle of diminishing marginal eagerness to trade," **Journal of Economic Theory** 18, 102-118.

YAARI, M.E. (1987): " The dual theory of choice under risk," **Econometrica 55**, 95-116.

ZAJAC, E.E. (1979): "Dupuit-Marshall consumer's surplus, utility, and revealed preference," **Journal of Economic Theory 20**, 260-270.