NORGES HANDELSHØYSKOLE BERGEN

A Contingent Claims Analysis of an Oil Reserve

by

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To my parents

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Chapter 1

INTRODUCTION

1.1 Objectives of the dissertation

This dissertation analyzes an undeveloped oil field under output price uncertainty and investment decision flexibility. The essence is to obtain the optimal management strategy and the value of the project. For these purposes we use the concept of contingent claims analysis.

Several alternative investment situations are considered. For each case, the optimal strategy is provided, stated in terms of an easily implemented break-even price rule. Our results indicate that management can increase the project value substantially when contingent claims analysis is applied.

Some possible regulations which may be imposed on an undeveloped oil field, e.g., through legislation, is also examined. The cost to the owner of the field is evaluated in the contingent claims framework by interpreting the regulation as a constraint on the set of feasible decision strategies, and by obtaining the induced opportunity loss.

The dissertation contains two results deserving particular interest: In a Black-Scholes economy, we derive the pricing function for two contingent claims, both providing a pay-off at the future maturity date described by the power function of the price of the underlying asset at that date. The future pay-off from one of the claims is made conditional on the price of the underlying asset at the maturity date being lower than some preset level. The future pay-off from the other claim is made conditional on the price of the underlying asset being below some preset level[1] for the entire period from the current date and until the maturity date.[2]

1.2 Methodology

The basic idea of contingent claims analysis[3] is that the value of the project must conform to the condition of no risk-free arbitrage opportunity.[4] The implication of this condition may briefly be explained as follows: Suppose that it is possible to replicate the future stochastic cash flow from the project by managing a portfolio of assets traded in the market according to a dynamic strategy. In that case, to rule out a risk-free arbitrage opportunity, the value of the project is required to be identical to the market value of the replicating portfolio.

The idea of evaluating a project by using the market prices of underlying assets, and by imposing the condition of no risk-free arbitrage, is not new.[5] Consider for instance the case of certainty, where the risk-free rate of interest is constant, and where the project pays one dollar at date one, and one dollar at date two. Suppose that two one-dollar discount bonds, maturing at date one and at date two, respectively, are traded in the market. The future cash flow from the project may then be replicated in the market by holding a portfolio consisting of one bond maturing at date one, and another bond maturing at date two. To rule out riskless arbitrage, the value of the project must be identical to the value of the replicating portfolio. Indeed, this corresponds to the value obtained by the net present value method when using the correct risk-free discount rate.

The idea may similarly be applied to the case of uncertainty. Con-

[1]The level is described by an exponential function of calendar time, and includes the special case of a constant level.

[2]The two results are presented in Chapter 8 and Chapter 9.

[3]For a survey of contingent claims analysis, see, e.g., Smith (1976) and (1979) and Mason and Merton (1985).

[4]A risk-free arbitrage opportunity may be interpreted as the existence of a "free lunch" or a "money pump" in the economy. The condition is further discussed in Chapter 2.

[5]For an introduction to the evaluation of assets by the no-arbitrage principle, see, e.g., Rubinstein (1987) and Varian (1987).

sider a project, at date one providing a cash flow corresponding to the stochastic price of a stock at that date, and at date two providing a similar cash flow dependent on the stock price at that date. Furthermore, suppose that the stock, which is traded in the market, pays no dividends during the two periods. Clearly, the future cash flow from the project may then be replicated in the market by acquiring two stocks immediately, and by selling one stock at each of the two future dates. To prevent riskless arbitrage, the current value of the project must be twice the current price of the stock.

The optimal exercise strategy when holding a European call option, written on a stock, is to exercise the option at the fixed future expiration date if and only if the stock price at that date exceeds the exercise price. The key insight of Black and Scholes (1973), leading to the famous pricing formula of the European call option, is that the future pay-off from the option in their model may be replicated in the market by a portfolio of risk-free bonds and the stock itself. In contrast to our two examples just above, the self-financing replicating strategy is in this case required to be dynamic and made contingent on the future development of the risky stock price. The reason for this complex replicating strategy is that the owner of the option faces both future uncertainty (the stock price at the expiration date) and future decision flexibility (whether to exercise or not at the expiration date).

Suppose that we use the net present value method to evaluate the European call option. With the Black-Scholes assumptions on the price dynamics of the stock, it is fairly straighforward to obtain the expected future cash flow.[6] To determine the risk-adjusted discount rate is, however, not an easy task. In general, it depends on the current stock price.

The European call option is a fairly simple asset. It may be exercised at one fixed future date only, and the value is zero if not exercised at that date. Thus, the optimal strategy at the future maturity date, is simply to compare the observable stock price at that date with the exercise price, and to exercise if and only if the former exceeds the latter. Consider for instance a more complex call option, where exercise is possible at several alternative preset dates. To maximize the value of the asset, the exercise strategy at each decision date requires a com-

[6]The expected cash flow corresponds to the expected value of a log-normal random variable.

parision of the intrinsic value and the option value, where the latter is computed by the net present value method. Clearly, the exercise strategy for each future decision date will affect the current assessments of both the expected future cash flow and the corresponding risk-adjusted discount rate, with consequences both to the current option value and the current optimal exercise decision.

The starting point of this dissertation is the evaluation of an undeveloped oil field. A project of this type is exposed to several sources of uncertainty, of which some are project specific. For instance, the total extractable quantity of oil, the total cost, and their distribution through time. Uncertainty may also be induced by more general factors, such as technological innovations, political changes, and fluctuation of market prices.

Contingent claims analysis provides a consistent conceptual framework in order to evaluate a future cash flow with future decision flexibility, and where the uncertainty may be related to the future development of market prices of traded assets. With the desire to exploit this comparative advantage of contingent claims analysis, we have chosen to focus the spot price of output (oil) as the source of uncertainty.

Other sources of uncertainty are not considered in this dissertation, in order to keep the basic structure of the models simple, providing decision rules to be fairly easily implemented to practical problems.[7]

Furthermore, when considering an undeveloped oil field, we particularly focus the flexibility related to the investment decision. The reason is that oil projects to a large degree are irreversible once undertaken, due to the cost structure.[8]

The basic idea of this dissertation is to interpret the undeveloped oil field as a contingent claim, with output as the underlying asset. We examine several degrees of investment decision flexibility, taking explicitly into account the opportunity to defer the investment decision itself, and make it at some later date dependent upon the spot output price at that date. The results are compared with more "traditional" models.

[7]It is well known, for instance from decision tree analysis, that adding on extra state variables increase the problem dramatically.

[8]Many oil projects are characterised by large investment and fixed production costs, as compared to the variable production costs.

1.3 Relations to existing literature

Myers (1984) notes the current gap between finance theory and strategic planning, and concludes that contingent claims analysis seems to be the most promising area of research for bridging this gap.[9] Mason and Merton (1985) divides the development of contingent claims analysis into "past", "present", and "future" applications, the latter representing the applications which are still in the development stage within academic research but which hold forth the promise of becoming a part of financial practice in the future. This dissertation attempts to contribute to this "future" category of contingent claims analysis.

McDonald and Siegel (1986) derives an analytical solution to the value of a perpetual investment opportunity, where the investment decision is irreversible. Both the investment cost and the value of the project if undertaken are described by a geometric Brownian motion.[10] In this dissertation we outline a model of the economy and of the oil field if developed, where holding a non-expiring licence to the oil reserve leads to a special case of the general evaluation problem solved in the mentioned article.

Majd and Pindyck (1987) considers the case where it takes time to invest. The investment is modeled as a sequence of decisions, where the manager at each point in time has the opportunity either to implement investments as scheduled or to wait. The optimal decision is made contingent on the stochastic value of a similar completed project ready for production.

Pindyck (1988) focuses on a producing company, and the interrelation between the capasity choice, the optimal production quantity, and the uncertainty with respect to the demand function of the output of the firm.

In this dissertation, we do not model the flexibility examined in the two latter articles. Rather, we assume that the project is irrevocable once undertaken, and that output is extracted according to a fixed preset production plan.[11]

[9]Overviews of the theory of contingent claims analysis are found in Smith (1976) and (1979).

[10]The geometric Brownian motion is discussed in Chapter 2.

[11]The latter assumption is relaxed in Chapter 7.

Tourinho (1979) is one of the first contributions implementing contingent claims analysis to evaluate a natural resource.

Brennan and Schwartz (1985) models a copper mine as a contingent claim with the stochastic spot output price as the source of uncertainty. The authors mainly consider operating flexibility, and characterize optimal strategies for closing and opening the mine temporarily, and the associated mine value.[12] This dissertation adapts a model of the economy similar to the one outlined in Brennan and Schwartz. Our focus, however, is on the investment decision.

Paddock, Siegel, and Smith (1988) considers an offshore petroleum lease as a contingent claim, using one unit of developed hydrocarbon reserve rather than one unit of output as the underlying asset. To obtain numerical results, the authors have to resort to a dubious "one-third" rule of thumb outside their model.[13]

Both Lund (1987) and MacKie-Mason (1987) examine the effect of a stochastic output price and a non-linear tax system on the project value and the optimal strategies. The former considers the case of the Norwegian petroleum sector, whereas the latter analyzes the case of mining. We do not take taxes into account in this dissertation.

1.4 Overview of the subsequent chapters

To implement the concept of contingent claims analysis, it is necessary to specify the model of the economy and to describe the characteristics of the project itself.

In Chapter 2 the economy is discussed. We make explicit assumptions on the spot price dynamics of output, and the value of a future claim on output relative to the current spot price.[14] The risk-free rate of interest is assumed constant and known. Furthermore, the economy is assumed to be frictionless and without risk-free arbitrage opportunies. This model of the economy leads to three evaluation rules characterizing the price of a contingent claim.

[12]Brennan and Schwartz (1986) is a popularized version of this article.

[13]Siegel, Smith, and Paddock (1987) is a popularized version of this article.

[14]The spot output price follows a geometric Brownian motion. The current value of a future claim on output is the current spot price, discounted back at the constant rate of return shortfall (convenience yield).

In Chapter 3, we describe the project in the case where the investment decision is already made, and assume there is no operating or abandonment flexibility. This means that the oil field may be represented by a fixed preset production schedule and a corresponding cost schedule. The value of the developed oil field is a linear function of the current spot output price. We also present our set of base case parameter values, which is used to illustrate the models derived in the later chapters.

We then consider the investment decision. Investment is modeled as undertaking a commitment to extract oil according to the fixed preset production schedule and to pay the future costs.

Chapter 4 deals with "traditional" investment decision models. First, we analyze the accept/reject case, where the manager at the decision date choose between initiating the project immediately or never. Second, we introduce the additional opportunity to choose - once and for all - the future date to start development. Break-even prices and project values are obtained for both cases.

In Chapter 5 we use the contingent claims framework to evaluate an opportunity to make an accept/reject investment decision at a fixed future date. In this case, the undeveloped oil field represents a future right, but no obligation, to acquire the future oil production by paying the future costs. By interpreting this investment opportunity as a European call option, we apply the famous Black-Scholes option pricing formula to obtain the current value of the oil field.

In Chapter 6, investment may be undertaken at any date, rather than on a fixed future date only. In the case where the investment opportunity is non-expiring, we present an analytical solution to the value of the undeveloped oil field and the trigger price indicating immediate investment.

In Chapter 7, we introduce operating managerial flexibility, and analyze the opportunity to costlessly switch the production on and off temporarily. By assuming exponentially declining production (if any), we provide the analytical solution to the field value and the trigger price associated with the optimal switching strategy. Our numerical results indicate that the future switching flexibility may be ignored when analyzing an investment opportunity.

In Chapter 8, we present our result evaluating a claim on a pay-off at a fixed future date, described by a power function of the price of the underlying asset at that date. The claim is contingent on the price of the underlying asset at the future date being lower than some preset level. This result has several interesting applications, and is used in this chapter to evaluate the costs of imposing a temporary freeze on development of an oil reserve.

In Chapter 9, we derive the value of a claim with a similar future pay-off as above. The pay-off is assumed contingent on the price of the underlying asset being lower than some preset level for the entire period from the current evaluation date and until the maturity date. The result is used to analyze a promise to develop an oil reserve before a fixed future date.

Chapter 10 contains some concluding remarks.

Chapter 2

THE ECONOMY

2.1 Introduction

This dissertation implements contingent claims analysis to evaluate an undeveloped oil field in the case of output price uncertainty and investment decision flexibility. It is then necessary to make assumptions on both the economy and the project. This chapter is devoted to the economy. We outline an economy similar to Brennan and Schwartz (1985), and present three implied evaluation rules of contingent claims.

2.2 Assumptions

Our *first* basic assumption is that the dynamics of the spot price of oil $S(t)$ is described by a geometric Brownian motion, defined by

$$\frac{dS(t)}{S(t)} = \alpha dt + \sigma dZ(t) \tag{2.1}$$

The term α represents the instantaneous trend (if any), and σ is the instantaneous standard deviation of the relative price change per time unit.[1] The term $dZ(t)$ corresponds to the increment of the standard Brownian motion.[2] In words, Eq. (2.1) states that successive relative

[1]We assume that both α and σ are constant through time.

[2]The increments of the standard Brownian motion are identically distributed normal random variables with $E(dZ) = 0$ and $E(dZ^2) = dt$, and uncorrelated

price changes are identically distributed normal random variables, and uncorrelated across time.

The assumed price dynamics may equivalently be stated in terms of the uncertain spot price at the future date t', given the spot price at the current date t. Eq. (2.1) then translates into

$$S(t') = S(t)e^{(\alpha-\frac{1}{2}\sigma^2)(t'-t)+\sigma Z(t'-t)} \tag{2.2}$$

where $S(t)$ is the current spot price and the term $Z(t)$ represents the standard Brownian motion.[3] From Eq. (2.2) it is easy to see that the logarithmic rate $\ln(S(t')/S(t))$ is normally distributed.[4] This means that the spot price $S(t')$ at some future date t', given the current spot price $S(t)$, is log-normally distributed.[5]

The assumed price dynamics implies that the spot price of oil follows a continuous sample path, with no jumps. The price path may thus be drawn without lifting the pencil from the paper.

Our *second* basic assumption is that there exists a traded asset in the market, with relative return perfectly correlated with the relative return of the spot price of oil, and with identical volatility σ.[6]

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across time, with $E[dZ(t)dZ(t')] = 0$ for all $t \neq t'$. For a brief introduction to stochastic calculus, see Smith (1979).

[3]The Brownian motion $Z(t)$ is normally distributed with expectation $E(Z) = 0$ and variance $E(Z^2) = t$.

[4]It is normal with parameters

$$\begin{aligned} E[\ln(S(t')/S(t))] &= (\alpha - \tfrac{1}{2}\sigma^2)(t' - t) \\ \text{Var}[\ln(S(t')/S(t))] &= \sigma^2(t' - t) \end{aligned}$$

[5]The exponent in Eq. (2.2) is normal. The spot price $S(t')$ at the future date t', given the current spot price $S(t)$, is thus by definition log-normally distributed, with expected value

$$E_t[S(t')] = S(t)e^{\alpha(t'-t)}$$

see, e.g., Ingersoll (1987) p. 14.

[6]This "twin asset" need not necessarily to exist as a separate traded asset in the market. It is sufficient that its value can be replicated in the market by a portfolio of traded assets equipped with an appropriate dynamic self-financing strategy.

The "twin asset" may for instance be interpreted as the marginal unit of the physical output (oil) in stock to a holder with positive optimal storage. Alternatively, the "twin asset" may be the value of a self-financing portfolio containing futures contracts on oil, or shares of stocks in oil companies, and riskless bonds.

The relation between the equilibrium required rate of return of this "twin asset", $\hat{\alpha}$, and the growth rate of the expected oil price, α, is

$$\delta = \hat{\alpha} - \alpha > 0 \tag{2.3}$$

Note that the rate δ is independent of whether both $\hat{\alpha}$ and α are measured in nominal or in real terms.[7]

The term δ may be interpreted as the "rate of return shortfall" related to oil.[8] It represents the rate of return forgone from holding oil if merely receiving return through the price change of oil. An analogy is δ being the continuous dividend pay-out rate on a stock.

Alternatively, δ may be considered as the "net marginal convenience yield" related to oil, reflecting the net benefit from the marginal unit of output in stock relative to a claim on future delivery of output.[9] With this interpretation, δ is similar to the liquidity premium from holding cash.

The assumption of $\delta > 0$ implies that there is an opportunity cost of having oil in the ground.[10]

The risk-free rate of interest $r > 0$ is constant and known. Riskless borrowing and lending at this rate r, are unrestricted.

Investors are assumed to prefer more to less, and to be risk averse. They are required to agree on the volatitily σ of the spot price of oil. Investors need not necessarily to agree on the size of α or $\hat{\alpha}$, but only on their difference $\delta \equiv \hat{\alpha} - \alpha$.

We assume that the economy is frictionless , with continuous trading, and no sources to imperfections such as taxes, transaction costs, or short sale restrictions. The economy is characterized by no risk-free arbitrage opportunities.

We now make some comments on the economy described just above. If the future cash flow from a project may be replicated by managing a portfolio of traded assets according to a dynamic self-financing strategy,

[7] We assume that both $\hat{\alpha}$ and δ are constant through time.

[8] See McDonald and Siegel (1984) and Pindyck (1988).

[9] Brennan and Schwartz (1985) use a constant rate δ when evaluating a copper mine. Brennan (1989) and Gibson and Schwartz (1989) analyze some alternative models of the convenience yield.

[10] This opportunity cost resolves the "extraction paradox", discussed in Tourinho (1979).

the no-arbitrage condition requires the value of the project to be equal to the value of the replicating portfolio. Suppose for the moment that a replicating strategy exists, and that arbitrage is possible. In that case, investors may earn a risk-free profit by selling the overpriced portfolio and buying the underpriced one. The actions of the arbitrageurs will force the prices to adjust. Equilibrium is reached when the arbitrage opportunity has vanished, that is, when the price of the project and the price of the replicating portfolio are equal.

In this economy, a negative current price $S(t)$ is ruled out. To see this, suppose for the moment that $S(t) < 0$. In that case, it is possible to obtain a risk-free profit $-S(t) > 0$ by "buying" the asset and abandoning it costlessly, thus violating the no-arbitrage condition. Furthermore, we note from Eq. (2.2) that the assumed price process rules out negative future prices $S(t')$.

From Eq. (2.2), it follows that a current price $S(t) = 0$ implies $S(t') = 0$ for all future dates $t' > t$ as well. Suppose that the stochastic price $S(t')$ at some future date t' is positive with a positive probability. With investors preferring more to less, the asset clearly must command a positive current price in the market, and we thus have $S(t) > 0$. Now, suppose instead that the price $S(t') = 0$ for all future dates $t' > t$, and that the current price $S(t) > 0$. In that case it is possible to make a risk-free profit of $S(t)$ by selling the asset short, and reversing the position without costs at the future date t', violating the no-arbitrage condition.

2.3 Three evaluation rules

To rule out riskless arbitrage opportunities, the prices in this economy must conform to the following evaluation rules: First, the value at the date t of receiving one riskless dollar at the future date t' is

$$V_t[Y(t') = 1] = e^{-r(t'-t)} \tag{2.4}$$

where $Y(t')$ is the certain future cash flow, and $V_t[\cdot]$ is a general evaluator.

Second, the value at date t of a claim on one unit of output at the future date t' is

$$V_t[Y(t') = S(t')] = e^{-\delta(t'-t)}S(t) \tag{2.5}$$

where $Y(t')$ is the random future cash flow, and $S(t)$ is the spot price at the evaluation date.[11] In this economy, the equilibrium futures price at date t of a hypothetical futures contract with delivery date t' is

$$F_t(S(t), t') = e^{(r-\delta)(t'-t)}S(t)$$

see Brennan and Schwartz (1985) and Ross (1978).

Third, consider an asset whose future pay-off can be written as a function of calendar time t and the price of output $S(t)$ only. The asset pays a continuous instantaneous cash flow to its holder $Ddt \equiv D(S,t)dt$. This asset may be interpreted as a contingent claim, with market value $U(S,t)$. To prevent arbitrage in our economy, its market value $U(S,t)$ must satisfy the partial differential equation

$$\tfrac{1}{2}\sigma^2 S^2 U_{SS} + (r-\delta)SU_S - rU + U_t + D = 0 \tag{2.6}$$

see, e.g., Merton (1977).[12] To obtain the market value of the asset, we must specify the boundary conditions.

The future claim on one dollar, and the future claim on one unit of output, may be interpreted as contingent claims. It is easy to verify that both Eqs. (2.4) and (2.5) satisfy the partial differential equation, Eq. (2.6).

[11]This evaluation rule, implied by the no-arbitrage condition, is consistent with the RADR-method. According to the latter, the current value of the claim is found by discounting back the expected future cash flow at the equilibrium rate of return. That is,

$$V_t[Y(t') = S(t')] = e^{-\hat{\alpha}(t'-t)}E_t[S(t')]$$

By inserting the expected future spot price, given in footnote 5, and by using the definition of δ, Eq. (2.3) above, we obtain Eq. (2.5).

[12]Eq. (2.6) is found by translating Eq. (1) in the mentioned article into our notation. Furthermore, the dividends D_1 related to the underlying asset in Merton's model is identical to $\delta S(t)$ in our economy.

Chapter 3

THE PROJECT

3.1 Introduction

In the previous chapter, we described and discussed the economy. In this chapter, we turn to the project.

We assume that the project when undertaken may be represented by a fixed preset production schedule and a corresponding fixed preset cost schedule. This means that there is no operating or abandonment flexibility related to the developed oil field. For many oil projects, this description is realistic due to the cost structure.

With our assumptions on the project we evaluate a commitment to initiate investments immediately. Finally, we present a numerical base case which will be used to illustrate our results in the next chapters.

3.2 Assumptions

Our basic assumption on the project is that, once undertaken, it may be described by a given production schedule, $q(\tau \mid t)$, and a given cost schedule, $b(\tau \mid t)$, where t is the initiation date and τ is the project time.

This means that the project is irreversible once undertaken.[1] More-

[1] Majd and Pindyck (1987) consider the case where it takes time to build, and where the investment decision is a sequence of decisions rather one irrevocable decision.

over, when initiated, there is no flexibility to reschedule the production, or to abandon the project.[2]

In several decision situations considered in the following chapters, the holder has the flexibility to initiate investments at future dates as well. We assume that if investments are initiated at some future date t' rather that at the current date t, the production schedule, considered as a function of project time τ, is unchanged. That is,

$$q(\tau \mid t) = q(\tau) \tag{3.1}$$

Furthermore, initiating at the future date t' rather that at the current date t causes the entire cost schedule to shift upwards at the exponential rate $\pi < r$. Stated formally, we assume

$$b(\tau \mid t') = e^{\pi(t'-t)} b(\tau \mid t) \tag{3.2}$$

where t' and t are alternative initiation dates.

3.3 The commitment value

Now, consider a project identical to the one described in Section 3.2, where the irrevocable investment decision just has been made. In this case, there is by assumption no decision flexibility left. The oil field may thus be interpreted as a claim on future delivery of oil according to the fixed production schedule in Eq. (3.1), combined with an obligation to repay a loan incurred according to the cost schedule in Eq. (3.2).

By using the two evaluation rules in Chapter 2, Eqs. (2.4) and (2.5), it is easy to verify that at the current date t, the value of a commitment to initiate immediately is

$$C(S(t), t) = AS(t) - B(t) \tag{3.3}$$

where we define

$$A \equiv \int e^{-\delta\tau} q(\tau) d\tau \tag{3.4}$$

and

$$B(t) \equiv \int e^{-r\tau} b(\tau \mid t) d\tau \tag{3.5}$$

[2]The latter assumption is relaxed in Chapter 7.

The constant A may be interpreted as the time-adjusted quantity of oil. It represents the quantity of oil received immediately that is equivalent to receiving the total quantity of oil in the field, $Q = \int q(\tau)d\tau$, according to the fixed production schedule $q(\tau)$. The constant $B(t)$ is the present value of future investment- and production costs at date t, given immediate development.

Note that the current value of the oil field, conditional on development being initiated immediately, is linear in the current spot price of oil $S(t)$. This is a concequence of no decision flexibility being left, and thus no opportunity for the manager to respond to oil price changes.

The value of the project $C(S(t), t)$ increases, ceteris paribus, with a higher current spot price of oil, with a lower rate of return shortfall δ, and with a higher riskless interest rate r.

3.4 A base case example

Throughout this dissertation, we illustrate our results with a numerical example. We use the following numerical values

Rate of return shortfall	δ	0.06
Risk-free interest rate	r	0.05
Cost escalation rate	π	0.00
Output price variance	σ^2	0.07

where all numbers are continuously compounded on an annual basis. The spot output price is quoted in terms of USD per barrel.

The project is described by

Discounted quantity of output	A	130	Mill. barrels
Discounted total costs	B	1040	Mill. USD

Chapter 4

TRADITIONAL MODELS

4.1 Introduction

In this chapter, we proceed to the investment decision. First, we consider the case where the investor is to choose between to accept the project immediately, or to reject it forever. Second, we introduce additional flexibility at the investment decision date by including the opportunity - once and for all - to fix a future date at which the project is to be initiated.

For both cases, we obtain the break-even price and the value of the undeveloped oil field given the optimal investment strategy. The results in this chapter are used as benchmarks when we later proceed to the contingent claims analysis.

4.2 Accept/reject

Consider an undeveloped oil field that, once undertaken, is similar to a commitment to extract oil (see Section 3.3). Suppose the holder at the current investment decision date is to choose between to accept the project or to reject it. The decision flexibility is thus whether to undertake a commitment to initiate the project immediately or never.

The current value of the undeveloped oil field, contingent on the optimal decision being made, is

$$V(S(t),t) = \max\{C(S(t),t), 0\} \tag{4.1}$$

where $C(S(t),t)$ is the value of a commitment to initiate investments immediately, given by Eq. (3.3). The optimal decision in this case is

$$\begin{array}{ll} \text{Reject} & \text{if } S(t) < S_{BE}(t) \\ \text{Accept} & \text{if } S(t) \geq S_{BE}(t) \end{array} \tag{4.2}$$

where $S_{BE}(t)$ is the accept/reject break even-price, defined by

$$S_{BE}(t) \equiv \frac{B(t)}{A} \tag{4.3}$$

In Eq. (4.3), A and $B(t)$ are given by Eqs. (3.4) and (3.5), respectively. It is straightforward to show that the break-even price increases with a higher rate of return shortfall δ and a lower risk-free interest rate r.

The value of the oil field is

$$V(S(t),t) = \begin{cases} 0 & \text{if } S(t) < S_{BE}(t) \\ AS(t) - B(t) & \text{if } S(t) \geq S_{BE}(t) \end{cases} \tag{4.4}$$

$V(S(t),t)$ represents a linear function of the current spot price of oil $S(t)$, with a kink at $S(t) = S_{BE}(t)$.

4.3 Optimal timing

In the accept/reject model above, there were only two decision alternatives available to the investor. Now, suppose that we in addition introduce the opportunity - once and for all - to fix a future date at which investments are to be initiated. This timing decision is irrevocable, as is the case with the accept and the reject alternatives. When fixed, the future initiation date T may *not* be changed, even if so desired.

At the fixed future date T, the undeveloped oil field represents a commitment to initiate immediately. From Eq. (3.3), we know that the value at the future date T of this commitment is $C(S(T),T)$, depending on the stochastic spot price $S(T)$ at that date. By using the evaluation rules in Eqs. (2.4) and (2.5), we find that the current value of fixing the future initiation date T is

$$\begin{aligned} C(S(t),T) &\equiv V_t[C(S(T),T)] \\ &= e^{-\delta(T-t)}AS(t) - e^{-(r-\pi)(T-t)}B(t) \end{aligned} \tag{4.5}$$

where $V_t[\cdot]$ is a general evaluator, A and $B(t)$ are defined by Eqs. (3.4) and (3.5), and π is the cost escalation rate.[1]

Given the optimal choice, the value of the undeveloped oil field is

$$V(S(t),t) = \max\left\{\max_{T \geq t} C(S(t),T), 0\right\}$$

where the future initiation date T is the decision variable.

By assumption, the rate of return shortfall δ is positive, c.f. Eq. (2.3).[2] Now, suppose for the moment that $r - \pi \leq 0 < \delta$. In that case, we see directly from Eq. (4.5) that the current value of the future commitment $C(S(t),T)$ is a decreasing function of the initiation date T. This means that fixing a later initiation date is inferior to initiate immediately. The decision thus collaps into the accept/reject situation examined above.

With the parameter values $0 < r - \pi = \delta$, the current value of a commitment to initiate at date T, Eq. (4.5), may be written as the value of immediate initiation, $C(S(t),t)$, discounted back at the positive rate $r - \pi$. The sign of $C(S(t),T)$ is then identical to the sign of $C(S(t),t)$, and fixing a later initiation date will reduce its absolute value. Thus, with $C(S(t),t)$ positive, immediate development is optimal. If $C(S(t),t)$ is negative, the latest possible initiation date is the "optimal" one, but inferior to rejecting the project. In this case, fixing a future initiation date $T > t$ is never the optimal choice, and we are thus left with the accept/reject decision situation.

With $0 < r - \pi < \delta$, the interior solution of the first-order condition of $C(S(t),T)$ wrt. T (if any) represents a minimum.[3] The optimal

[1] From Section 3.2, we recall that the entire cost schedule escalates exponentially with rate π if the initiation date is deferred, see Eq. (3.2). This assumption implies that

$$B(T) = e^{\pi(T-t)}B(t)$$

where T and t are alternative initiation dates.

[2] Suppose for the moment that $\delta \leq 0$, and $r - \pi > 0$. We then see from Eq. (4.5) that $C(S(t),T)$ is an increasing function of T, and it is thus optimal to delay the initiation as long as possible. With T unbounded, the optimal choice is $T^* = \infty$, leading to an infinitely large project value if $\delta < 0$, and to the project value $QS(t)$ if $\delta = 0$. In an economy with these parameter values, there is no incentive to turn "oil in the ground" into producing oil fields. This situation corresponds to the "extraction paradox" in Tourinho (1979).

[3] The first- and the second-order conditions are found in footnote 5 below.

initiation date is $T = t$ or $T = \infty$, and the holder is thus left with accept and reject as the relevant decision alternatives.[4]

Now, the case of $0 < \delta < r - \pi$ remains to be discussed. With no restrictions on the latest possible initiation date T, the optimal decision at date t, contingent on the current spot price of oil $S(t)$, is

$$\begin{array}{ll} \text{Initiate at date } T^* & \text{if } S(t) < S_{OT}(t) \\ \text{Initiate immediately} & \text{if } S(t) \geq S_{OT}(t) \end{array} \tag{4.6}$$

where both T^* and $S_{OT}(t)$ are defined below.

The optimal future initiation date $T^* \geq t$ is given by the expression

$$T^*(S(t)) = t + \max\left\{0, \frac{\ln(S_{OT}(t)/S(t))}{(r-\pi)-\delta}\right\} \tag{4.7}$$

and is a decreasing function of the current spot price $S(t)$.[5]

The critical price, indicating that $T^* = t$, is

$$S_{OT}(t) = \left(\frac{r-\pi}{\delta}\right) S_{BE}(t) \tag{4.8}$$

where the accept/reject break even-price $S_{BE}(t)$ is defined by Eq. (4.3). We note that the critical price $S_{OT}(t)$ indicating immediate investment, is higher than the break-even price S_{BE}.

---

[4]With $\delta > 0$, it is easy to see from Eq. (4.5) that

$$\lim_{T\to\infty} C(S(t), T) \leq 0$$

for all $r - \pi$, and thus inferior to rejecting the project.

[5]The optimal future initiation date T^* is determined by solving the first-order condition

$$\frac{\partial C(S(t),T)}{\partial T} = -\delta e^{-\delta(T-t)} AS(t) + (r-\pi)e^{-(r-\pi)(T-t)}B(t) = 0$$

with respect to T. In the case of $0 < \delta < r - \pi$, it is easy to see that we have

$$\frac{\partial^2 C(S(t),T^*)}{\partial T^2} = (\delta - (r-\pi))\delta e^{-\delta(T^*-t)} AS(t) < 0$$

and the solution $T^* > t$ (if any) thus represents a maximum.

The value of the oil field in the optimal timing case is

$$V(S(t),t) = \begin{cases} C(S(t),T^*) & \text{if } S(t) < S_{OT} \\ C(S(t),t) & \text{if } S(t) \geq S_{OT} \end{cases} \tag{4.9}$$

With $S(t) < S_{OT}(t)$, the current value of the oil field contingent on fixing the optimal initiation date T^*, is a convex function of the current spot price $S(t)$. It converges to zero when $S \to 0$. With $S(t) \geq S_{OT}$, accepting the project is optimal, and the value of the investment opportunity is identical to a commitment to initiate immediately.

4.4 Conclusions

When considering immediate investment, this action must be compared to the best possible alternative if the present value is to be maximized. In many cases, where uncertainty and flexibility are present, neither turning the project down forever nor fixing a future initiation date once and for all, represent this "best alternative". The two traditional models above, allowing for strategies that are determined by calendar time only, may lead to non-optimal decisions, possibly inducing a substantial opportunity loss.

Chapter 5

FUTURE ACCEPT/REJECT

5.1 Introduction

In this chapter, we consider an undeveloped oil reserve where the decision flexibility corresponds to a future accept/reject investment decision. This problem is similar to the evaluation of a European call option. With a suitable reinterpretation of parameters, we use the Black-Scholes option pricing formula to obtain the current value of the oil field. We show some practical applications of the result.

5.2 The future accept/reject decision

Consider an undeveloped oil field that, once undertaken, represents a commitment to extract oil, as discussed in Chapter 3. Assume that the decision flexibility corresponds to an accept/reject investment decision to be made at the fixed future date T.

This may be the situation, for instance, if the oil reserve is unprofitable to develop on its own, but where idle processing and transportation capacity from a neighboring oil field will be available at date T. Another example may be that the licence requires the holder to refrain from initiating development before the future date T, and that the licence expires if investments are not undertaken at that date.

If development is initiated at the future date T, the oil field represents by assumption a commitment to extract oil. However, the project will be undertaken at that date only if optimal, and the future value of the oil field is thus

$$Y_C(S(T), T) = \max\{AS(T) - B(T), 0\}$$

The optimal decision is to develop at date T if the future spot price $S(T)$ equals or exceeds the break-even price $S_{BE}(T) \equiv B(T)/A$. The future value of the investment opportunity is a linear function of the stochastic future spot price $S(T)$, with a kink at $S(T) = S_{BE}(T)$.

For the holder of the future accept/reject opportunity, it is possible though not optimal, to undertake an immediate commitment to initiate at the future decision date. Alternatively, he may promise today to reject the project at date T. The current value of the undeveloped oil field W_C is thus bounded from below by the value of choosing the "best" of the two mentioned non-optimal strategies. We may thus conclude that

$$W_C \geq \max\{C(S(t), T), 0\}$$

where $C(S(t), T)$ is defined by Eq. (4.5).

The two discounting rules in Chapter 3, evaluating future claims on one unit of output and one riskless dollar, are not appropriate to find W_C. By examining $Y_C(S(T), T)$, however, we see that the future value corresponds to the pay-off at the maturity date from A European call options, each written on one unit of output and with exercise price $S_{BE}(T)$.[1] By a suitable reinterpretation of the parameters in the Black-Scholes call option pricing formula, we obtain that the current value of the future accept/reject investment opportunity is[2]

$$W_C \equiv V_t[\max\{AS(T) - B(T), 0\}]$$

[1]It is easy to see that the value of the accept/reject opportunity at the future date T equivalently may be written

$$Y_C(S(T), T) = A\max\{S(T) - S_{BE}(T), 0\}$$

where $S_{BE}(T)$ is the break-even price at that date.

[2]First, we interpret one unit of output as the underlying asset (one share of stock), following a geometric Brownian motion. The rate of return shortfall δ then corresponds to a continuous dividend pay-out rate on this stock. Second, we interpret the future accept/reject break-even price $S_{BE}(T)$ as the exercise price of the

$$= e^{-\delta(T-t)}AS(t)N[d_1] - e^{-(r-\pi)(T-t)}B(t)N[d_2] \quad (5.1)$$

where $N[\cdot]$ is the cumulative normal probability function. The constants d_1 and d_2 are defined by

$$d_1 \equiv \frac{\ln(S(t)/S_{BE}(t)) + (r - \pi - \delta + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \quad (5.2)$$

$$d_2 \equiv d_1 - \sigma\sqrt{T-t} \quad (5.3)$$

In Section 8.4, we obtain Eqs. (5.1) - (5.3) as a special case.

The current value of the future accept/reject investment opportunity W_C is increasing and convex in the current spot price $S(t)$. With $S(t)$ being low, the probability that the oil field will be developed at date T is close to zero, and we have the limit

$$\lim_{S(t)\to 0} W_C = 0$$

With a high current spot price, the probability that the project will be accepted at date T approaches one. The future accept/reject investment opportunity then converges to a corresponding commitment to intitiate development. We have thus argued that

$$\lim_{S(t)\to\infty} W_C = C(S(t), T)$$

The value W_C increases with a lower rate of return shortfall δ, a higher riskfree rate of return r, a lower cost escalation rate π, and a higher volatility σ. The effect of extending the time to maturity $T-t$ is ambiguous. The direct effect of deferring T is positive, as is the case with a financial European option. However, the opportunity cost of having oil in the ground, caused by the rate of return shortfall δ and the cost escalation rate π, contributes negatively.[3]

---

option. Third, we insert the relationship $S_{BE}(T) = \exp\{\pi(T-t)\}S_{BE}(t)$.

The Black-Scholes call option pricing formula in the case where the underlying asset pays a constant dividend rate is found, for instance, in Kemna (1987) Eq. (6.3.4), McDonald and Siegel (1984), and Smith (1976). Some minor algebraic manipulations then lead to Eqs. (5.1) - (5.3).

[3]The comparative statics of W_C are obtained in Appendix A.

5.3 An abandonment option

Consider an undeveloped oil field, where the holder for some reason has undertaken a commitment to initiate investments at the future date T. Development will thus be started at date T, no matter what the spot price appears to be at that date. This situation may be the result, for instance, of an agreement between the holder and the Government.

Suppose it is possible for the holder to make an arrangement today, so that the project may be abandoned without costs at the future date T, if so desired.

The undeveloped oil reserve will be abandoned at the future date T only if optimal. The future value of this arrangement is thus

$$Y_P(S(T),T) = \max\{-AS(T) + B(T), 0\}$$

The opportunity to abandon the project is similar to a European put option. By rewriting the future pay-off $Y_P(S(T),T)$, and by using previous results,[4] we find that the current value of the opportunity to abandon the project at the future date T is

$$\begin{aligned} W_P &\equiv V_t[\max\{-AS(T) + B(T), 0\}] \\ &= -e^{-\delta(T-t)}AS(t)N[-d_1] + e^{-(r-\pi)(T-t)}B(t)N[-d_2] \end{aligned} \quad (5.4)$$

where $N[\cdot]$ is the cumulative normal probability function, and where d_1 and d_2 are defined by Eqs. (5.2) and (5.3), respectively.[5]

By combining Eq. (5.4) with Eqs. (4.5) and (5.1), we find that

$$C(S(t),T) + W_P = W_C \quad (5.5)$$

---

[4]The future value of the abandonment option $Y_P(S(T),T)$ may alternatively be written

$$\max\{-AS(T) + B(T), 0\} = -AS(T) + B(T) + \max\{AS(T) - B(T), 0\}$$

The first two terms on the right hand side correspond to the future value of a commitment to initiate immediately, whereas the last term represents the (negative) future value of an accept/reject opportunity. The future value $Y_P(S(T),T)$ may thus be evaluated by Eqs. (4.5) and (5.1). The symmetry of $N[\cdot]$ and some minor algebraic manipulations lead to Eq. (5.4).

[5]With $\delta = 0$, $\pi = 0$, and $A = 1$, Eq. (5.4) corresponds to the standard European put option, see, e.g., Ingersoll (1987) p. 320.

This equation states that commitment to initiate investments at the future date T, combined with the opportunity to abandon the project costlessly at that date, is equivalent to the future accept/reject investment opportunity at the future date T. With $\delta = 0$, $\pi = 0$, and $A = 1$, Eq. (5.5) boils down to the put-call parity for European options, see, e.g., Ingersoll (1987) p. 304.

5.4 Immediate versus future decision

Now, suppose the holder of the undeveloped oil field is to choose either an immediate accept/reject investment decision, or to defer the accept/reject investment decision to the fixed future date T. This may be the case, for instance, with Governmental regulations, or technological constraints.

From our discussion in Section 5.2, we know that the current value of a future accept/reject investment opportunity W_C is positive for all $S(t) > 0$. Rejecting the project immediately is thus inferior, and the relevant decision alternatives today are thus either to initiate development immediately, or to make the final investment decision at the future date T.

The optimal decision depends on the current spot price $S(t)$. The critical spot price, with indifference between initiating immediately and deferrring the investment decision to date T, is

$$S_E^*(t) \equiv \{S(t) : AS(t) - B(t) = W_C\} \tag{5.6}$$

where W_C is defined by Eqs. (5.1) - (5.3) above. In this case, immediate initiation of the project is compared to an alternative with a positive value, whereas in the accept/reject model, the alternative has zero value. It is thus obvious that the critical price $S_E^*(t)$ exceeds the accept/reject break even price $S_{BE}(t)$. Unfortunately, no closed form solution of $S_E^*(t)$ is available, and its value must be approximated by numerical methods for given parameter values.

The optimal decision in this case is

$$\begin{array}{ll} \text{Defer the decision} & \text{if } S(t) < S_E^*(t) \\ \text{Initiate immediately} & \text{if } S(t) \geq S_E^*(t) \end{array} \tag{5.7}$$

and the current value of the undeveloped oil field is

$$V(S(t),t) = \begin{cases} W_C & \text{if } S(t) < S_E^*(t) \\ AS(t) - B(t) & \text{if } S(t) \geq S_E^*(t) \end{cases} \quad (5.8)$$

given the optimal strategy.

We now illustrate this situation using our base case parameter values.[6] In addition, we assume that the time to the future decision date T is $T - t = 4$ years.

The traditional accept/reject break-even price S_{BE} is 8 USD/barrel in this case.[7] Figure 5.1 confirms that the value of the immediate accept/reject investment opportunity is linear in the current spot output price, with a kink at $S(t) = S_{BE}$.

The critical price, indicating indifference between initiating immediately and deferring the final investment decision to the future date T, is $S_E^* = 10.6$ USD/barrel. We see from Figure 5.1 that the current value of the future accept/reject opportunity is increasing and convex in the spot output price $S(t)$.

With $S(t) = S_{BE}$, the current value of the future accept/reject decision is 158 mill. USD. This represents the opportunity loss from neglecting the flexibility to defer the final decision to date T. In Figure 5.1, we see that the opportunity loss[8] in the region $S_{BE} \leq S(t) \leq S_E^*$ is decreasing in $S(t)$, and is zero with $S(t) = S_E^*$.

5.5 Optimal timing of the decision date

In the optimal timing model considered in Chapter 4, the holder is assumed to fix a future date at which the project is to be initiated. This means that the project will be accepted at the future date T, without respect to the future spot price $S(T)$.

[6] Recall from Section 3.4 that $\delta = 0.06$, $r = 0.05$, $\pi = 0$, $\sigma^2 = 0.07$, $A = 130$, $B = 1040$.

[7] With our base case parameter values, we have $\delta > r - \pi$. This means that "optimal timing" collapses into the accept/reject situation, see Chapter 4.

[8] The opportunity loss in the region $S_{BE} \leq S(t) \leq S_E^*$, caused by initiating immediately rather than deferring the final decision to date T, is found by the vertical difference between the broken curve and the upward sloping line in Figure 5.1.

Now, suppose instead that the holder of the undeveloped oil field at the current date t is to fix - once and for all - the future date T at which the accept/reject investment decision is to be made. If the spot price of oil $S(T)$ at the future date T turns out to be lower than the break-even price $S_{BE}(T)$, the project will be rejected. The problem stated here in fact boils down to determining the optimal maturity date of the future accept/reject investment opportunity W_C considered above.

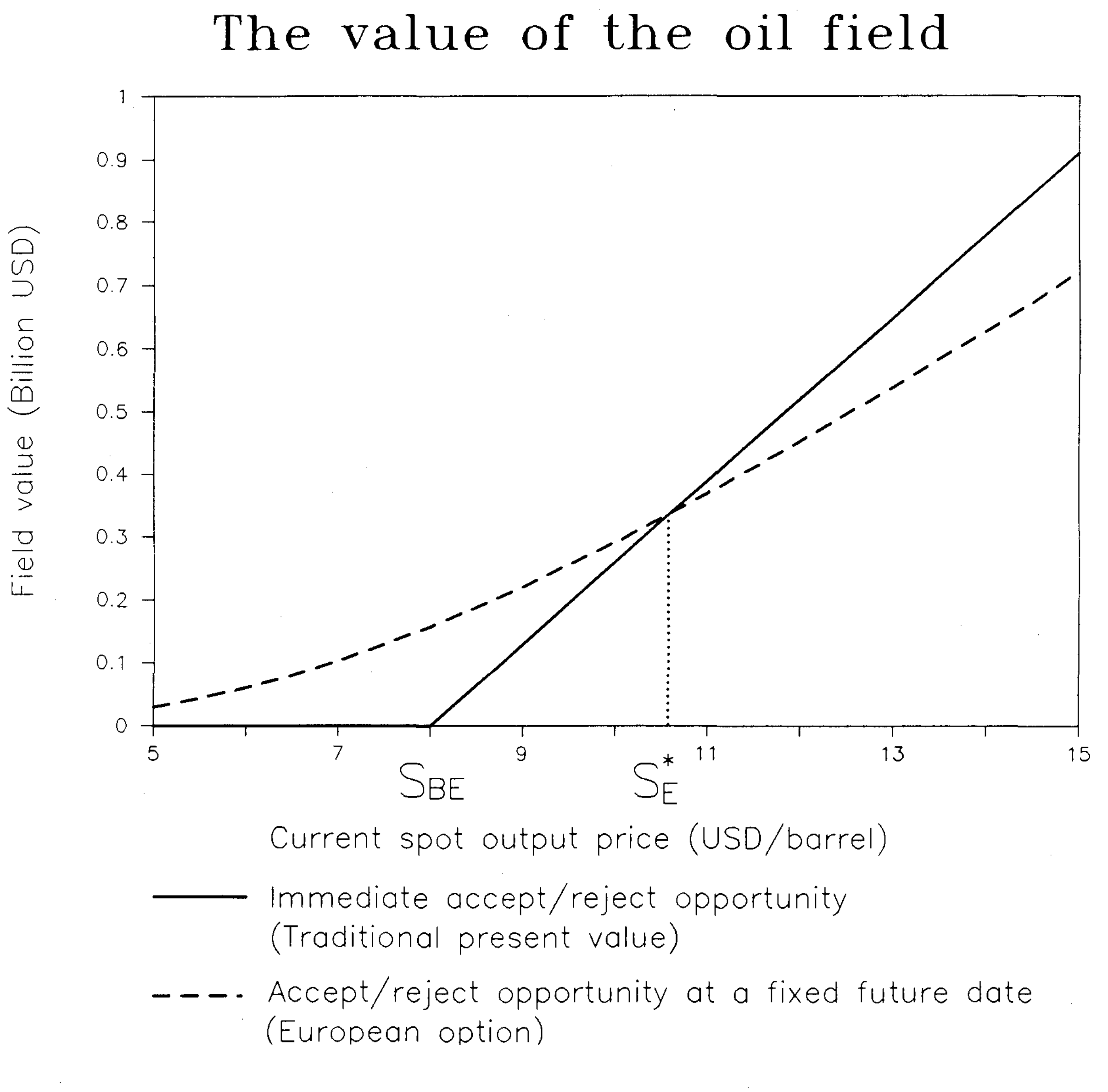
In the Black-Scholes case of a European call option written on a stock paying no dividends, the option value increases with the maturity date. Deferring the future accept/reject decision date of the investment opportunity, however, causes an opportunity loss through the rate of return shortfall on output as well as the escalating costs. From Eq. (5.1), we see directly that with $\delta > 0$ and $r - \pi > 0$, W_C converges to zero when T approaches infinity.

The current decision problem may be stated as

$$V(S(t), t) = \max_{T \geq t} W_C(S(t), T) \tag{5.9}$$

where the expression W_C is given by Eqs. (5.1) - (5.3). There exist no closed-form solution to this problem.

Figure 5.1


The value of the oil field
Field value (Billion USD)
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
5
7
9
11
13
15
SBE
S*E
Current spot output price (USD/barrel)
Immediate accept/reject opportunity
(Traditional present value)
Accept/reject opportunity at a fixed future date
(European option)

Appendix A

Comparative statics

In this appendix, we consider the comparative statics of the value of the future accept/reject investment opportunity, W_C. We may alternatively write W_C, presented in Eqs. (5.1) - (5.3), as

$$W_C = e^{-\delta(T-t)} A(\delta) f(S, K, k, \sigma, T)$$

where the function f is the standard Black-Scholes call option pricing formula

$$f(S, K, k, \sigma, T) = SN[d] - e^{-k(T-t)} K N[d - \sigma\sqrt{T-t}]$$

with

$$d \equiv \frac{\ln(S/K) + (k + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

and where we interpret the two arguments K and k as

$$K(\underbrace{\delta}_{+}, \underbrace{r}_{-}) \equiv S_{BE}(t) = B(r)/A(\delta)$$

$$k(\underbrace{\delta}_{-}, \underbrace{r}_{+}, \underbrace{\pi}_{-}) \equiv r - \pi - \delta$$

K and k corresponds to the exercise price and the interest rate in the standard option pricing formula.

The partial derivatives of f are[1]

$$
\begin{aligned}
\partial f/\partial S &= N[d] > 0 \\
\partial f/\partial K &= -e^{-k(T-t)}N[d-\sigma\sqrt{T-t}] < 0 \\
\partial f/\partial k &= (T-t)e^{-k(T-t)}KN[d-\sigma\sqrt{T-t}] > 0 \\
\partial f/\partial \sigma &= \sqrt{T-t}Sn[d] > 0 \\
\partial f/\partial T &= \underbrace{ke^{-k(T-t)}KN[d-\sigma\sqrt{T-t}]}_{\text{Depends on } k} + \underbrace{\frac{\sigma S}{2\sqrt{T-t}}n[d]}_{+}
\end{aligned}
$$

see, e.g., Jarrow and Rudd (1983) Eqs. (9-5.a) - (9-5.e) or Smith (1976) Eqs. (45) - (49).

By using the results above, we find the following partial derivatives of W_C

$$
\begin{aligned}
\frac{\partial W_C}{\partial S} &= Ae^{-\delta(T-t)}\underbrace{\frac{\partial f}{\partial S}}_{+} > 0 \\
\frac{\partial W_C}{\partial \delta} &= -(T-t)W_C + e^{-\delta(T-t)}\underbrace{\frac{\partial A}{\partial \delta}}_{-}f + e^{-\delta(T-t)}A\left(\underbrace{\frac{\partial f}{\partial K}}_{-}\underbrace{\frac{\partial K}{\partial \delta}}_{+} + \underbrace{\frac{\partial f}{\partial k}}_{+}\underbrace{\frac{\partial k}{\partial \delta}}_{-}\right) < 0 \\
\frac{\partial W_C}{\partial r} &= e^{-\delta(T-t)}A\left(\underbrace{\frac{\partial f}{\partial K}}_{-}\underbrace{\frac{\partial K}{\partial r}}_{-} + \underbrace{\frac{\partial f}{\partial k}}_{+}\underbrace{\frac{\partial k}{\partial r}}_{+}\right) > 0
\end{aligned}
$$

[1]The partial derivatives may be obtained by using

$$Sn[d] = e^{-k(T-t)}Kn[d-\sigma\sqrt{T-t}]$$

where $n[\cdot]$ is the standard normal density function. This equation follows from the definition of d and the standard normal density function.

$$\frac{\partial W_C}{\partial \pi} = e^{-\delta(T-t)} A \left(\underbrace{\frac{\partial f}{\partial k}}_{+} \underbrace{\frac{\partial k}{\partial \pi}}_{-} \right) < 0$$

$$\frac{\partial W_C}{\partial \sigma} = e^{-\delta(T-t)} A \underbrace{\frac{\partial f}{\partial \sigma}}_{+} > 0$$

$$\frac{\partial W_C}{\partial T} = -\delta W_C + e^{-\delta(T-t)} A \frac{\partial f}{\partial T}$$

The sign of the partial derivative of $\partial W_C / \partial T$ is not determined. With $\delta > 0$, the first term of the equation just above is clearly negative. The sign of the second term, however, depends on the sign and the value of $\partial f / \partial T$.

Chapter 6

INVESTMENT AT ANY TIME

6.1 Introduction

In the traditional investment decision model presented in Chapter 4, the owner is assumed to face an immediate choice between to accept the project, to reject the project, or to fix - once and for all - the future initiation date. With uncertainty present, however, deferring *the investment decision itself* is superior to fixing the initiation date in advance, as the future decision may take into account new information received in the meantime.

We now apply the contingent claim framework to incorporate the flexibility to defer the investment decision, by interpreting the undeveloped oil field as an American option. In the first section, we consider the case where the investment decision in principle may be deferred perpetually. An analytical solution to the field value and to the optimal investment strategy is presented. In the second section, we proceed to the case where the investment opportunity expires at a fixed future date. Finally, we provide a numerical example using our base case parameter values as input.

6.2 The perpetual investment opportunity

Assume that the investment decision may in principle be deferred perpetually. If deferred, no cash outlays are incurred, but the present value of the future costs $B(t)$ grows exponentially at the rate π. The project is irrevocable once undertaken.

6.2.1 The mathematical description

We interpret the investment opportunity as a contingent claim, with the output spot price S as the only risk source. The value of the perpetual investment opportunity, $U(S,t)$, satisfies the partial differential equation

$$\tfrac{1}{2}\sigma^2 S^2 U_{SS} + (r-\delta)SU_S - rU + U_t = 0 \tag{6.1}$$

where the subscripts indicate partial derivatives.[1]

We now turn to the boundary conditions. First, at any date t, there exists a trigger price $S^*(t)$ indicating that immediate initiation of development is optimal. When the current spot price equals this trigger price, the value of the investment opportunity is equal to the value of a commitment to initiate immediately. Formally,

$$U(S^*(t),t) = AS^*(t) - B(t) \tag{6.2}$$

Second, given the optimal investment strategy, the "high contact" or "smooth pasting" condition is met.[2] The condition states that when the spot price equals the optimal trigger price, the sensitivity of the option value, and the sensitivity of the intrinsic value, both with respect to the spot price, are equal. In this case the condition translates into

$$U_S(S^*(t),t) = A \tag{6.3}$$

Third, suppose that the current output spot price is zero. We then know from Chapter 2 that the spot price will be zero in the economy for all future dates as well. Initiating the project at date t' will then

[1] Eq. (6.1) is found by inserting $D=0$ in Eq. (2.6).

[2] See Merton (1973), footnote 60.

induce a loss $B(t')$. In this case, it will never be optimal to use the investment opportunity, and we thus have

$$U(0,t) = 0 \tag{6.4}$$

Eqs. (6.1) - (6.4) determine the value of the investment opportunity $U(S,t)$ and the optimal trigger price strategy $S^*(t)$.

The problem stated here corresponds to a perpetual American call option written on a stock that pays a continuous proportional dividend, see Samuelson (1965). McDonald and Siegel (1986) analyze a similar option written on two stochastic assets. If exercised, the holder receives the value of the first underlying asset by giving up the value of the latter.

6.2.2 The solution

The solution to our problem may be found by reinterpreting the parameters in one of the models of the two articles just mentioned. The trigger price, indicating immediate investment, is

$$S^*(t) = \frac{\beta}{\beta - 1} S_{BE}(t) \tag{6.5}$$

where the parameter $\beta > 1$ is defined by Eq. (6.8) below. We see that the trigger price at date t, $S^*(t)$, corresponds to the accept/reject break-even price at that date, $S_{BE}(t)$, adjusted upwards with the factor $\beta/(\beta - 1) > 1$. The optimal decision rule is thus

$$\begin{array}{ll} \text{Defer} & \text{if } S(t) < S^*(t) \\ \text{Accept} & \text{if } S(t) \geq S^*(t) \end{array} \tag{6.6}$$

and thus made conditional on the current spot price.

The value of the investment opportunity is

$$U(S(t),t) = \begin{cases} \alpha(t)S(t)^\beta & \text{if } S(t) < S^*(t) \\ AS(t) - B(t) & \text{if } S(t) \geq S^*(t) \end{cases} \tag{6.7}$$

where the parameters $\beta > 1$ and $\alpha > 0$ are defined by Eqs. (6.8) and (6.9), respectively. As we would expect, the value $U(S,t)$ is positive, and increasing in the current spot price of oil. If the current spot price equals or exceeds the trigger price, immediate investment is optimal, and the value of the investment opportunity is identical to a commitment to start development at once.

The exponent β is defined by

$$\beta \equiv \left(\frac{1}{2} - \frac{(r-\pi)-\delta}{\sigma^2}\right) + \sqrt{\left(\frac{(r-\pi)-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + 2\frac{r-\pi}{\sigma^2}} \tag{6.8}$$

and the constant α is

$$\alpha(t) \equiv \left(\frac{B(t)}{\beta - 1}\right)(S^*(t))^{-\beta} \tag{6.9}$$

The condition of the rate of return shortfall $\delta > 0$ ensures that the exponent $\beta > 1$.[3]

With $\delta \leq 0$, it is never be optimal to initiate investments before the terminal date. [4] The perpetual investment opportunity will thus never be used.[5]

In Appendix A, we show that the solution of $U(S,t)$ satisfies the general partial differential equation of a contingent claim, Eq. (6.1). It is easy to verify that the three boundary conditions are met.

6.2.3 The flexibility factor ϕ

The optimal investment strategy according to the contingent claim model is stated in terms of a break-even decision rule. From Eq. (6.5), we see that the trigger price $S^*(t)$ may be written as the accept/reject break-even price $S_{BE}(t)$, adjusted upwards with a flexibility factor. This factor ϕ, defined by

$$\phi \equiv S^*(t)/S_{BE}(t) = \frac{\beta}{\beta - 1} \tag{6.10}$$

---

[3]See Appendix B.

[4]It is well known from the financial literature that premature exercise of an American call option written on a stock paying no dividends is non-optimal, see, e.g., Ingersoll (1987) p. 305.

[5]This is consistent with what we find in the optimal timing model, see footnote 2 in Chapter 4.

indicates how much the accept/reject break-even price must be adjusted because of the flexibility to make the investment decision at any time.

From Eq. (6.8) we see that time does not enter into the expression of β. We may thus conclude that the flexibility factor ϕ is independent of t. In Appendix C, we show that we have

$$\phi = \phi(\underbrace{\delta}_{-}, \underbrace{r}_{+}, \underbrace{\pi}_{-}, \underbrace{\sigma^2}_{+})$$

where the sign below each argument indicates the sign of the partial derivative wrt. each argument, respectively.

We see that the flexibility factor ϕ depends on economy-wide parameters only: δ, r, π, and σ^2. It does not depend on project specific characteristics, such as the production schedule or the cost schedule.

6.3 The finite horizon investment opportunity

In many cases, the investment opportunity is not perpetual, but expires at some future date if not used before this date. One example may be that licence requires the oil company to return the oil field back to the owner, say the Government, at some fixed future date if investments has not been initiated before that date. Another example may be that the oil reserve represents a sattelite field, where development is dependent on a neighboring field being in production.

In this section, we consider the finite horizon investment opportunity. First, we use our previous results to obtain bounds to the value of the undeveloped oil field. Second, we describe the problem in terms of the PDE with appropriate boundary conditions. Unfortunately, no analytical solution to this problem is known. Several numerical methods are, however, available to approximate the value for a given set of parameters.[6]

Now, suppose that the investment decision may be deferred, but that the investment opportunity expires at the fixed future date T'. If the investment decision is deferred, no costs are incurred, but the

[6] For a survey of numerical methods, see Geske and Shastri (1985).

present value of the future costs $B(t)$ escalates at the exponential rate π. The project is irrevocable once undertaken.

6.3.1 Upper and lower bounds

It is possible for the holder of the finite investment opportunity to undertake an immediate commitment to initiate at date $T \in [t, T']$. Alternatively, he may reject the project. Fixing such a strategy, however, is not optimal and we may thus conclude that the value of the finite investment opportunity $V(S(t), t)$ is bounded from below by

$$\max\left\{\max_{T\in[t,T']} C(S(t), T), 0\right\} \leq V(S(t), t)$$

where $C(S(t), T)$ is defined by Eq. (4.5).

On the other hand, the future cash flow from the finite investment opportunity may be replicated by acquiring a similar non-expiring investment opportunity, and by following a non-optimal strategy. The value of the finite investment opportunity is thus bounded from above by $U(S(t), t)$.

Hence, we argue that the current value of the finite investment opportunity $V(S(t), t)$ is bounded by

$$\max\left\{\max_{T\in[t,T']} C(S(t), T), 0\right\} \leq V(S(t), t) \leq U(S(t), t) \tag{6.11}$$

where $C(S(t), T)$ and $U(S(t), t)$ are given by Eq. (4.5) and Eq. (6.7), respectively.

We note that the lower bound is non-negative. Furthermore, if the current output price $S(t) \geq S^*(t)$, defined by Eq. (6.5) above, we know from Eq. (6.7) that the value of the perpetual investment opportunity is equal to the value of initiating immediately. In this case, the lower and the upper bound of $V(S(t), t)$ coincide, and we thus have

$$V(S(t), t) = AS(t) - B(t) \text{ when } S(t) \geq S^*(t)$$

This means that the trigger price in the finite case is bounded from above by $S^*(t)$.

6.3.2 The mathematical description

The value of the finite horizon investment opportunity V satisfies the general partial differential equation of a contingent claim[7]

$$\tfrac{1}{2}\sigma^2 S^2 V_{SS} + (r - \delta) S V_S - rV + V_t = 0 \tag{6.12}$$

We next turn to the boundary conditions.

First, if the investment has not been initiated before the expiration date T', the holder faces an accept/reject decision situation at this future date. The value of the investment opportunity at the future date T' is thus

$$V(S(T'), T') = \max\{AS(T') - B(T'), 0\} \tag{6.13}$$

Second, at each date t, there exists a trigger price $S_F^*(t)$, indicating that immediate initiation is optimal. When the current spot output price equals this trigger price, we have

$$V(S_F^*(t), t) = AS_F^*(t) - B(t) \tag{6.14}$$

It can be seen from the first boundary condition, Eq. (6.13), that the trigger price at the terminal date T' is

$$S_F^*(T') = B(T')/A = e^{(r-\pi)(T'-t)} S_{BE}(t)$$

The trigger price at the expiration T date is identical to the break-even price of an opportunity to make an accept/reject investment decision at the future date T.

Third, from the previous section, we know from the boundary conditions of the perpetual investment opportunity that $U(0, t) = 0$. This represents an upper bound to $V(0, t)$. Moreover, we know that V in general is non-negative. We may thus conclude that

$$V(0, t) = 0 \tag{6.15}$$

Eqs. (6.12) - (6.15) define the value of the finite investment opportunity $V(S, t)$ and the optimal trigger price function $S_F^*(t)$. The finite

[7]Note that Eqs. (6.12) and (6.1) are similar.

investment opportunity is similar to an American call option with time to maturity $T' - t$, written on a stock that pays a positive continuous constant dividend rate.

There is no known analytical solution available in this case. The value of this contingent claim may be approximated numerically for a given set of parameter values. For a survey of numerical methods, see Geske and Shastri (1985).

The binomial method may be phrased in a decision tree framework, and represents the most intuitive approach, see, e.g., Ekern (1988). Boyle (1977) develops a Monte Carlo simulation method for obtaining the option value, whereas Brennan and Schwartz (1978) describes a finite difference method. Barone-Adesi and Whaley (1987) presents a solution algoritm for evaluating finite American call and put options.

6.4 A numerical example

In this section, we provide a numerical example using our base case parameter values.[8] In addition, assume that the time to the expiration of the finite horizon investment opportunity is $T' - t = 4$ years.

The results are shown in Figure 6.1. The kinked line, with value zero when $S \leq S_{BE}$, and sloping upwards in the region $S \geq S_{BE}$, represents the value of the oil reserve in the accept/reject situation, c.f. Figure 5.1 in the previous chapter. Figure 6.1 in this chapter illustrates that the value of the project , when taking into account the opportunity of deferring the investment decision, is convex in the current spot price.

In the case of a non-expiring investment opportunity, we find that the exponent in Eq. (6.8) is $\beta = 2$ and the constant in Eq. (6.9) is $\alpha = 4\frac{1}{16}$. The flexibility factor is $\phi \equiv S^*/S_{BE} = 2$. By recalling that the accept/reject break-even price is $S_{BE} = 8$ USD/barrel, we thus have that the trigger price of the non-expiring investment opportunity is

$$S^* = 16.0 \text{ USD/barrel}$$

In the case where the investment opportunity expires in 4 years, the

[8]We recall from Section 3.4 that $\delta = 0.06$, $r = 0.05$, $\pi = 0$, $\sigma^2 = 0.07$, $A = 130$, and $B = 1040$. The set of parameter values implies $S_{BE} = 8$ USD/barrel.

trigger price is[9]

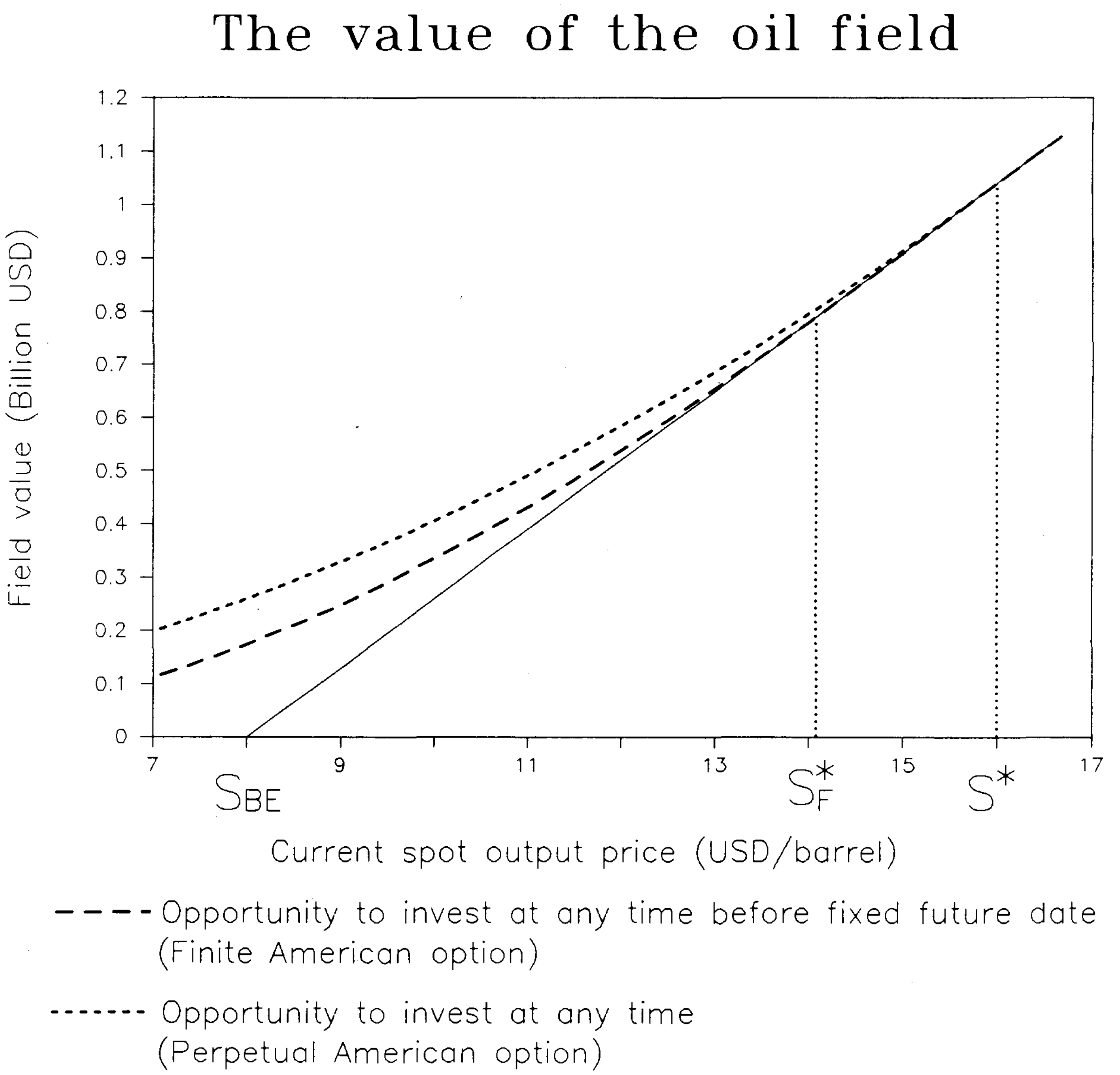
$$S_F^* = 14.1 \text{ USD/barrel}$$

Both S^* and S_F^* exceed the traditional break-even price S_{BE} by a large fraction.

Consider the vertical difference between the upper curve and the kinked line in Figure 6.1. This difference represents the opportunity loss to the owner of a perpetual investment opportunity from managing the project according to the traditional break-even price rule, developing immediately if $S \geq S_{BE}$, and abandoning the oil reserve if $S < S_{BE}$. The opportunity loss is at its maximum when $S = S_{BE}$, with $U(S_{BE}) =$ 260 mill. USD. Our example indicates that the opportunity loss can be substantial when the flexibility to defer the investment decision is ignored.

[9] We implement the explicit finite difference method to approximate the value of the oil reserve and the trigger price. See Brennan and Schwartz (1978) for a discussion of this method.

Figure 6.1


The value of the oil field
Field value (Billion USD)
1.2
1.1
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
7
9
11
13
15
17
SBE
SF*
S*
Current spot output price (USD/barrel)
Opportunity to invest at any time before fixed future date (Finite American option)
Opportunity to invest at any time (Perpetual American option)

Appendix A

U satisfies the PDE

In this appendix, we show that the value of the perpetual investment opportunity when "alive", presented in Section 6.2, satisfies the general partial differential equation of a contingent claim. We recall from Eqs. (6.7), (6.9), and (6.5) that the function is given by

$$U(S,t) = \alpha(t)S^{\beta} \text{ when } S(t) < S^{*}(t)$$

where

$$\alpha(t) \equiv \left(\frac{B(t)}{\beta - 1}\right)(S^{*}(t))^{-\beta}$$

$$S^{*}(t) = \frac{\beta}{\beta - 1} S_{BE}(t)$$

It can be verified that the partial derivatives of $U(S,t)$ are

$$\begin{aligned} U_S &= \beta U S^{-1} \\ U_{SS} &= \beta(\beta - 1) U S^{-2} \\ U_t &= -\pi(\beta - 1) U \end{aligned}$$

By inserting the partial derivatives above into the general partial differential equation, given by

$$\tfrac{1}{2}\sigma^2 S^2 U_{SS} + (r - \delta) S U_S - rU + U_t = 0$$

we obtain the equivalent expression

$$\left(\tfrac{1}{2}\sigma^2\beta^2 + (r - \pi - \delta - \tfrac{1}{2}\sigma^2)\beta - (r - \pi)\right) U = 0 \qquad \text{(A.1)}$$

The solution $U = 0$ is not of interest here. The first factor on the left hand side represents a square expression in β. It can be verified that

$$\beta = \left(\frac{1}{2} - \frac{(r-\pi)-\delta}{\sigma^2}\right) + \sqrt{\left(\frac{(r-\pi)-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + 2\frac{r-\pi}{\sigma^2}}$$

stated in Eq. (6.8), is a solution to Eq. (A.1). We may thus conclude that the solution presented in Section 6.2 satisfies the general partial differential equation of a contingent claim.

Appendix B

$\delta > 0$ a sufficient condition

We now show the condition of the rate of return shortfall $\delta > 0$ is sufficient to ensure that we have the exponent $\beta > 1$. We define for the moment the two parameters

$$a \equiv \frac{r - \pi}{\sigma^2}$$

and

$$b \equiv \frac{(r - \pi) - \delta}{\sigma^2}$$

The condition $\delta > 0$ then translates into

$$a > b$$

Using the two parameters defined above, the exponent β may be written

$$\beta = \left(\tfrac{1}{2} - b\right) + \sqrt{\left(b - \tfrac{1}{2}\right)^2 + 2a}$$

The first term inside the square root is clearly positive. We now substitute a for b in the expression above. Recalling that $a > b$, we then have

$$\beta > \left(\tfrac{1}{2} - b\right) + \sqrt{\left(b - \tfrac{1}{2}\right)^2 + 2b}$$

with strict inequality. The terms inside the square root represents a complete square expression in b. By rearranging, we obtain

$$\beta > \left(\tfrac{1}{2} - b\right) + \sqrt{\left(b + \tfrac{1}{2}\right)^2}$$

The right hand side of this inequality, representing a strict lower bound to the exponent β, equals one. We may thus conclude that a positive "rate of return shortfall"

$$\delta > 0$$

is a sufficient condition to ensure that the exponent $\beta > 1$.

Appendix C

Comparative statics of ϕ

The exponent β is a function of δ, r,π, and σ^2. For notational convenience, we define the parameter

$$\zeta \equiv r - \pi$$

and write the exponent β

$$\beta = \left(\frac{1}{2} - \frac{\zeta - \delta}{\sigma^2}\right) + \sqrt{\left(\frac{\zeta - \delta}{\sigma^2} - \frac{1}{2}\right)^2 + 2\frac{\zeta}{\sigma^2}}$$

We recall from the previous appendix that the condition $\delta > 0$ ensures $\beta > 1$.

Proposition 1 *The sign of the partial derivative of β w.r.t. the "rate of return shortfall" δ is*

$$\frac{\partial \beta}{\partial \delta} > 0$$

Proof: The partial derivative in this case is

$$\frac{\partial \beta}{\partial \delta} = \frac{\beta}{\sigma^2 \sqrt{\left(\frac{\zeta - \delta}{\sigma^2} - \frac{1}{2}\right)^2 + 2\frac{\zeta}{\sigma^2}}} \tag{C.1}$$

Knowing that $\beta > 1$, the proposition is easily verified. □

Proposition 2 *The signs of the partial derivative of β wrt. the interest rate r, and the cost escalation rate π, are*

$$\frac{\partial\beta}{\partial r} < 0$$

and

$$\frac{\partial\beta}{\partial\pi} > 0$$

respectively.

Proof: The partial derivative of β wrt. the parameter ζ is

$$\frac{\partial\beta}{\partial\zeta} = \frac{1-\beta}{\sigma\sqrt{\left(\frac{\zeta-\delta}{\sigma^2}-\frac{1}{2}\right)^2+2\frac{\zeta}{\sigma^2}}} \tag{C.2}$$

With $\beta > 1$, we have

$$\frac{\partial\beta}{\partial\zeta} < 0$$

The parameter is defined by $\zeta \equiv r - \pi$, and the sign of the partial derivative of β with respect to the interest rate r, and the cost escalation rate π, are thus

$$\frac{\partial\beta}{\partial r} < 0$$

and

$$\frac{\partial\beta}{\partial\pi} > 0$$

respectively. □

Proposition 3 *The sign of the partial derivative of β wrt. the volatility σ^2 is*

$$\frac{\partial\beta}{\partial(\sigma^2)} < 0$$

Proof: In this case, we obtain the partial derivative

$$\frac{\partial\beta}{\partial(\sigma^2)} = \frac{(\zeta-\delta)\beta-\zeta}{(\sigma^2)^2\sqrt{\left(\frac{\zeta-\delta}{\sigma^2}-\frac{1}{2}\right)^2+2\frac{\zeta}{\sigma^2}}} \tag{C.3}$$

The sign of the partial derivative depends on the sign of the nominator. We may alternatively express the nominator n by

$$n \equiv (\zeta - \delta)\beta - \zeta = (\beta - 1)\zeta - \beta\delta$$

For convenience, the problem is divided into several regions, depending on the relationship between the two parameters δ and ζ.

Region 1: $\zeta < 0 < \delta \Rightarrow n = \underbrace{(\beta - 1)}_{+}\underbrace{\zeta}_{-} - \underbrace{\beta\delta}_{+} < 0$

Region 2: $\zeta = 0 < \delta \Rightarrow n = \underbrace{(\beta - 1)}_{+}\underbrace{\zeta}_{0} - \underbrace{\beta\delta}_{+} < 0$

Region 3: $0 < \zeta < \delta \Rightarrow n = \underbrace{(\zeta - \delta)}_{-}\underbrace{\beta}_{+} - \underbrace{\zeta}_{+} < 0$

Region 4: $0 < \delta = \zeta \Rightarrow n = \underbrace{(\zeta - \delta)}_{0}\underbrace{\beta}_{+} - \underbrace{\zeta}_{+} < 0$

Region 5: $0 < \delta < \zeta$
We start with the condition of a positive "rate of return shortfall"

$$\delta > 0$$

and subtract $\zeta - \delta$ on each side

$$\zeta > \zeta - \delta$$

By multiplying through with the positive factor $\frac{\zeta}{(\zeta-\delta)^2}$, and by simplifying, we obtain

$$\left(\frac{\zeta}{\zeta - \delta}\right)^2 > \left(\frac{\zeta}{\zeta - \delta}\right)$$

We now add the same expression on each side of the inequality

$$\left(\frac{\zeta}{\zeta - \delta}\right)^2 - \left(\frac{\zeta}{\zeta - \delta}\right) + 2\left(\frac{\zeta}{\sigma^2}\right) > 2\left(\frac{\zeta}{\sigma^2}\right)$$

The two last terms on the left hand side may alternatively be written

$$\left(\frac{\zeta}{\zeta-\delta}\right)^2 + 2\left(\frac{\zeta}{\zeta-\delta}\right)\left(\frac{\zeta-\delta}{\sigma^2}-\frac{1}{2}\right) > 2\left(\frac{\zeta}{\sigma^2}\right)$$

To obtain a complete square expression on the left hand side, we add the same expression on each side

$$\left(\frac{\zeta}{\zeta-\delta}\right)^2 + 2\left(\frac{\zeta}{\zeta-\delta}\right)\left(\frac{\zeta-\delta}{\sigma^2}-\frac{1}{2}\right) + \left(\frac{\zeta-\delta}{\sigma^2}-\frac{1}{2}\right)^2 > 2\left(\frac{\zeta}{\sigma^2}\right) + \left(\frac{\zeta-\delta}{\sigma^2}-\frac{1}{2}\right)^2$$

The expression on the left hand side of the inequality is now a complete square expression. Hence, we have

$$\left(\frac{\zeta}{\zeta-\delta} + \left(\frac{\zeta-\delta}{\sigma^2}-\frac{1}{2}\right)\right)^2 > 2\left(\frac{\zeta}{\sigma^2}\right) + \left(\frac{\zeta-\delta}{\sigma^2}-\frac{1}{2}\right)^2$$

The right hand side is clearly positive. By taking the square root on each side, and by some rearranging, we obtain

$$\frac{\zeta}{\zeta-\delta} > \left(\frac{1}{2}-\frac{\zeta-\delta}{\sigma^2}\right) + \sqrt{\left(\frac{\zeta-\delta}{\sigma^2}-\frac{1}{2}\right)^2 + 2\frac{\zeta}{\sigma^2}}$$

The expression on the right hand side is identical to our exponent β, and we thus have established that

$$\frac{\zeta}{\zeta-\delta} > \beta$$

By rearranging the terms, we have the result

$$n = (\zeta-\delta)\beta - \zeta < 0$$

In all regions, we have found that the nominator n is negative. We may thus conclude that the partial derivative of the exponent β with respect to the volatility σ^2 is negative. That is,

$$\frac{\partial\beta}{\partial(\sigma^2)} < 0$$

□

Proposition 4 *The comparative statics of the flexibility factor ϕ are*

$$\phi = \phi(\underbrace{\delta}_{-}, \underbrace{r}_{+}, \underbrace{\pi}_{-}, \underbrace{\sigma^2}_{+})$$

where the sign below each argument indicates the sign of the partial derivative.

Proof:
The flexibility factor ϕ is given by

$$\phi = \frac{\beta}{\beta - 1}$$

and the partial derivative of ϕ with respect to the exponent β is

$$\frac{\partial \phi}{\partial \beta} = -\frac{1}{(\beta - 1)^2} < 0 \qquad \text{(C.4)}$$

The partial derivative of the exponent β, and the partial derivative of the flexibility factor ϕ, with respect to the same argument, have opposite signs. By using this property, and the results above, the proposition is easily verified. □.

Chapter 7

A PRODUCTION SWITCH

7.1 Introduction

In the previous chapters, we assumed the oil field, once developed, represents a commitment to extract oil according to a fixed preset schedule. If the oil price drops sufficiently low, however, it may be desirable to close down the production temporarily, or even to abandon the entire project.

In this chapter, we take into account the opportunity to switch the production on and off. Balancing realism and simplicity, we tilt our explicit assumptions in favour of the latter, yielding a model with tractable complexity.

The chapter is organized as follows: First, given some additional assumptions, we reconsider the developed oil field in the case of no production switch. Second, we introduce the production switch technology, and present the optimal production policy and the associated field value.

The existence of a production switch flexibility will also affect the investment decision. Third, we analyze a perpetual investment opportunity, where the oil field if developed will be equipped with a production switch. Assuming no time lag between the development decision and start-up of production, we determine the trigger price for investment as well as the option value of the investment opportunity.

Finally, we use the base case parameter values to illustrate the effects of the production switch technology on the optimal strategy and the project value, and compare with our previous results. The numerical example indicates that the error caused by ignoring the switching flexibility is negligible when considering the investment decision.

7.2 The commitment value

In this section, we reconsider the commitment to produce oil according to a fixed preset schedule, given some additional assumptions. Our first additional assumption is that the instantaneous extraction of oil is given and proportional to the remaining quantity of oil in the ground.[1] This is equivalent to writing the production $q(t)$ at date t during a short time interval dt as

$$q(t)dt = \gamma Q(t)dt \tag{7.1}$$

where $Q(t)$ represents the total quantity of oil in the ground at that date, and the parameter $\gamma > 0$ is the given extraction rate. With no flexibility to switch on and off the production, the remaining quantity of oil Q in the field at the future date t' is deterministic, and given by the function

$$Q(t') = e^{-\gamma(t'-t)}Q(t) \tag{7.2}$$

We see that Q is an exponentially declining function of calendar time.

The second additional assumption is that the project is perpetual. At first sight, this may seem like a restrictive assumption. With a reasonable high rate γ, however, the resource will be extracted quite rapidly, and the model outlined in this chapter may thus serve as a fairly good approximation.

The third additional assumption is that the unit variable cost of production, c, is independent of both time and production. Furthermore, there are no fixed production costs.

From Chapter 3, we know that the current value of the commitment to extract oil is linear in the current spot output price $S(t)$. With

[1] Paddock, Siegel, and Smith (1988) notes that this is a standard assumption in the literature on petroleum extraction and reflects geological constraints on extraction.

Brennan and Schwartz (1985) considers switching flexibility in the case of a copper mine. In their specific model, the authors assume that the produced quantity (if any) per time unit is constant, and that the total quantity of copper is infinite.

our additional assumptions, it is easy to verify that the value of the developed oil field is

$$C(S(t), t) = AS(t) - B_p \tag{7.3}$$

where the time-adjusted quantity of oil A is

$$A = \frac{\gamma}{\delta + \gamma} Q(t) \tag{7.4}$$

and where the present value of the future variable production costs B_p is

$$B_p = \frac{\gamma}{r + \gamma} cQ(t) \tag{7.5}$$

Now, consider a hypothetical accept/reject decision situation, with an immediate choice between undertaking a commitment to produce according to the fixed preset schedule and to abandon the field costlessly. From Eqs. (7.3) - (7.5), we obtain that the break-even price, S_p, indicating indifference between the two alternatives, is

$$S_p = \frac{B_p}{A} = \frac{\delta + \gamma}{r + \gamma} c \tag{7.6}$$

We note that S_p differs from the unit cost of production c whenever $\delta \neq r$.

7.3 The developed oil field

7.3.1 Assumptions

Suppose the developed oil field considered in Section 7.2 is equipped with a production switch technology, so that the production may be costlessly switched on and off at any date. Furthermore, assume that the project in principle is perpetual.

If the field is in production, the extraction rate γ and the unit cost c are both constant and given (see above). If the oil field is temporary shut down, no costs are incurred,[2] and that the total extractable quantity $Q(t)$ remains constant.[3]

[2]It is easy to take into account fixed costs that are determined by calendar time only, and that are independent of whether the field is producing or not.

[3]We do not consider the switching flexibility as a mean to increase the total

7.3.2 The partial differential equation

With the flexibility to switch on and off the production, the developed oil field represents a right, but no obligation, to extract oil. We interpret the field as a contingent claim, with the output spot price as the only source of uncertainty. The value of the developed oil field $H(S,t)$ satisifes the general partial differential equation[4]

$$\tfrac{1}{2}\sigma^2 S^2 H_{SS} + (r-\delta)SH_S - rH + H_t + D = 0 \qquad (7.7)$$

where D represents "cash dividends" to the holder of the contingent claim.

If the oil reserve is not producing, the project is not generating cash, and we have $D = 0$. Furter, with no time horizon, the partial derivative of the field value V wrt. calendar time is zero. The value of the non-producing oil field V thus satisfies the PDE

$$\tfrac{1}{2}\sigma^2 S^2 V_{SS} + (r-\delta)SV_S - rV = 0 \qquad (7.8)$$

If the oil field is producing, the extraction is proportional to the remaining quantity of oil in the ground. With a constant unit production cost, c, the instantaneous net cash revenue from operations at date t is

$$D(S(t),t)dt = \gamma Q(t)\,(S(t)-c)\,dt \qquad (7.9)$$

Eq. (7.9) may be interpreted as the cash dividends received by the holder of the contingent claim.

As there is no time horizon related to this problem, the only way the calendar time affects the value of the producing oil field is through the decline in the future extractable quantity of oil. Our assumption of a constant exponential extraction rate, Eq. (7.2), then implies that the partial derivative of the field value U wrt. calendar time t is

$$U_t = \frac{\partial U(S,Q)}{\partial Q}\frac{\partial Q(t)}{\partial t} = -\gamma Q(t) U_Q \qquad (7.10)$$

where U_Q is the partial derivative of the field value wrt. the total remaining quantity of oil Q.

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extractable quantity of oil, but merely focus on the output price uncertainty.

[4]See the discussion in Chapter 2.

By inserting the expressions of the "net cash revenue" and the time derivative, Eqs. (7.9) and (7.10), into the general partial differential equation, Eq. (7.7) above, we have that the value of the producing oil field U satisfies the PDE

$$\tfrac{1}{2}\sigma^2 S^2 U_{SS} + (r-\delta)SU_S - rU - \gamma Q U_Q + \gamma Q(S-c) = 0 \qquad (7.11)$$

This equation may alternatively be found by inserting our assumptions into Eq. (15) of Brennan and Schwartz (1985).

To obtain the particular solution to our problem, boundary conditions need to be specified. This is the topic to which we now turn.

7.3.3 The boundary conditions

If the spot price of oil approaches infinity, the probability that the production will be switched off within, say ten years, is almost zero. This means that the developed oil field is more or less equivalent to a commitment to produce according to the fixed profile. By using the results from the Section 7.2, our first boundary condition reads

$$\lim_{S\to\infty} U(S) = AS - B_p \qquad (7.12)$$

On the other hand, if the spot price of oil approaches zero, it is unlikely that the production will be switched on in the near future. In this case, the oil field is similar to an abandoned one. The value of the production opportunity will then be close to zero, and we thus have the limit

$$\lim_{S\to 0} V(S) = 0 \qquad (7.13)$$

Somewhere in between these two extreme cases, there exists a critical spot price of oil, for which the holder of the oil field is indifferent between currently producing and not. We then have

$$U(S_p^*) = V(S_p^*) \qquad (7.14)$$

where S_p^* is the critical price indicating indifference.

With investors maximizing value, the critical price, representing a decision variable, is chosen to maximize the value of the oil reserve. This leads to the maximization problem

$$H(S) = G(S, S_p^*) = \max_x G(S,x) = \begin{cases} V(S,x) & \text{when } S \le x \\ U(S,x) & \text{when } S \ge x \end{cases}$$

where the function $H(S)$ is the value of the oil field given the optimal choice of the critical price $x = S_p^*$.[5] The maximization problem, and the optimality conditions presented below, are discussed in more detail in Appendix A.

The first optimality condition of the maximization problem above states that the critical price x is to be chosen so that the partial derivative of the function G wrt. this argument is zero. That is,

$$G_x(S, S_p^*) = \begin{cases} V_x(S, S_p^*) = 0 & \text{when } S \leq x \\ U_x(S, S_p^*) = 0 & \text{when } S \geq x \end{cases} \tag{7.15}$$

With the decision variable x being independent of the spot price of the underlying asset, the optimal critical price S_p^*, is independent of the current spot price S as well.

Suppose the optimal critical price is chosen, $x = S_p^*$, and the spot price of oil equals this critical price, $S = S_p^*$. Then, the second optimality condition states that the value of the field if not producing, and the value of the field if producing, have equal sensitivity wrt. the spot price of oil, S. Formally,

$$U_S(S_p^*, S_p^*) = V_S(S_p^*, S_p^*) \tag{7.16}$$

This is known as the "high contact" condition, see Samuelson (1965) and Merton (1973).

From Eq. (7.14) and Eq. (7.16), we see that at the boundary $S = S_p^*$, both the value of the developed oil field, and the risk exposure of the field value wrt. spot price uncertainty, are independent of whether the oil field is currently producing or not. In that case, U and V thus are "identical" assets, with equal value and equal risk.

7.3.4 The optimal switching strategy and field value

The partial differential equations, Eqs. (7.8) and (7.11), together with the boundary conditions, Eqs. (7.12) - (7.16), represent a mathematical

[5] Both functions $U(S, x)$ and $V(S, x)$ are by assumption continuous and twice differentiable wrt. the two arguments. They are concave wrt. x. The decision variable x is independent of the price S.

description of the value of the developed oil field. This system has an analytical solution, obtained in Appendix B. We now present and comment the solution.

The critical spot price of oil S_p^*, indicating indifference between producing and not producing, is given by

$$S_p^* = \frac{\beta_1 \beta_4}{(\beta_1 - 1)(\beta_4 - 1)} \frac{B_p}{A} \tag{7.17}$$

The optimal switching policy thus consists of a comparision between current spot price of oil, S, and the critical price, S_p^*. We have the following "if - then"-switching strategy:

$$\begin{array}{ll} \text{Do not produce if} & S \leq S_p^* \\ \text{Produce if} & S \geq S_p^* \end{array} \tag{7.18}$$

If it is optimal to produce ($S \geq S_p^*$), the value of the developed oil field U is

$$U(S) = \alpha_7 S^{\beta_4} + AS - B_p \tag{7.19}$$

where α_7 and β_4 are defined below. We recognize the two last terms as the value of the corresponding commitment produce according to the fixed preset schedule. The first term in this equation then represents the value of the flexibility to switch on and off the production in the future as the spot price of oil changes. If the current spot price is just above the critical price, the probability that the option to switch off will be used within, say 2 years, is higher than if the current spot price is high. We thus expect the pure option value to be a decreasing function of current spot price, which indeed is the case.

If it is optimal not to produce ($S \leq S_p^*$), the value of the developed oil field V is

$$V(S) = \alpha_1 S^{\beta_1} \tag{7.20}$$

where α_1 and β_1 are defined below. The value of the oil field when not producing, V, reflects the value of the flexibility, at any future date, to turn the non-producing oil field into a producing one that is equipped with a production switch. The probability that this "swithing on"-option will be used is higher the closer the current price of oil is to the critical price S_p^*. Hence, V is an increasing function of current spot price.

The exponents contained in the equations above, are defined by

$$\beta_1 = \left(\frac{1}{2} - \frac{r-\delta}{\sigma^2}\right) + \sqrt{\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + 2\frac{r}{\sigma^2}} \tag{7.21}$$

$$\beta_4 = \left(\frac{1}{2} - \frac{r-\delta}{\sigma^2}\right) - \sqrt{\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + 2\frac{r+\gamma}{\sigma^2}} \tag{7.22}$$

By comparing Eq. (7.21) and Eq. (6.8), we see that with the parameter $\pi = 0$ in Eq. (6.8), the two exponents β and β_1 are identical. With our restrictions on the parameter values, we have $\beta_1 > 1$ and $\beta_4 < 0$.[6] The constants are defined by

$$\alpha_1 = \frac{\beta_4 B_p}{(\beta_1 - 1)(\beta_4 - \beta_1)(S_p^*)^{\beta_1}} \tag{7.23}$$

$$\alpha_7 = \frac{\beta_1 B_p}{(\beta_4 - 1)(\beta_4 - \beta_1)(S_p^*)^{\beta_4}} \tag{7.24}$$

and are both positive.

7.4 The investment opportunity

In this section, we proceed to an investment opportunity, where the oil reserve, if developed, will be equipped with a production switch technology as discussed above. We assume that the investment opportunity is perpetual, and that the development of the oil field is instantaneous.[7]

If the investment decision is deferred, the present value of the future investment costs B_i is constant.[8] There are no costs associated with holding this non-developed oil field.

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[6] We recall from Appendix B in Chapter 6 that $\delta > 0$ implies $\beta > 1$. This result includes the special case of $\pi = 0$. Furthermore, we noted just above that $\beta = \beta_1$ when $\pi = 0$. By combining the results, we thus have $\beta_1 > 1$.

We assume both $r > 0$ and $\gamma > 0$, see Section 2.2 and Section 7.2. It is easy to show that $r + \gamma > 0$ implies $\beta_4 < 0$.

[7] In Chapter 8, we consider a similar investment problem where there is a fixed time lag between the initiation date of the project and the first date of production.

[8] The cost escalation rate π is thus zero.

We interpret the undeveloped oil reserve as a contingent claim. With no cash "dividends", constant exercise price, and no time horizon, the value of this asset W satisfies the partial differential equation

$$\frac{1}{2}\sigma^2 S^2 W_{SS} + (r-\delta)SW_S - rW = 0 \tag{7.25}$$

We next turn to the boundary conditions.

The probability that the oil field will be developed within, say ten years, decreases with the spot price of oil. The value of the investment opportunity will then converge towards zero, and we thus have the limit

$$\lim_{S\to 0} W(S) = 0 \tag{7.26}$$

By comparing Eqs. (7.25) and (7.26), with Eqs. (7.8) and (7.13), we see that the partial differential equation and the "low price" boundary condition of the investment opportunity are identical to the corresponding conditions of the developed non-producing oil field. If we for the moment assume that there are no investment costs, the investment opportunity and the developed non-producing oil field are in fact identical assets. In that case, the critical price S_i^*, indicating immediate "investment", is identical to the switching price S_p^*.

From this hypothetical discussion, it is obvious that the presence of investment costs implies that the critical investment price, S_i^*, exceeds the critical switching price, S_p^*. If the current spot price of oil indicates immediate investment, it will also be optimal to switch on the production on the immediately developed oil field.[9] To obtain the critical investment price S_i^*, we must thus compare the value of the investment opportunity W with the value of the corresponding producing developed oil field U. At the trigger price S_i^*, indicating immediate investment, we thus have

$$W(S_i^*) = U(S_i^*) - B_i \tag{7.27}$$

In this case, the "high contact" condition reads

$$W_S(S_i^*) = U_S(S_i^*) \tag{7.28}$$

[9] Recall from the assumptions stated in the beginning of this section that development is instantaneous.

The partial differential equation, Eq. (7.25), and the three boundary conditions, Eqs. (7.26) - (7.28), define the value of the investment opportunity and the corresponding trigger price. The system is solved in Appendix C. Here we present the results.

Unfortunately, there is no closed form solution to the value of the trigger price, S_i^*, indicating that immediate investment is optimal. Appendix C.2 demonstrates that the the critical price S_i^* is unique. Appendix C.1 shows that it is defined implicitly by the equation

$$(\beta_4 - \beta_1)\alpha_7 \cdot (S_i^*)^{\beta_4} - (\beta_1 - 1)AS_i^* + \beta_1(B_i + B_p) = 0 \qquad (7.29)$$

The value of S_i^* is found for given parameter values by using an iterative procedure.

In Appendix C.3, we show that the trigger price S_i^*, indicating that immediate investment is optimal, is bounded by

$$\left(1 + \frac{B_i}{B_p}\right) S_p^* < S_i^* < S^*$$

where S_p^* and S^* are defined by Eqs. (7.17) and (6.5), respectively. By knowing the project parameters A, B_p, and B_i, and the exponents β_1 and β_4, we may thus obtain bounds to S_i^*.

The lower bound may be interpreted as the trigger price of an undeveloped oil field similar to the one considered, except that the investment cost is zero, and that the unit production cost is $((B_i + B_p)/B_p)c$ rather than c. The trigger price indicating "investment" then coincides with the critical price indicating that producing is optimal. The upper bound corresponds to the trigger price of the project considered without the flexibility to switch on and off.

The value of the investment opportunity W is[10]

$$W(S) = \alpha_8 S^{\beta_1} \qquad (7.30)$$

where the exponent β_1 is given by Eq. (7.21) above, and where the positive[11] constant α_8 is defined by

$$\alpha_8 = \frac{(\beta_4 - 1)AS_i^* - \beta_4(B_i + B_p)}{(\beta_4 - \beta_1)(S_i^*)^{\beta_1}} \qquad (7.31)$$

[10]See Appendix C.1.

[11]See Appendix C.4.

We have already noted that the PDE and the "low price" condition of the investment opportunity and the developed non-producing oil field are indentical. It is no surprise, then, that the two formulas, Eqs. (7.20) and (7.30), expressing the value of the two assets, are similar but with different constants.

In Appendix C.5, we analyze the partial derivative of S_i^* wrt. the investment costs B_i, conditional on the total costs $B_i + B_p$ being constant. We find that

$$\left.\frac{dS_i^*}{dB_i}\right|_{B_i+B_p=\overline{B}} > 0 \tag{7.32}$$

From the definition of the traditional accept/reject break-even price S_{BE} as $(B_i + B_p)/A$, it is easy to see that

$$\left.\frac{dS_{BE}}{dB_i}\right|_{B_i+B_p=\overline{B}} = 0 \tag{7.33}$$

Now, consider two projects that are identical, except for the distribution of the total costs $\overline{B}$ into investment costs B_i and production costs B_p. From Eqs. (7.32) and (7.33), we may conclude that even though the two projects have equal break-even prices S_{BE}, the trigger price S_i^* of the one with the highest ratio (B_i/B_p) will exceed the trigger price of the other. In Appendix C.5 it is shown that this translates into the field with the highest ratio having the lowest value. All others equal, we thus prefer a project where the investment costs B_i counts for a small fraction of total costs $\overline{B}$.

We may interpret the ratio B_i/B_p as being inversely related to the degree of the total investment and operating flexibility of the undeveloped oil field. A higher ratio (B_i/B_p) thus means a lower degree of flexibility, and, all others equal, a lower field value.

7.5 A numerical example

In this section, we provide a numerical example using our base case parameter values.[12] Furthermore, suppose that

[12] We recall from Section 3.4 that $\delta = 0.06$, $r = 0.05$, $\pi = 0$, $\sigma^2 = 0.07$, $A = 130$, and $B = 1040$.

Extractable quantity	Q	190	Mill. barrels
Investment costs	B_i	669.5	Mill. USD
Extraction rate	γ	0.13	
Unit production cost	c	2.7	USD/barrel

By inserting the numerical values of γ, δ, r, c, and Q into Eqs. (7.4) and (7.5), we find

Time-adjusted quantity	A	130	Mill. barrels
Production costs	B_p	370.5	Mill. USD

The total costs are then $B = B_i + B_p = 1040$ mill. USD.

Now, consider the developed oil field. The break-even price, indicating indifference between to abandon the project once and for all, and to undertake a commitment never to switch off the production, is $S_p = B_p/A = 2.85$ USD/barrel.

The value of the two exponents in Eqs. (7.21) and (7.22) are $\beta_1 = 2$ and $\beta_4 = -1\frac{5}{7}$. By inserting the numerical values of β_1, β_4, B_p, and A, into Eq. (7.17), we obtain that the trigger price, indicating switching on and off production, is $S_p^* = 3.6$ USD/barrel. S_p^* thus exceeds the unit production cost c.

Table 7.1 shows the value of the developed oil field. The first column contains the current spot price. The next column shows the value of a commitment to extract oil according the fixed preset schedule, see Section 7.2. In this case, the project value is a linear function of the current spot price.

The last column represents the value of the developed oil field with the flexibility to switch production on and off, c.f. Section 7.3. We see that the value of the project is convex in S. With the current spot price close to zero, the field value is also close to zero. In this case, we expect that it will be a long time before switching on production is optimal, and the project is thus similar to an abandoned one. If the current spot price is high, we see from Table 7.1 that the value of the opportunity to extract oil converges to the value of a pure commitment. In this case, the field is expected to be producing for a long time before it will be optimal to switch off production for the first time.

Table 7.2 shows the value of the undeveloped oil field as a function of the current spot output price. The first column represents the accept/reject investment decision situation, see Section 4.2. The break-even price is $S_{BE} \equiv B/A = 8$ USD/barrel.

In the next column, we find the value of a non-expiring investment opportunity, where the oil field, if developed, corresponds to a pure commitment to extract oil, discussed in Section 6.2. We recall from the numerical example of Chapter 6 that the trigger price, indicating immediate development, is $S^* = 16.0$ USD/barrel.

The final column of Table 7.2 contains the value of a non-expiring investment opportunity in the case where the oil field, if developed, will be equipped with a production switch technology. The trigger price, defined by Eq. (7.29), is $S_i^* = 15.8$ USD/barrel.

By comparing the project values of the two last columns in Table 7.2, we may conclude that the additional value of the switching flexibility is negligible. Furthermore, the difference between the two trigger prices, S^* and S_i^*, is marginal. Our numerical results indicate that when analyzing a project with both investment and switching flexibility, and where the investment cost is large relative to production cost, the most important decision flexibility to model is the one related to the investment decision.

Table 7.1: The value of the developed oil field

Current spot price (USD)	Commitment to produce (Mill. USD)	Production switch (Mill. USD)
1	- 240	13
2	- 110	53
3	20	119
4	150	211
5	280	321
6	410	440
7	540	563
8	670	688
9	800	815
10	930	942
11	1060	1070
12	1190	1199
13	1320	1328
14	1450	1457
15	1580	1586
16	1710	1715

Table 7.2: The value of the undeveloped oil field

Current output spot price (USD)	Traditional accept reject (Mill. USD)	Investment flexibility only (Mill. USD)	Both investment and switching flexibility (Mill. USD)
1	0	4	4
2	0	16	16
3	0	37	37
4	0	65	65
5	0	102	102
6	0	146	147
7	0	199	200
8	0	260	261
9	130	329	331
10	260	406	409
11	390	492	494
12	520	585	588
13	650	687	690
14	780	796	801
15	910	914	919
16	1040	1040	1046
17	1170	1170	1175
18	1300	1300	1305
19	1430	1430	1434
20	1560	1560	1564
21	1690	1690	1694
22	1820	1820	1823

Appendix A

The optimality conditions

Consider the following maximization problem

$$G(S, S_p^*) = \max_x G(S, x) = \begin{cases} V(S, x) & \text{when } S \leq x \\ U(S, x) & \text{when } S \geq x \end{cases}$$

where the functions U and V are assumed to be continuous, twice differentiable wrt. both arguments, and concave in their second argument x. The decision variable x is independent of S.

We rewrite the problem above as a constrained maximization problem. Formally, we have the equivalent problem

$$G(S, S_p^*) = \max_{s,x} G(s, x) \text{ subject to } s = S$$

The corresponding Lagrange-function $L(s, x)$ is

$$\max_{s,x} L(s, x) = \begin{cases} \max_{s,x} V(s, x) - \lambda(s - S) & \text{when } s \leq x \\ \max_{s,x} U(s, x) - \lambda(s - S) & \text{when } s \geq x \end{cases}$$

The partial derivatives of the Lagrange function $L(s, x)$ wrt. the two arguments s and x, evaluated in the point $(s = S, x = S_p^*)$, are

$$\frac{\partial L(S, S_p^*)}{\partial s} = V_s(S, S_p^*) - \lambda = 0 \tag{A.1}$$

$$\frac{\partial L(S, S_p^*)}{\partial x} = V_x(S, S_p^*) = 0 \tag{A.2}$$

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in the region $s \leq x$. And in the region $s \geq x$, we have

$$\frac{\partial L(S, S_p^*)}{\partial s} = U_s(S, S_p^*) - \lambda = 0 \tag{A.3}$$

$$\frac{\partial L(S, S_p^*)}{\partial x} = U_x(S, S_p^*) = 0 \tag{A.4}$$

We now evaluate Eq. (A.1) and Eq. (A.3) at the boundary $s = S = S_p^*$. By equating the two left hand sides, and cancelling the shadow price λ, we obtain

$$U_s(S_p^*, S_p^*) = V_s(S_p^*, S_p^*) \tag{A.5}$$

This equation states that in the point $(s = S_p^*, x = S_p^*)$, the two functions have equal sensitivity wrt. S. This is known as the "high contact" condition.[1]

The two remaining equations state that the optimal choice of the decision variable, $x = S_p^*$, is the one at which the partial derivative of the function (U or V) wrt. x is zero. That is,

$$G_x(S, S_p^*) = \begin{cases} V_x(S, S_p^*) = 0 & \text{when } S \leq S_p^* \\ U_x(S, S_p^*) = 0 & \text{when } S \geq S_p^* \end{cases} \tag{A.6}$$

We recall that the decision variable x is independent of S. The optimal value of the decision variable S_p^*, obtained by evaluating Eq. (A.6) in $S = \overline{S}$, is thus valid for all values of S.

[1]See Samuelson (1965) and Merton (1973), footnote 60.

Appendix B

The developed oil field

B.1 The non-producing oil field

First, we consider the developed non-producing oil field. Its value V must satisfy the partial differential equation given by Eq. (7.8). The general solution to Eq. (7.8) is

$$V(S) = \alpha_1 S^{\beta_1} + \alpha_2 S^{\beta_2}$$

where we define the exponents

$$\beta_1 = \left(\frac{1}{2} - \frac{r-\delta}{\sigma^2}\right) + \sqrt{\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + 2\frac{r}{\sigma^2}} \tag{B.1}$$

and

$$\beta_2 = \left(\frac{1}{2} - \frac{r-\delta}{\sigma^2}\right) - \sqrt{\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + 2\frac{r}{\sigma^2}}$$

With $\delta > 0$ and $r > 0$, the two exponents have opposite signs, with $\beta_1 > 1$ and $\beta_2 < 0$. To meet the "low price" boundary condition in Eq. (7.13), stating that the value of the developed non-producing oil field converges towards zero as the spot price of oil approaches zero, we must have the constant, associated with the negative exponent, is

$$\alpha_2 = 0$$

The constant $\alpha_2 = 0$ implies that the value of the developed non-producing oil field $V(S)$ is given by the following power function of S

$$V(S) = \alpha_1 S^{\beta_1} \tag{B.2}$$

where β_1 is determined by Eq. (B.1), and where the constant α_1 is obtained below.

B.2 The producing oil field

The PDE in Eq. (7.11), that the value of the producing oil field U must satisfy, is somewhat more tricky to solve. However, with our assumed constant exponential extraction rate, it is possible to obtain an analytical solution.

The relation between the value of the producing oil field, U, and the value of one barrel of oil "in the ground" in the same field, u, is

$$U(S,Q) = u(S)Q \tag{B.3}$$

We thus have the following relationship between the partial derivatives

$$U_S = Qu_S \tag{B.4}$$

$$U_{SS} = Qu_{SS} \tag{B.5}$$

$$U_Q = u \tag{B.6}$$

By inserting Eqs. (B.3) - (B.6) above into the PDE in Eq. (7.11), cancelling Q, and rearranging, we obtain that the equivalent condition, stated in terms of the value of one unit of oil in the ground u, is

$$\tfrac{1}{2}\sigma^2 S^2 u_{SS} + (r-\delta)Su_S - (r+\gamma)u + \gamma(S-c) = 0 \tag{B.7}$$

see, e.g., Eq. (37) in Brennan and Schwartz (1985).

The general solution of this PDE is

$$u(S) = \alpha_3 S^{\beta_3} + \alpha_4 S^{\beta_4} + \alpha_5 S + \alpha_6 \tag{B.8}$$

where the two exponents β_3 and β_4 are given by

$$\beta_3 = \left(\frac{1}{2} - \frac{r-\delta}{\sigma^2}\right) + \sqrt{\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + 2\frac{r+\gamma}{\sigma^2}}$$

$$\beta_4 = \left(\frac{1}{2} - \frac{r-\delta}{\sigma^2}\right) - \sqrt{\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + 2\frac{r+\gamma}{\sigma^2}} \tag{B.9}$$

With $\delta > 0$, $r > 0$, and $\gamma > 0$, we have $\beta_3 > 1$ and $\beta_4 < 0$.

The boundary condition related to "high oil price", Eq. (7.12), translated into the value of one unit of oil "in the ground", reads

$$\lim_{S\to\infty} u(S) = \frac{A}{Q}S - \frac{B_p}{Q}$$

With the positive sign of the exponent β_3, it is necessarry to have the associated constant

$$\alpha_3 = 0$$

to meet this condition. From comparing the two last terms of Eq. (B.8) with the limit stated just above, we see that the two remaining constants α_5 and α_6 are

$$\alpha_5 = A/Q$$

and

$$\alpha_6 = -B_p/Q$$

We now insert the expressions of the constants α_3, α_5, and α_6 into the general solution of the value of one unit of oil in the ground u, Eq. (B.8). By multiplying through that expression with the total quantity of oil Q, we have that the value of the producing oil field U is given by

$$U(S) = \alpha_7 S^{\beta_4} + AS - B_p \tag{B.10}$$

where we have defined the new constant α_7 as

$$\alpha_7 \equiv \alpha_4 Q$$

The exponent β_4 is defined by Eq. (B.9) above, and the constant α_7 remains to be determined below.

B.3 The critical price S_i^*

So far, we have found that the functional form of the asset value, contingent on that the optimal critical price $x = S_p^*$ is chosen, is

$$G(S, S_p^*) = \begin{cases} V(S, S_p^*) = \alpha_1 S^{\beta_1} & \text{when } S \le S_p^* \\ U(S, S_p^*) = \alpha_7 S^{\beta_4} + AS - B_p & \text{when } S \ge S_p^* \end{cases}$$

where constants β_1, β_4, A, and B_p are determined above. In this subsection, by using the remaining boundary conditions in Subsection 7.3.3 above, the optimal critical price S_p^*, and α_1 and α_7 are found.

The partial derivative of the asset value wrt. the price of the underlying asset S, is in this case

$$G_S(S, S_p^*) \begin{cases} V_S(S, S_p^*) = \alpha_1 \beta_1 S^{\beta_1 - 1} & \text{when } S \leq S_p^* \\ U_S(S, S_p^*) = \alpha_7 \beta_4 S^{\beta_4 - 1} + A & \text{when } S \geq S_p^* \end{cases} \tag{B.11}$$

Provided that the critical price is chosen optimally, $x = S_p^*$, and that the price of the underlying asset equals this critical price, $S = S_p^*$, we know from Eq. (7.14) that the oil field has the same value whether producing or not producing. In this case, we thus have the condition

$$\alpha_1 \cdot (S_p^*)^{\beta_1} = \alpha_7 \cdot (S_p^*)^{\beta_4} + A S_p^* - B_p \tag{B.12}$$

Furthermore, at this point, the "high contact" condition, Eq. (7.16), stating that the value of the oil field if producing and the value of the oil field if not producing have equal sensitivities wrt. S, is also met. From Eq. (B.11), we see that the condition in this case translates into

$$\alpha_1 \beta_1 \cdot (S_p^*)^{\beta_1 - 1} = \alpha_7 \beta_4 \cdot (S_p^*)^{\beta_4 - 1} + A \tag{B.13}$$

Eqs. (B.12) and (B.13) imply that the α_1 and α_7 are given by

$$\alpha_1 = \frac{(\beta_4 - 1) A S_p^* - \beta_4 B_p}{(\beta_4 - \beta_1)(S_p^*)^{\beta_1}} \tag{B.14}$$

and

$$\alpha_7 = \frac{(\beta_1 - 1) A S_p^* - \beta_1 B_p}{(\beta_4 - \beta_1)(S_p^*)^{\beta_4}} \tag{B.15}$$

We see that both α_1 and α_7 depends on the value of the optimal critical price S_p^*.

The remaining optimality condition, Eq. (7.15), states that the partial derivative of the asset value wrt. the decision variable x, evaluated in the point $(S, x = S_p^*)$, is zero. Formally, we have

$$G_x(S, x = S_p^*) = 0 \tag{B.16}$$

We recall that the decision variable x is independent of S. It is thus sufficient to evaluate this condition with S fixed at an arbitrary chosen value.

Now, consider Eq. (B.16), where we fix $S = S_p^*$, and choose to focus on the non-producing oil field.[1] As S and x are independent, the remaining optimality condition, Eq. (B.16), is equivalent to

$$\frac{\partial \alpha_1(S_p^*, S_p^*)}{\partial x} = 0 \tag{B.17}$$

in this case.

From Eq. (B.14), we see that α_1 is expressed in terms of the critical price S_p^*. The derivative of α_1 along the boundary $S = x$, evaluated in the point $(S = S_p^*, x = S_p^*)$, may be written as

$$\left.\frac{d\alpha_1(S = S_p^*, x = S_p^*)}{dS_p^*}\right|_{S=x} = \frac{\partial \alpha_1(S_p^*, S_p^*)}{\partial S} + \frac{\partial \alpha_1(S_p^*, S_p^*)}{\partial x} \tag{B.18}$$

However, α_1 is independent of S, and the first term on the right hand side in Eq. (B.18) is zero. Moreover, according to the optimality condition, Eq. (B.17), the second term in Eq. (B.18) is zero as well. The optimal critical price S_p^* is thus determined by the condition

$$\frac{\partial \alpha_1}{\partial S_p^*} = 0$$

By obtaining the partial derivative on the left hand side of the equation just above, and by solving the equation with respect to S_p^*, we find that the optimal trigger price, indicating indifference between currently producing and not, S_p^*, is given by

$$S_p^* = \frac{\beta_1 \beta_4}{(\beta_1 - 1)(\beta_4 - 1)} \frac{B_p}{A} \tag{B.19}$$

as reported as Eq. (7.17) in the text. By inserting the solution of the trigger price into α_1 and α_7, we obtain the simplified expressions

$$\alpha_1 = \frac{\beta_4 B_p}{(\beta_1 - 1)(\beta_4 - \beta_1)(S_p^*)^{\beta_1}} \tag{B.20}$$

[1] With the producing oil field as the starting point of the analysis, we will arrive at the identical result by proceeding as below.

$$\alpha_7 = \frac{\beta_1 B_p}{(\beta_4 - 1)(\beta_4 - \beta_1)(S_p^*)^{\beta_4}} \tag{B.21}$$

Appendix C

The investment opportunity

C.1 The solution

We noted in Section 7.4 that the undeveloped oil field and the developed non-producing oil field have identical partial differential equations and "low price" conditions. From our discussion in Appendix B.1, it should then be obvious that the value of the non-developed oil field is given by a similar function described by

$$W(S) = \alpha_8 S^{\beta_1}$$

where exponent β_1 is defined above in Eq. (7.21).

The constant α_8 and the critical oil price S_i^* are determined by the two remaining boundary conditions in Eqs. (7.27) and (7.28). We insert the expressions of W and U, and their corresponding partial derivatives, into the two boundary conditions, respectively, and yield

$$\alpha_8 \cdot (S_i^*)^{\beta_1} = \alpha_7 \cdot (S_i^*)^{\beta_4} + AS_i^* - (B_i + B_p) \qquad \text{(C.1)}$$

$$\alpha_8 \beta_1 \cdot (S_i^*)^{\beta_1 - 1} = \alpha_7 \beta_4 \cdot (S_i^*)^{\beta_4 - 1} + A \qquad \text{(C.2)}$$

where we denote the trigger price indicating immediate investment by S_i^*. By multiplying through Eq. (C.1) with β_4, and through Eq. (C.2) with $-S_i^*$, and adding the left hand sides and the right hand sides, we have

$$(\beta_4 - \beta_1)\, \alpha_8 \cdot (S_i^*)^{\beta_1} = (\beta_4 - 1)\, AS_i^* - \beta_4 (B_i + B_p)$$

We solve this equation with respect to the constant α_8, and obtain

$$\alpha_8 = \frac{(\beta_4 - 1)AS_i^* - \beta_4(B_i + B_p)}{(\beta_4 - \beta_1)(S_i^*)^{\beta_1}} \tag{C.3}$$

By inserting the constant α_8 into the first boundary condition, Eq. (C.1), and rearranging, we have that the trigger price S_i^* is defined by the equation

$$(\beta_4 - \beta_1)\alpha_7 \cdot (S_i^*)^{\beta_4} - (\beta_1 - 1)AS_i^* + \beta_1(B_i + B_p) = 0 \tag{C.4}$$

There is no closed form expression of this critical price.

C.2 The critical price S_i^* is unique

In the hypothetical case without investment costs, the undeveloped and the developed non-producing oil field are identical assets. Thus, with investment costs present, the trigger price S_i^*, indicating immediate investment, strictly exceeds the "switching price" S_p^*. This is equivalent to stating that the ratio

$$\lambda_i^* \equiv \frac{S_i^*}{S_p^*} > 1$$

By inserting the definition of α_7, Eq. (7.24), into Eq. (C.4), and some rearranging, we find that the condition determining the trigger price S_i^* equivalently may be written

$$\frac{1}{\beta_4 - 1}\left(\frac{S_i^*}{S_p^*}\right)^{\beta_4} - \frac{\beta_4}{\beta_4 - 1}\left(\frac{S_i^*}{S_p^*}\right) + \left(1 + \frac{B_i}{B_p}\right) = 0$$

The condition just above may alternatively be stated in terms of the function

$$H(\lambda) \equiv \frac{1}{\beta_4 - 1}\lambda^{\beta_4} - \frac{\beta_4}{\beta_4 - 1}\lambda + \left(1 + \frac{B_i}{B_p}\right) \tag{C.5}$$

where $\lambda > 1$, and where the trigger price $S_i^* = \lambda_i^* S_p^*$ is given by

$$H(\lambda_i^*) = 0$$

We see from Eq. (C.5) that $H(\lambda)$ is continuous wrt. λ. The partial derivative of $H(\lambda)$ wrt. λ is

$$\frac{\partial H(\lambda)}{\partial \lambda} = \frac{\beta_4}{\beta_4 - 1}\left(\lambda^{\beta_4 - 1} - 1\right) < 0 \tag{C.6}$$

and $H(\lambda)$ is thus a decreasing function of λ. Furthermore, it is easy to verify that we have the two limits

$$\begin{aligned} \lim_{\lambda \to 1} H(\lambda) &= \frac{B_i}{B_p} > 0 \\ \lim_{\lambda \to \infty} H(\lambda) &= -\infty \end{aligned}$$

To sum up, we have found that $H(\lambda)$ is a continuous and decreasing function of λ in the region $\lambda > 1$. Its value is positive when the argument is "small", and negative when the argument is "large". We may thus conclude that λ_i^*, defined by

$$H(\lambda_i^*) = 0$$

is unique. Concequently, we have established that the trigger price $S_i^* = \lambda_i^* S_p^*$ is unique.

C.3 A lower and an upper bound to S_i^*

Suppose for the moment that the trigger price is

$$S_L = \left(1 + \frac{B_i}{B_p}\right) S_p^*$$

This corresponds to the ratio

$$\lambda_L = \left(1 + \frac{B_i}{B_p}\right)$$

By inserting λ_L into Eq. (C.5), and rearranging, we obtain

$$H(\lambda_L) = -\frac{1}{\beta_4 - 1}\left\{\left(1 + \frac{B_i}{B_p}\right) - \left(1 + \frac{B_i}{B_p}\right)^{\beta_4}\right\}$$

With positive investment costs B_i, the expression inside the curly brackets exceeds one. Recalling that the exponent β_4 is negative, the expression outside the curly bracets is negative. We thus have that

$$H(\lambda_L) > 0$$

The function $H(\lambda)$ is decreasing in λ, and λ_L thus represents a lower bound to the "true" ratio, λ_i^*. From this discussion we may conclude that

$$S_L = \left(1 + \frac{B_i}{B_p}\right) S_p^* < S_i^* \tag{C.7}$$

Now, suppose instead that the trigger price is

$$S_U = S^*$$

This translates into the ratio

$$\lambda_U = \frac{\beta_4 - 1}{\beta_4}\left(1 + \frac{B_i}{B_p}\right)$$

By inserting the expression of λ_U into the function $H(\lambda)$, given by Eq. (C.5) above, and rearranging, we have

$$H(\lambda_U) = \frac{1}{\beta_4 - 1}\left(\frac{\beta_4 - 1}{\beta_4}\right)^{\beta_4}\left(1 + \frac{B_i}{B_p}\right)$$

It is easy to see that

$$H(\lambda_U) < 0$$

and we conclude that

$$S_i^* < S^* = S_U \tag{C.8}$$

Collecting our results, stated in Eqs. (C.7) and (C.8), we have found that the trigger price S_i^* is bounded by

$$\left(1 + \frac{B_i}{B_p}\right) S_p^* < S_i^* < S^* \tag{C.9}$$

where S_p^* and S^* are given by Eqs. (7.17) and (6.5), respectively.

C.4 The constant α_8 is positive

By Eq. (7.31) above, the constant α_8 is given as

$$\alpha_8 = \frac{(\beta_4 - 1)AS_i^* - \beta_4(B_i + B_p)}{(\beta_4 - \beta_1)(S_i^*)^{\beta_1}}$$

Some manipulations lead to the equivalent expression

$$\alpha_8 = \frac{(\beta_4 - 1)A}{(\beta_4 - \beta_1)(S_i^*)^{\beta_1}} \left\{ S_i^* - \frac{\beta_1 - 1}{\beta_1} \left(1 + \frac{B_i}{B_p}\right) S_p^* \right\}$$

The factor outside the curly brackets is positive. As $\beta_1 > 1$, we have

$$0 < \frac{\beta_1 - 1}{\beta_1} < 1$$

Furthermore,

$$S_i^* > \left(1 + \frac{B_i}{B_p}\right) S_p^*$$

see Eq. (C.9). Thus, the expression inside the curly brackets is positive, and we have established that $\alpha_8 > 0$.

C.5 The effect of a change in cost mix

In this subsection, we consider the concequence on the trigger price S_i^*, and on the value of the undeveloped oil field W, of a change in the investment costs B_i, conditional on the total costs $\overline{B}$ being constant. Formally, we will find

$$\left.\frac{dS_i^*}{dB_i}\right|_{B_i+B_p=\overline{B}} = \left.\frac{d(\lambda_i^* S_p^*)}{dB_i}\right|_{B_i+B_p=\overline{B}}$$

This expression is equivalent to

$$\left.\frac{dS_i^*}{dB_i}\right|_{B_i+B_p=\overline{B}} = \left(\frac{\partial \lambda_i^*}{\partial B_i} S_p^* + \lambda_i^* \frac{\partial S_p^*}{\partial B_i}\right) - \left(\frac{\partial \lambda_i^*}{\partial B_p} S_p^* + \lambda_i^* \frac{\partial S_p^*}{\partial B_p}\right) \quad \text{(C.10)}$$

From Eq. (7.17), we easily find the partial derivatives

$$\frac{\partial S_p^*}{\partial B_i} = 0 \tag{C.11}$$

$$\frac{\partial S_p^*}{\partial B_p} = \frac{S_p^*}{B_p} \tag{C.12}$$

The ratio λ_i^* is determined by $H(\lambda_i^*) = 0$ where the function is given in Eq. (C.5) above. By using this relationship, we obtain

$$\frac{\partial \lambda_i^*}{\partial B_i} = -\frac{\partial H(\lambda_i^*)/\partial B_i}{\partial H(\lambda_i^*)/\partial \lambda} = -\frac{1}{B_p}\frac{1}{\partial H(\lambda_i^*)/\partial \lambda} \tag{C.13}$$

$$\frac{\partial \lambda_i^*}{\partial B_p} = -\frac{\partial H(\lambda_i^*)/\partial B_p}{\partial H(\lambda_i^*)/\partial \lambda} = -\frac{B_i}{(B_p)^2}\frac{1}{\partial H(\lambda_i^*)/\partial \lambda} \tag{C.14}$$

By inserting Eqs. (C.11) - (C.14) into Eq. (C.10), some rearranging yields

$$\left.\frac{dS_i^*}{dB_i}\right|_{B_i+B_p=\overline{B}} = -\frac{S_p^*}{(B_p)^2(\partial H(\lambda_i^*)/\partial \lambda)}\left\{B_p + B_i + B_p\lambda_i^*\frac{\partial H(\lambda_i^*)}{\partial \lambda}\right\} \tag{C.15}$$

As $\partial H/\partial \lambda < 0$, see Eq. (C.6), the expression outside the curly brackets is positive. Now, consider for the moment the expression inside the curly brackets of Eq. (C.15), here denoted by $\mathcal{C}$. By inserting the expression of the partial derivative $\partial H(\lambda_i^*)/\partial \lambda$, found in Eq. (C.6), into the last term of $\mathcal{C}$, we have

$$\mathcal{C} = B_p + B_i + B_p\frac{\beta_4}{\beta_4 - 1}(\lambda_i^*)^{\beta_4} - B_p\frac{\beta_4}{\beta_4 - 1}\lambda_i^* \tag{C.16}$$

The definition of λ_i^*, $H(\lambda_i^*) = 0$, implies that

$$-B_p\frac{\beta_4}{\beta_4 - 1}\lambda_i^* = -B_p\frac{1}{\beta_4 - 1}(\lambda_i^*)^{\beta_4} - (B_p + B_i) \tag{C.17}$$

By inserting Eq. (C.17) into Eq. (C.16), and simplifying, we find that the expression inside the curly brackets, $\mathcal{C}$, is equivalent to

$$\mathcal{C} = B_p(\lambda_i^*)^{\beta_4} > 0$$

We may thus conclude that

$$\left.\frac{dS_i^*}{dB_i}\right|_{B_i+B_p=\overline{B}} = -\frac{(\lambda_i^*)^{\beta_4} S_p^*}{(\partial H(\lambda_i^*)/\partial\lambda)B_p} > 0$$

Next, consider the effect on the value of the undeveloped oil field, W, when the cost structure changes as above. The field value W is described by Eqs. (7.30), (7.31), and (7.21). We note that the change in cost structure only affects α_8 through the trigger price S_i^*, and we thus have

$$\left.\frac{dW}{dB_i}\right|_{B_i+B_p=\overline{B}} = S^{\beta_1}\frac{\partial\alpha_8}{\partial S_i^*}\left(\left.\frac{dS_i^*}{dB_i}\right|_{B_i+B_p=\overline{B}}\right)$$

The sign of the derivative is thus determined by the midle factor. It is easy to verify that the partial derivative of α_8 wrt. S_i^* may be written

$$\frac{\partial\alpha_8}{\partial S_i^*} = -\frac{(S_i^*)^{-(\beta_1+1)}}{(\beta_1-1)(\beta_4-1)(\beta_4-\beta_1)}\left\{S_i^* - \left(1+\frac{B_i}{B_p}\right)S_p^*\right\} \tag{C.18}$$

The second term inside the curly brackets is identical to the lower bound of S_i^*, see Eq. (C.9). The expression inside the curly brackets is thus positive. With $\beta_1 > 1$ and $\beta_4 < 0$, the three terms in the denominator of the term outside the curly brackets are positive, negative, and negative, respectively. The nominator is positive. By collecting the results, and taking into account the negative sign, we may conclude that

$$\left.\frac{dW}{dB_i}\right|_{B_i+B_p=\overline{B}} < 0$$

Chapter 8

A FREEZE ON INVESTMENTS

8.1 Introduction

The principal contribution of this chapter is the pricing function evaluating a contingent claim with a future pay-off described by a truncated power function of the price of the risky asset at the maturity date. With this result, it is straightforward to evaluate some more complex contingent claims.

The first case to be considered is a perpetual investment opportunity, on which there for some reason is imposed a temporary freeze. We illustrate this model by a numerical example using our base case parameter values. In the second case, we look at a compound option model, where the holder at a fixed future date may acquire a perpetual investment opportunity. Finally, the perpetual investment opportunity in the case of a production switch technology, previously discussed in Chapter 7, is extended to allow for a fixed time lag between the development decision date and the first possible date of production.

8.2 The result

Consider a contingent claim related to the risky asset S, which prior to the future date T pays nothing, and at the future date T pays

$$\eta(S(T) \mid \overline{S}, \varepsilon) \equiv S(T)^{\varepsilon} I(S(T) < \overline{S}) = \begin{cases} S(T)^{\varepsilon} & \text{if} \quad S(T) < \overline{S} \\ 0 & \text{if} \quad S(T) \geq \overline{S} \end{cases} \tag{8.1}$$

where the exponent ε is a constant, $S(T)$ is the price of the risky asset at the future date T, and $I(\cdot)$ is the indicator function.

Theorem 1 *The current value of a claim on the pay-off $S(T)^{\varepsilon}$ at the future date T, conditional on $S(T) < \overline{S}$, is given by the function*

$$\Psi(S, t \mid \overline{S}, \varepsilon) \equiv V_t[S(T)^{\varepsilon} I(S(T) < \overline{S})] = e^{\lambda} S^{\varepsilon} N[-d] \tag{8.2}$$

where $V_t[\cdot]$ is a general evaluator, and where we define

$$\lambda(t \mid \varepsilon) \equiv [(\varepsilon - 1)r - \varepsilon\delta + \tfrac{1}{2}\varepsilon(\varepsilon - 1)\sigma^2](T - t) \tag{8.3}$$

$$d(S, t \mid \overline{S}, \varepsilon) \equiv \frac{\ln(S/\overline{S}) + [r - \delta + (\varepsilon - \frac{1}{2})\sigma^2](T - t)}{\sigma\sqrt{(T - t)}} \tag{8.4}$$

and where S is the current price of the risky asset.

Proof: For $t < T$, the function $\Psi(S, t)$ satisfies the general partial differential equation

$$\tfrac{1}{2}\sigma^2 S^2 \Psi_{SS} + (r - \delta) S \Psi_S - r\Psi + \Psi_t = 0 \tag{8.5}$$

The function $\Psi(S, t)$ has the limiting values

$$\lim_{S \to \infty} \Psi(S, t) = 0 \tag{8.6}$$

$$\lim_{S \to 0} \Psi(S, t) = \begin{cases} \infty & \text{if } \varepsilon < 0 \\ e^{-r(T-t)} & \text{if } \varepsilon = 0 \\ 0 & \text{if } \varepsilon > 0 \end{cases} \tag{8.7}$$

$$\lim_{t \to T} \Psi(S, t) = \begin{cases} S(T)^{\varepsilon} & \text{if } S(T) < \overline{S} \\ \frac{1}{2} S(T)^{\varepsilon} & \text{if } S(T) = \overline{S} \\ 0 & \text{if } S(T) > \overline{S} \end{cases} \tag{8.8}$$

see Appendix A.

The property stated in Eq. (8.5) means there exists a self-financing trading strategy, by which the future pay-off $\Psi(S,T)$ received at date T may be attained in the market today at the price $\Psi(S,t)$. By comparing the desired pay-off η at date T in Eq. (8.1) with the pay-off $\lim_{t\to T}\Psi(S,t)$ in Eq. (8.8), provided by the self-financing strategy associated with the value process Ψ, we see that the two future pay-offs are identical except for the case of $S(T)=\overline{S}$. With our price process, however, the event

$$\left\{S(T): S(T)=\overline{S} \mid S(t)\right\}$$

has zero probability. We thus have that

$$\Pr\left\{S(T): \Psi(S(T),T)=\eta(S(T)) \mid S(t)\right\}=1$$

almost surely.

We may now call upon the theory on pricing on contingent claims, originated in Harrison and Kreps (1979) and Harrison and Pliska (1981). One of the key results from this theory is that "a contingent claim X is said to be *attainable at price* π in our security market model if there exists a self-financing trading strategy ϕ with associated market value process V, such that $V_0=\pi$ and $V_T=X$, almost surely", see p. 220 in the latter article.

We may thus conclude that the contingent claim with future pay-off η is priced by arbitrage by

$$\Psi(S,t \mid \overline{S},\varepsilon)=V_t[\eta(S(T) \mid \overline{S},\varepsilon)]$$

and this completes the proof of Theorem 1. □

We see from Eq. (8.6) that the current value of the claim converges to zero when S approaches infinity. The intuition behind this result is that the probability of η providing a positive pay-off at date T is close to zero when S is large.

In the hypothetical case of $S=0$, we know from the price dynamics discussed in Section 2.3 that the price at the future date T will be $S(T)=0$. This observation implies that the event $S(T)<\overline{S}$ will occur with probability one. The asset thus corresponds to a riskless claim on the future pay-off $\lim_{S\to 0}S^{\varepsilon}$. The current value of the claim is thus found by discounting the risk-free future pay-off at the rate r, leading to Eq. (8.7).

8.2.1 A "pure" claim on the power function

Consider a claim on the future power function pay-off, that is *not* made contingent on the spot price at the maturity date T being lower than some level $\overline{S}$. The future pay-off from this claim may be written

$$\lim_{\overline{S}\to\infty} \eta = S(T)^{\varepsilon} \tag{8.9}$$

see Eq. (8.1). It is easy to verify from Eq. (8.4) that $\overline{S} \to \infty$ implies $d \to -\infty$, and $N[-d] \to 1$. The current value of this "pure" claim on the power function pay-off is

$$V_t[S(T)^{\varepsilon}] = \lim_{\overline{S}\to\infty} \Psi = e^{\lambda} S^{\varepsilon} \tag{8.10}$$

see Eqs. (8.2) - (8.4).

Now, suppose that we instead evaluate the future pay-off in Eq. (8.9) by the traditional RADR method, discounting back the expected future pay-off by the appropriate risk-adjusted discount rate ρ. To be consistent with market values, this method is required to give the same result as in Eq. (8.10). We thus have

$$e^{\lambda} S^{\varepsilon} = V_t[S(T)^{\varepsilon}] = e^{-\rho(T-t)} E_t[S(T)^{\varepsilon}]$$

By inserting the expected future pay-off[1] on the right hand side of this expression, and by using the definition of δ and λ, Eqs. (2.3) and (8.3) respectively, we find that the implicit market RADR ρ is

$$\rho = \varepsilon(\hat{\alpha} - r) + r$$

in this case. Note that the risk premium related to the power function, $\rho - r$, corresponds to the risk premium of S, $\hat{\alpha} - r$, adjusted with the exponent ε.

[1]The expected value of $S(T)^{\varepsilon}$, as viewed from date t, is

$$E_t[S(T)^{\varepsilon}] = e^{[\varepsilon\alpha + \frac{1}{2}\varepsilon(\varepsilon-1)\sigma^2](T-t)} S^{\varepsilon}$$

where S is the spot price at date t. The expression is found by applying Ingersoll (1987) p. 14 to obtain the expected value of the normally distributed future spot price, defined in Eq. (2.2).

8.2.2 A contingent claim on the riskless asset

With $\varepsilon = 0$, η describes a one dollar pay-off conditional on the future price of the risky asset $S(T)$ being lower than $\overline{S}$. The current value of this contingent claim on the riskless asset is

$$\begin{aligned} V_t[I(S(T) < \overline{S})] &= \Psi(S, t \mid \overline{S}, \varepsilon = 0) \\ &= e^{-r(T-t)} N[-d_2] \end{aligned} \tag{8.11}$$

where

$$d_2(\overline{S}) \equiv d(S, t \mid \overline{S}, \varepsilon = 0) = \frac{\ln(S/\overline{S}) + [r - \delta - \frac{1}{2}\sigma^2](T-t)}{\sigma\sqrt{(T-t)}} \tag{8.12}$$

see Eqs. (8.2) - (8.4). By inserting d_1, defined in Eq. (5.2), into Eq. (5.3), and rearranging, we find that d_2 defined in the case of the European option is similar to Eq. (8.12).

The limit

$$\lim_{\overline{S} \to \infty} \Psi(S, t \mid \overline{S}, \varepsilon = 0) = e^{-r(T-t)} \tag{8.13}$$

is the current value of receiving one dollar at date T, compare with Eq. (2.4).

8.2.3 A contingent claim on the risky asset

By inserting $\varepsilon = 1$ into Eq. (8.1), we see that the future pay-off η corresponds to a claim on a delivery of the risky asset if and only if the future price $S(T)$ is lower than $\overline{S}$. Note that $\lambda(t \mid \varepsilon = 1) = -\delta(T-t)$. The current value of this contingent claim on the risky asset is

$$\begin{aligned} V_t[S(T)I(S(T) < \overline{S})] &= \Psi(S, t \mid \overline{S}, \varepsilon = 1) \\ &= e^{-\delta(T-t)} S N[-d_1] \end{aligned} \tag{8.14}$$

where

$$d_1(\overline{S}) \equiv d(S, t \mid \overline{S}, \varepsilon = 1) = \frac{\ln(S/\overline{S}) + [r - \delta + \frac{1}{2}\sigma^2](T-t)}{\sigma\sqrt{(T-t)}} \tag{8.15}$$

Note that d_1 in Eqs. (5.2) and (8.15) are similar.

The limit

$$\lim_{\overline{S}\to\infty} \Psi(S,t \mid \overline{S}, \varepsilon = 1) = e^{-\delta(T-t)}S \tag{8.16}$$

is known as the forward identity. That is, the value at date t of a claim on a delivery of the risky asset at the future date T, see Eq. (2.5).

8.2.4 The European put option

Now, consider a European put option written on S with exercise price $\overline{S}$ and time to maturity $T-t$. With an optimal exercise of the option, the future pay-off from the option is

$$Y(S(T)) = \begin{cases} \overline{S} - S(T) & \text{if} \quad S(T) < \overline{S} \\ 0 & \text{if} \quad S(T) \geq \overline{S} \end{cases} \tag{8.17}$$

Note that the future pay-off from the put option $Y(T)$ may be decomposed into a claim on $\overline{S}$ dollars, and an obligation to deliver one unit of the risky asset, where both claims are made conditional on the future price $S(T)$ being lower than the exercise price $\overline{S}$. The two claims may be expressed by using the function η defined in Eq. (8.1). We thus write the future pay-off from the put option as

$$Y(S(T)) = \overline{S}\eta(S(T) \mid \overline{S}, \varepsilon = 0) - \eta(S(T) \mid \overline{S}, \varepsilon = 1) \tag{8.18}$$

where $\overline{S}$ represents a constant.

By using the evaluator Ψ, some rearranging lead to that the current value of the European put option is

$$W_{PUT} = e^{-r(T-t)}\overline{S}N[-d_2(\overline{S})] - e^{-\delta(T-t)}SN[-d_1(\overline{S})] \tag{8.19}$$

By comparing Eq. (8.19) and (5.4), we see that the expressions are equivalent. With $\delta = 0$, we have the Black-Scholes pricing formula for a European put option, see, e.g., Ingersoll (1987) p. 320.

8.2.5 The European call option

The future pay-off from a European call option written on S with exercise price $\overline{S}$ and time to maturity $T-t$ is

$$Y(S(T)) = \begin{cases} 0 & \text{if} \quad S(T) < \overline{S} \\ S(T) - \overline{S} & \text{if} \quad S(T) \geq \overline{S} \end{cases} \tag{8.20}$$

conditional on optimal exercise. By decomposing Y into pay-offs described by the η-function above, we have

$$Y(S(T)) = S(T) - \overline{S} - \eta(S(T) \mid \overline{S}, \varepsilon = 1) + \overline{S}\eta(S(T) \mid \overline{S}, \varepsilon = 0) \quad (8.21)$$

The two first terms represent the pay-off at date T from receiving one unit of the risky asset and paying the amount of $\overline{S}$ dollars. The two last terms represent the value of a commitment to deliver one unit of the risky asset and a claim on $\overline{S}$ dollars, both with maturity date T and made contingent $S < \overline{S}$. The two last terms thus represent the future pay-off from a put option with exercise price and time to maturity identical to the call option[2], see Eq. (8.18) above.

By using the evaluator Ψ above, and rearranging, we find that the current value of the European call option is

$$W_{CALL} = e^{-\delta(T-t)}SN[d_1(\overline{S})] - e^{-r(T-t)}\overline{S}N[d_2(\overline{S})] \quad (8.22)$$

With $\delta = 0$, we have the famous Black-Scholes formula, see, e.g., Ingersoll (1987) p. 314.

8.3 The future accept/reject decision

With our result above, we may evaluate directly the future accept/reject investment opportunity, discussed in Chapter 5. We recall that the

[2]By using this fact, and by evaluating both sides of Eq. (8.21) as viewed from date t, we obtain the famous put - call parity

$$W_{CALL} = e^{-\delta(T-t)}S - e^{-r(T-t)}\overline{S} + W_{PUT}$$

where both options have exercise price $\overline{S}$ and time to maturity $T - t$. The value of the call may be found by inserting Eq. (8.19) into this expression, and rearranging.

In the finance literature, see, e.g., Jarrow and Rudd (1983), the put-call parity with dividends is often expressed by the current stock price and the present value of future dividends, rather than by our term $e^{-\delta(T-t)}S$ just above. The current value of future dividends is identical to the value of holding one unit of the "twin asset", and selling one futures contract on the same asset with maturity date T. The current value of this strategy is $S - e^{-\delta(T-t)}S$, c.f., Eq. (2.5). By inserting this expression into Eq. (4-3) of Jarrow and Rudd (1983), we obtain the put-call parity as stated in this footnote.

value of the accept/reject investment decision at the decision date T is

$$Y_C(S(T),T) = \begin{cases} 0 & \text{if} \quad S(T) < S_{BE}(T) \\ AS(T) - B(T) & \text{if} \quad S(T) \geq S_{BE}(T) \end{cases} \tag{8.23}$$

where $S_{BE}(T) \equiv B(T)/A$ is the break-even price at the future decision date T. By using Theorem 1, we obtain that the value at date t of receiving an accept/reject investment opportunity at the future date T is

$$W = e^{-\delta(T-t)} ASN[d_1(S_{BE})] - e^{-r(T-t)} BN[d_2(S_{BE})] \tag{8.24}$$

where $B \equiv B(T)$ and $S_{BE} \equiv S_{BE}(T)$ both are related to the future date T. By using that both the total costs and thus the break-even price escalates exponentially at the rate π, it can be verified that Eq. (8.24) is equivalent to the results presented i Chapter 5, Eqs. (5.1) - (5.3).

8.4 A freeze on investments

In Chapter 6, we assumed that the perpetual investment opportunity can be exercised at any time. In some cases, however, there may exist a restriction with respect to the earliest possible initiation date. One example is the Government, due to political considerations, imposing a temporary freeze on undeveloped oil fields. Another example is where two neighboring undeveloped oil fields for some reason may not be developed simultaneously, even if so desired. The results derived in this chapter provides the tools required to deal with this kind of problems.

The restriction with respect to the earliest possible initiation date T represents a reduction of the feasible set of actions available to the holder of the investment opportunity. With the assumed price dynamics of oil, there is a positive probability that the spot output price will hit the trigger price before date T, indicating that immediate initiation is optimal. Due to the restriction, then, the oil field may be kept undeveloped, whereas immediate development, from the holders' point of view, is optimal. Hence, the value W of the delayed perpetual investment opportunity is less than the pure one U, and we thus have

$$W(S(t),t) < U(S(t),t)$$

On the other hand, it is clearly possible for the holder of the investment opportunity to undertake a commitment to initiate at the expiration date of the freeze. Similarly, he may promise never to develop. However, fixing the two mentioned strategies today are not optimal, and we may thus conclude that

$$W(S(t),t) > \max\{C(S(t),T),0\}$$

where the commitment value $C(S(t),T)$ is given by Eq. (4.5).

At the earliest possible initiation date T, the undeveloped oil field represents a pure perpetual investment opportunity. The future value of the oil field at the future date T is hence given by Eq. (6.7)

$$U(S(T)) = \begin{cases} \alpha S(T)^{\beta} & \text{if} \quad S(T) < S^* \\ AS(T) - B & \text{if} \quad S(T) \geq S^* \end{cases} \tag{8.25}$$

where α, B, and S^* are related to date T. If the oil price at the future date T equals or exceeds the trigger price S^*, development will be initiated immediately.

It is easy to verify that the future pay-off $U(S(T))$ alternatively may be written

$$\begin{aligned} U(S(T)) &= \alpha\eta(S(T) \mid S^*, \beta) \\ &+ A(S(T) - \eta(S(T) \mid S^*, 1)) \\ &- B(1 - \eta(S(T) \mid S^*, 0)) \end{aligned} \tag{8.26}$$

The first term is identical to the future value from the number of α claims on the power function pay-off, received if and only if the future price $S(T) < S^*$. The remaining terms represents the future pay-off from receiving the quantity of A risky assets and paying B dollars, combined with a position that cancels this transaction when $S(T) < S^*$.

By using our results above, and rearranging, we obtain that the current value of the delayed perpetual investment opportunity is

$$\begin{aligned} W &= \alpha e^{\lambda(t|\beta)} S^{\beta} N[-d(S, t \mid S^*, \beta)] \\ &+ Ae^{-\delta(T-t)} S N[d_1(S^*)] \\ &- Be^{-r(T-t)} N[d_2(S^*)] \end{aligned} \tag{8.27}$$

where α, B, and S^* are related to the future date T, and S is the current spot price. The first term in the expression above is the value of receiving the undeveloped oil field if and only if the future price $S(T)$ is lower than the trigger price S^*. The remaining terms reflect the value of receiving the undeveloped oil field if and only if the future price indicates immediate development.

Now, we consider a numerical example using our base case parameter values.[3] In Section 6.4, we found that the parameter values translate into $\beta = 2$, $\alpha = 4\frac{1}{16}$, and $S^* = 16$ USD/barrel. Furthermore, suppose that the remaining time to the expiration of the freeze is $T - t = 4$ years.

The results are illustrated in Figure 8.1. The kinked line represents the current value of an immediate accept/reject opportunity to undertake a commitment to start development at the future date T. If undertaken, the current value is given by Eq. (4.5).

The upper curve is the value of the oil field in the case of a non-expiring investment opportunity, which may be initiated immediately, if so desired. We discussed this decision situation in Section 6.2.

The remaining curve in Figure 8.1 reflects the field value when there is imposed a 4-year freeze on development of the non-expiring investment opportunity. The freeze prevents the project from being initiated before the initiation date T of the freeze. With a low current spot price, the probability that S will hit the trigger price from below before date T is low. The value of the project is then close to the field value without any freeze. With S being high, immediate investments are optimal, but not feasible. It is, however, likely that the spot price $S(T)$ at the future date T will indicate development at that date, and the project is thus similar to a future commitment.

The vertical difference between the two upper curves in Figure 8.1 represents the opportunity loss of the freeze. As we would expect, it increases with a higher spot price, and converges towards the difference between the value of the non-expiring investment opportunity and the current value of undertaking a commitment to develop at the future date T.

[3]We recall from Section 3.4 that $\delta = 0.06$, $r = 0.05$, $\pi = 0$, $\sigma^2 = 0.07$, $A = 130$, and $B = 1040$.

8.5 A compound investment opportunity

In the section above, we consider a claim on a future delivery of a perpetual investment opportunity. To complicate things, suppose that the project of interest gives the holder the right to acquire a perpetual investment opportunity at the fixed future date T by investing B_i dollars at that date. As viewed from the current date t, the project thus corresponds to a compound option, that is, a European option written on a perpetual American option.

The pay-off pattern at the future date T depends on the future price of oil $S(T)$. Moreover, it also depends on the relation between the critical price indicating exercise of the European option at the maturity date T, $\hat{S}$, and the critical price indicating immediate exercise of the perpetual American option at that date, S^*.

With $S^* \leq \hat{S}$, it is always optimal to "kill" the American option immediately if the European option is exercised. In this case, the value of the project at the future date T is

$$U(S(T)) = \begin{cases} 0 & \text{if} \quad S(T) < S^* \\ 0 & \text{if} \quad S^* \leq S(T) < \hat{S} \\ AS(T) - B_p - B_i & \text{if} \quad \hat{S} \leq S(T) \end{cases} \tag{8.28}$$

We see that $U(S(T))$ in Eq. (8.28) is identical to the value of an accept/reject investment opportunity with break-even price $\hat{S} = (B_i + B_p)/A$. The current value is thus found by substituting $\hat{S}$ for S_{BE}, and $B_i + B_p$ for B, in Eq. (8.24) above.

With $S^* > \hat{S}$, there is a positive probability that, given the European option is exercised, it might be optimal to keep the American option "alive". In this case, the future value of the project corresponds to

$$U(S(T)) = \begin{cases} 0 & \text{if} \quad S(T) < \hat{S} \\ \alpha S(T)^\beta - B_i & \text{if} \quad \hat{S} \leq S(T) < S^* \\ AS(T) - B_p - B_i & \text{if} \quad S^* \leq S(T) \end{cases} \tag{8.29}$$

where $\hat{S} = (B_i/\alpha)^{1/\beta}$. Eq. (8.24) is now insufficient. However, by using our evaluator presented above, we arrive at the following current project value

$$\begin{aligned} W &= \alpha e^{\lambda(t|\beta)} S^{\beta} \left(N[-d(S,t \mid S^*,\beta)] - N[-d(S,t \mid \hat{S},\beta)] \right) \\ &- B_i e^{-r(T-t)} \left(N[-d_2(S^*)] - N[-d_2(\hat{S})] \right) \\ &+ A e^{-\delta(T-t)} S N[d_1(S^*)] \\ &- (B_p + B_i) e^{-r(T-t)} N[d_2(S^*)] \end{aligned} \tag{8.30}$$

The term in the first line represents the value of α claims on a pay-off according to the power function, contingent on the future price $S(T)$ being between $\hat{S}$ and S^*. The term in the next line is the value of a future obligation to pay the amount of B_i dollars conditional on the same event. The term in the third line is the value of a future claim on delivery of A units of oil contingent on the future price $S(T) \geq S^*$, while the remaining term represents the value of an obligation to pay the amount of $(B_p + B_i)$ dollars in the same state.

With the investment cost B_i approaching zero, exercising the European option at date T will always be optimal, that is, $\hat{S} \to 0$. The future pay-off in Eq. (8.29) then converges towards Eq. (8.25), and we thus have the temporary freeze on investments, considered in the section above, as the limiting case.

8.6 Investment with a time lag

In Chapter 7, we analyzed an investment opportunity on an oil field with a production switch. We then implicitly assumed that the development is instantaneous. Now, suppose instead that there is a fixed time lag between the date of the irrevocable investment decision and the first possible production date.[4]

From Eqs. (7.19) and (7.20), we know that the value of the developed oil field at the first possible production date T is

$$U(S(T)) = \begin{cases} \alpha_1 S(T)^{\beta_1} & \text{if} \quad S(T) < S_p^* \\ \alpha_7 S(T)^{\beta_4} + AS(T) - B_p & \text{if} \quad S(T) \geq S_p^* \end{cases} \tag{8.31}$$

[4]Majd and Pindyck (1987) consider a case where it takes time to build, and where the holder has the flexibility to stop the development temporarily. If the development is stopped, both the investment costs and the first possible date of production are delayed.

where S_p^* is the trigger price indicating that producing at date T is optimal, given by Eq. (7.17).

Eq. (8.31) may be decomposed into a sum of pay-off functions $\eta(S(T) \mid \overline{S}, \varepsilon)$, see Eq. (8.1), which each is priced by the evaluator $\Psi(S, t \mid \overline{S}, \varepsilon)$ of Theorem 1. It can be verified that the value at the current decision date t of initiating development of the project (except for investment cost) is

$$\begin{aligned} V &= \alpha_1 e^{\lambda(t|\beta_1)} S^{\beta_1} N[-d(S, t \mid S_p^*, \beta_1)] \\ &+ \alpha_7 e^{\lambda(t|\beta_4)} S^{\beta_4} N[d(S, t \mid S_p^*, \beta_4)] \\ &+ A e^{-\delta(T-t)} S N[d_1(S_p^*)] - B_p e^{-r(T-t)} N[d_2(S_p^*)] \end{aligned} \quad (8.32)$$

The first term in the equation above represents the present value of a future claim on the developed oil field at date T if and only if it is non-optimal to start production at that time. The next term represents the present value of receiving the pure option value of the oil field at date T if and only if it is then optimal to start production immediately. The remaining terms represent the value of a contingent claim on a similar oil field without any production switch, conditional on the same event.

Above, we have a closed form solution describing the value of the oil field (before investment costs B_i) contingent on an immediate development decision. Now, suppose that the investment decision in principle may be deferred perpetually, and that there is no cost escalation. This perpetual investment opportunity represents a contingent claim, and its value W satisfies the familiar PDE[5]

$$\tfrac{1}{2}\sigma^2 S^2 W_{SS} + (r - \delta) S W_S - rW = 0 \quad (8.33)$$

Further, the boundary conditions are similar to the conditions describing the investment opportunity in Chapter 6. First, we have the "low price" condition

$$\lim_{S \to 0} W = 0 \quad (8.34)$$

Second, the condition describing the critical oil S_i^* indicating that immediate development decision is optimal

$$W(S_i^*) = V(S_i^*) - B_i \quad (8.35)$$

[5]The investment opportunity is by assumption perpetual. Furthermore, both the investment costs B_i and the production costs B_p are independent of the development decision date t. Thus, we argue that the partial derivative W_t is zero.

Third, the "high contact" condition, stating that at S_i^*, the two assets have equal risk

$$W_S(S_i^*) = V_S(S_i^*) \tag{8.36}$$

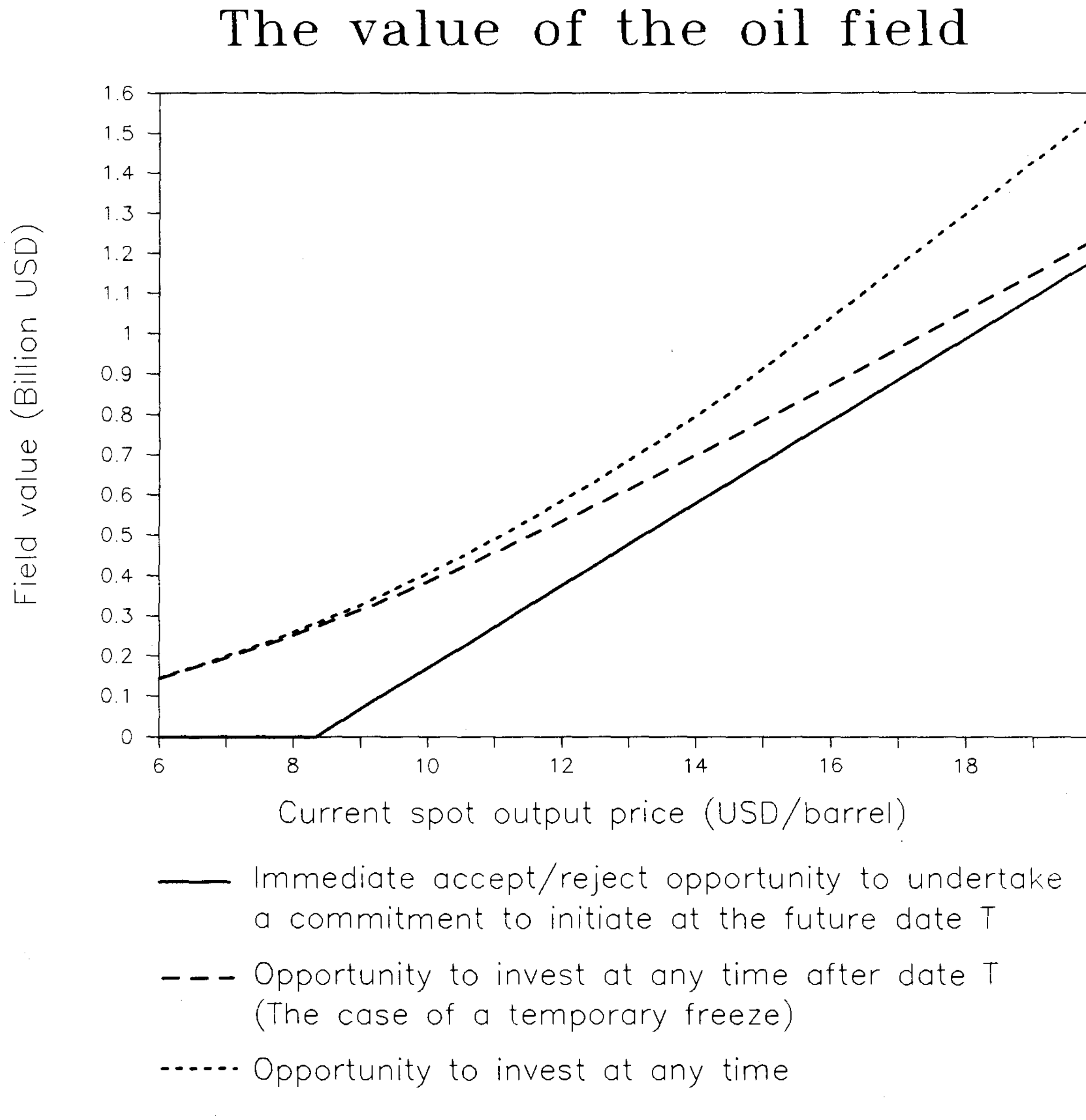
The PDE and the "low price" boundary condition, Eqs. (8.33) and (8.34), respectively, give us the expression of the value of the investment opportunity

$$W(S) = \alpha S^{\beta_1} \tag{8.37}$$

where the exponent β_1 is defined by Eq. (7.21). The constant α and the critical price indicating immediate investment, S_i^*, are determined by the two remaining boundary conditions.

No closed form solution exists in this case. For given parameter values, however, it is straightforward to find α and S_i^* by iterative procedures. The existence of a time lag will reduce the set of feasible actions available to the holder of the oil field. Hence, the critical price indicating immediate investment increases, and the value of the investment opportunity decreases, with an increased time lag.

Figure 8.1


The value of the oil field
Field value (Billion USD)
1.6
1.5
1.4
1.3
1.2
1.1
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
6
8
10
12
14
16
18
20
Current spot output price (USD/barrel)
Immediate accept/reject opportunity to undertake a commitment to initiate at the future date T
Opportunity to invest at any time after date T (The case of a temporary freeze)
Opportunity to invest at any time

Appendix A

The evaluator function Ψ

A.1 The partial differential equation

Proposition 5 *The function $\Psi(S,t)$ satisfies the partial differential equation*

$$\tfrac{1}{2}\sigma^2 S^2 \Psi_{SS} + (r-\delta)S\Psi_S - r\Psi + \Psi_t = 0 \qquad \text{(A.1)}$$

Proof: The partial derivatives of the function $\Psi(S,t)$ are

$$\Psi_S = (\varepsilon - F)\frac{\Psi}{S} \qquad \text{(A.2)}$$

$$\Psi_{SS} = \left\{\varepsilon(\varepsilon-1) - 2\varepsilon F + \frac{Fd}{\sigma\sqrt{T-t}} + F\right\}\frac{\Psi}{S^2} \qquad \text{(A.3)}$$

$$\begin{aligned} \Psi_t &= \Big\{-(\varepsilon-1)r + \varepsilon\delta - \tfrac{1}{2}\varepsilon(\varepsilon-1)\sigma^2 \\ &+ \left(r - \delta + (\varepsilon - \tfrac{1}{2})\sigma^2 - \frac{\sigma d}{\sqrt{T-t}}\right)F\Big\}\Psi \end{aligned} \qquad \text{(A.4)}$$

where we define

$$F = F(S,t \mid \overline{S},\varepsilon) \equiv \frac{n[d(S,t \mid \overline{S},\varepsilon)]}{\sigma\sqrt{T-t}N[-d(S,t \mid \overline{S},\varepsilon)]} \qquad \text{(A.5)}$$

and $d \equiv d(S,t \mid \overline{S},\varepsilon)$ is defined by Eq. (8.4) above. By inserting the partial derivatives, Eqs. (A.2) - (A.4), into the left hand side of Eq. (A.1) above, we find that the equation is satisfied. The proposition, stated as Eq. (8.5) in Theorem 1, is thus verified. □

A.2 The boundary conditions

Proof of Eq. (8.6) in Theorem 1: It can be seen from Eqs. (8.2) - (8.4) that we have the two limits

$$\lim_{S\to\infty} S^{\varepsilon} = \begin{cases} 0 & \text{if } \varepsilon < 0 \\ 1 & \text{if } \varepsilon = 0 \\ \infty & \text{if } \varepsilon > 0 \end{cases}$$

$$\lim_{S\to\infty} N[-d] = 0$$

In the case of $\varepsilon \leq 0$, by combining the two limits, we see directly that

$$\lim_{S\to\infty} \Psi(S,t) = 0$$

With $\varepsilon > 0$, we have by l'Hopitals rule that

$$\begin{aligned} \lim_{S\to\infty} \Psi(S,t) &= \lim_{S\to\infty} \left(e^{\lambda} S^{\varepsilon} N[-d]\right) \\ &= \lim_{S\to\infty} \left(e^{\lambda} \frac{\partial(S^{\varepsilon})}{\partial S} \frac{\partial(N[-d])}{\partial S}\right) \\ &= e^{\lambda} \lim_{S\to\infty} \left(\varepsilon S^{\varepsilon-1} n[-d] \frac{\partial(-d)}{\partial S}\right) \end{aligned}$$

By inserting the definition of the standard normal probability density function, and the partial derivative $\partial(-d)/\partial S$, some rearranging gives

$$\lim_{S\to\infty} \Psi(S,t) = -\frac{\varepsilon e^{\lambda}}{\sqrt{2\pi}\sigma\sqrt{T-t}} \lim_{S\to\infty} \left(e^{(\varepsilon-2)\ln(S)-\frac{1}{2}d^2}\right)$$

Note from Eq. (8.4) that the expression $\ln(S)$ enters in d. This means that the exponent in the last factor just above represents a square expression in $\ln(S)$. By completing the square, we obtain

$$\lim_{S\to\infty} \Psi(S,t) = k_1 \lim_{S\to\infty} \left(e^{-\frac{1}{2}(k_2 \ln(S)-k_3)^2}\right)$$

where k_1, k_2, and k_3 are constants. Clearly, the expression on the right hand side has the limit zero. We may thus conclude that

$$\lim_{S\to\infty} \Psi(S,t) = 0 \tag{A.6}$$

and Eq. (8.6) in Theorem 1 is verified. □

Proof of Eq. (8.7) in Theorem 1: It can be seen from Eqs. (8.2) - (8.4) that we have the two limits

$$\lim_{S\to 0} S^{\varepsilon} = \begin{cases} \infty & \text{if } \varepsilon < 0 \\ 1 & \text{if } \varepsilon = 0 \\ 0 & \text{if } \varepsilon > 0 \end{cases}$$

$$\lim_{S\to 0} N[-d(S)] = 1$$

By combining the two limits, and noting from Eq. (8.3) that $\lambda(t \mid \varepsilon = 0) = -r(T-t)$, we have the result

$$\lim_{S\to 0} \Psi(S,t) = \begin{cases} \infty & \text{if } \varepsilon < 0 \\ e^{-r(T-t)} & \text{if } \varepsilon = 0 \\ 0 & \text{if } \varepsilon > 0 \end{cases} \tag{A.7}$$

stated as Eq. (8.7) in Theorem 1. □

Proof of Eq. (8.8) in Theorem 1: From Eq. (8.4), it follows that

$$\lim_{t\to T} d = \lim_{t\to T}\left(\frac{\ln(S/\overline{S})}{\sigma\sqrt{T-t}} + \frac{[r-\delta+(\varepsilon-\frac{1}{2})\sigma^2]\sqrt{T-t}}{\sigma}\right)$$

The limit of the second term is zero. From the first term, we thus have

$$\lim_{t\to T} d = 0 \text{ if } S = \overline{S}$$

and

$$\lim_{t\to T} d = \lim_{t\to T}\left(\frac{\ln(S/\overline{S})}{\sigma\sqrt{T-t}}\right) = \begin{cases} -\infty & \text{if } S < \overline{S} \\ \infty & \text{if } S > \overline{S} \end{cases}$$

The result implies

$$\lim_{t\to T} N[-d] = \begin{cases} 1 & \text{if } S < \overline{S} \\ \frac{1}{2} & \text{if } S = \overline{S} \\ 0 & \text{if } S > \overline{S} \end{cases}$$

From Eq. (8.3) it is easy to see that $\lim_{t\to T}\lambda(t \mid \varepsilon) = 0$, and thus $\lim_{t\to T} e^{\lambda} = 1$. We may conclude that

$$\lim_{t\to T} \Psi(S,t) = \begin{cases} S^{\varepsilon} & \text{if } S < \overline{S} \\ \frac{1}{2}S^{\varepsilon} & \text{if } S = \overline{S} \\ 0 & \text{if } S > \overline{S} \end{cases} \tag{A.8}$$

which represents Eq. (8.8) in Theorem 1 □

Appendix B

The derivation of Ψ

In Section 8.2 and Appendix A above, we find that the current value of the future pay-off $\eta(S(T) \mid \overline{S}, \varepsilon)$, defined by Eq. (8.1), is

$$V_t[I(S(T) < \overline{S})S(T)^{\varepsilon}] = \Psi(S, t \mid \overline{S}, \varepsilon)$$

where Ψ is given by Eqs. (8.2) - (8.4). The proof above is based on knowing the solution Ψ in the first place, however, and by checking that this function satisfies the necessary conditions. In this appendix, we show how the function Ψ is derived.

From Eq. (2.2), we know that the spot output price at the future date T, conditional on the price at the current date 0 being S_0, may be written

$$S_T(X) = S_0 e^{(\alpha - \frac{1}{2}\sigma^2)T + X} \tag{B.1}$$

where the random variable $X \equiv \sigma Z(T)$ is normally distributed with expectation zero and variance $(\sigma^2 T)$. The riskfree rate of interest is r. Furthermore, the economy is "frictionless", and characterized by continuous trading and no riskfree arbitrage opportunities.

Our economy is generated by a geometric Brownian motion, and is thus similar to the model of Black-Scholes. Harrison and Kreps (1979) proves that this economy is complete. This result enables us to draw upon powerful theory for evaluating contingent claims.[1]

Harrison and Pliska (1981) describes a contingent claim as an asset providing a pay-off Y at the maturity date T, and no pay-off prior to

[1] See, e.g., Harrison and Kreps (1979), Harrison and Pliska (1981), and Aase (1988). For an application, see Ross (1989).

this date. The future pay-off Y is defined as a non-negative random variable, which at the horizon date T is measurable with respect to the price history of the underlying asset. Completeness means that there for each contingent claim exists a self-financing trading strategy, associated with a market value process V, so that $V_T = Y$ almost surely at the horizon date. The unique price at which the contingent claim may be attained in the market today is[2]

$$V_0[Y] = e^{-rT} E_0^*[Y] \tag{B.2}$$

where the expectations operator $E_0^*[\cdot]$ is associated with a particular probability measure.

Now, consider a contingent claim in our economy, with no pay-off prior ot the future date T, and with pay-off $Y(X)$ at date T.[3] As our economy is complete, this claim may be attained in the market today at an unique price, given by Eq.(B.2). By Riesz representation theorem, Eq. (B.2) may be written as

$$V_0[Y(X)] = e^{-rT} E_0[Y(X)\xi(X)] \tag{B.3}$$

where $E[\cdot]$ is the expectations operator associated with the true probability measure, and where $\xi(X)$ is unique.[4] Before presenting $\xi(X)$, we make some comments on the intuition behind the evaluation rule.

We may alternatively state Eq. (B.3) as

$$V_0[Y(X)] = e^{-rT} \int Y(x)\xi(x)f(x)dx \tag{B.4}$$

where $f(x)$ is the probability density function of X.[5]

[2]See Harrison and Pliska (1981) Eq. (1.15).

[3]We assume that $E[Y^2(X)] < \infty$, which may be interpreted as the future pay-off having a finite variance.

[4]$E[\xi^2(X)] < \infty$.

[5]The probability density function $f(x)$ is

$$f(x) = n\left[\frac{x}{\sigma\sqrt{T}}\right] = \frac{1}{\sqrt{2\pi\sigma^2 T}} \exp\left\{-\frac{1}{2}\left(\frac{x}{\sigma\sqrt{T}}\right)^2\right\}$$

where $n[\cdot]$ is the standard normal density function.

Note from Eq. (B.4) that the current value of the claim corresponds to an integral, discounted back at the riskless interest rate. The integral may be interpreted as the number of future claims on one dollar with maturity date T being equivalent to the risky future pay-off Y, given the current information set.

The product $\xi(x)f(x)dx$ in Eq. (B.4) translates a one-dollar claim received at date T conditional on $X \in [x, x + dx)$, into the equivalent numbers of future riskless claims on one dollar. The product $\xi(x)f(x)$ thus represents an implicit price function of claims made conditional on the random variable X, given the current information set.

We may interpret $X \in [x, x + dx)$ as a state. The future one-dollar claim, made conditional on this state, may be attained in the market today at the price $e^{-rT}\xi(x)f(x)dx$.

The random variable X is by assumption normally distributed, and we thus have the probability density function $f(x) > 0$ for all x. Suppose for the moment that there exists an outcome $X \in [x, x + dx)$ for which the product $\xi(x)f(x)$ is non-positive. In that case, a one-dollar claim made conditional on this future state, possibly providing a pay-off at the future date T, may be attained in the market today for free or at a "negative" price. With investors preferring more wealth to less, however, this situation does not represent an equilibrium. We thus require future claims made conditional on $X \in [x, x+dx)$ to command a positive current price. Moreover, to prevent arbitrage, this price must be unique.

Now, consider for the moment a riskless claim on receiving one dollar at the future date T. According to Eq. (B.4), the current value of this claim is

$$V_0[Y(X) = 1] = e^{-rT} \int \xi(x)f(x)dx$$

To rule out arbitrage, we see that the integral is required to be equal to one.

To sum up, we know that the product $\xi(x)f(x)$ is positive and unique for all x, and that its cumulative value is one. Hence, $\xi(x)f(x)$ may be interpreted as a market based certainty equivalent function, translating an uncertain future pay-off dependent on X into a riskless future pay-off. Alternatively, $\xi(x)f(x)$ may be given the interpretation of a risk-adjusted probability density.

For $V_0[\cdot]$ to serve as an evaluator of contingent claims in our economy, it is a necessary requirement that the evaluator provides consistent prices of both the future claim on one riskless dollar, and the future claim on one unit of output. By combining the evaluator from Eq. (B.3) with the two discounting rules, Eqs. (2.4) and (2.5), we have the two conditions

$$V_0[Y(X) = 1] = e^{-rT} \quad \Rightarrow \quad E_0[\xi(X)] = 1 \tag{B.5}$$

$$V_0[Y(X) = S_T(X)] = e^{-\delta T} S_0 \quad \Rightarrow \quad E_0[S_T(X)\xi(X)] = e^{(r-\delta)T} S_0 \tag{B.6}$$

These two equations boil down to restrictions on $\xi(x)$.

In our economy, the function $\xi(x)$ is uniquely determined by the two conditions above, and is given by[6]

$$\xi(X) = \exp\left\{-\frac{(\mu^* T)X}{\sigma^2 T} - \frac{1}{2}\frac{(\mu^* T)^2}{\sigma^2 T}\right\} \tag{B.7}$$

where we define

$$\mu^* \equiv \hat{\alpha} - r \tag{B.8}$$

[6] The underlying risky asset in our economy, with equilibrium expected rate of return, is the "twin asset". Consider a self-financing portfolio of "twin assets", with initial value S_0 at date 0. The value of this portfolio at date T, discounted back at the riskfree interest rate, is

$$\nu(T) = S_0 \exp\left\{(\hat{\alpha} - r - \tfrac{1}{2}\sigma^2)T + \sigma Z(T)\right\}$$

c.f. Eqs. (2.2) and (2.3).

Compare our discounted price process ν with the diffusion component in Eq. (3.2) in Aase (1988). The L_t-function in Eq. (3.3) in the mentioned article translates in our case into

$$\xi = \exp\left\{-\int_0^T \left(\frac{\hat{\alpha} - r}{\sigma}\right) dZ(t) - \frac{1}{2}\int_0^T \left(\frac{\hat{\alpha} - r}{\sigma}\right)^2 dt\right\}$$

We recall from Section 2.2 that $\hat{\alpha}$, r, and σ are constant. Furthermore, it follows from the definition of the Brownian motion that $\int_0^T dZ(t) = Z(T)$. We thus have

$$\xi = \exp\left\{-\left(\frac{\hat{\alpha} - r}{\sigma}\right) Z(T) - \frac{1}{2}\left(\frac{\hat{\alpha} - r}{\sigma}\right)^2 T\right\}$$

By inserting our definitions $\mu^* \equiv \hat{\alpha} - r$ and $X \equiv \sigma Z(T)$, and rearranging, we obtain Eq. (B.7) above. It is fearly easy to verify that Eq. (B.7) satisfies the conditions in Eqs. (B.5) and (B.6), and that $E[\xi^2] < \infty$.

The term μ^* represents the risk premium related to the "twin asset".

By inserting the normal density function $f(x)$, footnote 5 in this appendix, and the solution of $\xi(x)$, Eq. (B.7), into the implicit price function, some rearranging leads to

$$\xi(x)f(x) = n\left[\frac{x+\mu^*T}{\sigma\sqrt{T}}\right] \tag{B.9}$$

We see that the price function $\xi(x)f(x)$ corresponds to the normal density function with mean (μ^*T) and variance $(\sigma^2 T)$.

Now, consider the contingent claim $Y(X)$ defined by

$$Y(X) = I(X < \overline{X})y(X) \tag{B.10}$$

where[7]

$$\overline{X} \equiv \ln(\overline{S}/S_0) - (\alpha - \tfrac{1}{2}\sigma^2)T \tag{B.11}$$

By inserting the pay-off from the contingent claim, Eq. (B.10), and the implicit price function, Eq. (B.9), into the evaluation rule of Eq. (B.4), we obtain

$$V_0[Y(X)] = e^{-rT}\int I(x<\overline{X})y(x)n\left[\frac{x+\mu^*T}{\sigma\sqrt{T}}\right]dx \tag{B.12}$$

The integral just above may alternatively be written[8]

$$V_0[Y(X)] = e^{-rT}\int I(x<\overline{X}+\mu^*T)y(x-\mu^*T)n\left[\frac{x}{\sigma\sqrt{T}}\right]dx \tag{B.13}$$

The integral represents the expected value of a random variable generated by X, defined in the beginning of this appendix. We may thus write

$$V_0[I(X<\overline{X})y(X)] = e^{-rT}E_0[I(X<\overline{X}+\mu^*T)y(X-\mu^*T)] \tag{B.14}$$

[7]Formally, $\overline{X}$ is defined by

$$\overline{X} \equiv \{X : S_T(X) = \overline{S} \mid S_0\} = \ln(\overline{S}/S_0) - (\alpha - \tfrac{1}{2}\sigma^2)T$$

where $S_T(X)$ is defined in Eq. (B.1) above.

[8]Define for the moment a new variable $x' \equiv x + \mu^*T$. Use this definition, and the implied relationship $dx = dx'$, to express Eq. (B.12) in terms of the new variable x'. Note that the probability density function of x', $n[x'/\sigma\sqrt{T}]$, is identical to the probability density function of x, $f(x)$, in footnote 5. By substituting x' for x in the expression, we have Eq. (B.13).

where μ^* and $\overline{X}$ are defined by Eqs. (B.8) and (B.11), respectively, and the probability density function of X is given in footnote 5.

B.1 The contingent power function pay-off

In this subsection, we use the evaluation rule obtained just above to show that

$$\Psi \equiv V_0[I(S(T) < \overline{S})S(T)^{\varepsilon}]$$

is identical to the solution presented in Eqs. (8.2) - (8.4). According to the evaluation rule in Eq. (B.14) above, the current value of a future claim on a pay-off described by the power function of the risky asset at date T, conditional on $S(T) < \overline{S}$, is

$$\Psi = e^{-rT}E_0[I(X < \overline{X} + \mu^*T)S_T(X - \mu^*T)^{\varepsilon}] \tag{B.15}$$

where μ^*, $\overline{X}$, and $S_T(X)$ are defined by Eqs. (B.8), (B.11), and (B.1), respectively.

First, we insert the definition of $S_T(\cdot)$, with $X - (\hat{\alpha} - r)T$ as the argument, into Eq. (B.15). Some rearranging gives

$$\Psi = e^{\{(\varepsilon-1)r - \varepsilon\delta - \frac{1}{2}\varepsilon\sigma^2\}T}(S_0)^{\varepsilon}E_0[I(X < \overline{X} + \mu^*T)e^{\varepsilon X}]$$

To evaluate the expectation contained in this expression, we need the following result.

Proposition 6 *Consider a stochastic variable Z that is normally distributed with expectation μ and variance σ^2. We then have*

$$E[I(Z < \overline{Z})e^{\varepsilon Z}] = e^{\varepsilon\mu + \frac{1}{2}\varepsilon^2\sigma^2}N\left[\frac{\overline{Z} - (\mu + \varepsilon\sigma^2)}{\sigma}\right]$$

where $I(\cdot)$ is the indicator function, and $N[\cdot]$ is the standard cumulative normal distribution.

Proof: See Appendix B.2 below. □

By using this result, with a suitable reinterpretation of parameters, and by some rearranging, we obtain[9]

$$\Psi = e^{\{(\varepsilon-1)r-\varepsilon\delta+\frac{1}{2}\varepsilon(\varepsilon-1)\sigma^2\}T}(S_0)^{\varepsilon} N\left[\frac{\overline{X}+\mu^* T - \varepsilon\sigma^2 T}{\sigma\sqrt{T}}\right]$$

Finally, we insert $\overline{X}$ and μ^* into the argument of the cumulative normal distribution, and arrive at the desired result

$$\Psi \equiv V_0[I(S(T) < \overline{S})S(T)^{\varepsilon}] = e^{\lambda}(S_0)^{\varepsilon}N[-d] \qquad \text{(B.16)}$$

where

$$\lambda \equiv \{(\varepsilon-1)r - \varepsilon\delta + \tfrac{1}{2}\varepsilon(\varepsilon-1)\sigma^2\}T \qquad \text{(B.17)}$$

$$d \equiv \frac{\ln(S_0/\overline{S}) + (r - \delta + (\varepsilon - \frac{1}{2})\sigma^2)T}{\sigma\sqrt{T}} \qquad \text{(B.18)}$$

B.2 Proof of Proposition 6

Consider the expected value

$$E[I(Z < \overline{Z})e^{\varepsilon Z}] = \int I(z \le \overline{Z})e^{\varepsilon z}\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{1}{2}\left(\frac{z-\mu}{\sigma}\right)^2}dz \qquad \text{(B.19)}$$

where the stochastic variable Z is normally distributed with expectation μ and variance σ^2.

The exponents on the right hand side of Eq. (B.19) may be written as a constant and a square expression containing z. By collecting the terms in z, completing the square, and rearranging, it may be verified that we have the equivalent expression

$$-\frac{1}{2}\left(\frac{z-\mu}{\sigma}\right)^2 + \varepsilon z = -\frac{1}{2}\left(\frac{z-(\mu+\varepsilon\sigma^2)}{\sigma}\right)^2 + \varepsilon\mu + \frac{1}{2}\varepsilon^2\sigma^2$$

[9]We recall that X is normally distributed with zero expectation and variance $\sigma^2 T$.

Thus, by inserting the equivalent expression above into Eq. (B.19), and by rearranging the integral, we obtain

$$E[I(Z<\overline{Z})e^{\varepsilon Z}] = e^{\varepsilon\mu+\frac{1}{2}\varepsilon^2\sigma^2}\int I(z<\overline{Z})\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{1}{2}\left(\frac{z-(\mu+\varepsilon\sigma^2)}{\sigma}\right)^2}dz$$

The function inside the integral represents a truncated probability density function of a normally distributed random variable. The integral is the probability that the value of this random variable, with expectation $\mu+\varepsilon\sigma^2$ and variance σ^2, will be lower or equal to $\overline{Z}$. We thus have the result

$$E[I(Z<\overline{Z})e^{\varepsilon Z}] = e^{\varepsilon\mu+\frac{1}{2}\varepsilon^2\sigma^2}N\left[\frac{\overline{Z}-(\mu+\varepsilon\sigma^2)}{\sigma}\right] \tag{B.20}$$

See Ingersoll (1987) p. 15 for the special case $\varepsilon = 1$. □

Chapter 9

A PROMISE TO INVEST

9.1 Introduction

This chapter presents the pricing function for a contingent claim with a similar pay-off as the one considered in the previous chapter. In this case, however, the future pay-off is made conditional on the price of the underlying asset being lower than the trigger price for the entire period from the current date and until the future maturity date.

The starting point of this chapter is a situation where the Government for some reason has promised to take steps so that the development of a particular oil field, currently licenced to a commercial company, will be initiated before a fixed future date. We interpret the Governmental promise as a constraint on the set of possible strategies available to the present or to the future owner of the field. In this setting, the economic cost of the promise corresponds to the opportunity loss induced by the imposed constraint.

This chapter is organized as follows: In Section 9.2, we use the traditional decision model to analyze the promise. We recall from Chapter 4 that the owner in this case at the decision date is to choose between to accept the project, to reject it, or - once and for all - to fix a future date at which development will be initiated.

We next proceed to the framework of contingent claims analysis. In Section 9.3, we consider a situation where the perpetual investment opportunity, if not yet developed by the commercial company at the future date, is bought back by the Government at the market value, and investments initiated immediately. By calling upon a pricing function derived in the appendix, we obtain the cost of the promise.

In Section 9.4, we assume that the oil field is bought immediately by the Government at the fair market value, and transferred to the national oil company with an instruction to initiate development not later than the future date. In this case, the oil field corresponds to a commitment to initiate before the terminal date, combined with a "timing option". The flexibility reflects that the decision to initiate development is made contingent on the future spot output price, rather than fixing the initiation date in advance.

A numerical example is provided in Section 9.5. We conclude that the traditional model underestimates the loss induced.

9.2 The traditional model

In this section, we analyze the costs of the Governmental promise according to the traditional model. We recall from Chapter 4 that the owner of the oil field is facing the three following decision alternatives: To accept the project immediately, to fix - once and for all - a future date at which the project will be initiated, or to reject it immediately. The decision flexibility thus corresponds to an immediate choice between investment strategies that are uniquely determined by calendar time.

The value of the oil field without the Governmental promise, contingent on the optimal decision being made, is

$$V(S(t),t) = \max\left\{\max_{T\geq t} C(S(t),T), 0\right\} \tag{9.1}$$

where $C(S(t),T)$ is the current value of a commitment to initiate at the future date T, see Eq. (4.5).

Given the promise to initiate before or at the fixed future date $\overline{T}$, the value of the undeveloped oil field is

$$\overline{V}(S(t),t) = \max_{t\leq T\leq \overline{T}} C(S(t),T) \tag{9.2}$$

according to the traditional model.

Depending on the parameter values, we either have the optimal timing model or we are left with the simple accept/reject model.

9.2.1 Accept/reject

Suppose that $0 < r - \pi \leq \delta$. We then recall from Chapter 4 that the traditional model collapses into the accept/reject decision situation In the unconstrained case, the optimal decision rule is

$$\begin{array}{ll} \text{Reject} & \text{if } S(t) < S_{BE}(t) \\ \text{Accept} & \text{if } S(t) \geq S_{BE}(t) \end{array} \tag{9.3}$$

and the value of the undeveloped oil field is

$$V = \begin{cases} 0 & \text{if } S(t) < S_{BE}(t) \\ C(S(t),t) & \text{if } S(t) \geq S_{BE}(t) \end{cases} \tag{9.4}$$

In the constrained case, rejecting the project is not feasible. With the restrictions on the parameter values, fixing an initiation date $t < T < \overline{T}$ is never the superior decision[1] and the choice is thus between accepting the project immediately, or to undertake a commitment to initiate the project at the future date $\overline{T}$. The spot price, indicating indifference between the two alternative values $AS-B$ and $e^{-\delta(T-t)}AS-e^{-(r-\pi)(T-t)}B$, is

$$S(t) = gS_{BE}(t) \tag{9.5}$$

where the factor g is

$$g \equiv \frac{1 - e^{-(r-\pi)(\overline{T}-t)}}{1 - e^{-\delta(\overline{T}-t)}} < 1 \tag{9.6}$$

If the current spot price of oil is $S(t)$ is lower than $gS_{BE}(t)$, it is optimal to initiate the project at the future date $\overline{T}$. The value of the field in the constrained case is thus

$$\overline{V} = \begin{cases} C(S(t),\overline{T}) & \text{if } S < gS_{BE} \\ C(S(t),t) & \text{if } S \geq gS_{BE} \end{cases} \tag{9.7}$$

With $S < S_{BE}$, rejecting the project is optimal, but not feasible, and thus $\overline{V} < 0$.

The opportunity loss is defined by $V - \overline{V}$. It is non-negative, and zero if $S \geq S_{BE}$.

Our findings are contained in the following table:

[1] From Chapter 4, we know that the interior solution (if any) of the first-order condition in this case represents a minimum. The optimal value in the constrained case is thus found on one of the boundaries, $T = t$ or $T = \overline{T}$.

Parameter values: $0 < r - \pi \leq \delta$			
Spot price S	Field value V	$\overline{V}$	Opportunity loss $V - \overline{V}$
$S < gS_{BE}$	0	$C(S, \overline{T})$	$-C(S, \overline{T})$
$gS_{BE} \leq S < S_{BE}$	0	$C(S, t)$	$-C(S, t)$
$S_{BE} \leq S$	$C(S, t)$	$C(S, t)$	0

We see that the accept/reject decision model indicates an opportunity loss only when the current spot price $S(t)$ is below the break-even price $S_{BE}(t)$.

9.2.2 Optimal timing

Suppose that $0 < \delta < r - \pi$. In the unconstrained case, we recall from Chapter 4 that the optimal decision rule is

$$\begin{array}{ll} \text{Initiate at date } T^* & \text{if } S(t) < S_{OT}(t) \\ \text{Accept} & \text{if } S(t) \geq S_{OT}(t) \end{array} \tag{9.8}$$

and that the value of the undeveloped oil field is

$$V = \begin{cases} C(S(t), T^*) & \text{if } S(t) < S_{OT}(t) \\ C(S(t), t) & \text{if } S(t) \geq S_{OT}(t) \end{cases} \tag{9.9}$$

T^* is the optimal initiation date, and $S_{OT}(t)$ is the critical price indicating that $T^* = t$.[2]

The constraint, induced by the Governments promise that the development of the oil field will be initiated not later than the future date

[2]The optimal intiatation date T^* is defined by

$$T^*(S(t)) = t + \max\left\{0, \frac{\ln(S_{OT}(t)/S(t))}{(r - \pi) - \delta}\right\}$$

and the critical price S_{OT}, indicating $T^* = t$, is

$$S_{OT}(t) = \left(\frac{r - \pi}{\delta}\right) S_{BE}(t)$$

see Eqs. (4.7) and (4.8), respectively.

$\overline{T}$, is binding only if $T^* > \overline{T}$. The spot price, associated with $T^* = \overline{T}$, is

$$S(t) = fS_{OT}(t) \tag{9.10}$$

where the factor f is

$$f \equiv e^{-[(r-\pi)-\delta](\overline{T}-t)} < 1 \tag{9.11}$$

The value of the oil field according to the optimal timing model is thus

$$\overline{V} = \begin{cases} C(S(t), \overline{T}) & \text{if } S(t) < fS_{OT}(t) \\ C(S(t), T^*) & \text{if } fS_{OT}(t) \leq S(t) < S_{OT}(t) \\ C(S(t), t) & \text{if } S_{OT}(t) \leq S(t) \end{cases} \tag{9.12}$$

in the constrained case.

Our results are contained in the following table:

Parameter values: $0 < \delta < r - \pi$			
Spot price S	Field value V	$\overline{V}$	Opportunity loss $V - \overline{V}$
$S < fS_{OT}$	$C(S, T^*)$	$C(S, \overline{T})$	$C(S, T^*) - C(S(t), \overline{T})$
$fS_{OT} \leq S < S_{OT}$	$C(S, T^*)$	$C(S, T^*)$	0
$S_{OT} \leq S$	$C(S, t)$	$C(S, t)$	0

We see that the optimal timing model indicates an opportunity loss only when the current spot price $S(t)$ is lower than $fS_{OT}(t)$. If the spot price is higher, the model suggests that the promise is without costs.

9.3 Possible future intervention

In the following sections, we interpret the undeveloped oil field as a contingent claim. We assume that the commercial oil company in principle may defer the investment decision perpetually, if so desired. Furthermore, we assume that both parties have accepted that a possible transfer of the oil field is to be based on a "mutual fair value". On one hand, the Government is required to refrain from passing on new legislation that "changes the rules of the game". To force the company to initiate development, or to expropriate the field, is thus not possible. On the other hand, the company is not entitled to reject an offer from the Government to buy back the undeveloped oil field at the market value.

Suppose that the Government declares this strategy of buying back the oil field at the future date $\overline{T}$ if the field is still undeveloped at that date, and in that case it will undertake immediate development. With this arrangement, the future status of the oil field determines whether future costs are incurred or not, while the spot output price at date $\overline{T}$ determines the level of the future costs (if any).

The net future cost of the transaction at date $\overline{T}$, conditional on the field still being undeveloped at that date, is

$$Y(S(\overline{T}),\overline{T}) = U(S(\overline{T}),\overline{T}) - C(S(\overline{T}),\overline{T}) \tag{9.13}$$

The first term in the equation above is the future market value of the perpetual investment opportunity, given by Eq. (6.7). The last term represents the future value of a commitment to initiate the field at date $\overline{T}$, see Eq. (4.5).

By inserting Eqs. (6.7) and (4.5) into Eq. (9.13), we obtain the equivalent expression

$$Y(S(\overline{T}),\overline{T}) = \alpha(\overline{T})S(\overline{T})^{\beta} - AS(\overline{T}) + B(\overline{T}) \tag{9.14}$$

where $\alpha(\overline{T})$ and $B(\overline{T})$ are related to the future date T.

The oil company will not suffer any loss from this possible future transaction, as the price paid by the Government is equal to the future market price of the non-expiring licence. During the period from date t to $\overline{T}$, the commercial company will thus manage the investment opportunity as if it is a perpetual one. From Chapter 6 above, we know that the optimal strategy is to initiate development immediately whenever the spot output price $S(\tau)$ equals (or exceeds) the trigger price

$$S_{\tau}^{*} = e^{\pi(\tau-t)}S_{t}^{*} \tag{9.15}$$

In Eq. (9.15), the trigger price S_t^* at the current date t is defined by Eq. (6.5), and π is the escalation rate.

The status of the oil field at date $\overline{T}$ is thus dependent on whether the spot price of oil $S(\tau)$ hits the trigger price S_τ^* during the period $\tau \in [t, \overline{T}]$ or not. The current costs of the Governmental promise, L, may thus be expressed as

$$L(S,t) \equiv V_t\left[Y(S(\overline{T}),\overline{T}) \cdot I\left(\inf_{\tau\in[t,\overline{T}]}\left(e^{\pi(\tau-t)}S_t^* - S(\tau)\right) > 0\right)\right] \qquad (9.16)$$

where $V_t[\cdot]$ is a general evaluator, and where $Y(S(\overline{T}),\overline{T})$ is defined by Eq. (9.14).

To evaluate the future costs $L(S,t)$, we now call upon the following result:

Theorem 2 *The current value of a contingent claim with pay-off at the future date T described by the power function of the price S at that date, $S(T)^\varepsilon$, and made conditional on the price $S(\tau)$ being lower than S_τ^* for all dates $\tau \in [t,T]$, is*

$$\begin{aligned}\varphi(S,t \mid S_T^*,\varepsilon) &\equiv V_t\left[S(T)^\varepsilon \cdot I\left(\inf_{\tau\in[t,T]}\left(e^{\pi(\tau-t)}S_t^* - S(\tau)\right) > 0\right)\right] \\ &= \begin{cases} \Psi(S,t \mid S_T^*,\varepsilon) - (S_T^*)^{\varepsilon-\kappa}\Psi(S,t \mid S_T^*,\kappa) & \text{if } S(t) < S_t^* \\ 0 & \text{if } S(t) \geq S_t^* \end{cases} \qquad (9.17)\end{aligned}$$

where

$$\kappa(S,t \mid \varepsilon) = \varepsilon + 2\frac{\ln(S_t^*/S)}{\sigma^2(T-t)} \qquad (9.18)$$

The function Ψ is defined by Eqs. (8.2) - (8.4) in the previous chapter.

Proof: See Appendix A and Appendix B. □

With $S(t) \geq S_t^*$, the condition for the claim to provide a positive pay-off at the future date T is violated. The current value of the claim is thus zero, as stated in Eq. (9.17).

Next, consider the case of $S(t) < S_t^*$. We see from Eq. (9.17) that the value of the claim then is expressed by the function Ψ. Recall from Chapter 8 that Ψ represents the pricing function of a pay-off at the future date T according to a power-function of the spot price at that date, $S(T)^\varepsilon$, received if and only if $S(T) < S_T^*$.

The function φ is somewhat more complex than Ψ, as it evaluates a claim that is made conditional on the spot price $S(\tau)$ being lower than the trigger price S^*_τ for *all dates* $\tau \in [t, T]$. From Eq. (9.17), we see that φ is written as the value of its counterpart Ψ only dependent on the event $S(T) < S^*_T$, minus a discount. This discount reflects that φ provides no pay-off at the future date T if the spot price $S(\tau)$ hits the trigger price S^*_τ in the meantime.

By using Theorem 2 to evaluate each of the terms in Eq. (9.14), we find that the current cost of the promise is

$$\begin{aligned} L(S,t) &= \alpha(\overline{T})\varphi(S,t \mid S^*_{\overline{T}}, \beta) \\ &- A\varphi(S,t \mid S^*_{\overline{T}}, 1) \\ &+ B(\overline{T})\varphi(S,t \mid S^*_{\overline{T}}, 0) \end{aligned} \tag{9.19}$$

where $\alpha(\overline{T})$ and $B(\overline{T})$ are constants.

The function $L(S,t)$ satisfies the partial differential equation of a contingent claim, see Appendix A. Furthermore, we show in Appendix A that the function $L(S,t)$ has the following properties:

$$L(S,t) = 0 \text{ if } S(t) \geq S^*_t \tag{9.20}$$

$$\lim_{S \to 0} L(S,t) = e^{-r(\overline{T}-t)} B(\overline{T}) \tag{9.21}$$

$$\lim_{t \to \overline{T}} L(S,t) = \begin{cases} \alpha(\overline{T})S(\overline{T})^\beta & \\ - AS(\overline{T}) + B(\overline{T}) & \text{if } S(\overline{T}) < S^*_{\overline{T}} \\ 0 & \text{if } S(\overline{T}) \geq S^*_{\overline{T}} \end{cases} \tag{9.22}$$

We see from Eq. (9.20) that there is no opportunity loss when $S(t) \geq S^*_t$. In this case, it is optimal for the commercial company to initiate immediate development, and thus no need for future intervention.

If the spot price of oil is close to zero, the probability that the oil field still will be undeveloped at date $\overline{T}$ is close to one. Furthermore, the current value of the future production of oil is negligible. In this case, the the promise is thus similar to a commitment to pay the costs $B(\overline{T})$ at the future date $\overline{T}$, implying an opportunity loss as stated in Eq. (9.21).

Finally, with the spot price being lower than the trigger price, and the time to maturity $\overline{T} - t$ approaching zero, the promise boils down a commitment to implement the arrangement immediately, see Eq. (9.22).

9.4 Immediate intervention

The arrangement considered in the previous section causes the commercial company to behave somewhat "myopic", as the oil field is managed during the period from date t to $\overline{T}$ as if no promise exists. Obviously, if development is to be initiated before or at date $\overline{T}$, this fact should be taken into account when deciding upon the investment strategy.

Now, suppose instead that the Government buys the perpetual investment opportunity immediately at the current market value, and instructs the national oil company to start development before or at the future date $\overline{T}$. To the national oil company, the transferred oil field represents a commitment to initiate not later than date $\overline{T}$. The national company is not required to fix the initiation date in advance, however, but may choose an investment strategy made conditional on the future risky output price.

Given the Governmental instruction, the undeveloped oil field may be interpreted as a contingent claim. Clearly, its current value W, is bounded from below by the "traditional" field value in the constrained case, $\overline{V}$, where the initiation date $T \in [t, \overline{T}]$ is fixed in advance. On the other hand, W is bounded from above by the value of the perpetual investment opportunity. In short, we have argued

$$\overline{V}(S,t) \leq W(S,t) \leq U(S,t) \tag{9.23}$$

where $\overline{V}$ and U are given by Eqs. (9.2) and (6.7), respectively.

The value of the oil field, contingent on the Governmental instruction, $W(S,t)$, satisfies the partial differential equation

$$\tfrac{1}{2}\sigma^2 S^2 W_{SS} + (r-\delta)SW_S - rW + W_t = 0 \tag{9.24}$$

Furthermore, boundary conditions must be specified.

First, suppose the current spot price of oil is close to zero. In that case, the probability that the project will be initiated at the latest possible date approaches one, while the current value of the future production approaches zero. The obligation to initiate before or at the date $\overline{T}$ will thus converge towards an obligation to pay the investment costs at this future date. We have

$$\lim_{S\to 0} W(S,t) = -e^{-r(\overline{T}-t)}B(\overline{T}) \tag{9.25}$$

Second, if development has not been initiated before the future date $\overline{T}$, immediate development is required at that date, and thus

$$W(S,t=\overline{T}) = AS(\overline{T}) - B(\overline{T}) \tag{9.26}$$

Third, there exists a trigger price function $S^*_W(\tau)$, $\tau \in [t,\overline{T}]$, describing the investment strategy that maximizes the value of the oil field. This condition is given by

$$W(S^*_W(\tau),\tau) = AS^*_W(\tau) - B(\tau) \tag{9.27}$$

The partial differential equation and the three boundary conditions determine the field value and the optimal investment strategy.

Unfortunately, there exists no closed form solution to the field value in this case. For a given set of parameter values, however, the value $W(S,t)$ and the optimal investment strategy may be approximated by employing numerical methods.[3]

It is now easy to obtain the opportunity loss given this arrangement. We recall that the Government in this case acquires the perpetual investment opportunity immediately at the current market value $U(S,t)$, given by Eq. (6.7). The value of the undeveloped oil field when instructing the national oil company to initiate investments before or at date $\overline{T}$ corresponds to $W(S,t)$, examined just above. We may thus conclude that the opportunity loss induced by the Governments promise is

$$\text{Opportunity loss} = U(S,t) - W(S,t) \tag{9.28}$$

in this case.

The opportunity loss with this arrangement is less than the opportunity loss in the previous section. The reason for this is that the promise now is taken into account when managing the oil field during the period from date t to $\overline{T}$.

[3] For a survey of numerical methods, see Geske and Shastri (1985).

9.5 A numerical example

In this section, we provide a numerical example, using our base case parameter values.[4] Furthermore, suppose that the undeveloped oil field has to be initiated within $\overline{T} - t = 4$ years.

From the previous chapters, we recall that the accept/reject break-even price is $S_{BE} = 8$ USD/barrel, and that the trigger price indicating exercise of the perpetual investment opportunity, is $S^* = 16$ USD/barrel.

With the base case parameter values, the traditional optimal timing model collapses into the simple accept/reject decision case.[5] The critical price, indicating indifference between accepting the project and to undertake a commitment to initiate at the future date $\overline{T}$, is

$$gS_{BE} = 6.8 \text{ USD/barrel}$$

see Eq. (9.5) above.

In the case where the Government acquires the undeveloped oil field immediately, and transfers it to the national company, the trigger price indicating indifference between intiating immediately and deferring the decision is[6]

$$S_W^* = 9.3 \text{ USD/barrel}$$

This trigger price is a function of the remaining time to the maturity date of the promise, $\overline{T} - t$.

In the following table, we present the field values for several spot prices, and given different degrees of decision flexibility.

[4] We recall from Section 3.4 that $\delta = 0.06$, $r = 0.05$, $\pi = 0$, $\sigma^2 = 0.07$, $A = 130$, and $B = 1040$.

[5] See Section 4.3 and note that $\delta > r - \pi$.

[6] The trigger price S_W^* and the field value W are found by using the explicit finite difference method, see, e.g., Brennan and Schwartz (1978).

FIELD VALUE (Mill. USD)				
Spot	Trad. model		Option model	
price	V	$\overline{V}$	U	W
4.2	0	-425	71	-423
4.9	0	-350	98	-345
5.8	0	-261	135	-251
6.8	0	-157	188	-136
8.0	0	0	260	6
9.4	184	184	360	184
11.1	401	401	499	401
13.1	656	656	692	656
16.0	1040	1040	1040	1040
18.8	1408	1408	1408	1408

The second column shows the value V in the case of an accept/reject investment decision, c.f. Eq. (9.4). Note that $V = 0$ when the spot price equals or is lower than $S_{BE} = 8$ USD/barrel. In the third column, we find the field value $\overline{V}$ in the corresponding constrained case. The decision maker is here to choose between to develop immediately or to undertake a commitment to initiate at date $\overline{T}$, see Eq. (9.7). The two field values V and $\overline{V}$ are equal when the current price exceeds the break-even price S_{BE} (8 USD/barrel), as accepting the project then is optimal.

In the two last columns, the oil field is considered as a contingent claim, where the investment strategy may be made dependent on the development of the future spot output price. The fourth column represents the value in the case of a perpetual investment opportunity U, see Eq. (6.7), whereas the last one takes into account the promise to initiate before or at date $\overline{T}$. When the spot price exceeds the trigger price S^* = 16 USD/barrel, initiating the non-expiring investment opportunity is optimal, and the two field values U and W are equal.

We note that the difference between the asset value in the constrained case according to the traditional model and the contingent claim model, $\overline{V}$ and W, is small. This means that given the Governments instruction, the value of the flexibility to follow an investment strategy made contingent on the future spot price, rather that fixing the initiation date in advance, is negligible.

The opportunity losses are given in the following table:

OPPORTUNITY LOSS (Mill. USD)			
Spot price	Trad $V - \overline{V}$	Option-model L	$U - W$
4.2	425	496	494
4.9	350	447	443
5.8	261	395	387
6.8	157	340	324
8.0	0	282	254
9.4	0	223	176
11.1	0	161	98
13.1	0	95	35
16.0	0	0	0
18.8	0	0	0

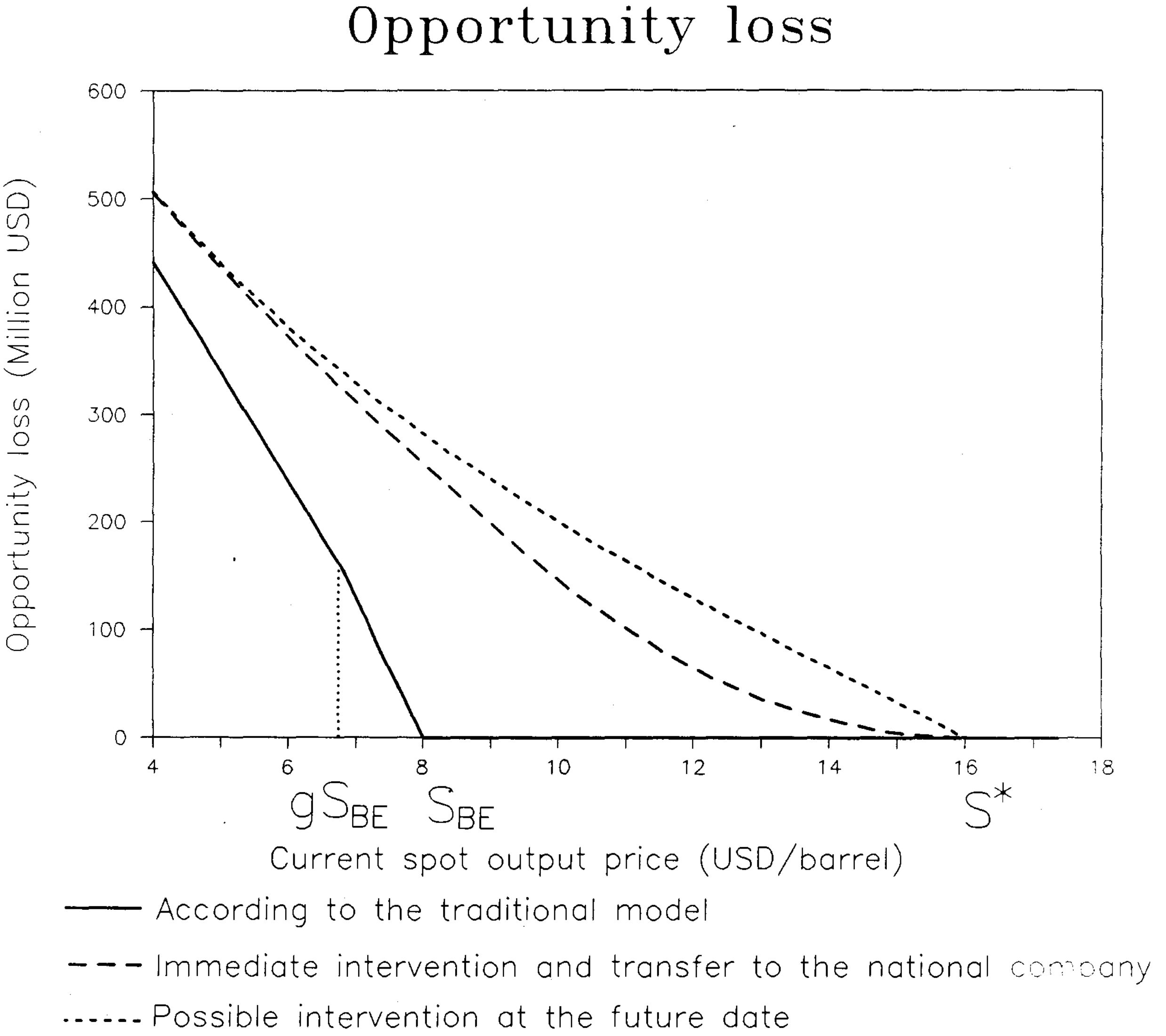
and is also illustrated in Figure 9.1.

According to the traditional model, an opportunity loss is incurred only if the current spot price is lower than the break even price S_{BE}. From Figure 9.1, we see that $V - \overline{V}$ (the linear curve) is kinked in the critical price gS_{BE}, indicating indifference between investing immediately and undertaking a commitment to initiate at date $\overline{T}$.

We see from the figure that the traditional model understates the opportunity loss induced by the development promise. From the previous table, we recall that the numerical values of $\overline{V}$ and W are approximately equal. This means that the difference $U - \overline{V}$ in this case may serve as an approximation to the "true" opportunity loss $U - W$, given by the contingent claim model. The difference in the opportunity loss between the traditional and the contingent claim model is thus mainly caused by the traditional model failing to evaluate properly the unconstrained investment opportunity.

The numerical results indicate that if the Government bases the decision to make a promise as considered above on on a traditional model, this may lead to a serious loss of value.

Figure 9.1


Opportunity loss
Opportunity loss (Million USD)
600
500
400
300
200
100
0
4
6
8
10
12
14
16
18
gS
BE
S
BE
S*
Current spot output price (USD/barrel)
According to the traditional model
Immediate intervention and transfer to the national company
Possible intervention at the future date

Appendix A

The evaluator function φ

Consider the contingent claim with a pay-off at the future date T, defined by

$$Y \equiv S(T)^{\epsilon} \cdot I\left(\inf_{\tau\in[t,T]}\left(e^{\pi(\tau-t)}S_t^* - S(\tau)\right) > 0\right) \tag{A.1}$$

The future pay-off represents a random variable that is dependent on the sample path of S from the current date t to the maturity date T. It provides no pay-off at date T if the spot price $S(\tau)$ equals or exceeds the associated trigger price for some date $\tau \in [t, T]$.

Define the stopping date τ_s by

$$\tau_s \equiv \inf\left\{\tau \in [t,\infty) : S(\tau) \geq e^{\pi(\tau-t)}S_t^*\right\} \tag{A.2}$$

The random variable τ_s represents the first date for which the spot price S equals or exceeds the corresponding trigger price. If the event

$$\{\tau_s \in [t,\infty) : \tau_s \leq T\}$$

happens, the contingent claim will provide no pay-off at date T, and we thus have $Y = 0$ with probability one.

In the following, we divide the evaluation problem into regions.

A.1 The interior region

Consider the region defined by

$$S(t) < S_t^* \text{ and } t < \min\{\tau_s, T\}$$

In this region, the claim is still "alive", with a positive probability of providing a positive pay-off at the future maturity date T.

Proposition 7 *In this region, the function $\varphi(S,t)$, defined in Eqs. (9.17) - (9.18), satisfies the partial differential equation*

$$\tfrac{1}{2}\sigma^2 S^2 \varphi_{SS} + (r-\delta)S\varphi_S - r\varphi + \varphi_t = 0 \tag{A.3}$$

and has the limiting values

$$\lim_{S\to S_t^*} \varphi(S,t) = 0 \tag{A.4}$$

$$\lim_{S\to 0} \varphi(S,t) = \begin{cases} \infty & \text{if } \varepsilon < 0 \\ e^{-r(T-t)} & \text{if } \varepsilon = 0 \\ 0 & \text{if } \varepsilon > 0 \end{cases} \tag{A.5}$$

$$\lim_{t\to T} \varphi(S,t) = \begin{cases} S(T)^\varepsilon & \text{if } S(T) < S_T^* \\ 0 & \text{if } S(T) = S_T^* \end{cases} \tag{A.6}$$

Proof of Eq. (A.3): It can be verified that the partial derivatives of φ are

$$\begin{aligned}\varphi_S &= \Psi_S(S,t \mid S_T^*, \varepsilon) - (S_T^*)^{\varepsilon-\kappa}\Psi_S(S,t \mid S_T^*, \kappa) \\ &- \left(2F - 2\frac{d}{\sigma\sqrt{T-t}}\right)\frac{\varphi}{S}\end{aligned} \tag{A.7}$$

$$\begin{aligned}\varphi_{SS} &= \Psi_{SS}(S,t \mid S_T^*, \varepsilon) - (S_T^*)^{\varepsilon-\kappa}\Psi_{SS}(S,t \mid S_T^*, \kappa) \\ &- \frac{2}{\sigma^2}\left(2F - 2\frac{d}{\sigma\sqrt{T-t}}\right)\left((\kappa-\tfrac{1}{2})\sigma^2 - \frac{\sigma d}{\sqrt{T-t}}\right)\frac{\varphi}{S^2}\end{aligned} \tag{A.8}$$

$$\begin{aligned}\varphi_t &= \Psi_t(S,t \mid S_T^*, \varepsilon) - (S_T^*)^{\varepsilon-\kappa}\Psi_t(S,t \mid S_T^*, \kappa) \\ &- \left(2F - 2\frac{d}{\sigma\sqrt{T-t}}\right)\frac{\ln(S/S_T^*)}{T-t}\varphi\end{aligned} \tag{A.9}$$

where

$$F \equiv F(S,t \mid S_T^*, \kappa) = \frac{n[d(S,t \mid S_T^*, \kappa)]}{\sigma\sqrt{T-t}N[-d(S,t \mid S_T^*, \kappa)]} \tag{A.10}$$

$$d \equiv d(S,t \mid S_T^*, \kappa) = \frac{\ln(S/S_T^*) + (r - \delta + (\kappa - \tfrac{1}{2})\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \tag{A.11}$$

By inserting Eq. (9.17) and Eqs. (A.7) - (A.9) into the partial differential equation, Eq. (A.3), and by using the result from the previous chapter that Ψ satisfies the partial differential equation, we find that the remaining terms cancel out. Thus, the evaluator φ satisfies the partial differential equation of a contingent claim. □

Proof of Eq. (A.4): We see from Eq. (9.18) that $\kappa(S_t^*, t) = \varepsilon$. By inserting $S = S_t^*$ and $\kappa = \varepsilon$ into Eq. (9.17), the limit stated in Eq. (A.4) is verified. □

Proof of Eq. (A.5): By taking the limit of each side of Eq. (9.17), we have

$$\begin{aligned}\lim_{S\to 0} \varphi(S,t \mid \varepsilon) &= \lim_{S\to 0} \Psi(S,t \mid S_T^*, \varepsilon) \\ &- \lim_{S\to 0} \left((S_T^*)^{\varepsilon-\kappa}\Psi(S,t \mid S_T^*, \kappa)\right)\end{aligned}$$

By comparing Eqs. (8.7) and Eq. (A.5) just above, we see that the limit on the left hand side and the limit of the first term on the right hand side of the equation just above are identical. To verify Eq. (A.5), we thus need to show that

$$\lim_{S\to 0}(S_T^*)^{\varepsilon-\kappa}\Psi(S,t \mid S_T^*, \kappa) = 0$$

By using the definition of Ψ, we may alternatively write the expression just above as

$$(S_T^*)^{\varepsilon-\kappa}\Psi(S,t \mid S_T^*, \kappa) = e^{-r(T-t)}(S_T^*)^{\varepsilon}e^{\theta\kappa}N[-d(S,t \mid S_T^*, \kappa)] \quad \text{(A.12)}$$

where

$$\theta \equiv [r - \pi - \delta + \tfrac{1}{2}(\varepsilon - 1)\sigma^2](T - t) \quad \text{(A.13)}$$

We note from Eq. (9.18) that

$$\lim_{S\to 0} \kappa(S,t \mid \varepsilon) = \infty$$

By inserting κ from Eq. (9.18) into the expression of d, Eq. (A.11), we find that

$$\lim_{S\to 0} d(S,t \mid S_T^*, \kappa(S)) = \infty$$

and thus

$$\lim_{S\to 0} N[-d(S,t \mid S_T^*, \kappa(S))] = 0$$

If $\theta < 0$, we see directly from Eq. (A.12) that the limit is zero, and Eq. (A.5) is thus verified. In the case of $\theta \geq 0$, using l'Hopitals rule and rearranging lead to

$$\lim_{S\to 0}(S_T^*)^{\varepsilon-\kappa(S)}\Psi(S,t \mid S_T^*, \kappa(S)) = -k_1 \lim_{\kappa\to\infty} e^{-\frac{1}{2}k_2\left[\frac{1}{2}\sigma^2(T-t)\kappa-\theta\right]^2} = 0 \tag{A.14}$$

where k_1 and k_2 are positive constants. We may thus conclude that the limit of $\varphi(S,t \mid \varepsilon)$ in this case is determined by the limit of the first term on the right hand side of Eq. (9.17), presented in Eq. (8.7). Eq. (A.5) above is thus confirmed. □

Proof of Eq. (A.6): We know that the limit of the first term of Eq. (9.17) is

$$\lim_{t\to T}\Psi(S,t \mid S_T^*, \varepsilon) = \begin{cases} S^\varepsilon & \text{if } S < S_T^* \\ \frac{1}{2}S^\varepsilon & \text{if } S = S_T^* \end{cases} \tag{A.15}$$

see Eq. (8.8) in the previous chapter. Now, consider the second term of Eq. (9.17). By some rearranging, we may write the limit of this term as

$$\lim_{t\to T}(S_T^*)^{\varepsilon-\kappa}\Psi(S,t \mid S_T^*, \kappa) = \lim_{t\to T} e^{-r(T-t)}(S_T^*)^{\varepsilon}e^{\theta\kappa}N[-d(S,t \mid S_T^*, \kappa)]$$

where θ is defined by Eq. (A.13). From the definitions of θ and κ, Eqs. (A.13) and (9.18), it follows that

$$\lim_{t\to T} e^{\theta\kappa} = e^{k_3 \ln(S_T^*/S)} > 0$$

where k_3 is a constant. Morover, we see that this limit is unity when $S = S_T^*$. By inserting κ from Eq. (9.18) into the definition of d, Eq. (A.11), and rearranging, we find that the limit of d is

$$\lim_{t\to T} d(S,t \mid S_T^*, \kappa) = \begin{cases} \infty & \text{if } S < S_T^* \\ 0 & \text{if } S = S_T^* \end{cases}$$

and thus

$$\lim_{t\to T} N[-d(S,t \mid S_T^*, \kappa)] = \begin{cases} 0 & \text{if } S < S_T^* \\ \frac{1}{2} & \text{if } S = S_T^* \end{cases}$$

We may conclude that the limit of the second term of Eq. (9.17) is

$$\lim_{t\to T}(S_T^*)^{\varepsilon-\kappa}\Psi(S,t \mid S_T^*,\kappa) = \begin{cases} 0 & \text{if } S < S_T^* \\ \frac{1}{2}(S_T^*)^{\varepsilon} & \text{if } S = S_T^* \end{cases} \tag{A.16}$$

By combining the results with respect to the two terms on the right hand side of Eq. (9.17), Eqs. (A.15) and (A.16), respectively, Eq. (A.6) is verified. □

Conclusion: From Eq. (A.3), we may conclude that the strategy associated with the evaluator φ is self-financing in the region considered here. Furthermore, Eq. (A.6) states that if S is lower than the trigger price for the entire period from date t and to date T, i.e., $T < \tau_s$, the value of the self-financing portfolio and the value of the contingent claim Y are identical at the maturity date T.

A.2 The boundary

Now, consider the boundary

$$S(t) = S_t^* \text{ and } t = \tau_s \leq T$$

This means that the price S at date $t \leq T$ hits the trigger price from below for the first time. This event causes the future pay-off from the contingent claim to be zero with probability one. The current value of the replicating portfolio is thus in this case required to be zero.

From Eqs. (A.4) and (A.6) above, we see that the function, defined in the interior region, has the limiting value

$$\lim_{S\to S_\tau^*} \varphi(S,\tau) = 0 \text{ for } \tau \in [t,T]$$

The value of the function in the interior region thus converges to the required boundary value. We thus assign

$$\varphi(S(t) = S_t^*, t = \tau_s) = 0$$

at the boundary.

A.3 A trivial boundary

Recall that the event, causing the contingent claim to provide no pay-off, is defined on the sample path S from date t and until date T. In the case where the price at the initial date t exceeds the corresponding trigger price, we have

$$S(t) > S_t^* \Rightarrow \tau_s = t$$

In this case, we know already at the initial date t that the contingent claim has zero pay-off at date T. We thus require the value of the replicating portfolio to be

$$\varphi(S, t) = 0$$

when $S(t) > S_t^*$ and $\tau_s = t$.

A.4 The remaining region

The remaning region corresponds to the case where the condition for the contingent claim to provide a future pay-off has been violated. In this case, we assign

$$\varphi(S, \tau) = 0$$

when $\tau_s < \tau \leq T$.

A.5 Conclusion

We have verified that the function $\varphi(S, t)$, stated in Eq. (9.17), satisfies the self-financing condition, and generates a pay-off at the maturity date T identical to Y, as defined in Eq. (A.1). We may thus conclude that the function $\varphi(S, t)$ is the pricing function of the contingent claim.

We recall from Eq. (9.19) that the opportunity loss from the promise is

$$\begin{aligned} L(S,t) &= \alpha(\overline{T})\varphi(S, t \mid S_{\overline{T}}^*, \beta) \\ &- A\varphi(S, t \mid S_{\overline{T}}^*, 1) \\ &+ B(\overline{T})\varphi(S, t \mid S_{\overline{T}}^*, 0) \end{aligned} \tag{A.17}$$

With $\alpha(\overline{T})$ and $B(\overline{T})$ constant, $L(S,t)$ is a linear combination of claims which each satisfies Eq. (A.3). We may thus conclude that the function $L(S,t)$ satisfies the partial differential equation of a continent claim.

Furthermore, it follows from Eqs. (A.4) - (A.6) above that the function $L(S,t)$ has the following properties:

$$L(S,t) = 0 \text{ if } S(t) \geq S_t^* \tag{A.18}$$

$$\lim_{S\to 0} L(S,t) = e^{-r(\overline{T}-t)}B(\overline{T}) \tag{A.19}$$

$$\lim_{t\to\overline{T}} L(S,t) = \begin{cases} \alpha(\overline{T})S(\overline{T})^{\beta} & \\ -\ AS(\overline{T}) + B(\overline{T}) & \text{if } S(\overline{T}) < S_{\overline{T}}^* \\ 0 & \text{if } S(\overline{T}) \geq S_{\overline{T}}^* \end{cases} \tag{A.20}$$

The results are stated in Section 9.3 above.

Appendix B

The derivation of the evaluator φ

In Appendix A, we show that the function φ satisfies the partial differential equation with suitable boundary conditions, and conclude that it represents the desired pricing function. This proof relies on knowing the function in the first place. Thus, in the following, we show how the evaluator function is obtained.

Consider the claim with future pay-off

$$Y \equiv S(T)^{\epsilon} I(S(T) < S_T^*) I(A_T) \tag{B.1}$$

where we define the event A_T

$$A_T \equiv \left\{ \inf_{\tau \in [0,T]} (e^{\pi\tau} S_0^* - S(\tau)) > 0 \right\} \tag{B.2}$$

We can see that the pay-off from the claim at the future date T is made conditional on the characteristics of the price path of S during the period $\tau \in [0, t]$. This contingent claim may be attained in our economy, see Harrison and Kreps (1979).[1] The intuition behind this interesting result is as follows: Suppose that the spot price $S(\tau)$ has

[1]The authors consider a general model where the economy is generated by a multi-dimensional diffusion process. They note (p. 396) that "every contingent claim may be expressed as a function of the vector price history over the interval $[0, t]$".

been lower than the trigger price S^*_τ for the period $[0, t]$. The possible event that the spot price will hit the trigger price from below during the time interval $[t, t + dt)$ is then dependent on the spot price at that date, $S(t)$, and the stochastic price increment $dS(t)$. The fact that the future pay-off from the claim is made conditional on the event A_T does not introduce any new dimension of uncertainty to our problem.

The unique price at which the claim Y may be attained today, is

$$\begin{aligned} \varphi &\equiv V_0\left[S(T)^\epsilon I(S(T) < S^*_T) I(A_T)\right] \\ &= e^{-rT} E_0\left[S(T)^\epsilon I(S(T) < S^*_T) I(A_T)\xi\right] \end{aligned} \tag{B.3}$$

where $V_0[\cdot]$ is a general evaluator, and where ξ represents the function presented in Appendix B in Chapter 8. By using conditional expectation, we may alternatively write

$$\varphi = e^{-rT} E_0\left[S(T)^\epsilon I(S(T) < S^*_T) E_0[I(A_T)\xi \mid S(T)]\right] \tag{B.4}$$

The inner expectation represents the certainty equivalent of a one-dollar claim contingent on the event A_T, as a function of the future price $S(T)$.

We recall from Appendix B in Chapter 8 that the unique function for evaluating contingent claims with pay-off at date T, is

$$\xi = \xi(X) = \exp\left\{-\frac{(\mu^* T)X}{\sigma^2 T} - \frac{1}{2}\frac{(\mu^* T)^2}{\sigma^2 T}\right\}$$

where the price at date T is

$$S_T(X) = S_0 e^{(\alpha - \frac{1}{2}\sigma^2)T + X}$$

It is important to notice that $\xi(X)$ only depends on the price $S_T(X)$ at the terminal date. This means that given the terminal price $S_T(X)$, ξ is independent of whether the event A_T has occurred or not.

The inner expectation of Eq. (B.4) may thus be written

$$E_0[I(A_T)\xi \mid S(T)] = \Pr\{A_T \mid S(T)\}\xi$$

By inserting this expression into (B.4), we obtain

$$\varphi = E_0\left[I(S(T) < S^*_T) \Pr\{A_T \mid S(T)\} S(T)^\epsilon \xi\right] \tag{B.5}$$

The only factor not yet known to us is the conditional probability $\Pr\{A_T \mid S(T)\}$.

Proposition 8 *The conditional probability is given by*

$$\begin{aligned}\Pr\{A_T \mid S(T)\} &\equiv \Pr\left\{\inf_{\tau\in[0,T]}(e^{\pi\tau}S_0^* - S(\tau)) > 0 \mid S_0, S_T\right\} \\ &= 1 - (S_T^*/S_T)^{\varepsilon-\kappa} \end{aligned} \tag{B.6}$$

where

$$\kappa \equiv \varepsilon + 2\frac{\ln(S_0^*/S_0)}{\sigma^2 T} \tag{B.7}$$

Proof: See Appendix B.1 below. □

Note that the conditional probability is independent of the drift term of S. Thus, even if the investors disagree upon the parameter value of α, they will agree upon the conditional probability.

Now, by inserting the conditional probability into the Eq. (B.5), and by rearranging, we obtain

$$\begin{aligned}\varphi &= e^{-rT}E_0\left[I(S(T) < S_T^*)S(T)^{\varepsilon}\xi\right] \\ &- (S_T^*)^{\varepsilon-\kappa}e^{-rT}E_0\left[I(S(T) < S_T^*)S(T)^{\kappa}\xi\right] \end{aligned} \tag{B.8}$$

We see that the two expectations on the right hand side may be interpreted as contingent claims on the power-function pay-off. By calling upon the result from Chapter 8, we have that the value of our path-contingent claim is

$$\varphi = \Psi(S,t \mid, S_T^*, \varepsilon) - (S_T^*)^{(\varepsilon-\kappa)}\,\Psi(S,t \mid S_T^*, \kappa) \tag{B.9}$$

where the parameter κ is defined by Eq. (B.7) just above.

B.1 The conditional probability

In this section, we find the conditional probability

$$P \equiv \Pr\left\{\inf_{\tau\in[0,T]}(e^{\pi\tau}S_0^* - S(\tau)) > 0 \mid S(0) = S_0, S(T) = S_T\right\} \tag{B.10}$$

We recall that the price process of the risky asset $S(\tau)$ is

$$S(\tau) = S_0 e^{(\alpha-\frac{1}{2}\sigma^2)\tau + X(\tau)}$$

where the term $X(\tau)$ is normally distributed with zero expectation and variance $\sigma^2\tau$.

It is easy to verify that the conditional probability P is equivalent to

$$P = \Pr\left\{\inf_{\tau\in[0,T]} Z(\tau) > 0 \mid Z(0) = Z_0, Z(T) = Z_T\right\}$$

where we define the transformed variable

$$Z(\tau) \equiv \ln(S^*_\tau / S(\tau)) \sim N(Z_0 + \mu\tau, \sigma^2\tau)$$

and where

$$\begin{aligned} \mu &\equiv \pi - \alpha + \tfrac{1}{2}\sigma^2 \\ Z_0 &= \ln(S^*_0/S_0) \\ Z_T &= \ln(S^*_T/S_T) \end{aligned}$$

We now need the following result

Proposition 9

$$\Pr\left\{\inf_{\tau\in[0,T]} Z(\tau) > 0 \mid Z(0) = Z_0, Z(T) = Z_T\right\} = 1 - \exp\left(-2\frac{Z_0 Z_T}{\sigma^2 T}\right)$$

Proof: See Appendix B.2 below. □

By making the necessary substitutions, we find that the conditional probability is

$$\begin{aligned} P &\equiv \Pr\left\{\inf_{\tau\in[0,T]} (e^{\pi\tau} S^*_0 - S(\tau)) > 0 \mid S(0) = S_0, S(T) = S_T\right\} \\ &= 1 - (S^*_T/S_T)^{\varepsilon-\kappa} \qquad \text{(B.11)} \end{aligned}$$

where we define

$$\kappa \equiv \varepsilon + 2\frac{\ln(S^*_0/S_0)}{\sigma^2 T} \qquad \text{(B.12)}$$

The result is stated as Eqs. (B.6) and (B.7) above.

B.2 The Brownian bridge

Consider a stochastic process $Z(\tau)$ described by

$$Z(\tau) = Z_0 + \mu\tau + \sigma B(\tau) \tag{B.13}$$

where $Z_0 > 0$ is the initial value of the process, and $B(\tau)$ represents the standard Brownian motion. As viewed from date 0, the process $Z(\tau)$ is normally distributed, that is, we have

$$Z(\tau) \sim N\left(Z_0 + \mu\tau, \sigma^2\tau\right) \tag{B.14}$$

We are interested in the conditional probability

$$P \equiv \Pr\left\{\inf_{\tau\in[0,T]} Z(t) > 0 \mid Z(0) = Z_0, Z(T) = Z_T\right\}$$

P represents the probability that the process $Z(\tau)$ will be positive for the period from date 0 to T, contingent on the positive initial value $Z(0) = Z_0$ and the positive terminal value $Z(T) = Z_T$. The process, restricted with respect to both the initial and the terminal value, is known as a "Brownian bridge".

We know that the probability density function of the process value at the future date T, conditional on the initial value being Z_0, is

$$\Pr\{Z_T \mid Z_0\} = n\left[\frac{Z_T - Z_0 - \mu T}{\sigma\sqrt{T}}\right] dz \tag{B.15}$$

where $n[\cdot]$ is the standard normal probability density function.

Furthermore, we know that the probability density function that the process ends up with the positive value Z_T at date T and follows a sample path where $Z(\tau)$ is positive from date 0 to T, given the positive initial value Z_0, is

$$\begin{aligned}
&\Pr\left\{Z_T \cap \left(\inf_{\tau\in[0,T]} Z(\tau) > 0\right) \mid Z_0\right\} \qquad \text{(B.16)}\\
= \; & n\left[\frac{Z_T - Z_0 - \mu T}{\sigma\sqrt{T}}\right] dz - \exp\left(\frac{-2\mu Z_0}{\sigma^2}\right) n\left[\frac{Z_T + Z_0 - \mu T}{\sigma\sqrt{T}}\right] dz
\end{aligned}$$

see, e.g., Ingersoll (1987) p. 352.

Now, according to the law of probability, we may express the desired conditional probability P as

$$P = \frac{\Pr\left\{Z_T \cap (\inf_{\tau\in[0,T]} Z(\tau) > 0) \mid Z_0\right\}}{\Pr\{Z_T \mid Z_0\}}$$

By inserting Eqs. (B.16) and (B.15) in the nominator and the denominator above, respectively, and by simplifying, we obtain that the desired probability is

$$\begin{aligned} P &\equiv \Pr\left\{\inf_{\tau\in[0,T]} Z(\tau) > 0 \mid Z(0) = Z_0, Z(T) = Z_T\right\} \\ &= 1 - \exp\left(-2\frac{Z_0 Z_T}{\sigma^2 T}\right) \qquad \text{(B.17)} \end{aligned}$$

With the product $Z(0) \cdot Z(T) = 0$, we see that the probability P is zero. Further, P is an increasing function of both $Z(0)$ and $Z(T)$, and converges towards unity as $Z(0) \cdot Z(T) \to \infty$.

We note that the probability is *independent* of the drift term μ of the process. Ceteris paribus, a higher variability $\sigma^2 T$ reduces the probability that the process will stay positive from date 0 to T.

Chapter 10

CONCLUDING REMARKS

The essence of this dissertation is the analysis of the investment decision under output price uncertainty, applied to the case of an undeveloped oil reserve. The aim has been to obtain a theoretical platform for the optimal management decision strategy and the corresponding project value. The methodology of contingent claims analysis has been applied.

We assumed that the spot price dynamics of output follows a geometric Brownian motion. Furthermore, we introduced the traded "twin asset" with equilibrium expected rate of return. The relationship between the price dynamics of output and the price dynamics of the "twin asset" was described by a constant "rate of return shortfall" on output. The assumptions lead to a fairly simple relationship between the current value of a future claim on output and the current spot price.

An oil reserve, if developed, has been interpreted as a commitment to extract oil according to a fixed preset production schedule, combined with a commitment to pay the future costs incurred according to a corresponding cost schedule. Investment has been assumed to be irreversible and modeled as undertaking a developed oil field.

Several investment decision situations have been analyzed. We started with the simple case of accept/reject, where the decision maker choose between to undertake the project immediately or to reject the project forever. Gradually, we allowed for more complex decision situations. We took explicitly into account the opportunity to defer the investment decision itself, and to make it at some later date conditional on relevant information received in the meantime. For each case, the optimal management strategy has been provided, stated in terms of a break-even price decision rule. Furthermore, the corresponding optimal project value was presented.

The contingent claims framework was also applied to evaluate the economic cost of some regulations which may be imposed on an undeveloped oil reserve. The regulations were interpreted as constraints on the feasible set of management strategies available to the owner of the field, and the costs were evaluated by the implied opportunity losses. In particular, we considered the case of a temporary freeze on development, and a case of a promise to develop not later than a fixed future date.

The results in this dissertation are not restricted to the case of oil. The most promising area for practical applications of contingent claims analysis, however, appears to be projects involving non-renewable resources: First, in the case of a non-renewable resource, only companies possessing a particular type of input (i.e., a reserve of the resource) may produce the physical output. Second, the total future production of output is more or less given by the characteristics of the reserve. Third, the output is fairly homogeneous, and is for most cases associated with markets where both the commodity itself and financial assets written on the commodity are traded.

Contingent claims analysis uses information, in principle derived from the market, in order to obtain the implicit market value of the project. One important challenge for further research is to which extent relevant information actually may be extracted from market data.

In the case of oil, one approach to extract market information may be to analyze stock prices of oil companies. A problem here, however, is that many oil companies represent in fact a portfolio of different oil reserves, rather than single projects. Furthermore, the stock prices may be distorted by the corporate tax system and regulations imposed by the Governments.

An alternative approach may be to analyze market prices of the commodity itself and financial assets written on the commodity. The assets for which market prices are available, however, are short-lived whereas most oil projects are long-lived assets.

A topic of particular interest to the models presented in this dissertation is the market behaviour of the spot output price and the rate of return shortfall (the convenience yield).[1] So far, one possible extension of the assumptions made in this dissertation may be to model the spot output price as a mixed diffusion and jump process, thus allowing the price path to be discontiuous.[2] Another extention may be to describe the rate of return shortfall as a function of time, or possibly as being stochastic.[3]

There is, however, clearly a trade-off between the "realism" of a model, and its possible applicability. The approach chosen in this dissertation has been to tilt the assumptions in favour of simplicity, and to adapt and develop analytical models that may readily be implemented to practical problems.

To conclude, the results derived in this dissertation demonstrate that management can increase project value substantially by implementing investment decision rules derived from contingent claims analysis.

Sucessful application of more sophisticated methods into the managerial sphere concurrently demand extended management knowledge of decision flexibility theory as well as suitable management tools.

The benefit of advanced methods and improved results are the obvious incentives. Therefore, academia as well as management may concurrently prosper form further development of theory as well as improved practical knowledge and means of application.

Within the field of contingent claims analysis there is undoubtedly room for further research and development.

[1]See Gibson and Schwartz (1989).

[2]See Aase (1988).

[3]See Brennan (1989).

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