PORTFOLIO CHOICE IN A THEORY OF SAVING.

By

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1. Introduction.

The theory of consumer saving is usually developed on the assumption that savings can be invested in one asset only, bearing a fixed rate of return. This is a natural assumption, since the theory is constructed for a world of certainty, and in such a world there should be no reason, at least in the absence of transactions costs, for a rational consumer to hold his savings in any other asset than the one yielding the highest rate of return. True, this might not always be the same asset, so that changes in rates of return might lead to adjustments of savings portfolios, but such adjustments would always take the form of complete switches, so that any consumer would always be holding one asset only.[1] For the economy as a whole, it should be noted, more than one asset might still willingly be held by the consumer sector, owing to differences of opinion concerning yields. Tobin ([8] , pp. 68-70) refers to this viewpoint as the Keynesian explanation of the smoothness of the aggregate liquidity preference schedule.

In reality, of course, savers can - and do - invest in more than one asset. The explanation of this must be sought in part by the uncertainty of the yield associated with some kinds of assets.[2] The modern theory of portfolio selection rationalizes asset choice in these terms, but it has not succeeded in integrating asset choice with the analysis of the consumption-saving decision. Now, it is of course true that, as Tobin has remarked ([9] , p. 28), there are great tactical advantages to the theorist in treating separately the decision on the total amount of saving to be made out of current income and the decision on how to allocate total portfolio resources between various kinds of assets. Still, since these decisions seem to have a high degree of interdependence in practice, an attempt to analyze them within a unified framework seems to be called for.

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1) A model of saving and portfolio choice under conditions of certainty has been analyzed by Roger F. Miller in [5] .

2) There are other explanations too. Money is demanded for transactions purposes, which we abstract from in this paper. Also, real assets like houses and cars are demanded because their consumption services cannot be fully enjoyed without ownership.

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It has been found convenient to start out in section 2 with a discussion of a simple model of saving under certainty. After a discussion in section 3 of some measures of risk aversion, section 4 analyzes a model where the assumption of one asset only is preserved, but where the rate of return to saving is a random variable. Sections 5 through 7 present a model with two assets, money and a risky asset, and analyze effects of changes in income and yield. Section 8 analyzes changes in the degree of riskiness, as measured by the variance of yield, in terms of a quadratic utility function. In section 9 we comment on the possibility of extending the model to allow for borrowing. Finally, section 10 contains some concluding remarks.

2. The Consumption-Saving Decision under Conditions of Certainty.

We shall make no attempt here to do full justice to the various theories that exist for the explanation of consumer saving behaviour. We assume simply that the consumer has a preference ordering over consumption undertaken in the period under consideration and his accumulated savings at the end of the period, hereafter referred to as final wealth. Such a model [1] obviously offers a simplified picture of the underlying decision process; however, it has been found sufficient to analyze the effects of changes in income and yield on current consumption and saving, which is basically what we are interested in for the purpose of descriptive economic analysis. Of course, if our aim is to construct a planning model, we are interested in the whole time shape of the consumption stream, extending far into the future, but the development of such models is not the task of the present paper.

We take the preference ordering of the consumer to be represented by the utility function

$$U^* = U^* (C, Y)$$

where C is consumption and Y is final wealth. At the beginning of the period the consumer can be imagined to split his total resources, Q, in two; one part being set aside for consumption during the period, and the other part being invested in the only asset to which he has access as

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It has been found convenient to start out in section 2 with a discussion of a simple model of saving under certainty. After a discussion in section 3 of some measures of risk aversion, section 4 analyzes a model where the assumption of one asset only is preserved, but where the return from saving is a random variable. Sections 5 through 7 present a model with two assets, money and a risky asset, and analyze effects of changes in income and yield. Section 8 analyzes changes in the [illegible] riskiness, as measured by the variance of yield, in terms of a quadratic utility function. In section 9 we comment on the possibility of extending the model to allow for borrowing. Finally, section 10 contains some concluding remarks.

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We take the preference ordering of the consumer to be represented by the utility function

$$U^0 = U^0(C, Y)$$

where C is consumption and Y is final wealth. At the beginning of the period the consumer can be supposed to split his total resources, C_0, in two: one part being set aside for consumption during the period, and the other part being invested in the only asset to which he has access at

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an investor. Throughout the paper we shall think of quantities of assets as being given in monetary units without specifying further the nature of the various types of assets. We shall also assume that prices of consumption goods are held constant, so that we may represent consumption by total expenditure on consumption goods.

Most writers on the theory of saving seem to have recognized that the above utility function is too general for their purpose. The awkward aspect of it is that it allows for inferiority, so that either consumption or final wealth may have negative income elasticities. This does not make much sense, and so we may feel entitled to restrict the form of the utility function in such a way as to preclude the possibility of inferiority. One way in which this can be done is to postulate a utility function of the form

$$(1) \qquad U^{*} = V(C) + W(Y)$$

with positive and declining marginal utilities, i.e. $V'(C), W'(Y) > 0$ and $V''(C), W''(Y) < 0$.

Final wealth is obtained as

$$(2) \qquad Y = (Q - C)(1 + X) \qquad 1 + X > 0$$

where Q is total resources, or income for short, and X is the rate of return on savings. This equation is the budget constraint, and the consumer is seen as maximizing (1) subject to (2). This leads to the first-order maximum condition

$$\frac{V'(C)}{W'(Y)} - 1 = X$$

which is analogous to Fisher's rule for optimal allocation over time: Equality between the marginal rate of time preference and the rate of interest.

From this model we can deduct the effect on consumption of a change in income (the marginal propensity to consume). It can be written as

$$(3) \qquad \frac{\delta C}{\delta Q} = \frac{(1 + X)^2 W''(Y)}{(1 + X)^2 W''(Y) + V''(C)}$$

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Most writers on the theory of saving seem to have recognized that the general utility function is too general for their purpose. The awkward aspect of it is that it allows for complementarity, so that either consumption or final wealth may have negative income elasticities. This does not make much sense, and ... may feel entitled to restrict the form of the utility function in such a way as to preclude the possibility of complementarity. One way in which this can be done is to postulate a utility function of the form

$$U = V(C) + W(Y) \tag{1}$$

with positive and declining marginal utilities, i.e. $V'(C), W'(Y) > 0$ and $V''(C), W''(Y) < 0$.

Final wealth is obtained as

$$Y = (A - C)(1 + X) \tag{2}$$

where C is total consumption, A is income or wealth, and X is the rate of return on savings. This equation is the budget constraint, and the consumer is seen as maximizing (1) subject to (2). This leads to the first-order maximum condition

$$X = \frac{V'(C)}{W'(Y)} - 1$$

which is analogous to Fisher's rule for optimal allocation over time: Equality between the marginal rate of time preference and the rate of interest.

From this model we can deduct the effect on consumption of a change in income (the marginal propensity to consume). It can be written as

$$\frac{\partial C}{\partial A} = \frac{(1 + X)^2 W''(Y)}{(1 + X)^2 W''(Y) + V''(C)} \tag{3}$$

which is positive and less than one. This follows, of course, directly from the assumption of no inferiority.

The effect on consumption of a change in the rate of return is obtained as

$$(4) \qquad \frac{\delta C}{\delta X} = \frac{1}{H} Y W''(Y) + \frac{1}{H} W'(Y)$$

where

$$H = (1 + X)^2 W''(Y) + V''(C) < 0$$

We have here the sum of a positive income effect and a negative substitution effect, so that the sign of the sum is indeterminate in the absence of further information on the utility function. This is a familiar result. But it is of considerable interest to examine the precise conditions under which the one or the other of the two effects dominates. We can express this as follows:

$\delta C / \delta X$ is greater than, equal to, or less than zero, according as the elasticity of the marginal utility of wealth, $- Y W''(Y) / W'(Y)$, is greater than, equal to, or less than unity.[1)]

This result, as it stands, is hardly very interesting, since the present analysis does not allow us to guess at the relevant value of the elasticity of the marginal utility of wealth. However, since it will be shown below that the value of this elasticity assumes a particular significance when uncertainty is introduced, the above result may serve as a useful point of reference.

3. Measures of Risk Aversion.

In the following sections we shall study the consumption-saving decision when the rate of return X is a random variable with density function f (X). In section 4 we analyze a model with one asset only, as a prelude to later sections, where asset choice is introduced.

The consumer, which is taken to obey the axioms laid down by von Neumann and Morgenstern for rational choice under

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which is positive and less than [illegible]. This follows, of course, directly from the interpretation of [illegible] as an [illegible].

The effect on consumption of a change in the rate of return is obtained as

$$\frac{\partial C}{\partial r} = \frac{1}{D}\left[\ldots V''(Y) + \ldots W'(Y) \right] \qquad (9)$$

where

$$D = \ldots (1 + r)^2 \, W''(Y) + V''(C) \ldots 0$$

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1) Thus, if the utility of wealth is logarithmic, substitution and income effects will cancel out, and no effect will be observed on consumption and saving of a change in the rate of return on saving.

uncertainty, maximizes expected utility, expressed by the function

$$U = V(C) + \int_{-1}^{\infty} W(Y) f(X) dX$$

or, introducing a convenient notation,

$$(5) \qquad U = V(C) + E\left[W(Y) \right]$$

where $U = E[U^*]$. The signs of the first and second order partial derivatives are as before. (It should be noted that the assumption of declining marginal utility of wealth now also serves to ensure risk aversion.) In his Helsinki lectures [1] K. J. Arrow shows that a utility function satisfying the conditions of the expected utility theorem must be bounded both from above and from below. This result is utilized in his discussion of measures of risk aversion. The following two measures[1)] are both invariant under positive linear transformations of the utility function.

Absolute risk aversion $R_A(Y) = -W''(Y) / W'(Y)$

Relative risk aversion $R_R(Y) = -Y W''(Y) / W'(Y)$

Arrow now advances specific hypotheses concerning the variation of these measures as Y changes.

First, absolute risk aversion is taken to decrease with Y. This amounts to saying that "the willingness to engage in small bets of fixed size increases with wealth, in the sense that the odds demanded diminish. If absolute risk aversion increased with wealth, it would follow that as an individual became wealthier, he would actually decrease the amount of risky assets held" ([1], p. 35). While the behaviour described in the last sentence of the quotation may not seem so completely absurd to everybody else as it does to Arrow[1)], one may easily agree with him that decreasing absolute risk aversion seems to be a hypothesis well worth exploring.

1) It will be noted that relative risk aversion is the same concept as the elasticity of the marginal utility of wealth.

2) After all, one may argue that risks are taken only to obtain higher expected return, and when the need for higher return is reduced (due to higher wealth), there seems to be good reason for an individual to become less of a risk taker than he was before.

uncertainty, maximum [illegible] expected utility [illegible] expressed by the function

$$U = v(C) [illegible]$$

We introduce [illegible]

$$[illegible] \qquad (8)$$

where [illegible]. The slope of [illegible] and [illegible] the assumption of diminishing marginal utility of wealth [illegible] K. J. Arrow [illegible] the conditions of the expected utility [illegible] This result is [illegible]

[illegible]

Relative risk aversion [illegible]

[illegible]

Thus, [illegible] with wealth, it would follow [illegible]

[illegible]

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Second, relative risk aversion is assumed to increase with Y. This implies that "if both wealth and the size of the bet are increased in the same proportion, the willingness to accept the bet (as measured by the odds demanded) should decrease" ([1], p. 36). Arrow now argues that it follows from the boundedness of the utility function that as wealth increases, the relative risk aversion cannot tend to a limit below one; further, as wealth falls toward zero the relative risk aversion cannot approach a limit above one. Relative risk aversion, therefore, nust "hover around 1, being, if anything, somewhat less for low wealths and somewhat higher for high wealths" ([1] , p. 37).

These measures and Arrow's hypotheses[1)] on their variation with wealth will be adopted in the following.

4. Extension of the One-Asset Model to the Case of Uncertainty.

The consumer now maximizes the utility function (5) subject to the budget constraint

(6) $$Y = (Q - C)(1 + X)$$

This gives the first-order maximum condition

$$V'(C) - E\left[W'(Y)(1 + X)\right] = 0$$

which is the analogue to Fisher's rule in the present model, and the second-order condition

$$J = V''(C) + E\left[W''(Y)(1 + X)^2\right] < 0$$

From this we can easily compute the marginal propensity to consume as

(7) $$\frac{\delta C}{\delta Q} = \frac{E\left[W''(Y)(1 + X)^2\right]}{E\left[W''(Y)(1 + X)^2\right] + V''(C)}$$

which is positive and less than one. (7) is seen to be the exact equivalent of (3), which gives the MPC for the certainty case.

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These measures and Arrow's hypotheses[1] of their variation ... will be adopted in the following.

4. Extension of the Consumer Model to the Case of Uncertainty

The consumer now maximizes the utility function (5) subject to the budget constraint

$$U = u[(Y - C)(1 + X)] + v(C) \qquad (5)$$

This gives as first-order maximization condition

$$v'(C) - E[u'(Y)(1 + X)] = 0$$

... the second-order condition

$$D = v''(C) + E[u''(Y)(1 + X)^2] < 0$$

From this we can easily compute the marginal propensity to consume as

$$\frac{\partial C}{\partial Y} = \frac{E[u''(Y)(1 + X)^2]}{E[u''(Y)(1 + X)^2] + v''(C)} \qquad (7)$$

which is positive and less than one. (7) is seen to be the exact equivalent of (3), which gives the MPC for the certainty case.

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How does the yield on savings affect the choice between consumption and accumulatinn? Since the rate of return is a random variable, the relevant parameter is now the probability distribution of X; we wish to examine the effect on consumption of a shift in the probability distribution which has no other effect than altering the expected value of X. Such a shift can be described geometrically as a paralell shift and is illustrated in fig. 1 below where the curve I is the original distribution and II is the curve after the paralell shift has taken place.

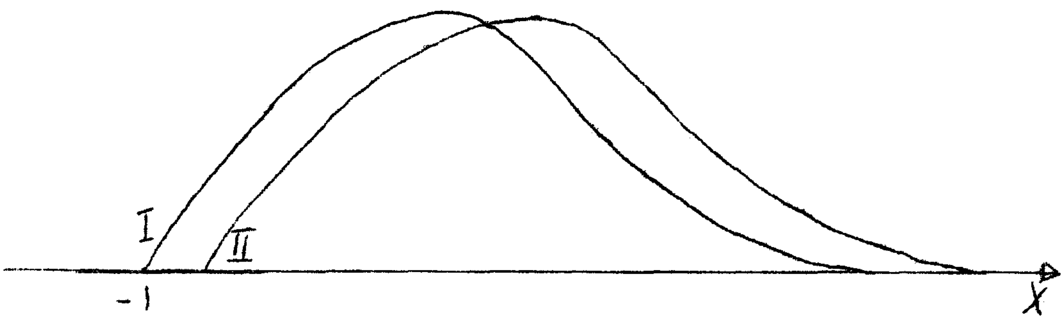

Fig. 1

Algebraically, we can examine the effects on consumption (and on saving) by such a paralell shift by introducing a shift parameter γ, into the utility function and the budget constraint, which will now be written as

$$U = V(C) + \int_{-1+\gamma}^{\infty} W(Y)\, f(X)\, dX$$

and

$$Y = (C_0 - C)(1 + X + \gamma)$$

where γ is a positive number. We may think of our original case with $\gamma = 0$ as our initial situation. An increase of γ will then be equivalent to such a paralell shift of the probability distribution as

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$$U = V(C) + \int_{-1}^{\infty} u(Y) f(X) dX$$

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$$Y = (1 - C)(1 + X + \gamma)$$

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is illustrated by fig. 1. The first and second order maximum conditions evaluated at $\gamma = 0$ are as before.

Taking now the derivative of C with respect to γ, we can write this as

$$\frac{\delta C}{\delta \gamma} = \frac{1}{J} (Q - C) E \left[W''(Y) (1+X+\gamma) + \frac{1}{J} E \; W'(Y) \right]$$

Since (Q - C) is not a random variable, we can rearrange this as

$$(8) \qquad \frac{\delta C}{\delta \gamma} = \frac{1}{J} E \left[W''(Y) Y \right] + \frac{1}{J} E \left[W'(Y) \right]$$

As in the certainty case, we have evaluated the effect of a change in yield as the sum of a positive income effect and at negative substitution effect. It is interesting to note that (8) can be obtained from (4) by simply taking expected values of each single term in the latter equation. So far, then, we have shown the following: The conclusions concerning the effect of changes in the rate of return on the consumption-saving decision which can be derived from the certainty model of section 2, in particular the conflicting tendencies of the income and substitution effects, are upheld by the present model. Moreover, the precise conclusions can be stated in essentially the same form.

However, the introduction of uncertainty actually allows us to go further than concluding that the total result is indeterminate. Equation (8) can be rewritten, after a little manipulation, as

$$(9) \qquad \frac{\delta C}{\delta \gamma} \qquad - \frac{1}{J} W'(Y) E \left[R_R (Y) - 1 \right]$$

so that the sign is determined by the value of the relative risk aversion, $R_R (Y)$. Since this is the same thing as the elasticity of the marginal utility of wealth, this is again the same conclusion as we presented for the certainty case. But accepting Arrow's argument, as outlined in section 3, we can now restate this conclusion in operational terms. Since the typical value of $R_R (Y)$ is one, the typical value of $\delta C / \delta \gamma$ is zero. Moreover, since $R_R (Y)$ increases with wealth, and therefore with income, $\delta C / \delta \gamma$ must be negative for "low" incomes and positive for "high" incomes, but the magnitude of the effect would probably be small.

[illegible] by Fig. 1. [illegible] maximum conditions [illegible]

[illegible] with respect to Y, we can [illegible]

$$[illegible]$$

Since (1 - C) [illegible], we can [illegible] this as

$$[illegible] \qquad (8)$$

As in the [illegible] case, we have evaluated the [illegible] a change [illegible] income and substitution effects. [illegible] (6) by [illegible] equation [illegible] the following. The conclusion [illegible] saving [illegible] 2, in particular [illegible] consequences of the income and substitution effects, [illegible] conclusion can be [illegible].

However, the [illegible] of uncertainty [illegible] us to go further [illegible] Equation (6) [illegible] as

$$[illegible] \qquad (9)$$

so that the sign is determined by the [illegible] of the relative [illegible] $R_A(Y)$. Since this is [illegible] the elasticity of [illegible] utility of wealth, this [illegible] the same conclusion as we presented for the [illegible] case. This [illegible] argument, as outlined in section 3, we can now [illegible] in operational terms. Since the typical value of $R_R(Y)$ [illegible] in [illegible]. Moreover, [illegible] $R_R(Y)$ increases with wealth, [illegible] with income, [illegible] for "high" incomes, but the magnitude of the effect would probably be small.

This is an interesting result. Economists have indeed been inclined to think that the effect on consumption of a change in the rate of return on saving is negligible, but their reasons have been that since the substitution and income effects work in opposite directions, the assumption of an all-over effect of zero has seemed the safest bet. We have here presented a theoretical argument which supports this intuitive conclusion. That an increase in yield serves to decrease consumption (increase saving) for low levels of wealth and income and to increase consumption (decrease saving) for high wealth and income levels is a result which may not correspond very closely to people's intuitive notions, but its theoretical foundations are, I think, quite strong.

5. A Two-Asset Model.

It is now time to introduce asset choice. Surely one of the most fundamental modifications of traditional saving theory which becomes necessary once we take account of uncertainty, is that the consumer will not generally hold his wealth in the form of one asset only. He has access to a wide variety of assets with different yield expectations and different degrees of risk. In our simplified model, the spectre of assets is reduced to two. One of them promises a yield of zero with complete certainty; this we shall refer to as money. The other asset is similar to the one discussed in section 4; we shall refer to it as "the risky asset".

Our utility function is as before

$$U = V(C) + E\left[W(Y)\right] \tag{10}$$

while the budget constraint is

$$Y = Q - C + aX \tag{11}$$

where a is the amount of risky assets held. [(11) is really a condensed version of the "real" budget constraint

$$Q - C = a + m$$

[illegible], economists have indeed been inclined to [illegible] of a change in the rate [illegible] have been that [illegible] in opposite directions, the [illegible] the other one. [illegible] which account this [illegible] con-[illegible] and [illegible] (borrowing) for [illegible] people [illegible] theoretical foundations [illegible]. I think, quite strong.

5. A [illegible] Model

Before [illegible] asset choices, [illegible] most fundamental [illegible] portfolio and saving theory [illegible] consumer [illegible] uncertainty, [illegible] assumes that he generally holds [illegible] only. [illegible] of assets with [illegible] and different degrees of [illegible] of assets [illegible] with consumption [illegible] this we shall [illegible] is similar to the one discussed in section [illegible] we shall make [illegible] "[illegible] asset".

Our utility function [illegible]

$$U = u(C) + E[v(A)] \qquad (10)$$

while the budget constraint is

$$Y = L - C + A \qquad (11)$$

where [illegible] is the amount of [illegible] held. [(11) is really a condensed version of the "real" budget constraint

$$Q - C = [illegible]$$

where m is the amount of money held, and the definition of final wealth

$$Y = m + a(1 + X)$$

Substitution of the latter equation in the former gives (11).]

Maximization of (10) subject to (11) gives the first-order conditions

$$(12) \quad \begin{cases} V'(C) - E\left[W'(Y)\right] = 0 \\ E\left[W'(Y)X\right] = 0 \end{cases}$$

and the second-order condition

$$(13) \quad D = V''(C)\,E\left[W''(Y)X^2\right] + E\left[W''(Y)\right]E\left[W''(Y)X^2\right] - \{E\left[W''(Y)X\right]\}^2 > 0$$

These conditions, together with the assumption of diminishing marginal utility, defines the consumer's optimum position.

6. Changes in Income.

In this section we shall evaluate the effects of changes in income on the optimum values on consumption and asset holdings. In our previous models, this exercise was not really very interesting, since the assumption of no inferiority is practically equivalent to postulating a MPC of a value between zero and one. From this it evidently also follows that the marginal propensity to buy assets is between zero and one. But in the present model we have two assets, so that a value of the MPC between zero and one is not sufficient to assure us that the income elasticity of one of the assets is not negative.

By implicit differentiation in equations (12) above, we can compute the following partial derivatives:

$$(14) \quad \frac{\delta a}{\delta \Omega} \quad - \frac{1}{D}\,V''(C)\,E\left[W''(Y)X\right]$$

$$\frac{\delta m}{\delta Q} = \frac{1}{D} V''(C) \{ E[W''(Y) X^2] + E[W''(Y) X] \} \tag{15}$$

$$\frac{\delta C}{\delta Q} = \frac{1}{D} \{ E[W''(Y)] \, E[W''(Y) X^2] - \{E[W''(Y) X]\}^2 \} \tag{16}$$

Of the terms occurring in these equations it is immediately clear that $E[W''(Y)]$ and $E[W''(Y)X^2]$ are both negative. Moreover, it can be shown that decreasing absolute risk aversion implies that $E[W''(Y) X]$ is positive and that increasing relative risk aversion implies that $E[W''(Y) X Y]$ is negative. Proofs of these assertions are set out in the appendix to this paper; we shall use them here to show that the partial derivatives (14)- (16) are all positive and less than one.

It should be noted that even though our model is very similar to the one discussed in the previous section, it is not self-evident that all "goods" should be superior goods. There are only two arguments in the utility function, viz, consumption and final wealth. Money and risky assets are only means to obtain an end, and it is not a priori clear that a positive propensity to save would imply positive propensities to buy for both assets.

However, the model does predict that the demand for both assets will increase with income. First, since $E[W''(Y) X] > 0$, it follows immediately that the risky asset is not an inferior good; i. e. $\delta a / \delta Q > 0$.

To show that $\delta m / \delta Q > 0$, we proceed as follows: Since $V''(C)/D$ is negative, $\delta m / \delta Q$ will have the opposite sign of

$$K = E[W''(Y) X^2] + E[W''(Y) X]$$

Now multiply K by (a + m) and add and subtract, on the right-hand side, the expression $a X E[W''(Y) X]$. After some rearrangement we then obtain

$$K(a + m) = m E[W''(Y) X^2] + (a+m+aX) E[W''(Y) X]$$

Since $a + m + a X = Y$, we can write

$$K(a + m) = m E[W''(Y) X^2] + E[W''(Y) X Y]$$

(15) [illegible] $\left[w''(Y) \ldots^{2}\right] + \ldots E[w'''(Y) X]$ [illegible]

(16) [illegible] $E[w''(Y) X^{2}] \ldots [w'''(Y) X]$ [illegible]

Of the terms [illegible] in equation [illegible] it is obviously clear that [illegible] $(Y)]$ [illegible] $[w''(Y)]$ are both [illegible] because [illegible]. It can be shown that decreasing absolute risk aversion implies that [illegible] relative risk aversion implies that [illegible] $(Y) X]$ is positive and [illegible] $[w'''(Y) X Y]$ is negative. The proofs of these assertions are set out in the appendix to this paper; we shall use them here to show that the partial derivatives (15)-(16) are all positive [illegible] here.

It should be noted that even though our model is very [illegible] to a [illegible] model discussed in the previous section, it is not [illegible] "goods" [illegible]. There are only [illegible] final wealth. [illegible] clear that a positive propensity to save would imply positive quantities [illegible] buy for each [illegible].

[illegible] for both assets will increase [illegible]. First, since [illegible] $[w''(Y) X]$ [illegible], it follows immediately that [illegible] are not inferior goods; i.e. [illegible] > 0.

[illegible], we proceed as follows: since $V''(C)$ [illegible] will have the opposite sign of

$$[\text{illegible}]\; E[w''(Y) X] \ldots [w''(Y) X]$$

Now multiply X by [illegible] $+ m)$ and add and subtract, on the right-hand side, the expression [illegible] $[w'''(Y) X]$. After some rearrangement we then obtain

$$X(n + m) = \ldots E[w''(Y) X^{2}] + (a+m+\ldots) E[w''(Y) X]$$

Since [illegible] Y, we can write

$$X(\ldots + m) = \ldots [w''(Y) X^{2}] + E[w''(Y) X Y]$$

where we have the sum of two negative terms on the right. K is therefore negative and $\delta m / \delta Q$ accordingly positive.

We now turn to the marginal propensity to consume, as written in (16). From the definition of D in (13) it is clear that the MPC is always less than one. To show that it is also positive, we examine the sign of the numerator

$$L = E[W''(Y)]\, E[W''(Y)\, X^2] - E[W''(Y)X]\, E[W''(Y)\, X]$$

Now add and subtract $E[W''(Y)]\, E[W''(Y)X]$ After a little manipulation we can then write

$$L = E[W''(Y)]\, K - E[W''(Y)X]\, \{E[W''(Y)] + E[W''(Y)X]\}$$

or

$$L = E[W''(Y)]\, K - E[W''(Y)\, X]\, E[(1 + X)\, W''(Y)]$$

K was shown above to be negative. The first term of this expression is therefore positive. Since X cannot take on values below -1, the last term is the product of a positive and a negative factor. L is accordingly positive, and so is the MPC. We have now shown that all three partial derivatives of equations (14) - (16) are positive and less than one.

This in itself may not be terribly interesting. However, we are now in a position to give an answer to the following question: How will an increase in income affect money's share in the portfolio? To see this, we have to evaluate the sign of the partial derivative

$$\frac{\delta}{\delta Q}\left(\frac{m}{a + m}\right) = \frac{1}{(a + m)^2}\left(\frac{\delta m}{\delta Q}\, a - \frac{\delta a}{\delta Q}\, m\right)$$

Substituting from (14) and (15) we have

$$\frac{\delta}{\delta Q}\left(\frac{m}{a + m}\right) = \frac{1}{(a + m)^2} \cdot \frac{1}{D}\, V''(C)\, \{aE[W''(Y)\, X^2]$$

$$+\ aE[W''(Y)\, X] + mE[W''(Y)\, X]\}$$

To the factor in braces, add and subtract $a\, X\, E[W''(Y)\, X]$. We can then write

$$(17) \qquad \frac{\delta}{\delta Q}\left(\frac{m}{a+m}\right) = \frac{1}{(a+m)^2} \cdot \frac{1}{D} V''(C)\, E\left[W''(Y)\, X\, Y\right]$$

Since the last factor is negative, the whole expression is positive. As income (and with it wealth) rises, money's share in the portfolio will increase.[1)]

A similar result has been given by Arrow in [1]. In terms of a pure portfolio model without consumption, Arrow finds that money has a "wealth elasticity" greater than one, wealth being defined as the initial value of the portfolio. He finds this result to conform with various empirical studies of the demand for money which agree in finding an <u>income</u> elasticity of the demand for money of at least one.

The result obtained by Arrow can easily be reconciled with that of the present paper. Since the long-run relationship between consumption and income has been found to be one of proportionality, the elasticity of money holdings with respect to <u>wealth</u> will be the same as money's <u>income</u> elasticity. This, of course, is the basic justification behind Arrow's procedure when he compares his wealth elasticity with empirical income elasticities.

Given a proportional consumption function, it is easy to show that the conclusion that money's portfolio share will increase with income is equivalent to a wealth elasticity of money greater than one. Let

$$A \equiv a + m$$

define wealth as the initial value of the portfolio. Letting ε_a and ε_m be the wealth elasticities of the risky asset and money, respectively, and α denote the risky asset's portfolio share, we have that, as an identity,

$$\varepsilon_a \alpha + \varepsilon_m (1 - \alpha) \equiv 1$$

---

1) We have shown that

$$\frac{\delta m}{\delta Q} a - \frac{\delta a}{\delta Q} \cdot m > 0$$

Multiplying by Q and dividing by a m, we can restate this as

$$\frac{\delta m}{\delta Q} \cdot \frac{C}{m} - \frac{\delta a}{\delta Q} \cdot \frac{Q}{a} > 0$$

The income elasticity of money is greater than the income elasticity of risky assets. This is simply an alternative way of stating the conclusion.

$$[illegible] \frac{1}{2} [illegible] \; u'''(y) \, m [illegible]$$

Since the last [illegible] positive. As income and wealth rise, [illegible] will increase.[1]

A similar result [illegible] in [illegible].

[illegible] that [illegible] defined as [illegible] with [illegible] in finding an income elasticity of the demand for money of [illegible] one.

[illegible] be reconciled with [illegible] between consumption and [illegible] utility, the [illegible] of money holdings with respect to wealth [illegible] money income [illegible] with complete income [illegible].

[illegible] consumption [illegible] to show that the conclusion that the [illegible] portfolio [illegible] will increase with income is equivalent [illegible] greater than one. Let

[illegible]

define wealth as the initial value of the portfolio, [illegible] and [illegible] by the wealth elasticities of the [illegible] and [illegible], respectively, and [illegible] denote the [illegible], we have that, [illegible],

$$[illegible]$$

---

1) We have shown that

$$[illegible]$$

Multiplying by Q and dividing by [illegible], we [illegible] state this as

$$[illegible]$$

The income elasticity of [illegible] is greater than the [illegible] of risky assets. This is simply an [illegible] way of stating the conclusion.

or, equivalently

$$\alpha (\varepsilon_a - \varepsilon_m) = 1 - \varepsilon_m$$

Since increasing share of money in the portfolio was shown above to imply $\varepsilon_a - \varepsilon_m < 0$ we must have $\varepsilon_m > 1$. The conclusion arrived at by Arrow and the one derived in this paper are therefore completely equivalent for the case of a proportional consumption function, and the empirical studies cited by Arrow[1] in support of his theoretical conclusion are equally relevant as evidence for the hypothesis advanced in this paper. The result does not, however, follow as a purely theoretical proposition, since there is nothing in the present model that assures us that the relationship between consumption and income will be one of proportionality.

One further comment on empirical work seems to be in order. The studies of the demand for money referred to by Arrow are all time-series analyses. However, there is a study based on cross-section data by Dorothy S. Projector [7] which presents very different results; the share of liquid assets, by any admissible difinition, seems to decline very pronouncedly with income. I suspect that, imperfections of measurement aside, these apparently contradictory results might be theoretically reconciled by extending the present model in two directions, viz. (1) to take account of transaction costs and (2) to introduce, in some way, a distinction between permanent and transitory income changes. We cannot go further into these matters here. Suffice it to say that since this paper ignores phenomena like transaction costs and transitory income changes, which may be of chief importance as short-run influences on saving and portfolio decisions, the evidence from time series studies, covering fairly long time periods, seems to be the most relevant data with which to confront the hypothesis. To the extent that this is true, the hypothesis accords fairly well with the data.[2]

---

1) See [1], pp. 44, where Arrow lists the well-known studies by Selden, Friedman, Latané and Meltzer.

2) The identification of riskless assets with real-world money holdings may, however, be of somewhat doubtful value in a world of changing price levels. In times of erratic inflation, money may seem to the individual saver a much more risky investment than common stock or real capital.

etc., equivalent [illegible]

[illegible]

Since income [illegible] shown above to imply [illegible] conclusion arrived [illegible] this paper [illegible] completely [illegible] of a permanent income [illegible] function, and the empirical [illegible] Arrow[1] [illegible] far theoretical conclusion [illegible] hypothesis advanced in this paper. [illegible] as a purely theoretical proposition, [illegible] in [illegible] model that assumes [illegible] that the total [illegible] consumption [illegible] will [illegible] proportional [illegible].

[illegible] were [illegible] earlier. [illegible] interesting analysis [illegible] study [illegible] theory [illegible] directly [illegible] the data [illegible] difficulty [illegible] very [illegible] fact, [illegible] of measurement [illegible] these [illegible] might be theoretically reconciled by [illegible] the present [illegible] directions, viz. (1) to take account of [illegible] and (2) to introduce, in some way, [illegible] distinction between [illegible] income changes. [illegible] must [illegible] these [illegible] the [illegible] this paper [illegible] changes, which [illegible] and portfolio [illegible] which [illegible] long [illegible] data will [illegible] the hypothesis. [illegible] hypothesis [illegible] with income.[4]

[1] [illegible] the [illegible] Friedman, [illegible]

[4] The [illegible] with cash-wealth [illegible] holdings may [illegible] of changing price [illegible] individual [illegible] capital.

7. Changes in Yield.

In this section we shall examine the effects, in the two-asset model, of changes in yield in the sense of a paralell shift of the probability distribution of X. The study of such changes is particularly interesting in this model, since there will be two types of effects at work. First, we would expect changes in yield to affect the choice between consumption and saving. Second, changes in yield should presumably lead to a redistribution of portfolio resources. Our attention will be centered on the question of how the second type of effect interacts with the first.

As above, we shall refer to such a shift of the probability distribution as a change in yield. Equation (11) now reads

$$Y = Q - C + a\,(X + \gamma)$$

Without loss of generality, we can evaluate our expressions at $\gamma = 0$. The first-order (12) and second-order (13) maximum conditions can then be utilized as they stand. We now differentiate with respect to γ in equations (12). This gives us

$$(18) \qquad \frac{\delta a}{\delta \gamma} = a\,\frac{\delta a}{\delta Q} - \frac{1}{D}\, E\left[W'(Y)\right] \{V''(C) + E\left[W''(Y)\right]\}$$

$$(19) \qquad \frac{\delta m}{\delta \gamma} = a\,\frac{\delta m}{\delta Q} + \frac{1}{D}\, E\left[W'(Y)\right] \{V''(C) + E\left[W''(Y)(1 + X)\right]\}$$

$$(20) \qquad \frac{\delta C}{\delta \gamma} = a\,\frac{\delta C}{\delta Q} - \frac{1}{D}\, E\left[W''(Y)\,X\right] E\left[W'(Y)\right]$$

All these expressions are written as the sum of an income effect and a substitution effect. In view of our previous results, all income effects are positive. What this means is essentially that it is now possible to increase both consumption and final wealth from the levels enjoyed before the change in yield.

Turning now to $\delta a/\delta\gamma$, the substitution effect is seen to be positive, reinforcing the income effect. But the substitution effect, in this case, is not solely the result of substitution of future for present goods, as in conventional saving models. It is also the result of a portfolio substitution of risky assets for money, since the relative desirability of the former has been increased.

7. Changes in [illegible]

[illegible] in this period, [illegible] works [illegible] the question [illegible] with the first. [illegible] distribution [illegible] change in yield. [illegible]

[illegible]

(18) [illegible]

(19) [illegible]

(20) [illegible]

All these expressions [illegible] income effect and a substitution effect. [illegible], all income effects are positive, [illegible] that it is now possible to increase both consumption [illegible] the levels enjoyed before the change in yield.

Turning [illegible] the substitution [illegible] be [illegible], indicating the income effect. [illegible] substitution [illegible] consumption and saving [illegible] relation [illegible] been [illegible].

In equations (19) and (20), the substitution effect pulls in the opposite direction of the income effect. As far as the demand for money is concerned, an increase in yield reduces money's attractiveness as an investment, and as for consumption, resources can now more profitably than before be carried over to the future. However, since $\delta a / \delta\gamma$ has been shown to be positive, at least one of $\delta m / \delta\gamma$ and $\delta C / \delta\gamma$ must be negative; this follows simply from the budget constraint. Therefore, if an increase in yield raises consumption demand, the demand for money will fall. On the other hand, if the larger yield leads to less consumption, the demand for money may rise or fall.

It is not difficult to extend the model so as to let money bear a non-random rate of interest. The effect of an increase in such a rate would clearly be to increase the demand for money, while for consumption and the risky asset income and substitution effects would be of opposite signs.

It should be remembered, however, that in drawing implications of the present analysis for macroeconomic models, the rate of interest figuring in such models should be identified with the random rate of return on the risky asset. This is clearly implied in e.g. Tobin's work [8] when he discusses Keynes' liquidity preference function in terms of a portfolio model. For the rate of interest relevant to the consumption function is assumed to be the same as the one which plays such a prominent role in the liquidity preference function. This in itself may well serve to point out the need for a simultaneous study of saving and portfolio decisions, such as the one we have attempted here.

8. Changes in Riskiness.

In the previous section we have associated the changes in the rate of return studied in deterministic models with paralell shifts in the probability distribution of the rate of return. Generally speaking, no simple measure can be found which describes fully the degree of riskiness attached to the portfolio. The most popular measure in the literature is, of course, the variance, and it is certainly of interest to examine the effects of changes in this measure on consumption and asset holdings. As a point of reference, one may keep in mind the simple risk-premium theory which states, roughly, that an increase in riskiness is equivalent to a fall in the expected rate of return.

[illegible] institutions [illegible] quite in the [illegible] money is concerned [illegible] yield reduces [illegible] attractiveness as an investment [illegible] more profitably than [illegible] [illegible] constraints. [illegible] demand [illegible] consumption [illegible]

[illegible] an increase [illegible] would clearly [illegible] money [illegible] motion and [illegible] risky [illegible] in [illegible]

[illegible] cash [illegible] rate of interest [illegible] rate of return [illegible] e.g. Hicks's work [illegible] liquidity [illegible] is [illegible] plays no [illegible] in itself [illegible] earning and [illegible] as the [illegible]

[illegible] Change in [illegible]

[illegible] changes in the [illegible] cash [illegible] rate of [illegible] measure [illegible] interest [illegible] and [illegible] roughly [illegible] expected [illegible]

We shall now work with the following utility function for wealth

$$W = \alpha Y^2 + \beta Y \qquad \beta > 0, \quad \alpha < 0$$

For general purposes, this utility function is not very satisfactory. Were we to use it to study effects of changes in income, we would find that it implies that the risky asset is an inferior good.[1] For the present purpose, however, it is well suited, since these awkward aspects of it are unimportant for the issues under discussion.[2]

Our general utility function is

$$U = V(C) + \int_{-1}^{\infty} (\alpha Y^2 + \beta Y)\, f(X)\, dX$$

which, upon integration, yields

$$(21) \qquad U = V(C) + \beta(Q - C) + \alpha (Q - C)^2$$
$$+ 2 \alpha a (Q - C) E[X] + \alpha a^2 E[X^2] + \beta a E[X]$$

The utility function can thus be written as quadratic in return and initial wealth $(Q - C)$.

The first-order maximum conditions are

$$(22) \qquad \begin{cases} V'(C) - 2\alpha (Q - C) - 2\alpha a E[X] - \beta = 0 \\ 2\alpha (Q - C) E[X] + 2\alpha a E[X^2] + \beta E[X] = 0 \end{cases}$$

and the second-order condition is

$$(23) \qquad H = (V''(C) + 2\alpha) 2\alpha E[X^2] - 4\alpha^2 \{E[X]\}^2 > 0$$

1) In terms of the measures of Arrow and Pratt, the quadratic function displays increasing absolute risk aversion. See [1], pp. 35 - 36.

2) The function which is most satisfactory according to the Arrow-Pratt measures, is the logarithmic function $W(Y) = \log Y$, which has decreasing absolute risk aversion and constant relative risk aversion equal to one. However, this function is very complicated computationally. But it can be (Footnote continued next page)

We shall now work with the following utility function for wealth:

$$U = [illegible]\, Y + [illegible]\, Y^2, \quad [illegible] > 0, \quad [illegible] < 0$$

For general purposes, this utility function is not very satisfactory. [illegible] were [illegible] to study [illegible] we would find that it [illegible] the (risky) asset [illegible] inferior [illegible] 1) [illegible] However, it will satisfy [illegible] those [illegible] aspects of it are unimportant [illegible] the [illegible] 2)

Our general utility function is

$$U = V(C) + [illegible] ([illegible] Y^2 + [illegible] Y) [illegible]$$

which, upon integration, yields

$$(21) \quad [illegible] = V(C) + [illegible]$$

$$[illegible]$$

The utility function can thus be written as quadratic in return and initial wealth ($[illegible] - C$).

The first-order maximum conditions are

$$(22) \quad \begin{cases} V'(C) - 2 [illegible] (\,[illegible] - C) - 2 [illegible] E[Y] [illegible] = 0 \\ 2 [illegible] (\,[illegible] - C)\, E[Y] + 2 [illegible] E[Y^2] + [illegible] E[Y] [illegible] = 0 \end{cases}$$

and the second-order condition is

$$(23) \quad [illegible] = (V''(C) + 2 [illegible])\, 2 [illegible] E[Y^2] - 4 [illegible] (E[Y])^2 [illegible] > 0$$

1) In terms of [illegible] measures of [illegible] Arrow and Pratt, the [illegible] utility function [illegible] increasing absolute risk aversion. See [illegible]

2) [illegible] function which is most satisfactory according to [illegible] Arrow-Pratt [illegible] the logarithmic function [illegible] (Y) = log Y, [illegible] decreasing [illegible] absolute risk aversion and constant relative risk aversion. [illegible] this function is very complicated computationally [illegible] (Footnote [illegible])

To find the derivatives of C, a and m with respect to the variance when the mean is held constant, we differentiate (22) with respect to $E[X^2]$ utilizing the well-known formula $\sigma^2 = E[X^2] - \{E[X]\}^2$.
The result is

(24) $$\frac{\delta C}{\delta \sigma^2} = -\frac{1}{H} 4 \alpha^2 a E[X]$$

(25) $$\frac{\delta a}{\delta \sigma^2} = -\frac{1}{H} 2 \alpha a [V''(C) + 2\alpha]$$

(26) $$\frac{\delta m}{\delta \sigma^2} = \frac{1}{H} 2 \alpha a [2\alpha (E[X] + 1) + V''(C)]$$

The signs of these expressions are easy to evaluate as being negative, negative and positive, respectively. That is to say, consumption will fall with increased riskiness (more will be saved), while the consumer will reduce his holdings of risky assets and increase his money holdings.

The part of this conclusion which may be somewhat surprising is that less will be consumed and more will be saved the higher is the degree of riskiness. However, the result does seem to be well in line with the basic assumption of risk aversion. The higher is the degree of riskiness, the more the rational consumer must save in order to be sure that the realized level of final wealth will not be too low. Also, since money will be substituted for risky assets in the portfolio, more will now have to be saved, at any given rate of return, to attain the same value of final wealth that was planned before the increase in riskiness.

If we compare our results in this section with those previously presented for changes in yield, they are found to conform only partially with the notions of risk-premium theory. It can be demonstrated that the effects of increases in expected yield are qualitatively the same as those presented for the general case in section 7, as far as the substitution effects are concerned.[1] As to the demand for asset holdings, increases in risk

Footnote from proceeding page continued:
shown that the marginal rates of substitution between expected yield and variance, $-dE[X]/d\sigma^2$ are essentially similar for the quadratic and the logarithmic utility finctions. Hence the former can be taken as an approximation to the latter for this particular problem.

1) It seems to me that the substitution effects offer the most relevant comparison. In any case, without restricting attention to them no clear-cut conclusions can be drawn.

To find the derivatives of a, n and m with respect to the variance when the mean is held constant, we differentiate (22) with respect to $E[X^2]$ utilizing the well-known formula [illegible] $E[X^2] - (E[X])^2$.
The result is

(24) [illegible]

(25) [illegible]

(26) [illegible]

The [illegible] of these [illegible] to evaluate [illegible] negative, [illegible] positive, [illegible] will [illegible] uncommon [illegible] will [illegible] holdings [illegible].

The [illegible] which may [illegible] is that [illegible] will be [illegible] in the degree of [illegible] will be in line with [illegible]. The higher [illegible] or [illegible] relations [illegible] since [illegible] will [illegible] will not [illegible] value of final wealth that was planned [illegible].

If we compare [illegible] with those [illegible] presented for changes in yields, they [illegible] with the notions of [illegible] theory. [illegible] effects of [illegible] yields are quantitatively [illegible] for the general case in section 5, as far as [illegible] effects [illegible].[1] [illegible] asset holdings, [illegible]

Footnote from preceding page continued:

[illegible] shows that the marginal [illegible] between expected yield and variance, with $[X]/d$ [illegible] than for the [illegible] and [illegible] has [illegible] utility function [illegible] this [illegible].

1) It seems to me that the [illegible] effects [illegible] important [illegible] in any case, without [illegible] can be [illegible].

and return have opposite effects; an increase in the variance leads the consumer to demand more money and less of the risky asset. But in the case of consumption the substitution effect of an increase in yield is negative, and so is the effect of an increase in riskiness.

9. Borrowing.

Throughout the paper, the two arguments in the utility function have been consumption and final wealth, both being taken as positive quantities. An alternative formulation is to let the utility function depend on present and "future" consumption. This is the formulation used in Irving Fisher's classic model in [3], which provides the standard exposition of the theory of saving found in most text-books. This formulation allows treatment of the case of consumers who plan to consume more than their income, i. e. who are net borrowers. Formally, in order to let our model cover the case of consumers who are net borrowers, we have to introduce future non-capital income. We can then let final wealth be negative without implying that the individual consumes a negative amount in the future. If future income is non-stochastic, such an extension of the model is not really very fundamental. If future income is a random variable, we shall have to work with joint probability distributions of future income and yield. There may be reasons for doubting that much can be gained by working with several kinds of uncertainty at a time.

In the Fisher model the consumer is seen as having access to a perfect capital market in which he can lend and borrow at the same rate of interest. The formal equivalent of this assumption is achieved, in this model, by letting a take on negative values; i. e. the consumer himself can issue bonds.

Explicit consideration of borrowers becomes necessary if, e. g., one studies the determination of interest rates and asset prices in a general equilibrium model. However, if one's main interest is the microeconomic foundations of aggregate relationships like the consumption function and the liquidity preference function, then the case of net lenders is the most interesting, since the consumer sector as a whole is treated as a lending sector in macroeconomic models. This is really the main justification for concentrating attention on the case of lenders.

10. Concluding Remarks.

Problems in economic theory become unmanageable unless one splits them up in some way. This is true also for saving decisions and portfolio decisions. However, one may suspect that these two types of decisions may be closely interrelated, so that one should at least once try to study them simultaneously. It is hoped that the approach of the present paper may have contributed toward a better understanding of the interrelationship between saving and portfolio decisions.

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Appendix.

We shall prove the following two propositions

I If $R_A(Y)$ is decreasing, then $E[W''(Y)X] \geqq 0$.

II If $R_R(Y)$ is increasing, then $E[W''(Y)XY] \leqq 0$.

I.

Assuming an inferior maximum for the choice of a (which we have done throughout the paper), we have from (12)

(A. 1) $$E[W'(Y)X] = 0$$

Let $A = Q - C$. Since $Y = A + aX$ and R_A is decreasing

$$R_A(Y) \leqq R_A(A) \quad \text{if } X \geqq 0$$

Substituting from the definition of R_A, we can write

(A. 2) $$-W''(Y)/W'(Y) \leqq R_A(A) \quad \text{if } X \geqq 0$$

Trivially

(A. 3) $$-W'(Y)X \leqq 0 \text{ if } X \geqq 0$$

We now multiply through in (A. 2) by $-W'(Y)X$. The inequality is then reversed.

(A. 4) $$W''(Y)X \geqq -R_A(A)W'(Y)X \quad \text{if } X \geqq 0$$

Suppose now that $X \leqq 0$. Then the inequalities (A. 2) and (A. 3) are both reversed, and sc (A. 4) holds for all X. Since $R_A(A)$ is not a random variable, we can take expectations of both sides of (A. 4) and write

(A. 5) $$E[W''(Y)X] \geqq -R_A(A)E[W'(Y)X] \quad \text{for all } X$$

APPENDIX A.

We shall prove the following two propositions:

I. If $R_X(Y)$ is decreasing, then [illegible] ≤ 0.

II. If $R_X(Y)$ is increasing, then [illegible] ≥ 0.

I

Depending on [illegible] for the choice of [illegible] (which we have used throughout this paper), we have from (12)

$$E[\ldots(Y)\,X] = 0 \qquad \text{(A.1)}$$

Let [illegible]. Since [illegible] and R_X is decreasing

[illegible]

Substituting from the definition of R_X, we can write

(A.2) [illegible]

Similarly

(A.3) [illegible]

Now multiply through in (A.2) by [illegible] $(Y)\,X$. The inequality is then reversed,

(A.4) [illegible]

Suppose now that X [illegible]. Then the inequalities (A.2) and (A.3) are both reversed, and so (A.4) holds for all X. Since $R_X(Y)$ is not a random variable, we can take expectations of both sides of (A.4) and write

(A.5) [illegible] for all X.

In view of (A. 1) the right-hand side is equal to zero. Hence proposition I has been proved.[1)]

II

The proof of the second proposition can be readily established by an analogous procedure.

Increasing R_R implies that

$$R_R(Y) \leqq R_R(A) \quad \text{if } X \geqq 0$$

or

$$\text{(A. 6)} \qquad -W''(Y)\, Y/W'(Y) \geqq R_R(A) \quad \text{if } X \geqq 0$$

Multiply through by $-W'(Y)X$, Using (A. 3) we obtain

$$\text{(A. 7)} \qquad W''(Y)\, X\, Y \leqq -R_R(A)\, W'(Y)\, X \quad \text{if } X \geqq 0$$

As before, if $X \leqq 0$, the inequality (A. 7) continues to hold, since inequality signs in both (A. 3) and (A. 6) are reversed. Taking expected values in (A. 7)

$$\text{(A. 8)} \qquad E\left[W''(Y)\, X\, Y\right] \leqq -R_R(A)\, E\left[W'(Y)\, X\right] = 0 \quad \text{for all } X$$

This proves proposition II.

1) The proof is due to K. J. Arrow, who has presented it in a personal communication to my colleague J. Mossin.

In view of (A.1) the right-hand side is equal to zero. Hence, proposition I has been proved.[1]

II.

The proof of the second proposition can be easily established by an analogous procedure.

[illegible] that

$$[illegible]$$

(A.6) [illegible] $= 0$

Multiply [illegible] we obtain

(A.7) [illegible] $= 0$

As before, if [illegible] inequality (A.7) continues to hold, the inequality signs in both (A.7) and (A.6) are reversed. Taking expected values in (A.6)

(A.8) [illegible] $= 0$ for all [illegible]

This proves proposition II.

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1) The proof is due to K.J. Arrow, who has presented it in a personal communication to my colleague J. Marschak.