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EIGHT ESSAYS IN THE THEORY OF OPTIMAL RESOURCE ALLOCATION OVER TIME

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PREFACE

The work reported in this collection of essays was started at the end of the academic year 1972 - 1973 when I was a visiting research associate in the Economics Department at the University of California, Berkeley, USA. In this period I benefited greatly from courses and seminars given by Professor Daniel McFadden and by Professor Karl Shell (at Stanford University).

Except for a leave of absence (Spring term 1976), the research has been done at the Institute of Economics, The Norwegian School of Economics and Business Administration, on a part-time basis together with my teaching and administrative duties as Lecturer/Senior Lecturer in Economics. Part of the Spring term 1976 was spent at the Unit for Research in the Economics of Education at the London School of Economics.

At various stages of the work expert typing has been provided by Turid Nygaard, Sissel Gullaksen, Grete Didriksen, Kirsten Herstad and Inger Meyer.

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Various personal acknowledgements are made in connection with the different articles. In addition I would like to mention Karl Göran Mäler for helpful advise during an earlier stage of the work.

To all these persons and institutions I express my gratitude.

Bergen, May 1979

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INTRODUCTION

The collection of essays presented here has two parts. Part one, consisting of four essays together with an appendix, deals with optimizing the allocation of labour between the educational sector and the rest of the economy over time. Part two,consisting of four essays, deals with problems connected with optimal extraction of an exhaustible resource in a small, open economy. The approach is dynamic in both parts, and the problems are studied from a macroeconomic point of view. Except for the appendix to Part I, the essays are purely theoretical. The aim has been to develop theories for the optimal dynamic management of the economy within these two areas.

The first article in Part I, "On the Optimal Allocation of Labor to the Educational Sector", was published in the Swedish Journal of Economics in 1975. In this article, the optimal allocation of labour to the educational sector is discussed within a simple macroeconomic model. The model consists of two sectors, one which produces knowledge - called educational capital in the model - and another which produces goods. The amount of labour allocated to the educational sector is optimized under the assumption that the level of knowledge enters the social welfare function. It is shown in this case that there is in general no unique steady-state optimum for the allocation of labour to the educational sector. Consequently, this assumption has been dropped in all subsequent essays. The case when education is regarded only as a means of increasing the production of goods is then discussed. Finally, the solution obtained is compared to the criterion for investment in education usually proposed in the "economics of education" literature. The distinction between the stock of educational capital and its corresponding flow is shown to be crucial.

The second essay in Part I, "Economic Growth and the Allocation of Labour between Education and Goods-Production: Positive and Normative Aspects", has been published before as Discussion

Paper 1/1976 from the Institute of Economics, The Norwegian School of Economics and Business Administration (revised 18.6. 1976). This essay begins with a positive analysis of the interdependence between the educational sector and the rest of the economy during a process of economic growth. Another difference from the first essay is that the labour-force is now assumed to increase over time.

Based on two different behavioural relationships between demand for education and the level of income in society, labour-allocation over time is analyzed within a positive model of economic growth incorporating the features of education. Both of them yield a unique, globally stable steady-state where "knowledge per capita" and the part of the labour-force going into the educational sector are constant over time. The allocation of labour over time is then optimized. It is shown that, independent of initial conditions in the economy, there exists a unique optimal path for the allocation of labour to the educational sector with an associated optimal development for the whole economy. A unique steady-state optimum exists, which is reached only asymptotically along the optimal path.The value of the variables in steady state is independent of the initial situation and depends on the rate of social time preference, the efficiency of the educational sector, the rate of depreciation of knowledge and the rate of increase in population. Along the optimal path the part of labour allocated to the educational sector should be falling towards its stationary level if the initial level of knowledge in society is less than the optimal level. Further, a lower initial level of knowledge will lead to a higher initial part of labour allocated to education. The optimal development of the economy is then confronted with the time-path implied by the positive model. Investment criteria for allocating labour to the educational sector are derived. Modifications in the structure of the model and their effect on the optimal path of the economy are considered. Special emphasis is here given to the effect of different rates of technological change between the educational sector and the goods-producing sector. It is shown that the allocation pattern along the optimal path may be reversed in periods with rapid technological progress in goods production.

The third essay in Part I, "Om optimal utvikling av ein kunnskapsbasert industrisektor" (in Norwegian), was published in Statsøkonomisk Tidsskrift in 1976. Whereas, in the two preceding essays, all goods production was aggregated into one sector, this model is more disaggregated since the production of goods and services now takes place in two sectors with different knowledge intensity. The problem in this article is therefore to find optimal paths for the allocation of labour between the educational sector and the two goods-producing sectors - the knowledge-intensive sector and the traditional sector. There may now be more regimes in the optimal policy and different optimal paths of labour allocation. It may now be optimal not to develop a knowledge-based industrial sector initially while at the same time the economy is building up its educational capital. When the level of knowledge has reached a certain level, time is ripe for beginning to allocate labour to a knowledge-based industrial sector as well. Another possibility is that if educational capital is initially abundant, it may be optimal to have a knowledge-based industrial sector initially even if it would not be optimal for ever.When it is optimal to allocate labour to all three sectors, the properties of the solution are fairly similar to those found in the second essay.

The fourth essay in Part I, "On the Optimal Development of Knowledge-Based Industries and the Educational Sector in a Small, Open Economy",was published in the International Economic Review in 1978. The differences between this essay and the third are that the instantaneous social welfare function is now based on less restrictive assumptions and that export and import of the educational-intensive and the traditional good is now allowed. As a result, complete specialization in the production of one of the goods is possible and may be optimal. If specialization to knowledge-based production is optimal, the solution is analogous to the solution in the second essay.It may also be optimal to specialize in traditional production. In this model there is then no reason to keep up an educational sector. When non-specialization is optimal, the results in this essay coincide pretty much with those of the third essay.

As an appendix to Part I I have included the article "Returns to Higher Education in Norway", published in the Swedish Journal of Economics in 1972. While obviously outside the mainstream of the argument in the first part of the essay collection, this article has been included because it is an illustration of the "returns to education" calculations undertaken in the economics of education literature mentioned to in the two first essays in Part I, where this article also has been referred to.

The first essay in Part II, "Optimal Savings and Exhaustible Resource Extraction in an Open Economy", was published in the Journal of Economic Theory in 1978. In this article,a macro-economic model for an open economy where optimal savings and exhaustible resource extraction can be determined simultaneously, is presented. The model is applicable to an economy with a considerable stock of exhaustible resources which are exported. The results are somewhat more general than those found in earlier contributions. The optimal extraction path depends on conditions in the rest of the economy, and the optimal path of capital accumulation depends on conditions in the resource sector. With constant prices and the capital intensity of the economy less than or equal to the modified golden rule, extraction is either constant for some initial period and then falling, or always falling, along any of the possible optimal policy sequences for the economy. When the price of the resource depends exponentially on time, it is optimal if, and only if, the rate of increase in the price of the resource is greater than some critical value, determined partly by the capital intensity of the economy, to depart from the optimal sequences mentioned above. In that case, resource extraction is increasing over time, and it may be optimal to leave the resource in the ground for some initial period. When the capital intensity of the economy increases, the price rise needed to make such a policy optimal is reduced. As the initial capital stock of the economy increases, the extraction period is lengthened and the extraction level is reduced for every t. The resource is exhausted when extraction ends and the extraction period is always finite. Extraction should be reduced gradually towards zero, where extraction ends. If a resource is discovered and exploited, compared to a situation without resource extraction, consumption gets an initial positive shift, while its relative rate of growth along the optimal path is reduced. Consumption and the capital stock will be higher also in the postextraction period. With constant marginal extraction costs - an assumption often made in the literature - an interior solution for savings and resource extraction at the same time cannot be optimal.

The second essay in Part II,"Resource Extraction, Financial Transactions and Consumption in an Open Economy", has been submitted to the Scandinavian Journal of Economics. At the present time I do not know whether or not it will be published there. Whereas in the first essay in Part II savings take the form of physical capital accumulation,this article presents a model of resource extraction in an open economy where borrowing or lending abroad is possible. Optimal strategies over time for consumption, financial transactions and resource use are derived. The properties of these time paths are compared to the results in earlier contributions. The main effect from allowing financial investment or disinvestment in a model of resource extraction in an open economy, is to separate the optimal consumption stream over time from the optimal path of resource extraction. If borrowing possibilities are unlimited, the separation will be complete. Without borrowing restrictions, optimal resource extraction is either zero or at its maximum. Resource extraction at less than the maximal rate can only be optimal if borrowing possibilities are exploited at its maximum. In that case there is in general no reason a priori to expect a falling optimal path of resource extraction in this model.

The third essay in Part II, "On Labour Allocation, Savings and Resource Extraction in an Open Economy", has been published before as Discussion Paper 7/78 from the Institute of Economics, The Norwegian School of Economics and Business

Administration. This paper presents a dynamic model for an open economy where labour allocation, savings and resource extraction can be optimized simultaneously. In the two preceeding essays, resource extraction is controlled directly by "turning the tap", whereas in this model extraction is controlled by the employment in the resource sector. Since labour must be released from the rest of the economy in order to extract resources, labour allocation over time between the two parts of the economy must be optimized. Marginal extraction costs are increasing due to the increasing alternative cost of labour. Properties of the optimal paths are derived, and their dependence on prices, parameters and initial conditions in the economy are examined. It is shown that cet. par. a poor country should extract a given resource faster than a rich country. Also, the widespread notion that total savings should increase when a new resource is discovered and exploited, is not substantiated in this model. The optimal savings rate and also the absolute amount of savings are always shifted down when exploitation of a new resource begins, so that total consumption increases by more than the value of the new resources extracted. The optimal pattern of economic development is therefore to slow down capital accumulation when resource extraction is started up and for the period extraction lasts, compared to a situation without resource extraction. When the resource extraction period is over, the stock of physical capital is therefore lower than it would have been at the same time without resource extraction, but it is higher than when resource extraction started.

The fourth essay in Part II, "On the Optimal Development of a Small, Open Economy With an Exhaustible Resource", is a revised version of Discussion Paper 15/78 from the Institute of Economics, The Norwegian School of Economics and Business Administration. Savings may now take the form of physical and/or financial capital accumulation. Borrowing abroad is also possible. The purpose of this paper is to provide a more general model of optimal resource use in an open economy where optimal paths of resource extraction, consumption, financial transactions and savings in physical capital can be determined simultaneously. We have distinguished between situations with and without borrowing restrictions. Without borrowing restrictions, the stock of physical capital is instantly adjusted so that its net marginal productivity equals the real rate of interest (given exogeneously). The depletion rate is then determined by the nominal rate of interest in the world financial markets and properties of the cost function in resource extraction. International credit rationing at the going market rate of interest may necessitate resource extraction for direct import purposes. A liberalization or removal of credit limits therefore slows down optimal resource use. A positive shift in the initial resource stock have similar effects since it increases the total debt a country may incur; it may also ease or remove existing borrowing constraints through improving the country's international creditworthyness. Price trends for the resource and for imported goods have been introduced in this model. The effects of these trends on extraction and consumption depend on whether borrowing restrictions are effective or not.

PART I

ON THE OPTIMAL ALLOCATION OF LABOR TO THE EDUCATIONAL SECTOR*

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Summary

The optimal allocation of labor to the educational sector is discussed within a simple macroeconomic model. The model consists of two sectors, one which produces knowledge—called educational capital in the model—and another which produces goods. The amount of labor allocated to the educational sector is optimized under the assumption that the level of knowledge enters the social welfare function. It is shown in this case that there is in general no unique steady-state optimum for the allocation of labor to the educational sector.

We then discuss sufficient conditions for uniqueness, the case when education is regarded only as a means of increasing the production of goods, dependence of the optimal policy on the initial situation in the non-uniqueness case and consistency of the optimal policy. Finally, the solution obtained is compared to the criterion for investment in education usually proposed in the "economics of education" literature. The distinction between the *stock* of educational capital and its corresponding *flow* is shown to be crucial.

I. Introduction

Despite its importance in the current debate, very few attempts have been made to discuss the optimal investment in education on a macroeconomic level. To my knowledge, the only paper that discusses this problem explicitly is one by Uzawa [7]. He analyzes a two-sector growth model where purely labor-argumenting technical change is produced in the educational sector, and where this output from the educational sector enters the goods-producing sector as an input. By using the welfare criterion of maximizing the discounted sum of consumption per capita, his problem is then to find the optimal allocation of labor between the two sectors and to choose the optimal savings path over time. Education is regarded only as a means of making labor more productive. The level of education or knowledge in society is irrelevant to social welfare.

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* The main part of this paper was written while I was a visiting research associate at the Economics Department, University of California, Berkeley. I am indebted to Karl Shell, K. P. Hagen, V. Norman and A. Uhde for useful comments.

We propose and analyze a model whose structure is somewhat different and less complex than Uzawa's although the objective function is perhaps more interesting, since it is assumed that society is interested in the level of knowledge as well as in the aggregate production of goods. In addition, it is easier to give our results a clear economic interpretation.

II. The Model

The model consists of two sectors—an educational sector that produces "knowledge", which we call "educational capital", and a sector that produces goods. Of course, we would have preferred a model with many types of education and many goods-producing sectors, but it seems very difficult to analyze such a general case.

The main additional simplification of the model is that we disregard physical capital, so that labor and educational capital are the only specified factors of production. There is not technical change in the model and the production period in the educational sector is not taken into account. The total amount of labor, $\bar{L}$, is given and constant over time. Labor in this context is understood to be completely uneducated labor.

The amount of labor employed in the educational sector is denoted by L_2 and the output in this sector is denoted by $J(t)$, so that at any point in time, t, output in the educational sector is given by

$$J(t) = h(L_2(t)) \tag{1}$$

where we assume

$h' > 0, h'' < 0.$

$J(t)$ is net in the sense that any output in the educational sector which is subsequently used as input (such as students who become teachers after graduation) is not included in $J(t)$.

The output in the goods-producing sector is assumed to be produced according to the following production function:

$$X(t) = f(L_1(t), E(t)). \tag{2}$$

$X(t)$ is total production of goods at time t, $L_1(t)$ is the amount of labor allocated to the production of goods, and $E(t)$ is the level of knowledge in society, a stock called the stock of educational capital.

In (2) we assume that $f(0, E(t)) = 0, \frac{\partial f}{\partial L_1(t)} = f_L > 0,$

$\frac{\partial f}{\partial E(t)} = f_E > 0, \frac{\partial^2 f}{\partial L_1(t)^2} = f_{LL} < 0, \frac{\partial^2 f}{\partial E(t)^2} = f_{EE} < 0,$

and

$$\frac{\partial^2 f}{\partial L_1 \partial E} = f_{LE} = 0.$$

In other words (2) says that the level of the production of goods in society depends on the level of knowledge, which we call educational capital since it is produced in the educational sector, and on the amount of labor allocated to the goods-producing sector. Production functions that express the same idea can be found in [2] (see e.g. p. 14 and model 7.1, p. 36). Labor in this instance refers to uneducated labor, so we have made the abstraction of completely separating the productivity of the level of education in society from the productivity of "primitive" labor in the production process.[1] The problems of measuring the educational capital in society will not be dealt with in detail. In principle they are similar to the problems of measuring the stock of physical capital. Several attempts to measure the stock of educational capital in different countries have been made, see e.g. those mentioned in [5] (Chapter 20, p. 523 and the discussion on p. 742). A detailed estimate of educational capital in Norway in 1950 and 1960 is presented in [1]. In this respect there seems to be a better empirical basis for (2) than for the production function used by Uzawa in [7].

The stock of educational capital is built up through gross addition to the existing stock $J(t)$, given by (1). On the other hand, it also depreciates since educated people die, knowledge becomes obsolete and people forget what they once learned. $E(t)$ is assumed to depreciate at a constant rate μ. The equation of motion for the state variable of the problem, $E(t)$, is therefore:

$$\dot{E}(t) = J(t) - \mu E(t) \tag{3}$$

where $\dot{E}(t) = dE(t)/dt$ is the net increase in $E(t)$ at point in time t. We assume that initially there is a stock of educational capital, E_0, so that

$$E(0) = E_0 \tag{4}$$

and that $E(\infty)$ is free.

Finally, since the total amount of labor is constant, we have that

$$L_1(t) + L_2(t) = \bar{L}. \tag{5}$$

The question now, is given the structure of this economy, described by equations (1)–(5), what is the optimal allocation of labor to the educational sector over time, i.e. what is the optimal trajectory of $L_2(t)$?

The answer to this problem obviously depends on the objective function.

[1] It thereby seems natural to assume $f_{LE} = 0$. It might be noted, however, that the analysis also holds under the weaker assumption that the Hessian matrix of (2) is negative semi-definite, i.e. $f_{LL} f_{EE} - (f_{LE})^2 \geq 0$. (This was pointed out to me by V. Norman.)

We analyze the problem under the assumption that the stock of educational capital in society and the aggregate production of goods enter the social welfare function. This assumption will probably strike the reader as reasonable. The stock of educational capital is synonymous to the level of knowledge in society and most societies aim at increasing the level of knowledge among its citizens.

The "instantaneous" social welfare function will therefore be of the form

$$u = u(X(t), E(t)), \tag{*}$$

where we assume $u_x > 0$, $u_E > 0$, $u_{xx} < 0$ and $u_{EE} < 0$.
(Here u_x means $\partial u/\partial X$, u_{xx} means $\partial^2 u/\partial X^2$, etc.)

Suppose, therefore, that the aim of society is to maximize the present value of its instantaneous utility function (*) from initial time, zero, and that its planning horizon is infinity. The welfare functional will then be

$$\int_0^\infty u(X(t), E(t))\, e^{-\varrho t} dt, \tag{6}$$

where $\varrho > 0$ is the social rate of discount (the social rate of time preference), assumed constant over time. The integral in (6) will converge since there is an upper bound on u which will be reached when the given amount of labor is so distributed between the production of goods and educational capital formation that u is maximized.

III. Optimizing the Allocation of Labor

The problem for society is to maximize (6), subject to the constraints (1) to (5). A possible solution would be that $L_2(t) = 0$ for some t, in which case there would not be any production in the educational sector. This possibility will be disregarded in what follows. The case where $L_1(t) = 0$ is also ruled out by the assumptions made with respect to the f function in connection with (2).

To simplify the problem, we can insert (5) into (1), so that

$$J(t) = h(\bar{L} - L_1(t)),$$

or that

$$J(t) = g(L_1(t)), \tag{7}$$

where $g' < 0$ and $g'' < 0$.
Finally, by substituting for $X(t)$ from (2) in (6), and using (7) in (3), we get the following optimal control problem:

$$\begin{cases} \text{Max} \int_0^\infty u(f(L_1(t), E(t)), E(t)) e^{-\varrho t} \, dt \\ \text{subject to} \\ \text{(i)}\ \dot{E}(t) = g(L_1(t)) - \mu E(t) \\ \text{(ii)}\ E(0) = E_0; E(\infty) \quad \text{is free} \\ \text{(iii)}\ 0 < L_1(t) < \bar{L}. \end{cases} \tag{8}$$

The control variable is now $L_1(t)$ and the state variable is $E(t)$. When the optimal trajectory for $L_1(t)$ is found, the optimal allocation of labor to the educational sector is given by (5).

In order to analyze (8), from the (current-value) Hamiltonian function

$$H(t, L_1(t), E(t), p(t)) = e^{-\varrho t}[u(f(L_1(t), E(t)), E(t)) + p(t)(g(L_1(t) - \mu E(t))]. \tag{9}$$

Necessary conditions[1] for a maximum of (6), subject to the constraints ((8), (i), (ii), and (iii)), are

$$\dot{E}(t) = g(L_1(t)) - \mu E(t) \tag{10}$$

$$\dot{p}(t) = -(u_x f_E + u_E) + (\mu + \varrho) p(t) \tag{11}$$

$$\frac{\partial H}{\partial L_1(t)} = u_x f_L + g' p(t) = 0. \tag{12}$$

In addition it will be seen that

$$\lim_{t \to \infty} e^{-\varrho t} p(t) = 0 \tag{13}$$

is satisfied for the optimal path in this problem, although in general it is not necessary for an optimal solution. (10) is merely a repetition of (3). (11) is the optimal path of the shadow price of educational capital. (12) is the optimality condition which says that for any t, the marginal product of labor allocated to the goods-producing sector, evaluated in terms of the social utility function, should equal the marginal product of labor allocated to the educational sector times the shadow price of educational capital at the same point in time, where the path of the shadow price is given from (11).

For any $p(t)$, $L_1(t)$ is determined implicitly by (12). Implicit differentiation yields

$$\frac{dL_1(t)}{dp(t)} = -\frac{g'}{u_{xx}(f_L)^2 + u_x f_{LL} + g'' p(t)}. \tag{14a}$$

According to (12), $p(t)$ must be positive.

From the assumptions with respect to the g and f functions, (14a) is there-

[1] The following analysis is based on the presentation of optimal control theory in [6], Chapter 19.

fore negative. The higher the shadow price of educational capital, the lower the allocation of labor to the goods-producing sector and, hence, the higher the allocation of labor to the educational sector.

$L_1(t)$ also depends on $E(t)$, so that

$$\frac{\partial L_1(t)}{\partial E(t)} = -\frac{u_{xx} f_E f_L + u_{Ex} f_L}{u_{xx}(f_L)^2 + u_x f_{LL} + g'' p(t)}, \tag{14b}$$

which is also negative for $u_{Ex} \leqslant 0$. The dependence of L_1 on p and E will be denoted by $L_1 = L(p, E)$, and (14a) and (14b) by L_p and L_E, respectively. Since the optimal $L_1(t)$ is given as a function of $p(t)$ and $E(t)$ by (12), (10) and (11) are two autonomous differential equations in two unknowns, $p(t)$ and $E(t)$.

We now want to see whether there is a unique rest point to the system of two differential equations (10) and (11). For $\dot{E}(t) = 0$, it is easy to establish that

$$\frac{dp(t)}{dE(t)} = \frac{\mu - g' L_E}{g' L_p}. \tag{15}$$

The slope of $\dot{E}(t) = 0$ is not determined from (15).

For the slope of the curve $\dot{p}(t) = 0$ in the phase space, we obtain the following, rather messy, expression:

$$\frac{dp(t)}{dE(t)} = \frac{u_{xx} f_E^2 + 2u_{Ex} f_E + u_x f_{EE} + u_{EE} + (u_{xx} f_L f_E + u_{Ex} f_L) L_E}{(\mu + \varrho) - (u_{xx} f_E + u_{Ex}) f_L L_p}. \tag{16}$$

The sign of this slope is not determined from the assumptions made with respect to the functions that enter it, or from a concavity condition on the u-function in (*). Thus, in general, there may be any number of stationaries to the two differential equations (10) and (11). This result is analogous to that obtained by Kurz when wealth effects were introduced into the standard model of optimal economic growth [4]. While it may be arguable whether the stock of physical capital should enter the social welfare function along with consumption, since this is in a sense "double counting", it would seem rather reasonable that the stock of educational capital should. The problem of a non-unique optimum would therefore seem to be more relevant to deciding on investment in education than on investment in physical capital. However, we postpone a discussion of the economic implications of non-uniqueness until we have treated the simpler case of a unique solution to the two differential equations (10) and (11).

Sufficient conditions for such a solution are

$$u_{xx} = u_{Ex} = 0. \tag{**}$$

When (**) holds, (14b) is zero so that (15) is positive and (16) negative. Clearly, the case where only $X(t)$ enters the social welfare function, i.e. when

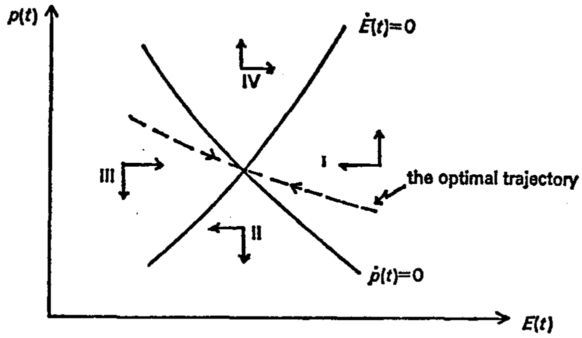

Fig. 1

education is regarded only as a means of making labor more productive, satisfies (**). Assuming that (**) holds, we can now make a graphic analysis of the system (10) and (11) from which the optimal trajectory of $L_1(t)$, and hence $L_2(t)$, will emerge.

Fig. 1 shows a phase diagram for the path of solutions to (10) and (11) when (**) holds.

Since now $\left.\frac{dp}{dE}\right|_{\dot{p}(t)=0} < 0$ and $\left.\frac{dp}{dE}\right|_{\dot{E}(t)=0} > 0$,

the curves $\dot{p}(t)=0$ and $\dot{E}(t)=0$ have a unique intersection and divide the $(E(t), p(t))$-space into four regions, labelled by roman numerals.

In order to determine directions of the movements of points in phase space, consider first the curve $\dot{E}(t)=0$. For given E, $\dot{E}(t)$ increases with p. So, $\dot{E}(t)>0(<0)$ for points above (below) $\dot{E}(t)=0$. The same applies to the curve $\dot{p}(t)=0$. For given $E, \dot{p}(t)$ increases with p, so that $\dot{p}(t)>0(<0)$ above (below) $\dot{p}(t)=0$. The movements of $E(t)$ and $p(t)$ in the different regions of the phase space are indicated by arrows.

The equilibrium of the system is represented by the intersection of $\dot{E}(t)=0$ and $\dot{p}(t)=0$. At the equilibrium, the stock of educational capital is constant over time and this level is denoted by E^{∞}. The same will hold for $p(t)$, whose equilibrium level is denoted by p^{∞}. p^{∞} determines an allocation of labor between the educational and goods-producing sectors, also constant over time.

It is easily realized that if we start from some arbitrary point in phase space, we do not generally approach the equilibrium. Consider a point in region IV. Here, both $p(t)$ and $E(t)$ are increasing. The only boundary of the region that might be hit is $\dot{E}(t)=0$, in which case the moving point would go back into region IV. Hence, a point in region IV will remain there, $p(t)$ and $E(t)$ will be steadily increasing and, according to (14a), $L_1(t)$ would tend toward zero. This is clearly inoptimal since, by assumption, $f(0, E(t))=0$.

Consider, then, a point in region II. Here both $p(t)$ and $E(t)$ are decreasing.

The only boundary that might be hit is $\dot{E}(t)=0$, but then we would be back in region II.

The two remaining regions can be analyzed as follows. In region III, $p(t)$ is decreasing and $E(t)$ is increasing. If the path remains in the region, it eventually has to approach limits which can only be the equilibrium values of $p(t)$ and $E(t)$. If it leaves the region, it can either cross $\dot{p}(t)=0$ and enter region IV, where it meets the fate described above. Or it can cross $\dot{E}(t)=0$ and enter region II, in which case its fate has also been described above. Similar reasoning applies to a path starting in region I; either it stays in the region and approaches the equilibrium value, or it enters region II or IV.

It is intuitively clear that (p^{∞}, E^{∞}) is a saddle point, i.e. there is one and only one path in the (p, E) space which converges to the equilibrium, so that to any given initial E_0 there corresponds a unique p_0, such that a path starting from the point (p_0, E_0) will converge to the equilibrium (p^{∞}, E^{∞}).[1]

That this path is optimal is clear from the following.[2]

a) *The Hamiltonian function (9) is concave in $E(t)$ and $L_1(t)$ simultaneously, for given $p(t)$ and t.*

This is because (1) the u-function in (9) is strictly concave in $E(t)$ and $L_1(t)$ when $f_{LE}=0$ and (**) holds. The Hessian matrix of u is then negative definite since its determinant is

$$u_x f_{LL}(u_x f_{EE} + u_{EE})$$

which is positive; (2) $g(L_1(t))$ is concave since $g''<0$ and $-\mu E$ is linear, hence concave. Finally, the sum of two concave functions is a concave function.

b)

$$\lim_{t\to\infty} e^{-\varrho t} p(t)(E(t)-E^{\infty})=0. \qquad (***)$$

This is so since, for $t\to\infty$, $p(t)$ approaches p^{∞} while $E(t)$ approaches E^{∞}. Hence (***) must approach zero for $t\to\infty$.

Since the path which satisfies (11) and (12) also satisfies a) and b), it is optimal.

We may therefore conclude so far that when (**) holds there exists a unique equilibrium, and to every initial E_0 there corresponds a unique p_0 so that the solution to (10), (11), and (12) with initial values (p_0, E_0) converges to the equilibrium (p^{∞}, E^{∞}). Assuming the functions (10), (11) and (12) known and E_0 given, the development of $(p(t), E(t))$ over time could be simulated for alternative values of p_0. By trial and error the unique p_0 that results in convergence of $(p(t), E(t))$ to (p^{∞}, E^{∞}) could then be found. Along this optimal path $(p^*(t), E^*(t))$ the allocation of labor to the educational sector will be

[1] This is shown formally in the Appendix.
[2] See [6], theorem 19.5, p. 528.

given as a function of $p^*(t)$ and the optimal trajectory for $L_2(t)$ can, in principle, be computed from (12) and (5). If $E(0) < E^\infty$, the optimal $L_2(t)$ must be decreasing over time.

An optimal policy will therefore be as follows. If E_0 happens to be equal to E^∞, choose $p_0 = p^\infty$ and the allocation of labor that corresponds to p^∞, $L_2(p^\infty)$. Keep this allocation indefinitely. If $E_0 \neq E^\infty$, find p_0 and reassign the allocation of labor to the educational sector continuously, using Equation (12). The amount of labor allocated to the educational sector will then approach the optimal amount $L_2(p^\infty)$ assymptotically.

Finally, let us examine the effects on the "steady-state" solution (p^∞, E^∞) and $L_2(p^\infty)$ due to changes in the parameters of the model, μ and ϱ.[1] By differentiating (10) and (11), with $L_1(t)$ given as a function of $p(t)$ from (12), we obtain the following results for changes in the social rate of discount ϱ (where the derivatives are evaluated at $\dot{p}(t) = \dot{E}(t) = 0$):

$$\frac{\partial E}{\partial \varrho} = \frac{1}{D} \quad (pg' L_p) \tag{17}$$

$$\frac{\partial p}{\partial \varrho} = \frac{1}{D} \quad (\mu p). \tag{18}$$

(The time argument in the functions is omitted from now on.)

Since $D = g'L_p(u_x f_{EE} + u_{EE}) - \mu(\mu + \varrho) < 0$, $\partial E/\partial \varrho < 0$ and $\partial p/\partial \varrho < 0$.

This means that the optimal steady-state level of educational capital is decreased (increased) if society chooses to evaluate the present, as opposed to the future, production of goods higher (lower). Obviously, the same applies with respect to the shadow price of educational capital $p(t)$, so that the amount of labor allocated to the educational sector decreases (increases) as the social rate of discount increases (decreases).

For changes in the rate of depreciation of educational capital μ, we find that

$$\frac{\partial E}{\partial \mu} = \frac{1}{D} \quad (g' L_p + (\mu + \varrho) E) \tag{19}$$

$$\frac{\partial p}{\partial \mu} = \frac{1}{D} \quad ((u_x f_{EE} + u_{EE}) E + \mu p). \tag{20}$$

$\partial E/\partial \mu$ is negative, so the optimal steady-state level of E is decreased (increased) if its rate of depreciation exhibits a postive (negative) shift. *A priori*, the effect on the shadow price of educational capital is ambiguous for $f_{EE} < 0$.

Whether or not the amount of labor allocated to the educational sector should rise or fall when the rate of depreciation changes is therefore not determined when $f_{EE} < 0$. If $f_{EE} \simeq 0$ in the relevant range, the allocation of labor to the educational sector should be reduced if the rate of depreciation of educational capital exhibits a positive shift.

[1] The equilibrium growth path (p^∞, E^∞) is a (special) steady-state growth path in the sense that all variables grow at the uniform rate of zero.

IV. On the Problem of Non-uniqueness

First of all, it should be noted that non-uniqueness may occur even if the sufficiency conditions are satisfied. The Hamiltonian function would still be concave if e.g. $u_{xx}<0$, although uniqueness is not guaranteed in this case. Any non-unique rest point to (10) and (11) therefore still satisfies the necessary and sufficient conditions for optimality as long as concavity of the Hamiltonian is ensured.

Depending on the form of the utility-function $u(E, X)$, phase diagrams can, in general, be constructed in which there will exist an arbitrarily large number of stationary points. In order to illustrate the optimal policy in the case of multiple equilibria, a phase diagram was drawn where we assume that $\left.\frac{dp}{dE}\right|_{\dot{E}=0} >0$ and that $\left.\frac{dp}{dE}\right|_{\dot{p}=0}$ is oscillating so that the curves $\dot{E}(t)=0$ and $\dot{p}(t)=0$ have four intersections.

For a stationary (p^{∞}, E^{∞}) to be a saddle-point, we require that

$$\left.\frac{dp(t)}{dE(t)}\right|_{\dot{E}(p^{\infty},\, E^{\infty})=0} - \left.\frac{dp(t)}{dE(t)}\right|_{\dot{p}(p^{\infty},\, E^{\infty})} >0.$$

In Fig. 2, E_2 and E_4 are saddle points while E_1 and E_3 are totally unstable points.

Directions of the movement of points in the phase space are again indicated by arrows. As shown by the arrows, the optimal allocation of labor to the educational sector is now of the following form. If the stock of educational capital initially happens to correspond to one of the totally unstable equilibria, as e.g. E_3 in Figure 2, keep it there. Otherwise, the stock of educational capital should converge to the value which corresponds to the nearest stationary with the saddle point property. This would be E_2 in Figure 2 if the initial situation was between E_1 and E_3. It would be E_4 if the initial situation was between E_3 and E_4. The conclusion here is that the optimal stock of educational capital and therefore also the optimal allocation of manpower depends on the initial situation.

This means that *ceteris paribus* a country with an initial stock of educational capital above the threshold level E_3 would optimally move towards the higher level E_4, whereas a country with educational capital initially less than E_3 would move towards E_2.

Although it constituted a special example, this conclusion is clearly a feature of all phase diagrams with multiple stationaries, when the planning horizon is infinity.

This indicates that *ceteris paribus* it will not be "worth the effort" for a country with low educational capital initially to try to reach the level of educational capital that is optimal for a country with a higher stock of educational capital initially. This somewhat discomforting conclusion is arrived at

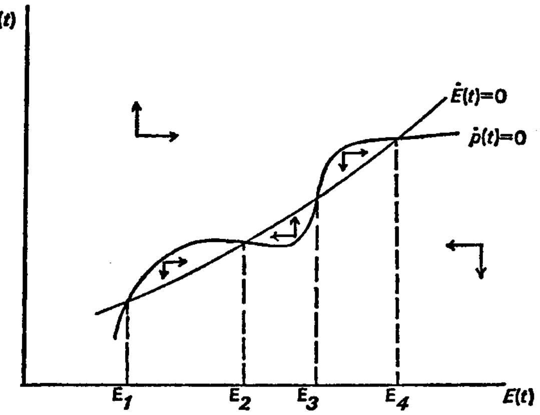

Fig. 2

when the stock of educational capital is introduced into the social welfare function. When education is regarded only as a means of increasing the production of goods, identical countries would optimally move towards the same level of educational capital, regardless of the initial situation.

With multiple equilibria, the optimal policy will still be consistent as long as the instantaneous utility function is unchanged over time. This means that there will not be any motive for revising the policy once it has been found. This can be explained by thinking—for the sake of simplicity—in terms of a discrete formulation. The reason is essentially that since the discount factor takes the (discrete) form $(1+r)^{-t}$, the marginal rate of substitution between the production of goods (or educational capital) in a pair of adjacent periods is independent of the time at which they are viewed. This means that if an optimal plan is found at point in time t_0, and checked with a view to a possible revision at $t_1 > t_0$, the marginal rate of substitution between the production of goods in a pair of future periods is the same viewed from either t_0 or t_1. Hence the plan is still optimal at t_1 and no revision is made. (This argument is discussed at length in Heal [3], Ch. 10.)

V. Interpretation. Relation to Cost-benefit Analysis

Let us examine the necessary and sufficient conditions for optimality in the case of a unique solution to the problem (8) more closely than was done in connection with (11) and (12). An attempt will also be made to relate these conditions to the criteria proposed for investment in education in the "economics of education" literature.[1]

For this purpose it will be useful to begin with the special case when education

[1] See e.g. [8], p. 276.

is regarded only as a means of increasing the production of goods, i.e. when $u_E \equiv 0$ and, for simplicity, $u_x \equiv 1$. (Of course this special case satisfies the assumptions sufficient for a unique solution to problem (8).) After eliminating $p(t)$, we obtain from (11) and (12) that, along the steady-state

$$f_L = -\frac{1}{\mu+\varrho} f_E g'. \tag{21}$$

By making the intellectual experiment that at point in time zero we are in the stady-state, (21) is equal to

$$f_L = -\int_0^\infty f_E g' e^{-(\mu+\varrho)t} dt. \tag{22}$$

(22) says that along the steady-state growth path the production of goods forgone by allocating labor to the educational sector at time zero should, at the margin, equal the present value of the increased production of goods due to the increase in educational capital brought about by the marginal amount of labor allocated to the educational sector. In other words, along the steady-state path, labor is allocated between the two sectors in such a way that the marginal cost of "investment in education" is equal to its marginal benefits. When we are not in the steady-state optimum, this means that at a given point in time the stock of educational capital should be increased or decreased according to whether

$$f_L \lesssim \int_0^\infty f_E \cdot g' e^{-(\mu+\varrho)t} dt. \tag{23}$$

Outside the steady-state, the exact path for the allocation of labor to the educational sector is given by the optimal trajectory in Figure 1, represented by the dotted line through the intersection of $\dot{p}(t)=0$ and $\dot{E}(t)=0$. Along the optimal trajectory, the optimal $L_2(t)$ is given as a function of the optimal $p(t)$.

In conclusion, let us compare (23) with the cost-benefit criterion most often used in the economics of education. This requires some simplifications.

Assume therefore that $\mu=0$, g' is a constant and that educational capital is computed as the total man years spent in the educational sector by the work force. If, at point in time zero, we consider the question of whether or not to educate one "marginal" person for one year, $g'=1$. Whether or not this is a profitable investment project depends on whether

$$f_L(0) \lesssim \int_0^\infty f_E(t) e^{-\varrho t} dt. \tag{24}$$

In cost–benefit analyses performed in the economics of education literature, f_L is set equal to earnings forgone and f_E is estimated as the difference between earnings with and without the extra education (estimated from cross-section data which is then also supposed to be applicable in the future). This difference is then discounted over the rest of the individual's lifetime. The project is profitable if earnings forgone are less than the present value of future increases in earnings. Apart from the difficult—perhaps impossible—task of measuring productivity from earnings data, the above shows that the criterion used in cost–benefit analysis in education is based on the assumption that the stock of educational capital, and therefore its marginal productivity, is unchanged over time. So, for a single marginal "dose" of new educational capital, the criterion used in cost–benefit analyses in education is consistent with our model and its objective function.

At most, however, the criterion indicates whether the actual stock of educational capital exceeds or falls short of the optimal stock. The cost–benefit criterion cannot give any indication of the optimal path outside the steady-state optimum. The distinction between the *stock* of educational capital and the *flow* of additions to this stock is crucial here. The cost–benefit criterion does not say anything about the relation between the actual flow and the optimal flow at a given point in time. It does not follow for instance, that the actual flow of new educational capital should be increased permanently if the actual stock of educational capital falls short of the optimal stock.

If we now return to the case where the stock of educational capital enters the social welfare function, the expression corresponding to (22) will be:

$$f_L = -\int_0^\infty \left(f_E + \frac{u_E}{u_x}\right) g' e^{-(\mu+\varrho)t}\, dt. \tag{25}$$

Not surprisingly, (25) shows that *ceteris paribus* benefits from education, and accordingly, the optimal investment in education, would now be higher due to the positive term u_E/u_x. (25) is a formalization of the informal "rule" often encountered in the economics of education literature that a term which represents the "consumption benefits" from higher knowledge should be "added to the monetary returns". Obviously, *a priori* knowledge of the marginal rate of substitution of goods for educational capital for all values of X and E would be required in order to make such a procedure operational.

VI. Conclusions

a) In the case where the sole *raison d'être* of the educational sector is its ultimate contribution to the production of goods in society, there is a unique and constant stock of educational capital that is optimal. This steady-state level of knowledge can only be reached assymptotically.

b) This feature of the solution is changed when the stock of knowledge in itself enters the social welfare function. Under this assumption, there is in general no unique steady-state stock of educational capital that is optimal. In the case of multiple equilibria, the optimal stock of educational capital will in general depend on the initial stock. An optimal policy, however, will still be consistent in the sense that once an optimal policy has been found at the beginning of the planning period, it will not be revised later on in the planning period.

c) It has been shown that, properly understood, the cost–benefit criterion usually proposed in the literature on the "economics of education" is consistent with the optimality condition in our model. It must be stressed, however, that this criterion deals only with the optimal stock of educational capital in relation to the actual stock—it does not say anything about the optimal allocation of labor to the educational sector when the actual stock of educational capital diverges from the optimal one.

Appendix

To establish the saddle point property of (p^{∞}, E^{∞}), make a linear expansion of (10) and (11) around (p^{∞}, E^{∞}):

$$\dot{p} \simeq (\mu+\varrho)(p-p^{\infty}) - (u_z f_{EE} + u_{EE})(E-E^{\infty})$$

$$\dot{E} \simeq g' L_p (p-p^{\infty}) - \mu(E-E^{\infty}).$$

The behavior of the system around (p^{∞}, E^{∞}) is determined by the characteristic roots of the matrix

$$\begin{pmatrix} \mu+\varrho & -(u_z f_{EE}+u_{EE}) \\ g' L_p & -\mu \end{pmatrix}$$

i.e. the roots of the equation

$$\lambda^2 - (\mu+\varrho-\mu)\lambda + ((\mu+\varrho)(-\mu) - (-(u_z f_{EE}+u_{EE}) g' L_p)) = 0,$$

viz.

$$\tfrac{1}{2}[\varrho \pm \sqrt{\varrho^2 - 4\{-\mu(\mu+\varrho) + (u_z f_{EE}+u_{EE}) g' L_p\}}].$$

As the expression under the square root sign is positive and greater than ϱ, both roots are real but have opposite signs. Consequently, (p^{∞}, E^{∞}) is a saddle point.

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ECONOMIC GROWTH AND THE ALLOCATION OF LABOUR BETWEEN EDUCATION AND GOODS PRODUCTION: POSITIVE AND NORMATIVE ASPECTS.

By Jostein Aarrestad

1. Introduction.

The total resources of a country may conventionally be grouped as follows:

(a) Physical Capital
(b) Natural Resources
(c) Population
(d) Educational Capital

All these resources may be increased or reduced over time, consciously or unconsciously. While there is a vast literature relating to the optimal development of (a) over time and a growing literature regarding (b) and (c), very little has so far been published on the optimal management of a society's educational capital over time. Traditionally, economists have regarded the production factor "labour" as homogenous. The quality of labour has been assumed constant over time. However, by altering the allocation of resources to e.g. the educational sector, the quality of labour may be consciously changed. Society is then faced with a dynamic optimization problem since to the extent resources are allocated to the educational sector <u>now</u>, in order to make labour more productive <u>later</u>, these resources cannot be used for producing goods and (other) services <u>now</u>.

Special aspects of this problem have been treated by Razin [5] and [6] and Dobell and Ho [1]. A more general analysis has been given by Uzawa [8] and, recently, by Manning [4]. This paper, which is a generalization of [10], also attempts a general approach to the problem. A problem in Uzawa's paper is that he is not explicitly concerned with the optimal management of the <u>stock</u> of educational capital over time, but with the rate of improvement in labour efficiency, which in his formulation can never be negative. In our context this would mean that the stock of educational capital, i.e. the stock of knowledge in society, can never be reduced, even if there is no activity in the educational sector. This does not seem entirely realistic, and we will analyze a model where this is avoided. The model can also be used to give a positive analysis of the interdependence between the educational sector and the rest of the economy during a process of economic growth. So far, according to Wan [9], p. 231, "there has been no "positive" growth model incorporating the features of education". The solution in the positive model can then be confronted with the optimal solution. Other new features in this model are: (i) The effect on the optimal allocation of labour over time from different rates of technological progress in the educational sector and the rest of the economy is analyzed, and (ii) instead of being used rather ad hoc as in the "economics of education" literature, investment criteria for the allocation of labour to the educational sector can now be derived from an explicit dynamic optimization model.

2. The Model.

Since the main problem is to find optimal paths for the allocation of labour over time, a dynamic model is needed. The model consists of two sectors - an educational sector producing "knowledge" which we shall call "educational capital" and a goods-producing sector. The main simplifications are that we disregard physical capital, so that labour and educational capital are the only specified factors of production. The model is therefore more relevant to an economy rich in physical capital and where labour is a "bottle-neck", as e.g. Norway. The production period in the educational sector is overlooked. Labour is here to be understood as completely uneducated labour, so we have made the abstraction of completely separating the productivity of educational capital from the productivity of "raw" labour in the production process. The amount of "raw" labour employed in the educational sector is denoted by L_2 and the output in this sector is denoted J(t). At any point in time, t, output in the educational sector is given by

$$(1) \qquad J(t) = G(L_2(t)); \qquad G' > 0, \ G'' \leq 0.$$

When $G'' = 0$, G' will be denoted by α. J(t) is net in the sense that any output in the educational sector that is subsequently used as input (as when students become teachers after graduation) is not included in J(t).[1] The output in the goods-producing sector is produced according to the following production function:

$$(2) \qquad X(t) = F(L_1(t), E(t)).$$

Here X(t) is total production of goods at time t, $L_1(t)$ is the amount of "raw" labour allocated to goods production, and E(t) is the level of knowledge in society, a stock called the stock of educational capital. F is assumed to be strictly concave with positive and diminishing marginal productivities. Also $F(L_1(t), 0) = F(0, E(t)) = 0$. In other words (2) says that the level of goods production in society depends on the level of knowledge, which we have called educational capital, since it is produced in the educational sector, and on the amount of labour allocated to the goods-producing sector. Production functions expressing the same idea can be found in [2]. (See e.g. page 14 and model 7.1 page 36.) We shall not here dwell upon the problems of measuring the educational capital in society. In principle they are similar to the problems of measuring the stock of physical capital. Several attempts to measure the stock of educational capital in different countries have been made, see e.g. those mentioned in [7], in chapter 20, p. 523 and the discussion on page 742. A detailed estimate of the educational capital in Norway in 1950 and 1960 is presented in [3]. There seems to be a better empirical basis for (2) than for the production functions used by Uzawa in [8]. The stock of educational capital is built up through the gross addition to the existing stock J(t), given by (1). On the other hand, it depreciates since educated people die, knowledge gets obsolete and people forget what they once learned. E(t) is assumed to depreciate at a constant rate μ. The equation of motion for the state variable of the problem, E(t), is therefore:

$$(3) \qquad \dot{E}(t) = J(t) - \mu E(t),$$

where $\dot{E}(t) = \frac{dE(t)}{dt}$ is the net increase in E(t) at point in time t. We assume that there is initially a stock of educational capital, E_0, so that

$$(4) \qquad E(0) = E_0.$$ [2)]

Finally, there is a fixed proportion between the population and labour force L(t).L grows at the exponential rate n, so that

$$(5) \qquad L(t) = L_0 e^{nt},$$

and we have full employment, i.e.

$$(6) \qquad L_1(t) + L_2(t) = L(t).$$

Assume now that F is homogenous of degree one in its two arguments, so that

$$(7) \qquad \frac{X}{L_1} = F\left(\frac{E}{L_1}, 1\right) = f\left(\frac{u}{l_1}\right)$$

where

$u = \frac{E}{L}$ is the aggregate educational capital ratio, and

$l_1 = \frac{L_1}{L}$ is the part of the available labour force allocated to goods production.

Similarly, l_2 is the part of the total labour force allocated to the educational sector, so that

(8) $l_1 + l_2 = 1$.

(7) says that the average labour productivity in the goods-producing sector, i.e. total production of goods per man-year worked, is an increasing function of the level of knowledge per worker - which seems very reasonable. From (7) we now have

(9) $x = l_1 f(\frac{u}{l_1})$.

Since $\frac{\partial X}{\partial E} = f'(\frac{u}{l_1})$, we have from the properties of (2) that

(10a) $f'(\frac{u}{l_1}) > 0,\ f''(\frac{u}{l_1}) < 0$.

Also

(10b) $\frac{\partial X}{\partial L_1} = f(\frac{u}{l_1}) - f'(\frac{u}{l_1})\frac{u}{l_1}$

which by the assumptions on (2) is positive. For reasons to become clear later on, we also assume that

(11) $f'(0) > \frac{\mu+n}{\alpha v}$

where α and v will be defined later. The development of the aggregate educational capital/labour ratio u over time is given by

(12) $\dot{u}(t) = h(t) - \lambda u(t)$

where

$h = \frac{J}{L}$ [3], $\frac{\dot{L}}{L} = n$ and $\lambda = \mu + n$.

Assume proportionality between output and input in the educational sector, so that $J = \alpha L_2$ or, dividing by L and using (8),

(13) $h = \alpha(1-l_1)$,

where α is a constant factor of proportionality. (9), (12) and (13) are three relations between four time functions, x(t), u(t), h(t) and $l_1(t)$, hence the model is not yet determined. It can be "closed" either

(i) by a description of how l_1 is actually determined in an economy (a descriptive model), or

(ii) by choosing $l_1(t)$ such that the development over time of the economy from a given initial situation is optimized (an optimization model). Let us treat the two possibilities in turn.

3. Descriptive Theories.

a) It is a well-known fact that demand for education and the level of income in a society tend to vary in the same direction. At the individual level, the reason may be that education gets more profitable as income per capita in society increases and/or education is a normal consumption good. Paralell to the treatment of total savings in relation to total income in the neoclassical model of economic growth, the simplest way to formalize this relation is to assume a fixed relation between the part of the population that, at any time, is undertaking education and production per capita, so that

(14) $$l_2 = v l_1 f(\frac{u}{l_1})$$

where v is a constant. (8), (9), (12), (13) and (14) is now a determined model in x, u, h, l_1 and l_2. To study the development over time of this system from a given initial situation, we substitute (14) in (13) and (13) in (12) which yields

$$\dot{u} = \alpha v l_1 f(\frac{u}{l_1}) - \lambda u$$

or, dividing by l_1:

(15) $$\frac{\dot{u}}{l_1} = \alpha v f(\frac{u}{l_1}) - \lambda \frac{u}{l_1}.$$ [4)]

To analyze (15), define $z = \frac{u}{l_1}$.

Instead of (15) we then have

(15b) $$\dot{z} + \frac{\dot{l}_1}{l_1} z = \alpha v f(z) - \lambda z.$$

From (14)

$$\dot{l}_1 = - \frac{v l_1 f'(z)\dot{z}}{1+vf(z)} .$$

Using this expression in (15b) we obtain

(15c) $$\dot{z} = \frac{1+vf(z)}{1+v\phi(z)}[\alpha v f(z) - \lambda z]$$

where

$$\phi(z) = \frac{\partial X}{\partial L_1} = f(z)-f'(z)z > 0,$$

which means that

$$\frac{1+vf(z)}{1+v\phi(z)} > 0 \text{ for all } z.$$

For this reason and because of (10a) and (11), *there exists a unique and globally stable steady-state educational capital intensity* $z^* = (\frac{u}{l_1})^*$, *such that*

(16) $$\alpha v f(z^*) = \lambda z^*.$$

This solution is illustrated in figures 1 and 2 below.

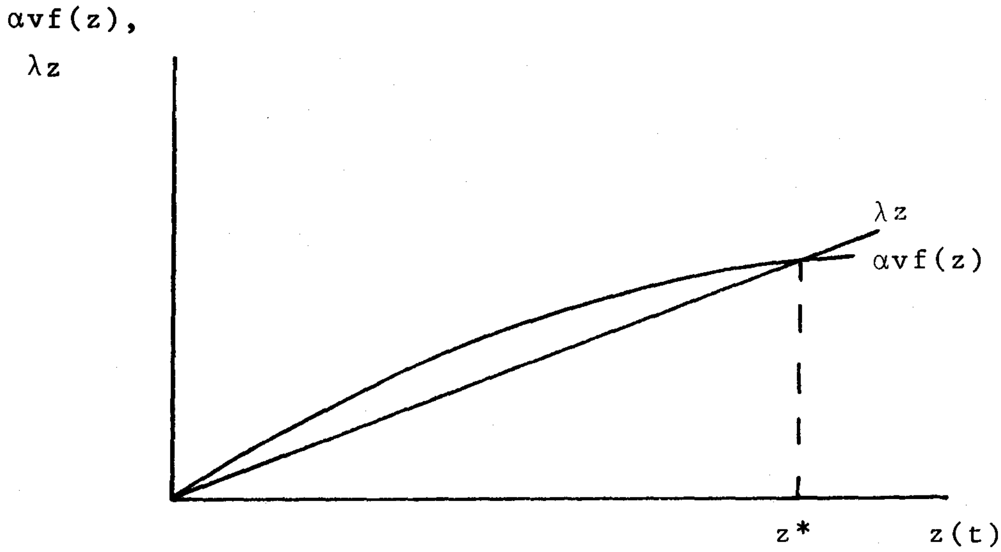

Figure 1

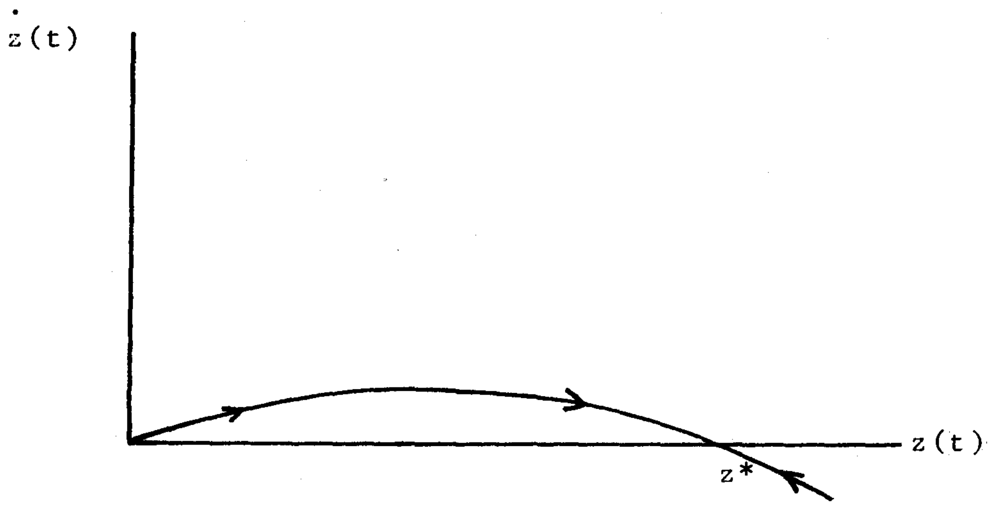

Figure 2

From the definition of z:

$$\dot{u} = \frac{l_1(1+v\phi(z))\dot{z}}{1 + vf(z)}$$

which means that

$$(17)\quad \begin{cases} \dot{z} = 0 \iff \dot{u} = 0, \text{ since } \dfrac{l_1(1+v\phi(z))}{1+vf(z)} \neq 0 \\ \text{and} \\ \text{sign } \dot{z} = \text{sign } \dot{u}, \text{ since } \dfrac{l_1(1+v\phi(z))}{1+vf(z)} > 0. \end{cases}$$

(17) shows that the steady-state level of z is also a steady-state level of u, with the same stability properties as the steady-state level of z. This means that $\frac{E}{L}$ is constant or that the total educational capital in society grows at the same rate as the exogenously given growth rate of population. Also, since in this steady state both u and $\frac{u}{l_1}$ are constant, l_1 must be constant as well, i.e. a constant fraction of the population will be allocated to the educational sector. From (9) it then follows that $\frac{X}{L}$ will be constant which means that total goods production in society grows at the same rate as the population.

b) Another possible assumption as regards the allocation of labour to the educational sector would simply be that, at any point in time, a constant fraction of the population is in the educational sector. Such a relation might e.g. be the result of a conscious educational policy. Denoting the constant l_2 by $\bar{l}_2$, we would then from (12) have that $\dot{u} = \alpha\bar{l}_2 - \lambda u$ which has the solution

$$(18)\qquad u(t) = (u_0 - \frac{\alpha}{\lambda}\bar{l}_2)e^{-\lambda t} + \frac{\alpha}{\lambda}\bar{l}_2,$$

where u_0 is the initial educational capital intensity.

Provided $u_0 < \frac{\alpha}{\lambda}\bar{l}_2$, u(t) in (18) will rise asymptotically towards a steady-state level given by $u = \frac{\alpha}{\lambda}\bar{l}_2$.

Therefore, under the above two behaviour assumptions, which do not seem unreasonable, the economy would move towards a stable steady state with a constant level of educational capital per worker. The level of this steady-state educational capital intensity depends on α, μ, n and $\bar{l}_2$ (or v under assumption a)). When $\bar{l}_2 = 1$ we will asymptotically reach the highest possible sustainable u, denoted by $\tilde{u}$, where $\tilde{u}$ is given by $\tilde{u} = \frac{\alpha}{\lambda}$. A natural question now is: What is the "best" steady-state level of educational capital per worker? For given α and λ, the problem is then to find the "best" value of l_2 (or l_1).

4. Normative Analysis.

a) A "golden rule" - type of analysis.

The golden rule in standard growth theory is that steady-state capital intensity (and associated savings ratio) that maximizes consumption per capita in steady state. In this model, where consumption does not enter explicitly, we would naturally among the possible steady states choose that which maximizes production per capita x, given by (9). Using $\bar{l}_1$ as a control variable in case b) above, given steady-state $u = \frac{\alpha}{\lambda}(1-\bar{l}_1)$, we get the following first-order condition for a maximum of x in a steady state:

$$(19) \qquad \frac{\partial x}{\partial l_1} = f(\frac{u}{l_1}) - \frac{\alpha}{\lambda} f'(\frac{u}{l_1}) \frac{1}{l_1} = 0$$

To interpret (19), multiply by l_1. In addition we have that

$$f'(\frac{u}{l_1}) = \frac{\partial x}{\partial u} = \frac{\partial X}{\partial E}$$

and $\frac{dE}{dL_2} = \frac{\alpha}{\lambda}$ (in steady state). We can now rewrite (19) as

$$\frac{X}{L} = \frac{\partial X}{\partial E} \frac{dE}{dL_2}$$

which says simply that the part of the labour force in the educational sector should be expanded until, in a stedy state, the average product per worker x is equal to the indirect marginal product of labour, that is the marginal product of educational capital times the effect on the educational capital of a marginal increase in the part of the labour force that is in the educational sector. That (19) is a maximum condition is clear since

$$(20) \qquad \frac{\partial^2 x}{\partial l_1^2} = \left(\frac{\alpha}{\lambda}\right)^2 \frac{1}{l_1^3} f''\left(\frac{u}{l_1}\right) < 0.$$

The optimal l_1 in a steady state given by (19) is a function of the fraction $\frac{\alpha}{\lambda}$. To simplify notation, denote this fraction by Z. The effect on the optimal l_1 in a steady state from a change in Z is then given by

$$(21) \qquad \frac{\partial l_1}{\partial Z} = \frac{l_1 f' + Z\left(\frac{1}{l_1} - 1\right) f''}{\left(\frac{Z}{l_1}\right)^2 f''}$$

Somewhat surprisingly, perhaps, the direction of this effect is in general not determined. We see, however, that as l_1 approaches 1, (21) will be negative. This means that when the part of the labour force allocated to the educational sector initially is "low", an increase in α, due to e.g. more efficient teaching methods would increase the part of the labour force allocated to the educational sector, while an increase in n - the growth rate of the population, or in μ - the rate of depreciation of the educational capital, brought about by e.g. a rise in the death rate, would reduce the part of the population allocated to the educational sector. On the other hand, if l_1 approaches zero, that is, the part of the labour force in the educational sector is "high", then the opposite conclusion would hold. It must be remembered that a golden rule - type of analysis leads to normative results of rather doubtful relevance, since the initial situation and the development over time towards a possible stationary state is disregarded. The problem is reduced to a choice between alternative hypothetical steady states with alternative hypothetical constant l_1's. To take the initial situation into account and find the possible optimal paths over time of $l_1(t)$ from a given starting point, a dynamic analysis is needed.

b) Optimization over time.

To find the optimal path of l_1 over time, assume that society wants to maximize the present value of production per capita. Suppose that the planning horizon is infinity. The objective is then to maximize

$$(22) \qquad \int_0^\infty x(t)e^{-\rho t}dt,$$

where ρ is the social rate of time preference, assumed positive and constant. This integral will converge provided the rate of growth in x is less than ρ. Substituting for x from (9) and from (13) in (12) we get the following optimal control problem with $l_1(t)$ as a control variable:

$$(23) \qquad \begin{cases} \max \int_0^\infty l_1 f(\frac{u}{l_1})e^{-\rho t}dt \\ \text{s.t.} \\ \qquad \dot{u} = \alpha(1-l_1) - \lambda u \\ \qquad\quad 0 < l_1 \leq 1 \\ \qquad u(0) = u_o \text{ (given)} \\ \lim_{t\to\infty} u(t) \text{ is free} \end{cases}$$

(Since production would be zero if $l_1 = 0$, we restrict l_1 to be positive). To solve problem (23), form the (present value) Hamiltonian function

$$(24) \qquad H = e^{-\rho t}\{l_1 f(\frac{u}{l_1}) + p[\alpha(1-l_1) - \lambda u]\}$$

Necessary conditions for a solution to (23) are that there exists a continuous function p(t) such that

(25) $\dot{p}(t) = -f'(\frac{u}{l_1}) + (\rho+\lambda)p(t)$

(26) $\dot{u}(t) = \alpha(1-l_1(t)) - \lambda u(t)$

(27) $\frac{\partial H}{\partial l_1} = f(\frac{u}{l_1}) - f'(\frac{u}{l_1})\frac{u}{l_1} - \alpha p \geqq 0$ and $= 0$ if $l_1 < 1$.

p(t) can be interpreted as the "shadow-price" of u, i.e. p(t) expresses the increase in the optimal value of the objective function obtained from adding "one extra unit" of educational capital per capita to the stock of educational capital per capita at point in time t. (25) shows the optimal path of this shadow price. (26) is merely a repetition of (12). (27) is the optimality condition for l_1 at any t, saying that, for an interior solution, the marginal product of labour in goods production - $\frac{\partial x}{\partial l_1}$ - should equal the marginal product of labour allocated to the educational sector - α - times the shadow price of u, that is p. If (27) holds with strict inequality we have a boundary solution and $l_1 = 1$. Since $\frac{\partial x}{\partial l_1}$ and α are both positive, p(t) must always be positive when the solution is interior. (27) shows that we may have two types of solutions, depending on whether $l_1(t)$ is interior or not. Let us denote the region where l_1 is interior by N (for nonspecialization) and the other by S (for specialization) and analyze them in turn, beginning with N.

(i) The N-region.

(27) now determines an optimal l_1, which, for a given α, depends on u and p. That is

(28) $l_1 = l_1(u,p)$.

The essential feature of (28) is that

(29) $$\frac{\partial l_1}{\partial u} = \frac{l_1}{u}$$

and

(30) $$\frac{\partial l_1}{\partial p} = \frac{\alpha l_1^3}{u^2 f''} .$$

Since the optimal l_1 is given as a function of p and u from (28), (25) and (26) are two autonomous differential equations in u and p. This permits a two-dimensional graphic analysis of the system from which the optimal trajectory of $l_1(t)$ will emerge. The graph of $\dot{p}(t) = 0$ in the (p,u) phase-plane is a horisontal line since

(31) $$\frac{dp}{du}\Big|_{\dot{p}=0} = 0.$$

For the slope of the graph of $\dot{u}(t) = 0$ we obtain

(32) $$\frac{dp}{du}\Big|_{\dot{u}=0} = -\frac{\lambda + \alpha\frac{\partial l_1}{\partial u}}{\alpha\frac{\partial l_1}{\partial p}}$$

which is positive. Before we discuss the solution graphically, consider region S.

(ii) The S-region.

To study the boundary of the N-region, denote the set of (p,u) values where $l_1 = 1$ is an interior solution to (27) by B, so that

$$B = \{(p,u) \mid f(u) - f'(u)u = \alpha p\}.$$

Along B we then have that

$$\frac{dp}{du} = -\frac{1}{\alpha}\{f''(u)u\} > 0$$

so that the boundary of the N-region has a positive slope. Further, in region S we have

$$\dot{u} = -\lambda u < 0$$

There is therefore no non-trivial stationary point for u in S.

$$\dot{p} = 0 \text{ when}$$

$$p = \frac{1}{\rho+\lambda}\{f'(u)\}$$

so that

(33) $$\frac{dp}{du}\Big|_{\dot{p}=0} = \frac{1}{\rho+\lambda}\{f''(u)\} < 0.$$

In figure 3, the above analysis of regions N and S is illustrated in a phase-diagram i the (p,u) - plane. The S-region is shaded, and the movements of u(t) and p(t) in the different regions of the phase-plane are indicated by arrows.

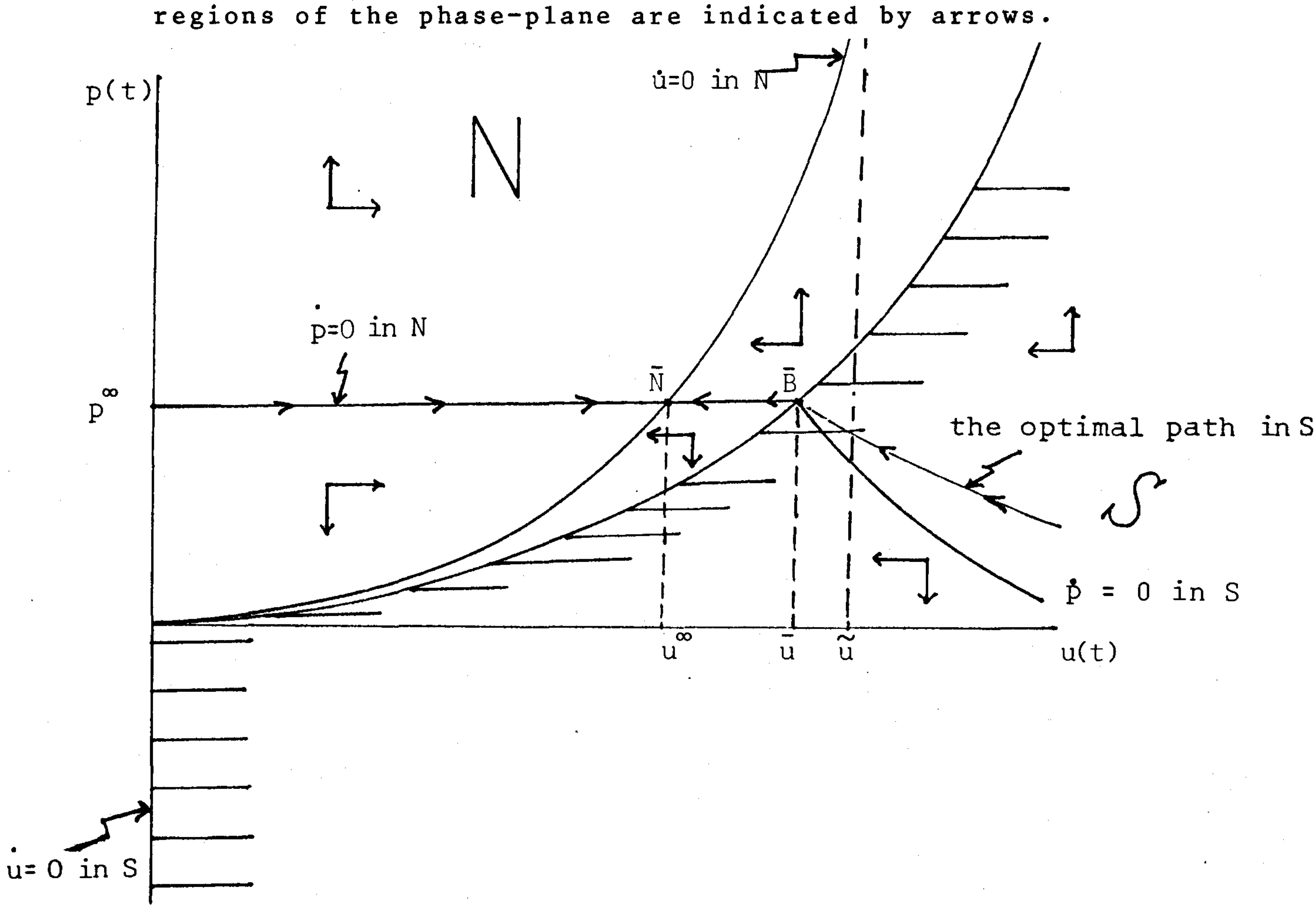

Figure 3.

Some additional information on the phase-diagram:

i) B must start out from (0,0) since from the definition of B

$p = 0 \Rightarrow u = 0$ and $u = 0 \Rightarrow p = 0$.

ii) A point on B will, for a given p, always be to the right of a point on $\dot{u}=0$. This follows from (29).

iii) The curve $\dot{u}=0$ must also start out from (0,0), since from (26) and (27) $p=0 \Rightarrow u=0$ and $u=0 \Rightarrow p=0$.

iv) As u approaches $\tilde{u}$, $\left.\frac{dp}{du}\right|_{\dot{u}=0}$ approaches infinity. This is so since $\left.\frac{dp}{du}\right|_{\dot{u}=0}$ is always positive and for $u > \tilde{u}$, $\dot{u}$ is always negative.

The equilibrium of the system is represented by the intersection of $\dot{p}(t)=0$ and $\dot{u}(t)=0$. Thus it can be stated that *there must exist a unique optimal stationary state for p and u* - (p^{∞}, u^{∞}) - *such that*

$$(*) \qquad u^{\infty} = \frac{\alpha}{\lambda} l_2(p^{\infty}, u^{\infty})$$

and

$$(**) \qquad p^{\infty} = \frac{1}{\rho+\lambda} f'\left(\frac{u^{\infty}}{l_1(p^{\infty}, u^{\infty})}\right)$$

This point is shown in the phase-diagram. Since p and u are stationary in this point, l_i (i=1,2) will also be constant. The equilibrium of the system is therefore a state of proportional growth, where the absolute value of all variables grow at the rate n over time. This state is reached only asymptotically. To interprete the equilibrium values of u and p, we see that (**) multiplied by α is the present value of a marginal allocation of labour to the educational sector. In the stationary state this value must, according to (27), be equal to the (instantaneous) marginal value of allocating labour to the goods-producing sector. It is easily realized that if we start from some arbitrary point in the phase-plane we do not generally approach the equilibrium, and it is straight forward to show that (p^{∞}, u^{∞}) is a saddlepoint.

c) The Optimal Policy.

Since (p^{∞}, u^{∞}) is a saddlepoint, there is one and only one path in the (p,u)-space converging to the equilibrium, so that given an initial u_o there corresponds a unique p_o, which for all u_o in N is $p_o = p^{\infty}$, such that a path starting from the point (p_o, u_o) will converge to the equilibrium (p^{∞}, u^{∞}). Along this path,[5] the optimal l_1 is given as a function of u alone and the optimal l_1 can in principle be computed.

It remains to discuss possible optimal policies. Consider first initial points in region N. If u(0) happens to be equal to u^{∞}, then $l_1(0)$ can be chosen equal to the optimal constant level of l_1, and the optimal policy is to keep u and l_1 constant. If $u(0) < u^{\infty}$, find that $l_1(0)$ which satisfies (28) with p^{∞} inserted and reassign $l_1(t)$ continuously to satisfy (28). Since, by assumption, we are now on the optimal path, u(t) must increase, and from (26) we see that u(t) approaches u^{∞} asymptotically. Since p(t) is constant and l_1 is an increasing function of u through (28), $l_1(t)$ will increase as u(t) increases. This means that when $u(0) < u^{\infty}$ the initial l_1 must be set below the stationary value of l_1. *The part of the labour force that is allocated to the educational sector will therefore decrease as u increases, and the lower the initial u is, the higher will the initial l_2 be.* An economic explanation for this pattern can be found in the fact that the opportunity cost, in terms of goods-production forgone, of allocating labour to the educational sector is lower as long as u is "low". This is because L_1 and E are complementary $(\frac{\partial^2 F}{\partial L_1 \partial E} > 0)$ in goods production, since F is homogenous of degree one. Since u is reached only asymptotically and $\frac{\partial^2 l_1}{\partial u^2} < 0$ from (29), the optimal constant l_1 will also be reached only asymptotically. Opposite conclusions will hold for the case when the initial educational capital intensity is higher than u^∞. In that case l_2 would increase over time from its initial level which would be less than l_2. Again, the stationary values for both u and l_1 would be reached only asymptotically. Optimal paths for $l_2(t)$, assuming $u(0) < u^\infty$, from two initial situations are illustrated in figure 4.

There is a striking contrast between the optimal pattern of labour allocation over time shown in this figure and the time-form of $l_2(t)$ observed in actual life. In Norway, the part of the population allocated to the educational sector on full-time basis has been rising steadily over time, from e.g. 0,16 in 1955/56 to 0,21 in 1972/73 and the trend is the same in other countries.

There is a very good correspondence between the actual pattern of labour allocation over time and the time form of $l_1(t)$ implied by the first positive model presented in the earlier part of this paper. For $z < z^*$ this model implies an increasing part of the population in the educational sector over time. For reasons outside this model such a development need not be inoptimal. Education may e.g. be undertaken for consumption purposes. At the end of this paper it will also be shown that the optimal time form of $l_1(t)$ is reversed in periods with rapid technological progress in the goods-producing sector relative to the educational sector. Except for such periods, however, the actual development of $l_2(t)$ observed in most countries is not consistent with the aim of maximizing some present value of production per capita. Without going into the matter in detail here, an optimal time form of $l_2(t)$ could perhaps be obtained in the positive model by using taxes and/or subsidies to make the "educational prospensity" v a decreasing function of production per capita.

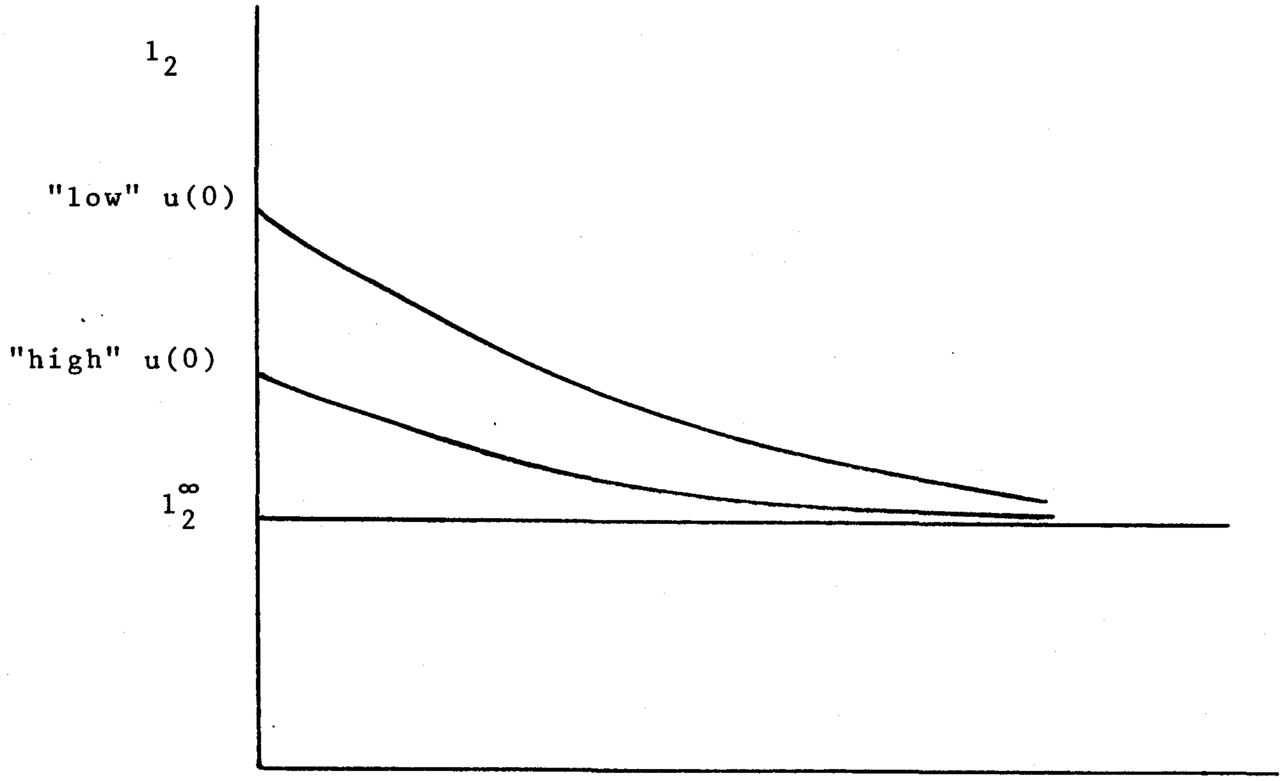

Figure 4.

An extreme variant of the case where $u(0) > u^\infty$ would be an initial point in region S, that is (for) initial u's to the right of $\bar{u}$ in figure 3. In S, the initial educational capital intensity is so high that the whole labour force would be allocated to goods production. Given an u_o in S, the optimal policy is to assign a p_o on the unique optimal trajectory in S leading to $(p^\infty, \bar{u})$. From the directions of movement of p and u in S, indicated by the arrows, it follows that along the optimal trajectory u must be decreasing and p increasing until $(p^\infty, \bar{u})$ is reached. As that point in time, the system switches into region N and follows the optimal policy outlined above for the case $u_0 > u^\infty$. That the "candidate" optimal policies discussed above are really optimal, is clear from the following:

a) The Hamiltonian function (24) is concave in u(t) and $l_1(t)$ simultanously, for given p(t) and t.

b)

$$\lim_{t\to\infty} e^{-\rho t} p(t)(u(t)-u^\infty) = 0.$$

This is so since p(t) is constant and as $t\to\infty$ u(t) approaches

u^∞. Since the path satisfying (25) and (27) also satisfies a) and b) above, it is indeed optimal.

Before we make a more thorough economic interpretation of the optimality conditions, it may be of some interest to study how the optimal steady state u and l_1 depend on the parameters of the problem. By (28), l_1^∞ depends on u^∞ and p^∞, and by (25) and (26), u^∞ and p^∞ depend on the values of α, n, μ and ρ. Differentiating in (25) and (26) we get the following effects on the optimal steady-state u (where the derivatives are evaluated at $\dot{p}(t) = \dot{u}(t) = 0$):

$$(34) \qquad \frac{\partial u^\infty}{\partial \alpha} = \frac{1}{D} \{-(1-l_1)[\alpha\frac{l_1}{u} + \rho + \lambda] \}$$

$$(35) \qquad \frac{\partial u^\infty}{\partial \lambda} = \frac{1}{D} \{u(\alpha\frac{l_1}{u} + \rho + \lambda) - \frac{\alpha^2 l_1^3}{u^2 f''} p\}$$

$$(36) \qquad \frac{\partial u^\infty}{\partial \rho} = -\frac{1}{D} \{ \frac{\alpha^2 l_1^3}{u^2 f''} p\}$$

where $\qquad D = -(\alpha\frac{l_1}{u} + \rho + \lambda)(\alpha\frac{l_1}{u} + \lambda) < 0.$

From these expressions we see that an increase in λ (in n and/or μ)reduces u^∞; that an increase in ρ also reduces u^∞ and that an increase in α increases u^∞. These results will be explained when we have found similar expressions for the partials of p^∞:

$$(37) \qquad \frac{\partial p^\infty}{\partial \alpha} = 0$$

$$(38) \qquad \frac{\partial p^\infty}{\partial \rho} = \frac{\partial p^\infty}{\partial \lambda} = \frac{1}{D} \{(\alpha\frac{l_1}{u} + \lambda)p\} < 0$$

From (34) and since $\frac{\partial l_1^\infty}{\partial \alpha} = \frac{\partial l_1^\infty}{\partial p^\infty}\frac{\partial p^\infty}{\partial \alpha} + \frac{\partial l_1^\infty}{\partial u^\infty}\frac{\partial u^\infty}{\partial \alpha} > 0$, it is clear that

in the steady state, the higher α is, i.e. the more "efficient" the educational sector is, the greater is the part of the labour force engaged in goods production and the greater is the educational level per worker. For the effect of a change in ρ on l_1^∞ we obtain after some manipulation

$$(39) \qquad \frac{\partial l_1}{\partial \rho} = \frac{1}{D} \{\frac{\alpha l_1^3 \lambda \rho}{u^2 f''} > 0.$$

This means that an increase in the social rate of time preference would reduce the stationary-state part of the labour force employed in the educational sector, and from (36) it would also reduce the optimal "education intensity" in steady state. Finally,

$$(40) \qquad \frac{\partial l_1}{\partial \lambda} = \frac{\partial l_1}{\partial \rho} - \frac{u}{(\frac{\alpha l_1}{u} + \lambda)}$$

which is in general indeterminate in sign. If f" approaches zero, however, $\frac{\partial l_1}{\partial \rho}$ approaches infinity so that for small absolute values of f", (40) is positive, and the allocation of labour to the educational sector should be reduced.

To summarize, the lower the rate of growth of the population in a country is, the higher is the optimal level of education among its citizens. Also if we associate the rate of depreciation of human capital μ with the death rate, we see that the higher the death rate is, the lower is the optimal level of knowledge in the population. Since the effect of λ on l_1 in (40) is not clear, it is possible to imagine two countries of which one has a higher death rate and rate of population growth than the other and where this country optimally allocates a larger part of its labour force to education only to obtain a lower level of knowledge among its population. This shows the relevance of demographic factors for optimal educational policies. It shows also a vicious circle, since *as long as industrially underdeveloped countries have a higher λ than developed ones, it is, ceteris paribus, optimal for them to have a lower level of knowledge in their work force then developed countries.*

(36) and (39) show, not surprisingly, that the optimal educational intensity and the part of the labour force allocated to the educational sector are increased (reduced) if society chooses to evaluate present, as opposed to future, goods production, lower (higher). Finally, increased efficiency in the educational sector effective through a higher α, leads optimally to a higher level of education per capita, and to a smaller part of the population in the educational sector at any time.[6)]

5. Interpretation.

Let us examine the necessary conditions for optimality more closely than was done in connection with (25), (26) and (27). Combining (25) and (27), we obtain in N

$$(41) \qquad -f'\left(\frac{u}{l_1}\right) + (\rho+\lambda)\left(\frac{1}{\alpha}\frac{\partial x}{\partial l_1}\right) = 0$$

or

$$(42) \qquad \frac{\frac{\partial X}{\partial L_1}}{\frac{\partial X}{\partial E}} = \frac{\alpha}{\mu+n+\rho}.$$

The MRS of L_1 for E depends on the relative quantities of E and L_1 only. In the (E,L_1)-plane depicted in figure 5, (42) is therefore a ray through the origin, connecting all points on the isoquants with slope $\frac{\alpha}{\mu+n+\rho}$, labelled by MGR (for "Modified Golden Rule").

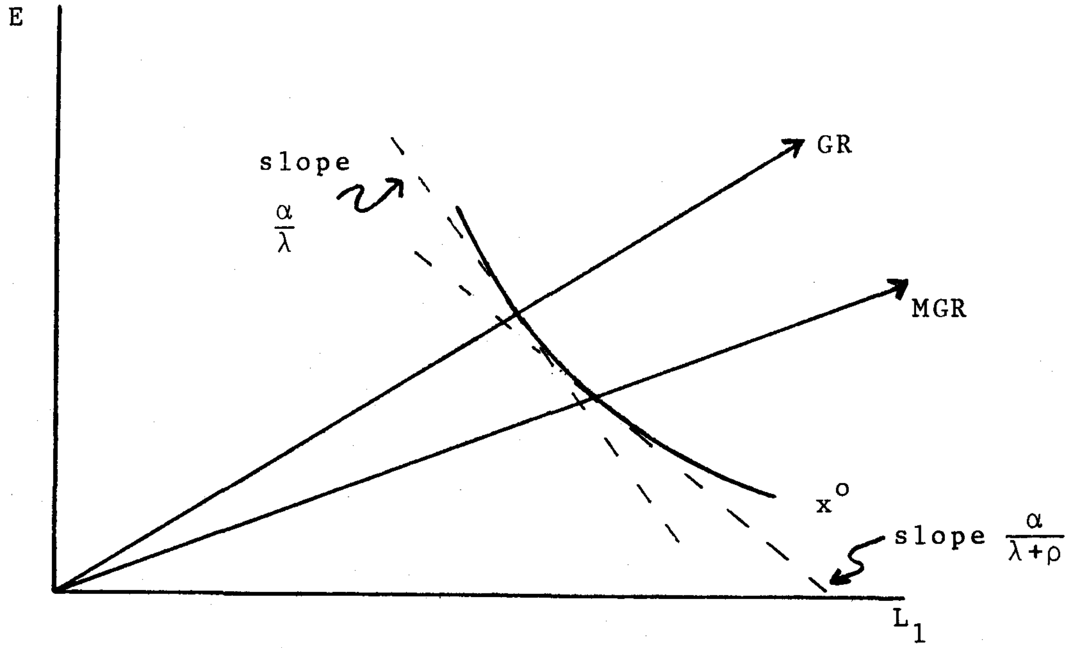

Figure 5

This ray shows the optimal expansion over time for E and L_1, in the stationary state (which is only reached asymptotically). The other ray, labelled GR, is the expansion in Golden Rule (when $\rho=0$). Figure 5 shows that there will be too much educational capital in Golden Rule and that the difference between the GR and MGR levels of educational capital will increase through time. The reason is, of course, that the "waiting costs", expressed by ρ, of the "roundaboutness" of production in this model is overlooked in GR. Substituting $\alpha = \frac{dJ}{dL_2}$ in (42) we obtain $\frac{\partial X}{\partial L_1} = (\frac{1}{\rho+\lambda} \frac{\partial X}{\partial E}) \frac{dJ}{dL_2}$. Since $\frac{\partial X}{\partial E} = f'(\frac{u}{l_1})$ is constant in steady state, the expression above can be written as

$$(43) \qquad \frac{\partial X}{\partial L_1}(0) = \int_0^\infty \frac{\partial X}{\partial E} \cdot \frac{dJ}{dL_2} e^{-(\rho+\mu+n)t} dt.$$

The LHS of (43) is the instantaneous loss of expanding the number of students (and thereby reducing the number of workers) with "one unit". The RHS gives the everlasting benefits (in steady state) of expanding the number of students with one unit. The investment criterion in steady state for allocating labour to the educational sector should therefore be that the instantaneous marginal cost of expanding the number of students equals the present value of the everlasting marginal benefit. The effect of population growth and depreciation of human capital is taken care of in the discount factor, which for this reason is greater than the social rate of time preference.

The cost/benefit criterion cannot tell anything about the optimal path outside the steady-state optimum. It does not follow for instance, that the actual flow of new educational capital should be increased if the actual stock of educational capital falls short of the optimal stock. Outside the steady state, the exact path for the allocation of labour to the educational sector is determined by the optimal trajectory in figure 3, represented by the horizontal line through $p(t) = p^{\infty}$. Along this optimal trajectory the optimal $l_2(t)$ is given as a function of u, and some of the properties of the optimal time-form of $l_1(t)$ have been discussed earlier, and illustrated in figure 4.

6. Modifications and extensions.

A number of extensions and modifications of the basic model are possible. We shall consider two.

6.1. Variable labour-force participation ratio.

Consider first the case where the labour-force participation ratio is a variable. Let

> P(t) be the size of the population at t, and
> L(t) the labour force at t.

Per capita variables must now be redefined: $u = \frac{E}{P}$, $h = \frac{J}{P} = \frac{\alpha L_2}{P} = \alpha l_2$ where $l_2 = \frac{L_2}{P}$. Defining l_1 in the same way, we have

$$(44) \qquad l_1 + l_2 = \frac{L_1}{P} + \frac{L_2}{P} = \frac{L}{P}.$$

The crucial assumption we shall make now is that the labour-force participation ratio depends positively on the level of education in society, so that

(45) $L(t) = y(u)P(t)$

which together with (44) means that

(46) $l_1 + l_2 = y(u).$

As before $\frac{\dot{P}}{P} = n$, so that $\dot{u} = \alpha l_2 - \lambda u$, where $\lambda = \mu + n$.
Also $x = l_1 f(\frac{u}{l_1})$.

Concerning the labour-force participation ratio, it seems reasonable to think that it has the form shown below in figure 6.

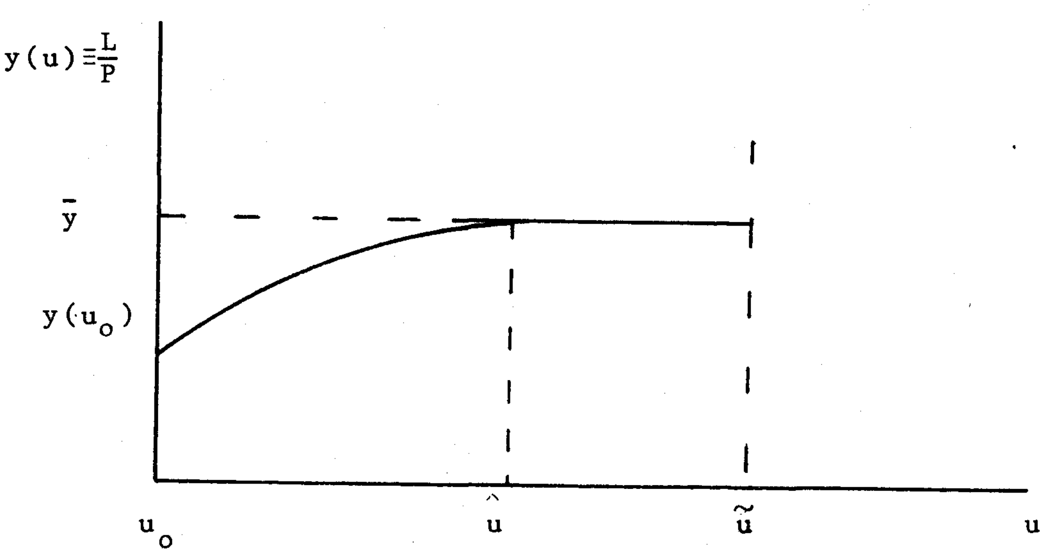

Figure 6

In words: From a given initial level of labour-force participation in the population, determined by the initial level of education, labour-force participation tends to increase monotonously with u, but at a decreasing rate, until, at some educational level in the population, $\hat{u}$, less than the largest sustainable level $\tilde{u}$, the labour-force participation ratio reaches a maximum. The main justification for this relation is to be found in the fact that female labour-force participation increases with the level of education. The value of $\bar{y}$ will depend on factors exogenous to this model, especially the age distribution of the population. When the effect of education on labour-force participation is recognised, it can be shown that the solution has the following properties:

a) If $\dot{u} = 0$ and $\dot{p} = 0$ intersect for $u \geq \hat{u}$ the equilibrium values of p and u are the same as in the basic model, and the constant optimal value of l_1 is the same. The optimal path towards the equilibrium point for $u < \hat{u}$ will, however, be different when the effect of u on labour participation is recognized. This is so since now the optimal trajectory for any given $u < \hat{u}$ lies above the optimal trajectory in figure 3. This means that the optimal $l_2(t)$ is higher now for all $u < \hat{u}$. Therefore: Until the level of education is reached where the labour-force participation ratio is maximized, a greater part of the labour force should now at any time be allocated to the educational sector compared to the case where the labour-force participation ratio is regarded as exogenous.

b) The other possibility is that $\dot{u} = 0$ and $\dot{p} = 0$ intersect at $u < \hat{u}$, so that the optimal equilibrium values of p and u, which need not be unique in this case, are both above the corresponding values in the basic model. It means that if the optimal level of knowledge is reached before the labour-force participation has reached its maximum, both the optimal level of knowledge and the optimal constant part of the labour force allocated to the educational sector is higher than in the basic model. The paths towards these levels have the same properties compared to the basic model as those discussed above when $u < \hat{u}$.

For the equilibrium level of p we now obtain

$$(47) \quad p = \frac{1}{\rho+\lambda}\{f'(\frac{u}{1_1}) + wy'(u)\}$$

where w is the shadow price of labour. The present value of a marginal allocation of labour to the educational sector - αp - now consists of two parts. The first: $\frac{\alpha}{\rho+\lambda} f'(\frac{u}{1_1})$ is the present value of the marginal product of educational capital in equilibrium multiplied by the marginal product of labour in producing educational capital - α. The second: $\frac{\alpha}{\rho+\lambda} wy'(u)$ is the present value of the gain of available labour due to a marginal increase of educational capital in equilibrium, multiplied by α.

6.2. Technical Change.

Let now

$$X(t) = F(L_1, E)e^{\varepsilon_1 t}$$

and

$$J(t) = \alpha L_2 e^{\varepsilon_2 t}$$

where ε_i is the rate of exogenous technical change in sector i (i = 1,2). In intensive terms:

$$(48) \quad x(t) = 1_1 f(\frac{u}{1_1})e^{\varepsilon_1 t}$$

and

$$(49) \quad h(t) = \alpha 1_2 e^{\varepsilon_2 t}$$

Using (48) and (49) instead of (9) and (13) in (23) we obtain

(50) $\dot{u} = \alpha(1-l_1)e^{\varepsilon_2 t} - \lambda u$

(51) $\dot{p} = -f'(\frac{u}{l_1})e^{\varepsilon_1 t} + (\rho+\lambda)p.$

(50) and (51) now depend explicitly on time, which means that the system is not autonomous any more, so that in general there will not be any stationary points in the (p,u)-plane. Still, some information on the optimal development of the system when technical change is present can be obtained from a phase-plane where time is regarded as a variable. Consider first the case where there is technological progress in goods production only, i.e., $\varepsilon_1 > 0$, whereas $\varepsilon_2 = 0$. The scenario is set out in figure 7 below.

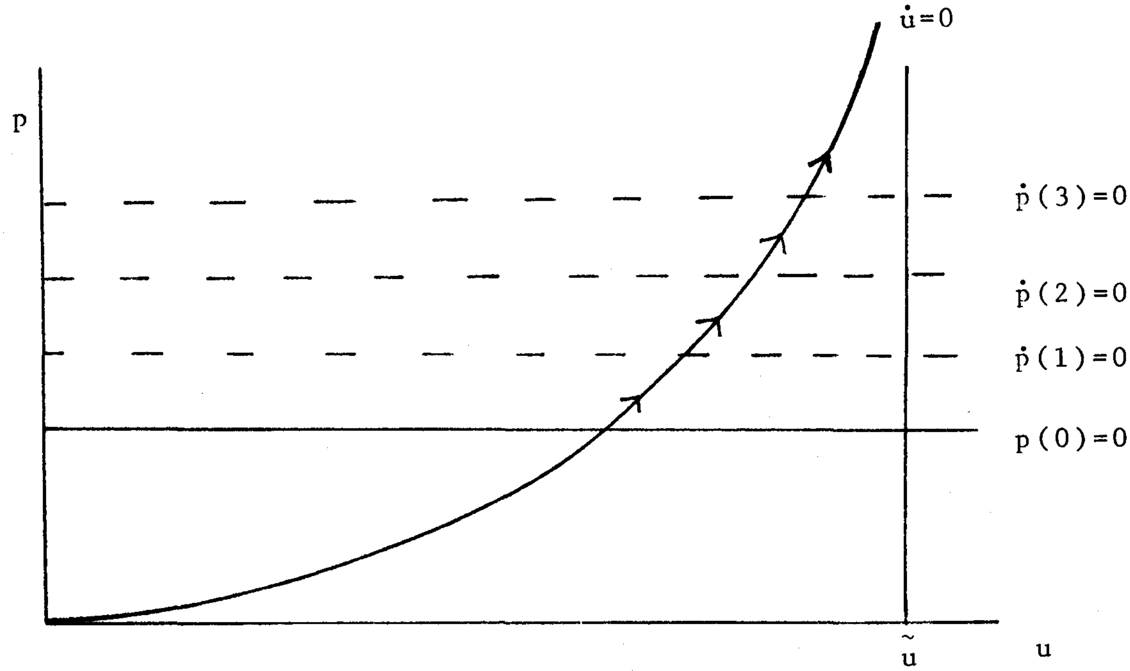

Figure 7

For t=0 the $\dot{u}=0$ and $\dot{p}=0$ curves are well known. As t increases, the $\dot{u}=0$ curve is not affected, since $\varepsilon_2=0$. For a given u, however, the first term on the RHS of (51) shows increasingly higher negative values as time elapses. To satisfy the equation, p must therefore increase over time and the $\dot{p}=0$-curve will therefore shift upwards over time as indicated in the figure by the dotted curves. Over time, the system would therefore follow the arrow in the figure, with p and u steadily increasing and u approaching $\tilde{u}$. Without technological progress in the educational sector, a steady increase in u is only possible if the part of the population allocated to education is increasing over time, so *a situation with no technical progress in the educational sector, but with positive exogenous technical change in goods-production leads optimally to an over-increasing part of the population being employed in the educational sector.* The scenario of the opposite case, with $\varepsilon_2 > 0$ and $\varepsilon_1 = 0$ is set out in figure 8.

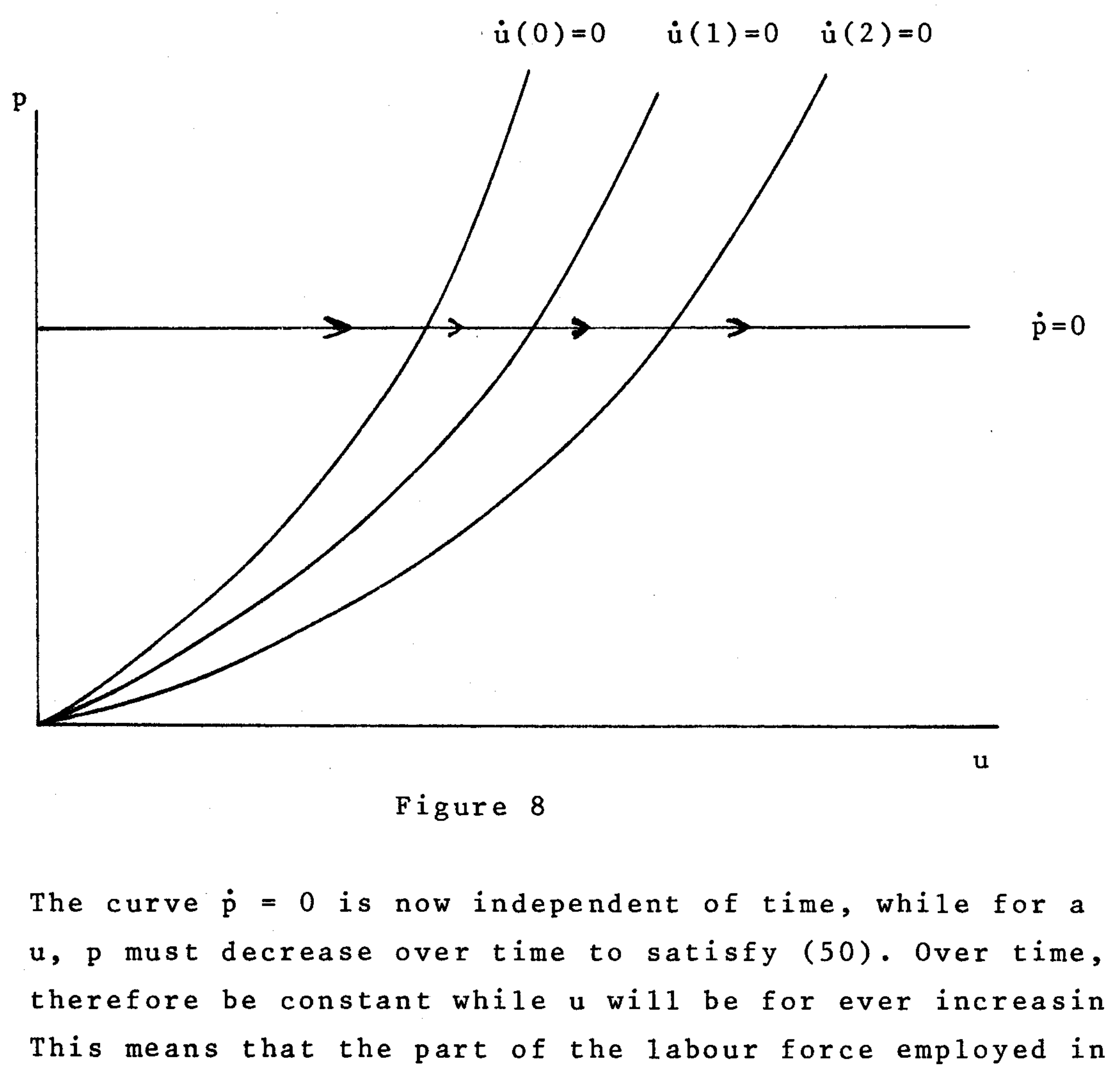

Figure 8

The curve $\dot{p} = 0$ is now independent of time, while for a given u, p must decrease over time to satisfy (50). Over time, p will therefore be constant while u will be for ever increasing. This means that the part of the labour force employed in goods production will be steadily increasing. With l_1 and u increasing, x(t) will also be ever-increasing. A combination of the two cases, so that both ε_1 and ε_2 are positive, would obviously also lead to an ever-lasting increase in both p and u. The optimal path of $l_1(t)$ cannot generally be determined in this case.

As to the relevance of the above results, it seems that a typical feature in the relation between the educational sector and the goods-producing sector in a modern society is that technical progress in the educational sector is rather slow compared to the goods-producing sector. In periods with rapid technological progress in the goods-producing sector, effects of the type discussed in the extreme case where $\varepsilon_1 > 0$ and $\varepsilon_2 = 0$ may therefore be relevant. In such periods the conclusion in the basic model that the optimal $l_2(t)$ should decrease over time must be reversed. Such a development for the whole future is hardly optimal since p(t) is steadily increasing. The opposite case, that $\varepsilon_1 = 0$ and $\varepsilon_2 > 0$ seems less relevant.

7. Summary.

Two positive models, based on reasonable behavioural assumptions of labour allocation over time, are analyzed. Both of them yield a unique, stable steady state where "knowledge per capita" and the part of the labour force going into the educational sector are constant over time. A golden rule for labour allocation to the educational sector is considered. The allocation of labour to the educational sector is then optimized over time. A unique steady state optimum exists, which is reached only asymptotically along the optimal path. The values of the variables in steady state are independent of the initial situation and depends on the rate of social time preference, the efficiency of the educational sector, the rate of depreciation of knowledge and the rate of increase in population. Along the optimal path the part of labour allocated to the educational sector should be falling towards its stationary level if the initial level of knowledge in society is less than the optimal level. Further, a lower initial level of knowledge will lead to a higher initial part of labour allocated to education. The allocation pattern along the optimal path may be reversed in periods with rapid tecnological progress in goods production. Finally, criteria for investment in education which have been used rather ad hoc in the "economics of education" literature, may now be derived from an explicit dynamic model.

Appendix.

Denote the stock of educational capital used in the educational sector by E_2 and gross investment by J_2. Define H as total gross production of educational capital.

(A1) $H = J + J_2$,

where J is defined in the text. Assume that H is produced by a constant returns to scale production function F_2 with the usual properties

(A.2) $H = F_2(L_2, E_2)$.

Efficient factor-combination implies a fixed relation between L_2 and E_2 for all H, so that, say $E_2 = kL_2$, which means that along the efficiency locus in the factor-plane H can be expressed as a function of L_2 alone: $H = F_2^*(L_2, kL_2)$. From (A.1) we now have $J = F_2^*(L_2, kL_2) - J_2$.

If we introduce depreciation also in this sector, we have that

$$\dot{E}_2 = J_2 - \mu E_2$$

or

$$k\dot{L}_2 = J_2 - \mu k L_2$$

so that

(A.3) $J = F_2^*(L_2, kl_2) - \mu k L_2 - k\dot{L}_2$

or

(A.4) $J = G(L_2) - k\dot{L}_2$

where

$$G(L_2) = F_2^*(L_2, kL_2) - \mu k L_2.$$

(A.4) is approximated by (1) in the text which therefore holds when the product $k\dot{L}_2$ is "sufficiently small".

As an example regard L_2 as pupils and E_2 as teachers and specify H as

$$H = \min (L_2, \frac{E_2}{b}).$$

We now obtain k = b. b is the teacher/pupil ratio, say 1/20. Instead of (A.4) we now get

$$J \doteq (1 - \frac{\mu}{20})L_2 - \frac{1}{20}\dot{L}_2.$$

Historically $\dot{L}_2$ has been "small" compared to L_2. In periods with drastic changes in L_2, however, (1) may not be good approximation to (A.4)

Notes.

* I am indebted to S. Strøm at the University of Oslo and my colleague K.P. Hagen for extremely valuable comments. This work was begun in 1973 when I was a visiting research associate at the Economics Department, University of California, Berkeley. Financial assistance from the Norway-America Association is gratefully acknowledged.

1) The derivation of the G-function, which is a first-order approximation, is discussed in the appendix.

2) It might be objected that the treatment of educational capital in (2), (3) and (4) is formally identical to the treatment of physical capital in models of economic growth. However, as long as a capital concept is used, this can (and should) not be avoided. The point is that this model - in contrast to models of optimal savings-focuses on the optimal allocation of labour to education, and the model is constructed so as to make this analysis as explicit as possible.

3) To avoid confusion with dotted variables, we have used h instead of j, which would otherwise have been the natural symbol to use here.

4) Because l_1 enters (15), this differential equation is not formally identical to the so-called "fundamental differential equation of economic growth".

5) This path is the candidate optimal path-"candidate" since we have not yet considered sufficiency.

6) This means that if e.g. α is decreased at t', l_1 would make a negative jump at t'. After t', l_1 would again increase towards the new and lower optimal stationary level.

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OM OPTIMAL UTVIKLING AV EIN KUNNSKAPSBASERT INDUSTRISEKTOR

Av Jostein Aarrestad*

1. *Innleiing.*

Tradisjonelt har økonomane betrakta produksjonsfaktoren «arbeidskraft» som homogen. Arbeidskraftas kvalitet har vore føresett konstant. Men ved ymse tiltak, spesielt ved å byggja ut utdanningsnivået i eit samfunn, kan ein medvete påverka kvaliteten av arbeidskrafta. Ein annan måte å seia dette på, er at ein byggjer ut utdanningskapitalen i samfunnet. Problemet som då melder seg, er kor mye av samfunnets ressursar det er optimalt å satsa for å byggja ut og vedlikehalda utdanningskapitalen i samfunnet. Det oppstår her eit optimaliseringsproblem av dynamisk art, fordi i den grad ressursar nå blir allokert til utdanningssektoren for å auka utdanningskapitalen med sikte på å kunne produsera meir i framtida, vil desse ressursane ikkje kunne brukast til produksjon nå.

Inn under dette optimaliseringsproblemet fell ein velkjend påstand frå norsk industripolitisk debatt etter krigen: «Vi må satsa på intelligensindustri». Nemninga «kunnskapsbasert industri» er også brukt i den seinare tid. Ein mulig måte å presisera og diskutera påstanden på, er den følgjande analyse av ein enkel makroøkonomisk modell for optimal utvikling av ein kunnskapsbasert industrisektor i eit samfunn.[1]

Trass i den vekt utdanningsnivået i eit samfunn blir tillagt som forklaringsvariabel for den økonomiske utviklinga, og trass i den store mengd ressursar som til ei kvar tid går inn i utdanningssektoren i eit samfunn, eksisterer det i litteraturen få arbeid som diskuterer den optimale ressursbruk i utdanningssektoren frå eit dynamisk synspunkt. Dette i motsetnad til den omfattande litteraturen om optimal akkumulering av realkapital. Ulike, men relaterte problem er tatt opp av

* Eg vil takka dosent Steinar Strøm, Universitetet i Oslo, for ei rekke nyttige merknader til ein tidlegare versjon av denne artikkelen.

[1] Alternativt kan modellen tolkast som ein modell for optimal utvikling av ein «moderne» sektor, i motsetnad til den «tradisjonelle» sektor, i eit u-land.

Uzawa [8], Razin [6], Dobell og Ho [1] og nyleg av Manning [5]. Vi skal i denne artikkelen sjå på ei generalisering av modellen framstilt i [9], sidan, mellom anna,

a) folkemengda nå ikkje er konstant, og
b) økonomien nå er delt i tre sektorar, noe som gjer at det kan vera fleire regime i den optimale politikk.

Hovudproblema i denne artikkelen er å finna optimale banar for allokeringa av arbeidskrafta over tid mellom dei tre sektorane i økonomien. For å kunne svara på slike spørsmål trengst det ein dynamisk modell. Vi oppnår enklast mulig matematiske uttrykk ved ei kontinuerlig formulering av modellen.

2. *Modellen.*

Modellen består som sagt av tre sektorar — ein utdanningssektor som produserer kunnskap, som vi vil kalla utdanningskapital, og to sektorar for vareproduksjon. Den første vareproduksjonssektoren brukar utdanningskapital i produksjonen, la oss kalla den sektoren intelligensindustrien. Den andre vareproduksjonssektoren brukar ikkje utdanningskapital i produksjonen. Ei slik forenkling er ei tilnærming til det faktiske forhold at utdanningskapitalen per arbeidar varierer ganske mye mellom dei forskjellige næringane, og også mellom dei forskjellige industrigreinene, sjå t.d. [2].

Hovudforenklinga er elles at realkapitalen ikkje er spesifisert i modellen, slik at arbeid og utdanningskapital er dei einaste produksjonsfaktorane. Modellen vil derfor vera mest relevant for ein økonomi rik på realkapital, der arbeidskrafta er ein «flaskehals» (som Norge?). Produksjonsperioden i utdanningssektoren er også oversett. «Arbeid» må forstås som fullstendig uutdanna arbeidskraft, fordi vi har gjort den abstraksjon å skilja utdanningskapitalens produktivitet fullstendig frå det «rå» arbeidets produktivitet i produksjonsprosessen.[1] Vi går da over til å spesifisera modellen.

Mengda «rå» arbeidskraft sysselsett i utdanningssektoren kallar vi L_3 og produksjonen i denne sektoren kallar vi J. For å forenkla går vi

[1] Meir fruktbart i problemstillinga om intelligensindustri er det å tolka «rå» arbeidskraft som arbeidarar med bare elementær, obligatorisk utdanning.

ut frå at det er eit fast forhold mellom produksjon og innsett mengd arbeidskraft i denne sektoren, slik at på eit kvart tidspunkt t er

(1) $$J(t) = aL_3(t),$$

der a er ein konstant.

J er netto i den forstand at eit kvart produkt frå utdanningssektoren som sidan blir sett inn igjen i utdanningssektoren, som t.d. når ein student blir lærar etter eksamen, ikkje er inkludert i J.

I «intelligensindustrien» er produktfunksjonen

(2) $$X_1(t) = F[L_1(t), E(t)],$$

der

$X_1(t)$ = totalproduksjon i denne sektoren

$L_1(t)$ = den mengd arbeidskraft som er sysselsett i intelligensindustrien

$E(t)$ = den totale utdanningskapital i samfunnet

F antar vi er konkav med positive og avtakande grenseproduktivitetar. E og L er komplementære, dvs. $\partial^2 F/\partial E \partial L_1 > 0$.

Vidare er

(2a) $$F(0, E) = F(L_1, 0) = 0$$ $$F_E(0, E) = 0.$$[1]

(2) seier altså at produksjonen i intelligensindustrien avheng av kunnskapsmengda i samfunnet og den mengd arbeidskraft som er i denne sektoren. Produktfunksjonar som uttrykker same idé, kan ein finna i [3], t.d. modell 7.1. på side 36. Vi skal ikkje her diskutera problema med å måla utdanningskapitalen. Fleire freistnader på å gjera det i ulike land er gjort, sjå t.d. [7], kapittel 20 og diskusjonen s. 742. I [4], har E. Hoffmann berekna utdanningskapitalen i Norge i 1950 og 1960.

Behaldninga av utdanningskapital får til ei kvar tid ein brutto tilvekst J. På den andre sida har vi også ei form for kapitalslit her, sidan utdanna folk døyr, visse typar kunnskap blir økonomisk mindre verdifulle på grunn av den tekniske utvikling, og dessutan gløymer

[1] Her er $F_E = \partial F / \partial E$.

folk til stadighet det dei eingong lærde. Om kapitalslitet antar vi enkelt at det utgjer ein konstant del μ av behaldninga til ei kvar tid, slik at

$$\dot{E}(t) = J(t) - \mu E(t). \tag{3}$$

Initialt antar vi ei gitt behaldning av utdanningskapital lik E_0.

I vareproduksjonssektor 2 tenkjer vi oss at produksjonen skjer utelukkande ved hjelp av arbeidskraft. For å få mest mulig enkle uttrykk går vi ut frå at det også i denne sektoren er eit fast forhold mellom produksjonen og den innsette mengd arbeidskraft, slik at

$$X_2 = \beta L_2 \tag{4}$$

der
X_2 er produsert mengd i denne sektoren,
L_2 er den mengd arbeidskraft som er sysselsett her og
β er ein positiv konstant.

Vidare går vi ut frå at det er eit fast forhold mellom folkemengda og totaltilgangen på arbeidskraft, $L(t)$, og at $L(t)$ veks med ein eksogen og konstant tilvekstrate n, slik at

$$L(t) = L_0 e^{nt};\ L_0 \text{ er gitt.} \tag{5}$$

Den samla sysselsetting i dei tre sektorane kan ikkje overstiga totaltilgangen av arbeidskraft, dvs.

$$L_1 + L_2 + L_3 \leq L. \tag{6}$$

Anta nå at F er homogen av grad 1 i L_1 og E slik at

$$X_1/L_1 = F(E/L_1, 1) = f(u/l_1) \tag{7}$$

der $u = E/L$ er den aggregerte utdanningsintensiteten, dvs. utdanningskapitalen per arbeidar i økonomien, og $l_1 = L_1/L$ er den delen av totaltilgangen på arbeidskraft som er allokert til intelligensindustrien.

(7) seier at den gjennomsnittlege arbeidsproduktivitet i intelligensindustrien, dvs. totalproduksjonen per årsverk, er ein stigande funksjon av kunnskapsnivået per arbeidar — noe som ikkje verkar urimeleg. X_1/L_1 er den totale produksjon per arbeidar i sektor 1. Vi er meir

interessert i totalproduksjonen i sektor 1 per arbeidar *totalt* i økonomien, $x_1 = X_1/L$. Frå (7) får vi då

(8) $$x_1 = l_1 f(u/l_1).$$

Sidan

$$\partial X_1/\partial E = f'(u/l_1)$$

har vi frå eigenskapane ved (2) at

(9) $$f' > 0 \text{ og } f'' < 0.$$

Dessutan er

(10) $$\partial X_1/\partial L_1 = f - (u/l_1)f' > 0.$$

Utviklinga av den aggregerte utdanningsintensiteten u over tid er gitt ved

$$\frac{d}{dt}(u(t)) = \dot{u} = \frac{d}{dt}\left(\frac{E}{L}\right) = \frac{J(t) - \mu E(t)}{L(t)} - \frac{\dot{L}(t)}{L(t)} u(t),$$

slik at

(11) $$\dot{u}(t) = h(t) - \lambda u(t),$$

der

$$h = J/L^1,$$

$$\dot{L}/L = n$$

og

$$\lambda = \mu + n.$$

Divisjon med L i (1) gir

(12) $$h(t) = al_3$$

der $l_3 = L_3/L$ dvs. den delen av totaltilgangen på arbeidskraft som er allokert til utdanningssektoren.

Dividerer vi så med L i (4) får vi

(13) $$x_2 = \beta l_2$$

[1] For å unngå forvirring m.o.t. variablar med prikk over skal vi bruka h i staden for j, som elles ville vore mest naturleg.

der

$x_2 = X_2/L$ dvs. gjennomsnittsproduksjonen av vare 2 per arbeidar totalt i økonomien, og

$l_2 = L_2/L$ er den delen av totaltilgangen på arbeidskraft som blir allokert til sektor 2. Til sist vil (6) på intensiv form bli

$$l_1 + l_2 + l_3 \leq 1. \tag{14}$$

(8), (11), (12), (13) og (14) er fem relasjonar i sju funksjonar, $x_1(t)$, $x_2(t)$, $u(t)$, $h(t)$, $l_1(t)$, $l_2(t)$ og $l_3(t)$. Modellen som den står har altså nå to fridomsgrader.

Men ved å velja t.d. $l_1(t)$ og $l_2(t)$ slik at utviklinga av økonomien frå ein gitt initialsituasjon blir optimalisert, vil modellen bli determinert. Implisitt i eit slikt optimalt forløp av økonomien vil det då vera eit svar på om, og under kva slags forhold, det lønner seg å «satsa» på intelligensindustri. Oppgava er altså å finna den optimale allokering av arbeidskrafta mellom dei tre sektorane, gitt (8), (11), (12), (13), (14), E_0 og L_0.

Samfunnets velferd, W, vil til ei kvar tid avhenga av dei produserte mengdene per capita av dei to varene, dvs.

$$W = W(x_1, x_2).$$

Vi skal anta W additiv, slik at

$$W = U(x_1) + V(x_2)^1 \tag{15}$$

der

$$U' > 0,\ U'' < 0,$$

$$V' > 0,\ V'' < 0$$

og der

$$\lim_{x_2 \to 0} V'(x_2) = \infty$$

Vidare skal vi anta at samfunnet har ein uendelig lang planleggingshorisont, at det ikkje er noen restriksjonar på verdien for u på noe tidspunkt og at den sosiale diskonteringsraten er ein konstant $\rho > 0$.

[1] Det er altså bare nytten til eit representativt individ som tel i W-funksjonen. Alternativt kunne folkemengda trekkjast inn ved at t.d. $W = L(t)[U(x_1) + V(x_2)]$. Argumenta for dette er etter mi meining ikkje overtydande.

3. *Analyse av modellen.*

For å letta oversikta samlar vi relasjonane ovanfor. Vi får då det følgjande problem i optimal kontrollteori:

$$\text{Maks} \int_0^\infty [U(x_1) + V(x_2)]e^{-\rho t}dt$$

når

$$x_1 = l_1 f(u/l_1)$$
$$x_2 = \beta l_2$$
$$\dot{u} = h - \lambda u$$
$$h = \alpha l_3$$
$$l_1 + l_2 + l_3 \leq 1 \qquad (16)$$
$$0 \leq l_i < 1;\ i = 1,3$$
$$0 < l_2 \leq 1$$
$$u(0) = u_0 \text{ (gitt)}$$
$$\lim_{t \to \infty} u(t) \text{ er fri}$$

På grunn av føresetnaden om $V(x_2)$ vil det alltid vera produksjon i sektor 2. A priori treng det derimot ikkje vera optimalt å ha produksjon i intelligensindustrien og/eller utdanningssektoren.

Vi vil derfor ha fire mulige regime i den optimale løysinga, skjematisk oppstilt i tabell 1:

Tabell 1.

Regime	l_1	l_2	l_3
A	>0	>0	>0
B	>0	>0	0
C	0	>0	>0
D	0	1	0

La oss først studera den optimale utvikling i økonomien under regime A, dvs. når ei indre løysing er optimal.

3.1 Indre løysing

Det vil nå vera optimalt å allokera arbeidskraft til alle tre sektorane. Ved å danna Lagrange-funksjonen L, der

$$L = e^{-\rho t}\{U(l_1 f(u/l_1)) + V(\beta l_2) + p(\alpha l_3 - \lambda u) + w(1 - l_1 - l_2 - l_3)\},$$

finn vi dei følgjande nødvendige vilkår for å løysa (16):

Det eksisterer ein kontinuerlig $p(t)$, slik at

(17) $$\dot{p} = -U'[f'(u/l_1)] + (\rho + \lambda)p, \text{ og}$$

(18) $$U'[f - (u/l_1)f'] = V'\beta = \alpha p = w.$$

Endelig må (11) sjølsagt halda. (Banar som tilfredsstiller (17), (18) og (11) og tilsvarande vilkår for andre regime er *kandidatar* til den optimale politikk, korvidt dei verkeleg er optimale, skal vi koma tilbake til). w er her arbeidskraftas skuggepris, mens p er den adjungerte variabelen til rørslelikninga.

(18) seier at for å ha eit optimum, må til ei kvar tid grensenytten av produksjonen multiplisert med arbeidets grenseprodukt i begge vareproduksjonssektorane vera lik, og lik skuggeprisen på arbeidskraft. Denne felles storleiken må igjen vera lik αp, dvs. skuggeprisen på utdanningskapital multiplisert med α — arbeidskraftas grenseproduktivitet i å produsera utdanningskapital. Meir fullstendig: $p(t)$ kan noe upresist tolkast som auken i den optimale verdi av kriteriefunksjonen av å leggja til «ei eining ekstra» av utdanningskapital til behaldinga av utdanningskapital på tidspunkt t. Det framgår av (18) at p og w begge må vera positive. Full sysselsetting er derfor alltid optimalt, og (14) må alltid halda med likskapsteikn. Langs den optimale bane for p må (17) vera tilfredsstilt.

Ved ei indre løysing er w, l_1, l_2 og dermed l_3 implisitt gitt som funksjonar av u og p frå (18). Definér

(19) $$q(l_1, u) \equiv U[l_1 f(u/l_1)]$$

Fordi $l_1 f(u/l_1)$ er konkav i l_1 og u, og fordi U er ein stigande og konkav funksjon av x_1, er q konkav.

Implisitt derivasjon i (18) gir, ved bruk av (19), at[1]

$$(20)\qquad \begin{cases} \partial l_1/\partial u = -q_{lu}/q_{ll};\ \partial l_1/\partial p = a/q_{ll}, \\ \partial l_2/\partial u = 0;\ \partial l_2/\partial p = a/V''\beta^2, \\ \partial w/\partial u = 0;\ \partial w/\partial p = a. \end{cases}$$

Med w og l_i $(i = 1, 2, 3)$ som funksjonar av p og u, er (11) og (17) to autonome differensiallikningar i u og p. Vi kan då foreta ein to-dimensjonal grafisk analyse av dette systemet. Ved å setja inn frå (20) finn vi frå (11) at langs kurva for $\dot{u} = 0$ er

$$(21)\qquad \frac{dp}{du} = -\frac{\lambda + a(\partial l_1/\partial u)}{a(\partial l_1/\partial p + \partial l_2/\partial p)}$$

og frå (17) at langs $\dot{p} = 0$ er

$$(22)\qquad \frac{dp}{du} = \frac{q_{lu}(\partial l_1/\partial u) + q_{uu}}{\rho + \lambda - q_{lu}(\partial l_1/\partial p)}$$

Ved å setja inn for $(\partial l_1/\partial u)$ kan (22) skrivast som

$$\frac{dp}{du} = \frac{(1/q_{ll})[-(q_{lu})^2 + q_{ll}q_{uu}]}{-q_{lu}(\partial l_1/\partial p) + (\rho + \lambda)}$$

Sidan q er konkav, er teljaren i (22) negativ, men bortsett frå det, strekk ikkje konkaviteten til for å bestemma forteiknet på (21) eller (22). Det problematiske leddet er q_{lu} som vi må anta positivt, noe det vil vera hvis t.d. krumninga på U-funksjonen er «tilstrekkeleg liten».

I så fall er nemnaren i (22) positiv, slik at (22) er negativ, mens (21) er positiv.

Den høgaste utdanningskapitalen som kan oppretthaldast, $\tilde{u}$, følgjer av

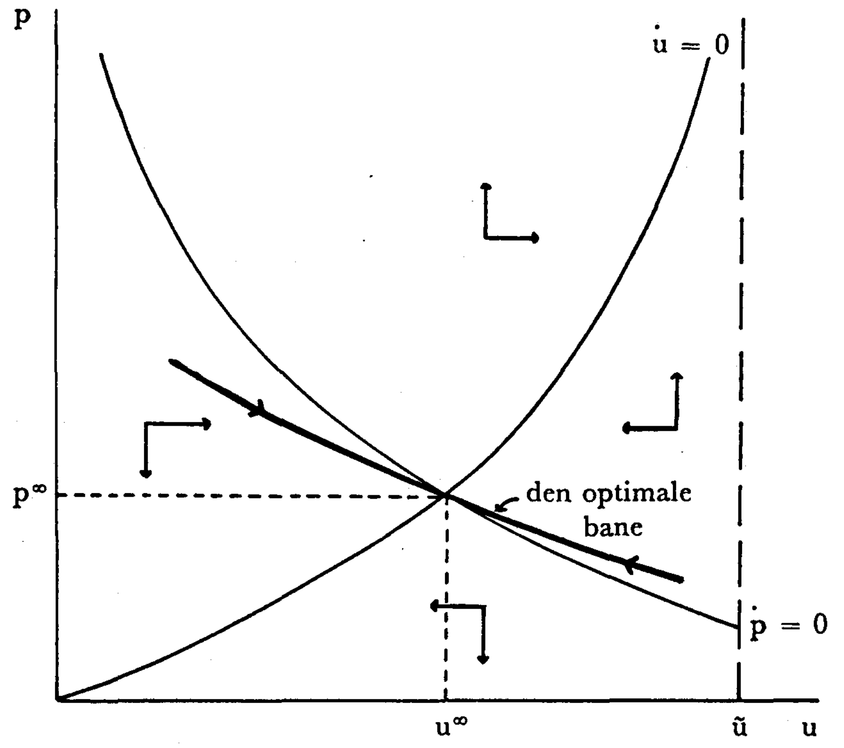
$$al = \lambda\tilde{u},$$

dvs.

$$\tilde{u} = a/\lambda.$$

I (p,u)-planet framstilt i figur 1, må kurva for $\dot{u} = 0$ starta frå (0,0). Dette fordi (11) og (20) betyr at $u = 0 \Rightarrow p = 0$ for at $\dot{u} = 0$. Om-

[1] Her er $q_{lu} = \partial^2 q/\partial l \partial u$ osb.



Figur 1.

vendt vil $\dot{p} = 0 \Rightarrow u = 0$. Når u nærmar seg $\tilde{u}$, må $\dot{u} = 0$ bli «loddrett». Dette fordi kurva alltid er stigande for $u < \tilde{u}$, mens for $u > \tilde{u}$ er $\dot{u}$ alltid negativ.

Forma på $\dot{u} = 0$ i (11) er derfor som vist i figur 1, der også $\tilde{u}$ er avsett.

Kurva for $\dot{p} = 0$ er fallande mot høgre i diagrammet, og p er alltid positiv.

Jamvel om den eksakte form på kurvene for $\dot{u} = 0$ og $\dot{p} = 0$ er vanskeleg å finna, veit vi nå nok til å fastslå at *det må eksistera eit og bare eit stasjonærnivå for p og u — (p^∞, u^∞) — slik at*

(23) $$u^\infty = (a/\lambda) l_3(p^\infty, u^\infty)$$

og

(24) $$p^{\infty} = \frac{1}{\lambda + \rho}\left\{U'(x_1)f'\left[\frac{u^{\infty}}{l_1(u^{\infty}, p^{\infty})}\right]\right\}$$

Dette er vist i figur 1.

Sidan p og u er stasjonære i dette punktet, vil også l_i, $(i = 1, 2, 3)$ vera konstante over tid når p og u er lik sine stasjonærverdiar. (24) multiplisert med a er den neddiskonterte verdien av ei marginal allokering av arbeidskraft til utdanningssektoren. I stasjonærtilstanden skal denne i følge (18) vera lik den marginale verdi av å allokera arbeidskraft til dei to andre sektorane.

Pilene viser dei dynamiske kreftene som verkar på systemet i dei forskjellige regionane i fase-planet. Det er visuelt intuitivt, og kan lett visast at (p^{∞}, u^{∞}) er eit sadelpunkt. Det vil seia at det er ein og bare ein bane i (p, u)-planet som fører til (p^{∞}, u^{∞}) slik at til ein kvar initial u_0 korresponderer det ein eintydig p_0, slik at ein bane som startar i (p_0, u_0), konvergerer mot (p^{∞}, u^{∞}). Denne banen er innteikna i figuren. Vi skal seinare visa at denne banen verkeleg er optimal.

Løysinga av optimumsproblemet når vi har ei indre løysing, er altså å finna den initiale p_0 og så følgja den optimale banen mot (p^{∞}, u^{∞}). Kva karakteriserer så denne optimale banen? Frå figuren ser vi at for $u_0 < u^{\infty}$ er $p(t)$ fallande over tid, mens $u(t)$ er veksande. Kunnskapsmengda i samfunnet vil altså vera monotont stigande mot u^{∞}, som bare vil nåast assymptotisk.

Kva så med den optimale allokering av arbeidskraft over tid til dei tre sektorane langs den optimale banen? Frå (18) har vi at

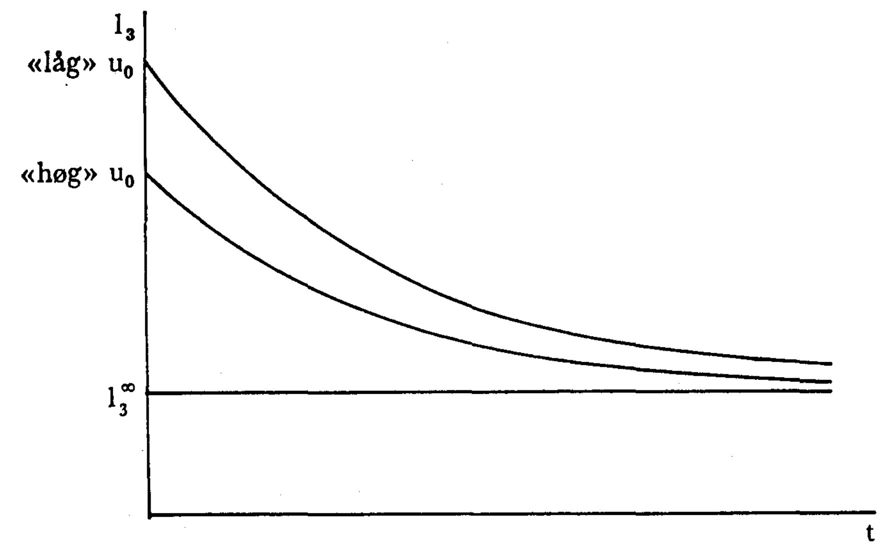
(25) $$\dot{l}_1 = (\partial l_1/\partial p)\dot{p} + (\partial l_1/\partial u)\dot{u}.$$

Hvis $u_0 < u^{\infty}$, er $\dot{p} < 0$ og $\dot{u} > 0$. Av (25) ser vi då at *langs den optimale banen er $l_1(t)$ stigande, dvs. den delen av totaltilgangen på arbeidskraft som er allokert til intelligensindustrien, skal stiga over tid.* Tilsvarande for l_2 får vi at

$$\dot{l}_2 = (\partial l_2/\partial p)\dot{p}$$

slik at for $u_0 < u^{\infty}$ *skal også allokeringa av arbeidskraft til vareproduksjonssektor 2 stiga over tid langs den optimale banen.*

Det følgjer då at $l_3(t)$ dvs. *den delen av totaltilgangen på arbeidskraft som optimalt blir allokert til utdanningssektoren, må vera fallande over tid.* Vidare



Figur 2.

ser vi frå figur 1 at di lågare u_0 er, di høgare vil p_0 vera, og di høgare vil den initialt optimale l_3 vera (med tilsvarande lågare utgangspunkt for l_1 og l_2). Den optimale bane for l_3 frå to ulike initialsituasjonar for u er illustrert i figur 2.

Motsette konklusjonar vil halda hvis $u_0 > u^\infty$. Då vil l_3 tilta over tid, mens l_1, l_2 og u vil falla over tid langs den optimale banen. Endeleg, hvis tilfeldigvis $u_0 = u^\infty$, kan $l_i(0)$ veljast lik sine optimale konstante verdiar, og den optimale politikk er å halda l_i konstant. I teorien kan det også tenkjast at u_0 er så høg at det er optimalt å setja $l_3(0) = 0$. Dette skal vi kommentera seinare saman med alternativet at den optimale $l_1(t) = 0$ for noen t. Foreløpig held vi oss til ein situasjon der ei indre løysing for l_i er optimal, noe som truleg også er mest relevant.

Resultata ovanfor gjeld for gitte, konstante verdiar av parametrane i modellen. Det kan då vera av ei viss interesse å studera korleis det optimale nivå for p, u, l_i, x_1 og x_2 til ei kvar tid avheng av parametrane i problemet, α, λ og ρ. Ved å derivera implisitt i (17) og (11) med $l_i = l_i(p, u)$ får vi dei følgjande verknader på dei optimale stasjonære verdiane av u og p:

Den optimale stasjonære utdanningskapitalen per arbeidar vil gå ned når samfunnets tidspreferanserate aukar, dvs. når samfunnet vel å leggja meir vekt på produksjon idag, i motsetnad til i morgon. Vidare ser vi at eit positivt skift i utdanningssektorens produktivitet, a, på grunn av til dømes effektive læremidlar og læremetodar, fører til ein høgare optimal utdanningskapital. Ein raskare vekst i befolkninga og/eller depresisering av utdanningskapitalen ved t.d. at eksisterande kunnskap blir raskare økonomisk verdilaus, vil ha den motsette verknad. Utdanningskapitalens skuggepris går ned når tidspreferanseraten aukar, og den går også ned når utdanningssektorens effektivitet aukar. Verknaden på skuggeprisen av ei endring i λ er uklar.

Korleis blir så allokeringa av arbeidskraft mellom dei tre sektorane påverka av skift i parametrane? Frå (22) avheng l_i^∞ av p^∞ og u^∞ som igjen er funksjonar av ρ, λ og a. Ved å setja inn i

$$(\partial l_i^\infty/\partial a) = (\partial l_i^\infty/\partial p^\infty)(\partial p^\infty/\partial a) + (\partial l_i^\infty/\partial u^\infty)(\partial u^\infty/\partial a) \quad (i = 1,2,3),$$

ser vi at når effektiviteten i utdanningssektoren, a, går opp, vil l_1 og l_2 begge gå opp, dvs. allokeringa av arbeidskraft til utdanningssektoren går ned, mens allokeringa av arbeidskraft til begge dei vareproduserande sektorane går opp.[1] Tilsvarande hvis ρ aukar, går den optimale mengda av arbeidskraft i sektor 2 opp, mens den går ned i utdanningssektoren. Det er uklart kva som vil skje med allokeringa av arbeidskraft til intelligensindustrien. Når a går opp, vil produksjonen av begge varer auka. Når ρ går opp (ned), vil produksjonen i sektor 2 auka (avta). Dermed må produksjonen i intelligensindustrien avta (auka). Frå formlane er det uklart kva som vil skje når λ aukar. Den totale produksjonskapasitet i økonomien vil då falla. Eit rimelig resultat er at «tapet» blir spreidd på sektor 1 og 2 slik at l_2 avtar, w stig og l_1 avtar. l_3 tiltar derfor, men mindre enn det som trengst for å kompensera for auken i μ, sidan den optimale u fell. Dette altså ved ei indre løysing for alle tre sektorane.

Til nå har det ikkje vore spørsmål *om* det skal satsast på intelligensindustri, men *kor mye* det bør satsast. Svaret på dette avheng altså mellom anna av verdiane på n, μ, ρ og a. Spesielt kan ein merka seg

[1] Dette er å forstå slik at hvis t.d. a går ned, vil l_1 gjera eit negativt sprang for så, etter det, igjen å stiga mot det nye, lågare, stasjonærnivået.

relevansen av demografiske forhold. Befolkningstilvekstens rolle har vi alt nemnt. Vidare vil det vel vera rimeleg å assosiera utdanningskapitalens depresieringsrate med dødsraten i befolkninga, slik at di høgare denne er, di lågare er det optimale kunnskapsnivå i samfunnet. Eit tiltak som reduksjon i pensjonsalderen vil, ved sida av å redusera den yrkesaktive del av befolkninga, også gi μ eit positivt skift og dermed redusera det optimale kunnskapsnivå i befolkninga og påverka den optimale fordelinga av arbeidskrafta mellom dei tre sektorane. Vedrørande forholdet «i-land»/«u-land» viser dette ein vond sirkel. Så lenge «u-land» har ein høgare λ enn «i-land» vil det, alt anna like, vera optimalt for dei å ha eit lågare kunnskapsnivå per arbeidar enn i «i-land», samtidig som det kan tenkjast å vera optimalt for dei å allokera ein større del av arbeidskraftstilgangen til utdanningssektoren.

Kva så med den optimale utvikling i økonomien hvis l_1 og/eller l_3 er lik null for noen t?

3.2 Andre regime

I tillegg til A kan vi som nemnt i samband med tabell 1, ha dei følgjande 3 regime:

B: $l_3 = 0$ er optimalt hvis $ap < w = V'\beta = U'[f - (u/l_1)f']$, slik at arbeidskraftas grenseprodukt i å produsera utdanningskapital multiplisert med utdanningskapitalens skuggepris er mindre enn skuggeprisen på arbeidskrafta brukt i dei to andre sektorane. «Grenselinja» b mellom A og B, innteikna i figur 3, dvs. der $l_3 = 0$ er ei indre løysing, vil i figur 1 liggja til høgre for kurva $\dot{u} = 0$. Dette fordi

$\partial l_3/\partial u < 0$ frå (20), slik at

$l_3 \varepsilon \mathrm{A} > l_3 \varepsilon \mathrm{b} \Leftrightarrow u \varepsilon \mathrm{A} < u \varepsilon \mathrm{b}$ for gitt p.

Dessutan må $u \varepsilon \mathrm{b} \leq \tilde{u}$.

På grunn av at $\partial l_3/\partial u < 0$ og $\partial l_3/\partial p > 0$ frå (20) må b vera stigande i (p,u)-planet.

I B vil u heile tida vera fallande fordi $\dot{u} = -\lambda u$ og p stigande fordi $\left.\frac{dp}{du}\right|_{\dot{p}=0}$ er negativ i B. Det vil seia at l_1 minkar over tid, mens l_2 aukar langs den optimale banen.

C: $l_1 = 0$ hvis $U'[f - (u/l_1)f'] < w = ap = V'\beta$, dvs. når det marginale bidrag til samfunnets velferd av å sysselsetja folk i intelligensindustrien er mindre enn arbeidskraftas skuggepris.

Grenselinja c mellom A og C, også innteikna i figur 3, dvs. der $l_1 = 0$ er ei indre løysing, vil i figur 1 liggja til venstre for kurva $\dot{u} = 0$. Dette fordi $(\partial l_1/\partial u) > 0$ frå (20) slik at

$l_1 \varepsilon A > l_1 \varepsilon c \Leftrightarrow u \varepsilon A > u \varepsilon c$ for gitt p.

Frå (20) har vi at $(\partial l_1/\partial u) > 0$ og $(\partial l_1/\partial p) < 0$ slik at c også må vera stigande i (p,u)-planet.

På grunn av (2a) får vi i staden for (17) at

(26) $$\dot{p} = (\lambda + \rho)p \text{ i C},$$

dvs.

(27) $$p(t) = p_0 e^{(\lambda+\rho)t};\ p > 0,$$

slik at p vil vera eksponensielt stigande i C.

Fordi

$$V'\beta = ap_0 e^{(\lambda+\rho)t}$$

må l_2 stadig falla i C. l_3 må derfor stadig stiga, sidan $l_1 = 0$. Med l_3 stadig stigande må u (før eller seinare) auka i C. For $p = 0$ er vi opplagt ikkje i C fordi ap då er mindre enn w.

I figur 3 er regionane B og C og «grenselinjene» b og c teikna inn saman med A. På same måte som for $\dot{u} = 0$ kan det visast at b og c må starta frå (0,0).

D: l_1 og $l_3 = 0$ samtidig hvis og bare hvis både

$ap < w = V'\beta$ og

$U'[f - (u/l_1)f'] < w = V'\beta$.

I så fall er det produksjon bare i sektor 2.

I D har vi også på grunn av (2a) at

(26) $$\dot{p} = (\lambda + \rho)p.$$

I D er (26) tilfredsstilt for $p = 0$ for alle t.

Alternativt kunne ein tenkja seg at $p \neq 0$ og stigande som i C. Jamvel om det er vanskeleg å visa at dette ikkje kan vera tilfelle når vi har ein uendelig planleggingshorisont, er det klart at dette alternativet ikkje kan forekoma i tilfellet med ein endelig planleggingshorisont, $T < \infty$. Då er transversalitetsvilkåret, når vi har fritt endepunkt, at

$$e^{-\rho T}p(T) = 0,$$

som bare er oppfylt når

(28) $$p(T) = 0.$$

I så fall er det klart at den einaste $p(t)$ som tilfredsstiller både (26) og (28) er $p(t) = 0$ for alle t. Sidan $p(t) \equiv 0$ for alle $T < \infty$, er det rimelig å gå ut frå at det også vil gjelda når T går mot uendelig. u vil vera fallande i D.

Når vi tar med alle regimene A, B, C og D, vil det å fastleggja den optimale politikk også inkludera problemet å finna dei overgangane mellom A, B, C og D som kan vera optimale over tid, dvs. å finna den optimale rekkefølgje mellom dei 4 regimene. Dette betyr å finna ut kva slags overgangar («switches») som er mulige ut frå optimumsvilkåra. Dernest vil l_i, x_i, p og u ha eit optimalt forløp innafor kvart regime, som t.d. allerede utførleg diskutert under A. Kva andre regime kan så eventuelt A gå over i? Sidan $l_i(p^\infty, u^\infty) > 0$ for alle i, kan ikkje A gå over i noe anna regime langs den optimale banen. Dette fordi u alltid går mot u^∞ i A.

Ser vi på B, må denne politikken åpenbart slå over i A langs den optimale bane fordi i B er u fallande, og p stigande over tid slik at det vil vera eit tidspunkt t' der den optimale $l_3(t)$ går over frå å vera null til å bli positiv og vi er i A. B kan ikkje slå over i C fordi B betyr $p < (1/a)U'[f - (u/l_1)f']$, mens C betyr $p > (1/a)U'[f - (u/l_1)f']$.

For at B skulle slå over i C måtte altså p gjera eit sprang, men det er umulig ifølge maksimumsprinsippet. B kan også slå over i D.

Politikk C må slå over i A. Dette for det første fordi det er meiningslaust å oppretthalda aktivitet i utdanningssektoren utan at det nokon gong blir produksjon i «intelligensindustrien». For det andre kan ikkje C slå over i B av same grunn som B ikkje kunne slå over i C. C kan heller ikkje slå over i D fordi p då måtte gjera eit sprang.

Vedrørande D kan denne politikken ikkje slå over i noen annan politikk. I D er $p = 0$ og w konstant og lik $V'(\beta)\beta$. Med p konstant må $ap < w$ initialt halda permanent og «switch» til andre regime er derfor ikkje mulig.

Vi står då igjen med dei følgjande mulige sekvensar for den optimale politikk:

A
B → A
C → A
D
B → D

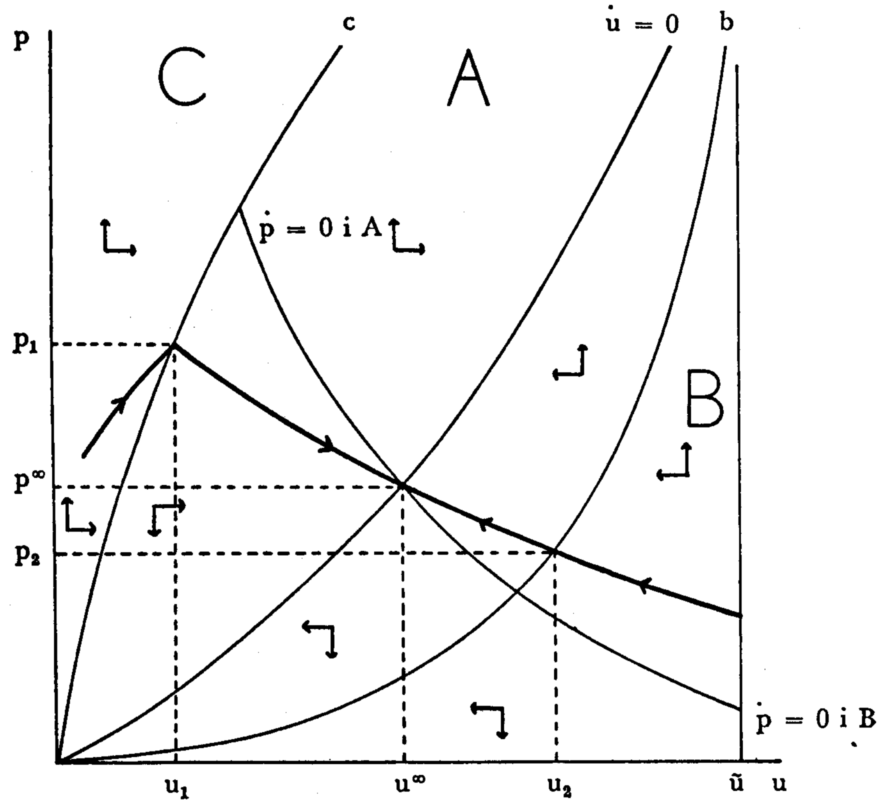
B og C er regime som ikkje kan oppretthaldast langs den optimale banen, mens A og D er dei regima som før eller seinare blir etablert. Når A eller D er etablert, vil dette regimet oppretthaldast for alltid, så lenge data i problemet er uendra. Den optimale politikk hvis økonomien initialt er i A, er alt utførleg diskutert.

Ved hjelp av figur 3 basert på eigenskapane ved regimene B og C — og «grenselinjene» mellom desse regionane og A — skal vi nå gjera greie for den optimale politikk hvis økonomien initialt er i B eller C.[1]

Optimal politikk er nå: Hvis $u_0 < u_1$: Vel p_0 slik at (p_0, u_0) ligg på den eintydige banen i (p, u)-planet som fører til (p_1, u_1). Når dette punktet er nådd, gå så fram som gjort greie for tidlegare under A når $u_0 < u^\infty$. Hvis $u_0 > u_2$: Vel p_0 slik at (p_0, u_0) ligg på den eintydige banen som fører til (p_2, u_2). Når dette punktet er nådd, er den optimale politikk gjort greie for under A med $u_0 > u^\infty$.

Anta at t.d. økonomien initialt er i B, dvs. $l_3 = 0$. u er fallande og

[1] Regime D i fasediagrammet vil vera samanfallande med u-aksen.



Figur 3.

p er stigande. Den mengd arbeidskraft som optimalt skal allokerast til «intelligensindustrien», er avtakande mens mengda av arbeidskraft til sektor 2 aukar inntil $p = p_2$ og $u = u_2$ slik at økonomien slår over i regime A med $u > u^\infty$, der $l_3 > 0$ nå tiltar mot l_3^∞, mens l_1 og l_2 avtar mot l_1^∞ og l_2^∞. Det kan synast noe kunstig at økonomien initialt skulle ha «for mye» utdanningskapital, men det kan også tenkjast at ei endring i data, t.d. eit positivt skift i ρ, samfunnets tidspreferanserate, kan føre til at den aktuelle utdanningskapitalen i samfunnet vil vera for høg, slik at det er optimalt for samfunnet å slå over frå A til B med ei etterfølgjande tilpassing som ovanfor skissert.

Anta så at økonomien initialt er i C, den optimale $l_1 = 0$, l_2 fell og l_3 stig langs den optimale banen. C vil gå over i A når $p = p_1$ og

$u = u_1$ med $l_1 > 0$ og stigande, l_2 nå stigande og l_3 nå fallande mot stasjonærnivået. Om vi går utafor modellen, kan vi sjå på dette forløpet som ein tilnærma beskrivelse av den aktuelle politikken vis a vis oljesektoren. Det oppstod her «spontant» ein ny produksjonssektor med behov for større kunnskapsmengde enn det som var til disposisjon innanlands i utgangspunktet. Dei innanlandske arbeidskraftressursane har så for størstedelen blitt satsa på å bygga opp den nødvendige kunnskapsmengda før norsk ekspertise i særleg grad skal gå inn i og i det vesentlege overta drifta i oljesektoren.

At den optimale politikken som fører til enten A eller D verkeleg er optimal, er klart frå det følgjande:

(29)
i) Lagrangefunksjonen er konkav i u og l_i, $(i = 1, 2, 3)$, for gitt $p(t)$ og t.
ii) $\lim_{t \to \infty} e^{-\rho t} p(t)(u(t) - u^\infty) = 0$

Dette fordi i A går $p(t)$ mot p^∞ når $t \to \infty$, mens $u(t)$ går mot u^∞. I D er $p(t) = 0$ og ii) er dermed også oppfylt i D.

4. *Avslutningsmerknader*

4.1 Oppsummering.

Dei viktigaste dynamiske resultata fra modellen er

1: Hvis det frå eit visst tidspunkt og for «all framtid» er lønnsamt å «satsa på ingelligensindustri», vil det vera ei eintydig, stasjonær allokering av arbeidskraft mellom dei tre sektorane: Utdanningssektoren, «intelligensindustrien» og annan vareproduksjon som er optimal. Frå ein gitt initialsituasjon blir denne allokeringa bare nådd assymptotisk, og den er uavhengig av initialsituasjonen. Den optimale bane mot denne stasjonære, optimale allokeringa, hvis det initiale kunnskapsnivå er mindre enn det optimale, er karakterisert ved at den delen av den tilgjengelige arbeidskrafta som blir allokert til utdanningssektoren skal *avta* over tid, mens den for både intelligensindustrien og annan vareproduksjon skal *tilta* over tid. Vidare, di lågare kunnskapsnivået initialt er, di høgare skal den initiale allokering av arbeidskraft til utdanningssektoren vera, og di lågare blir allokeringa til begge dei to andre sektorane.

Den optimale stasjonære allokeringa, og dermed den optimale allokering av arbeidskraft til ei kvar tid langs den optimale bane, er bestemt av mellom anna samfunnets tidspreferanserate, utdanningssektorens effektivitet, tilveksttakta i folkemengda og kor fort kunnskapen depresierast.

2: Det kan også vera optimalt ikkje å satsa på intelligensindustrien initialt, mens ein i den første perioden bare driv «tradisjonell» produksjon og oppbygging av «kunnskapskapitalen». Når denne så har nådd eit visst nivå, er tida komen for å begynna å allokera arbeidskraft til intelligensindustrien også. I den første perioden har den optimale allokering av arbeidskraft over tid det følgjande mønster: Den delen av arbeidskrafta som går til utdanningssektoren aukar over tid langs den optimale banen, mens den delen som går til tradisjonell industri skal avta.

3: Endelig er det tenkelig at det ikkje er optimalt på noe tidspunkt å ha aktivitet i intelligensindustrien. Det er då i denne modellen heller ingen grunn til å oppretthalda noen utdanningssektor. Ein slik situasjon i økonomien, der det ikkje er lønnsamt å satsa på intelligensindustri — eller utdanning av arbeidskrafta — vil, for gitte data, vera eit permanent trekk ved økonomien, og denne situasjonen må også gjelda initialt.

4: Det kan tenkjast å vera optimalt å ha ein intelligensindustri initialt jamvel om det ikkje vil vera optimalt for alltid. Det ville då ikkje vera noen aktivitet i utdanningssektoren, og aktiviteten i «intelligensindustrien» skulle nedtrappast inntil denne sektoren blir nedlagt.

4.2 Modifikasjonar og utvidingar av modellen

Det er lett å peika på trekk ved modellen ovanfor som gjer den «urealistisk». Mellom anna medfører det ingen tilpasningskostnader t.d. å leggja ned utdanningssektoren for ein periode, mens dette faktisk ville vera forbunde med store problem. Å modifisera modellen på dette punktet er truleg ikkje enkelt.

Derimot er det mulig å utvida modellen til å

i) modifisera føresetnaden om at bruttoproduksjonen i utdanningssektoren er proporsjonal med arbeidskraftsinnsatsen i denne sektoren,

ii) ha med (ulik) eksogen teknisk framgang i dei tre sektorane, og

iii) trekkja inn utlandet, slik at eksport og import av begge varene er mulig, kanskje også eksport og import av arbeidskraft og utdanningskapital («brain drain»).

Modellen ovanfor er dessutan basert på at alle avgjerder blir tatt sentralt. Det vil også vera av interesse å diskutera korvidt den optimale utvikling i ein økonomi av denne typen kan realiserast ved hjelp av desentralisering av avgjerdene til den enkelte utdanningssøkar. La oss til slutt sjå litt nærare på punkt ii) ovanfor: Eksogen teknisk framgang i utdanningssektoren vil vera det same som at a stadig skifter oppover over tid. Frå før veit vi at dette ville bety at u stadig aukar, mens p fell over tid, dvs. l_1 og l_2 vil heile tida stiga langs den optimale bane, mens l_3 vil vera fallande.

Verknaden av eksogen teknisk framgang i intelligensindustrien kan studerast på liknande vis, ved å la første ledd i (17) få eit positivt skift, som vi så kan anta skjer kontinuerlig over tid.

Ved same framgangsmåte som for ei endring i a finn vi at u og p nå begge vil stiga over tid, men sidan det ikkje er noe teknisk framsteg i utdanningssektoren vil u gå mot $\tilde{u}$, mens p ikkje har noen øvre grense. Dette betyr at den delen av arbeidskrafta som blir allokert til utdanningssektoren, nå må stiga over tid langs den optimale banen — dette i motsetnad til resultatet i den opphavelige modellen. l_2 må avta. Ein kombinasjon der vi har teknisk framsteg i begge sektorane ville åpenbart leia til ei stadig stigning i både u og p langs den optimale banen, mens det er uråd å seia noe generelt om $l_i(t)$ skal avta eller tilta langs den optimale banen.

Det er vel grunn til å tru at den tekniske framgangen i utdanningssektoren er heller liten samanlikna med vareproduksjonssektorane. I periodar med stor skilnad i teknisk framgang mellom dei to sektorane kan verknader av typen «teknisk framgang bare i intelligensindustrien» derfor vera relevant. *I slike periodar kan derfor konklusjonen frå den opp-*

havelege modellen om at l_1 og l_2 skal stiga over tid, mens l_3 skal falla langs den optimale banen, bli reversert.

Ei slik utvikling for all framtid kan neppe vera optimal sidan $p(t)$ er stadig stigande, noe som betyr at eit vilkår tilsvarande ii) i (29) ikkje treng vera oppfylt.

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ON THE OPTIMAL DEVELOPMENT OF KNOWLEDGE-BASED INDUSTRIES AND THE EDUCATIONAL SECTOR IN A SMALL OPEN ECONOMY*

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1. INTRODUCTION

Traditionally, economists have regarded the quality of labor as constant over time. However, by altering the allocation of resources to, e. g., the educational sector, the productivity of labor may be consciously changed. Society is then faced with a dynamic optimization problem since, to the extent resources are allocated to the educational sector *now*, in order to make labor more productive *later*, these resources cannot be used for producing goods and (other) services *now*.

Considerable resources today are allocated to the educational sector in most countries. Even so, little theoretical attention has been given to the problem of optimal allocation of resources to the educational sector from a dynamic point of view. This is in contrast to the vast literature on optimal accumulation of physical capital. Exceptions are Uzawa [1965], Razin [1972], Dobell and Ho [1967] and, recently, Manning [1975, 1976]. This paper is a generalization of Aarrestad [1975], where the problem was to find the optimal allocation of labor to the educational sector in a centrally planned closed economy where all production was aggregated into one sector. The following model is more disaggregated since in addition to the educational sector, the production of goods and services now takes place in two sectors with different "knowledge-intensity," which means that there may be more regimes in the optimal policy. The present model is also more general since it allows for export and import of the two types of goods. It is hoped that the theory may throw some light on how to find the optimal level over time of general education and technical "know-how" in a (homogeneous) work-force. The model may be given two "real-world" interpretations:

i) to study the optimal development of a "knowledge-based" industrial sector and the educational sector in a developed economy, or
ii) to study the optimal development of the "modern" vs. the "traditional" sector and the educational sector in a less developed economy.

The main problem in this paper is to find optimal paths for the allocation of labor over time to the three sectors of the economy. To answer such questions

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a dynamic model is needed.

2. THE MODEL

The model consists of three sectors — two sectors for the production of goods, and one educational sector producing general knowledge and technical "know-how". The accumulated level of general knowledge and technical "know-how" in society will be called "educational capital." Of course, measuring a variable of this kind is difficult. As a point of departure, a possible proxy would be the total number of years of education embodied in the labor force. Concentrating on the role of formal education, other possible sources of accumulated "know-how" as, e.g., "learning by doing" are neglected. Educational capital is used in sector 2 only, which is the knowledge-based industrial sector. To simplify the analysis, labor is the only input in the production of good 1.[2] Educational capital is sector specific, but not worker specific — it increases the productivity of all workers in sector 2 only. In this model, therefore, education alters the quality, but not the composition, of the labor force in sector 2 over time, so that there are no skill margins within the labor force in sector 2; i.e., labor is homogeneous.[3] We have made the abstraction of specifying (i) the level of education and "know-how" and (ii) "raw" labor as two separate factors in the production process. These are the only specified factors of production — physical capital is disregarded. Also, the production period in the educational sector is overlooked and there are no "vintage"-effects. "Raw" labor is to be understood as completely uneducated labor if the model is applied to a LDC, while in a DC-context it is probably more fruitful to think of "raw" labor as labor with only compulsory elementary schooling.

The amount of "raw" labor employed at time t in the educational sector will be denoted by $L_3(t)$ and the output of this sector by $J(t)$, given by

$$J(t) = \alpha L_3(t); \qquad \alpha \text{ is a constant} > 0. \tag{1}$$

$J(t)$ is net in the sense that any output in the educational sector that is subsequently used as input (as when students become teachers after graduation) is not included in $J(t)$.

In the knowledge-based industry the production-function is

$$X_2(t) = F(L_2(t), E(t)) \tag{2}$$

where

$X_2(t)$ = the total production in this sector

[2] While this is a simplification made to avoid the problem of allocating educational capital optimally between the two sectors, it is not inconsistent with the fact that there are great differences in educational capital per worker between industrial sectors.

[3] A heterogeneous labor force, diversified according to educational background, would be unmanageable in this model.

$L_2(t)$ = the amount of "raw" labor employed in this sector, and
$E(t)$ = some index of the accumulated level of education and technical "know-how" in the economy.

L_2 and E are assumed complementary in production, i.e. $\frac{\partial^2 F}{\partial E \partial L_2} > 0$. F is assumed to be homogeneous of degree one and strictly concave with positive and diminishing marginal productivities. Further

(2a) $$F(L_2, 0) = F(0, E) = F_E(0, E) = 0$$ [4]

(2) says that the level of goods-production in sector 2 depends on the accumulated level of education and know-how, and on the amount of labor allocated to this sector. (Production functions expressing the same idea can be found in Haavelmo [1954]. See, e.g., page 14 and model 7.1 page 36.) The stock of educational capital is built up through the gross addition to the existing stock $J(t)$, given by (1). On the other hand, it depreciates since knowledge becomes obsolete and people forget what they once learned. $E(t)$ is assumed to depreciate at a constant rate μ. We then get

(3) $$\dot{E}(t) = J(t) - \mu E(t).$$

Initially there is a stock of educational capital, E_0, i.e.

(4) $$E(0) = E_0.$$

In sector 1 "raw" labor is the only input in the production process so that

(5) $$X_1 = G(L_1); \quad G' > 0, \quad G'' \leq 0,$$

where

X_1 = total output in sector 1
L_1 = the amount of labor employed in this sector.

Further, there is a fixed proportion between the population and the total labor force, $L(t)$. L is assumed constant, so that,

(6) $$L(t) = \bar{L} \text{ (given)}.$$ [5]

Finally, employment in the three sectors cannot exceed the total labor force,

(7) $$L_1 + L_2 + L_3 \leq L.$$

Given the structure of the economy, described by equations (1)–(7) we want to maximize social welfare. Implicit in the optimal development that emerges, there

[4] $F_E = \frac{\partial F}{\partial E}$, etc.

[5] This assumption can be relaxed, and it can be shown that the effects of a constant growth-rate in population are identical to the effects of the rate of depreciation of educational capital. However, with population growth, (5) must be linear, and the solution of the model will then be singular, which is the reason why we prefer the present model.

will be answers to questions such as whether, under what conditions and to what extent the knowledge-based industrial sector and/or the educational sector should be developed.

Assume now that we are dealing with a small country with an open economy, so that both goods can be traded internationally at prices given in the world-market. The price of good 2 in terms of good 1 is p ($p_1=1$, $p_2=p$), assumed given and constant. Let the instantaneous social welfare function be

$$U = U(c_1, c_2)$$

where

$$c_i = \text{total consumption of good } i; \qquad i = 1, 2.$$

U is assumed to be strictly concave with positive and diminishing first order partial derivatives. The budget constraint applied at every instant is that the value of total production in terms of world-market prices equals the value of total consumption, so that

$$c_1 + pc_2 = X_1 + pX_2 \tag{8}$$

Assume further that the social rate of discount is a constant $\rho>0$, that there are no restrictions on E at any time and that society's planning horizon is infinity. We then have the following problem in optimal control theory:

$$(9) \quad \begin{cases} \max\limits_{c_1, c_2, L_1, L_2, L_3} \int_0^\infty U(c_1, c_2)e^{-\rho t}dt \\ \text{s.t.} \quad c_1 + pc_2 = X_1 + pX_2 \\ \qquad X_1 = G(L_1) \\ \qquad X_2 = F(L_2, E) \\ \qquad \dot{E} = J - \mu E \\ \qquad J = \alpha L_3 \\ \qquad L_1 + L_2 + L_3 \leq \bar{L} \\ \qquad 0 \leq L_i \leq \bar{L}; \qquad i = 1, 2, 3 \\ \qquad E(0) = E_0 \text{ (given)} \\ \qquad \lim\limits_{t\to\infty} E(t) \text{ is free} \\ \qquad L_i \ (i = 1, 2, 3) \text{ piecewise continuous.} \end{cases}$$

3. ANALYSIS OF THE MODEL

To solve (9) form the Lagrange expression,

$$(10) \qquad L = e^{-\rho t}\{U[G(L_1) + pF(L_2, E) - pc_2, c_2] + q(\alpha L_3 - \mu E) + w(\bar{L} - L_1 - L_2 - L_3)\},$$

where we have inserted for c_1, X_1 and X_2 and where w is the shadow price of "raw" labor (the shadow wage) and $q(t)$ is the costate variable associated with the equation of motion (3). Necessary conditions for a solution to (9) are that there exists a continuous $q(t)$ such that:

$$(11) \qquad \dot{q} = -U_1 pF_E(L_2, E) + (\rho + \mu)q$$

$$(12) \qquad U_1 pF_L(L_2, E) - w \leq 0 \text{ and } = 0 \text{ if } L_2 > 0$$

$$(13) \qquad U_1 G'(L_1) - w \leq 0 \text{ and } = 0 \text{ if } L_1 > 0$$

$$(14) \qquad \alpha q - w \leq 0 \text{ and } = 0 \text{ if } L_3 > 0$$

$$(15) \qquad \frac{U_2}{U_1} = p$$ [6]

Consumption is governed by (15). Obviously (7) always holds as an equality, since $U_1 G' > 0$. For feasibility we require $X_1 + pX_2 > 0$ for all t. A number of regimes are possible in the optimal solution. They are enumerated in Table 1.

TABLE 1

Regime	Control variables		
	L_1	L_2	L_3
A	>0	>0	>0
B	>0	>0	0
C	>0	0	>0
D	$\bar{L}$	0	0
H	0	$\bar{L}$	0
I	0	>0	>0

3.1. *Regime A (The Interior Solution).* It is now optimal to allocate labor to all three sectors. From (12), (13) and (14) we then have,

$$(16) \qquad U_1 pF_L = U_1 G' = \alpha q = w.$$

(16) says that for an interior optimum the social value of the marginal product of labor in both production sectors must at any time be equal to the shadow wage which in turn must be equal to αq, the shadow price of educational capital multiplied by α — the marginal product of labor in producing educational capital. To be more complete: $q(t)$ may be interpreted as the increase in the optimal value of the objective function obtained from adding "one extra unit" of educational capital per capita to the stock of educational capital per capita at time t. q is

[6] Here $U_1 = \frac{\partial U}{\partial c_1}$, etc.

positive from (14). Along the optimal path for $q(t)$, (11) must be satisfied.[7] Implicit differentiation in (12)–(14) yields

$$(17)\qquad \begin{cases} \dfrac{\partial L_1}{\partial E} = 0 & \dfrac{\partial L_1}{\partial q} = \dfrac{\alpha}{U_1 G''} \\ \dfrac{\partial L_2}{\partial E} = -\dfrac{F_{LE}}{F_{LL}} & \dfrac{\partial L_2}{\partial q} = \dfrac{\alpha}{U_1 p F_{LL}} \end{cases}$$

With the optimal L_i ($i=1, 2, 3$) now given as functions of q and E, (3) and (11) are two autonomous differential equations in E and q. This permits a two-dimensional graphic analysis of the system from which the optimal trajectories of L_i ($i=1, 2, 3$) will emerge. For the slope of the graph of $\dot{q}(t)=0$ in the (q, E) phase-plane we obtain

$$(18)\qquad \left.\frac{dq}{dE}\right|_{\dot{q}=0} = \frac{U_1 p\left(F_{EL}\dfrac{\partial L_2}{\partial E} + F_{EE}\right)}{-U_1 p F_{EL}\dfrac{\partial L_2}{\partial q} + \rho + \mu}.$$

From (17), the denominator in (18) is positive. Inserting for $\frac{\partial L_2}{\partial E}$ from (17) and rearranging, the nominator can be written as $\frac{U_1 p}{F_{LL}}(F_{EE}F_{LL}-(F_{EL})^2)$ which is negative since F is concave. (18) is therefore negative. The slope of the graph of $\dot{E}(t)=0$ is given by

$$(19)\qquad \left.\frac{dq}{dE}\right|_{\dot{E}=0} = \frac{\mu - \alpha\dfrac{\partial L_3}{\partial E}}{\alpha\dfrac{\partial L_3}{\partial q}},$$

which from (7) and (17) is positive. The highest possible sustainable E, $\bar{E}$ is given by

$$(20)\qquad \bar{E} = \frac{\alpha\bar{L}}{\mu}.$$

In the (q, E)-plane shown in Figure 1, the curve for $\dot{E}=0$ must start from $\left(\frac{w}{\alpha}, 0\right)$, since from (3) and (17) $q=\frac{w}{\alpha}\Rightarrow E=0$ and also $E=0\Rightarrow q=\frac{w}{\alpha}$. As $E\rightarrow\bar{E}$, $\left.\frac{dp}{dE}\right|_{\dot{E}=0}\rightarrow\infty$. This is because the slope of $\dot{E}=0$ is always positive for $E<\bar{E}$, while for $E>\bar{E}$, $\dot{E}$ is always negative. The form of $\dot{E}=0$ is therefore as shown in Figure 1, where $\bar{E}$ and the graph of $\dot{q}=0$ are shown as well.

The graph of $\dot{q}=0$ is falling to the right in the (q, E)-plane and q is always positive. We therefore know enough to state that *in regime A there must exist*

[7] In competitive price adjustment terms (11) has the following interpretation: In an economy in which educational capital "rental" is rewarded by its marginal social value product, the price of a unit of educational capital must change so as to reward the "rentier" for waiting less the value of net rentals received $U_1 pF_L - \mu q$.

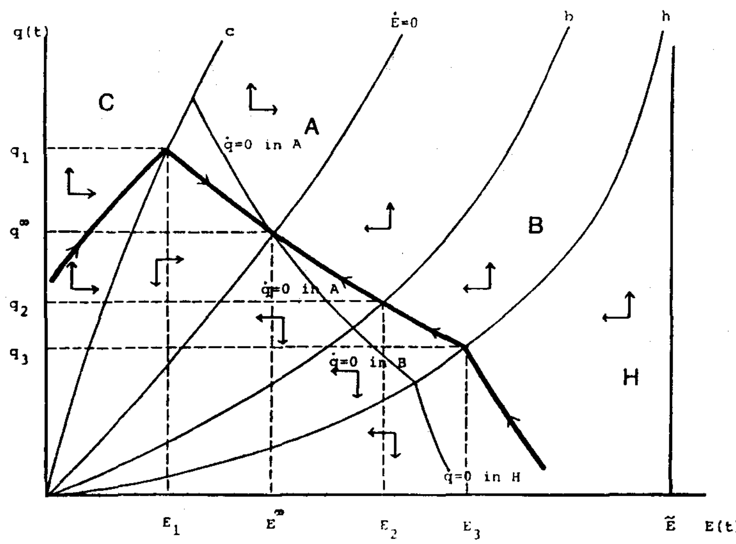


FIGURE 1

a unique stationary state for q and E— (q^∞, E^∞) *such that*

$$E^\infty = \frac{\alpha}{\mu} L_3(q^\infty, E^\infty) \tag{21}$$

$$q^\infty = \frac{U_1 p}{\rho + \mu} p F_E(L_2(q^\infty, E^\infty), E^\infty)\,. \tag{22}$$

This point is shown in the phase-diagram. Since q and E are stationary in this point, L_i $(i=1, 2, 3)$ will also be constant. The equilibrium of the system is therefore a stationary state where the values of all variables are constant over time. This state is reached only asymptotically. To interpret the equilibrium values of E and q, we see that (22) multiplied by α is the present social value of a marginal allocation of labor to the educational sector. In the stationary state this value must, according to (16) be equal to the (instantaneous) marginal value of allocating labor to the two other sectors. The arrows show the dynamic forces working on the system in the different regions of the phase-plane. It is intuitive and can easily be shown that (q^∞, E^∞) is a saddle-point. That is: There is one and only one path in the (q, E)-plane leading to (q^∞, E^∞) such that to each initial E_0 there corresponds a unique q_0 such that a path starting from (q_0, E_0) converges to (q^∞, E^∞). This "candidate" optimal trajectory is shown in the figure. It will be shown later that this path is indeed optimal. The solution to the optimization problem when we have an interior solution is therefore to find the initial q_0 and follow the optimal trajectory towards (q^∞, E^∞). What, then, characterizes this optimal trajectory? From the figure it can be seen that for $E_0 < E^\infty$, $q(t)$ is falling over time, while $E(t)$ is growing. The level of education will therefore be growing over time along the optimal path, until E^∞ is reached (asymptotically).

Since

$$\frac{dL_i}{dt} = \frac{\partial L_i}{\partial q}\frac{dq}{dt} + \frac{\partial L_i}{\partial E}\frac{dE}{dt}; \qquad i = 1, 2, 3. \tag{23}$$

it follows from (17), (23) and Figure 1 that for $E_0 < E^\infty$, *the part of the work-force allocated to the knowledge-based industrial sector, and the part of the labor-force employed in sector* 1 *should increase over time along the optimal path.* Accordingly, *the part of the labor-force that is optimally allocated to the educational sector must be falling over time when* $E_0 < E^\infty$.

The lower the initial E is, the higher must the initial q and therefore L_3 be, with a correspondingly lower initial level for L_2 and L_1.

While the initial L_3 must be set above its optimal stationary value when $E_0 < E^\infty$, opposite conclusions will hold for the case when the initial educational intensity is higher than E^∞. $L_3(t)$ should then increase over time, while L_1, L_2 and E will be falling over time along the optimal path. If finally, by accident $E_0 = E^\infty$, $L_i(0)$ $(i = 1, 2, 3)$ should be chosen equal to their optimal constant values and the optimal policy is to keep L_i $(i = 1, 2, 3)$ constant over time.

3.2. *Other regimes.* As shown in Table 1, in addition to A, five other regimes are possible as part of the optimal solution. Let us treat them in turn.

B: $L_3 = 0$ is optimal if (14) holds with inequality sign, so that $\alpha q < w = U_1 G' = U_1 p F_L$. The marginal product of labor in producing knowledge multiplied by the shadow price of knowledge is less than the shadow price of labor employed in the other two sectors. The "border-line", b, between A and B, i.e., the locus of all points where $L_3 = 0$ is an interior solution (14) will in Figure 1 lie to the right of the curve $\dot{E} = 0$ in A. This is because, from (17) $\frac{\partial L_3}{\partial E} < 0$ so that $L_3 \in A > L_3 \in b \Leftrightarrow E \in A < E \in b$ for a given q. In addition $E \in b \leq \tilde{E}$. Since $\frac{\partial L_3}{\partial E} < 0$ and $\frac{\partial L_3}{\partial q} > 0$ from (17), b must have a positive slope in the (q, E)-plane. Also, in B, E will have no non-trivial stationary since $\dot{E} = -\mu E$ so that E will be falling in B. $\left.\frac{dq}{dE}\right|_{\dot{q}=0}$ in B is given by (18). q is therefore increasing along the optimal path in B. This means that the optimal L_2 decreases in B, which again must imply that L_1 increases over time in this regime, since $L_3 = 0$.

C: $L_2 = 0$ if (12) holds with an inequality sign so that $U_1 p F_L < w = \alpha q = U_1 G'$ in which case the value of labor's marginal product in the knowledge-based industrial sector is less than the shadow-wage. From (2a) and (11)

$$q(t) = q_0 e^{(\mu+\rho)t} \quad \text{in} \quad C. \tag{24}$$

We then have $U_1 G'(L_1) = \alpha q_0 e^{(\mu+\rho)t}$ so that L_1 is decreasing and L_3 increasing in C. For "large" t, L_3 approaches $\bar{L}$ as L_1 approaches zero. E must therefore be increasing in C. The "borderline", c, between A and C, i.e., the locus of all points such that $L_1 = 0$ is an interior solution to (12), will be to the left of the curve

$\dot{E}=0$ in Figure 1. This is because $\frac{\partial L_1}{\partial E}>0$ from (17) so that $L_1 \in A > L_1 \in c \Leftrightarrow u \in A > u \in c$ for a given q. In addition we have from (17) that $\frac{\partial L_1}{\partial q}<0$ so that also c must have a positive slope in the (q, E)-plane.

D: $L_2=L_3=0$ at the same time if and only if

$$\left.\begin{matrix} \alpha q \\ U_1 p F_L \end{matrix}\right\} < w = U_1 G'(\bar{L}).$$

In that case there is activity in sector 1 only. Due to (2a) q in D is given by (24). In D, q need not be positive so that (24) is satisfied for $q=0$ for all t. $q<0$ is impossible, since from the economic interpretation of q, a negative q would imply a negative marginal productivity of educational capital, which contradicts the assumptions on (2). Also $q>0$ is impossible. When D is the final policy this is so since

$$\lim_{t\to\infty} q(t) = \infty$$

so that the economy would sooner or later be in C which contradicts the assumption that D is the final policy. If D was assumed *not* to be a final policy, $G'(\bar{L})$ is a constant, q is then growing exponentially. Assume then, that L_3 switches to a positive number at $t'>t_0$. If the horizon is finite, such a policy cannot be optimal since the marginal loss $G'(\bar{L})$ is constant over time, while the marginal gain due to an increased educational capital is falling over time because the pay-off period is shrinking over time. If the horizon is infinity, the pay-off period will also be infinite, so that both the marginal loss and the marginal gain are constant over time. If a marginal reallocation from sector 1 to sector 3 is profitable at t', it is so also at $t''<t'$, including t_0 and we are in C. Consequently $q \underset{t}{\equiv} 0$ in D.[8]

H: The optimal L_2 is now equal to $\bar{L}$, so all activity is concentrated to the educational intensive industry. E will be decreasing in H, since $\dot{E}=-\mu E$ and $\dot{q}=0$ when $q=\frac{u_1}{\rho+\mu}\{pF_E(\bar{L}, E)\}$ so that

$$\text{(25)} \qquad \left.\frac{dq}{dE}\right|_{\dot{q}=0} = \frac{U_1}{\rho+\mu}\{pF_{EE}(\bar{L}, E)\} < 0.$$

q is therefore increasing in H. The borderline h of H, i.e. the locus of (q, E)-values such that $L_2=\bar{L}$ is an interior solution to (12), is the set

$$h = \{(q, E) | U_1 p F_L(\bar{L}, E) = \alpha q\}.$$

[8] When D is the final policy and the horizon, T, is finite this follows from the transversality condition

$$e^{-\rho T} q(T)=0$$

which, together with (24), is only satisfied for $q=0$ for all $T<\infty$.

Along this borderline we then have

$$\frac{dq}{dE} = \frac{U_1}{\alpha} pF_{LE} > 0, \tag{26}$$

so that the boundary of H has a positive slope in the (q, E)-plane.

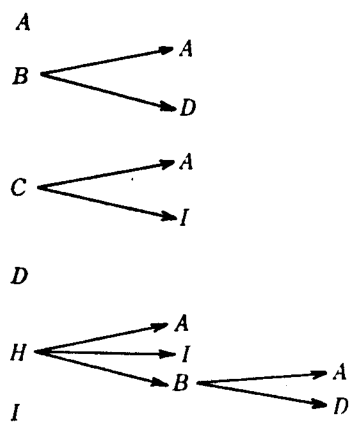
I: The optimal L_1 is equal to zero if $U_1 G'(L_1) < w = U_1 pF_L = \alpha q$, i.e., if the value of the marginal product of labor in sector 1 is less than its value in alternative employment. As in regime A, the slopes of $\dot{q}=0$ and $\dot{E}=0$ are given by (18) and (19). Because of this there will be an intersection of $\dot{q}(t)=0$ and $\dot{E}(t)=0$ in I, as in A, which is also a stationary state with the saddlepoint property. Except that $L_1=0$ in I, it is fairly obvious that the optimal path and the optimal policy in I have the same properties as in A.

3.3. *Optimal Policies.* The problem of finding the optimal policy also includes the problem of finding those switches between the six regimes which are compatible with the conditions for optimality. In other words the problem is to find the optimal sequences between the regimes over time. Within each regime there is then an optimal development of L_i, X_i, q and E as already extensively discussed under regime A.

What other regimes, then, may A switch into? Since q and E always approach q^∞ and E^∞ in A, A cannot switch into any other regime. Regarding B, E is falling in this regime so that B must switch into A. Since in B: $q < \frac{U_1 p}{\alpha} F_L$; while in C: $q > \frac{U_1 p}{\alpha} F_L$, B cannot switch to C because that would mean a jump in q, which is impossible by the maximum principle. A switch from B to D is possible. B cannot switch to H since by (17) L_2 is falling in B. Neither can B switch to I since $q < \frac{U_1}{\alpha} G'$ in B while in I $q > \frac{U_1}{\alpha} G'$ so that a jump in q would be required, which is impossible. Regime C must switch into A or I, since it would not make economic sense to keep up the activity in the educational sector if no production in sector 2 were to take place in the future. C cannot switch into B, D or H since that would require a jump in q. D cannot switch into any other regime. In D, $q=0$ for all t, so that $\alpha q < w$ initially must hold permanently. In regime H, E is falling so that q is increasing along the optimal path. H cannot switch to either C or D, since that would require a jump in q. Since E is falling, $U_1 pF_L(\bar{L}, E)$ must be falling over time in H. $G'(0)$ is a constant. We must therefore have one of the following switches for H:

i) to I if $\alpha q = U_1 pF_L(\bar{L}, E)$ while still $U_1 G'(0) < w$
ii) to B if $U_1 G'(0) = U_1 pF_L(\bar{L}, E)$ while still $\alpha q < w$
iii) to A if, by accident, αq and $U_1 pF_L(\bar{L}, E)$ "reach" $U_1 G'(0)$ simultaneously.

Finally, regime I cannot switch to any other regime for the same reasons that A cannot. We are then left with the following possible policy sequences for the optimal policy:



B, C and H are regimes that cannot be sustained along the optimal path, whereas A, D or I are final regimes — regimes that sooner or later will be established. When one of the final regimes has been established, it will be everlasting for constant values of the parameters including p. If regime I is established finally the economy specializes in producing "knowledge-intensive" goods. If D is the final regime, the economy specializes in goods produced in sector 1. If A is the final regime, it is optimal for the economy to produce both goods, i.e., non-specialization is optimal.

The optimal policy if the economy initially is in A has already been extensively discussed. With the help of Figure 1, based on the properties of regimes B, C and H, — already explained, — the optimal path if the economy initially is in B, C or H will now be discussed. Let us assume that A, and not I or D, will be the final regime. The optimal policy is now: If $E_0 < E_1$, choose q_0 such that (q_0, E_0) is on the unique path in the (q, E)-plane that leads to (q_1, E_1). When this point is reached, proceed as explained earlier under regime A when $E < E^{\infty}$. If the economy initially is in B, that is if $E_2 < E_0 < E_3$, choose $q_2 < q < q_3$, such that (q_0, E_0) is on the optimal path leading to (q_2, E_2) from which the optimal policy is as under A. If, finally, the economy initially is in H, choose q_0 such that (q_0, E_0) is on the unique optimal path in H that leads to (q_3, E_3), from which the optimal path is as explained under B. (As mentioned earlier, H might switch directly to A, in which case the optimal path does not pass through B).

To be more detailed: Assume that the economy is initially in H, an extreme case with a superabundant knowledge level in the "intelligence-industry" sector. All available labor is allocated to this sector, E is falling and q increasing along the optimal path. When q reaches q_3, while still $U_1 G'(0) > \alpha q$, labor is now allocated to both production sectors such that the part of the available labor force that is optimally allocated to the knowledge-based industrial sector is decreasing, while the part allocated to the rest of the goods-producing sector is increasing. This is the optimal process until $E = E_2$ at which point the system switches into its interior mode, A, with L_i $(i = 1, 2, 3) > 0$ where L_3 now increases towards L_3^{∞}, while

L_1 and L_2 decrease towards L_1^∞ and L_2^∞. Assume next that the economy is in C. Initially it is then non-optimal to have an industrial sector based on knowledge, but it is profitable to invest in expanding the level of education in society at the same time as goods-production in the traditional sector is carried on. In this phase, the part of labor allocated to the educational sector should be increasing over time, while L_1 is falling along the optimal path. C will switch into A when q reaches q_1 and $E=E_1$. At this point production in the knowledge-based sector is started up and the part of labor that is allocated to this sector should be steadily increasing towards its stationary value L_2^∞, while now the parts of labor allocated to the educational sector and to traditional goods-production should be decreasing towards their stationary levels.

That the "candidate" optimal policies leading to A, I or D are really optimal, is clear from the following

(27)
1) The Lagrangean (10) is concave in E, c_i $(i=1, 2)$ and L_i $(i=1, 2, 3)$ for a given $q(t)$ and t.
2) $\lim_{t\to\infty} e^{-\rho t}q(t)(E(t) - E^\infty) = 0$

This is so since in A and I, q approaches q^∞ when $t\to\infty$ while E approaches E^∞. $q\equiv 0$ in D so that 2) is also satisfied in D.

The analysis up to now is based on given constant values of the parameters of the model. We shall now study how the results found depend on these parameters. Some effects of exogenous technical change will also be discussed.

4. EFFECTS OF CHANGES IN PARAMETERS, PRICES AND TECHNOLOGY

Assume now that we are in regime A. To study how the optimal levels of q, E, L_i $(i=1, 2, 3)$, and X_i $(i=1, 2)$ at any time depend on the parameters of the model: α, μ and ρ, we differentiate implicitly in (3) and (11). Evaluating the derivatives at $\dot{q}=\dot{E}=0$, we get the following effects on the optimal steady-state E and q, E^∞ and q^∞:

$$\frac{\partial E^\infty}{\partial \rho} = \frac{1}{D}\left\{\alpha q \frac{\partial L_3}{\partial q}\right\} \tag{28}$$

$$\frac{\partial E^\infty}{\partial \alpha} = \frac{L_3}{D}\left\{U_1 pF_{EL}\frac{\partial L_2}{\partial q} + \rho + \mu\right\} \tag{29}$$

$$\frac{\partial E^\infty}{\partial \mu} = \frac{1}{D}\left\{\alpha q\frac{\partial L_3}{\partial q} - U_1 pEF_{EL}\frac{\partial L_2}{\partial q} + \rho + \mu\right\} \tag{30}$$

$$\frac{\partial q^\infty}{\partial \alpha} = -\frac{L_3}{D}\left\{U_1 pF_{EL}\frac{\partial L_2}{\partial E} + F_{EE}\right\} \tag{31}$$

$$\frac{\partial q^\infty}{\partial \rho} = -\frac{q}{D}\left\{\alpha\frac{\partial L_3}{\partial E} - \mu\right\} \tag{32}$$

$$\frac{\partial q^\infty}{\partial \mu} = \frac{1}{D}\left\{U_1 pE\left(F_{EL}\frac{\partial L_2}{\partial E} + F_{EE}\right) - q\left(\alpha\frac{\partial L_3}{\partial E} - \mu\right)\right\} \tag{33}$$

where

$$D = U_1 \alpha p \frac{\partial L_3}{\partial q}\left(F_{EL}\frac{\partial L_2}{\partial E} + F_{EE}\right) - \left(\alpha\frac{\partial L_3}{\partial E} - \mu\right)\left(U_1 pF_{EL}\frac{\partial L_2}{\partial q} - (\rho + \mu)\right) < 0.$$

From (16) L_i^∞ $(i=1, 2, 3)$ depends on q^∞ and E^∞, which again are functions of ρ, μ and α. Inserting for L_i $(i=1, 2, 3)$ in

$$\frac{\partial L_i^\infty}{\partial \alpha} = \frac{\partial L_i^\infty}{\partial q^\infty}\frac{\partial q^\infty}{\partial \alpha} + \frac{\partial L_i^\infty}{\partial E^\infty}\frac{\partial E^\infty}{\partial \alpha}; \qquad i = 1, 2, 3 \tag{34}$$

we can study how the allocation of labor among the three sectors depends on the value of α, and similarly for ρ and μ. From (29) and (31), it is clear that a positive shift in the productivity of the educational sector, α, due to, e.g., more efficient training methods, leads to a higher optimal educational capital. q is reduced. By (17) this means that an increase in α leads to a higher L_2 so that a larger part of the work-force is allocated to the knowledge-intensive production sector. L_1 is also increased, which implies that L_3, or the part of the labor force that is allocated to the educational sector is decreased when α increases.[9] An increase in α would therefore lead to a higher optimal stationary level of E, X_1 and X_2. From (28) and (32) it follows that the optimal stationary educational capital is decreased if the social rate of discount gets a positive shift, i.e., if society chooses to evaluate production today higher, relative to production tomorrow. q^∞ is decreased when ρ increases. From (17) we see that L_1 increases when ρ increases, so that the part of the labor force allocated to sector 1 is increased when the social rate of discount gets a positive shift. Since E^∞ is reduced, the part of the labor force allocated to the educational sector must be reduced when ρ increases. The effect on the allocation of labor to the knowledge-intensive sector from an increase in ρ is not clear. From this it follows that production in sector 1 increases when ρ increases and, consequently, production in sector 2 must fall. (30) and (33) show that a faster depreciation of educational capital leads to a lower optimal knowledge-level and to a lower q. In turn this means that L_1 is increased, so that the part of the labor force going to the two remaining sectors must decrease. The distribution of the reduced part of the labor force between these two sectors depends on how much the marginal product of labor in the knowledge-intensive sector is reduced when educational capital goes down. In any case production in sector 1 will increase so that production in sector 2 must fall.

If we associate the rate of depreciation of human capital μ with the death-rate,

[9] This means that if, e.g., α is increased at t', L_1 would make a negative jump at t'. After t', L_1 would again increase towards the new and lower optimal stationary level.

it is interesting that the higher the death-rate is, the lower is the optimal level of knowledge in the population. Since the effect of μ on L_2 and L_3 is not clear, it is possible to imagine two countries of which one has a higher death rate than the other and where this country optimally allocates a larger part of its labor force to education only to obtain a lower level of knowledge among its population. This shows the relevance of demographic factors for optimal development of a knowledge-based industrial sector, and for optimal educational policies. *Therefore, as long as industrially underdeveloped countries have a higher μ than developed ones, it is, ceteris paribus, optimal for them to have a lower level of knowledge in their work force than developed countries.*[10]

Proceeding as above, it is easily seen that a rise in the relative price of the knowledge-intensive good raises the optimal educational capital and its shadow price, q. From (17) it then follows that fewer people are allocated to sector 1. Since E^{∞} is increased, a greater part of the work-force must be allocated to the educational sector. The effect on L_2 is not clear. Not unexpectedly, the production of the knowledge-intensive goods must rise in response to an increase in its price, since X_1 falls.

This paper is about endogenous technical change. Technical change may also be exogenous as some technological progress in a small country consists in copying new inventions. Exogenous technical progress in the educational sector would mean a steadily rising α over time in (3), which means a steadily increasing E and falling q, so that L_1 and L_2 are increasing and L_3 falling over time. Exogenous technical progress in the knowledge-based sector only is from (11) equivalent to a steadily increasing relative price of the knowledge-intensive good, p. Over time, q and E are then steadily increasing with E approaching $\bar{E}$. A steady increase in E is only possible if the part of the population allocated to education is increasing over time, so *a situation with no technical progress in the educational sector, but with exogenous technical progress in the knowledge-based production sector leads optimally to an even increasing part of the population being employed in the educational sector.*

In periods with rapid technological progress in the knowledge-intensive sector, effects of the latter type may be relevant. In such periods the conclusion in the basic model that when $E_0 < E^{\infty}$ the optimal $L_3(t)$ should decrease over time must be reversed. Such a development for the whole future is hardly optimal since $q(t)$ is steadily increasing, which means that a condition corresponding to 2) in (27) above need not to be satisfied.

5. CONCLUSION

The principal dynamic results from the analysis are:

(1) If, from a given point in time, it is profitable to develop a knowledge-

[10] In a model with population growth, a constant, exogenous growth-rate in population would have the same effect.

based industrial sector, there will be a unique, stationary allocation of the workforce between the three sectors that is optimal. This includes the case where specialization to knowledge-based production is optimal. From a given initial situation this stationary state, where all the variables are constant over time, is reached only asymptotically along a unique optimal path. The optimal stationary allocation of labor between the sectors is independent of the initial conditions. Provided the initial level of knowledge in society is less than the optimal level, the optimal path towards this state has the following properties:

(i) If both goods are produced, the part of the available labor force that is allocated to sector 1, which does not use educational capital in production, should be increasing over time, so that production in this sector is always increasing.

(ii) The part of the labor force allocated to the educational sector should fall over time towards its optimal stationary level, whereas the part allocated to the knowledge-based industrial sector should increase towards its optimal stationary level.

(iii) As a corollary, the lower the initial level of education is, the higher should the initial allocation of labor to the educational sector be, with a correspondingly lower part going to the goods-producing sector.

(2) Conclusion (ii) above may be temporarily reversed in periods with fast technical progress in the knowledge-intensive industry, relative to the educational sector.

(3) It may be optimal not to develop a knowledge-based industrial sector initially. In this phase the economy specializes in producing the "traditional" good while at the same time building up the educational capital. When the level of knowledge has reached a certain level, time is ripe for beginning to allocate labor to a knowledge-based industrial sector as well. During this initial phase the allocation of labor over time has the following optimal pattern: The part going into the educational sector increases over time, so that the part going into the "traditional sector" decreases along the optimal path.

(4) Permanent specialization in production of good 1 may also be optimal. In this model there is then no reason for keeping up an educational sector. If educational capital is initially abundant it may be optimal to have a knowledge-based industrial sector initially even if it would not be optimal for ever. There would then be no activity in the educational sector, while to utilize the existing, but shrinking, educational capital, activity in the knowledge-based industry is phased out over a period until it is finally shut down. After that the economy specializes in good 1.

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RETURNS TO HIGHER EDUCATION IN NORWAY[1]

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Summary

In the first part of this paper, a theory of educational choice is sketched, assuming that the individuals regard the choice of education as an investment decision. Based on earning on the 1st of September, 1966, private returns to 17 types of higher education in Norway, compared to secondary education, are then calculated. The observed pattern of returns is discussed in relation to the theory sketched. Finally, "social" returns are calculated and some comments are made on their relevance for policy purposes.

1. Introduction

This paper has two parts: the first deals with private returns and the second with "social" returns. The article is based on the method pioneered by T. W. Schultz [7] and elaborated on by G. Becker [2].

In two respects, however, this article differs from similar works on returns to education:

(*a*) Whereas in the theoretical part of [2] G. Becker analyzed investment in education mainly from the point of view of the firms, the emphasis in the first part of this article is on the educational decisions of the individuals.

(*b*) The problem in this type of work has usually involved calculating returns to moving from one educational level to another (e.g. from high-school to college). In this paper returns to different *types* of higher education have been calculated.

2. Private Returns

The purpose of this part of the paper is twofold. The first aspect is to answer the matter-of-fact questions of whether there are positive returns to higher education in Norway and whether there are significant differences in returns between the different categories of higher education. The second aspect involves the question of whether the observed pattern of returns may be rationalized economically. In order to answer the second question we need a theory of educational choice based on economics, the implications of which may be tested against the observed pattern of returns.

[1] I am grateful to Agnar Sandmo for valuable comments on the manuscript.

2.1. *Sketch of an Economic Theory of Educational Choice*

Generally, the different characteristics of e.g. education k may be represented by a vector

$$\{V_{jk}, x_{1k}, ..., x_{nk}\}$$

V_{jk} is some measure of individual j's expected lifetime income in educational category k and $x_{1k}, ..., x_{nk}$ are different non-monetary characteristics of going through education k, during and after the educational period. Individual j will choose education k if

$$U_k^j(V_{jk}, x_{1k}, \ldots, x_{nk}) > U_f^j(V_{jf}, x_{1f}, \ldots, x_{nf}); k, f = 1, \ldots, r.$$

(U_k^j denotes individual j's utility from education k). We assume that an individual regards the choice of education as an investment decision. This means that

$$U_k^j(\quad) > U_f^j(\quad)$$

if

$$V_{jk} > V_{jf}.$$

This assumption therefore means that the effects on the choice of education of non-monetary differences between different types of higher education are negligible compared to the effects of differences in lifetime earnings.

Obviously this does not represent the "whole truth" about an educational decision. The purpose is to deduce observable hypotheses from such a behaviour assumption in order to see how far these hypotheses are able to "explain" reality when confronted with the pattern of returns. Assume therefore an individual who, having finished his secondary education, is faced with several educational alternatives including no further education. An expected future age-income profile corresponds to each of these alternatives.

Let

$w_t^{jk}(0)$ = expected income in educational category k for individual j in year t after commencing his education, evaluated at the point in time of calculation 0. ($k=1$ denotes no further education. $k=2, ..., r$ denotes different categories of higher education).

$c^{jk}(0)$ = the corresponding expected private cost of undertaking education (books, fees, etc.).

Individual j's expected *differential* returns from choosing some type of higher education instead of entering the labour market at once will then be given by

$$V^{jk}(0) = \sum_{t=1}^{T} v_t \{(w_t^{jk}(0) - w_t^{j1}(0) - c_t^{jk}(0)\}; \quad \begin{matrix} j = 1, \ldots, s \\ k = 2, \ldots, r \end{matrix}$$

v_t is the discount factor $(1+i)^{-t}$ where i is the subjective rate of discount.[1] Since $V^{jk}(0)$ is a subjective estimate, another individual may expect different returns from the same type of education.

If the individual regards the choice of education as an investment decision,[2] the decision rule for individual j is:

1. If $V^{jk}(0) < 0$ for $k = 2, ..., r$ do not undertake higher education.
2. If $V^{jk}(0) > 0$ for some of the ks choose the alternative with the highest $V^{jk}(0)$.

Observed Age-income Profiles

Let

$w_t^k(0)$ = the average yearly income for persons with education $k (k = 2, ..., r)$ in year t after commencing their education, observed at time 0.

$c_t^k(0)$ = the corresponding cost of education.

$w_t^1(0)$ = the average yearly income for persons with secondary education in year t after entering the labour force.

From cross-section data we can then observe

$$V^k(0) = \sum_{t=1}^{T} v_t \{w_t^k(0) - w_t^1(0) - c_t^k(0)\}; \quad k = 2, \ldots, r$$

$V^k(0)$ may be observed for alternative values of the rate of discount.

Due to the possibilities of substitution between labour of different "vintages" we may assume that all $w_t^k(0)$s depend on the number of persons in educational category k at time 0, denoted by $N^k(0)$. Thus we get

$$w_t^k = f_t^k(N^k(0)), \quad t = 1, \ldots, T$$

For $V^k(0)$ we have accordingly that

$$V^k(0) = f^k(N^k(0))$$

$V^k(0)$ will be a decreasing function of $N^k(0)$ since

$$\frac{dV^k(0)}{dN^k(0)} = \sum_{t=1}^{T} v_t \frac{d(w_t^k(0) - w_t^1(0))}{dN^k(0)}$$

and, assuming decreasing marginal productivity of labour, each term in this sum will be negative.

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[1] The reason why a subjective rate of discount, and not a market determined rate of interest, is used, is that no perfect loan market exists.

[2] It might perhaps be noted if the individual derives no utility from the non-monetary aspects of education, this kind of behaviour is consistent with utility maximization over time, see e.g. Irving Fisher's combined saving and investment model as presented by Sandmo in [5].

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On the Relation between Expected and Observed Magnitudes

It is reasonable to assume that an individual planning to undertake higher education bases his expectations as to future earnings on observed earnings. This can be represented by the expectation functions

$$w_t^{jk}(0) = e_t^{jk}(w_t^k(0)), \quad t = 1, \ldots, T$$

These expectation functions have to be specified in order to establish a relation between expected and observed magnitudes. The simplest alternative would be

$$w_t^{jk}(0) = w_t^k(0) \tag{1}$$

A less far-reaching simplification would be

$$w_t^{jk}(0) = w_t^k(0)\left(1 + \frac{u_j}{100}\right)^t \tag{2}$$

where u_j is the percentage growth in income per year expected by person j.

For the present value of undertaking education k (1) would mean that

$$V^{jk}(0) = V^k(0) \tag{3}$$

whereas (2) would mean that

$$V^{jk}(0) = V^k(0)\left(1 + \frac{u_j}{100}\right)^t \tag{4}$$

assuming the percentage growth in costs per year also equals u_j.

(3) or (4) would be the case for all individuals only if they were identical with respect to the subjective rate of discount, earning capacity and expectations of future growth in income. This is not very reasonable, even as a simplifying assumption.

A more plausible assumption regarding the relation between expected and observed returns would be the following

$$V^{jk}(0) > V^{jf}(0) \quad \text{for all } j$$

if

$$V^k(0) > V^f(0)$$

This means that at time 0 all individuals expect higher future returns in category k than in f if observed returns from cross-section data at time 0 are higher in k than in f. As an assumption especially regarding choice between alternative higher educations this does not seem too unrealistic. Of course there may be individuals who for special reasons expect higher returns in the category with lower observed returns but presumably such individuals will constitute a decreasing minority as the difference $V^k(0) - V^f(0)$ increases.

The Pattern of Returns

We now have

(a) $V^{jk}(0) > V^{jf}(0)$ for all j

if

$V^k(0) > V^f(0)$

(b) The decision rule for all j:
If $V^{jk}(0) < 0$ for all $k(k = 2, ..., r)$, do not undertake higher education.
If some $V^{jk}(0) > 0$, choose k instead of f if

$V^{jk}(0) > V^{jf}(0)$

that is if

$V^k(0) > V^f(0)$

(c) $V^k(0)$ is a decreasing function of $N^k(0)$.

Due to the decision rule education k will attract students as long as $V^k(0) > 0$. Assuming no shifts in demand for labour with different educational backgrounds we should expect the stream of students into the different types of higher education to result in a development where the differential returns for all categories tended towards zero. The speed of adjustment of the "market" will depend on the length of the educational period and on how free the choice of education is.

As regards the last question we may distinguish between the case with excess supply of all types of higher educational services so that choice of education is perfectly free, and the case with excess demand for all or some of the types of higher education so that choice has to be restricted in some way.

Case I: The individual is now a "quantity-adjuster" in the sense that he can choose freely between a number of $V^{jk}(0)$ which his own decisions will not affect noticeably.

Assume that the length of the educational period in category k is θ and that the demand curve for this type of educational skill is unchanged over time. If now initially, $V^k(0) > 0$ greater and than all other $V(0)$, N^k will increase so that at point in time $0 + \theta$ it will be equal to $N^k(0 + \theta)$. As a result V^k decreases to $V^k(0 + \theta)$. We therefore have

$N^k(0 + \theta) > N^k(0)$

$V^k(0) > V^k(0 + \theta)$

If we denote the point in time $(0+\theta)$ θ_1 we will in θ_1 have a new

$w_1^k(\theta_1), w_2^k(\theta_1), \ldots, w_n^k(\theta_1)$

and a corresponding new $V^k(\theta_1)$. If still $V^k(\theta_1)>0$ and greater than all other $V(\theta_1)$ we will have a new increase in N^k from $N^k(\theta_1)$ to $N^k(\theta_2)$, where again $\theta_2=\theta_1+\theta$. A new $V^k(\theta_2)<V^k(\theta_1)$ will correspond to $N^k(\theta_2)$ and so on as the process continues.

The decision rule will imply that

$N^k(0)$ stays constant as long as $V^k(0)=0$
$N^k(0)$ increases as long as $V^k(0)>0$
$N^k(0)$ decreases as long as $V^k(0)<0$

0 must be regarded here as an arbitrary "running" point in time of calculation. During the process the different educational categories will continually change place with one another in the "returns hierarchy" and the process will go on until observed average differential returns are (approximately) equal and equal to zero for all k.

Assume that this happens at time θ_L, i.e. that

$$V^k(\theta_L)=0; \quad (k=1, \ldots, r) \tag{5}$$

A situation characterized by (5) may then be called an equilibrium situation. Since

$$V^k(\theta_L)=\sum_{t=1}^{T} v_t\{w_t^k(\theta_L)-w_t^1(\theta_L)-c_t^k(\theta_L)\}=0 \text{ for all } k$$

an equilibrium situation implies that the present value of lifetime earnings in all higher educational categories must equal the present value of lifetime earnings without higher education plus the private costs of undertaking higher education.

Case II. We now have excess demand for all or some types of higher education. The excess demand may be temporary in the sense that the net inflow into an educational category is sufficient to increase the number in the category to the equilibrium level. But in this case it is reasonable to assume that the duration of the equilibrating process will be longer than in the case with excess supply. The point is, however, that as long as the supply restrictions are effective they will imply positive differential returns in the category.

Excess demand may also be permanent if the net inflow into a category is too small to increase the number in the category to the equilibrium level. Thus excess demand will be permanent.

This means that excess demand for some types of higher education, be they temporary or permanent, implies positive observed differential returns in these educational categories. It also means that no equilibrating mechanism exists in this case to even out returns between these educational categories.

Hypotheses: From an investment point of view we get the following hypoteses on private returns to higher education: (1) Observed average differential returns to all types of higher education with excess supply will be equalized and equal to zero in equilibrium. (2) Observed average differential returns to all types of higher education with excess demand will be positive and will not be equalized.

What are the possibilities of testing such hypotheses in the "market"? These hypotheses are based on static equilibrium conditions assuming an unchanged demand curve for different educational skills. The demand for educational services may of course also adjust to changes in expected returns due to shifts in demand for educational skills. But to the extent such shifts make expected returns more unstable, it will be more difficult to *reach* the equilibrium situation, even if the "market" always tends towards it. For this reason and due to the rather long "production period" in higher education it may be doubtful whether an equilibrium situation will prevail in the categories with free entry at the point in time of observation. On the other hand: The supply-conditions within higher education in Norway are stable in the sense that the fields of study to which entry is restricted to-day have had excess demand during the whole post-war period. Therefore it seems that the returns to education in different educational categories have stabilized on or fluctuates around different levels according to whether excess supply or excess demand prevails.

2.2. *Data and Method*

Data on Incomes and Costs

Incomes on September 1st, 1966, in different higher educational categories according to age and education, including employees with secondary education in banking and insurance, have been gathered from official publications on wage-statistics as well as from the earnings statistics of different professions. Since the age-income profiles are very space-consuming, they are omitted here.[1] Problems regarding differing earnings concepts, representativity and corrections so that all incomes refer to 1.9.1966 are all discussed in [1], pp. 26–38. The profiles are based on mens' earnings only. The two categories with secondary education are included to provide information on earnings foregone when studying and to represent the alternative income without higher education. Since no information on incomes during the educational period exists, they are disregarded. Grants and/or loans are also disregarded.

Incomes foregone while studying are directly observable from the age-

[1] They may be found in [1], Table 3, p. 34.

income profiles. Private costs for books, fees, etc. in higher education are omitted here due to lack of space.[1]

Some Problems of Method

It is well known that returns to education may be expressed either as a rate of return or as a present value. We will remark briefly on the well-known question of which concept is the more fruitful when commenting on the results. At this stage it suffices to say that both rates of return and present values have been computed. As to the choice of discount-rate, the main point is that without a perfect loan market lifetime income cannot be discounted using a market rate of interest (see e.g. Sandmo [5].) The rate of discount will depend on each individual's subjective rate of time preference, and the discount rate will vary positively with the strength of preference for consumption in the educational period. The present values have been computed for alternative discount rates of 4, 6 and 8 %. The reason why these values of the rate of discount have been chosen is discussed in detail in [1], pp. 40–44.

An observed age-income profile from cross-section data to-day will differ from a future age-income profile starting to-day due to increasing real income per capita over time. Returns calculations assuming a growth in real income per capita of 3 % per year are therefore also presented. The figure 3 % is based on a growth in NNP of 4 % per annum, a growth in population of 1 % and unchanged relations between the returns to different educations.

Uncertainty may enter the expected returns with respect to the length of the educational period, the drop-out possibility and also with respect to the dispersion of earnings within a profession. The data used for this article did not allow calculation of any measure of these types of uncertainty. The returns figures are therefore based on graduation at normal time and on the arithmetic mean of incomes in all age groups within each profession.

2.3. *Results and Comments*

Average differential returns to certain types of higher education before and after taxes are shown in Tables 1 and 2.[2] The returns are expressed either as a rate of return or as a present value calculated for alternative discount rates of 4, 6 and 8 %. The returns are calculated either directly from cross-section data or assuming a future rate of growth in income per capita of 3 % per year. Returns are given separately for privately and publicly employed where this information is available.

Some comments on the tables:

(*a*) Two figures are given in the column for the rate of return. They may be regarded as the upper and lower limit for the rate of return in the category. The upper limit means differential returns relative to the category "secondary

[1] May also be found in [1], Table 4, p. 39.
[2] For the way taxes have been computed, see [1], pp. 45–46.

Table 1. *Private average differential returns to higher education, before taxes, per 1.9.1966*

Educational category	Observed from cross-section data: Rate of return	Observed: Present value at alternative discount rates 4 %	6 %	8 %	Assuming a rate of growth of 3 % per year in real income per capita: Rate of return	Assuming growth: Present value at alternative discount rates 4 %	6 %	8 %
Arts, higher degree	2.6/6.9	71 518	15 926	−15 525	5.7/10.1	257 225	118 964	43 762
Arts, lower degree	—/5.7	22 820	−2 540	−17 940	—/8.9	99 551	43 365	10 378
Science, higher degree	5.7/9.4	134 309	61 115	18 903	8.9/12.7	376 315	196 288	97 890
Science, lower degree	1.7/9.9	81 631	39 856	14 799	4.8/13.2	214 068	116 132	61 030
Teachers' college	—/—	−20 394	−12 024	−9 070	—/—	−62 584	−29 536	−15 827
Law (private)	8.4/11.7	213 514	111 153	53 003	11.7/15.0	559 463	301 245	162 379
Law (public)	—/5.1	23 442	−11 743	−30 028	1.7/8.3	155 270	55 429	5 505
Economics	6.3/10.8	136 936	69 116	30 275	9.5/14.1	368 908	195 233	103 064
Medicine (public)	3.9/7.5	95 120	28 352	−7 540	7.0/10.7	337 387	154 655	61 320
Dentistry (private)	21.1/25.0	387 224	274 037	193 155	24.8/28.9	725 727	488 363	337 419
Dentistry (public)	8.7/15.3	101 252	69 177	45 705	12.0/18.8	174 187	123 487	86 299
Agricultural sciences (private)	—/13.5	86 399	53 339	31 335	—/16.9	175 002	111 367	70 558
Business administration	16.5/20.1	413 934	251 142	156 702	20.0/23.8	956 524	552 067	332 923
Engineering (private industry)	9.2/12.9	224 318	121 439	62 779	12.5/16.3	547 258	312 713	172 899
Engineering (state)	8.1/10.9	122 580	64 136	29 134	11.4/14.3	306 784	170 738	93 768
Engineering (private other than industry)	10.5/14.3	244 847	140 216	78 882	13.9/17.8	585 240	332 584	192 969
Engineering (municipality)	9.4/14.2	175 683	103 248	59 400	12.7/17.6	403 122	235 147	140 038

Source: [1], p. 50.

Table 2. *Private average differential returns to higher education, after taxes, per 1.9.1966*

Educational category	Observed from cross-section data				Assuming a rate of growth of 3 % per year in real income per capita			
	Rate of return	Present value at alternative discount rates			Rate of return	Present value at alternative discount rates		
		4 %	6 %	8 %		4 %	6 %	8 %
Arts, higher degree	0.6/4.8	12 037	−14 703	−29 660	3.6/7.9	101 321	34 900	−1 333
Arts, lower degree	—/3.9	−1 289	−14 904	−23 092	—/7.0	40 289	9 773	−7 980
Science, higher degree	3.5/7.1	48 089	12 429	−8 193	6.6/10.3	164 750	78 134	51 772
Science, lower degree	—/7.7	33 793	11 777	−1 573	2.1/10.9	102 332	51 800	36 021
Teachers' college	—/—	−15 132	−10 229	−8 528	—/—	−39 798	−20 494	−12 453
Law (private)	5.9/9.1	84 492	37 147	10 070	9.1/12.4	243 176	124 872	60 881
Law (public)	—/3.6	−5 211	−22 263	−31 091	—/6.7	58 037	10 233	−13 900
Economics	4.1/8.5	56 855	23 170	3 529	7.2/11.7	168 819	85 368	40 114
Medicine (public)	2.0/5.5	25 368	−6 369	−23 512	5.1/8.6	138 632	53 443	9 336
Dentistry (private)	15.0/18.6	172 788	115 338	77 335	18.4/22.2	330 269	216 626	145 184
Dentistry (public)	5.6/9.8	48 422	25 330	9 972	8.8/13.1	112 374	66 079	37 330
Agricultural sciences (private)	—/11.5	41 691	24 400	12 620	—/14.8	85 735	54 431	33 457
Business administration	12.3/15.7	171 623	101 584	60 237	15.7/19.2	399 747	230 292	136 929
Engineering (private industry)	6.5/10.0	90 496	43 220	16 107	9.7/13.3	248 590	32 973	45 318
Engineering (state)	3.3/8.4	49 435	20 030	2 377	6.4/11.6	141 190	73 553	34 957
Engineering (private other than industry)	7.5/11.1	101 348	52 989	24 394	10.7/14.4	256 344	141 579	77 431
Engineering (municipality)	6.2/10.8	75 676	39 826	17 972	9.4/14.1	186 486	104 875	58 076

Source [1], p. 51.

education employed in banking", and the lower limit means relative to the category "secondary education employed in insurance'. Present values are only given for the upper limit in order to simplify the presentation. Doing so means that returns to higher education will perhaps be a bit exaggerated, but the relation between the returns to the different categories of higher education will not be affected.

(*b*) "—" in the rate of return columns means that no positive rate of return exists. In all cases where the rate of return is positive it is unique since there is only one sign change in the accumulated income streams. From the tables we see that a ranking according to returns between the different educational categories sometimes differs depending on whether rates of return or present value at alternative discount rates are used. An example of this is the category "science", where the rate of return is higher for the lower degree, while the present value for all discount rates used is higher for the higher degree. This shows that it is meaningless to speak of the "returns to education" without specifying the rate of discount, i.e. how future income is evaluated relative to income to-day. When this evaluation (the discount rate) is given, the rate of return will only tell whether the returns to an education is positive or negative. The present values just be used to obtain a ranking between the alternatives, since a larger sum of money to-day is always preferred to a smaller.

Interpretation of the Results

1. Are there positive private returns to higher education in Norway?

Using a discount rate of 6 %, Table 1 shows that present values before taxes from cross-section data are positive for all categories except Arts (lower degree), Law (public employment) and teachers' college. Calculating with an expected growth in real income per capita of 3 % per year all categories except teachers' college will have a positive present value before tax even using a discount rate of 8 %. After tax this will be the case when using a discount rate of 6 % (Table 2).

Returns after taxes including probable future increase in incomes would seem to be the concept of greatest interest to an investor in education. Thus it is fair to say that with a reasonable discount-rate (of 6 %) it is profitable to go through the types of higher education we have examined, except teachers' college.

But the tables show that there are great differences in return between the different types of higher education.

2. The Pattern of Returns

The question here is whether the pattern of returns is compatible with the hypotheses from the theory sketched at the beginning of this article.

(*a*) Are observed differential returns to all types of higher education with free entry equal and equal to zero? Strictly speaking, this question is difficult to answer because the individual discount rate is subjective. We must therefore simplify and choose a discount rate equal for all individuals.

Studies with free entry are Arts, Sciences, Law and Economics.[1] From the tables it is obviously difficult to find support for the hypothesis that the individuals have chosen education in such a way that returns between these educations have been evened out. This is so regardless of which discount rate is used. Even if measures of significance are lacking in this primitive "test" we may legitimately conclude that our calculations do not support the hypothesis that returns to studies with free entry should be approximately equal. On the other hand we have found positive returns to these studies which should indicate that none of them has been so attractive *per se* that the monetary returns have been pressed below a reasonably profitable level.

(*b*) Are the observed differential returns to all types of higher education with restricted entry positive and greater than the returns to the types with free entry?

The answer here is yes, with some minor qualifications, depending on the way of ranking. We have already mentioned the studies with free entry. Entry to the others listed in Tables 1 and 2 are restricted.[2] Disregarding for the moment teachers and medical doctors, a ranking based on the rate of return from cross section data before taxes in Table 1 shows that *all* categories from studies with restricted entry top the list, the only exception being lawyers in private service (in the 8th place). The same way of ranking after tax would add the category "Economics" to the exceptions. A ranking according to present values at discount rates of 6 or 8 % from cross-section data before or after taxes gives similar results. The inclusion of a 3 % rate of growth in incomes per year does not change the picture noticeably.

Returns from the types of higher education that traditionally have had restrictions on entry are markedly higher than returns from studies with free entry. This result is unquestionable.

An apparent exception is Medicine, but the income concept used in the statistics for this profession probably underestimates the earnings so that this category is probably no exception to the rule. This question is discussed in [1], pp. 64–65.

The difference in returns between studies with restricted and free entry would have been even greater were it not for the considerable number of engineers, business graduates, dentists and doctors that complete their studies abroad and return to Norway every year. If this additional supply had not existed, pay and returns to education in these categories had been even higher.

[1] Free entry means that every one who has passed the matriculation examination may begin studying these subjects.
[2] This means that only a fixed number of those wishing to begin studying these subjects are admitted. Admission is usually based on the marks obtained in the matriculation examination.

The most paradoxical result from the calculations is the negative returns to teachers' training. The returns are negative at all discount rates, but demand for this type of education is far in escess of supply. One possible reason for this popularity may be the fact that demand is dominated by women, who find that economic prospects are favourable for them in teaching, relative to other possible occupations, since the difference in pay between men and women are less in teaching than in most other occupations. (More on this question in [1], pp. 68–70.) A similar pattern is found in Germany [6].

2.4. *Conclusions*

We have found that going through all types of higher education with free entry in Norway is profitable at a "reasonable" rate of discount, but the calculations do not support the hypothesis that returns are "evened out" between these educational categories. A possible interpretation of this might be that there is no reason for believing that the choice between studies with free entry is made solely on economic grounds. On the other hand the positive returns to all these studies might indicate that the choice whether or not to undertake higher education is based on a profitability calculation. In any case these studies have not had such an attraction *per se* that the returns have been pressed below a reasonably positive level.

There are markedly higher returns to studies where entry is restricted than to studies with free entry. This difference is compatible with the assumption that the individuals choose the type of education with the highest economic returns. (This means *in general;* of course there are individual exceptions to the rule.) One might ask whether such a difference in returns between studies with free or restricted entry could have come into existence for reasons other than the one we have assumed.

Suppose therefore that the choice of education were not an investment decision. An education with restricted entry would still be a study where demand for educational services was in excess of supply. But the demand would now depend on the "utility" of the study in question. Since there would be no a priori connection in this case between the total demand for an education and returns from this education, a study with restricted entry might imply negative as well as positive returns, and the same would be the case for studies with free entry. Thus, without assuming that the choice of education is an investment decision it is impossible to say that studies with restricted entry will imply greater returns than studies with free entry.

3. Social Returns

So far we have attached no social significance to the returns from higher education. The question is now whether returns calculations similar to those already made may be of some help in the problem of allocating resources optimally to and within the educational sector. In other words: Is it possible to establish criteria for where and how much to invest in higher education?

Such criteria must be derived from some social welfare function. Although rarely made explicit, the social welfare function usually postulated when calculating social returns to education, is

$$W(0)=\sum_{k=2}^{r}\sum_{t=1}^{T} v_t\,[w_t^k(N^k(0))\,N^k(0)-c_t^k\,N^k(0)] \tag{6}$$

This means that the educational policy at the point in time of calculation zero should aim at maximizing the expected total net present value from higher education.

Here v_t is the discount factor, now based on the social rate of discount; w_t^k is expected average *differential* income before taxes in educational category k (compared to secondary education) in year t after zero, and c_t^k is the expected average social cost of education k in year t, assumed constant. $N^k(0)$ is the number of persons with education k at time zero.

The rationale for this welfare function is that earnings express social benefits and that monetary costs express real resource costs.

$W(0)$ is maximized when

$$\sum v_t\left[w_t^k+N^k(0)\frac{dw_t^k}{dN^k(0)}-c_t^k\right]=0, \quad k=2,\ldots,r \tag{7}$$

This means that at the "running" point of time of calculation zero we have an optimal number of persons in an educational category when the expected social net present value from the marginal person in all higher educational categories equals zero. This investment criterion therefore says that the expected social marginal differential returns should equal the expected social marginal cost in all educational categories. Due to measurement problems the second term in the brackets is usually ignored. We then get

$$\sum v_t\,[w_t^k-c_t^k]=0$$

or

$$\sum v_t\,w_t^k=\sum v_t\,c_t^k$$

This means that the present value of expected average differential earnings should equal the present value of expected average social costs in all higher educational categories. Since measurement is possible in this case, this is the criterion used. In practice the investment criterion therefore simplifies to: Invest in those categories where the present value of expected future differential incomes (judged from cross section data to-day) exceed the present value of expected future costs.

Table 3. *Public expenditures per student per year in 1966 (norw. crowns)*

Teacher's colleges	7 500
University of Oslo	
Faculty of Law	2 600
Faculty of Arts	3 800
Faculty of Science	9 900
Faculty of Medicine	25 500
Faculty of Dentistry	26 300
Faculty of Social Sciences	4 300
Norwegian School of Economics and Business Administration	7 400
The Technical University of Norway	13 500
The Agricultural College of Norway[a]	46 800

Source: [1], p. 105.

[a] The part comprising research expenditures in this figure is extraordinarily large.

Data and Results

Earnings and costs are obtained in the same way as before, by observing them from cross-section data and correcting for expected future income growth. The only difference between a calculation of private returns before taxes and a calculation of social returns is that the costs not borne by the individual undertaking education have to be added to private costs of education. To this end public expenditure per student per year in Norway in 1965 is presented in Table 3.

From Table 3 we see that public expenditures per student vary considerably between the different types of higher education. Adding these costs to private costs of education gives the best possible estimate obtainable on social costs of higher education in Norway.

Using this cost concept, "social" returns to higher education have been calculated. The results are presented in Table 4. (In this table rates of return are presented for the upper limit only.)

The ranking according to returns in this table is somewhat different from the ranking in Tables 1 and 2. The reason is, of course, the considerable differences in social costs between the educational categories, caused by the introduction of public expenditures on higher education. The relation between high returns and studies with restrictions on entry is not so clear in this case as in the case of private returns. But the important question in connection with Table 4 is: Can we attach any normative significance to the returns figures in the table?

Some Comments on Policy

It is well-known that calculations of "social" returns to education have been severely critizised. The critique falls naturally into two parts:

(*a*) Questioning whether the economy functions in the way necessary for identifying earnings with social benefits and monetary costs with real resource costs, and

Table 4. *Average "social" differential returns to higer education, per 1.9.1966*

Educational category	Observed from cross-section data: Rate of return	Present value at alternative discount rates: 4 %	6 %	8 %	Assuming a rate of growth of 3 % per year in real income per capita: Rate of return	Present value at alternative discount rates: 4 %	6 %	8 %
Arts, higher degree	5.7	47 798	−6 560	−36 892	8.9	231 381	94 519	20 584
Arts, lower degree	4.3	5 227	−19 507	−34 326	7.5	81 312	25 410	−6 942
Science, higher degree	6.4	80 337	9 514	−30 525	9.6	318 325	140 936	44 955
Science, lower degree	6.2	44 258	3 493	−20 614	9.4	175 036	78 181	24 096
Teachers' college	—	−35 106	−26 600	−23 515	—	−77 512	−44 324	−30 480
Law (private)	10.6	201 476	99 544	41 791	14.0	546 711	288 961	150 528
Law (public)	4.5	11 404	−23 352	−41 229	7.7	142 517	43 145	−6 346
Economics	8.9	117 028	49 916	11 733	12.2	347 818	174 917	83 465
Medicine (public)	3.1	−43 910	−104 572	−134 861	6.2	188 099	12 072	−75 035
Dentistry (private)	12.6	296 598	177 437	99 078	16.0	622 035	387 546	239 302
Dentistry (public)	5.0	22 728	−17 553	−43 519	8.2	138 480	54 200	3 232
Agricultural science (private)	2.2	−42 773	−71 068	−88 597	5.5	38 008	−20 525	−56 452
Business administration	16.6	391 123	229 414	135 947	20.1	931 896	528 649	310 600
Engineering (private industry)	8.7	169 771	69 621	13 694	12.0	515 117	256 585	119 545
Engineering (state)	6.6	67 960	12 249	−20 232	9.8	247 561	114 534	40 344
Engineering (private other than industry)	9.6	190 227	88 330	29 516	12.9	526 018	276 380	139 545
Engineering (municipality)	8.7	121 132	51 425	10 093	12.0	343 976	179 015	86 681

Source: [1], p. 107.

Table 5.

Downward bias	Upward bias
1. Positive marginal external effects 2. Departures from perfect competition 3. Education also a consumption good	4. Average return greater than marginal return 5. "Returns to education" do not represent returns to education alone

(*b*) questioning whether the "returns to education" really are returns to education alone, and not also to a number of other factors.

This is not the place to repeat and evaluate the critique in detail,[1] but the following classification shows how the main points of criticism, if justified, would tend to bias the estimates of "social" returns given in Table 4.

Some very brief explanations of the points in Table 5:

1. If positive marginal external effects exist in an educational category, personal earnings will not measure the full social benefits from the education, and hence the returns estimates in Table 4 will be biased downwards.

2. If we regard a person with a special education (e.g. an engineer) as a factor of production, the price of the factor will, in equilibrium, be equal to the value of its marginal product only if the economy is perfectly competitive. Departing from perfect competition in the product and/or factor market, the price of the factor will be lower than the value of its marginal product. The earnings will in such cases not reflect the full social benefits from this type of education so that also for this reason the figures in Table 4 will be biased downwards.

3. If a type of education were also a consumption good, people would be willing to pay for this type of education without getting any monetary returns, and consequently the figures in Table 4 would be biased downwards for this reason as well.

4. Ignoring the second term in (7), i.e. basing the returns figures on average instead of marginal returns gives the figures in Table 4 an upward bias.

5. Obviously higher education is not the only factor that affects personal earnings. If there is a positive relation between the level of education and some of these other factors (as ability, parental status, race and sex) the figures in Table 4 will be biased upwards.

It should be obvious by now that the figures in Table 4 cannot simply be taken as the social returns to education; each of them has to be evaluated in the light of the objections summarized in Table 5. Thus every statement on the social returns to any investment in education will be disputable. Therefore the reader is left to draw his own conclusions from Table 4. The value of

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[1] Merret [4] is representative of the criticism, whereas Blaug [3] contains a defence for "social" returns calculations.

the figures in that table is, in my opinion, that they may form a precise point of departure for such discussion, not that they give a definite answer.

Whether the figures in Table 4 are accepted as a point of departure for discussions on the social returns to educational investments depends of course on whether it is accepted that the objective of the educational policy should be to maximize the welfare function (6). To construct welfare functions for the educational policy other than (6) is no simple task. Other possible aims of educational policy besides economic efficiency are

(*a*) equalization of educational opportunities,
(*b*) "self-realization" for the individuals in the educational system,
(*c*) free choice of education, and
(*d*) that the educational system should have a critical function in society.

We shall not attempt to solve the problem of how these targets might enter a more complex welfare function for the educational sector together with the efficiency target. Note, however, that if the efficiency consideration enters into a more complex welfare function, a policy which maximizes solely with respect to the efficiency variable will in general not lead to a maximum of the more comples welfare function.

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PART II

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Optimal Savings and Exhaustible Resource Extraction in an Open Economy

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1. INTRODUCTION

Economic analysis of natural resources has traditions going back to Malthus and Ricardo. The basis for the modern treatment of the best use of an exhaustible natural resource is the classical article by Hotelling [3]. Lately the problem has also been analyzed from a macroeconomic point of view by, e.g., Koopmans [5] and Vousden [8]. In these models either all consumption in the economy is provided from the resource, or an additional source of consumption, outside the model, is postulated. The assumption of no alternative sources of consumption is extreme and unrealistic. The assumption of an alternative, exogenous source of consumption has been introduced by Vousden in [8] as "a convenient simplification of the relevance of the rest of the economy to the resource-use decision." But except for foreign aid, and the most primitive subsistence agriculture, the time path of the alternative consumption will depend on the stock of physical capital in the economy and the savings ratio together with the growth in labor supply and technological progress. It seems reasonable to think that, e.g., the optimal savings ratio will depend on the availability of natural resources in the economy. On the other hand it does not seem reasonable to assume that the optimal path of resource depletion will be completely independent of, e.g., the stock of physical capital in the economy and the resulting potentiality for consumption from sources other than the current resource extraction. This shows the need for an integrated model of the economy where optimal savings and resource extraction can be determined *simultaneously*. The purpose of this paper is to present and analyze such a model for a small, open economy. The interrelationship between the optimal rate of investment and the optimal depletion of natural resources is explored by Heal and Dasgupta [2]. However, their model is rather different from the following model, which has the recent petroleum discoveries in the North Sea as its background. We consider an open economy where the resource good is exchanged for other goods in the

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world market. The problem of optimal extraction of a nonrenewable resource in an open economy has earlier been analyzed by Vousden [9], Kemp and Suzuki [4], Van Long [7], and Strøm [6]. Common to all these contributions is the fact that there is no physical capital accumulation in their models, so that the problem of determining the optimal accumulation of capital, together with resource extraction, disappears. In this paper the central issues therefore are:

(1) What is the optimal intertemporal pattern of physical capital accumulation in an open economy with a considerable stock of an exhaustible resource?

(2) What is the optimal intertemporal pattern of extraction of this resource?

(3) How are the decisions implicit in (1) affected by conditions in the resource sector?

(4) How are the decisions implicit in (2) affected by conditions in the rest of the economy?

2. The Model

The following variables are used:

$c(t)$ Total consumption per capita
$\bar{c}$ An exogenous source of consumption
$v(t)$ Resource extraction per capita
$s(t)$ The (average) savings ratio
$k(t)$ Physical capital per capita
$f(k)$ Production per capita, exclusive of resource extraction
$p(t)$ The price of the resource relative to the "price" of other goods in the world market
$b(t)$ Total extraction costs per capita
U Social welfare
ρ The social rate of discount (constant)
n The rate of growth in total population
μ The rate of depreciation of physical capital
$x(t)$ The stock of the resource per capita
$\Pi(t)$ Net proceeds per capita from resource extraction

The problem is then

$$\max \int_0^\infty U(c(t))\, e^{-\rho t}\, dt$$

s.t.

(i) $c(t) = (1 - s(t))[f(k(t)) + \Pi(t)] + \bar{c}$,
(ii) $\dot{k}(t) = s(t)[f(k(t)) + \Pi(t)] - \lambda k(t)$,
(iii) $\Pi(t) = p(t)\, v(t) - b(v(t))$,
(iv) $\lambda = n + \mu$,
(v) $-\dot{x}(t) = v(t) + nx(t)$,
(vi) $0 \leqslant s(t) \leqslant 1$,
(vii) $0 \leqslant v(t) \leqslant \bar{v}$,
(viii) k_0, x_0 given,
(ix) $\lim_{t\to\infty} x(t) \geqslant 0$, $\lim_{t\to\infty} k(t)$ free,
(x) ρ, n, μ, $\bar{c}$, and $p(t)$ exogenously given.

Stated in words, the problem posed is to find such paths over time for resource extraction and total savings that the present value of total social welfare is maximized. The planning horizon is infinity. Instantaneous welfare depends on consumption per capita, and we assume that $U' > 0$ and $U'' < 0$. Total population is assumed to grow at the same rate as the labor force. Consumption per capita is given by (i), where Π is defined in (iii). All net earnings from resource extraction are used for import, so that the current account is always balanced. $p(t)$ is assumed to be independent of the amount exported ("small country" argument). Relation (ii), the expression for the increase in capital intensity, is familiar from ordinary growth theory. (iii) expresses net earnings from resource extraction, where $b' > 0$ and $b'' \geqslant 0$. (v) says that the stock of the resource per capita is reduced by the extraction per capita v, and is also diluted by nx because of the growth in population. By (vi) s must be nonnegative and it cannot exceed one. (vii) says that the resource extraction is irreversible and that there is some upper bound $\bar{v}$ on extraction per unit of time, due to, e.g., limited pipeline capacity or limited loading capacity for tankers at the production platforms, caused by climatic and/or geographical conditions. In addition to the assumption of a balanced current account, the structure of the model above also assumes:

(a) No search activity for new resources.

(b) No uncertainties. In particular the future relative price of the resource is assumed known.

(c) The stock of the resource does not affect social welfare or the extraction conditions, except that it restricts total extraction.

(d) External effects are disregarded. Examples might be pollution due to oil spill, blowouts, reduced fishing possibilities, or the fact that two (or more) countries are extracting petroleum from the same reservoir.

(e) The producing country does not use the resource as an input.

Assumptions (a) to (e) are not trivial. Still, this model contains aspects from "real life" not found in any of the contributions quoted above.

3. Optimality Conditions and Policy Regimes

To analyze the problem, form the (present value) Hamiltonian function

$$H = e^{-\rho t}\{U[(1 - s(t))(f(k(t)) + \Pi(t)) + \bar{c}] + q_1(t)(s(t)[f(k(t)) + \Pi(t)] - \lambda k(t)) - q_2(t)(v(t) + nx)(t))\}, \quad (1)$$

where we have substituted for c in U from (i). $q_1(t)$ and $q_2(t)$ are the so-called co-state variables, associated with $k(t)$ and $x(t)$, respectively. According to Pontryagin's maximum principle, a solution to the problem posed must satisfy the following necessary conditions[1] :

(a) There exist continuous functions of time, q_1 and q_2, such that

$$\dot{q}_1 = -U'(1 - s)f' - [sf' - (\rho + \lambda)]\, q_1\,, \quad (2)$$
$$\dot{q}_2 = (\rho + n)\, q_2\,. \quad (3)$$

(b) For all t, H is maximized in s and v, so that

$$\begin{aligned} s(t) &= 1 && \text{if } q_1(t) > U', & v(t) &= \bar{v} && \text{if } q_2(t) < q_1\Pi' \\ &= \epsilon[0, 1] && \text{if } q_1(t) = U', & &= \epsilon[0, \bar{v}] && \text{if } q_2(t) = q_1\Pi' \\ &= 0 && \text{if } q_1(t) < U'; & &= 0 && \text{if } q_2(t) > q_1\Pi' \end{aligned} \quad (4)$$

A number of policies are therefore available to the economy; see Table I.

Regimes G, I, and J are of limited economic relevance and will not be referred to any more.

Equation (3) yields

$$q_2(t) = q_{2,0}e^{(\rho+n)t}; \qquad q_{2,0} > 0.^2 \quad (3')$$

q_2 is the shadow price of the resource, so that $q_2(t)$ denotes the addition to the optimal value of the criterion function of leaving the marginal unit of the resource at t unexploited. q_1 is the shadow price of physical capital per worker. By (4), if the solution for v is interior, the shadow price of the resource, q_2, is equal to the marginal proceeds from extraction times the shadow

[1] Such a solution will be really optimal, since (a) the Hamiltonian is concave in k, x, s, and v for given $q_1(t)$, $q_2(t)$, and t, and (b) it is shown later that $x(t)$ will be exhausted in finite time and that $q_1(t)$ and $k(t)$ approach finite limits as t goes to infinity.

[2] $q_{2,0}$ is of course not an exogenously given constant—it is determined in the optimization process. $q_{2,0} < 0$ is disregarded as economically uninteresting.

TABLE I

Regimes	Variables				
	s	v	c	q_1	q_2
A	1	$\bar{v}$	$\bar{c}$	$> U'$	$< \Pi'(\bar{v})q_1$
B	1	$\in [0, \bar{v}]$	$\bar{c}$	$> U'$	$= \Pi'(v)q_1$
C	1	0	$\bar{c}$	$> U'$	$> \Pi'(0)q_1$
D	$\in [0, 1]$	$\bar{v}$	$(1 - s)(f(k) + \Pi(\bar{v})) + \bar{c}$	$= U'$	$< \Pi'(\bar{v})q_1$
E	$\in [0, 1]$	$\in [0, \bar{v}]$	$(1 - s)(f(k) + \Pi(v)) + \bar{c}$	$= U'$	$= \Pi'(v)q_1$
F	$\in [0, 1]$	0	$(1 - s)(f(k)) + \bar{c}$	$= U'$	$> \Pi'(0)q_1$
G	0	$\bar{v}$	$f(k) + \Pi(\bar{v}) + \bar{c}$	$< U'$	$< \Pi'(\bar{v})q_1$
I	0	$\in [0, \bar{v}]$	$f(k) + \Pi(v) + \bar{c}$	$< U'$	$= \Pi'(v)q_1$
J	0	0	$f(k) + \bar{c}$	$< U'$	$> \Pi'(0)q_1$

price of capital. For an interior solution for s, this shadow price is equal to the marginal utility of consumption per worker. In that case, the shadow price of the resource is equal to the marginal utility of resource extraction in terms of consumption per worker $\Pi' U'$. Given k, q_1, and q_2, (4) determines the optimal $s(t)$ and $v(t)$ so that total consumption and capital-accumulation are determined. Let us study each policy in turn, assuming for the moment that $p(t)$ is a constant p.

3.1. *The Interior Solution* (*Regime E*)

Since $q_1 = U'$ in E, it follows from (2) that

$$\dot{q}_1 = (-f' + \rho + \lambda)\, q_1 \tag{5}$$

so that

$$\begin{aligned} \dot{q}_1 &> 0 \quad \text{if } k > k^*, \\ &= 0 \quad \text{if } k = k^*, \\ &< 0 \quad \text{if } k < k^*, \end{aligned} \tag{6}$$

where k^*, the modified golden rule capital intensity, is uniquely defined by $f'(k^*) = \rho + \lambda$, *irrespective of conditions in the resource-extraction sector.* Differentiating $q_1 = U'$ with respect to t, and using (5),

$$\dot{c} = (U'/U'')[-f' + \rho + \lambda]. \tag{7}$$

(7) is familiar from optimal growth theory and says that along the optimal path consumption is increasing (decreasing) as long as the capital intensity of the economy is below (above) the optimal steady state capital intensity k^*,

defined above. Regarding resource extraction, differentiating $\Pi'(v)\, q_1 = q_2$ with respect to time and solving for $\dot{v}$, we obtain

$$\dot{v} = (1/q_1\Pi'')(\dot{q}_2 - \Pi'\dot{q}_1).$$

Inserting for $\dot{q}_2$ and $\dot{q}_1$ from (3) and (5), respectively, and using (4), this simplifies to

$$\dot{v} = (\Pi'/\Pi'')(f' - \mu). \tag{8}$$

Due to the assumptions on the cost function, (8) says that, along the optimal path, resource extraction should decrease (increase) as long as k is less (greater) than $\hat{k}$, where $\hat{k} > k^*$ is defined by $f'(\hat{k}) = \mu$. In addition to the cost structure in the extraction sector *the optimal extraction path also depends on the capital intensity of the economy.* If the economy is growing ($k < k^*$), the shadow price of capital (equal to the marginal utility of consumption) is falling and the user cost of the resource is increasing, both contributing to a falling optimal path of resource extraction. In a contracting economy ($k > k^*$), the shadow price of capital (equal to the marginal utility of consumption) is increasing, counteracting the growing user cost of the resource. For $k = \hat{k}$ the relative rates of growth in q_1 and q_2 are equal, thus the rise in user cost is exactly offset by the rise in the shadow price of capital. For $k \gtrless \hat{k}$, $\dot{q}_1/q_1 \gtrless \dot{q}_2/q_2$. In a contracting economy resource extraction is therefore increasing for $k > \hat{k}$, it is constant for $k = \hat{k}$, and it is decreasing for $k < \hat{k}$. Furthermore, when $k < \hat{k}$, (8) shows that when $b'' \to 0$, $\dot{v}(t) \to -\infty$ which means that *when marginal extraction costs are constant for the permissible values of v, the interior solution cannot last for more than "an instant of time"*[3]; i.e., this regime cannot be part of an optimal policy sequence when $b'' = 0$. From (5) and (6) it is intuitively plausible that there is a unique equilibrium situation (q^*, k^*) in this regime. This equilibrium is a saddle-point; i.e., to every intial k_0 there is a unique optimal path leading to (q^*, k^*). (The proof is given in presentations of the one-sector optimal growth model; see, e.g., Burmeister and Dobell [1, Chap. 11]). Along this optimal path (7) and (8) hold, so, for $k_0 < k^*$, c is increasing while v is decreasing. While this equilibrium is reached only asymptotically, regime E will last only for a finite time until extraction stops and the resource is exhausted. From (4), optimal resource extraction in E is governed by

$$(p - b'(v(t))\, U'[(1 - s(t))(f(k(t) + \Pi(v(t))) + \bar{c}] = q_{2,0}e^{(\rho+n)t}. \tag{9}$$

In (9), $\lim_{t\to\infty}$ RHS $= \infty$, while LHS$(v) <$ LHS$(0) < \infty$. Thus (9) cannot hold for $t \to \infty$. There is some finite t, T, where v goes to zero and the extrac-

[3] This can be seen more clearly by assuming $\Pi'(v)q_1 = q_2$, with $b'' = 0$, to last for some interval of time. This is only possible for $f' = \mu$. But k has no stationary in $\hat{k}$, hence E cannot last for more than "an instant of time."

tion period is over. With a finite T, the transversality condition for $x(T)$ is

$$e^{-\rho T}q_2(T)\,x(T) = 0; \qquad e^{-\rho T}q_2(T) \geqslant 0. \tag{10}$$

Since $q_2(t)$ in E is always positive from (4), (10) is only satisfied for $x(T) = 0$. The same conclusions hold for regimes B and I. *At T the resource will therefore be exhausted.*

3.2. *Other Regimes*

B: In this regime the only source of consumption is the exogenous component, which provides the society with a subsistence level of consumption. At the same time, capital scarcity is extreme; this is why all production is saved. Equations (5), (6), and (8) hold also in this regime, and again, if extraction costs are constant, this regime cannot be part of an optimal policy.

A, C: These policies are boundary policies in the sense that the values of the control variables are at their boundary points. For these policies the exact behavior of the system may be inferred from these boundary values together with the initial conditions on k and x. Since $s = 1$, (5) and (6) hold also in these regimes.

D, F: In D resource extraction is maximal. In F there is no extraction; (5), (6), and (7) hold in both cases. As in regime E it can be shown that there is a unique equilibrium, which is a saddlepoint, for each of the regimes D and F.

Having studied the behavior of the economy within each possible policy regime, we now proceed to an analysis of possible switches between these regimes to find optimal policy sequences over time.

4. Optimal Policy Sequences

The necessary conditions for the various switches to take place between the different policies are summarized in Table II. Piecing the different policy regimes together is particularly simple in this model, even if we have two co-state variables, since q_2 is growing exponentially. As an illustration, follow the possible optimal sequence from an initial situation in regime A. Such an economy is extremely poor in physical capital. $k_0 < k^*$ and q_1 is falling. The shadow price of physical capital capital is higher than the marginal utility of consumption. All production is accumulated in order to expand the capital stock of the economy. On the other hand the economy is rich in the resource, which is extracted at a maximal rate. Since $c = \bar{c}$ is a constant in A, $U'(c)$ is a constant. Over time q_1 is falling and q_2 is increasing so that U' and $q_2/\Pi'(\bar{r})$ are approached "from above" by q_1. Policy A therefore cannot be sustained.

TABLE II

Necessary Conditions for Policy Switches

Switches from \ Switches to	B	C	D	E	F
A	$q_1 > U'(\bar{c})$ $\frac{q_2}{\Pi'(\bar{v})} = q_1$	0	$q_1 = U'(\bar{c})$ $\frac{q_2}{\Pi'(\bar{v})} > q_1$	$q_1 = U'(\bar{c})$ $\frac{q_2}{\Pi'(\bar{v})} = q_1$	0
B	·	$q_1 > U'(\bar{c})$ $\frac{q_2}{\Pi'(0)} = q_1$	0	$q_1 = U'(\bar{c})$ $\frac{q_2}{\Pi'(v)} = q_1$	$q_1 = U'(\bar{c})$ $\frac{q_2}{\Pi'(0)} = q_1$
C	0	·	0	0	$q_1 = U'(\bar{c})$ $\frac{q_2}{\Pi'(0)} > q_1$
D	0	0	·	$q_1 = U'[(1-s)(f(k) + \Pi(\bar{v})) + \bar{c}]$ $\frac{q_2}{\Pi'(\bar{v})} = q_1$	0
E	0	0	0	·	$q_1 = U'[(1-s)(f(k) + \Pi(0)) + \bar{c}]$ $\frac{q_2}{\Pi'(0)} = q_1$

If, at some t, $U'(\bar{c}) = q_1$, while still $q_2 < \Pi'(\bar{v})\, q_1$, the economy switches to D, where now consumption is above subsistence level, while the resource is extracted at a maximal rate. Total consumption is now increasing over time and $q_1 = U'$ is falling. The economy is accumulating capital also in this regime. Capital scarcity is less extreme than in regime A, and the resource is abundant. Of course, the initial situation of a real economy is usually in D or E. A plausible example of an initial situation in D is the United Kingdom after the recent petroleum discoveries in the North Sea. Policy D is also a transient policy since $q_2/\Pi'(\bar{v})$ is increasing while $U' = q_1$ is falling. Sooner or later the economy will therefore switch to E, with resource extraction at less than a maximal rate. Consumption is still increasing even if resource extraction is now decreasing towards zero over time. An example of an economy where the initial situation seems to be in this regime is Norway, where initial oil production is well below the possible maximum. When extraction ends, the resource is exhausted and the economy switches to F which is the final policy, which, once established, will last forever. In F the optimal development is the same as in the one-sector optimal growth model, with consumption and the capital intensity increasing towards their optimal steady-state levels.

Alternatively, if, at some t, $q_2 = \Pi'(\bar{v})\, q_1$, with $q_1 > U'(\bar{c})$, the economy switches from regime A to B, where now resource extraction is less than its maximum and falling, while still all production is used for accumulation purposes so that consumption per capita equals $\bar{c}$. B may of course also be the initial situation. B is also a transient regime. q_2 will be increasing over time while q_1 is falling. From B the economy must switch to either E or C, or, by accident, directly to F. If q_1 "reaches" $U'(\bar{c})$ while $q_2 = \Pi'(v)\, q_1$, the economy switches to E, with the optimal path as explained above. The other possibility is that $q_1 > U'(\bar{c})$ while $q_2 = \Pi'(0)\, q_1$. In that case the economy switches from B to C where resource extraction has ended while capital accumulation is still maximal. Since q_1 is falling towards $U'(\bar{c})$, a switch from C to F must eventually take place. Finally, A may also by accident switch directly to E, if q_1 happens to be equal to $U'(\bar{c})$ exactly at the point in time where maximal resource extraction stops. From an initial situation in A. the optimal policy sequence will therefore be one of the following:

(a) *ADEF*, (b) *ABEF*, (c) *ABCF*, (d) *AEF*, (e) *ABF*.

In all sequences extraction is gradually reduced. Maximal extraction until extraction stops is inoptimal, since such a policy would require a jump in one of the co-state variables, which is impossible by the maximum principle. In Fig. 1 the sequence *DEF* is illustrated graphically by $q_1{}^0$ and $q_2{}^0$. The economy is in regime D until t' with $q_1 > q_2/\Pi'(\bar{v})$. From t' to t'' it it is in E, with $q_1 = q_2/\Pi'(v)$, after which it is in F with $q_2/\Pi'(0) > q_1$. Resource extraction

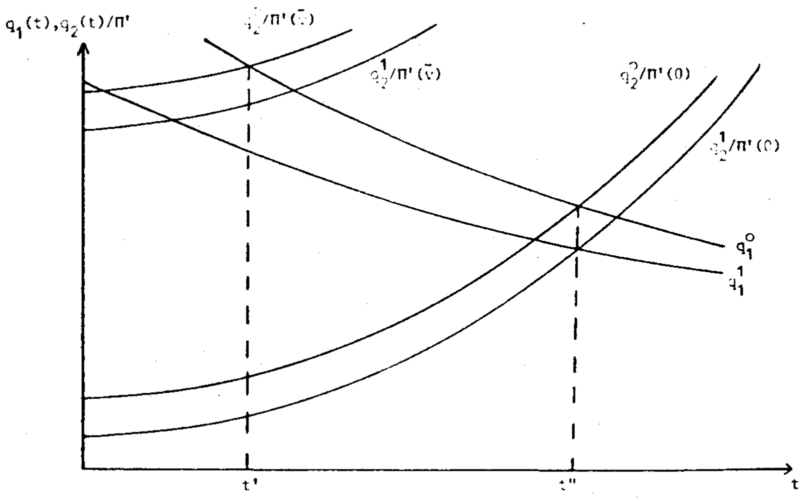

FIGURE 1

ends at t_1, when the resource is exhausted. An interesting question is now: How does the optimal extraction policy depend on the initial situation?

Consider first a positive shift in k_0, with everything else equal. From the one-sector optimal growth model we know that a higher k_0 would imply a lower $q_1(t)$ everywhere. A higher k_0 would cet. par. also mean a lower $q_{2,0}$. If not, the intersection of $q_1(t)$ and $q_2(t)/\Pi'(0)$ would take place for $t < t''$ so that the resource extraction period would be shortened. From the figure it is seen that the time period when extraction is maximal (regime D) is reduced when cer. par. q_1 shifts down. Also, for $q_1 = q_2/\Pi'(v)$ to hold for every t with a lower $q_1(t)$, $\Pi'(v)$ must increase; i.e., $v(t)$ must be reduced for every t. But a shorter extraction period with reduced extraction everywhere cannot exhaust the resource, hence it is inoptimal by (10). $q_{2,0}$ must therefore fall when k_0 gets a positive shift. Suppose then that $q_{2,0}$ falls so that the extraction period is unchanged. The new situation is illustrated by q_1^1 and q_2^1 in the figure. $q_1^1(t)$ is less steep than $q_1^0(t)$; this follows from (5). $\dot{q}_2/q_2$ is unchanged. Because of this, going backwards from t'', the difference between the two curves is less in the new situation than in the old; i.e., $q_1^0 - q_2^0/\Pi'(0) > q_1^1 - q_2^1/\Pi'(0)$ for all $t < t_1$, and the inequality increases as t goes towards zero. Thus, for $q_1 = q_2/\Pi'(v)$ to hold, this means that $1/\Pi'(v)$ is less in the new situation, or that the extraction level is less now for all $t < t''$, except in the period when D is operative. But the time period when this regime is operative must fall—this follows from the same argument, and is clear from the figure. Again this would mean that the resource is not exhausted at t_1, *hence the extraction period must be lengthened when the initial capital intensity gets a positive shift.* Regarding a positive shift in x_0, it can be shown along the same lines that cet. par., $q_1(t)$ and $q_2(t)$ must both fall when x_0 increases. Resource extraction and consumption increase and the extraction period is increased.

A more general question is: Under what initial conditions in the economy are each of the sequences (a)–(c), or parts of them, optimal? Suppose initially that EF is optimal and that k_0 now gets a negative shift. We then know that $v(t)$ shifts up and the period when E is operative is reduced. Over time $v(t)$ is falling, so for a sufficiently great negative shift in k_0, $v(t)$ would be equal to $\bar{v}$ for some timeperiod, regime D would be operative, and the optimal sequence would be DEF. For still lower k_0, regime A might be optimal initially so that the whole sequence (a) would be relevant. For even lower k_0, q_1 might not have reached $U'(\bar{c})$ when extraction is reduced from its maximal level and (b) would be the optimal sequence. Finally, (disregarding (d) and (e)), sequence (c) would be optimal for the lowest k_0.

Consider next the effects of changes in ρ, the social rate of time preference. In the capital sector, an increase in ρ leads to a fall in k^*. From the theory of optimal growth in the one-sector model we know that, for a given k, this leads to a lower q_1 and to a lower s for each k. Also increased social preference for consumption "today" relative to "tomorrow" would tend to concentrate resource extraction more towards the beginning of the planning period and to reduce the extraction period. This is accomplished by keeping the economy in E for a shorter period when ρ increases. As ρ increases still more and t_1 decreases, the optimal policy sequence may change from EF to DEF, and so on. Changes in the exogenously given rate of growth in population, n, have the same effects as changes in ρ.

Finally, consider the effects of an exponential trend in $p(t)$ so that $p(t) = p_0 e^{\beta t}$. Instead of (8), we now obtain

$$\dot{v} = (\Pi'/\Pi'')(f' - \mu) - \dot{p}/\Pi'' \tag{11}$$

from which it follows that, cet. par. *an exponential rise (fall) in the relative price of the resource would tend to reduce (increase) the rate of fall in extraction along the optimal path.* For sufficiently high rates of increase in the relative price of the good the optimal extraction path may be rising. Inserting for $\dot{p}$ and Π' and rearranging, (11) shows that

$$\dot{v}(t) \gtreqless 0 \qquad \text{as } f'(k) \lesseqgtr \beta/(1 - b'/p) + \mu. \tag{12}$$

From (4), $0 < b'/p < 1$, so that the RHS of (12) is always greater than $\beta + \mu$. As long as the marginal productivity of capital in the economy is greater than the RHS of (12), the optimal devlopment of the economy will in principle be the same as when $\beta = 0$. If the economy is more capital intensive, however, resource extraction will rise over time along the optimal path when there is a positive exponential trend in the relative price of the resource. In general (12) shows that the higher this trend is, the lower must the capital intensity be for falling resource-use to be optimal, or conversely, *the higher the capital intensity of the economy, the lower the price rise needed to make a rising extraction-path optimal.* The economic explanation is that when the relative price of the resource is rising, the increasing user cost of the resource and falling shadow price of capital are counteracted by the price rise. If capital is "scarce," however, its shadow price may fall so fast that it offsets the rate of increase in the resource price over and above the user cost. In that case optimal extraction would still be falling over time. Conditions for this case are given in (12).

5. Comparing the Results with Earlier Models

(a) *The Resource Model*

The results obtained on the optimal path of resource-use in this combined model are somewhat more general than those found in earlier contributions. The optimal extraction path now depends on conditions in the rest of the economy. When the relative price of the resource is given and marginal extraction costs are constant, we have shown that an interior solution to v and s simultaneously cannot be optimal. If the solution for the savings rate is interior, resource extraction should either be zero or at its maximum; and vice versa. This is so even if the social welfare function in the model is concave. A comparison with the results in the resource model by Vousden [8] is therefore not completely straightforward. In his model marginal extraction costs are constant. In a growth-model context, the alternative constant source of consumption postulated in his model must be interpreted as a steady-state consumption level, associated with some interior constant savings rate s^*. At the same time v is also interior and falling in his model, which is incompatible with the necessary conditions for optimality in the combined model, where v is either zero or $\bar{v}$ in a steady state. When marginal extraction costs are rising, however, an interior solution for s and v simultaneously is relevant in the combined model. With constant prices and the capital intensity less than or equal to the modified golden rule capital intensity, extraction is either constant for some initial period and then falling, or always falling. The extraction period is finite and the resource is always exhausted when extraction ends. All these results are similar to those found by Vousden in [8]. However, the result on the effect on the optimal extraction path from changes in the initial capital or resource stock bears little resemblance to the result in [8] that "the optimal time of exhaustion will increase as the alternative source of consumption, $\bar{C}$, falls."

Since the welfare function is the same in both models, the contrast in the solutions must be due to the difference in supply conditions. In the resource model the resource may be used for consumption purposes only. Extracting the resource is the only source of consumption in addition to the exogenous component. Since consumption from the resource is subject to a finite upper bound on cumulative extraction, society must—in order to survive—stretch out the use of the resource when the alternative source of consumption falls. In the combined model, the resource extracted may also be used for capital-formation purposes, and survival can be secured on basis of the physical capital stock alone. The future benefits to be derived from an extra unit of the resource extracted for investment purposes is higher the smaller the capital stock already attained. The optimal resource-use in the combined model is therefore slowed down when the initial capital intensity gets a positive shift.

Also, with an initial capital-intensity higher than the modified golden rule, a rising extraction is possible for some time. In such a contracting economy, an increasing shadow price of capital (equal to the marginal utility of consumption), which makes investment and consumption more valuable in the future, tends, cet. par., to make it optimal to postpone extraction; this effect must be balanced against a rising user cost of the resource. The combined effect may well be to keep back production for some time, in contrast to the ordinary case where a falling shadow price of capital and an increasing user cost of the resource both lead to a higher extraction now than in the future. While a contracting economy may be of limited practical interest, the relevance of the capital intensity on optimal resource-use is obvious in the case of a rising price of the resource. In that case the increasing user cost is counteracted by the rising resource price. In a model without capital, extraction rises if the rate of growth in price is higher than in the user cost. In a model with capital, this pattern is accentuated if capital is above its optimal steady-state level, since then the social value of capital (or consumption) over time increases as well. If capital is scarce, however, its shadow price is falling, and it may fall so fast that it offsets the rate of increase in the resource price over and above the user cost. In that case optimal extraction would still be falling over time. Such a country is too poor to afford to wait for the higher prices, at least for some initial period until the capital stock of the economy is built up.

(b) *The Capital Model*

The problem now is how the optimal path of consumption and capital accumulation in an economy is affected by conditions in the resource sector. To follow Vousden [8], assume that the economy initially, before resource extraction is started up at t_0, is in a steady state; i.e., $k = k^*$ and $s = s^* = \lambda k^*/f(k^*)$. Assume further that after extraction has started up, the economy is in regime E, so that extraction starts at less than its maximum. At the outset we may safety assume that consumption is a normal good for all t so that there is *some* increase in consumption when extraction starts. From (4) this means a negative shift in q_1 at t_0. Also, as in the one-sector optimal growth model it can be shown from (4) that $\partial s/\partial q_1 > 0$, so that *the savings rate falls when extraction begins.* But since the value of total production in the economy increases when extraction begins, *total* savings may a priori be increased or reduced (or stay constant). To show that total savings must rise, suppose the opposite. If they fall for $t = t_0 + \epsilon$ ($\epsilon > 0$ and sufficiently small), k falls below its steady-state level k^*. But from the one-sector optimal growth model a reduced k must imply a higher shadow-price q_1, so this is a contradiction. By the same argument, k cannot stay constant when q_1 gets a negative shift. *Hence total savings rise when extraction begins.* k increases above its steady-state level, so by (6), q_1 increases over time, which means that c is falling over time along the optimal path. Over time, k and c approach their optimal steady-state values asymptotically from above, while q_1 increases asymptotically towards its optimal stationary value. The optimal development of c, q_1, and k is illustrated in Fig. 2. At t_1, when extraction ends, c and k are both above their optimal stationary values.

Since the optimal steady state is unlikely to materialize in any actual economy (it is reached only asymptotically), the above thought-experiment is somewhat illegitimate. Consider therefore the case when $k < k^*$ when a new resource, like a petroleum reservoir, is discovered and exploited. We may

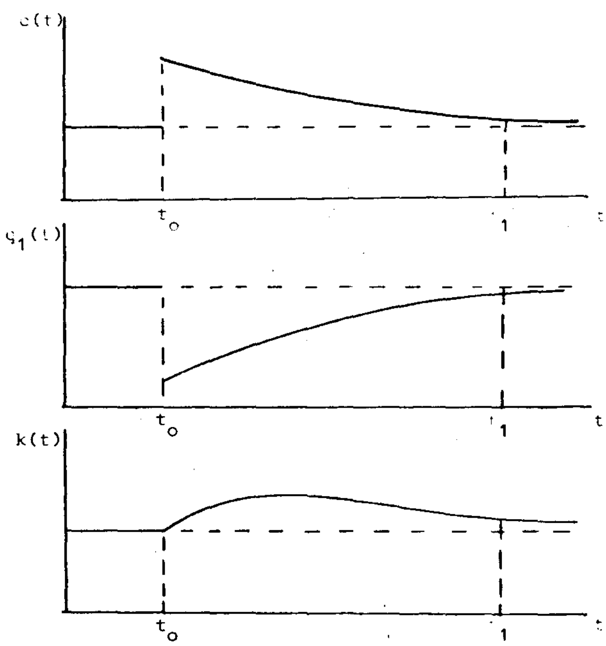

FIGURE 2

again safely assume that consumption gets a positive shift when extraction begins so that $q_1(t)$ shifts down. By (6), $q_1(t)$ must be falling, since $k < k^*$ also after extraction begins, which means that $c(t)$ is increasing. Two cases may be distinguished. The case in which $c(t) < c^*$ for all t is illustrated in Fig. 3, where the optimal paths of c, q_1, and k are shown. If, however, k_0 is sufficiently close to k^* and/or the new resource is sufficiently rich, $c(t)$ and $k(t)$ may increase above their optimal steady-state levels; see Fig. 4.

In this case, as in the case discussed in connection with Fig. 2, consumption shifts up above its optimal steady-state level when extraction begins, and, correspondingly, q_1 shifts below q_1^*. k is increasing. Since now $k(t_0) < k^*$, however, q_1 falls and $c(t)$ increases for some period after t_0, until, at $\hat{t}$, $k = k^*$. In $\hat{t}$, $\dot{q}_1 = 0$, and immediately after, when $k > k^*$, q_1 is increasing. This means that $c(t)$ has a maximum in $\hat{t}$. For $t > \hat{t}$ the optimal development of the economy is identical to the case when $k_0 = k^*$, discussed in connection with Fig. 2. (7) may be rewritten as

$$\dot{c}/c = (-f' + \rho + \lambda)/\breve{\omega}, \tag{13}$$

where $\breve{\omega} = cU''/U'$ is the elasticity of marginal utility. When extraction starts, total savings increase so that k increases faster than before the extraction period. With $\breve{\omega}$ (approximately) constant in the relevant range, it follows from (13) that the relative growth in consumption is reduced when

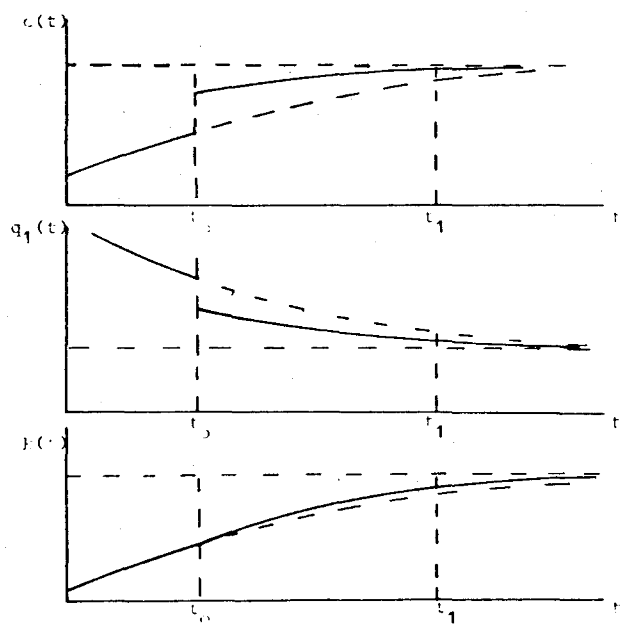

FIGURE 3

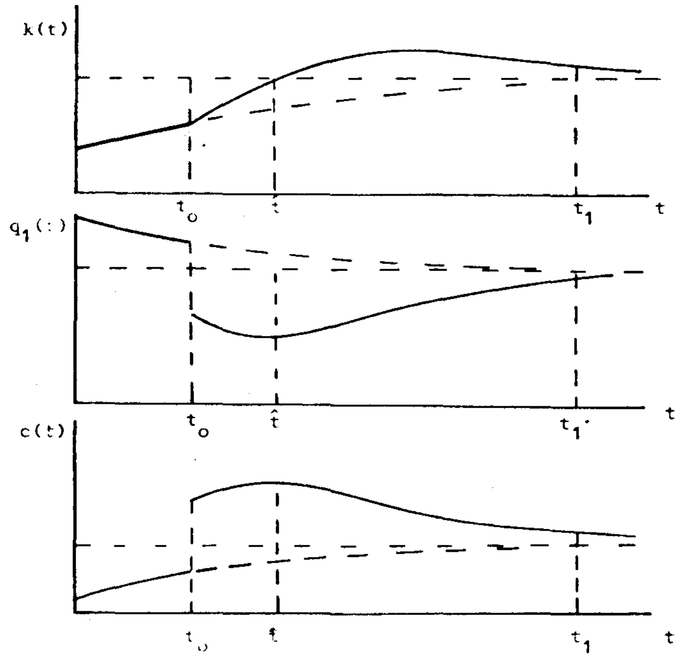

FIGURE 4

extraction starts. *The optimal reaction to a newly discovered resource is therefore a positive shift in initial consumption combined with a reduced relative growth in consumption.* Figures 2–4 show that *with* resource extraction $q_1(t)$ is less than $q_1(t)$ *without* extraction for all t, including $t > t_1$. This means that *also in the post-extraction period society will enjoy higher levels of capital and consumption per capita than it would have done without a resource extraction period.*

6. CONCLUSIONS

(i) A dynamic model for an open economy where savings and resource use can be optimized simultaneously has been analyzed. The results are somewhat more general than those found in earlier contributions. With constant prices and the capital intensity of the economy less than or equal to the modified golden rule, however, extraction is either constant for some initial period and then falling, or always falling, along any of the possible optimal sequences for the economy.

(ii) When the price of the resource depends exponentially on time, it is optimal if, and only if, the rate of increase in the price of the resource is greater than some critical value, determined partly by the capital intensity of the economy, to depart from the optimal sequences mentioned above. In that case resource extraction is increasing over time, and it may be optimal to leave the resource in the ground for some initial period. When the capital intensity of the economy increases, the price rise needed to make such a policy optimal is reduced.

(iii) It has been shown that as the initial capital stock of the economy increases, the extraction period is lengthened and the extraction level is reduced for every t. The resource is exhausted when extraction ends and the extraction period is always finite. Extraction should be reduced gradually towards zero, where extraction ends.

(iv) If a resource is discovered and exploited, the optimal savings rate in the economy falls, while total savings increase. Compared to a situation without resource extraction, consumption gets an initial positive shift, while its relative rate of growth along the optimal path is reduced. Consumption and the capital stock will be higher also in the postextraction period.

(v) Finally, it should be noted that with constant marginal extraction costs—an assumption often made in the literature—an interior solution for savings and resource extraction at the same time cannot be optimal.

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RESOURCE EXTRACTION, FINANCIAL TRANSACTIONS AND CONSUMPTION IN AN OPEN ECONOMY

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Abstract

Macroeconomic models of optimal resource use over time assume either a closed economy or an open economy without borrowing or lending abroad. A model of resource extraction in an open economy with borrowing/lending abroad is presented in this paper. When financial transactions are possible, the optimal path of resource extraction is separated from the optimal consumption stream, which is brought about by financial transactions. Optimal strategies over time for consumption, financial transactions and resource use are derived. The properties of these time paths are compared to the results of earlier studies.

I. Introduction, the Model and Optimality Conditions

Optimal extraction of exhaustible natural resources over time has been analyzed from a macroeconomic point of view by e.g. Koopmans (1973), Vousden (1973) and Heal & Dasgupta (1974). These contributions assume a closed economy. Optimal resource extraction in open economies has been analyzed by Vousden (1974), Strøm (1974), van Long (1974), Kemp & Suzuki (1975) and Gehrels (1975). While different in several respects, all of these studies, except Gehrels (1975), have one common feature—explicitly or implicitly the current account is always balanced, i.e. no borrowing or lending abroad is assumed to take place. Gehrels recognizes this possibility. His setup is interesting, but given the complexity of his model, which also includes physical capital accumulation, the analysis is rather cursory. For this reason there is scope for a more explicit treatment of the interrelationship between optimal resource extraction and financial transactions over time in an open economy. This paper has the recent petroleum and natural gas discoveries in the North Sea as its background, but the theory is applicable to any society with a substantial stock of an exhaustible natural resource which is mostly exported, but where the volume of export is too small to influence world prices of the commodity.

* I am indebted to a referee for well-taken criticism on an earlier draft.

The following variables will be used:

$c(t)$ = total consumption per capita
$\bar{c}(t)$ = an exogenous consumption per capita, from a source outside the model
$c_v(t)$ = consumption per capita financed by the sale of the resource abroad or by borrowing or reducing the country's stock of financial assets
$v(t)$ = resource extraction per capita
$p_v(t)$ = the net price of the resource on the world market (net of a constant marginal extraction cost)
$p_c(t)$ = the "price" of consumption goods on the world market
$b(t)$ = stock of foreign bonds or, if negative, foreign debt per capita, in nominal units
$x(t)$ = stock of the resource per capita
U = instantaneous social welfare
ϱ = social rate of discount
n = rate of growth in total population
μ = rate of change in the exogenous component of consumption
α = rate of return on foreign bonds or rate of interest on foreign debt
γ = rate of change in the resource price over time
β = rate of change in the price of consumption goods over time
T = time-horizon, finite.

The problem is then

$$\text{Max} \int_0^T U(c(t))\, e^{-\varrho t}\, dt$$

s.t.

(i) $c(t) = c_v(t) + \bar{c}(t)$

(ii) $p_c(t) c_v(t) = p_v(t) v(t) + \alpha b(t) - n b(t) - \dot{b}(t) \geqslant 0$

(iii) $-\dot{x}(t) = v(t) + n x(t)$

(iv) $\dot{b}(t) = \dot{b}(t)$

(v) $\dot{b}(t) \geqslant -\bar{z}$ for $b < 0$

(vi) $v(t) \leqslant \bar{v}$ (1)

(vii) $v(t) \geqslant 0$

(viii) $p_v(t) = p_{v,0} e^{\gamma t}$

(ix) $p_c(t) = p_{c,0} e^{\beta t}$

(x) $\bar{c}(t) = \bar{c}_0 e^{\mu t}$

(xi) $x(0) = x_0$, $b(0) = 0$

(xii) $x(T)$, $b(T) \geqslant 0$

(xiii) ϱ, n, α, γ, β, $p_{c,0}$, $p_{v,0}$, μ, c_0, $\bar{z}$, $\bar{v}$ and T exogenously given constants.

Verbally, the problem is to find such paths over time for resource extraction and foreign borrowing/lending that the present value of total social welfare is maximized. The planning horizon is finite. It may be 20, 50, 100 years or more. A long, but finite planning horizon corresponds to the view that the time period when the resource is extracted will be merely an "epoch" in the history of the society in question, so that in the longer view other sources of consumption are more important. Instantaneous welfare depends on consumption per capita and we assume that $U'>0$ and $U''<0$. The size of the total population does not affect social welfare. Consumption per capita is, according to (i), the sum of an exogenous component and consumption goods imported, paid for either by selling the resource in direct exchange for consumption goods or by selling bonds abroad as shown by (ii).

From a financial point of view, (ii) says that the increase in bond holdings per capita, $\dot{b}$, equals the value of the resource sold, $p_v v$, plus the dividend on the stock of bonds, αb, minus the value of imports, $p_c c_v$, and the reduction in bond holdings per capita due to population growth, nb. b and $\dot{b}$ may be positive or negative. If b is negative, the debt per capita will increase by the import and the interest on the debt, minus the value of the resource sold and the population effect. According to (iii) the stock of the resource per capita, which is initially given, is reduced by the extraction per capita, and is also diluted by nx due to population growth. (vi) says that there is some upper bound $\bar{v}$ on extraction per unit of time for e.g. technical reasons, and according to (vii) resource extraction is irreversible. (v) says that, when the country has debts, there is an upper bound on the debt increase. This is due to existing practices in financial circles as to how much a country of a given size may borrow abroad during e.g. one year and the national government's fear of losing control of economic policy due to pressure from abroad. According to (iv), the debt increase per capita is directly controllable. (viii) and (ix) express the assumption of constant exponential growth in the prices of the resource and of consumption goods on the world market. (xi) says that the initial stock of resources per capita is given and that initially the country has no debt or claims abroad. To make the problem economically interesting, assume that $x_0<\bar{v}T$, so that the rate of extraction cannot be maximal throughout the entire planning period. Finally, (xii) says that the stock of resources must be nonnegative and that there must be no debt at the end of the planning period. b and x are the state variables of the problem. v and $\dot{b}$ are the control variables.

In order to analyze the problem, form the (present value) Lagrangean

$$L=e^{-ct}\Big\{U\Big(\frac{1}{p_c}\,[p_v v+(\alpha-n)\,b-\dot{b}]+\bar{c}\Big)+q_1\dot{b}-q_2(v+nx)$$
$$-\lambda_1(p_v v+(\alpha-n)\,b-\dot{b})+\lambda_2(\bar{z}+\dot{b})+\lambda_3(\bar{v}-v)+\lambda_4 v\Big\}, \qquad (2)$$

where we have substituted for c from (i) and (ii). Necessary and sufficient[1] conditions for a solution to (1) are that there exist continuous functions of time q_1 and q_2 such that

$$\dot{q}_1 = -U'(\ )(\alpha-n)/p_c - (\alpha-n)\lambda_1 + \varrho q_1 \tag{3}$$

$$\dot{q}_2 = (\varrho+n)q_2; \quad \text{i.e.} \quad q_2(t) = q_{2,0}e^{(\varrho+n)t} \tag{4}$$

$$U'(\ )p_v/p_c - q_2 + p_v\lambda_1 - \lambda_3 + \lambda_4 = 0 \tag{5}$$

$$-U'(\ )1/p_c + q_1 - \lambda_1 + \lambda_2 = 0 \tag{6}$$

$$\lambda_1 \geqslant 0; \quad \lambda_1[p_v v + (\alpha-n)b - \dot{b}] = 0 \tag{7}$$

$$\lambda_2 \geqslant 0; \quad \lambda_2[\bar{z} + \dot{b}] = 0 \tag{8}$$

$$\lambda_3 \geqslant 0; \quad \lambda_3(\bar{v} - v) = 0 \tag{9}$$

$$\lambda_4 \geqslant 0; \quad \lambda_4 v = 0 \tag{10}$$

$$e^{-\varrho T}q_1(T) \geqslant 0; \quad e^{-\varrho T}q_1(T)b(T) = 0 \tag{11}$$

$$e^{-\varrho T}q_2(T) \geqslant 0; \quad e^{-\varrho T}q_2(T)x(T) = 0. \tag{12}$$

(11) and (12) are the transversality conditions.

q_1 and q_2 are the costate variables associated with the equations of motion (iv) and (iii), respectively. q_2 is the shadow price of the resource per capita so that $q_2(t)$ denotes the addition to the optimal value of the criterion function brought about by leaving the marginal unit of the resource unexploited at t. q_2 is often called the "user cost" of the resource. q_1 is the shadow price of financial capital. $q_1(t)$ denotes the addition to the optimal value of the criterion function from buying a marginal unit of foreign bonds at t. Suppose now that $c_v > 0$ for all $t \in [0, T]$, so that $\lambda_1 = 0$. From (6) and (8) we then get

$$-\dot{b}\begin{Bmatrix} = \bar{z} & \text{if} & U'(\ )/p_c > q_1 \\ \leqslant \bar{z} & \text{if} & U'(\ )/p_c = q_1. \end{Bmatrix} \tag{13}$$

For an interior solution for $-\dot{b}$, borrowing abroad, the shadow price of financial capital is equal to the marginal utility of borrowing in terms of consumption per worker, U'/p_c. If the shadow price is lower, maximal borrowing is optimal. Similarly, from (5), (9) and (10) we obtain

$$v\begin{Bmatrix} = \bar{v} & \text{if} & U'(\ )p_v/p_c > q_2 \\ \in [0, \bar{v}] & \text{if} & U'(\ )p_v/p_c = q_2 \\ = 0 & \text{if} & U'(\ )p_v/p_c < q_2. \end{Bmatrix} \tag{14}$$

(14) says that for an interior solution for resource extraction, the user cost of the resource, q_2, should be equal to the marginal utility of resource extraction

---

[1] The conditions are sufficient since the Lagrangean is concave in x, b, v and $\dot{b}$ for given q_1, q_2 and t.

Table 1

v ... $\dot{b}$	$\bar{v}$	$\in[0, \bar{v}]$	0
$\dot{b} = -\bar{z}$	*A*	*B*	*C*
$\dot{b} > -\bar{z}$	*D*	*E*	*F*

in terms of consumption per worker, $U'(\ )p_v/p_c$. If the user cost is higher, no extraction should take place; if it is lower, maximal extraction is optimal. From (13) and (14), the policy regimes shown in Table 1 are possible in the optimal solution.

II. Analysis

No Borrowing Restrictions

When $\dot{b} > -\bar{z}$, $U' = p_c q_1$ from (13). Inserting for U' in (14) we then obtain

$$v \begin{cases} = \bar{v} & \text{if} \quad p_v q_1 > q_2 \\ \in [0, \bar{v}] & \text{if} \quad p_v q_1 = q_2 \\ = 0 & \text{if} \quad p_v q_1 < q_2. \end{cases} \tag{14'}$$

Also, when $U' = p_c q_1$, (3) simplifies to

$$\dot{q}_1 = (\varrho + n - \alpha) q_1; \quad \text{i.e.} \quad q_1 = q_{1.0} e^{(\varrho + n - \alpha)t}. \tag{3'}$$

From (3′), (4) and (viii), the relative rate of change in $p_v q_1$ is greater (less) than the relative rate of change in q_2 if γ is greater (less) than α; that is, if the percentage increase in the price of the resource is greater (less) than the nominal rate of interest on bonds. Optimal extraction of the resource when there are no restrictions on borrowing therefore depends only on the development of the *nominal* price of the resource and on the *nominal* rate of interest on bonds. The optimal extraction path in this case is independent of the social rate of discount *and* the development of import prices. Disregarding the case where $\gamma = \alpha$, (14) also shows that *without borrowing restrictions, optimal resource extraction is either zero or at its maximum.* Regime *E* lasts only for "an instant of time", hence it cannot form part of any optimal policy sequence.

Consider first the case where $\gamma > \alpha$ (case I). If $p_{v.0} q_{1.0} > q_{2.0}$ then $p_v q_1 > q_2$ for all $t \in [0, T]$, and $v = \bar{v}$ for all t. But this possibility is excluded by the assumption that $\bar{v}T > x_0$. Alternatively, if $p_{v.0} q_{1.0} < q_{2.0}$ and $p_v q_1 < q_2$ for all t, the resource is not extracted at all. But the transversality condition (12) is only satisfied for $x(T) = 0$. Hence this policy is inoptimal and $p_v q_1$ and q_2 must have an intersection. Paths for $p_v q_1$ and q_2 compatible with the optimal conditions are shown in Fig. 1.

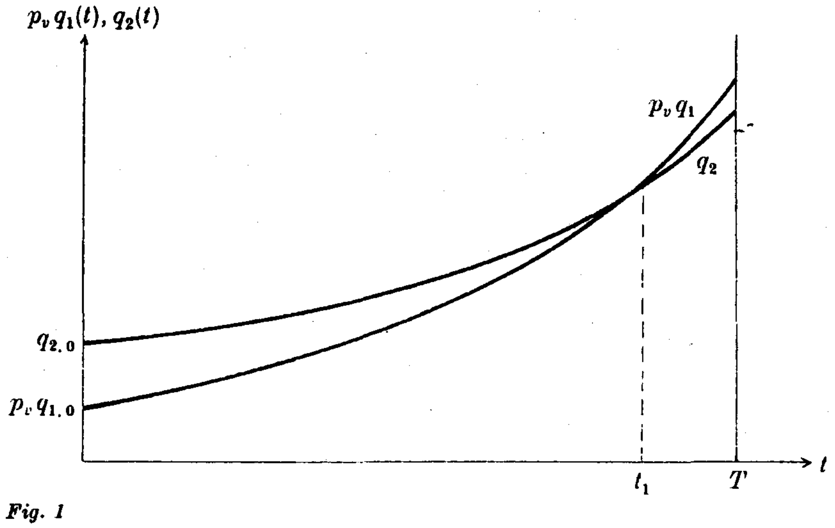

Fig. 1

For $t > t_1$ it follows from (14′) that $v = \bar{v}$. $v = 0$ for $t < t_1$. According to (12), the resource must be exhausted at T. To study the optimal development of consumption, differentiate $U'(c) = p_c q_1$ from (13) with respect to t. Using (ix), (3′) and (13) we obtain

$$\dot{c} = U'(\ )(-(\alpha-\beta)+\varrho+n)/U''. \tag{15}$$

(15) shows the absolute rate of change in real consumption per capita along the optimal path. It follows from this expression that when borrowing restrictions are absent, the time form of the optimal consumption path does *not* depend on the development of the resource price. Since β is the percentage rate of change in import prices, $\alpha-\beta$ is the *real* rate of interest, or the real rate of return on bonds. (15) therefore says that when the real rate of return on bonds is higher (lower) than the social rate of discount (plus any relative rate of increase in population), consumption is increasing (decreasing) along the optimal path. Denote the case where $\alpha-\beta<\varrho+n$ by I.1 and the case where $\alpha-\beta>\varrho+n$ by I.2. We may now study the optimal time path of financial transactions. Since $c_v>0$ for all t and $v=0$ for $t<t_1$, the country must borrow for $t<t_1$, i.e. $\dot{b}(t)<0$ for $t<t_1$. From (i) and (ii) we have $-\dot{b} = p_c(c-\bar{c})-(\alpha-n)b$. Differentiating with respect to t,

$$\ddot{b} = p_c[-(\dot{c}-\mu\bar{c})-\beta(c-\bar{c})+(\alpha-n)\dot{b}/p_c]. \tag{16}$$

(16) shows that the absolute rate of change in borrowing along the optimal path consists of three components: the change in the value of resource-based consumption along the optimal path, the price increase, if any, on resource-based consumption along the optimal path of real consumption and interest on the debt increase. Due to the first term, in particular, the sign of $\ddot{b}$ is gener-

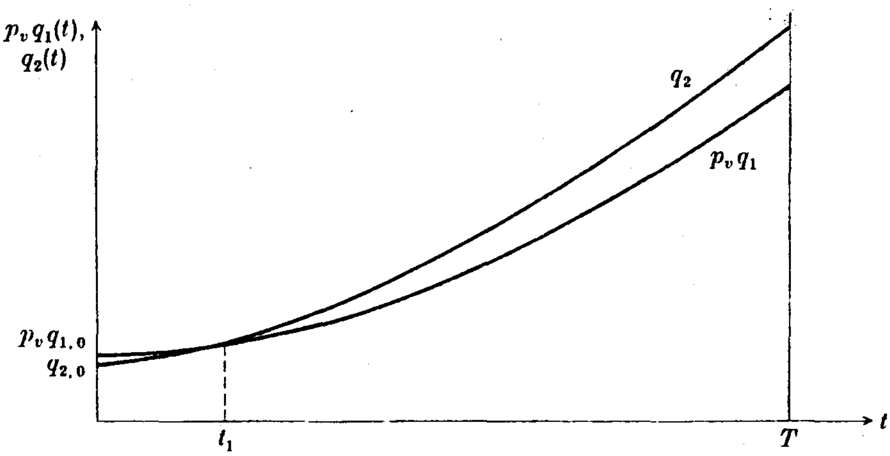

Fig. 2

ally ambiguous. Suppose therefore, for simplicity, that the exogenous component in consumption is constant, so that $\mu = 0$. In case I.2 real consumption is increasing along the optimal path. In this case, $\ddot{b} < 0$ for $\beta \geq 0$. For increasing or constant import prices, the debt would therefore be increasing for $t < t_1$ and there is also some margin for a fall in import prices. For all practical purposes borrowing is therefore increasing for $t < t_1$ in case I.2. At t_1, v jumps from zero to $\bar{v}$. From the expression for c, given in (i) and (ii), and the fact that q_1 must be continuous, it follows from (13) that $\dot{b}$ *must make a positive jump at* t_1. In fact $\dot{b}$ must be positive for $t > t_1$, otherwise the debt would continue to rise also during the extraction period and until T, which is not compatible with the transversality condition (11). Whether the repayment of the debt takes place at an increasing or decreasing rate, however, cannot be seen from (16) in this case. The optimal borrowing path in case I.1 is less clear. Real consumption then falls over time. With constant prices, borrowing for this purpose therefore falls over time. On the other hand the debt is increasing, so *cet. par.* borrowing for this purpose must increase. The total effect on the rate of change in borrowing is therefore ambiguous. As in case I.1, there will be a positive jump in $\dot{b}$ at t_1. In the repayment period, it follows from (16) that unless there is an extremely steep increase in import prices, repayment of the debt takes place at an increasing rate along the optimal path.

Next we consider case II where $\gamma < \alpha$, which includes the case where the price of the resource is constant. For the same reasons as in case I, the optimal paths for $p_v q_1$ and q_2 must intersect, as illustrated in Fig. 2. In this case, however, $v = \bar{v}$ for $t < t_1$ and $v = 0$ for $t > t_1$. (15) and (16) still hold. Denote the case where $\dot{c} < 0$ by II.1 and the case where $\dot{c} > 0$ by II.2.

By applying the same argument as in case I, there must now be a negative jump in $\dot{b}$ at t_1. Since $c_v > 0$ for all t, financial capital must be positive when extraction ends, i.e. $b(t_1)>0$. Also $b(T)=0$ according to (11), so that financial capital is positive in the post-extraction period, except at $t=T$. In case II.1, (16) shows that unless β is "very high", lending abroad increases over time in the extraction period. $\dot{b}$ must be falling towards zero at T. The sign of $\ddot{b}$ is ambiguous. In case II.2 the same pattern is to be expected for $t<t_1$, although real consumption now increases over time. If $\dot{b}<0$ for $t>t_1$, it follows from (16) that $\ddot{b}<0$ in this case in the post-extraction period, which means that the country reduces its holdings of foreign bonds at an increasing rate.

Borrowing Restrictions

In case I, the possibility of borrowing restrictions emerges. Suppose that such a situation exists, and that $v \in [0, \bar{v}]$. In this situation the optimal path of v (and c) is determined by (14). Differentiating $U'p_v = q_2 p_c$ and solving for $\dot{c}$ we now obtain, using (4) and (14):

$$\dot{c} = U'(\ )(-(\gamma-\beta)+\varrho+n)/U''. \tag{17}$$

In this situation (17) shows that the absolute rate of real consumption per capita does *not* depend on the rate of interest. Instead the percentage rate of change in the price of the resource enters the expression. (17) therefore says that when the real rate of return from keeping the resource in the ground is higher (lower) than the social rate of discount (plus any relative rate of increase in population), real consumption is increasing (decreasing) along the optimal path. This is merely a reflection of the fact that savings now take the form of reduced resource use instead of reduced borrowing. Using (4), (17) and the fact that $c=[p_v v+(\alpha-n)b+\bar{z}]/p_c+\bar{c}$ in this case, we obtain the following expression, from (14), for the rate of change in resource extraction:

$$\dot{v} = \dot{c}p_c/p_v+(\alpha-n)\bar{z}/p_v-\gamma v+\beta(c-\bar{c})p_c/p_v. \tag{18}$$

(18) says that the absolute rate of change in resource extraction along the optimal path depends on four factors: the change in consumption along the optimal path, increased interest on the debt since borrowing takes place throughout at a maximum rate and changes in the price over time, if any, of the resource and of imported consumption goods along their optimal paths. (18) shows that *in general there is no reason a priori to expect a falling optimal path of resource extraction in this model.* This conclusion contrasts with the result in models of optimal resource use in open economies where financial transactions are assumed away.

Optimal Policy Sequences

To obtain somewhat more specific results about the optimal paths of finance capital, the case where the elasticity of marginal utility $U''\cdot c/U'$ equals a constant $\breve{\omega}$ and where the exogeneous component in consumption is constant, is analyzed in the Appendix. Based on the conclusions of the Appendix, we obtain the following results for optimal policy sequences:

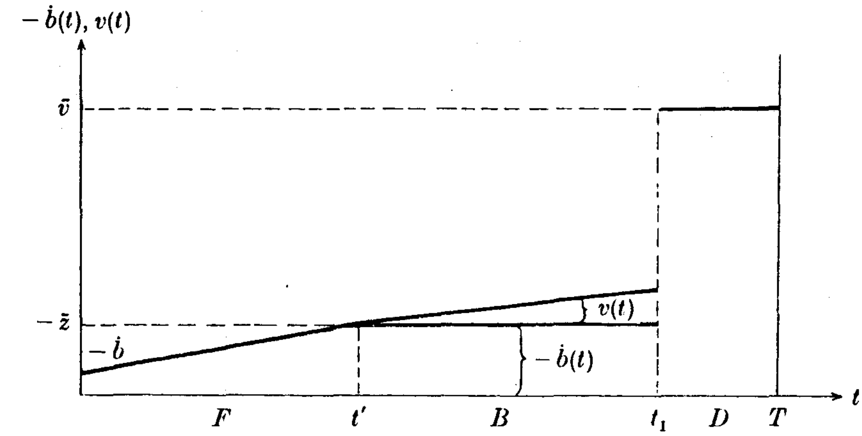

Fig. 3

For the time being, the values of α and γ are such that case I does not seem very relevant.[1] However, it is of general interest and may become relevant in the future. In the Appendix it is shown that regardless of whether or not consumption per capita is increasing along the optimal path, borrowing is increasing until resource extraction begins, except under extremely unrealistic conditions regarding the values of ϱ, α and β. The optimal policy sequence is then FD. Since borrowing is increasing in F it is also possible that borrowing reaches its "ceiling" at $t'<t_1$, so that at t' the economy is in C. For $t'<t<t_1$ the optimal policy is B, where the resource is extracted for direct import purposes as a supplement to debt-financed consumption (which is decreasing over time in this regime, due to the servicing of the debt). The optimal policy sequence is then $FCBD$, as illustrated in Fig. 3.

Reduced borrowing over time in case I is possible only in an extreme deflationary situation where import prices are falling and the nominal rate of interest on bonds is very low (see Appendix).

In case II finance capital is always positive, and the optimal policy sequence is DF. Borrowing does not take place and the resource is never extracted for direct import purposes. For the time being, this is the economically most relevant case.[2]

III. Discussion of the Results

1. Some policies will always be nonoptimal:

(i) In a resource-exporting economy, which is not confronted with any international borrowing restrictions, resource extraction at less than the maximal rate is always inoptimal.

[1] While this was true when the first draft was written, the reverse is true at the time of the final revision (June, 1979).
[2] See footnote 1.

(ii) If resource extraction takes place at less than its maximal level, resource use for financial accumulation purposes or for debt repayment cannot be optimal. Conversely, if financial investment is optimal, resource extraction at less than the maximal rate is always inoptimal.

(iii) When resource extraction takes place in order to import consumption goods, it is inoptimal to borrow less than the maximum amount of money abroad at any time. Equivalently, it is nonoptimal to borrow less than the possible maximum abroad as long as the resource good is exchanged for consumption goods abroad. It follows that in a situation with no (effective) upper bound on borrowing abroad, resource use in exchange for consumption import cannot be optimal.

2. When the rate of growth in the price of the resource is greater than the rate of interest on financial claims (case I), it pays to keep the resource in the earth as long as possible. The optimal policy sequences therefore have the following typical properties:

(i) Resource extraction always takes place at a maximal rate at the end of the planning period.

(ii) Borrowing abroad in the first part of the planning period and repayment at the end is optimal. The foreign debt is increasing throughout until repayment begins and decreasing throughout the repayment period until it is zero at the time horizon.

(iii) The broad aspects of optimal resource use and financial transactions mentioned above are independent of the value of the social rate of discount, the rate of population growth and the rate of growth in the price of consumption goods.

(iv) If the social rate of discount plus the rate of increase in population is greater (less) than the rate of increase in the price of the resource (or the rate of interest on bonds if there is no effective upper limit on borrowing) minus the rate of growth in the price of consumption goods, the optimal path of consumption per capita is decreasing (increasing) over time.

(v) Except under extreme economic conditions, borrowing increases during the borrowing period, and the borrowing restrictions may become effective before the repayment period begins.

(vi) Resource extraction may or may not be optimal during the borrowing period. If it is optimal, it is for the purpose of direct import of consumption goods only. Such a policy can only be optimal if borrowing possibilities are exploited at their maximum.

3. When the rate of growth in the price of the resource is less than the rate of return on financial assets (case II), it pays to shift the resource into financial assets as fast as possible. The optimal policies then have the following typical properties:

(i) Resource extraction takes place at a maximal rate at the beginning of the planning period until the resource is exhausted.

(ii) During the resource extraction period the country lends abroad. When this period is over, the proceeds on, and the foreign bonds themselves, are spent for consumption purposes.

(iii) The optimal consumption over time will be decreasing (case II.1) or increasing (case II.2) depending on whether the social rate of discount plus the rate of growth in population is greater or less than the rate of interest on bonds minus the rate of increase in the price of consumption goods.

4. To gain more insight into the conditions in the economy under which different policies will be optimal, consider partial changes in the data of the problem. An increase in the initial exogenous consumption $\bar{c}_0$ would in all cases "lift up" the consumption profiles over time. An increase in the initial value of the resource $p_{v,0}x_0$ or a fall in the initial price of the consumption good $p_{c,0}$, would have the same effect. A higher social rate of discount ϱ would, according to (15) and (17), increase the rate of decline in consumption in cases I.1 and II.1 and reduce the rate of growth in consumption in cases I.2 and II.2. In both cases, initial consumption would increase. An increase in the rate of population growth n has the same effects on consumption. The higher the upper bound on borrowing $\bar{z}$ is, the less likely it is that resource extraction for direct import purposes takes place as part of the optimal policy. If there is no upper bound on the rate of resource extraction v, repayment of the debt in case I would take place immediately before T with $v(t) \to \infty$, or in case II, the regime in which $v > 0$ would be infinitely short at the start of the planning period with an infinitely high extraction level. An increase in the growth rate of the price of consumption goods β leads, according to (15) and (17), to a faster fall for c in cases I.1 and II.1, or a slower increase in cases I.2 and II.2. The effect of a change in β on initial consumption is *a priori* ambiguous. The reason is that a change in β has two opposite effects, similar to a substitution and an income effect. If β is e.g. reduced, future income in real terms is increased. This is the income effect. However, a reduced β also increases the "price" of initial consumption, relative to consumption in the future. This is the substitution effect. The substitution effect would tend to reduce initial consumption, while the income effect works in the opposite direction. (This result was originally reported in Strøm (1974) and is elaborated there.) A fall (rise) in β would therefore not automatically lead to reduced (increased) initial consumption and borrowing in cases I.1 or II.1, although this is the best guess. Similar reasoning holds for cases I.2 and II.2. Without borrowing restrictions, the extraction path is not affected by a change in β. When the restrictions are effective, a rise in β would, according to (18), affect the optimal rate of change in extraction in two ways: negatively through the falling growth rate of consumption and positively through a steeper rise in the price of imports

relative to the resource price. The total effect is therefore unpredictable on the basis of the theory. An increase in the rate of change in the price of the resource, γ, may change the optimal policy from case II to case I. The rate of change in consumption along the optimal path will be affected if, and only if, borrowing possibilities are exploited at their maximum. Then, c falls less fast in case I.1 and increases faster in case I.2. Regarding initial consumption, the effects will be of a type similar to those for a change in β. When γ increases, future income increases, whereas the "price" of initial consumption increases, and there is an income and a substitution effect working in opposite directions. If borrowing restrictions are effective, $\dot{v}$ is, according to (18), affected positively from a positive shift in γ through the term $\dot{c}$ and negatively through the term $-\gamma v$, since less resources are then needed to pay for a given amount of consumption. The total effect on the rate of change in resource extraction is therefore again ambiguous. Finally, an increase in the rate of interest on bonds α may change the optimal policy from case I to case II. Without borrowing restrictions, a higher α leads, according to (15), to a slower decline in c in cases I.1 and II.1 and a faster increase in cases I.2 and II.2. A higher rate of interest on bonds would therefore tend to postpone consumption in both cases, either to evade some of the higher costs of borrowing (case I) or to profit from the higher yield on bond holdings (case II). When borrowing restrictions are effective, the rate of change in consumption along the optimal path is not affected by a change in the interest rate.

5. Compared with the earlier studies mentioned in the introduction, the main effect of allowing financial investment or disinvestment in a model of resource extraction in an open economy, is to separate the optimal consumption stream over time from the optimal path of resource extraction. If borrowing possibilities are unlimited, the separation will be complete. The results with respect to the optimal path of resource extraction in a closed economy as obtained by e.g. Vousden (1973) or in an open economy without borrowing possibilities as obtained by e.g. Strøm (1974) are then no longer valid. These results are typically that resource extraction (and consumption) should be decreasing along the optimal path, while in this model optimal resource extraction is always either zero or at its maximum. Resource extraction takes place in the beginning or at the end of the planning period, depending solely on whether the rate of interest on foreign bonds is higher or lower than the rate of increase in the resource price. This is in contrast to Vousden (1973), where the optimal path of resource extraction depends on the social rate of discount and on the properties of the utility function, or Strøm (1974), where in addition the rate of increase in the relative price of the resource also enters the resource extraction function. An optimal consumption path is then brought about by financial transactions. In addition to the properties of the social welfare function and the social rate of discount, which determine the optimal consumption path in Vousden (1973), this path is now—when there is no

effective limit on borrowing—also determined by the rates of change in the world-market prices of consumption goods combined with the rate of interest on bonds and population growth.

When borrowing possibilities are fully exploited, the result given in (18) does not correspond to the result obtained by Strøm (1974) for an open economy without financial transactions. The reason is the servicing of the debt. Because of this, the change in the value of resource extraction over time is not equal to the change in the value of consumption. It should also be noted that in contrast to the results in the contributions mentioned above, there is, in this model, no reason to expect *a priori* that optimal resource use is falling over time.

In conclusion, compare the relative rate of increase in total consumption, *after* the existence of the resource has been acknowledged in the optimal plan, with the growth rate *before* the resource was taken into account. From (15′) in the Appendix we solve for $\alpha-\beta$:

$$\alpha-\beta = g_1(-\check{\omega})+\varrho+n. \tag{19}$$

The expression corresponding to the RHS of (19) before the resource is acknowledged in the plan, is

$$\mu(-\check{\omega})+\varrho+n.$$

Whether $g_1 \gtrless \mu$ therefore depends on whether $\alpha-\beta \gtrless \mu(-\check{\omega})+\varrho+n$. Reasonable values for the constants (see Appendix) indicate quite clearly that $g_1<\mu$. The optimal reaction to a newly discovered resource is therefore a *positive* shift in *initial* consumption combined with a *reduced* relative rate of growth in consumption, as compared to the situation before the resource was discovered.

Appendix

Assume that the elasticity of marginal utility $\check{\omega}=U''\cdot c/U'$ is a constant. (15) may then be written as

$$\dot{c}=g_1 c(t) \quad \text{where} \quad g_1=(-(\alpha-\beta)+\varrho+n)/\check{\omega} \tag{15′}$$

so that

$$c(t)=c(0)e^{g_1 t}.$$

When $v=0$, we have that $\dot{b}-(\alpha-n)b=-p_c(c-\bar{c})$. To simplify, suppose that the exogenous component in consumption is constant. It may then, without loss of generality, be set equal to zero. In case I, $b(0)=0$ and the solution for $b(t)$ is

$$\text{(i)} \quad b(t)=\frac{p_{c,0}c(0)}{k}(e^{(\alpha-n)t}-e^{(g_1+\beta)t}); \quad t<t_1,$$

where $k=g_1+\beta-(\alpha-n)$.

When $-\dot{b}=\bar{z}$, which may occur only in case I, (18) now simplifies to

$$\dot{v}(t)=(g_2+\beta)c_v p_c/p_v+(\alpha-n)\bar{z}/p_v-\gamma v$$

where

$$g_2=(-(\gamma-\beta)+\varrho+n)/\check{\omega}.$$

Assuming $g_1+\beta\neq\alpha-n$, (i) shows that $b(t)<0$ as long as $v=0$. From (i)

$$\dot{b}=\frac{p_{c,0}c(0)}{k}(\alpha-n)e^{(\alpha-n)t}-(g_1+\beta)e^{(g_1+\beta)t}),$$

so that $\dot{b}<0$ and the foreign debt is always increasing as long as $v=0$. Further

$$\ddot{b}=\frac{p_{c,0}c(0)}{k}(\alpha-n)^2e^{(\alpha-n)t}-(g_1+\beta)^2e^{(g_1+\beta)t}).$$

$\alpha-n$ is positive. $\ddot{b}$ is then always negative, except if $g_1+\beta$ is negative and greater than $\alpha-n$ in absolute value. Disregarding n, this means $g_1+\beta<-\alpha$. Inserting for g_1 and using -2 as a reasonable guess for $\check{\omega}$, this implies $\varrho>3\alpha+\beta$. "Realistic values" for ϱ, α and β are 0.02–0.04, 0.07–0.09 and 0.05–0.07, respectively. We may therefore safely assume that $\ddot{b}<0$, so that borrowing increases over time in case I.

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ON LABOUR ALLOCATION, SAVINGS AND RESOURCE EXTRACTION IN AN OPEN ECONOMY

By Jostein Aarrestad

1. Introduction.

Dating back to Ramsey's classical article [6], the problem of optimal accumulation of physical capital has been given much attention in the literature. Especially since the midsixties, wide ramifications and generalizations of the Ramsey model have been given. Natural resources do not enter these types of models in an explicit way. On the other hand, economic analysis of natural resources has traditions back to Malthus and Ricardo. The basis for the modern treatment of the best use of an exhaustible natural resource is the classical article by Hotelling [3]. Lately the problem has also been analyzed from a macroeconomic point of view by e.g. Koopmans [5] and Vousden [9]. In these models either all consumption in the economy is provided from the resource, or an additional source of consumption, outside the model, is postulated. The assumption of no alternative sources of consumption is extreme and unrealistic. The assumption of an alternative, exogenous, source of consumption has been introduced by Vousden in [9] as "a convenient simplification of the relevance of the rest of the economy to the resource-use decision". But except for foreign aid and the most primitive type of subsistence agriculture, the time path of the alternative consumption will depend on the amounts of labour and physical capital in the rest of the economy and the savings ratio together with the growth in labour supply and technological progress. It seems reasonable to think that e.g. the optimal savings ratio will depend on the availability of natural resources in the economy. On the other hand it does not seem reasonable to assume that the optimal path of resource depletion will be completely independent of e.g. the stock of physical capital in the economy and the resulting potentiality for consumption from sources other than the current resource extraction. This shows the need for an intergrated model of the economy where optimal savings and resource extraction can be determined simultaneously. The purpose of this paper is to present and analyze such a model for a small, open economy.

The interrelationship between the optimal rate of investment and the optimal depletion of natural resources has been explored by Heal and Dasgupta [2]. However, their model is rather different from the following model, which has the recent petroleum discoveries in the North-Sea as its background. As Strøm [7], we shall consider an open economy where the resource good is exchanged for other goods in the world market. Also, resource extraction is controlled directly in [2] by "turning the tap", whereas in this model extraction is controlled by the employment in the resource sector. Since labour must be released from the rest of the economy in order to extract resources, labour allocation over time between the two parts of the economy must be optimized. Finally, marginal extraction costs are constant in [2], whereas in this model they are increasing due to the increasing alternative cost of labour. The problem of optimal extraction of a non-renewable resource in an open economy has earlier been analyzed by Vousden [10], Kemp and Suzuki [4], van Long [8] and Strøm [7]. Common to all these contributions are that there is no physical capital accumulation in their models, so that the problem of determining the optimal accumulation of capital, together with resource extraction disappears. In this paper the central issues will therefore be:

(1) What is the optimal intertemporal pattern of physical capital accumulation in an open economy with a considerable stock of an exhaustible resource,
(2) What is the optimal intertemporal pattern of labour allocation and resource extraction when extraction depends on the amount of labour released from the rest of the economy,
(3) How are the decisions implicit in (1) affected by conditions in the resource sector, and
(4) How are the decisions implicit in (2) affected by conditions in the rest of the economy.

2. The Model.

The following variables will be used:

c = total consumption per capita

y = total production per capita in market value

s = the (average) savings ratio

y_1 = production per capita, exclusive of resource extraction, in physical units (and market value)

y_2 = resource extraction per capita in physical units

k = physical capital per capita

l_i = the part of labour allocated to sector i (i=1,2)

p = the net price of the resource relative to the "price" of all other goods in the world market

x = the stock of the resource per capita

U = instantaneous social welfare

ρ = the social rate of discount (constant and positive)

n = the rate of growth in total population

μ = the rate of depreciation of physical capital

α = the marginal (and average) productivity of labour in resource extraction (a constant).

The problem is then:

(1) maximize $J = \int_0^\infty U(c(t))e^{-\rho t}\,dt$

subject to

(2) $c(t) = (1-s(t))y(t)$

(3) $y(t) = y_1(t) + p(t)y_2(t)$

(4) $y_1(t) = l_1(t)f(k(t)/l_1(t))$

(5) $y_2(t) = \alpha l_2(t)$

(6) $\dot{k}(t) = d\,k(t)/dt = s(t)y(t) - \lambda k(t)$

(7) $\lambda = n + \mu$

(8) $-\dot{x}(t) = y_2(t) + nx(t)$

(9) $0 \leq s(t) \leq 1$

(10) $0 \leq y_2(t) \leq \bar{y}_2$

(11) $l_1(t) + l_2(t) \leq 1$

(12) $l_1(t) \geq 0,\ 0 \leq l_2 \leq \bar{l}_2$

$k(0) = k_0$ (given); $x(0) = x_0$ (given)

$\lim_{t\to\infty} x(t) \geq 0, \quad \lim_{t\to\infty} k(t)$ free

ρ, n, μ, α exogenously given constants, $p(t)$ exogenously given.

Stated in words, the problem is to find such paths over time for total savings and labour allocation between the two sectors that the present value of social welfare is maximized. The planning horizon is infinity and intantaneous welfare depends on consumption per capita. In (1) we assume that $U'>0$, $U''<0$ and $\lim_{c\to 0} U'(c)=\infty$. Total production, evaluated at world-market prices, is divided between consumption and saving. All net earnings form resource extraction are used for import. In (4), f is invariant over time and we assume $f'>0$, $f''<0$ and $f'(0)>\lambda$. (4) is derived from a production function in capital and labour homogeneous of degree one. By (5), resource extraction depends on labour only. (5) is derived from a production function with fixed coefficients for both labour and physical capital, assuming that labour is never underemployed. The reason is that oil production in an open economy is easily financed abroad. Necessary capital equipment is rented abroad and need not be financed out of domestic savings. The problem of allocating physical capital optimally between the two sectors is therefore avoided. $p(t)$ is net of capital costs per unit of the resource and assumed independent of the amount exported ("small country" argument). In contrast, labour is assumed not to be imported. Thus domestic labour allocated to oil production (actual production, catering, supply, administration, government supervision, etc.) must be released from the rest of the economy.

(6) is the expression for the change in the capital intensity, well-known from growth theory. (8) says that the stock of the resource per capita is at any time reduced by the extraction per capita and is also diluted by nx because of the growth in population. By (9), s must be non-negative, and it cannot exceed one. Resource extraction is by (10) irreversible. An upper bound $\bar{y}_2$ on resource extraction is also assumed, due to e.g. limited pipe-line capacity or loading facilities. The point is that the transport capacity had been created without being optimized from the point of view of the producing nation. Another justification for an upper bound is membership in a production cartel with production quotas for each member. For simplicity it is assumed that this upper bound is independent of time. By (11) and (12) employment in the two sectors cannot exceed the total labour

force and employment in each sector must be non-negative. Also, due to (10) there is an upper bound $\bar{l}_2$ on l_2, such that $\bar{y}_2=\alpha\bar{l}_2$. Finally, the initial stocks of capital and the resource are given and there is no restriction on the capital stock at any time, whereas the stock of the resource must always be non-negative.

In addition to the assumption of a balanced current account, the structure of the model above also assumes:

a) No search activity for new resources.
b) No uncertainties. In particular the future relative price of the resource is assumed known.
c) The stock of the resource does not affect social welfare or the extraction conditions, except that it restricts total extraction.
d) External effects are disregarded. Examples might be pollution due to oil-spill, blow-outs, reduced fishing possibilities or the fact that two (or more) countries are extracting petroleum from the same reservoir.
e) The producing country does not use the resource as an input.

Assumption a) to e) are not trivial. Still, this model contains aspects from "real life" not found in any of the countributions quoted above.

3. Optimality conditions and policy regimes.

To solve the above problem, form the Lagragean expression[1)]

$$L = e^{-\rho t}\{U\,[(1-s)(l_1 f(k/l_1) + p\alpha l_2)\,] + q_1[s(l_1 f(k/l_1) + p\alpha l_2)-\lambda k]-q_2(\alpha l_2 + nx) + w(1 - l_1 + l_2)\}, \tag{13}$$

where we have inserted for c, y, y_1 and y_2. q_1 and q_2 are co-state variables associated with the equations of motion (6) and (8) and w is the shadow price of labour (the shadow wage).

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1) The time argument in the functions will from now on usually be dropped.

Necessary conditions[1)] for a solution to the problem are then that there exist continuous $q_1(t)$ and $q_2(t)$ such that

(14) $\dot{q}_1 = -U'(1-s)f' - q_1 sf' + (\rho+\lambda)q_1$

(15) $\dot{q}_2 = (\rho+n)q_2$

(16) $-U' + q_1 \begin{cases} \leq 0 \text{ if } s = 0 \\ = 0 \text{ if } 0 < s < 1 \\ \geq 0 \text{ if } s = 1 \end{cases}$

(17) $U'[(1-s)(f-(k/l_1)f')] + q_1 s(f-(k/l_1)f') - w \leq 0$ and $= 0$ if $l_1 > 0$

(18) $U'[(1-s)p\alpha] + q_1 sp\alpha - q_2\alpha - w \leq 0$ and $= 0$ if $l_2 > 0$.

From (1), (16) and (17), $w > 0$ so that (11) always holds as an equality. A number of regimes are possible in the optimal solution, see table 1, where the value of the main variables in the different regimes are given, assuming that $p(t)$ is a constant, for simplicity equal to one. Given k, q_1 and q_2, (16), (17) and (18) determine $s(t)$, $l_1(t)$ and $l_2(t)$ so that production, consumption, resource extraction and capital accumulation are determined. We shall study each regime in turn.

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1) Such a solution will really be optimal, since

a) (13) is concave in k, x, s, v, l_1 and l_2 for given q_1, q_2 and t, and

b) it will be shown later that $x(t)$ will be exhausted in finite time and that q_1 and k approach finite limits as t goes to infinity.

Table 1

Variables / Regime	s	l_1	l_2	q_1	q_2	c	$\dot{k}$	$\dot{x}$
A	>0	1	0	$=U'$	$>q_1$ $[\alpha-(f-kf')]/\alpha$	$(1-s)f(k)$	$sf(k)-\lambda k$	$-nx$
B	>0	$1-\bar{l}_2$	$\bar{l}_2$	$=U'$	$<q_1$ $[\alpha-(f-(k/\bar{l}_1)f')]/\alpha$	$(1-s)[\bar{l}_1 f(k/\bar{l}_1)+\alpha\bar{l}_2]$	$s[\bar{l}_1 f(k/\bar{l}_1)+\alpha\bar{l}_2]-\lambda k$	$-\alpha\bar{l}_2-nx$
C	>0	>0	>0	$=U'$	$=q_1$ $[\alpha-(f-(k/l_1)f')]/\alpha$	$(1-s)[l_1 f(k/l_1)+\alpha l_2]$	$s[l_1 f(k/l_1)+\alpha l_2]-\lambda k$	$-\alpha l_2-nx$
D	$=0$	1	0	$<U'$	$>q_1$ $[\alpha-(f-kf')]/\alpha$	$f(k)$	$-\lambda k$	$-nx$
E	$=0$	$1-\bar{l}_2$	$\bar{l}_2$	$<U'$	$<q_1$ $[\alpha-(f-(k/\bar{l}_1)f')]/\alpha$	$\bar{l}_1 f(k/l_1)+\alpha\bar{l}_2$	$-\lambda k$	$-\alpha\bar{l}_2-nx$
F	$=0$	>0	>0	$<U'$	$=q_1$ $[\alpha-(f-(k/l_1)f')]/\alpha$	$l_1 f(k/l_1)+\alpha l_2$	$-\lambda k$	$-\alpha l_2-nx$

At the outset regimes A, B and C are analyzed together since they are based to a considerable extent on (14) and (16) which are valid in all three regimes. Since $q_1 = U'$ in these regimes it follows from (14) that

$$(19) \qquad \dot{q}_1(t) = (-f' + \rho + \lambda)q_1(t)$$

so that

$$(20) \qquad \dot{q}_1 \lesseqgtr 0 \text{ if } k \lesseqgtr l_1 k^*;$$

where k*, the "modified golden rule" capital intensity, is defined by $f'(k^*) = \rho + \lambda$. As long as the economy is in any of these regimes, the overall capital intensity of the economy, k, approaches the optimal value $l_1 k^*$. q_1 is stationary for the same capital intensity. Stationaries for k in the (q_1,k)-plane are found from inserting for $s = s(q_1,k)$ from (16) in (6) with $\dot{k} = 0$. Implicit differentiation then yields

$$(21) \qquad \left.\frac{dq_1}{dk}\right|_{\dot{k}=0} = \frac{-f'+\lambda}{y\frac{\partial s}{\partial q_1}} .$$

This expression is well-known from the standard optimal growth model, as presented in e.g. [1]. For $f' >(<) \lambda$, $\dot{k} = 0$ has a negative (positive) slope in the (q_1,k)-diagram. The phase-diagram is given in figure 1, where $\tilde{k}$ is the maximum sustainable capital/labour ratio.

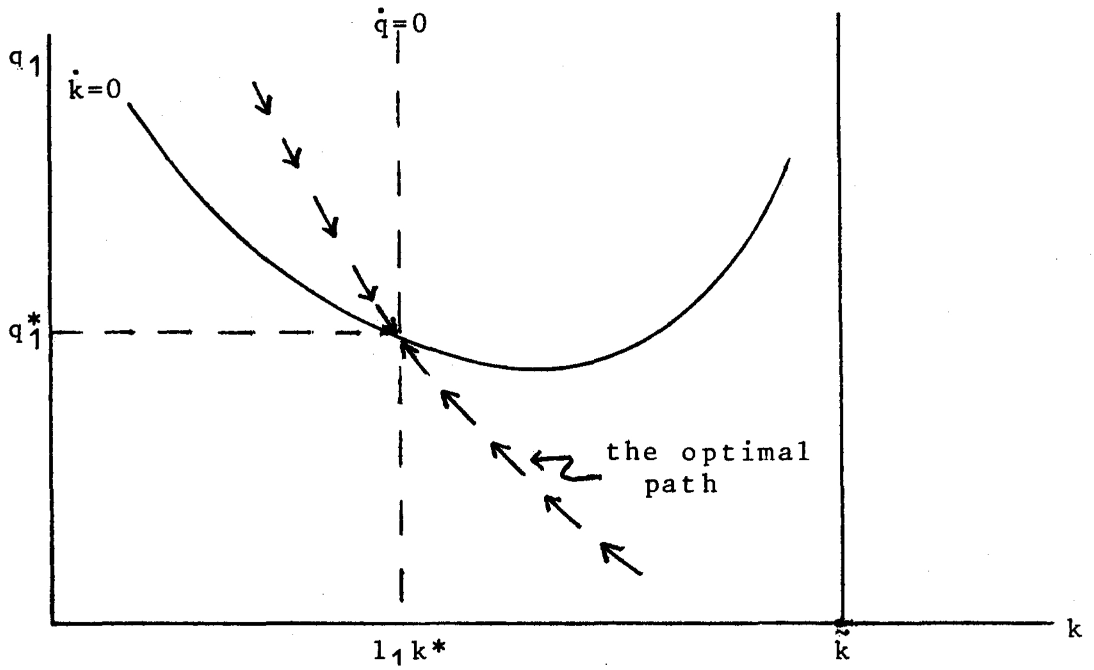

Figure 1.

$\dot{k} > 0$ (<0) for points above (below) the $\dot{k} = 0$ curve. It is intuitive (and easy to show) that (q_1^*, l_1k^*) is a saddlepoint in the (q_1,k)-plane. The optimal policy in these regimes are therefore:

1) If k_o is such that $k_o = l_1k^*$, find $q_1 = q_1^*$ and the system will be stationary, i.e. all variables are constant over time.

2) If k_o is such that $k_o \gtrless l_1k^*$, find that unique $q_{1,o}$ that leads the system towards (q_1^*, l_1k^*). For each k_o the optimal $q_{1,o}$ must lie on the "separatrix" marked the optimal path in the figure.

Consider now each of the regimes A, B and C.

A: In this regime no resource extraction takes place, since the social value of the marginal product of labour in the rest of the economy - $U'(f-kf')$ - is greater than the social value of the marginal product of labour in resource extraction - $(U'-q_2)\alpha$. q_2 is the user cost of the resource, reflecting the fact that what is extracted "today" cannot be used "tomorrow", since the resource stock is finite. q_2 denotes the addition to the optimal value of the criterion function of leaving the marginal unit of the resource at t unexploited. The user cost must be subtracted from the marginal utility of consumption to obtain the social value of resource extraction. In regime A, $l_1 = 1$ and the optimal policy is identical to the standard one-sector growth model, see e.g. [1] ch. 11.

B: In B resource extraction is at its maximum and the social value of the marginal product of labour is higher in this sector than in the rest of the economy. As long as the economy is in B, $k/\bar{l}_1$, where $\bar{l}_1 = 1 - \bar{l}_2$, approaches k^*. q_1 is stationary for the same value of k. Thus, as time elapses in B, k approaches a stationary value together with s and c. The higher $\bar{l}_1$ is, the higher is this stationary value. Since the resource stock is finite the time period where the economy is in B must also be finite.

C: (The Interior Solution):

In C resource extraction takes place, but at less than its maximal rate. From (17) and (18) the allocation of labour over time between the resource extraction sector and the rest of the economy is governed by

$$(22) \quad q_1[f-(k/l_1)f'] = (p(t)q_1 - q_2)\alpha. \text{ [1]}$$

Rearranging (22) as

$$(23) \quad q_1[p\alpha - (f-(k/l_1)f'] = \alpha q_2$$

it can be seen that, because of the user cost, for resource extraction to take place, the monetary value of the marginal product of labour in resource extraction, $p\alpha$, must exceed the monetary value of the marginal product of labour in the rest of the economy, $f-(k/l_1)f'$. The social value of this difference - $q_1[p\alpha - (f-(k/l_1)f']$ - must be equal to the user cost - αq_2 - of employing labour in resource extraction. The discrepancy increases through time, since the user cost grows exponentially.

With k_o given initially when regime C becomes effective, we may now distiguish two cases:

1\. $k_o < l^*_{1,o}k^*$ with $l^*_{1,o}$ given by (22). By the reasoning in connection with fig. 1, k/l^*_1 must in this case increase along the optimal path. From (22) the optimal $l_1(t)$, $l^*_1(t)$, depends on $q_1(t)$, $q_2(t)$ and $k(t)$. To study the optimal allocation of labour over time, differentiate (22) with respect to time and solve for $\dot{l}^*_1(t)$:

$$(24) \quad \dot{l}^*_1 = \frac{1}{q_1(k^2/l^{*3}_1)f''}\left\{\dot{q}_1[\alpha-(f-(k/l^*_1)f')] + q_1\alpha\dot{p} - \alpha\dot{q}_2\right\} + (l^*_1/k)\dot{k}.$$

We want to show that in this case l^*_1 must be increasing in C.

---

1) A time- dependent p(t) has now been re-introduced.

Consider the case where $k/\bar{l}_1 < k^*$ in B. k is then increasing in B. At t', say, the economy switches to C where $l_2 < \bar{l}_2$. By continuity of the co-state variables in (22) and (16) there can be no jump in l_2 or k at t', so that $\dot{k}(t' + \varepsilon) > 0$ for $\varepsilon > 0$ and sufficiently small. To show that k must be increasing everywhere in C, assume the opposite, i.e. that k reaches a maximum at $t'' > t'$, after which it decreases. But according to (24), l_1 would then be increasing in t", hence k/l_1 cannot be increasing in t". This is a contradiction and k must be increasing everywhere in C. From (24) it then follows that l^*_1 increases over time in C so that <u>resource extraction falls over time</u> in this regime when $k_o < l^*_{1,o}k^*$. When the initial regime in the economy is C, it will be shown below that eventually the economy must switch to A. In that case the above result can be shown in a similiar way by going backwards in time from A. Since q_1 falls in this case, c increases.

2. $k_o > l^*_{1,o}k^*$, $\dot{q}_1 > 0$ and k/l^*_1 must decrease. c falls. From (24) the sign of $\dot{l}^*_1$ is generally undetermined in such an economy. If the initial k/l_1 is very high, so that q_1 increases very fast, an *initial phase of increasing resource extraction over time cannot be excluded.* In the long run, however, q_2, which increases exponentially, will dominate the right-hand side of (22) so that sooner or later the allocation of labour to the resource sector must decrease - eventually towards zero - also in this case.

Regime C will last only for a finite time until extraction stops and the resource is exhausted. From (23), optimal allocation of labour between the two sectors in regime C is governed by

$$(25) \qquad [p\alpha-(f-(k/l_1)f'] \cdot U'\{(l_1 f(k/l_1)+\alpha l_2] \quad \} = \alpha q_{2,o} e^{(\rho+n)t}$$

In (25), the limit of the RHS as $t\to\infty$ is plus infinity, while the LHS for a positive l_2 is less than the LHS for $l_2 = 0$ which in turn is less than plus infinity. Thus (25) cannot hold for $t\to\infty$ and *there is some finite t, T, where l_2 goes to zero* and the extraction period is over. With a finite T, the transversality condition for x is

(26) $e^{-\rho T} q_2(T)x(T) = 0, \; e^{-\rho T} q_2(T) \geq 0.$

Since $q_2(t)$ is always positive, (26) is only satisfied for $x(T) = 0$. The same conclusions hold for regime F.

D,E,F: $\dot{q}_1 = 0 \Rightarrow q_1 = U'f'/\rho+\lambda$ so that:

(27) $$\left.\frac{dq_1}{dk}\right|_{\dot{q}_1=0} = \frac{f''U' + U''f'}{\rho + \lambda} < 0.$$

Since k is falling, q_1 must be increasing along the optimal path. In these regimes physical capital per worker is so abundant initially that its shadow price is less than the marginal utility of consumption per worker. Hence saving (and investment) is inoptimal. In F, (24) also holds.

Having studied the optimal behaviour of the economy within each policy regime, we now proceed to an analysis of possible switches between these regimes to find optimal policy sequences over time. Since situations where the initial capital intensity of the economy is less than the modified golden rule capital intensity are probably most relevant attention will be focused on them. Switches from initial situations in D, E and F will therefore not be considered.

4. Optimal policy sequences with constant prices.

The necessary conditions for the various swiches to take place are summarized in table 2 below where, again, $p = 1$. For simplicity, g(.) is defined as f(.) - (.)f'(.). A zero means that no switch is possible. Even if there are two co-state variables in this model, piecing the different policy regimes together is simple since q_2 is growing exponentially.

Table 2: Necessary conditions for policy switches.

Switch to / Switch from	A	B	C
A	.	0	0
B	0	.	$q_1 = U'[(1-s)[\bar{l}_1 f(k/\bar{l}_1) + \alpha(1-\bar{l}_1)]$ $q_1[g(k/\bar{l}_1)] = (q_1 - q_2)\alpha$
C	$q_1 = U'[(1-s)f(k)]$ $q_1[g(k)] = (q_1 - q_2)\alpha$	0	.

Consider the possible optimal sequences from an initial situation in B. The shadow price of physical capital is equal to the marginal utility of consumption and falling so that consumption is increasing over time. The economy is accumulating capital. The resource is extracted at a maximal rate, since, at the margin, the social value of the productivity of labour in resource extraction is higher than in the rest of the economy. A plausible example of an actual economy that might correspond to this picture is the UK after the recent petroleum and gas discoveries in the North-Sea. Policy B is transient. Since the user cost of the resource is exponentially increasing, thereby steadily reducing the social value of resource extraction, policy B cannot be sustained ad infinitum. Sooner or later the economy therefore switches to C, where the resource is extracted at less than its maximal rate and where resource extraction over time is reduced. This is accomplished by allocating a steadily increasing part of the labour force away from the resource sector into the rest of the economy. During the process, consumption always increases. Again, the initial situation may also be in C. The Norwegian case, where initial oil and gas production is deliberately kept lower than the maximum possible, seems to correspond well to this regime. When the social value of the product of "the last person" employed in resource extraction is less than its value in the rest of the economy, resource extraction ends and the whole labour force is employed in the rest of the economy. Extraction goes gradually towards zero and it ends in finite time as the resource is completely exhausted. The economy then switches to A, the final policy, which, once established will last for ever as the economy approaches the optimal steady state, identical to the optimal development in the standard one-sector optimal growth model. The optimal policy sequence is therefore either BCA or CA. It is always optimal to reduce extraction gradually before it stops altogether. Maximal extraction until extraction stops is non-optimal. Such a policy would require a jump in one of the co-state variables, which is impossible by the maximum principle.

5. Effects of changes in data, including p(t).

Consider first the effects of a positive shift in k_o, everything else being equal. Write (23) as

$$(23b) \qquad q_1(t) = \alpha q_2(t)/[\alpha - g(k/l_1)]$$

where $g = f(\cdot)-(\cdot)f'(\cdot)$.

(23b) must hold in regime C. Sequence BCA is illustrated in fig. 2 by variables with superscrip zero.

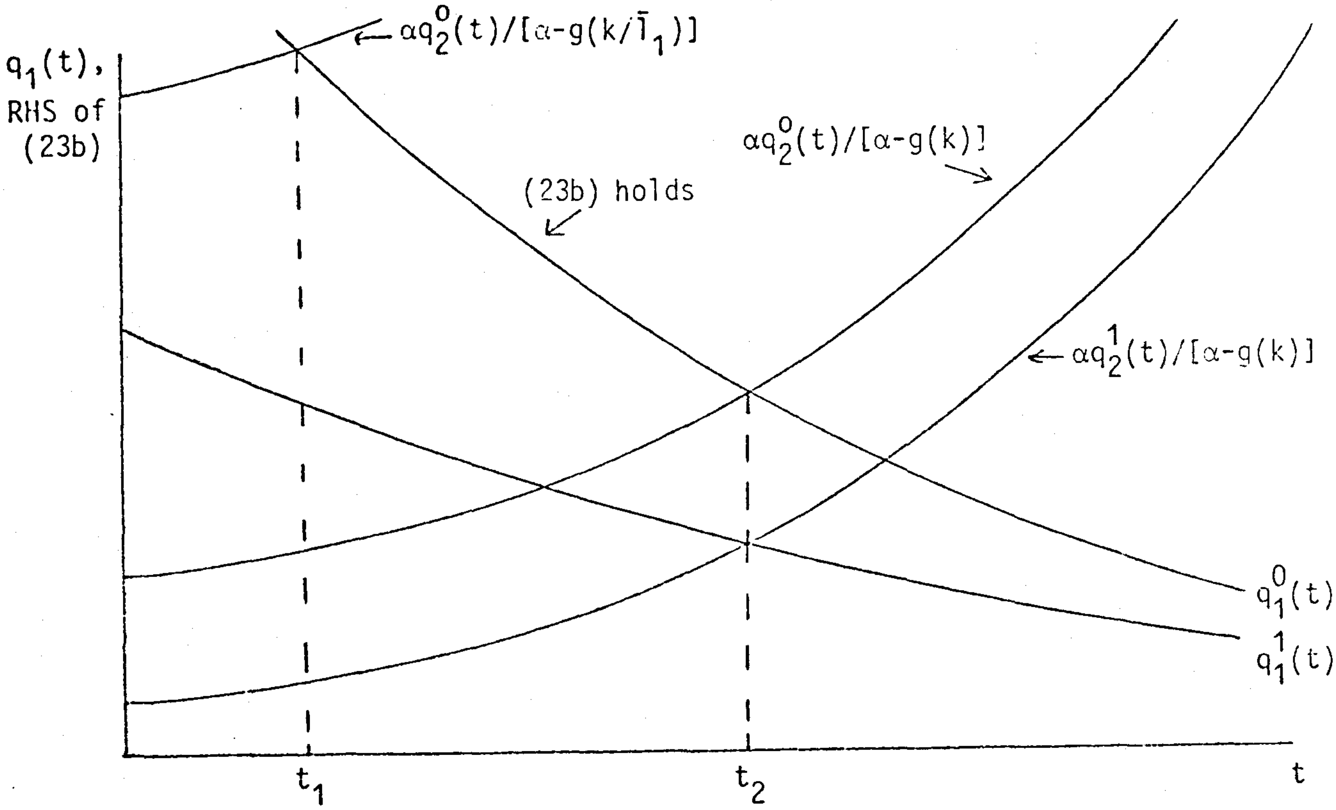

Figure 2.

Until t_1, the economy is in B and resource extraction is maximal. From t_1 to t_2 it is in C, after which it is in A. In B and A the RHS of (23b) is always increasing since αq_2 is always increasing, and the denominator is always decreasing since k increases along the optimal path. A higher k_o would imply a lower $q_1(t)$ everywhere. If not, c would decrease for some t which is clearly inoptimal. This is illustrated by $q_1^1(t)$ in the figure. $q_1^1(t)$ also falls less steeply over time than $q_1^o(t)$ - this follows from (19). In addition, a higher k_o also means a higher k everywhere along the optimal path (except in the optimal stationary state), so that the curve $\alpha q_2^o(t)/[\alpha-g(k)]$ is shifted upwards if k_o gets a positive shift. From the figure it is then clear that a higher k_o would, cet. par., also mean a lower $q_{2,o}$. If not, the intersection of $q_1^1(t)$ and $\alpha q_2^o(t)/[\alpha-g(k)]$ would take place for $t < t_2$ so that the extraction period would be shortened. Also, for (23b) to hold in this period, with a lower $q_1(t)$ everywhere, the denominator of the RHS of (23b) must increase for every t, so that l_2 must be reduced for every t. But reduced extraction for every t and a shorter extraction period cannot exhaust the resource, hence it is inoptimal by the transversality condition (26).

Suppose then that $q_{2.o}$ falls so that the extraction period is unchanged, indicated by $q_2^1(t)$ in the figure. Going backwards from t_1, since $q_1^1(t)$ is less steep than $q_1^o(t)$, the difference between the two curves - $q_1(t)$ and $\alpha q_2/[\alpha-g(k)]$ - is less in the new situation, except for $t=t_2$, and the discrepancy increases as we approach zero. Therefore, when (23b) holds, the denominator of the RHS of (23b) must be greater for all t. But with an extraction level everywhere less than before k_o was shifted up, the resource cannot be exhausted at t_2, *hence the optimal extraction period is lengthened when the initial capital intensity of the economy gets a positive shift*. Regarding a positive shift in x_o, it can be shown along the same lines that $q_1(t)$ and $q_2(t)$ must fall. Resource extraction and consumption increase and the extraction period is increased.

Consider next the effects of changes in the social rate of discount ρ. An increased ρ has two effects: (i) the modified golden rule level of capital per worker - with or without activity in the resource sector - is reduced, and this reduces q_1 for every given k (or k/l_1) so that the optimal s is lower for every k. (ii) The shadow price of the resource, $q_2(t)$, rises faster over time. The latter effect leads to a faster extraction of the resource by concentrating extraction more towards the beginning of the planning period. Consumption is also increased for some time in the first part in the planning period, but as the economy approaches steady state, consumption will decrease as ρ increases. An increase in the rate of growth in population has the same effects.

Consider finally the effects of an exponential trend in p(t) so that $p(t) = p_o e^{\beta t}$. (22) may now be written as

$$q_1[f-(k/l_1)f'] = (p_o q_1 e^{\beta t} - q_{2,o}e^{(\rho+n)t})\ \alpha$$

which shows that for $\beta < \rho+n$, the optimal development in the economy will, in principle, be the same as when $\beta = 0$. *For rates of increase in the price of the resource higher than the social rate of discount plus the growth rate in population, (24) shows that a rising resource extraction over time cannot be excluded, especially if $\dot{q}_1$ is small, i.e. when k/l_1 is "near" k^*.* An extreme variant of this case is that it is optimal to leave the resource in the ground for some time to profit from its increasing value over time. Whether a society can "afford" such a strategy, depends on how abundant the initial capital intensity of the economy is.

6. Comparing the results with earlier models.

The results obtained on the optimal path of resource use in this combined model are somewhat more general than those found in earlier contributions. With constant prices and the capital intensity less than or equal to the modified golden rule capital intensity, extraction is either constant for some initial period and then falling, or always falling. The extraction period is finite and the resource is always exhausted when extraction ends. All these results are similar to those found by Vousden in [9]. However, the result on the effect on the optimal extraction path from changes in the initial resource-stock does not bear any resemblance to the result in [9] that "the optimal time of exhaustion will increase as the alternative source of consumption, $\bar{C}$, falls". Since the welfare function is the same in both models, the contrast in the solutions must be due to the difference in supply conditions. In the resource model the resource may be used for consumption purposes only. Extracting the resource is the only source of consumption in addition to the exogenous component. Since consumption from the resource is subject to a finite upper bound on cumulative extraction, society must - in order to survive - stretch out the use of the resource when the alternative source of consumption falls. In the comined model, the value of the resource extracted may also be used for capital-formation purposes, and survival can be secured on basis of the physical capital stock alone. The future benefits to be achieved from an extra unit of the resource extracted for investment purposes is higher the smaller the capital stock already attained. The optimal resource use in the combined model is therefore slowed down when the initial capital intensity gets a positive shift. Also with an initial capital intensity higher than the modified golden rule, a rising extraction is possible for some time. In such a contracting economy, an increasing shadow price of capital (equal to the marginal utility of consumption), which makes investment and consumption more valuable in the future, tends, cet. par., to make it optimal to postpone extraction; this effect must be balanced against a rising user cost of the resource. The combined effect may well be to keep back production for some time, in contrast to the ordinary case where a falling shadow price of capital and an increasing user cost of the resource both lead to a higher extraction now than in the future. While a contracting economy may be of limited practical interest, the relevance of the capital intensity on optimal resource use is obvious in the case of a rising price of the resource. In that case the increasing user cost is counteracted by the rising resource price. In a model without capital, extraction rises if the rate of growth in price is higher than in the user cost. In a model with capital, this pattern is accentuated if capital is above its optimal steady-state level, since then the social value of capital over time increases as well. If capital is scarce, however, its shadow price is falling and it may fall so fast that it offsets the rate of increase in the resource price over and above the user cost. In that case optimal extraction would still be falling over time. Such a country is too poor to afford to wait for the higher prices, at least for some initial period until the capital stock of the economy is built up.

In the opposite direction, the question of how the optimal savings and consumption plan of the economy is affected by conditions in the resource sector can be studied by assuming that a new source of natural resources, like a new petroleum resevoir, is discovered and exploited at t_o.
Possible optimal sequences from t_o have been explored earlier. Assume that extraction begins at a maximal rate at t_o so that immediately after t_o, $l_2 = \bar{l}_2$. The capital intensity in the rest of the economy is then $k_o/\bar{l}_1$ where k_o was the capital intensity of the economy immediately before resource extraction started. The optimal paths of, s, c, and k from t_o then depend on whether $k_o/\bar{l}_1 \gtrless k^*$.

(a) If, by accident, $k_o/\bar{l}_1 = k^*$,

$\dot{q}_1 = 0$ as long as $l_2 = \bar{l}_2$. k is constant over time.
s is a constant $\hat{s}$ given by

$$\hat{s} = \frac{\lambda k_o}{\bar{l}_1 f(k_o/\bar{l}_1) + \alpha \bar{l}_2}$$

and c is a constant $\hat{c}$ given by

$$\hat{c} = (1-\hat{s})(\bar{l}_1 f(k_o/\bar{l}_1) + \alpha\bar{l}_2)$$

in this interval. Assume then that the system switches to an interior solution l_2 at $\hat{t}$. Immediately after $\hat{t}$, $k_o/l_1 < k^*$ and q_1 falls which means that c increases for $t > \hat{t}$. This development continues beyond t_1, when extraction ends. Assuming the economy was on an optimal growth path before t_o, k was then less than k^*, and c and k were increasing. Since total production per capita jumps up at t_o, while k is kept constant for $t \in [t_o, \hat{t}]$, total savings and the optimal savings rate must shift down at t_o. The optimal consumption path is illustrated below.

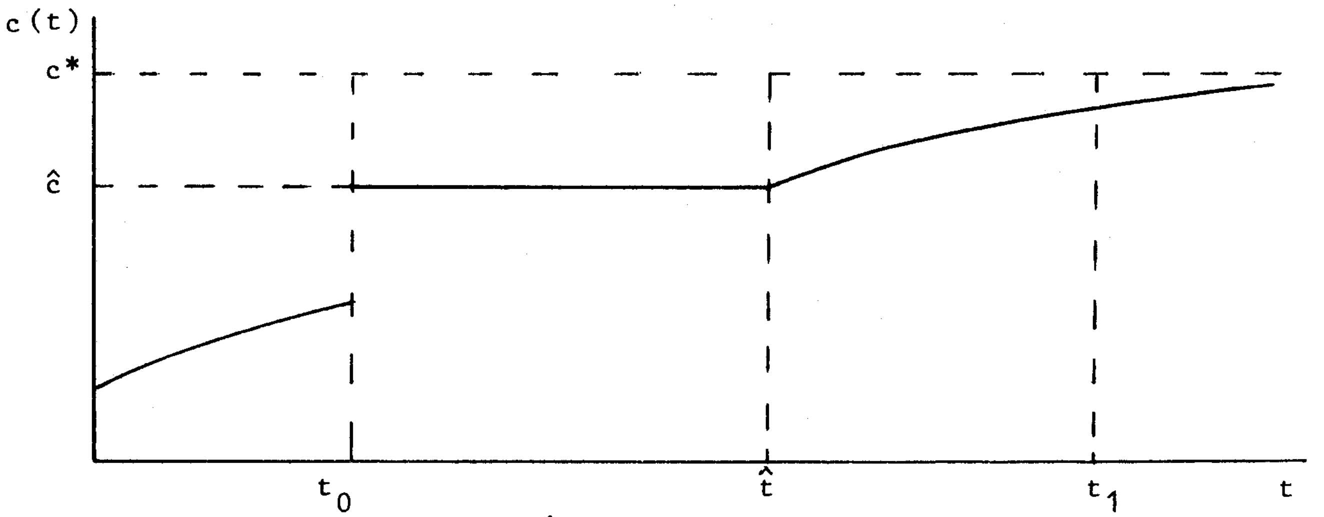

Figure 3.

For $t < t_o$, c increases along the optimal growth path of an economy without natural resources. At t_o, when the resource is discovered and exploited maximally for some period, c jumps up to a constant level which is kept as long as extraction is maximal. The "extra" c enjoyed in this period includes not only the extra production due to the resource; in addition the absolute amount of savings per capita falls at t_o. As the economy reaches the point $\hat{t}$ where maximal extraction stops, it has exactly the same amount of physical capital as in t_o, which means that it is poorer in <u>physical</u> capital than it would have been without the resource (where k always increases along the optimal path if $k_o < k^*$). After $\hat{t}$, c and k increases over time, both in the period where the resouce is extracted at a decreasing rate, and after the resource is exhausted, when c and k again approach their steady-state levels c^* and k^* asymptotically.

(b) If $k_o/\bar{l}_1 > k^*$, $\dot{q}_1 > 0$ and k is falling for $t \in [t_o, \hat{t}]$.

Since $q_1 = U'$, this means that c is falling in this period. Since k falls, absolute savings also falls at t_o compared to the level before the resource was discovered and exploited, so the extra consumption enjoyed is greater than the value of the resource extraction. As $k/\bar{l}_1$ approaches k^*, consumption and savings per capita approach the constant levels $\hat{c}$ and $\hat{s}$ found in case (a). At $\hat{t}$ the economy enters the stage where resource extraction decreases, with $k/l_1 > k^*$. Provided $k_o \leq k^*$ there must therefore be a point $t^* > \hat{t}$ where $k/l_1 = k^*$ so that $\dot{q}_1 = \dot{k} = 0$ in t^*. For $t > t^*$, $\dot{q}_1$ is therefore negative and c and k increase towards their optimal steady-state levels. c and k have a minimum in t^*. The optimal consumption path when $k_o/\bar{l}_1 > k^*$ is illustrated below.

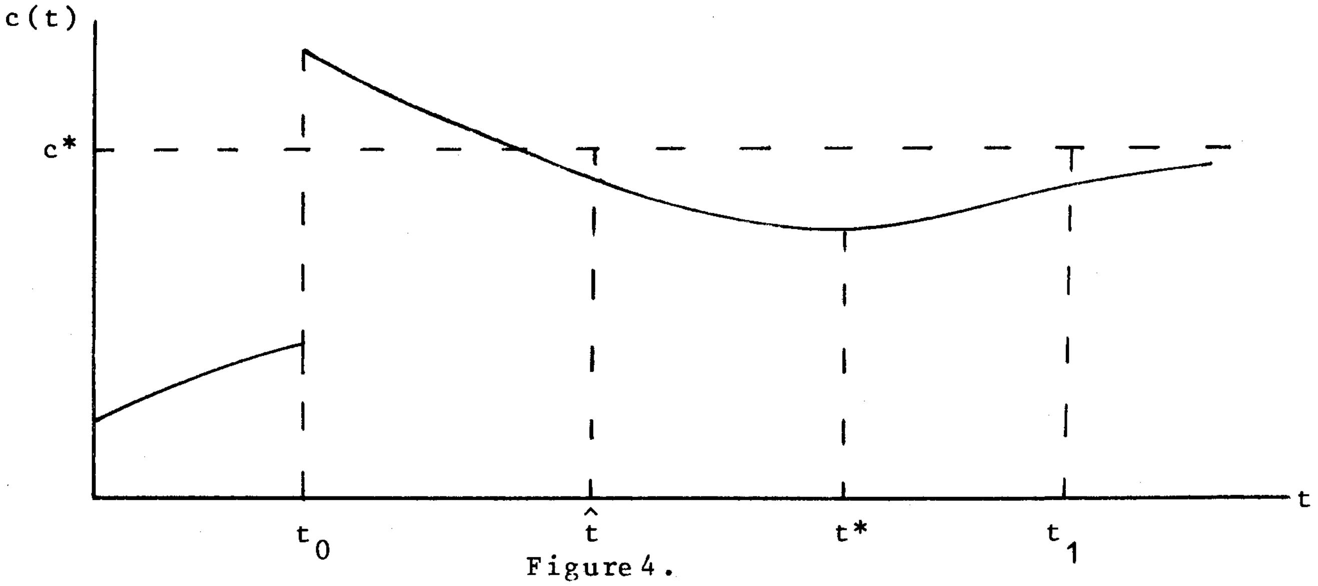

Figure 4.

An initial situation where the economy is in the neighbourhood of the optimal steady state before t_o is also included here, which shows that consumption may well be above the longrun optimal level in most of the resource extraction period in this case. The time profile of consumption will have a maximum at the beginning of the extraction period. The aggregative capital intensity of the economy will now be lower after the period of maximal resource extraction than it was initially, and for an initial capital intensity near k*, the actual capital intensity may well be less after the extraction period is over and for some time in the post-extraction period.

(c) $k_o/\bar{l}_1 < k^*$. In this case q_1 is decreasing, c is increasing and k is increasing for all $t \geq t_o$, including the period where resource extraction is maximal. The capital intensity in the economy when resource extraction is maximal is now less than the optimal capital intensity when $l_2 = \bar{l}_2$. Even so, there will be a positive jump in consumption per capita in t_o. This is because, for $t \in [t_o, \hat{t}]$, the economy "aims for" a lower capital intensity than it did for $t < t_o$. This means that the optimal absolute amount of savings is less for $t \geq t_o$ than for $t < t_o$, so that c shifts up at t_o. The optimal consumption path for this case is illustrated below.

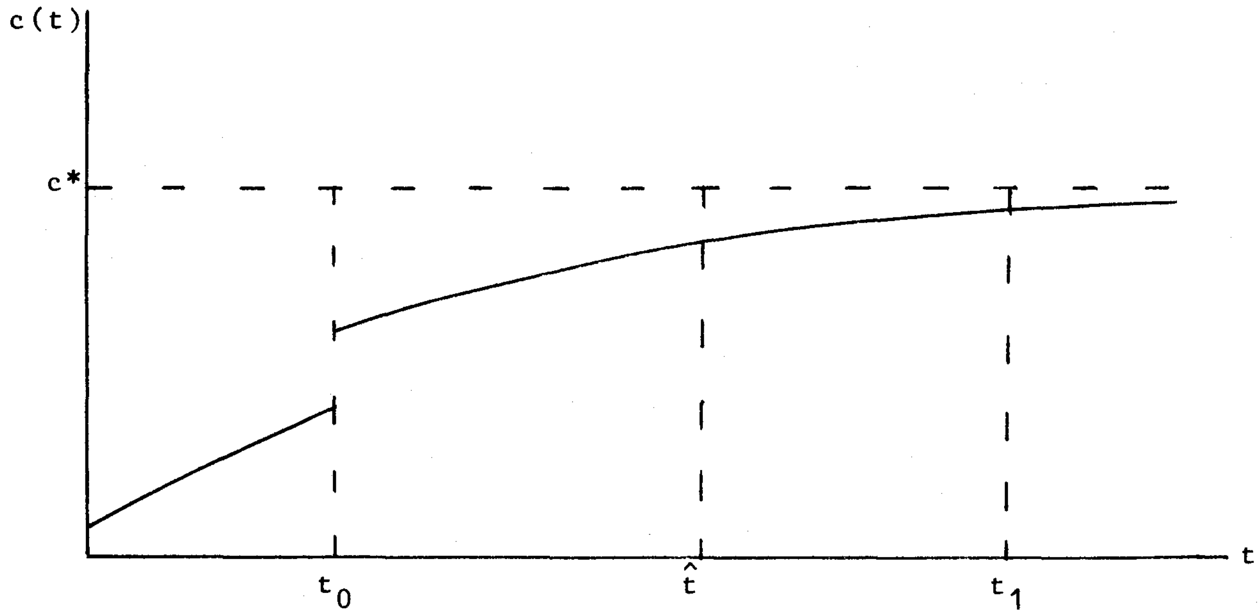

Figure 5.

To summarize: The optimal savings rate and also the absolute amount of savings are always shifted down when exploitation of a new resource begins, so that total consumption increases by *more* than the value of the new resources extracted.

Optimal capital accumulation should be slowed down, even if the long run optimal steady-state capital intensity remains the same. With resource extraction, a given level of capital intensity will be reached after a longer period. *The optimal pattern of economic development is therefore to slow down capital accumula-*

tion when resource extraction is started up and for the period extraction lasts, compared to a situation without resource extraction. When the resource-extraction period is over, the stock of physical capital is therefore lower than it would have been at the same time without resource extraction, but it is higher than when resource extraction started if the capital intensity then was less than, or equal to, the modified golden rule capital intensity. Thus, the widespread notion that savings and capital accumlation should increase when a natural resource, like petroleum, is discovered and exploited, is not substantiated in this model.

In cases (a) and (b) above it is obvious from the figures that the relative (and absolute) rate of change in consumption is reduced when extraction starts. Concerning case (c), differentiate $q_1 = U'$ with respect to time and use (19). The relative rate of change in consumption along the optimal path may thus be written as

$$(28) \qquad \dot{c}/c = (-f' + \rho + \lambda)/\check{\omega}$$

where $\check{\omega} = c \cdot U''/U'$ is the elasticity of marginal utility. When extraction starts, l_1 is reduced so that f' is reduced. With $\check{\omega}$ (approximately) constant in the relevant range, (28) shows that the relative rate of growth in consumption is reduced when extraction starts. *The optimal reaction to a newly discovered resource is therefore a positive shift in initial consumption combined with a reduced relative rate of growth in consumption,* compared to the growth rate before the resource was discovered.

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On the Optimal Development of a Small, Open Economy with an Exhaustible Resource

by

Jostein Aarrestad

1. Introduction

Optimal extraction of exhaustible natural resources over time has been analyzed from a macroeconomic point of view by e.g. Koopmans [6], Vousden [7] and Heal & Dasgupta [3]. These contributions assume a closed economy. Optimal resource extraction in open economies has been analyzed by Vousden [8], Kemp and Suzuki [5], Aarrestad [1] and Heal, Dasgupta & Eastwood [4]. In [5] and [8], however, accumulation of physical or financial capital is disregarded, i.e. all consumption in the economy is provided from the resource and no borrowing and lending abroad is assumed to take place, which clearly resistricts the relevance of the models for actual decision making in resource rich open economies. In [1], a model for an open economy is presented where optimal savings and resource extraction can be determined simultaneously. Since there is no financial capital in the model, savings take the form of physical capital accumulation only. In [4] financial capital is also incorporated. The purpose of this paper is to provide an alternative, and in some respects more general, model of optimal resource use in an open economy where optimal paths of resource extraction, consumption, financial transactions and savings in physical capital can be determined simultaneously. After having presented the model and related it to [4], the analysis of optimal policies and an economic discussion of the results follow. A comparison with earlier results will be given at the end.

2. The Model and Optimality Conditions

The folowing variables will be used:

$c(t)$	=	total consumption per capita
$v(t)$	=	total resource extraction per capita
$k(t)$	=	physical capital per capita
$f(k)$	=	production per capita, exclusive of resource extraction
$m(t)$	=	import per capita
$b(t)$	=	stock of foreign bonds or, if negative, foreign debt per capita
$x(t)$	=	the stock of the resource per capita
$p_v(t)$	=	the price of the resource in the world market
$p_m(t)$	=	the "price" of import in the world market
$C(v)$	=	total extraction costs per capita in real terms
$\Pi(v)$	=	net proceeds from resource extraction per capita
U	=	social welfare
ρ	=	the social rate of discount
n	=	the rate of growth in total population
μ	=	the rate of depreciation of physical capital
γ	=	the rate of change in the resource price over time
β	=	the rate of change in the price of import over time
r	=	the rate of return on foreign bonds or the rate of interest on debt.

The problem is then

$$\text{Max} \int_{t_o}^{\infty} U(c(t))e^{-\rho t}dt$$

s.t.

$$\text{(i)} \quad \dot{k}(t) = f(k(t)) + m(t) - c(t) - \lambda k(t)$$

$$\text{(ii)} \quad -\dot{x}(t) = v(t) + nx(t)$$

$$\text{(iii)} \quad \dot{b}(t) = rb(t) + \Pi(v(t)) - p_m(t)m(t) - nb(t)$$

$$\text{(iv)} \quad \Pi(v(t)) = p_v(t)v(t) - p_m(t)C(v(t))$$

$$\text{(v)} \quad \lambda = \mu+n$$

$$\text{(vi)} \quad p_v(t) = p_{v,o}e^{\gamma t}$$

$$\text{(vii)} \quad p_m(t) = p_{m,o}e^{\beta t}$$

$$\text{(viii)} \quad c(t) \geq 0$$

$$\text{(ix)} \quad m(t) \geq 0$$

(x) $\dot{b}(t)/p_m(t) \geq -\bar{z}$ when $b < 0$

(xi) $v(t) \begin{cases} \geq 0 \text{ when } C'' > 0 \\ \in [0, \bar{v}(t)] \text{ when } C'' = 0 \end{cases}$

(xii) $k(0) = k_o$, $x(0) = x_o$, $b(0) = 0$

(xiii) $\lim_{t \to \infty} k(t)$ is free, $\lim_{t \to \infty} x(t) \geq 0$, $\lim_{t \to \infty} b(t)/p_m(t) \geq 0$

(xiv) $\rho, n, \mu, \gamma, \beta, r, p_{v,o}, p_{m,o}$ exogenously given constants.

Stated in words, the problem is to find such paths over time for resource extraction, physical capital formation and foreign lending/borrowing that the present value of total social welfare is maximized. The control variables of the problem are c, v and m and the state variables are k, x and b. The planning horizon is infinity. Instantaneous welfare depends on consumption per capita only and we assume that $U' > 0$, $U'' < 0$ and that $\lim_{c \to 0} U'(c) = +\infty$.

The total population is assumed to grow at the same rate as the labour force. Consumption per capital is given implicitly in (i), the equation for the increase in the capital intensity of the economy, $\dot{k}$. (i) says that the capital intensity of the economy increases by the domestic production and import per capita minus depreciation and consumption per capita. (ii) says that the stock of the resource per capita is reduced by the extraction per capita and is also diluted because of population growth. By (iii) financial investment per capita equals the interest on foreign assets plus any net proceeds from resource extraction minus the value of imports minus a population-effect, all in per capita terms. $\dot{b}$ may also be nagative, in which case there is a negative financial investment. b may be negative as well, in which case the society owes money abroad. The only good exported is the resource, which is extracted for export purposes only. Net proceeds from resource extraction is given by (iv). p_v is independent of the amount exported ('small country' argument). The real extraction costs consist of input of the "macro-good". Expressed in value, the costs are then $p_m(t)C(v(t))$. The analysis is carried out with two different assumptions on extraction costs:

(a) that marginal extraction costs are constant ($C'>0$, $C''=0$) - an assumption implicit in most of the macroeconomic literature on optimal resource use, or (b) marginal extraction costs are rising when extraction per unit of time increases ($C'>0$, $C''>0$) - a more realistic assumption.

By (xi), when marginal extraction costs are constant, we shall assume there is some upper bound on per capita extraction per unit of time due to technical reasons as e.q. limited pipe-line capacity or loading facilities for tankers.[1)] The point is that the transport capacity has been created without being optimized from the point of view of the producing nation. A different justification for an upper bound on production is memebership in a production cartel with production quotas for each member. In addition (xi) says that resource extraction is irreversible. By (viii) and (ix) consumption and imports are also non-negative [2)] and by (vi) and (vii) the prices of the resource and of the macro-good grow each at a constant rate which may be positive, negative or zero.[3)] (x) says that when the country has debt, there is an upper bound on the real value of the debt-increase per capita per unit of time. The reason is existing convensions in financial circles on how much a country of a given size may borrow abroad during e.g. one year and a fear on the part of the national government of losing control of economic policy due to pressure from abroad. By (xii) the stocks of physical capital and the resource are given initially, and initially the country has no debt or claims abroad. Finally, by (xiii), the stocks of the resource and of the real value of foreign claims must be non-negative as time approaches infinity, while no restriction is needed on the stock of physical capital. The main simplifications of the model are:

a) No search activity for new resources
b) No uncertainties. In particular the future relative price of the resource is assumed known.
c) The stock of the resource does not affect social welfare or the extraction conditions, except that it restricts total extraction.

d) External effects are disregarded. Examples might be pollution due to oil-spill, blow-outs, reduced fishing possibilities or the fact that two (or more) countries are extracting petroleum from the same reservoir

e) The producing country does not use the resource as an input.

Assumption e) is avoided in [4]. Uncertainty is also introduced in one variant of the model, which causes considerable complications. On the other hand, it is in [4] assumed constant extraction costs, constant prices, no borrowing restrictions and instant and costless transformation of financial capital into physical capital and vice versa. Specific functional forms are also used to obtain unambigous results. In those respects the model in [4] is less general than the model presented above.

To analyze the problem, form the (present value) Lagrangean:

$$(1)\quad L = e^{-\rho t}\ \{U(c(t))+q_1(t)[f(k(t))+m(t)-c(t)-\lambda k(t)] +q_2(t)(-v(t)-nx(t))+q_3(t)[rb(t)+\Pi(v(t))-p_m(t)m(t) -nb(t)]+\mu_1 c(t)+\mu_2 m(t)+\mu_3\ [\overline{z}+rb(t)+\Pi(v(t))-p_m(t)m(t) -nb(t)]+\mu_4 v(t)+\mu_5(\overline{v}-v(t)\}$$

where q_1, q_2 and q_3 are co-state variables associated with k, x and b respectively,[4)] and where μ_i (i=1,..,5) are Lagrangean multipliers associated with the constraints (viii) - (xi). It is easily verified that necessary conditions for a solution to the problem are

$$(2)\qquad U' - q_1 + \mu_1 = 0$$

$$(3)\qquad -q_2+q_3\Pi'+\mu_4-\mu_5+\mu_3\Pi' = 0$$

$$(4)\qquad q_1-p_m q_3+\mu_2-\mu_3 p_m = 0$$

$$(5)\qquad \dot{q}_1 = (-f'+\rho+\lambda)q_1$$

$$(6)\qquad \dot{q}_2 = (\rho+n)q_2$$

$$(7)\qquad \dot{q}_3 = (-r+\rho+n)q_3 -\mu_3(r-n)$$

(8) $\mu_1 \geq 0, \quad \mu_1 c = 0$

(9) $\mu_2 \geq 0, \quad \mu_2 m = 0$

(10) $\mu_3 \geq 0, \quad \mu_3[\bar{z}+rb+\Pi-p_m m-nb] = 0$

(11) $\mu_4 \geq 0, \quad \mu_4 v = 0$

(12) $\mu_5 \geq 0, \quad \mu_5[\bar{v}-v] = 0$

3. Policy Regimes

3.1 No Borrowing Restrictions.

Due to the assumptions on the U-function, c is always positive, so that $\mu_1 = 0$. Consider now the case with no restrictions on borrowing so that $\mu_3 = 0$. Assume also that imports are positive so that $\mu_2 = 0$. Instead of (3), (4) and (7) we then get

(3') $-q_2+q_3\Pi'+\mu_4-\mu_5 = 0$

(4') $q_1-p_m q_3 = 0$

(7') $\dot{q}_3 = (-r+\rho+n)q_3.$

a) When marginal extraction costs are constant, write p for the "net price" $p_v - p_m C'$, so that $\Pi' = p$. With this cost structure, (3'), together with (11) and (12), show that

$$(13)\quad v^*(t) = \begin{cases} \bar{v} \text{ if } q_2 < pq_3 \\ \in [o,\bar{v}] \text{ if } q_2 = pq_3 \\ 0 \text{ if } q_2 > pq_3 \end{cases}$$

so that *optimal extraction is either maximal or zero.*

Differentiating (4') with respect to t and using (5) and (7'), we obtain

(14) $f'-\mu = r-\beta.$

(14) is an obvious condition for optimality: Marginal returns

on the two types of assets considered there should be equalized. The marginal product of physical capital minus its rate of depreciation, i.e. the net marginal productivity of physical capital should be equalized to the real rate of interest on bonds (the nominal interest on foreign assets minus the percentage rise in prices on imported goods). Optimal use of the third asset - the resource - is governed by (13). Consider first the case when prices are constant. Then, from (6) and (7') the relative rate of change in q_2 is greater than in q_3. This means that initially $q_2 < pq_3$[5)] and the resource is extracted at a maximal rate. At T, say, $q_2 = pq_3$ and the optimal solution for v is singular. For t>T, however, $q_2 > pq_3$ and resource extraction stops. Since the interior solution does not last more than "an instant of time", *the resource is extracted at a maximal rate until extraction stops.* With a finite stock of resources initially, x_o, T must be finite. The transversality condition for x(t) is then

$$(15)\quad e^{-\rho T} q_2(T)x(T) = 0; \quad e^{-\rho T} q_2(T) \geq 0.$$

Since $q_2(t)$ is always positive, (15) can only be satisfied for $x(T) = 0$. *At T the resource will therefore be exhausted.*

With constant prices there is no positive return on the resource-asset. With constant extraction costs, it is therefore optimal to extract the resource as fast as possible and convert it into assets with a positive return. This need not be the case when prices are rising. Suppose the "net price" $p(t) = p_v(t) - p_m C'$ increases at a constant rate α so that $p(t) = p_o e^{\alpha t}$. Of course, if p_m is a constant, $\alpha = \gamma$. Then the relative rate of change in q_2 is greater than the relative rate of change in $p(t)q_3$ as long as $\alpha \leq r$, i.e. when the percentage rise in the net price of the resource is less than the nominal rate of interest. From (13) optimal extraction policy with fixed extraction costs is then the same as in the case with constant prices. If, however, $\alpha > r$, the rate of growth in the net price of the resource is greater than the rate of interest on financial claims. It then pays to keep the resource in the ground as long as possible. A model with infinite planning horizon is not well suited to analyse this case. In a model where the planning horizon is finite, however, it can be shown that extraction should not take place until the end of the planning period, where the resource is extracted at a maximal rate until it is exhausted at the terminal date. In this case the country would obviously always be a net borrower.

Using (4') in (2), (2) and (7') leads to

(16) $\dot{c} = (U'/U'')(\rho+n-r+\beta)$

which shows that, in the absence of any change in import prices ($\beta=0$), consumption per capita will be steadily increasing (decreasing) if r is greater (less) than $\rho+n$. This merely reflects the fact that unless $r-\beta = \rho+n$, the capital intensity of the economy in this model does not approach the modified golden rule level k* defined by $f'(k^*)-\mu = \rho+n$, as in the standard model of optimal economic growth,[6)] but a level given by (14). Suppose that initially the stock of physical capital is below this level; *it is then instantly adjusted to this optimal level*[7)] since there is no upper bound on import (or debt-increase) in this regime - provided the value of the resource stock is sufficient to permit such an increase in k. Depending on the discrepancy between the initial capital stock and the optimal stock and the rate of maximal extraction, the society may be a net borrower or lender initially in this case. Due to (xiii), however, the country must be free from debt when extraction ends, since the resource is the only export-good. In the opposite initial situation when the real rate of interest on bonds is higher than the net productivity of capital, the country will be a net lender and the capital intensity of the economy must shrink until $f' = \mu+r-\beta$. An extreme variant of this case is when (4) holds with an inequality sign. Imports are then zero and the stock of foreign assets is built up at a maximal rate. Of course, the country may also enter the "post-extraction" period (t>T) with a stock of foreign assets.

If import prices are rising, (16) shows that consumption will be increasing along the optimal path as long as the percentage rise in import prices is less than the (positive) difference between the rate of interest and the social rate of discount, plus the rate of growth in population. If the rise in import prices is greater, consumption will fall along the optimal path.

b) Alternatively, when C">0, there is no upper bound on extraction, so that $\mu_5=0$. When v*(t) is positive, $\mu_4=0$. Differentiating (3') with respect to t we then obtain

$$\dot{v}^*(t) = \frac{1}{\Pi'' q_3}(\dot{q}_2 - \Pi'\dot{q}_3)$$

Inserting for $\dot{q}_2$ and $\dot{q}_3$ and using (3') this simplifies to

(17) $$\dot{v}^*(t) = r\Pi'/\Pi''.$$

(17) shows that *with constant prices and increasing marginal costs, extraction is falling over time as long as the resource is extracted.* Writing out (3') in full,

(18) $$(p_v - p_m C'(v))U'[f(k)+m-k-\lambda k] = q_{2,o}e^{(\rho+n)t}.$$

In (18), $\lim_{t\to\infty}$ RHS = ∞, while cet. par. LHS(v) < LHS (o)<∞(since $P_v - p_m C'(o) < \infty$).

c is always positive. Thus (18) cannot hold for t → ∞ and *there is some finite t where v goes to zero and the extraction period is over. At this time the resource is exhausted.* This follows from the same argument as when extraction costs were constant. When marginal extraction costs are increasing, (14) and (16) still hold. Optimal policies with respect to physical and financial capital are therefore in principle unchanged, except that the optimal resource extraction path is modified due to another cost structure. v*(t) is somewhat different now since the gains in interest of converting the resource into financial capital must be balanced against the increasing extraction costs per unit of time, as evidenced by (16), which shows that the optimal rate of extraction is determined by the rate of interest on financial claims and the properties of the cost-function.

When marginal extraction costs are rising and prices are functions of time, differentiation of (3') yields

(17') $\dot{v}^* = [r\Pi' - \dot{\Pi}' \text{ (v const.)}]/\Pi''$

where $\dot{\Pi}'$ (v const.) $= \dot{p}_v - \dot{p}_m C'$ is the rate of change in marginal proceeds from resource extraction at a constant-output rate. (17') shows that a change in the marginal proceeds at a constant-output rate now also affects the optimal resource use. Cet.par. an exponential rise (fall) in marginal proceeds would tend to reduce (increase) the rate of fall in extraction along the optimal path. Inserting for Π' and rearranging, (17') shows that when p_m is a constant,

(19) $\dot{v}^* \gtrless 0$ as $r \lessgtr \gamma/(1-p_m C'/p_v)$.

Since $o < p_m C'/p_v < 1$, the RHS of (19) is always greater then γ. When marginal extraction costs are rising, the price rise on the resource needed to make an increasing path of resource use optimal may therefore be somewhat less than the rate of interest on bonds. Again (14) and (16) hold.

3.2 Restrictions on Borrowing.

Consider first the case when extraction costs are constant. When imports are positive, $\mu_2 = 0$. Instead of (4'), we now have

(4") $q_1 - p_m q_3 - \mu_3 p_m = 0$

(2), (3), (5), (6) and (7) still hold.

From (4"), using (5) and (7), it can be seen that

(14') $f' - \mu > r - \beta$.

In the presence of borrowing restrictions, the net marginal productivity of physical capital will exceed the real rate of interest. The reason is, of course, that a restriction on borrowing in turn implies a restriction on imports. The stock of physical capital can therefore not be adjusted instantly to the level where marginal returns on the two assets are equalized. Inserting for μ_3 in (3) from (4"), the analogue to (13) is now

$$(13') \qquad v^*(t) = \begin{cases} \bar{v} \text{ when } p_m q_2 < p q_1 \\ \in [o, \bar{v}] \text{ when } p_m q_2 = p q_1 \\ o \text{ when } p_m q_2 > p q_1 \end{cases}$$

Assuming again that the "net price" of the resource, p(t), increases at the constant relative rate α, it follows from (vii), (5) and (6) that the relative rate of change in $p_m q_2$ is greatest if, and only if, $f' - \mu > \alpha - \beta$. In that case the resource is extracted immediately at a maximal rate until it is exhausted. If not, the resource is extracted at a maximal rate at the end of the planning period, and it is exhausted at the terminal date. Consequently, *if and only if there are effective restrictions on borrowing is the optimal extraction path affected by the physical capital intensity of the economy.* The rate of interest is then irrelevant for the resource-use decision. From (2), the analogue to (16) is now

$$(16') \qquad \dot{c} = (U'/U'')(\rho + \lambda - f')$$

where $f' - \mu$ has replaced $r - \beta$ in (16). (16') is identical to the formulae for the absolute growth of consumption per capita in the standard optimal growth model.

When extraction costs are rising, differentiating $q_2 = q_1 \Pi'/p_m$ yields, after some manipulations

$$(17'') \qquad \dot{v}^* = [(f' - \mu + \beta)\Pi' - \dot{\Pi}' \text{ (v const.)}]/\Pi''$$

With borrowing restrictions and constant prices ($\beta = \gamma = o$), (17") shows that the optimal rate of extraction is determined by the net marginal productivity of physical capital together with the properties of the cost function. The rate of interest is again irrelevant. With changing prices an exponential rise (fall) in the marginal proceeds from resource extraction, or an exponental fall (rise) in the price of imports, would cet.par. tend to reduce (increase) the rate of fall in extraction along the optimal path. Inserting for Π' and rearranging, it follows from (17") that, for a constant p_m,

$$(19') \quad \dot{v} \gtrless 0 \text{ as } f'(k) \lesseqgtr \gamma/(1-p_m C'/p_v)+\mu.$$

(14') and (16') hold also when extraction costs are rising.

The question remains whether the "candidate" optimal policies analyzed above are really optimal. First of all, the Lagrangean (1) is concave in k, x, b, m, v and c. When extraction is falling over time, $\lim e^{-\rho t} q_1(t)(k(t)-k^*) = 0$, x(t) will be exhausted in finite time and (with one exception to be mentioned later) b(t) will also go to zero in finite time. In that case the solution to the problem is really optimal. If optimal extraction is increasing over time, and the horizon is infinite, conditions ensuring that the candidate policies are really optimal are not satisfied. With a finite horizon, however, the necessary conditions are also sufficient for optimality due to the concavity of the Lagrangean.

3.3 Effects of Changes in Data.

Based on the previous analysis of the workings of the model, consider partial changes in the data of the problem. A higher social rate of discount, ρ, does not affect the extraction path. It reduces the rate of growth in consumption, when consumption is growing along the optimal path and increases the fall rate when consumption is falling. In both cases, initial consumption would increase. An increase in the rate of growth in population has the same effects. An increase in the nominal rate of interest on bonds, r, would by (14), lead to a lower capital intensity in the economy. Without borrowing restrictions, the rate of change in consumption is affected as from a fall in the social rate of discount. With borrowing restrictions, the rate of change in consumption is not affected by a change in r. Without borrowing restrictions, optimal resource extraction depends on the rate of interest such that a higher rate of interest leads to a faster extraction. When borrowing restrictions are effective, the rate of interest does not affect extraction policy. An increase in the relative rate of growth in the price of the resource, γ, always affects the optimal extraction policy such that extraction tends to be postponed. The rate of change in consumption is not affected. An increase in the rate of growth in the price of imported goods, β, leads to a lower real rate of interest and therefore to a higher capital intensity in the economy. The optimal extraction policy is affected differently by a change in β depending on whether there are restrictions on borrowing or not. With free borrowing, an increase in β would, cet.par. reduce net proceeds from the resource over time through the cost term. By (17') this would tend to speed up resource use by concentrating extraction more towards the beginning of the planning period. With borrowing restrictions, this effect is still present. In addition the optimal rate of depletion is now, by (17'), also affected by β per se. Thus, in this case, even if net proceeds from resource extraction were constant over time an increase in the rate of change in import prices would increase the optimal fall rate of extraction. The rate of change in consumption is affected by β in the absence of borrowing restrictions only. By (16), the rate of increase in c is reduced when β increases or the rate of fall is increased. If the initial physical capital intensity, k_o, is increased, the discrepancy between the initial net productivity of physical capital, $f'-\mu$, and the real rate of interest, $r-\beta$, is reduced. Provided $f'-\mu>r-\beta$, borrowing needs are therefore reduced, or the loan potential is increased. When k_o increases, the situation may change from one where restrictions on borrowing are effective to one with free borrowing. In that case it follows from the discussion in the next section, that resource extraction would be postponed. Also if borrowing restrictions are still relevant, a reduction in the difference between $f'-\mu$ and $r-\beta$ brought about by a positive shift in k_o would postpone extraction, as shown by (13') and (17"). Hence *a positive shift in the initial capital intensity of the* *economy leads to a less intense resource exploitation at the beginning of the planning period and to a lengthening of the resource extraction period if extraction costs are increasing. If extraction costs are constant, the resource-extraction period may be shifted from the beginning to the end of the planning period.* By (16') the rate of change in consumption is reduced (increased) if consumption is increasing (falling), if, and only if, there are borrowing restrictions. Of course, there is also a wealth effect from a positive shift in k_o, so that the consumption profiles over time are always "lifted up". An exogenous positive shift in the value of the initial stock of the resource, $p_{v,o}x_o$, would also "lift up" the consumption profile over time. If $p_{v,o}x_o$ shifts up, the economy may borrow a greater total amount. This means that the difference between $f'-\mu$ and r when borrowing must stop is reduced, and the economy may also enter the phase where the constraint in borrowing is no longer effective. By (13') and (17"), extraction of the resource is then less intense in the extended part of the borrowing period made possible by the shift in $p_{v,o}x_o$, and a fortioro if borrowing restrictions are no longer effective. If $\overline{z}$, the ceiling on borrowing, is increased, effects on resource extraction of a somewhat similar character results. This is so since the ceiling may then no more be effective, and even if it is, $f'-\mu$ falls faster towards $r-\beta$. In both cases the rate of extraction is slowed down. If the ceiling on borrowing is a function of the value of the resource stock of the economy, which is not unreasonable, the effects mentioned above of a shift in the initial resource stock will be strenghtened due to increased international "creditworthyness".

In general, therefore, *a positive shift in the initial value of the resource stock of the economy would lead to a less intense exploitation of the resource.*

4. Discussion.

a) With constant marginal extraction costs, an interior solution for resource extraction cannot be optimal for more than "an instant of time". When there are no borrowing restrictions internationally, the resource is extracted as fast as possible until the resource is exhausted if, and only if, the rate of increase in the "net price" of the resource is less than the nominal rate of interest on bonds (type I path). If not,exctraction is postponed until the end of the planning period, when, again, the resource is extracted at a maximal rate until it is exhausted at the terminal point in the planning period (type II path). The stock of physical capital is instantly adjusted so that its net marginal productivity equals the real rate of interest (given exogenously). We have here a situation with three assets, two of which with exogenous marginal returns. The asset with lowest (highest) return - the resource - is therefore converted into another asset as fast (slowly) as possible. Consumption per capita is permanently increasing (decreasing) if the real rate of interest on financial assets - equal to the net marginal productivity of physical capital - is less (greater) than the social rate of discount plus the rate of growth in population. If the extraction path is of type II, the country will initially be a net borrower, if it is of type I, the country may initially be a net borrower depending on β, r, k_o and $p\bar{v}$. If $f'(k_o)-\mu<r-\beta$, then obviously the initial stock of physical capital is "too high", and no borrowing is needed. If, however, $f'(k_o)-\mu>r-\beta$, an immediate expansion of the capital intensity of the economy is undertaken, whether this results in an initial loan depends on $p\bar{v}$. The higher the discrepancy between marginal returns to physical and financial capital is initially, the more probable it is that the country borrows money, even if it exports the resource at a maximal rate.

When the country is a net borrower, the possibility of borrowing restrictions emerges, in which case net marginal returns to physical capital will exceed the real rate of interest on bonds. With borrowing restrictions, type I (II) path above will be optimal if, and only if, the net marginal productivity on physical capital exceeds (is less than) the percentage rate of change in the net price of the resource relative to the percentage rate of change in the price of imported goods. This is because borrowing restrictions imply import restrictions, and the resource is in such a situation extracted for direct import purposes. The returns to converting the resource to physical capital through imports, $f'-\mu$, must then be compared to how much more a unit of the resource will command in terms of import goods when kept in the ground, $\alpha-\beta$.

It might be noted here that the conditions for the fastest possible resource extraction are less stringent when borrowing restrictions are effective in the sense that type I path may then be optimal under conditions which would make resource use of type II optimal in a situation without such restrictions. With borrowing restrictions, when $\beta=0$, $r<f'-\mu$, so that $\alpha<r$ would imply $\alpha<f'-\mu$ and type I extraction-path. With $\beta>0$, $\alpha-\beta<f'-\mu$ a fortioro. The other way around this means that with borrowing restrictions the fastest possible resource use is optimal in situations where the percentage price rise for the resource is higher than the rate of interest on bonds. First of all it is necessarily the case as long as $\alpha-\beta\leq r$, i.e. $\alpha\leq r+\beta$, and secondly, even if $\alpha>r+\beta$, type I extraction path may be optimal since $r<f'-\mu$. To what extent the percentage price rise for the resource may exceed the rate of interest on loans in a situation where type I resource extraction is optimal therefore depends on the percentage increase in import prices and on how seriously the borrowing restrictions are felt.

The absolute rate of change of consumption along the optimal path with a borrowing constraint is given in (16'). Since $f'-\mu>r-\beta$, consumption will be steadily increasing for $r-\beta>\rho+n$. For $r-\beta<\rho+n$, however, there may be an initial period of increasing consumption (as long as $f'-\mu>\rho+n$) before consumption culminates and then decreases, when $f'-\mu<\rho+n$. When borrowing is restricted, the optimal path of consumption is related to the capital intensity of the economy. These relations will be explored in full in the final section.

When the country borrows maximally, two possibilities exist. Either the optimal stock of physical capital has not been reached when repayment of the loan must begin, or the optimal capital stock is reached at, say t^*. For $t>t^*$ restrictions on borrowing are then not effective. If the optimal extraction path with borrowing restrictions is of type I, it is then possible that when the restrictions cease to be effective at t^*, the optimal extraction path will be of type II. This means an optimal policy sequence such that extraction is maximal as long as borrwoing restrictions are effective. When this regime ends, however, it is optimal to leave the rest of the resource in the ground until it is exploited maximally at the end of the planning period. The optimal time profile of resource extraction is then maximal extraction at the beginning and at the end of the planning period.

b) When marginal extraction costs are rising and there are no restrictions on borrowing, the optimal depletion rate is determined by the (exogenously given) nominal rate of interest on bonds, the relative rate of change in the net price of (or marginal proceeds from) the resource and the properties of the cost function. If the net price of the resource is constant, optimal extraction is always falling over time. A rising net price of the resource reduces the optimal fall rate in extraction. If the extraction path is falling over time, it can be shown that extraction goes to zero in finite time when the extraction period is over. At that time the resource is exhausted.

Except that the gains in interest from converting the resource into financial capital must now be balanced against increasing extraction costs, asset management is the same as when extraction costs are constant. This means that, if possible, the stock of physical capital is instantly adjusted to its optimal level, where $f'=\mu+r-\beta$. Again if this implies borrowing, a situation with borrowing restrictions may be relevant. In that case the depletion rate is independent of the rate of interest, whose role is now taken over by the net marginal returns to physical capital. In addition to the percentage rate of change in the marginal proceeds from the resource, the percentage rate of change in the price of imported goods *per se* now also affects the optimal depletion rate for the same reasons as when extraction costs are constant. From (17") it can be seen that cet.par. extraction is slowed down if the capital intensity of the economy increases, if the increase in marginal proceeds from the resource shifts upwards and if the rate of change in import prices shifts down.

The combined effect of price changes may be to keep back production for some time, in contrast to a model with constant prices where a falling shadow price of capital and an increasing user cost of the resource both lead to a higher extraction now than in the future. The relevance of the capital intensity on optimal resource use is obvious in the case of a rising price of the resource relative to the price of import-goods. In that case the increasing user cost is counteracted by the rising relative resource price. In a model without capital, extraction rises over time if the rate of growth in the relative price is higher than in the user cost. In a model with capital, this pattern is accentuated if capital is above its optimal level, where $f'-\mu<\rho+n$, since then the shadow price or the social value of capital (or consumption) over time increases as well ($\dot{q}_1>o$ from (5)). If capital is "scarce", however, its shadow price is falling and it may fall so fast that it offsets the rate of increase in the relative resource price over and above the user cost. In that case optimal extraction would still be falling over time, even if the percentage rate of growth in the relative price of the resource is higher than the percentage rate of growth in the user price of the resource. *Such a country is too poor to afford to wait for the higher prices, at least for some initial period until the physical capital stock of the economy is built up.*

In general (19') shows that the higher the relative price trend is, the lower must the capital intensity be for falling resource use to be optimal, or conversely; *the higher the capital intensity of the economy is, the lower is the relative price-rise needed to make a rising extraction path optimal.*

5. Comparing the Results with Earlier Models.

a) Resource Models

The results obtained on the optimal path of resource use in this model are somewhat more general than those found in earlier contributions. The optimal extraction path now depends on conditions in the rest of the economy, in particular on the stock of physical capital in the economy and on the borrowing possibilities. A main effect from allowing financial investment or disinvestment in a model of resource extraction in an open economy, is to separate the optimal consumption stream over time from the optimal path of resource extraction. If borrowing possibilities are unlimited, the separation will be complete. As noted in [4], this is to be expected as an analogue to the standard result in static trade theory, to the effect that an open economy's optimum production point is independent of its preferences, and determined entirely by world prices. When marginal extraction costs are constant, we have shown that resource extraction should either be zero or at its maximum. This is so even if the social welfare function in the model is concave. A comparison with the results in the model by Vousden [8] is therefore not completely straightforward. In his model marginal extraction costs are constant. At the same time v is interior and falling, which is incompatible with the necessary conditions for optimality in this model. When marginal extraction costs are rising, however, an interior solution for v is relevant also in this model. With constant prices and the capital intensity less than or equal to the modified golden-rule capital intensity, extraction is always falling. The extraction period is finite and the resource is always exhausted when extraction ends. All these results are similar to those found by Vousden in [8]. But whereas in his model the depletion rate is determined by the social rate of discount and properties of the instantaneous utility function, the depletion rate is now independent of these factors. Instead it is - in the absence of price changes and borrowing restrictions - determined by the nominal rate of interest in the world financial markets and properties of the cost function in resource extraction, factors that are easier to deal with from an empirical point of view.

However, the result on the effect on the optimal extraction path in this model from changes in the initial capital or resource stock when borrowing restrictions are effective, bears little resemblance to the result in [8] that "the optimal time of exhaustion will increase as the alternative source of consumption, $\overline{C}$, falls".

Since the welfare function is the same in both models, the contrast in the solutions must be due to the difference in supply conditions. In the resource model the resource may be used for consumption purposes only. Extracting the resource is the only source of consumption in addition to the exogeneous component. Since consumption from the resource is subject to a finite upper bound on cumulative extraction, society must - in order to survive - stretch out the use of the resource when the alternative source of consumption falls. In this model, the resource extracted may also be used for physical and/or financial capital formation purposes, and survival can be secured on basis of the capital stocks alone. The future benefits to be derived from an extra unit of the resource extracted for physical investment purposes is higher the smaller the capital stock already attained. The optimal resource use in this model with effective borrowing restrictions is therefore slowed down when the initial physical capital intensity gets a positive shift.

In contrast to [4], we have distinguished between situations with and without borrowing restrictions. This distinction is essential since the international borrowing possibilities affect extraction policies. International credit rationing at the going market rate of interest may necessitate resource extraction for direct import purposes. A liberalization or removal of credit limits therefore slows down optimal resource use. A positive shift in the initial resource stock have similar effects since it increases the total debt a country may incur; it may also ease or remove existing borrowing constraints through improving the country's international creditworthyness. Finally, also in contrast to [4], price trends for the resource and for imported goods have been introduced in this model. It is worth noting that the effects of these trends on extraction and consumption depend on whether borrowing restrictions are effective or not.

b) Optimal Growth Models

Consider finally the optimal development of consumption and the physical capital intensity in this model in relation to optimal development in the standard model for optimal exonomic growth. In the sequel the reasonable accumption of a decreasing resource exploitation over time is made. Recall the definition of the modified golden rule capital intensity, k*, as

$$f'(k^*)-\mu=\rho+n\,.$$

From (5) and (16'), it follows that $\dot{c} = \dot{k} = 0$ when k = k*, so that k = k* is also a stady-state in this model. Since the capital intensity in this model in the absence of borrowing restrictions is determined by the exogenously given real rate of interest on finance capital, a distinction must be made as to whether

$$r-\beta \gtrless \rho+n\,.$$

Define $\hat{k}$ by $f'(\hat{k})-\mu = r-\beta$. Assume safely that $k_o < \hat{k}$. If $r-\beta= \rho+n$ and there are no borrowing restrictions, k is instantly adjusted to k*, and k and c are kept constant over time. With restrictions on borrowing, k is gradually adjusted towards k*. If k* is reached before the borrowing possibilities are exhausted, optimal development from then on is to keep k and c constant. Till then c also increases, according to (16'). If borrowing possibilities are exhausted while k < k*, c is always increasing by (16'). Repayment begins while k is increasing asymptotically towards k*. This is so since to reduce the stock of physical capital in order to repay debt would obviously be inoptimal because the net productivity of physical capital is higher than the real rate of interest. The remaining alternative, to keep k constant, implies from (i) that $\dot{c} = \dot{m}$ (since $\dot{k} = 0$). Since c is increasing, imports must then increase, which contradicts the assumption of debt repayment. Hence k must increase in the repayment period. When repayment is over, k increases asymtotically towards k* as in the standard optimal growth model. c increases also.

If $r-\beta<\rho+n$, then k is adjusted towards $\hat{k}>k^*$. Without borrowing restrictions, adjustment to $\hat{k}$ is instant. By (16), c is then decreasing, while k is kept constant at $\hat{k}$. The reduction in c must therefore take place through a fall in imports over time. Whether the country initially was a net borrower or not, there must be some point in time, t', where net finance capital is non-negative. The economy may also at T enter the "post-extraction phase" with positive finance capital. Since c is falling, and the marginal returns to financial capital is constant, finance capital is gradually reduced towards zero at, say $\hat{t}$. For $t>\hat{t}$, $m\equiv b\equiv v\equiv x\equiv o$ so that the model collapses into a standard model of optimal economic growth with the initial capital intensity $\hat{k}$ greater than the optimal steady-state capital intensity k^*. For $t>\hat{t}$ the capital intensity and consumption in the economy will therefore decrease asymptotically towards their steady-state values. The optimal development of the economy under these assumptions and with negative financial capital initially is illustrated below in fig. 1. t_o denotes the time when extraction begins.

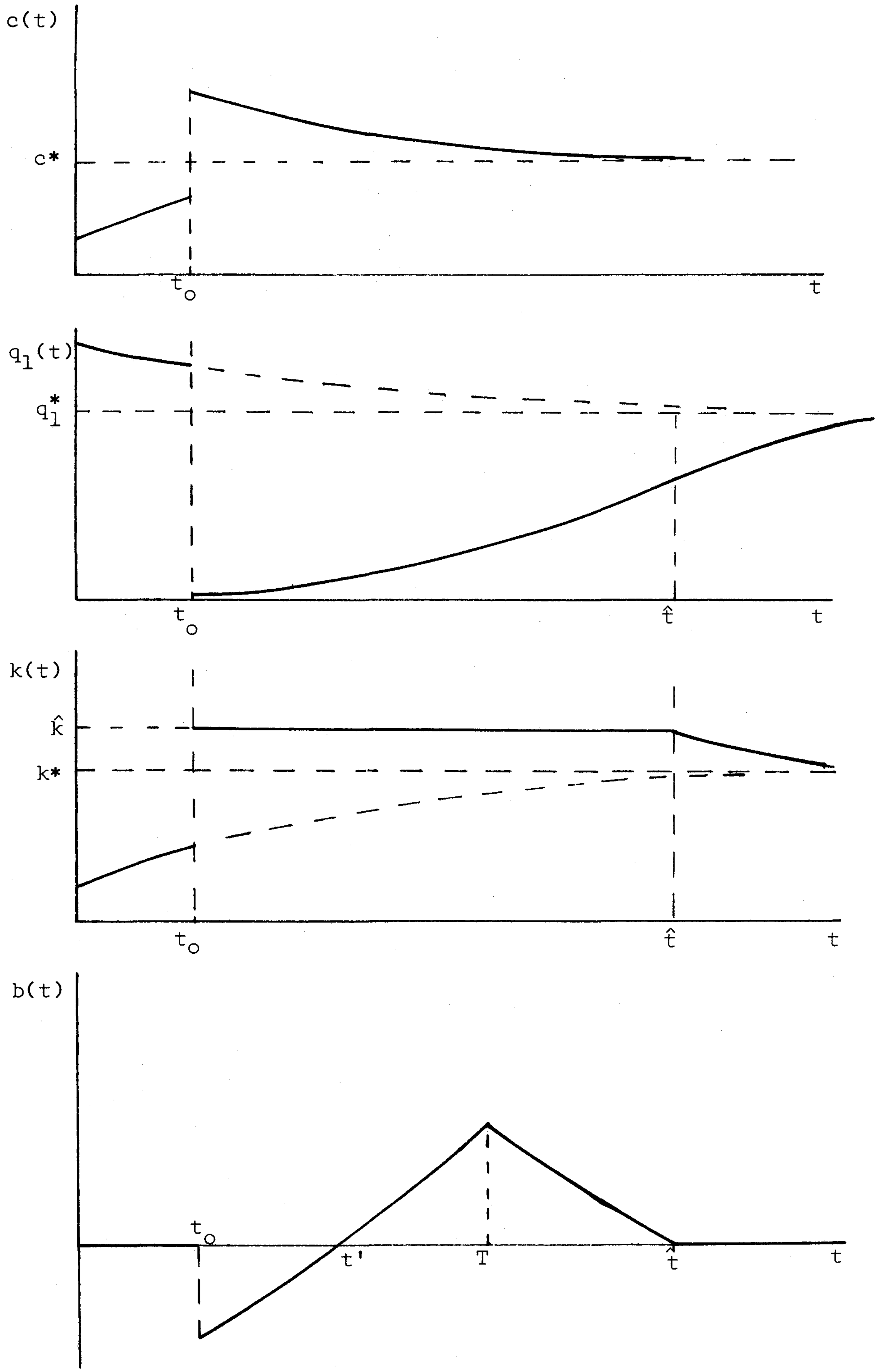

Figure 1.

With borrowing restrictions, the optimal development under the same assumption regarding the real rate of interest on bonds, would be somewhat modified, since k cannot be instantly adjusted to its optimal level. If $k_o<k^*$, there is therefore a first phase where $k<k^*$, and c is growing, according to (16'). As k increases and $k=k^*$ is reached, at $t=t_1$, $\dot{c}$ is zero. For $k>k^*$, c decreases as k approaches its optimal value $\hat{k}$. If $\hat{k}$ is reached at $t=t_2$, the optimal development from then on is as explained when borrowing restrictions were ineffective. The optimal development of the economy in this case is illustrated in fig. 2 below.

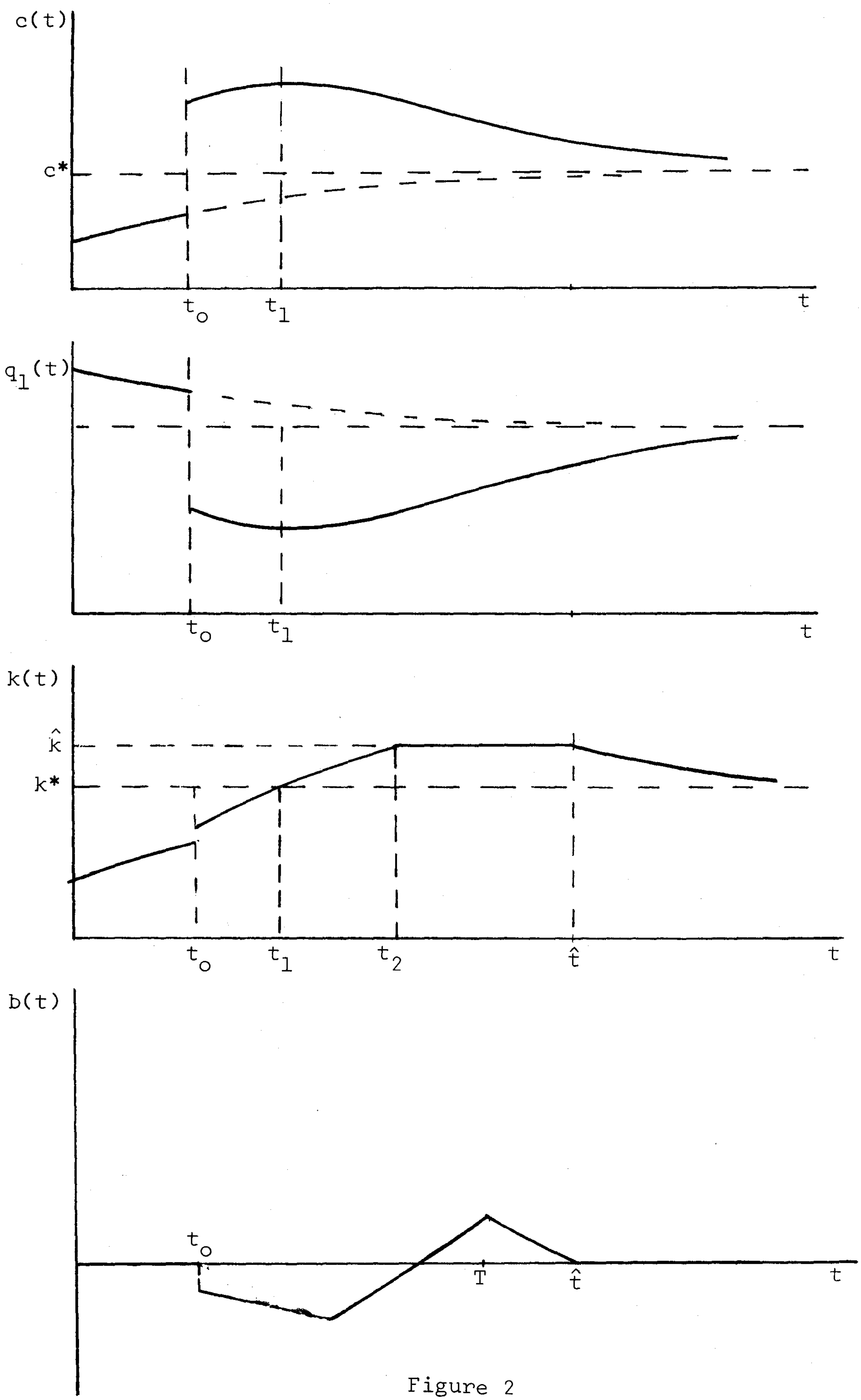

Figure 2

If the borrowing potential is exhausted while $k^*<k<\hat{k}$, k does not reach its optimal level. Since $f'(k)-\mu>r-\beta$, k is non-decreasing while the debt is being repayed. When repayment is over, k begins to shrink asymptotically towards k^*. c is decreasing as long as $k>k^*$. The borrowing potential may also be exhausted while $k<k^*$, in which case the optimal development of the economy is as described in the similar case when $r-\beta=\rho+n$.

The final possibility is $r-\beta>\rho+n$, in which case $\hat{k}<k^*$. Consumption grows according to (16), while the physical capital intensity is constant at $k=\hat{k}$. The growth in consumption must therefore originate in the proceeds from a steadily increasing finance capital, also in the post-extraction period.

(16') may be rewritten as

$$(20) \quad \dot{c}/c = (-f' + \rho + \lambda)/\check{\omega}$$

where $\check{\omega} = cU''/U'$ is the elasticity of marginal utility. When extraction starts, provided $k_O<\hat{k}$, k increases faster than before the extraction period. With $\check{\omega}$ (approximately) constant in the relevant range, it follows from (20) that the relative growth in consumption is reduced when extraction starts. *The optimal reaction to a newly discovered resource is therefore a positive shift in initial consumption combined with a reduced relative growth in consumption.* Figures 1 and 2 show that <u>with</u> resource extraction $q_1(t)$ is less than $q_1(t)$ <u>without</u> extraction for all t, including $t>T$. This means that *also in the post-extraction period society will enjoy higher levels of consumption per capita than it would have done without a resource extraction period.*

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Notes.

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1) Suppose that for this reason there is a constant upper bound $\overline{V}$ on total extraction per unit of time. In that case $\overline{v}(t) = \overline{V}/L(t)$, where $L(t)$ is total population. Since $L(t) = L_o e^{nt}$, it follows that $\overline{v}(t) = \overline{v}_o e^{-nt}$ where $\overline{v}_o = \overline{V}/L_o$.

2) One might object here that export of the macro-good should be possible. However, since we want to focus on the consequences of exploiting a resource for export purposes, the sharpest results are obtained by assuming that the resource is the only export-good in the economy. In part, an economic justification for this would be that, except for the "epoch" connected with the exploitation of the resource, it is the aim of the Government always to balance the current account.

3) An alternative would be to use the macro-good as a numeraire so that $p_m \equiv 1$. But because it is of interest to distinguish between real and nominal terms in this analysis, this approach has not been taken, although it would simplify some of the expressions.

4) The time argument in the functions will from now on usually be dropped.

5) If initially $q_2 > pq_3$, the resource would not be used at all, which would make the problem economically uninteresting.

6) See e.q. the presentation given in (2), ch. 11.

7) This shows that initial jumps in the state variables k and b are a feature of the optimal solution to the problem. However, it can be shown that for $t > t_o$ no such jumps can occur. A mathematically more complicated formulation of the control problem that allows for initial jumps in the state variables can be given. This formulation yields the same necessary conditions for $t > t_o$ as the formulation used here. Hence our simpler formulation assumes that the initial values of k and b have been adjusted properly.