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STUDIES IN RISK AND BOND VALUES

THE NORWEGIAN SCHOOL OF ECONOMICS AND
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"We believe that it is necessary to know as much as possible about the behaviour of the individual and about the simplest forms of exhange."

John von Neumann, and
Oskar Morgenstern

PREFACE

In the preparation of this dissentation I have had the valueable advice of many friends, of whom I would particularly like to thank: Karl Borch, Olav Harald Jensen, Edmond Malinvaud, Pascal Mazodier, Jan Mossin and Agnar Sandmo. Grants were received from Norges almenvitenskaplig Forskningsråd and from Bergens Privatbanks fond til Norges Handelshøyskoles lærere.

Bergen, August 1974
Cornelius M. Schilbred

C O N T E N T S

reasonable assumptions about market prices on uncertain investments. Chapter IV gives the study on market price formation.

The first paper was published in The Swedish Journal of Economics and reports an experiment with bond experts. The second paper is a study of the yields on Norwegian Government bonds, which was published in Statsøkonomisk Tidsskrift. This study was made to give a more consistent times series of interest rates to be used in econometric studies of monetary relations in Norway. It discusses relationship between transactions and market prices for bonds, and the applied means of approximation, when going from market prices to transaction prices. The paper thus deals with the problem of time being a variable allowed to take on real values. The computational and estimation technique applied to construct yield curves for Norwegian Government bonds, is used also in the following paper on Italian Government bond yields.

In the third paper, which was published in Annales d l'INSEE (French) and also in The Review of Economic Studies (English), attention is on a possible market price of risk. The prior studies reported give a set of assumptions on which to base a mean/variance equilibrium model for the market prices of risky assets. This model is tested on data from the Italian bond market. The data are observed market prices for lottery bonds. There appears to exist a market price of risk. This price was stable over both short and medium term, i.e. over a period of five years and also over one year subperiods. In the French version, which is included in Appendix II, it is also shown how the above results may be useful in practice.

A typical situation will be for a potential borrower in the bond market, who makes an initial study of possible prices at which the market will accept different combinations of coupons and repayment plans of a loan. Finally is shown how the chosen formulation of the mean/variance equilibrium model reconfirms some well known results on firm's market values and investment criteria.

Chapter I.*)

Bond Evaluation as a Decision under Certainty or Uncertainty.

1. Introduction.

The series of studies which are presented in the chapters following, focus on the valuation and market price formation on lottery bonds. This chapter will give an introductory description of these bonds and demonstrate the properties which make lottery bonds particularly suitable for empirical investigations within the economics of uncertainty.

The lottery bonds are bearer's bonds. This means that a bond is the property of the holder. When a large loan is to be made, the borrower will consider various possible lenders, such as banks and insurance companies, as well as the sale of bearer's bonds through the bond market. The loan may exceed the lending capacity of potential, individual leders, and so the bond market may be chosen. The selling of bearer's bonds may also be the cheapest means of financing large loans, when all costs such as legal fees, guarantee provisions the the cost of printing are taken into account, as well as the straight interest or coupon of the loan.

When a loan is floated through the bond market, the money is raised through the selling of bonds. The sum total of the face values of the bonds is then equal to the amount of the loan. The amount of the loan may, however, not be the same amount as that which is received by the borrower. The bonds may have been sold at a premium or discount from their face values, and also the loan expenses will be deducted before the borrower receives his money.

The loan agreement, which is the basis for the issuing and selling of bonds, also lays down the repayment plan for the loan. One of the following three forms of repayment is usually chosen.

*) This chapter, which gives an introduction to the empirical studies, draws in part on my papers [6] and [8]. It also takes into account the note by A. Buse [9] and my reply [10].

If the total amount of the loan will be repaid at a fixed future date; the loan is said to have a fixed maturity. The loan may be repaid through installments, so that the total amount paid per period is contant, and covers both interest and repayment of capital. This loan is an annuity loan. Thirdly the borrower may have agreed to make constant repayments per period of the capital in the loan. The loan is then a series loan.

If the loan has a fixed maturity date, all bonds will be redeemed at that date. From the point of view of an investor, the cash flow from the holding of a bond will consist of one or more payments for coupon, and at the end a coupon payment plus the redemption price of the bond. The redemption price is in most cases equal to the par or face value of the bond.

A fixed maturity loan has the bulk of its payments at the end of the loan period, when the loan capital is repaid. For the borrower it may, however, be desirable with a program of more even future payments. Such a program may permit a better timing between the payments to the bond holders and the receipts from the investments which the borrowed funds finance. Normally the market rate of interest will increase with increasing time to maturity for a loan. So that a loan with an even stream of repayments has a lower, average interest cost than a loan with the same total loan period, but fixed maturity.

Into the consideration of choice between a fixed maturity loan and an annuity loan or series loan, which are loans with more even cash flows, also enters the expectation about future changes in interest rates. With a fixed maturity loan, the borrower may feel it necessary to build up a capital fund, with which he will repay the loan at maturity. Such a sinking fund is quite often a provision of the loan contract. Future interest rates, therefore, determine the amounts to be invested in order to build up a sufficient fund at the end of the loan period.

2. A bond with fixed maturity.

To evaluate a bond with fixed maturity, the investor knows that the bond will produce constant interest payments at regular time intervals, and that it will be redeemed at a fixed price at maturity.

Coupon payments are received at the end of each period and the final coupon coincides with the redemption of the bond. For the time being it will further be assumed that investor evaluates a bond at the beginning of a period, and that the future payments to him which the bond will give rise to, are discounted at a constant rate of interest, i. The assumption of a constant rate of interest underlies the construction of bond values tables currently in use. The most common coupon period is six months, with annual periods as the second most important.

The value of a bond may be expressed as the present value of the future interest payments plus the present value of its redemption price.

$$B_t = \sum_{j=1}^{t} \frac{D \cdot r}{(1+i)^j} + \frac{D_R}{(1+i)^t}$$

(1.a)

$$B_t = D \cdot r \cdot \frac{(1+i)^t - 1}{i(1+i)^t} + \frac{D_R}{(1+i)^t}$$

Where

B_t = present value of a bond with fixed maturity at time t.
D = face value of bond
D_R = bond's redemption price
r = the coupon rate of interest (a percentage of the par value).
t = time to maturity (measured in number of coupon periods).
i = rate of interest or yield required by the investor
v = $(1+i)^{-1}$ = discount factor

As a bond will normally be redeemed at its par value $D_R = D$, the common bond value formula found in bond values tables is:

$$(1) \quad B_t = 100\left(\frac{r}{i} + \frac{i-r}{i} v^t\right).$$

D = 100 in the tables, so that values are given in percentage of the face value of a bond.

3. A lottery bond.

When the loan is either a series loan or an annuity loan, the loan contract will lay down the amount of capital to be repaid on each future repayment date. At the end of the loan period the total loan capital will have been repaid to the bond holders.

One may think of three ways to arrange for the repayment of series and annuity loans. The first is to split the loan up into a set of fixed maturity loans, and to issue bonds for each sub-loan. These bonds would have fixed maturities. The second method would be to arrange for the repayment of each bond through installments, so that the installments match the repayment plan of the total loan capital. The third possibility, and the one which is most commonly found in practice, is to issue lottery bonds. Each bond is then given a separate number, and a lottery is arranged before each repayment date, so as to select the bonds to be redeemed. A bond in a loan where the redemption of bonds is determined through lotteries, is called a lottery bond.

If, for instance, the total capital outstanding in a loan is N.kr. 100 mill., and the loan is a 10 year series loan, the annual repayment of capital will be N.kr. 10 mill. The bonds may have face values of N.kr. 1,000.-, which means that a total of 100,000 bonds are held by the investors. 10,000 bonds will be drawn for redemption in each of the ten lotteries, and the probability of a bond being drawn in the first, forthcoming lottery is $\frac{10.000}{100.000} = \frac{1}{10}$. The probability of not being drawn is $\left(1 - \frac{1}{10}\right) = \frac{9}{10}$.

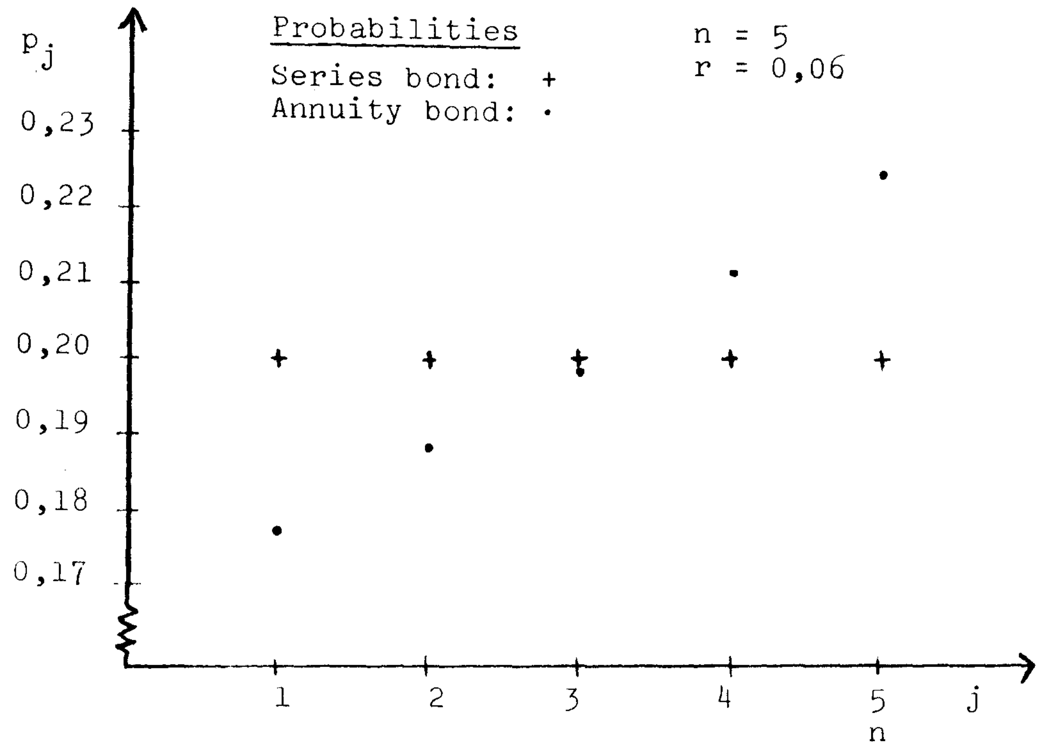
The second lottery will be arranged one year later, when 90,000 bonds will be outstanding. Again 10,000 bonds are to be drawn for redemption. The probability of being drawn in the second lottery, after having survived the first, is then $\frac{10.000}{90.000} = \frac{1}{9}$. The probability of a bond surviving the first lottery and being drawn for redemption in the second is consequently $\frac{9}{10} \cdot \frac{1}{9} = \frac{1}{10}$.

The probabilities, which may be taken as given data make lottery bonds an interesting study in the economics of uncertainty. One may ask how the information on the probabilities are used by investors in the evaluation of bonds, and then see if there exists a price of risk in the bond market. Before turning to these empirical questions, a more general description of the lottery bond will be given.

4. Probabilities

All bonds within one series of lotteries are of equal denomination. Each time a lottery is arranged, a specified amount of the bond numbers still outstanding are drawn, with all nonredeemed numbers having equal probability of occurring. The numbers drawn are published.

A probability distribution is objectively given as a consequence of the redemption agreement in the bond issue. The distribution changes every time a lottery has been arranged. Consider then an investor who evaluates a bond at the beginning of a year. The probability of the bond being drawn for redemption in the lottery at the end of the first year is equal to the ratio between the installment to be made in the lottery and the total amount of the issue outstanding during the first year. The probability of a bond being drawn for redemption in the second year's lottery, is the combined event of the bond not being drawn in the first lottery and being drawn in the second, where the probability of being drawn in the second lottery is equal to the ration between the installment to be made in that lottery, and the total amount outstanding during the second year. This argument may be used to derive the probability of a bond being drawn in a particular of each of the remaining lotteries.



p_j
0,23
0,22
0,21
0,20
0,19
0,18
0,17
Probabilities
Series bond: +
Annuity bond: ·
n = 5
r = 0,06
1
2
3
4
5
n
j

Fig. 1

From formula (1) we have that the difference in value between two certainty alternative bonds maturing in two consequtive years is:

$$B_j - B_{j-1} = \frac{r - i}{(1+i)^j}$$

For a required yield smaller than the nominal rate of interest the difference is positive, so that the bond values are monotonically increasing with increasing number of years left to maturity. If the yield is equal to the nominal rate of interest, the difference is zero, and the bond values are at par, independently of j. Finally the yield may be higher than the nominal rate of interest, leaving the difference negative, so that the bond values are decreasing monotonically with increasing number of years to maturity. From the denominator of the difference it may be noted that the absolute value of the difference decreases with increasing values of j. Two typical curves are drawn in figur 2.

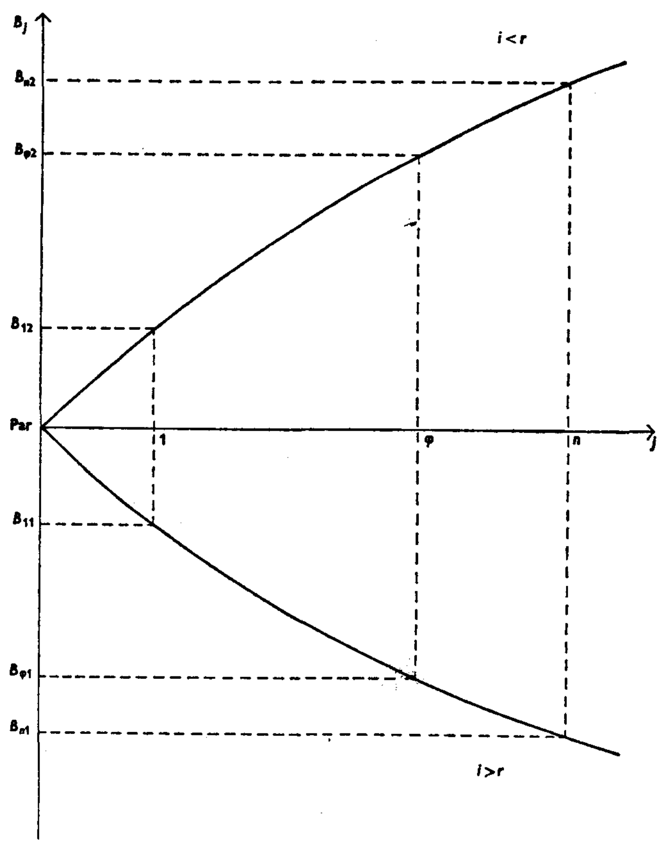


Fig. 2.

Along the j-axis are the future lotteries to be arranged, so that $j = 1$ denotes the first forthcoming lottery, and B_{12} is the present value of a bond with fixed maturity on the day of this lottery if $i<r$. As j can only take on integer values, the curves are not solid lines, but have been drawn as such to facilitate the exposition.

Take the case when the required yield is higher than the nominal rate of interest, $i>r$, and the certainty alternative bonds to a lottery bond are all valued below par. If there are n more lotteries to be arranged, on one of which the lottery bond will be redeemed, the most cautious investor attitude will be to deem the lottery bond equal to a certainty alternative bond with fixed maturity on the day of the n'th lottery, and assign to it the value B_{n1}. The investor is cautious because he assigns to the lottery bond the value of its cheapest certainty alternative bond. Should the bond indeed be drawn for redemption in the n'th lottery, his expectations are exactly met; but if the bond is drawn earlier, the investor's expectations are more than fulfilled. His most optimistic attitude will be to deem the lottery bond equal to a certainty alternative bond with fixed maturity on the day of the first forthcoming lottery, and evaluate it at B_{11}.

These are both cases of extreme cautiousness or optimism, and it may be reasonable to look at the case when the investor sets the maturity of the certainty alternative bond at ϕ, where $1<\phi<n$. Should the lottery bond be drawn for redemption in one of the lotteries arranged before the ϕth, the investor will experience a gain, because the lottery bond is indeed worth more to him on the day of evaluation if such a future event occurs. If, on the other hand, the lottery bond is redeemed on one of the lotteries arranged after the ϕth, he will have a loss. The set of gains or losses are the positive or negative differences between: the present values of the set of certainty alternative bonds and the value assigned to the lottery bond on the day of evaluation.

Turn, then, to the case when the required yields is lower than the nominal rate of interest, i<r, and the set of certainty alternative bonds to the lottery bond are all quoted above par. As is seen from Figure 2, the picture of relative values is reversed. In this case the investor of extreme caution would assume the lottery bond to be redeemed at the first forthcoming lottery, and evaluate it at B_{12}. Whereas the extreme optimist would assume the lottery bond to be drawn for redemption in the n'th and last lottery, and set the value at B_{n2}.

6. Two observed decision rules

It should be stressed at the outset that the rules to be reproduced are the result of a cursory investigation and discussion with bond brokers. The observed decision rules are only indications of direction. They should be read as rules of thumb, and not as exact evaluation procedures. Their observation does, however, point to interesting empirical questions. The two rules of thumb observed for series bonds and annuity bonds, respectively, should thus give an idea as to how one may set up an experiment to study investors' valuation behaviour. In the next chapter is reported an experiment which was carried out with the aim to study investors' treatment of risk when they evaluate lottery bonds.

When there are n more lotteries to be arranged and ϕ is the lottery in which the lottery bonds is assumed to be drawn for redemption, the observed decision rule for a series-bond is:

(5) $$\phi = \frac{n+1}{2},$$

which is linear, and easy to apply when investor uses a book of bond tables.

The median lottery is such that it is about as likely that a lottery bond will be drawn for redemption before the median lottery, as it is that it will be drawn later. The median bond to a lottery bond is the certainty alternative bond with fixed maturity in the median lottery. To deem a lottery bond equal to its median bond, the investor, therefore, behaves so as to hold the probability that he will experience a loss about equal to the probability that he will experience a gain.

For a series-bond the median bond can be determined as the certainty alternative bond with fixed maturity on the day of the ϕ'th lottery, where ϕ satisfies

$$\sum_{j=1}^{\phi-1} \frac{1}{n} = \sum_{j=\phi+1}^{n} \frac{1}{n}$$

$$(\phi-1)\frac{1}{n} = (n-\phi)\frac{1}{n}$$

$$\phi = \frac{n+1}{2} .$$

From this it may be inferred that if the investor behaves so as to hold the likelihood of a gain about equal to the likelihood of a loss, he may use the above decision rule in the case of equal probabilities. He will only get integer solutions for ϕ when n is an odd number. So if n is even, an investor will take the two integers closest to the solution for ϕ, and use the resulting bond prices as guide points to determine the value of the lottery bond

For an annuity-bond the observed decision rule is:

(6) $\phi = \frac{2}{3} \cdot n.$

Again determining the median lottery

$$\sum_{j=1}^{\phi-1} \frac{r(1+r)^{j-1}}{(1+r)^{n}-1} = \sum_{j=\phi+1}^{n} \frac{r(1+r)^{j-1}}{(1+r)^{n}-1}$$

$$\frac{(1+r)^{\phi-1}-1}{(1+r)-1} = \frac{(1+r)^{n-\phi}-1}{(1+r)-1} \cdot (1+r)^{\phi}$$

$$(1+r)^{\phi} = \{(1+r)^{n}+1\}\frac{(1+r)}{(2+r)} \quad (7)$$

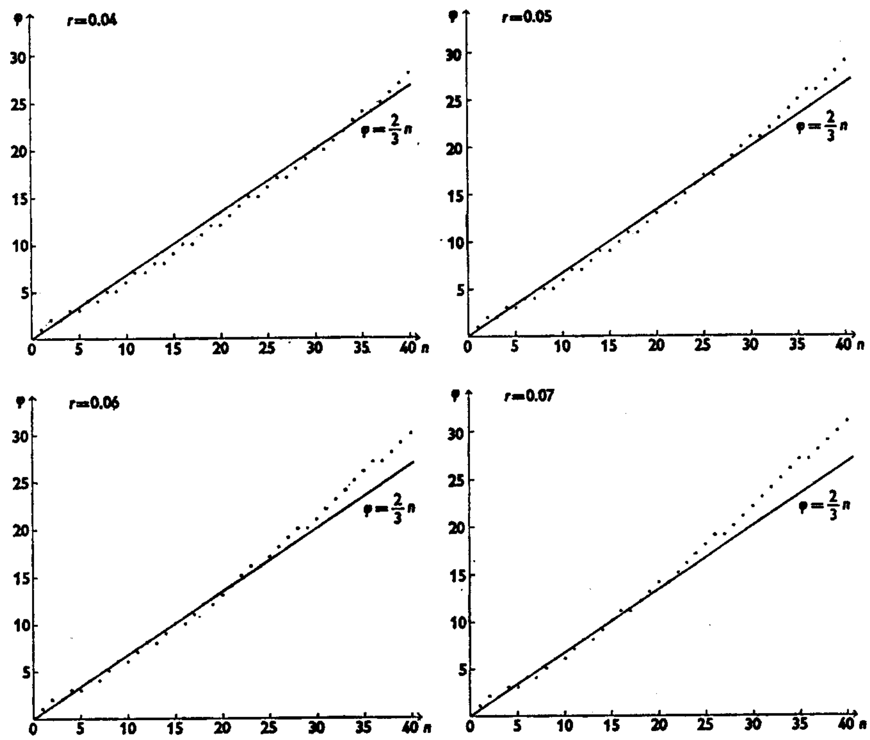


Fig. 2.

Fig. 3.

As is seen, the equation between ϕ and n in (7) is not linear, not even the logarithms. The most simple approach is to take som typical values for r, and for varying n choose ϕ the integer that minimizes the difference between the two sides of equation (7). This has been done in Figure 3, and the straight line of the decision rule has been drawn for comparison. The observed decision rule seems to give a reasonably good fit. A second indication is therefore given that the investors who apply the above rules, are looking at some measure of the center of each of the two probability distributions. It may be, however, that the measure of center is not the median, but the mathematical expectation or mean. So that the two decision rules are tools whereby an investor gets a close approximation to the respective means.

7. The two means

The mean value of a lottery bond is the weighted average of the set of present values of certainty alternative bonds, where the objectively given probabilities operate as weights. For a bond with t years left to the last lottery and n more lotteries to be arranged the mean will be:

$$\bar{B} = \sum_{j=1}^{n} \sum_{æ=1}^{t} p_j \cdot a_{jæ} \cdot (1+i)^{-æ},$$

where $a_{jæ}$ denotes the payment that the lottery bond gives rise to $æ$ years from the day of evaluation if the bond is drawn for redemption in the j'th lottery.

Once the period of grace is over, and a lottery bond is evaluated at a time when there are n lotteries left to be arranged, the mean will be:

$$(8) \qquad \bar{B} = \sum_{j=1}^{n} p_j \cdot B_j = \sum_{j=1}^{n} p_j \left\{ r \cdot \frac{(1+i)^j - 1}{i(1+i)^j} + \frac{1}{(1+i)^j} \right\},$$

because n is equal to t.

A series bond has the mean:

$$\bar{B}_S = \sum_{j=1}^{n} \frac{1}{n} \left\{ r \cdot \frac{(1+i)^j - 1}{i(1+i)^j} + \frac{1}{(1+i)^j} \right\}$$

$$= \frac{1}{n} \left\{ \frac{r}{i} \cdot \sum_{j=1}^{n} \frac{(1+i)^j}{(1+i)^j} + \left(1 - \frac{r}{i}\right) \sum_{j=1}^{n} \frac{1}{(1+i)^j} \right\}$$

$$(9) \qquad \bar{B}_S = \frac{r}{i} + \frac{1}{n} \cdot \frac{i-r}{i} \cdot \frac{(1+i)^n-1}{i(1+i)^n} .$$

This is a well known result. It is usually obtained by assuming the investor to be someone holding a large number of bonds in the lottery bond issue, so that he may, on the average, expect 1/n of his holdings to mature in each of the future lotteries. For the present paper, however, this line of argument is not applicable, as it is the evaluation of a single, risky asset that is studied, independently of other risky assets.

Similarly for an annuity-bond. Once the number of lotteries left to be arranged coincides with the number of years to final maturity of the issue, the mean will be:

$$\bar{B}_A = \sum_{j=1}^{n} \frac{r(1+r)^{j-1}}{(1+r)^n-1} \left\{ r \cdot \frac{(1+i)^j-1}{i(1+i)^j} + \frac{1}{(1+i)^j} \right\}$$

$$= \frac{r}{(1+r)^n-1} \sum_{j=1}^{n} \left\{ \frac{r}{i}(1+r)^{j-1} - \left(\frac{r}{i} - 1 \right) \frac{(1+r)^{j-1}}{(1+i)^j} \right\} ,$$

$$= \frac{r}{(1+r)^n-1} \left\{ \frac{(1+r)^n-1}{i} - \left(\frac{r-i}{i} \right) \left(\frac{1}{(1+i)} \cdot \frac{(1+r)^n(1+i)^{-n}-1}{(1+r)(1+i)^{-1}-1} \right) \right\}$$

$$= \frac{r}{(1+r)^n-1} \cdot \frac{(1+r)^n(1+i)^n-(1+r)^n}{i(1+i)^n} = \frac{r(1+r)^n}{(1+r)^n-1} \cdot \frac{(1+i)^n-1}{i(1+i)^n}$$

$$(10) \qquad \bar{B}_A = a^{-1}_{\overline{n}|r} \cdot a_{\overline{n}|i} ,$$

which is also a well known result. Again investor may be assumed to hold a large number of lottery bonds in the issue. For each "kroner" in face value of bonds investor may expect to receive approximately $a^{-1}_{\overline{n}|r}$ from debtor in cover of interest and redeemed bonds. These constant future receivables are discounted by $a_{\overline{n}|i}$ and $\bar{B}_A$ is obtained. Both expressions for the means (9) and (10) can be found in textbooks, and are thus known to well informed investors.

8. Bonds with optional redemption

The lottery bonds described above leaves debtor with no other possibilities but to arrange a lottery before each repayment of capital in the loan. One may, however, not uncommonly find other bond issues where debtor is given an optional strategy. He may either arrange a lottery or purchase bonds in the market to fulfil the repayment plan. The bonds thus have optional redemption.

An investor's evaluation of an option bond will depend on what future redemption decisions he expects debtor to make. Debtor's strategy may be a mixed strategy, and one which depends on the market bond yield on the day of evaluation, as well as his expectation of future changes in the rate of interest. For simplicity, we shall give a small example, and assume that the investor expects the ruling bond yield to remain in the future. Or alternatively, that his evaluation is only based on debtor's first, forthcoming decision, and on the present yield.

Two observations have been made. If the yield is higher than the nominal rate of interest on the bond, the valuation of an option bond is made on the assumption that debtor will buy bonds in the market to cover the installments. If, on the other hand, the yield is lower than the nominal rate of interest, valuation is made on the assumption that debtor will arrange a series of lotteries in the future.

Had the investor made his valuation on a particular day, and written his figure on a piece of paper to hide until debtor has made his decision, the above problem could have been described in the context of a two-person zero-sum game between investor and debtor.[1] Option bonds are, however, traded in markets where

[1] It will be seen from the two payoff matrixes soon to be worked out, that investor may well keep his valuation decision secret, and act as if in a strictly competitive game. The valuation decisions he would make by using the minimax theorem are the same as investors are observed to make, and both payoff matrixes have equilibrium pairs.

prices are made public, so that debtor has information on at least some investors' valuation decisions at the time when he has to make a decision. It may be rational for investors, therefore, to turn to decision theory under uncertainty and seek the assistance of the maximin criterion.[1)]

If debtor decides to buy bonds in the market, this increase in demand may increase the bond price. For an investor, however, his bond will then have a fixed maturity at the end of the loan period.

For a yield higher than the nominal rate of interest, i>r, it has already been observed that a lottery bond will be valued higher than a certainty alternative bond with fixed maturity on the day of the last lottery. An example will be that investor values the option bond at 80 if he expects debtor to arrange lotteries, and 70 if he expects debtor to purchase bonds in the market.

The payoff matrix is then:

Investor evaluates on the assumption of	Debtor decides to	
	arrange lotteries	purchase
lotteries	0	-10
purchase	10	0

The elements of the matrix are gains (positive) or losses (negative). A gain or loss is the difference between the value investor assignes to the option bond and the value he would have given the bond if he had known debtor's future decision. The maximin criterion advises investor to seek out the minima of each row, and choose the row with the highest minimum. Which leads investor

1) See Luce & Raiffa [7, p. 278]

to make his evaluation of the option bond on the assumption that debtor is going to purchase bonds in the market to cover his obligations, if $i > r$. This is also what investors are observed to do. Their reasoning when making this assumption being essentially the same as the one underlying the maximin criterion, namely that debtor will always choos the action that is the least favourable one to investor, simply because it minimizes debtor's payments.

When the required yield is lower than the nominal rate of interest, $i<r$, a lottery bond will be evaluated lower than a certainty alternative bond with fixed maturity on the day of the last lottery. So that the example may be that investor values the option bond at 120 if he expects debtor to arrange a series of future lotteries, and 125 if he expects debtor to buy bonds in the market.

Investor evaluates on the assumption of	Debtor decides to	
	arrange lotteries	purchase
lotteries	0	5
purchase	-5	0

The maximin criterion would advise the investor to make his evaluation on the assumption that debtor is going to arrange a series of future lotteries. This again is indeed the assumption that investors are observed to make. By arranging lotteries debtor is able to redeem the bonds at par, whereas he would have to pay a higher price should he decide to buy bonds in the market when $i<r$. Investor, therefore, bases his evaluation of the option bond on the assumption that debtor will choose the cheapest alternative.

9. Other studies of bond values under uncertainty

Bonds with optional redemption and lottery bonds are examples of

pure debt instruments which give rise to decision problems under uncertainty.
The sources of uncertainty are provisions in the loan contracts, which comes as an addition to the general economic uncertainty due to all contracts which run into the future. A further example of such uncertainty is given by the call privilege. Such a privilege exists if the debtor has been given the option to call the entire bond issue. An option will usually be exerciseable over a limited period of time, say three to five years before the final maturity of the loan. The price at which the call privilege may be exercised will normally start above the par value of a bond, and decline towards par as the bond approaches maturity. The possibility that debtor may call a loan gives rise to a source of uncertainty which is difficult to quantify in terms of probabilities. The problem has been studied by Hess and Winn [3]. . Their method was to interview institutional investors and to analyzé the market for high grade U.S. corporate bonds over the period 1926-1959. In general they found no significant relationship between the call features and bond yields, except for 1959. Although institutional investors claimed that the call privilege was valuable, Hess and Winn did not find that this was reflected in the market place. Later market studies by Jen and Wert [4,5] appear to confirm Hess and Winn's findings. Jen and Wert found, however, that yield on callable bonds issued in periods of high interest rates relative to their cyclical pattern, were high in comparison with bonds of similar grading offered at other times. This would be due to the expectation of a future decline in interest rates, and the consequent refunding of callable, high coupon loans.

Most empirical studies on bonds are focused on the term structure of interest rates, which will be discussed in chapter III. Of other bond related problems, Gelting [2] studies the effect of monetary policy and the behaviour of financial institutions on bond prices. Fisher [1] in a study of risk premiums on corporate bonds, seeks to isolate the effects of the firm's default risk and the marketability of the bonds on bond yields. He uses the three variables: past variability in earnings, past period of solvency and present equity/debt ratio as proxy variables for a firm's default risk. Marketability is measured by the total market value of the publicly traded bonds the firm has outstanding. The regressions support the stated hypotheses on risk premiums. Fisher concludes that by design it could not be tested whether investor behaviour is rational or stable, although he found elasticities to be reasonably stable over time.

Fisher uses variables reflecting default risk to explain risk premiums in the corporate bond market, but he expresses some concern as to whether the variables chosen reflect the true measure of default risk. In other words it is asked whether the variables chosen reflect the probability distribution with which the investors feel that they are faced. It is this dilemma that the lottery bonds solves. All relevant information on the lottery risk is available to investors, so that the question becomes one of isolating the lottery risk, in order to study its effect on bond valuation.

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Chapter II

An Experiment with Bonds and Risk.

Preface

In the foregoing chapter the valuation of lottery bonds was classified as a decision under uncertainty. The observed rules of thumb could in themselves not be used as a confirmation of the expected utility hypothesis. A natural further direction of study, therefore, was to set up and perform an experiment, with the intention of obtaining more exact information on individual behaviour.

In the experiment to be reported,[1] 30 bond experts were asked to evaluate a set of series-bonds. Series bonds were chosen because of the uniform probabilities, so that the probability distributions are two-parametric. The use of annuity bonds was excluded because their probability distributions are skew, and the Arrow-Pratt risk aversion measure to be used is formulated in the mean and variance of each probability distribution.

The bond experts participating were active bond dealers. This raises the problem of whether the bond values which they gave, were in effect what the bonds would be worth to themselves according to their personal risk preferences. This is what the experiment aimed at. The fact that one wished to have their personal assessments of values was stressed in the written instruction, which each bond expert was given, and also in the personal conversations. But the question remains, whether the bond experts were so used to think of bond values in terms of what the bonds would trade for in the market, that it was too much to ask them to try to disregard this aspect.

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1) A detailed description of the experiment is given in: Cornelius M. Schilbred, An Experiment with Bonds and Risk, Bergen, 1969.

Furthermore, one might have wished to study the stability of preferences over time. This might have been done by revisiting each bond expert, and ask him to evaluate different sets of bonds. The purpose of the experiment was, however, to get indication of reasonable assumptions to make on investor behaviour. From these assumptions one may formulate an hypothesis on the formation of market prices for risk assets. It was felt that the stability tests should be performed on a possible market price of risk, as this raises fewer questions of measurement, than does estimates of parameters in individual preference functions. The market price of risk is also more interesting and useful quantity, in the practical application of the economics of uncertainty.

AN EXPERIMENT WITH BONDS AND RISK

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Summary

Thirty bond experts evaluated a set of Government bonds with maturities given through a series of lotteries. The bonds had varying coupons, equal maturity provisions and the experts had access to a bond market, so that the yield on alternative, certain investments was given. The runs of signs of observed risk premiums discriminated between the von Neumann and Morgenstern theory of economic behavior, an alternative behavior of maturity fixing or no systematic behavior.

Representative experts are well-described by the von Neumann and Morgenstern theory. These experts displayed risk aversion and their absolute risk aversion functions increased over the interval studied.

1. Introduction

Empirical studies of individual behaviour under uncertainty usually attempt to derive underlying structures which may explain observations. The observations may be in the form of commonly observable facts, such as, for instance, the willingness of people to buy insurance and lottery tickets. Other studies derive data through controlled experiments, and still others from markets where risky assets are traded. The most well-known works within the above cathegories are probably the ones by Friedman & Savage, Mosteller & Nogee and Farrar.[1] This paper is a report on an experiment, carried out with the participation of Danish and Norwegian bond experts in an attempt to obtain information on how risk is treated in risk-taking situations.

The bonds, the evaluation of which is to be studied, are the so-called series-bonds. These are regular bearer's bonds. From the point of view of risk, the interesting aspect of a series-bond is the fact that it has an uncertain maturity date, with the probability of a particular maturity objectively given through information in the loan agreement.

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[1] Friedman, M. and Savage, L.: The utility of choices involving risk. *Journal of Political Economy*, 1948, pp. 279–304.
Mosteller, F. and Nogee, P. An experimental measurement of utility, *Journal of Political Economy*, 1951, pp. 371–404.
Farrar, D. E.: *The Investment Decision under Uncertainty*, Englewood Cliffs, N. J., 1962.

2. Series-bonds and Characteristics of the Experiment

When a loan issue is floated through the sale of series-bonds, the debtor undertakes to pay interest at regular time intervals, and to repay the loan through a series of constant future installments, payable on specified dates. In order to determine the particular series-bonds to be redeemed on each date, debtor also agrees to provide for a series of future lotteries to be arranged, usually under the auspices of some official authority. The debtor's obligation to repay in equal amounts and to arrange lotteries, provides the bonds with objectively given uniform probabilities. The probability that a series-bond will mature on the date of the jth future installment payment, is

$$p_j = \frac{1}{n}; \quad j = 1, \ldots, n$$

where n is the number of installments outstanding in the loan issue.[1]

Evaluating a series-bond, an investor is faced with three sources of uncertainty: (*a*) the lotteries, (*b*) the risk of debtor's default of payments, and (*c*) the likelihood of future shifts in the market term structure of interest rates, which will reflect itself as capital gains or losses on bonds.

The idea of the experiment was to establish a situation where the risk of default and also the market uncertainty would be absent. This leaves only the uncertainty owing to the lottery provision, and this uncertainty was controlled in the experiment.

The bonds chosen for the experiment were government bonds. This ensures that there will be no risk of default of payments, as it is within the powers of governments to create money. The market uncertainty was obviated by stating the market condition that a government series-bond with 5 % coupon and the same maturity provisions as the other bonds in the experiment would sell at par. The 5 % bond would continue to sell at par in the forseeable future. The participants were thus faced with a market for government bonds characterized by a horizontal yield curve for payments to be received with certainty, this yield being 5 % per annum.

The two hypotheses of the experiment are that the probabilities p_j are used by an investor in the process of evaluating a bond, and that he either forms a probability distribution of future maturity dates, or a probability distribution of present values.

In using a well-known type of security and letting bond experts be the participants in the experiment, an attempt was made to obtain knowledge of the considerations of men who have as their daily task to make decisions under risk. Summers has criticized various earlier experiments for representing highly artificial situations and yielding trifling pecuniary rewards or punish-

[1] Schilbred, C. M.: Bond evaluation as a decision under certainty, risk or uncertainty. *The Swedish Journal of Economics*, 1968, pp. 43–56.

By asking an investor to evaluate different risks, it is possible to derive indirect observations on his absolute risk aversion function with the aid of (5). The absolute risk aversion function may then be estimated, and the estimate fed into

$$u(x)=\int \exp\left(-\int R_a(x)\,dx\right)dx \tag{6}$$

in order to get the utility function for money, which is then determined up to a positive linear transformation.

Provided that a series-bond is under redemption, the first two moments of the probability distribution of present values required by (5) are given by (7) and (8).

$$E(B)=\sum_{j=1}^{n}\frac{1}{n}B_j$$

$$E(B)=\left(\frac{r}{i}+\frac{1}{n}\frac{i-r}{i}\frac{1-v^n}{i}\right)D, \tag{7}$$

$$\sigma_B^2=\sum_{j=1}^{n}\frac{1}{n}(B_j-E(B))^2$$

$$\sigma_B^2=(i-r)^2\left[\frac{D^2}{i^2}\left\{\frac{v^2(1-v^{2n})}{n(1-v^2)}-\left(\frac{v(1-v^n)}{n(1-v)}\right)^2\right\}\right] \tag{8}$$

The two partial derivatives with respect to the nominal rate of interest are

$$\frac{\partial E(B)}{\partial r}=k_1 \quad \text{and} \tag{9}$$

$$\frac{\partial \sigma_B^2}{\partial r}=-2(i-r)\,k_2 \tag{10}$$

with k_1 and k_2 as follows:

$$k_1=\frac{ni-1+v^n}{ni^2}D \quad \text{and}$$

$$k_2=\left[\frac{D^2}{i^2}\left\{\frac{v^2(1-v^{2n})}{n(1-v^2)}-\left(\frac{v(1-v^n)}{n(1-v)}\right)^2\right\}\right]$$

Equations (9) and (10) show that if in an experiment i, n and D are held constant, and chosen so that k_1 and k_2 are both different from zero, it is possible to face an investor with different risks by asking him to evaluate series-bonds which have different values for r. From (9) it is seen that the expected present value is then a linear function of r. The variance has the sign of the expression within the right hand brackets of (8), or k_2. As the variance is always nonnegative, so is k_2, and the variance thus describes a parabola with a minimum value of zero for $r = i$.

An investor who behaves in accordance with the expected utility hypothesis is usually classified either as risk averter, risk neutral or risk lover. The classification depends on whether he demands positive, zero or negative risk premiums throughout. This property may be used to discriminate between the two hypotheses proposed. For an investor who behaves according to the first hypothesis (a maturity fixer), the value of a bond is B_t as defined by (3), so that $Y_0 = B_t$. The observed risk premium for a maturity fixer will therefore be

$$\prod = E(B) - B_t$$

$$\prod = (i - r)\frac{(1 - v^n - niv^t)D}{ni^2} \tag{11}$$

If for a set of series-bonds, n is kept constant, a maturity fixer will arrive at one value for t, which is when he assumes that the bonds will mature. With i and D also constant, his behaviour will cause the value of the right hand fraction in (11) to stay fixed. So that by letting the nominal rates of interest on the bonds run from smaller to larger values than i, the observed risk premiums for a maturity fixer will change sign as r passes i.

The level of the wealth at which the change in sign of risk premiums occurs is given by $Z + D$. D may in turn be chosen at will by the experimenter, so that the change in sign of risk permiums can be made to occur at an arbitrarily chosen level of the investor's wealth. This effect cannot be produced by any ordinary utility function for money which assigns one, and only one, utility to each amount of money. In the experiment D was held constant, so that only two runs of signs would be produced by maturity fixers. Otherwise it would have been difficult to distinguish their signs from signs produced at random, and to isolate the maturity fixers. Their behaviour may in itself be worth verifying. If, however, one wanted to set up an experiment solely to confirm or reject the von Neumann and Morgenstern theory, D might be varied so that the signs of risk premiums produced by a maturity fixer might equally well have been produced by a random process.

4. The Experiment

Seventeen Norwegian and 13 Danish bond experts participated in the experiment. They were all experienced bond dealers, and actively engaged in trading with bonds. Each expert was asked to assume that he had won or received as a gift one of 14 different government bonds. He was then asked to state what he felt each bond was worth, i.e. to state the cash

Table 1. *Expected present values, variance and median values of the bonds in the experiment*

$n = 16.5 \cdot 2 = 33,\ i = 0.05/2 = 0.025,\ D = \text{kr. } 10\ 000$

Bond no.	Nominal rate of interest	Expected present value (kr)	Variance	Median value (kr)
1	2 1/2	8 378	623 739	8 286
2	2 3/4	8 540	505 229	8 457
3	3	8 702	399 193	8 629
4	3 1/4	8 864	305 632	8 800
5	3 1/2	9 027	224 546	8 972
6	3 3/4	9 189	155 934	9 143
7	4	9 351	99 798	9 314
8	4 1/4	9 513	56 136	9 486
9	4 1/2	9 676	24 949	9 657
10	5	10 000	0	10 000
11	5 1/4	10 162	6 237	10 171
12	5 1/2	10 324	24 949	10 343
13	5 3/4	10 487	56 136	10 514
14	6	10 649	99 798	10 686

amount that would have to be offered to him in order for him to be indifferent between receiving the cash or the bond.[1]

All bonds were government series-bonds, had 16½ years to final maturity of their respective issues, and par value kr 10 000. The nominal rate of interest ran from 2½ % to 6 % per annum, with installments and interest being payable semi-annually. The market condition was that bond no. 10, which carried a nominal rate of interest of 5 % per annum, traded at par value, and the experts were asked to use this bond as a basis when evaluating the other 13 bonds.

The characteristics of the bonds are given in Table 1. Bond no. 10 could be found on the Norwegian market, and it was quoted at par on the Oslo stock exchange during the 12 days from March 8th to March 19th, 1968, when the Norwegian experts were interviewed. None of the other 13 bonds were traded at the time on the Norwegian market. The Danish experts were visited during the days from May 7th to May 16th, 1968. None of the bonds which they were asked to evaluate were Danish government bonds on the market at that time. A 4 % series-bond with 16 4/12 years to maturity, semi-annual interest payments and with installments payable annually was, however, quite comparable to bond no. 7 in the experiment. It was quoted at 72.50 % of par value, thus yielding approximately 8.90 % per year. So the 5 % market situation, which the Danish experts were asked to assume, was quite different from the market situation they were experiencing in Copenhagen at that time.

[1] The experts were asked to disconsider possible differences between the tax liability on income from interest payments and capital gains or losses.

5. Two Special Hypotheses

An earlier investigation gave that a series-bond was deemed equal to its median bond, which has a fixed maturity in the φth lottery or installment payment,[1] where

$$\varphi = \frac{n+1}{2} \tag{12}$$

The median bond thus matures on the expected value of the future maturity dates. An investor who uses (12), as a decision rule would be classified as a maturity fixer. Equation (12) was taken as an indication that the experts looked at some measure of the center of the uniform probability distribution when assessing the value of a bond. It was suggested, however, that they might indeed be aiming for the expected present value, and that (12) could be used to get a value close to this. Two special hypotheses suggest themselves from these considerations. The significance level in the statistical analysis is 5 %.

The first special hypothesis is that if a bond expert is chosen at random and asked to evaluate a series-bond, he will evaluate it to be equal to its median bond. With $\varphi = (33+1)/2 = 17$, the median bond matures in $8\frac{1}{2}$ years. The present values of the median bonds to the series-bonds in the experi- ment are given in Table 1. Multivariate statistical analysis is used. Assuming the observations come from a normal distribution $N(\mu, \Sigma)$, the first hypothesis is that the unknown mean vector is the vector of median values, $\mu = \mu_m$. A value for the statistic $F_{13,\,17} = 1.67$ implies that this hypothesis cannot be rejected.

The second special hypothesis is that the bond expert will evaluate the series-bond at its expected present value. Testing now the hypothesis that the unknown mean vector is the vector of expected present values in Table 1, $\mu = \mu_{E(B)}$, the computed statistic $F_{13,\,17} = 1.69$ says that this hypothesis also cannot be rejected.

This experiment thus yields that predictions on bond expert valuations would be about equally good from either the median or the mean hypothesis.

6. Risk Premiums and Expert Classification

The experts are classified on the basis of runs of signs of their risk premiums, and the null hypothesis is that the signs have been produced at random. If an expert's values lead to rejection of the null hypothesis, he will be classified, otherwise he will be left unclassified.

First, it may be noted that for a risk averter or risk lover all risk premiums will be of equal sign. With the probability of a plus equal to that of a minus,

[1] *Bond Evaluation*, pp. 49–50.

Table 2.

Experts	Maturity fixers	Risk averters	Unclas-sified	Total
Norwegians	10	5	2	17
Danes	3	5	5	13
Total	13	10	7	30

the probability of observing a risk averter, risk neutral or risk lover, is 0.0001 under the null hypothesis. Such behaviour would, therefore, strongly reject the null hypothesis.

As $r<i$ for the first 9 bonds and $r>i$ for the last 4, a maturity fixer will have 9 signs of one kind followed by 4 of the other, i.e. exhibit 2 runs. An occurrence of this event under the null hypothesis has the probability 0.0028, so that a maturity fixer will also be eligible for classification in the experiment. Both the hypothesis for a maturity fixer and the expected utility hypothesis would predict few runs of signs, so that the test is one-tailed.

The classification procedure for an expert is first to substitute sign indicators for the risk premiums of the observation vector and apply the one-sample runs test.[1] If the expert is cleared for classification, his vector of sign indicators is compared with each of four classification vectors, one for risk averters, one for risk lovers and two for maturity fixers (one for signs running from positive to negative, and one for negative to positive). The expert is then classified into the group from which he has the smallest sum of squared differences of sign indicators.

Table 2 shows the results of the expert classification.

Testing the hypothesis that these is no difference behaviour between Danes and Norwegians, a $\chi^2=5.4$, indicates that there was no significant difference. This may mean that both groups have understood the experiment about equally well, and that they have accepted its market conditions.

On the average, positive risk pre- *) miums were demanded for all 13 bonds. If one were to choose a representative bond expert, this would favour the choice of a risk averter. If instead, the nominee represents the majority, a maturity fixer would be the candidate, as there were 13 maturity fixers and 10 risk averters participating.

Table 3 shows that most of the participants held senior positions in their firms, and that among those holding senior positions the majority were risk averters. If a senior position is indicative of a more mature comprehension of bond

[1] Negative risk premiums were given 0-indicator and nonnegative 1-indicator. Risk neutrals will thus be classified as risk averters, but they will be isolated in section 7.

*) In the article reference is made to table 2, which was omitted in the final edition.

Table 3.

	Maturity fixers	Risk averters	Unclas-sified	Total
Directors & h.o.d.				
Partners	7	9	5	21
Brokers	6	1	2	9
Total	13	10	7	30

evaluation problems, this would mean selection of a risk averter. The probability is that their considerations carry heavier weight in the actual price bargaining on the exchange.

7. Absolute Risk Aversion and Utility Functions for Money

The absolute risk aversion function should be evaluated at the points $(Z+E(Y))$, i.e. initial wealth plus the expected value of the risk. However no information on initial wealth was obtained. The functions were, therefore, only studied over the interval $[Z+\text{kr. } 8\,378, Z+\text{kr. } 10\,649]$.

In order to draw conclusions about the economics of uncertainty, it is often enough to know whether the absolute risk aversion function is increasing, constant or decreasing with increasing wealth. The absolute risk aversion functions were studied over an interval of kr. 2 271, which is likely to represent about half of a month's salary after tax for most of the risk averters participating. If it is assumed that the absolute risk aversion function is linear, or that it may be approximated with a linear function over this interval, it is possible to obtain information on its slope. The estimates are given in Table 4.

Table 4. *Estimate of* $R_a = C + D(E(B) - 8\,378)$

Expert no.	Constant term C	$F(1,11)$	Regression coefficient D	$F\ (1,11)$	$F\ (2,11)$ joint test on C and D estimates
5	−0.00064	0.03	0.0000051	4.20	5.60
6	−0.00483	0.58	0.0000119	6.01	5.51
9	−0.00349	0.30	0.0000097	3.96	3.86
10	0.00057	0.06	0.0000032	3.29	6.46
17	0.00008	0.33	0.0000002	6.93	15.36
18	−0.00140	0.32	0.0000036	3.83	3.62
19	−0.00141	0.06	0.0000129	9.65	13.17
21	0.00195	0.54	0.0000054	7.22	17.40
26	−0.00482	0.78	0.0000119	8.17	7.49
27	0.00000	0.01	0.0000000	0.87	1.63

The three hypotheses tested are whether the constant term C, the regression coefficient D, or both, are equal to zero. Looking first at the joint hypothesis on C and D, the hypothesis that they are both zero cannot be rejected for experts nos. 9, 18, and 27. The three are risk neutral, and might be described by the utility function for money $u(x)=x$. For the 7 other experts the joint hypothesis is rejected. None of the constant terms are significant, nor are the regression coefficients for nos. 5 and 10. Nos. 6, 17, 19, 21 and 26 do, however, have significant regression coefficients. The slopes of their absolute risk aversion functions are all positive, indicating increasing absolute risk aversion for these 5 experts.

Returning finally to the selection of a representative bond expert, and feeling that he should be a risk averter, the average risk premiums of Table 2 suggest that his utility function for money should have the property of increasing absolute risk aversion.[1] If in fact the absolute risk aversion is linear, $R_a(x)=Dx$, Borch[2] has shown that the utility function is the cumulated normal distribution.

$$u(x)=e^C\int_{-\infty}^{x} e^{-Dy^2/2}dy$$

where C is a constant of integration. This utility function is bounded from below and from above, and thus satisfies Menger's conditions.[3]

8. Conclusion

In the evaluation of series-bonds an investor is given information on a set of uniform probabilities through the redemption agreement of the loan.

Two hypotheses on investor behaviour are proposed and tested. One is that the investor uses the probabilities to form a distribution of future maturity dates, and derives an assumed maturity from this distribution. The series-bond is then given the value of a bond with fixed maturity on the assumed maturity date. The second hypothesis the is that investor behaves in accordance with the von Neumann and Morgenstern theory of economic behaviour. The two hypotheses are distinguished by means of runs of signs of risk premiums.

Thirty bond experts participated in the experiment, and each was given 13 bonds to evaluate. The general result was that good predictions are made by assuming that experts fix the maturity of a series-bond to its expected maturity, or that they evaluate the bond at its expected present value.

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1 $\hat{C}=0.00005$, $F^{C}_{(1,11)}=0.00$; $\hat{D}=0.0000024$, $F^{D}_{(1,11)}=6.75$; $F^{C,D}_{(2,11)}=10.90$.

2 Borch, Karl.: Decisions rules depending on the probability of ruin, *Oxford Economic Papers*, vol. 20, no. 1, March 1968, pp. 1–10.

3 Menger, K.: Das Unsicherheitsmoment in dėr Werlehre, *Zeitschrift für Nationalökonomie*, 1934, pp. 459–485.

An Experiment with Bonds and Risk

For experts whose behavior is explained by the von Neumann and Morgenstern theory of economic behaviour, it was attempted to derive utility functions for money. The two utility functions, which seem to suggest themselves are the linear and the cumulated normal distribution. A representative expert is characterized by the normal distribution.

Chapter III

Yields on Norwegian Government Bonds - Quarterly 1958-1971

Preface

When forming their assessments of bond values, the investors are assumed to have access to a market for risk free investments. In the experiment this was the market for a Government bond with no lottery risk. The yield on this Government bond was thus given to be the yield on alternative, risk free investments open to investors in lottery bonds. In the experiment, the yield curve, which gives the relationship between yield and time to maturity for bonds trading on a day, was thus given to be a constant 5% per year.

The rest of this book will be a report on studies of market prices, and how these are formed. Still the investors are assumed to have access to a market for risk free investments in Government bonds. But in the market, the yield curve for yields on Government bonds can not be assumed a constant function of time $i(t) = a$, where i = yield to maturity, t = time to maturity and a = constant.

Time to maturity, t, was set to be an integer in the experiment. When studying the market, however, time to maturity must be allowed to be a real number. The paper uses a standard prosedure to allow for this, so as to explain the transaction prices for bonds. But when setting the market prices for bonds, bond dealers use bond values tables which contain what is formally a mathematical error. The error is to let the variable time in the formula for bond values, take on real values, whereas the formula is only valid for integer values of t. It is explained how bond dealers correct for this error, through the established practise of adding simple interest to the market prices quoted, in order to obtain the transaction prices.

Yield curves give a condensed description of the interest rate structure on a particular trading day. The paper explains the construction of yield curves through regressions. The purpose of the paper is to show how one may construct a consistent times series of interest rates for Norwegian Government bonds. This same technique is then in turn used for Italian Government bonds, as part of the study to be reported in chapter IV on market prices for risk investments.

Særtrykk av Statsøkonomisk Tidsskrift nr. 1, 1972.

YIELDS ON NORWEGIAN GOVERNMENT BONDS — QUARTERLY 1958—1971*

BY CORNELIUS M. SCHILBRED

1. *Introduction.*

This study discusses the procedure of yield computation for bonds. The purpose is to support other studies of Norwegian monetary relations with more satisfactory interest rate data than those which are at present available.[1] These series are such as to make comparisons over time doubtful, and the following suggests how this may be rectified.

The yield data presently published by the Norwegian Central Bureau of Statistics are the yields on three government bonds. Two of these have medium or short term to maturity and one is a long term bond. The same bonds are, however, used in the series over a period of time, so that their terms to maturity decrease. This may in itself be an undersirable property in a time series of interest rates. Another problem is that the bonds in the series will have to be replaced at regular intervals. This may cause shifts in the time series published[2], buth such shifts may not necessarily reflect shifts in the market rate of interest.

In the following it will be explained how the bond yields have been

* This is one of a series of monetary studies at the Norwegian School of Economics and Business Administration, supported with grants from the Bank of Norway's Fund for Economic Research and the Research Fund of the Norwegian School of Economics and Business Administration.

I wish to thank Thore Johnsen for programming assistance, and Norges Bank for providing most of the quarterly price data before 1969.

[1] See Teigen (1972), and Teigen, Schilbred and Thore (1971).

[2] See OECD (1970) p. 35. There is, for example, an upward shift in long term yield in 1962 IV caused by a new bond being entered in the series. From figure 4 is seen that the long rate appears to have fallen in this period according to the present series.

computed from observed market prices. From the computed yields a set of yield curves has been estimated. These yield curves were then used to construct time series of interest rates, one short term and one long term yield.

2. *Market and Transactions prices.*

Before addressing the question of yield computation, a brief account of how bond prices are computed may be in order. A standard bond is a promise by the debtor to pay to the holder of the bond one or more amounts of money in coupon payments at a definite point or points in time, and to redeem the bond at its maturity. The value of a bond is equal to the present value of the amounts of money payable over its life:

$$B_n = \sum_{j=1}^{n} D \cdot r \cdot v^j + D \cdot v^n \qquad (1')$$

B_n = value of bond maturing at time n
r = coupon paid per period (per cent)
i = yield per period (per cent)
v = $(1+i)^{-1}$ = discount factor
D = par value of the bond = 1 in the following.

Taking the sum on the right hand side of (1′) and ordering)

$$B_n = \frac{r}{i} + \frac{i-r}{i} v^n \qquad (1)$$

(1) is the standard formula used in the computation of bond values tables.

As n is a summation index, formula (1) is only valid for integral values of n. If, say there are annual coupon payments in a bond issue, this means that formula (1) can only be used to compute the value of a bond when it has one, two, three or more years to maturity.

In order to derive a more general formula, (1) may be used to obtain the value of a bond at its last coupon payment. The bond then had n periods outstanding to maturity and its value was B_n. Define the part of the coupon period which has elapsed since the last coupon payment to be α, so that $1-\alpha$ is the part of the coupon period outstanding to the next payment. Time to maturity as on the day of computation is t. With α and t being real numbers, n will thus be the first integer larger than t, $n=[t]$, and $t=n-\alpha$. The coupon payments are spaced with equal time intervals between them, usually one half or one year, with the last coupon payment being made on the day of redemption of the bond. Time from the day of computation to the day of a coupon payment, s, may therefore be written $s=j-\alpha$, where $j=[s]$.

Assuming a constant yield, i, per period, and that the discount factor is a continous function of time, $v(s)$, the present value of a bond maturing at time t will be

$$B_t = \sum_{j=1}^{n} rv^s + v^t ,$$

and, as $s=j-\alpha$ and $t=n-\alpha$,

$$B_t = \sum_{j=1}^{n} rv^{j-\alpha} + v^{n-\alpha} ,$$

$$B_t = v^{-\alpha}\left\{\sum_{j=1}^{n} rv^j + v^n\right\} ,$$

$$B_t = (1+i)^\alpha\left\{\frac{r}{i} + \frac{i-r}{i}v^n\right\} \tag{2}$$

because $v^{-\alpha}=(1+i)^\alpha$.

Formula (2) is the general formula for the present value of a bond with t periods to maturity. It will be shown that by current practice of bond brokers, the transactions prices between buyers and sellers of bonds are close approximations to (2). A clear distinction must, however, be made between market prices and transactions prices.

The market prices are computed from formula (1). Bond values tables [3] currently in use are based on formula (1), where n is allowed to take non-integral values. This, of course, is incorrect and yields

[3] See for instance: *Comprehensive Bond Values Tables*, Computed and Compiled by Financial Publishing Company, Boston.

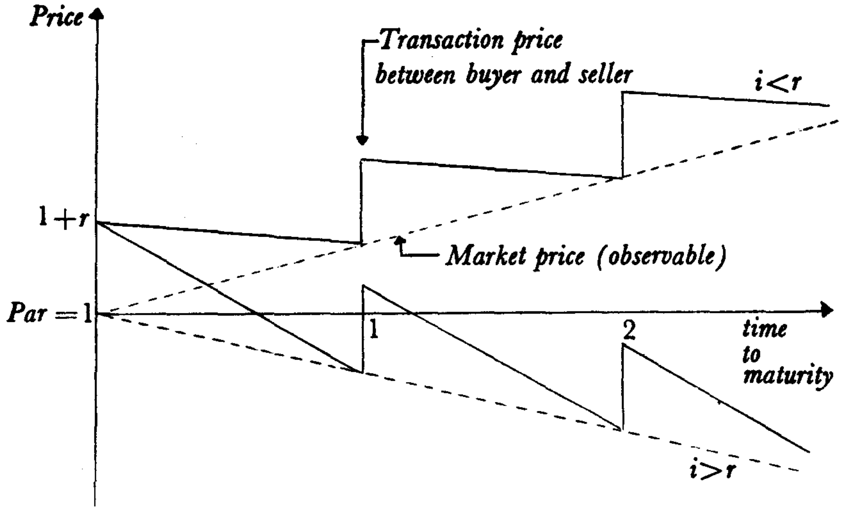

Figure 1.

observable market prices which are economically meaningless.[4] The standard procedure of setting a market price appears to be the following. First a desired or market yield to maturity is set, then a bond value table is used to get the market price.

Figure 1 gives the relationship between time to maturity and price. The transaction prices are given by formula (2) and the market prices by formula (1). Coupon payments are made at the points in time 0,1,2.[5]

[4] Such prices may be commonly observed in, among other, the following countries: Denmark, England, France, Italy, Norway, Sweden and the U.S.A.

[5] The curves have been drawn as straight lines, which is an approximation. The market price curve is concave to the origin if the yield is below the coupon, and it is convex to the origin if the yield is above the coupon; see Malkiel (1962). The transaction price curve is a step function. The steps occur because the value of a bond falls by the amount of the coupon payment on each coupon date. a on that day goes from 1 to 0, so that the transaction and the market price coincide when the coupon has been paid. n is a constant between two coupon payments, when a runs from 0 to 1, and $(1+i)^a$ therefore from 1 to $1+i$, exponentially. Thus the transaction price curve is stepwise convex to the origin.

It is seen from figure 1 why the market prices are economically meaningless. Take, for simplicity the case where yield to maturity is constant over time, and this yield is smaller than the coupon rate for a particular bond, $i<r$. The market price will then be above par, and it will decrease as the bond approaches maturity. This means that, for instance, the bond will be priced lower when it has 1/2 of a coupon period left to maturity than when it has 3/4 of a period, say a year, left. But with positive time preference, $i>0$, one should indeed be willing to pay more for an amount payable in six months than for the same amount payable in nine months.

The difference between the transactions price (2) and the observable market price (1), for a given time to maturity t, is:

$$(1+i)^{\alpha}\left\{\frac{r}{i}+\frac{i-r}{i}v^{n}\right\}-\left\{\frac{r}{i}+\frac{i-r}{i}v^{t}\right\}=$$

$$(1+i)^{\alpha}\left\{\frac{r}{i}+\frac{i-r}{i}v^{n}\right\}-\left\{\frac{r}{i}+\frac{i-r}{i}v^{n-\alpha}\right\}=\frac{r}{i}\{(1+i)^{\alpha}-1\} \tag{3'}$$

Bond brokers have, however, established a practice of "adding simple interest at the coupon rate" to the market price in order to obtain the price to be used in the transaction between buyer and seller. The work is done back in the broker's office, where the amount αr is added to the market price. This yields a difference between the theoretical transaction price of formula (2) and the real transaction price

$$\frac{r}{i}\{(1+i)^{\alpha}-1\}-\alpha r\,. \tag{3}$$

Expanding $(1+i)^{\alpha}$ in a Taylor's series and substituting

$$\frac{r}{i}\left\{1+\alpha i+\sum_{k=2}^{\infty}\frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}i^{k}-1\right\}-\alpha r$$

$$\frac{r}{i}\sum_{k=2}^{\infty}\frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}i^{k} \tag{4}$$

(4) gives the difference between the theoretical transaction price and the real transaction price. In general (4) will be very small. If for

instance a bond pays 6% annual coupon and yields 6% per annum, a kr. 1000 bond will be traded at a price which is kr. 0.437 too high, if there are 6 months outstanding to the next coupon payment.

Equation (4) shows that by adding simple interest to the market price, the bond brokers are correcting an error committed in the computation of bond values tables. The resulting transaction price is a close approximation to the theoretical price. The problem remains, however, that the market prices as usually quoted, are meaningless in an economic sense. It would be an improvement if the market price would be the sum of the price read from bond values tables and simple interest. Simple interest appears to be "included in the market price" for a few bonds on some exchanges.[6] This should, however, be the rule and not the exception.

3. *The computation of yield to maturity.*

The yield computations were made for prices quoted for Norwegian government bonds on the Oslo exchange on the last trading day of each quarter. For the prices observed, formula (1) was used to obtain the yield to maturity.[7] A search algorithm was used. Given its time to maturity, the value of a bond by formula (1) is decreasing monotonically with increasing values for the yield, i.[8] By trying different values for i and computing the market value from formula (1), a binary search method was used to obtain a value for i, which was such as to yield a market price sufficiently close to the price observed. The criterion for sufficient accuracy was that the yield tried should not give a price which differed from the observed price by more than $\pm$ kr. 0.025. The market prices are given with intervals of kr. 0.05, so that this accuracy criterion exhausts the accuracy of the given market price. This means, for instance, that if the market price

[6] There are no bonds on the Oslo exchange where simple interest is included in the market price. Although no thorough investigation has been made, simple interest is included in the market price on some Italian bonds, and also on some U.S. bonds.

[7] Information on coupon and redemption plan was obtained from *Håndbok over norske obligasjoner og aksjer*, published by Carl Kierulf & Co. A/S, Oslo.

[8] B. Malkiel (1962).

quoted was kr. 100.00, the search would stop if the yield tried gave a market value between kr. 99.975 and kr. 100.025.

For the bonds which do not have a fixed maturity but are redeemed through a series of lotteries, the time to expected maturity was used in the calculations above.[9] A series-bond was expected to mature on the day of its lottery number

$$\sum_{j=1}^{k} \frac{1}{k} \cdot j = \frac{k+1}{2},$$

where k is the number of future lotteries to be arranged. For an annuity-bond the expectation for the lotteries is

$$\sum_{j=1}^{k} \frac{r(1+r)^{j-1}}{(1+r)^k - 1} \cdot j = \frac{k(1+r)^k}{(1+r)^k - 1} - \frac{i}{r}.$$

There also exist government loans where the government has the option either to arrange lotteries or to purchase bonds in the market in order to meet the repayment schedule. Such bonds were treated as fixed maturity bonds if the price was below par, and as lottery bonds if the price was above par.[10]

4. *Yield curves.*

The purpose of a yield curve is to convey in a simple manner the relationship between the market yield and term to maturity.[11] David Durand (1942) is the classic in the construction of yield curves. Durand suggests four different yield curves as typical. Three of them are shown

[9] The assumption is that bond brokers behave so as to represent a bond with an uncertain maturity with the median bond to the lottery bond. See C. M. Schilbred (1972).

[10] This is a standard market convention. See for instance J. Grant (1964), the U.S. Treasury Bulletin and C. M. Schilbred (1968).

[11] Term to maturity is the usual measure of the duration of a loan. F. Macaulay (1938) has, however, suggested a more composite measure, which also takes into account the coupon rate. For two bonds with the same time to maturity but with different coupons, the bond with the highest coupon is defined by Macaulay to have the shortest duration, as this is the bond where the weight of the total payments is the earliest from today.

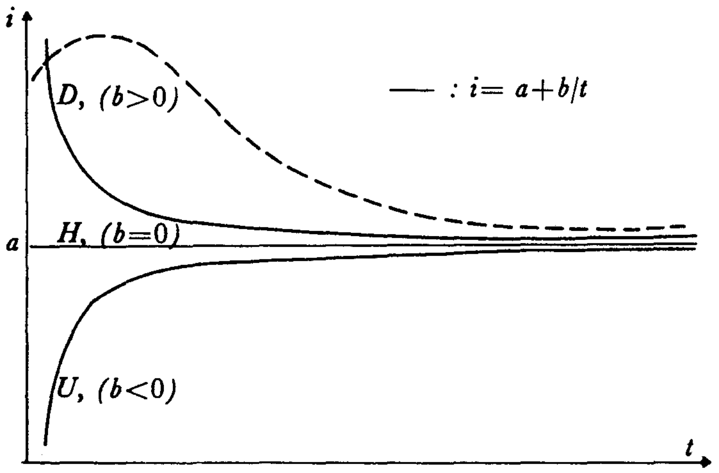

Figure 2.

as curves *D*, *H* and *U* in figure 2. Durand's fourth curve has the form of the dotted curve in figure 2.

Durand fitted his yield curves by freehand to the observed yields. An alternative technique is to choose a set of yields which span the maturity range studied, and to use linear interpolation to draw the yield curves. This was done by Grant (1964). Still another method is to use linear regression analysis to estimate the yield curves. K. Cohen, R. Kramer and W. H. Waugh (1966) used this approach for U.S. government securities, and D. Fisher (1966) used it for British data.

In the construction of yield curves, one has to treat large amounts of data. With the storage facilities of present computers, it is a simple task to update a time series of yield curves based on regression analysis, once the initial programs have been developed. As it is the purpose of the present work that it shall be simple to obtain interest rate data for monetary studies, it was therefore, chosen to use regression analysis to estimate the yield curves.

The functional form chosen was the curves *D*, *H* and *U* of figure 2. These are obtained by estimation of the rectangular hyperbola

$$i = a + b/t + e \qquad (5)$$

where i = yield per year
t = years to maturity
e = the estimation error.

The estimated constant term, $\hat{a}$, may be seen as "the long term rate of interest". Formally speaking a is the rate of interest on a perpetual bond, but no such bonds exist in Norway. By letting $t = 1$ in (5), it is seen that the sum of the coefficients $a + b$, give the rate of interest on a one-year bond.

In the regressions, bonds with optional redemption were deleted from the data. Their yield moved out of line with the other yields during the period. The government issued no loans with optional redemption in the later years, and most of such existing loans matured during the period. The market for these loans became rather inactive, and at times the government found it difficult to obtain through the market the amount of bonds necessary to fulfill the redemption obligations. For this reason the loans with optional redemption were not included in the estimated time series of interest rates.

Table 1. Estimate of: $i = a + b/t$.

End of quarter	$\hat{a}$	Std.err. $\hat{a}$	$\hat{b}$	Std.err. $\hat{b}$	R^2	No. of obs.
58 I	4.901*	0.061	−1.779*	0.102	0.9433	20
II	4.673*	0.090	−1.374*	0.120	0.8727	21
III	4.619*	0.119	−1.301*	0.164	0.7681	21
IV	4.501*	0.120	−1.041*	0.130	0.7785	20
59 I	4.715*	0.306	−2.170*	0.668	0.3829	19
II	4.691*	0.279	−2.514*	0.621	0.4766	20
III	4.635*	0.286	−1.992*	0.623	0.3617	20
IV	4.623*	0.248	−1.748*	0.529	0.3771	20
60 I	4.570*	0.263	−1.435*	0.548	0.2759	20
II	4.600*	0.280	−1.181*	0.558	0.1987	20
III	4.545*	0.248	−0.977	0.473	0.1911	20
IV	4.518*	0.233	−0.916*	0.417	0.2110	20
61 I	4.727*	0.165	−2.070*	0.302	0.7342	19
II	4.803*	0.147	−1.852*	0.216	0.8108	19
III	4.743*	0.204	−1.466*	0.374	0.4894	18
IV	4.818*	0.204	−1.798*	0.338	0.6531	17

(Table 1 cont.)

End of quarter	$\hat{a}$	Std.err. $\hat{a}$	$\hat{b}$	Std.err. $\hat{b}$	R^2	No. of obs.
62 I	4.554*	0.237	−1.081*	0.396	0.4030	13
II	4.824*	0.160	−2.175*	0.293	0.8459	12
III	4.956*	0.126	−2.180*	0.193	0.9267	12
IV	4.711*	0.108	−0.943*	0.201	0.7094	11
63 I	4.621*	0.098	−1.068*	0.182	0.7571	13
II	4.592*	0.104	−0.782*	0.181	0.6279	13
III	4.531*	0.101	−0.511*	0.162	0.4741	13
IV	4.657*	0.096	−0.993*	0.168	0.7772	12
64 I	4.558*	0.166	−0.459	0.346	0.1637	11
II	4.590*	0.133	−0.381	0.269	0.1815	11
III	4.665*	0.102	−0.702*	0.202	0.5724	11
IV	4.620*	0.094	−0.327	0.180	0.2673	11
65 I	4.640*	0.097	−0.485*	0.181	0.4175	12
II	4.730*	0.107	−0.279*	0.188	0.1663	13
III	4.977*	0.158	−2.045*	0.335	0.7562	14
IV	4.960*	0.168	−1.760*	0.353	0.6739	14
66 I	4.946*	0.138	−1.849*	0.293	0.7397	16
II	4.890*	0.144	−1.513*	0.303	0.6398	16
III	4.869*	0.125	−1.544*	0.263	0.6960	17
IV	4.799*	0.125	−1.146*	0.259	0.5499	18
67 I	4.883*	0.112	−1.292*	0.224	0.6488	20
II	4.822*	0.082	−1.061*	0.153	0.7257	20
III	4.754*	0.060	−0.819*	0.100	0.7445	25
IV	4.906*	0.118	−1.527*	0.243	0.6205	26
68 I	4.883*	0.110	−1.318*	0.221	0.5678	29
II	4.855*	0.099	−1.064*	0.190	0.5367	29
III	4.756*	0.081	−0.727*	0.140	0.4975	29
IV	4.687*	0.064	−0.465*	0.082	0.5408	29
69 I	4.745*	0.082	−0.791*	0.144	0.5154	30
II	4.732*	0.073	−0.666*	0.103	0.5991	30
III	5.290*	0.228	0.785*	0.346	0.1459	32
IV	5.590*	0.249	0.617	0.412	0.0741	30
70 I	5.401*	0.239	1.791*	0.376	0.4302	32
II	5.748*	0.129	−0.013	0.200	0.0001	35
III	5.612*	0.114	0.508*	0.156	0.2414	35
IV	5.720*	0.085	−0.216	0.106	0.1145	34
71 I	5.413*	0.234	2.552*	0.330	0.6810	30
II	5.664*	0.113	0.160	0.157	0.0359	30
III	5.617*	0.116	0.523*	0.150	0.3104	29

Table 1 reports the regressions. The level of significance chosen is 5%, and significant estimates are marked with asterisks. Table 2 below shows in a condensed form the results from table 1.

Table 2.

	Constant term $\hat{a}$	Regression coeff. $\hat{b}$		
		neg.	pos.	total
Significant at 5% level	55	41	5	46
Not significant at 5% level	0	7	2	9
Total	55	48	7	55

All the constant terms were significantly different from zero. The estimated values for $\hat{a}$ should thus give a good indication of the yield on a long term bond at the end of each quarter. Long term may, for practical purposes, be taken to mean maturities longer than 12 years.

Of the estimates on regression coefficients 46 out of 55 were significant; 41 of these were negative, which means that the yield curve was of the form U, or upward sloping. 5 had positive and significant regression coefficients, saying that the yield curves were downward sloping, D. In 9 quarters the estimated regression coefficients were not significantly different from zero. The yield curves in these cases were horizontal lines, H.

The coefficient of determination, R^2, varied between 0.00 (1970 II) and 0.94 (1958 I). In 1970 II the yield curve was horizontal. The estimated constant term was 45 times its standard error. The R^2 expresses the correlation between the yield and time, and when the yield curve is horizontal, there is no such correlation. This means that the R^2 is only an interesting statistic in the case of an upward or a downward sloping yield curve. Figure 3 shows the estimated yield curves for 1958 I and 1970 I together with the computed yields and maturities. For 1970 II one sees how the horizontal yield curve appears to be representative of the market situation, although the $R^2 = 0.00$. In 1958 I the yield curve is clearly upward sloping, and the R^2 is a meaningful statistic.

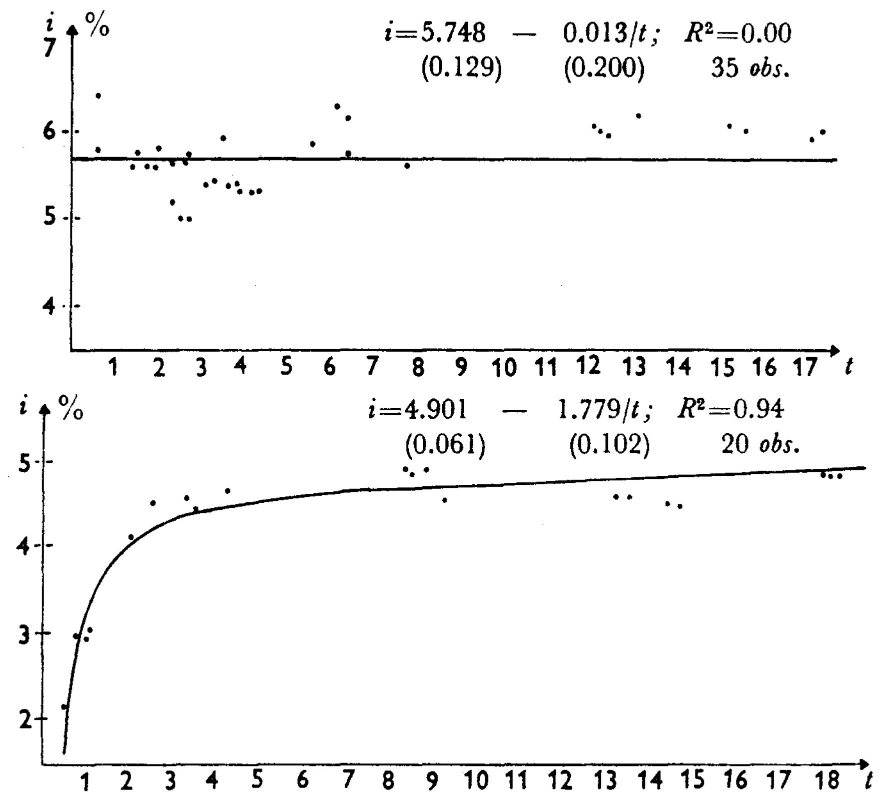

Figure 3.

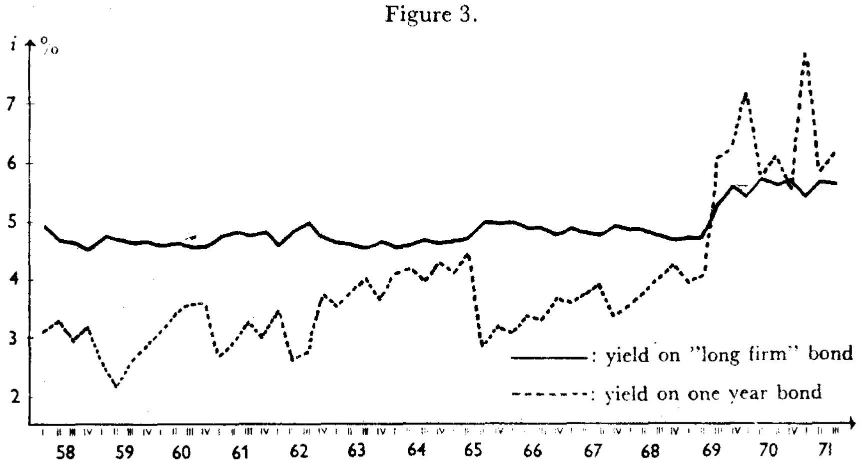

Figure 4.

Chapter IV

The Market Price of Risk.

Preface

The main result of the experiment in chapter II was that the von Neumann and Morgenstern theory of economic behaviour may be used to describe the investor's treatment of risk. The investors were typically risk averse, and their absolute risk aversion increasing with increasing wealth over the interval studied. A cumulated normal distribution is among the utility functions for present wealth which fit the observed behaviour of investors. Another utility function which displays increasing absolute risk aversion is the quadratic $u(x) = x-cx^2$.

The experiment was set up on the assumption that the two first moments, the mean and variance, contain the essential relevant information to investors. This assumption is maintained in the study of market prices, which is reported in the following papers.[1)] The equilibrium model to be formulated as the hypothesis for observable market prices, will thus be a mean/variance equilibrium model. The means and variances are over probability distributions of present values for the lottery bonds.

The choice of present values was made because the possible cash flows from each lottery bond have different time lengths. So that if one were to choose the standard formulation that investors' maximize the expected utility of end of period wealth,

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1) The market price study was first published in French: Cornelius M. Schilbred, "Le prix du risque", Annales de l'INSEE, no. 9, Jan.-April 1972, pp. 89-118. A condensed version was subsequently published in The Review of Economic Studies, Vol. XL(2), April 1973, pp. 283-292. The English RES version is included in this chapter to give a consistent presentation in language. The Annales paper is given in Appendix 2.

the determination of the length of this period itself poses a difficult and critical problem. The present value model avoids this. It also falls in with the established investment criterion under certainty, which is to choose investments so as to maximize present wealth.

One of the fundamental assumptions of equilibrium theory under uncertainty, is that investors have homogeneous information on the probability distributions. Lottery bonds offer a unique property in this respect, as the probabilities may be taken to be objectively given.

Other empirical tests of various mean/variance models for securities markets nave been based on data from the stock market.[2)]

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2)

F. Black, M. Jensen, M. Scholes, "The Capital Asset Pricing Model: Some Empirical Results", Studies in the Theory of Capital Markets, edited by Michael Jensen, Praeger, New York, 1972.

G.W. Douglas, "Risk in the Equity Markets: An Empirical Appraisal of Market Efficiency", Yale Econ. Essays 9 (Spring 1969), pp. 3-45.

Eugene F. Fama and James D. MacBeth, "Risk, Return and Equilibrium: Empirical Tests", Journal of Political Economy, 1973, pp. 607-636.

I. Friend and M. Blume, "Measurement of Portfolio Performance under Uncertainty", American Economic Review, V.60 (September 1970), pp. 561-575.

Terje Hansen, "A Quarterly Portfolio Allocation Model", Discussion paper (mimeograpted), The Norwegian School of Economics and Business Administration, Bergen, 1969.

M. Miller and M. Scholes, "Rates of Return in Relation to Risk: A Re-Examination of some Recent Findings", Studies in the Theory of Capital Markets", edited by Michael Jensen, New York, Praeger 1972.

Investors have been assumed to make their forecasts on future rates of return on the basis of historical data. It is well accepted that this is a doubtful assumption. Given the stock market data, it is, however, probably the most reasonable assumption to make.

The conclusions of the empirical tests have not been in support of the mean/variance formulation of the equilibrium model, or as it is sometimes called, the Sharpe-Lintner-Mossin model. An alternative "two parameter portfolio model" appears to fare somewhat better.[3)] The hypotheses are both two-parametric, and it is hard to see that the data are precise enough to justify a discrimination between the two. Results which support a "two parameter portfolio model", are also felt therefore, to give some indication that the S-L-M model is a reasonable hypothesis.

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3) E.Fama, op.cit. p. 633.

Reprinted from THE REVIEW OF ECONOMIC STUDIES, Vol. XL (2), April, 1973, CORNELIUS M. SCHILBRED, pp. 283-292.

The Market Price of Risk [1, 2]

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A mean-variance equilibrium model is tested against data from the Italian bond market. General equilibrium models under uncertainty were first constructed by Allais [1] and Arrow [2]. The model tested is of the class formulated by Borch [3], on the basis of works by Markowitz [8] and Tobin [18], and developed further by Sharpe [17], Lintner [6] and Mossin [10].

The bonds used in the test are annuity bonds. These are bearer bonds in loans, where the loan is repaid by a constant amount per period in cover of repayment of capital and interest on the loan. A lottery is arranged before each payment date, in order to determine which bonds shall be redeemed by the forthcoming repayment of capital. The bonds not drawn for redemption then participate in the next lottery, which is held one period later. A sequence of such lotteries is arranged, until all bonds in the loan have been redeemed, and the full amount of capital in the loan has been repaid. An annuity bond thus has an uncertain maturity. Its probability of redemption on a future date is given by the repayment plan for the capital of the loan.

The certainty alternative which is considered relevant to the investors is investment in Government bonds. With the market for Government bonds assumed to be in equilibrium, this property is used to take account of the time dimension of investment in the annuity bonds. The investors are then assumed to behave in accordance with the von Neumann and Morgenstern theory [11], and establish preferences over probability distributions of present wealth. Such a behavioural assumption was tested earlier in an experiment [15], and gives a reasonable description of investor behaviour. Italian data were chosen, because investors paid no taxes on income from capital gains and interest payments during the period studied.

I. THE MEAN/VARIANCE EQUILIBRIUM MODEL

The mean/variance model assumes that investors choose between probability distributions on the basis of their mean and variance. Borch [4] has then shown, that if the investors behave according to the von Neumann and Morgenstern theory, investors have quadratic utility functions for money or present wealth $u_i(x) = x - c_i x^2$, where x is money and c_i is a constant for each investor i. Further, Samuelson [13] has shown that a quadratic utility function gives a good two-moment approximation to any utility function. Expected utility may then be written as

$$E_i[u_i(x)] = E_i - c_i(E_i^2 + S_i^2) \qquad \ldots(1)$$

where E_i is the mathematical expectation and S_i^2 is the variance of investor i's portfolio.

The economy is Walrasian, so that investors consider prices as given and the set of equilibrium prices is reached through a tâtonnement process. The n investors come to the market with given amounts of initial wealth W_i ($i = 1, \ldots, n$). There are m different

1 *First version received July* 1972; *final version received November* 1972. (*Eds.*)
2 This study was made during a stay at INSEE, Paris. A more extensive report is published in Annales de l'INSEE. I wish to thank Edmond Malinvaud and Pascal Mazodier for most valuable discussions and suggestions. The data was kindly provided by Banca d'Italia, Roma.

securities, labelled j, on the market, and investor i holds a fraction z_{ij} of security j in equilibrium. The market clearing conditions are

$$\sum_i z_{ij} = 1, \quad (j = 1, \ldots, m) \qquad \ldots(2)$$

so that the total number of securities on the market is given. With the security prices p_j, investors have the budget constraints

$$\sum_j z_{ij}p_j - W_i = 0 \quad (i = 1, \ldots, n). \qquad \ldots(3)$$

The investors then behave so as to choose a portfolio of securities which maximizes the expected utility of quadratic utility functions. The mathematical expectation of a portfolio is

$$E_i = \sum_j z_{ij}\mu_j \quad (i = 1, \ldots, n) \qquad \ldots(4)$$

where μ_j is the expected present value of security j. The total variance of a portfolio is

$$S_i^2 = \sum_j \sum_k z_{ij}z_{ik}\sigma_{jk} \quad (i = 1, \ldots, n) \qquad \ldots(5)$$

where σ_{jk} is the covariance between the present values of securities j and k.

To maximize expected utility, subject to the budget constraints (3), form the Lagrangeans:

$$L_i = \sum_j z_{ij}\mu_j - c_i[\sum_j \sum_k z_{ij}z_{ik}\sigma_{jk} + (\sum_j z_{ij}\mu_j)^2] + \lambda_i[\sum_j z_{ij}p_j - W_i] \quad (i = 1, \ldots, n).$$

Setting the partial derivatives equal to zero:

$$\frac{\partial L_i}{\partial z_{ij}} = \mu_j - 2c_i[\sum_k z_{ik}\sigma_{jk} + \mu_j \sum_j z_{ij}\mu_j] + \lambda_i p_j = 0 \quad \begin{matrix}(i = 1, \ldots, n)\\ (j = 1, \ldots, m).\end{matrix}$$

Dividing by c_i, summing over i and using the market clearing conditions (2):

$$\mu_j \sum_i \frac{1}{c_i} - 2\sum_k \sigma_{jk} - 2\mu_j \sum_j \mu_j + p_j \sum_i \frac{\lambda_i}{c_i} = 0 \quad (j = 1, \ldots, m). \qquad \ldots(6)$$

The hypothesis is that there exists a security such that its value is given with certainty. Money may be such a security, or numeraire. Taking this to be the security labelled m, $p_m = \mu_m$ and $\sigma_{mk} = 0$ for all k, so that the equation for security m is:

$$\sum_i \frac{1}{c_i} - 2\sum_j \mu_j + \sum_i \frac{\lambda_i}{c_i} = 0, \qquad \ldots(7)$$

and substitution from (7) for $\sum_i \lambda_i/c_i$ in the other $m-1$ equations (6) yields:

$$\mu_j \sum_i \frac{1}{c_i} - 2\sum_k \sigma_{jk} - 2\mu_j \sum_j \mu_j + p_j \left[2\sum_j \mu_j - \sum_i \frac{1}{c_i}\right] = 0.$$

This gives the equilibrium prices for the m securities:

$$p_j = \mu_j - \frac{1}{\sum_i \frac{1}{2c_i} - \sum_j \mu_j} \sum_k \sigma_{jk} \quad (j = 1, \ldots, m)$$

or

$$p_j = \mu_j - \gamma \sum_k \sigma_{jk} \quad (j = 1, \ldots, m) \qquad \ldots(8)$$

where

$$\gamma = \frac{1}{\sum_i \frac{1}{2c_i} - \sum_j \mu_j}.$$

γ is the market price of risk. With given utility functions and probability distributions, γ is a given quantity, and it is the same for all securities on the market.

The reinsurance formulation of (8) can be found in [3, p. 433]. In the Sharpe-Lintner-Mossin one-period model, the quantities on the right-hand side of (8) are for end of period distributions and utility functions. Their model discounts the right-hand side of (8) by $(1+R)^{-1}$, where R is the risk-free rate of interest. This can be seen as discounting the certainty-equivalent of the end of period distributions to the present, and gives the link between the two formulations of the mean/variance equilibrium model.

II. PROBABILITY DISTRIBUTIONS AND THE ITALIAN BONDS

The mean/variance equilibrium model will be tested against the transactions prices for bonds issued by Instituto Mobiliare Italiano (IMI), and bonds issued by the Italian Treasury, the Bueno del Tesoro Poliennali or BTP bonds. IMI raises funds for its operations on the Italian bond market, and operates in close understanding with the Italian Government [12]. When IMI opens a new loan, it issues and sells bonds on the market. The sale of bonds ceases when the total amount of the loan has been raised.

IMI's loans are annuities, which means that a constant amount is paid per year in interest and repayment of capital. There will normally be a large number of bonds on the market, and only a fraction of these are redeemed by IMI on each payment date. The fraction of the outstanding bonds to be redeemed in a forthcoming capital repayment, is equal to the ratio between the amount to be repaid and the total amount of the loan still outstanding. Every IMI bond on the market, therefore, carries a number, and a lottery is arranged before each payment date, in order to determine which bonds IMI shall be obliged to redeem. The numbers drawn are published, so that the bond holders may deliver their bonds and receive an amount to cover interest and the face value of the bonds. Bonds which have not been drawn for redemption receive only the coupon rate of interest and participate in the next lottery.

The fact that the bonds are redeemed through lotteries where the amounts to be drawn are fixed in advance, provides each bond with objectively given probabilities as to its future date of redemption. If a bond carries the coupon r per period, and there will be q future instalments paid, the probability that the annuity bond will be drawn for redemption at the s-th lottery is [14, p. 47]:

$$g_s(r, q) = \frac{r(1+r)^{s-1}}{(1+r)^q - 1}, \quad (s = 1, 2, \ldots, q). \qquad \ldots(9)$$

It is this lottery risk for which we shall attempt to estimate the market price. The ideal situation for measurement would have been if IMI had also issued bonds with fixed maturities, in a sufficient number so that there would be one loan maturing on each instalment day of the annuity loans, and with coupons matching those of the annuities. The set of possible transaction prices for each annuity bond could then be obtained directly as the transaction prices of the corresponding fixed maturity, or certainty equivalent bonds, and with the probabilities (9) the first two moments required for estimation of γ in (8) could have been computed.[1]

IMI does not, however, issue bonds with fixed maturities. The certainty alternative to IMI bonds, which is considered relevant to Italian investors, is investment in BTP bonds. The Italian Treasury issues BTP bonds with about yearly intervals, and it only issues bonds with fixed maturities.

[1] The transaction price is equal to the quoted market price plus simple interest at the coupon rate. If α is the fraction of the coupon period which has elapsed since the last coupon payment simple interest is equal to αr. By adding simple interest to the market price, bond brokers are in fact correcting for a mathematical error made in the computation of bond values tables. The resulting transactions price between buyer and seller is a close approximation to the present value of the bond, where a constant rate of interest i, per period is used in the discounting. See [16].

The maturity dates of the BTP bonds do not in general coincide with the dates of the instalment payments for IMI bonds. In order to get the transaction prices for the hypothetical BTP bonds with the proper maturities, yield curves were estimated on the basis of the observed BTP prices.

For a bond with fixed maturity at time t, $[t]$ coupon payments will be made, where $[t]$ is the closest integer larger than t. At time t the bond holder will receive payment for the par value of the bond plus coupon. The present value of a bond is then present value of the coupon payments and the par value:

$$B(r, i, t) = (1+i)^{\alpha}\left[\sum_{\beta=1}^{[t]} r\left(\frac{1}{1+i}\right)^{\beta} + 1\cdot\left(\frac{1}{1+i}\right)^{[t]}\right]100$$

$$= (1+i)^{\alpha}\left[\frac{r}{i} + \frac{i-r}{i}\left(\frac{1}{1+i}\right)^{[t]}\right]100. \qquad \text{...(10)}$$

Where 100 = par value of bonds quoted.

t = time to maturity measured in units of coupon period (real number).

$\alpha = [t]-t$, so that $(1-\alpha)$ is the fraction of coupon period until the first forthcoming coupon payment (see note, p. 285).

i = yield per coupon period of bond with fixed maturity at time t (per cent).

r = coupon paid per period (per cent).

$B(r, i, t)$ = present value of a bond.

To derive the yield to maturity for a BTP bond on an observation date, an algorithm was used. This would obtain a value for i, such as to give a bond value by (10), sufficiently accurate to fall within the accuracy limits of the observed price for the BTP bond.[1] The number of BTP bonds on the market at any time was 8 or 9, and the time to maturities between 0 and 9 years. A rectangular hyperbola was fitted to the set of computed yields giving a yield curve

$$i = b_1 + b_2/t + \varepsilon \qquad \text{...(11)}$$

where ε is the normally distributed error term with expectation zero and given variance.

This yield curve for Government bonds represents the market yield on relatively certain future payments on an observation date. The estimated yield curve and (10) were used to find what the market price would have been for a lottery bond if it had been known with certainty that it would mature on a particular future redemption date. With this set of possible present values which the lottery bond may have on a day, and the probabilities, the expected present value of a lottery bond is:

$$\mu_B = \sum_{s=1}^{q} B(r, i, t)\cdot g_s(r, q) \qquad \text{...(12)}$$

where s is the lottery arranged before the instalment date t. The variance of the lottery is

$$\sigma_B^2 = \sum_{s=1}^{q} [B(r, i, t)-\mu_B]^2\cdot g_s(r, q). \qquad \text{...(13)}$$

The IMI bonds had maturities between 0 and 19 years, with expected maturity of 12 years for the bond with the longest maturity observed. This means that the BTP yield curves used were well outside the observed value for time to maturity for BTP bonds. Estimated $\hat{b}_1$ in (11) may be taken as the long term yield for Government bonds, and the standard errors for $\hat{b}_1$ were small, with the lowest t-statistic $t_6 = 49{\cdot}63$. As the corresponding theoretical value for $t_6(0{\cdot}05) = 2{\cdot}45$, $\hat{b}_1$ should give a reliable indication of the market long-term yield for relatively certain future payments.

[1] The computational procedure and estimation of yield curves is discussed in [16]. [7] and [9] have studied the construction of yield curves for BTP bonds in more detail.

As regards the covariances required by hypothesis (8), these are for outcomes in lotteries for different loan issues. There is no link between the series of lotteries, so all lottery covariances will be zero. There is, however, one source of covariation between the payout in the different loans, which comes from the possible uncertainty of IMI's ability to meet its future payment obligations. This source of uncertainty has to be taken into account because the observable market prices for alternative, relatively certain claims are for BTP (Government) bonds with fixed maturities, and not for IMI bonds. The lottery risk may, therefore, not be the only source of relative uncertainty. There may also be a difference in default risk between IMI and BTP bonds. By issuing and selling bonds, IMI gives claims to its probability distribution of cash flow. If this probability distribution is given, the sum of the amounts paid to the holders of securities issued by IMI must be equal to the amount paid by IMI under a particular event:

$$a_{jtw} = \sum_{l} a_{jltw} \qquad \text{...(14)}$$

where a_{jtw} = amount payable from firm j (in this case IMI) at time t if the event w occurs.

a_{jltw} = amount payable by firm j to the holder of security l at time t in the event w.

Given the market for claims to be paid with certainty, the market price of £1 payable at time t, v_t, is given. A set of payments then has a present market value $a_{jlw} = \Sigma_t a_{jltw} v_t$, and likewise $a_{jw} = \Sigma_t a_{jtw} v_t$. From (14) then follows

$$a_{jw} = \sum_{l} a_{jlw}, \qquad \text{...(15)}$$

i.e. that the present market value of firm j if the cash flow w occurs, is equal to the sum of the present market values under the event w of the l securities issued by firm j.

Using (15) it can be shown that

$$\sum_{l} \mu_{jl} = \mu_j \qquad \text{...(16)}$$

μ_j = expected present value of firm j.

μ_{jl} = expected present value of security l issued by firm j.

From (15) and (16) it follows in turn that

$$\sum_{l} \sum_{o} {}_j\sigma_{lo} = \sigma_j^2 \qquad \text{...(17)}$$

${}_j\sigma_{lo}$ = covariance between the present values of securities l and o issued by firm j.

(17) says that the sum of the variances and covariances between the probability distributions for payments to the holders of securities issued by firm j is equal to the total variance of firm j's cash flow distribution. So that by issuing securities, the firm is distributing its cash flow variance to its securities. If there is uncertainty associated with firm j's cash flow, therefore, some or all the variances and covariances between its securities will be non-zero.

For IMI as a bond issuer, the variances and covariances will be equal for all bonds if there is an immediate danger of illiquidity for IMI. Such default will mean that the same percentage amount of the face values of the bonds are paid to the bond holders, as all bonds have equal priority to IMI's assets. A more long-term possibility of default will, however, mean that the total variance of IMI's cash flow distribution is spread unevenly over its bonds, with the long-term loans getting a larger proportion of the default risk. In this case it is probable that all the possible cash flows foreseen for IMI are such that they permit the institution to meet its short term payment obligations, and then there will be no default risk for bonds in loans with a short time to final maturity.

If there is uncertainty as to the future cash payments to be made by IMI, and there are also other securities on the market with uncertain future cash flows, it may be that not all the covariances between the cash flow distributions on the market are zero. The covariance between the present market values of firms j and k is σ_{jk}, and by (15) and (16) is given as

$$\sum_{l} {}_{j}\sigma_{lk} = \sigma_{jk}, \quad \ldots(18)$$

which says that the sum of the covariances for firm j's securities l with firm k, is equal to the covariance between firms j and k.

In the case of default risk for IMI, it is the distribution of the total risk, σ_{jk} (all j and k), over its securities, ${}_{j}\sigma_{lk}$, which enters in the determination of the relative market prices of the securities. This default risk enters together with the lottery risk, so that the measured uncertainty in terms of lottery variances may not be a measure of the total uncertainty required by the hypothesis of equilibrium prices (8).

One may, however, have that a high (low) lottery risk occurs together with a low (high) default risk. This depends on the spread in the market values of the certainty equivalent bonds, the time to final maturity of the loan, and whether there is an immediate or more long-term risk of default. Without knowing the investors' perception of the future of the Italian Government and IMI on each observation date, it is, therefore, not possible to say, *a priori*, how default risk affects the estimation of the market price of risk.

III. EMPIRICAL RESULTS

The years 1958 to 1963 were chosen as the observation period.[1] This should be a long enough period to permit a test of the stability over time of the equilibrium price hypothesis (8). Because they are the most active trading months, May, June and September of each year were used as the months from which to draw a random sample of a total of 40 trading days.

For the IMI loans outstanding on these days, the loan information [5] published by IMI was used together with the estimated yield curves for BTP bonds, to compute the expected present values μ_{jd} and variances σ^2_{jd} for bond j on observation date d. Simple interest at the coupon rate was added to the observed market prices to obtain the transaction prices p_{jd}.

As the equilibrium model is specified for a given amount of securities, a sales dummy

$sd = 1$: if bonds in the issue are on sale from IMI,

$sd = 0$: otherwise

was used to account for a possible effect of bond sales on the observed market prices. To test the hypothesis of a market price for risk γ_d, it is useful to introduce the concept of risk premium $\pi_{jd} = \mu_{jd} - p_{jd}$ and to rewrite (8) as $\pi_{jd} = \gamma_d \sigma^2_{jd}$. As the lotteries may not be the only source of uncertainty, we shall assume that a difference in default risk between the BTP bonds and IMI bonds affects all IMI bonds alike. So that a difference in default risk will manifest itself through a non-zero intercept in the regression:

$$\pi_{jd} = a_d + \gamma_d \sigma^2_{jd} + h_d sd_{jd} + \varepsilon.$$

Using all observations to test this hypothesis, the following estimates were obtained:[2]

$$\begin{array}{c} \pi = \underset{(0{\cdot}184)}{2{\cdot}46} + \underset{(0{\cdot}027)}{0{\cdot}34\sigma^2} + \underset{(0{\cdot}252)}{3{\cdot}07sd} \\ (R^2 = 0{\cdot}50,\ 361 \text{ obs., ESS} = 1857{\cdot}89) \end{array} \quad \ldots(19)$$

This result tends to confirm the mean/variance equilibrium model. The estimated market

[1] The Italian tax laws were changed in 1971, and the observation period should be chosen well before any effects of expected future changes in tax laws influenced relative bond prices.

[2] The level of significance chosen is 5 per cent. The standard error is given in parenthesis below each estimate.

price of risk is significantly different from zero. The significance of the sales dummy suggests that IMI on the whole experienced three percentage points lower prices for bonds in issues which were on sale, compared to the prices for bonds closed series. The significant intercept means that IMI bonds traded at prices which were about two and a half percentage points lower than the prices which would have been paid for similar bonds if they had been issued by the Italian Treasury.

The regression (19) explains about half of total variance, so that there may also be other factors which affect the observed risk premiums. The positive intercept points to the question of a possible default risk, which has not been accounted for in the computed means and variances for IMI bonds.

It may, therefore, have affected the estimated market price of risk that the yield curves were for BTP bonds, and not for IMI bonds with fixed maturities. The form of the BTP yield curve should reflect expectations about future changes in interest rates, commodity prices, BTP default risk, and other relevant economic factors. A difference in default risk between BTP and IMI bonds would affect the computed expected present values and variances through (10). The effect would depend on the coupon r, and time to maturity, t, of the annuity bonds. As both t and r have already been used in the computations of μ_{jd} and σ_{jd}^2, there is no direct way to study a possible difference in default risk. To enter t and r as explanatory variables together with σ_{jd}^2 causes an identification problem. The question of default risk is, however, important and the introduction of dummy variables for the coupons of the different IMI bonds was chosen as the method of studying it. During the period, IMI had outstanding bonds with 5%, 5·5% and 6% coupons, and the two dummy variables are:

$$C_1 = 1: \text{ if } 6\% \text{ coupon}$$
$$C_1 = 0: \text{ otherwise}$$
$$C_2 = 1: \text{ if } 5\% \text{ coupon}$$
$$C_2 = 0: \text{ otherwise.}$$

This gave the regression

$$\begin{array}{l} \pi = 2{\cdot}06 \quad +0{\cdot}47\sigma^2 - 0{\cdot}87C_1 + 2{\cdot}38C_2 + 2{\cdot}95sd. \\ \quad\;\; (0{\cdot}429)\;(0{\cdot}023)\;(0{\cdot}409)\;(0{\cdot}447)\;(0{\cdot}192) \\ (R^2 = 0{\cdot}72,\ 361 \text{ obs., ESS} = 1048{\cdot}51) \end{array} \qquad \ldots(20)$$

It is evident from (20) that the use of BTP yield curves poses a problem, as both coupon dummies are significantly different from zero, and the coefficient of determination has increased considerably. The introduction of coupon dummies has increased the estimated market price of risk. Studying the three subsamples:

$$\begin{array}{l} 6\% \text{ coupon:} \quad \pi = 1{\cdot}65 \quad +0{\cdot}50\sigma^2 + 0{\cdot}38sd \\ \qquad\qquad\qquad (0{\cdot}085)\;(0{\cdot}013)\;(0{\cdot}137) \\ (R^2 = 0{\cdot}88,\ 264 \text{ obs., ESS} = 199{\cdot}90) \end{array} \qquad \ldots(20.1)$$

$$\begin{array}{l} 5{\cdot}5\% \text{ coupon:} \quad \pi = 0{\cdot}68 \quad +0{\cdot}21\sigma^2 + 9{\cdot}68sd \\ \qquad\qquad\qquad (0{\cdot}216)\;(0{\cdot}228)\;(2{\cdot}541) \\ (R^2 = 0{\cdot}99,\ 19 \text{ obs., ESS} = 8{\cdot}94) \end{array} \qquad \ldots(20.2)$$

$$\begin{array}{l} 5\% \text{ coupon:} \quad \pi = 1{\cdot}65 \quad +3{\cdot}78\sigma^2 + 4{\cdot}12sd. \\ \qquad\qquad\qquad (0{\cdot}502)\;(0{\cdot}278)\;(0{\cdot}516) \\ (R^2 = 0{\cdot}74,\ 78 \text{ obs., ESS} = 273{\cdot}5) \end{array} \qquad \ldots(20.3)$$

The estimated market prices of risk are quite different for the three bond groups, and for the 5·5% coupons the $\hat{\gamma}_{5\cdot5\%}$ is no longer significant. There were only 5·5% bonds on the market on 19 of the 40 days of observation. Of the two 5·5% issues, one issue matured in 1961 and a new issue was opened in 1963. The sale of the new issue depressed the price considerably, whereas the lottery risk apparently played an insignificant role in price determination.

As regards the 5% coupon bonds, the number of such loans on the market varied between 1 and 3 during the observation period. Three new 5% loans were issued, and the sale of these had a significant effect on market prices. The $\hat{\gamma}_{5\%}$ is particularly high, 11 times larger than the estimated market price of risk for the sample as a whole. The $\hat{b}_1$ for the estimated yield curves for BTP bonds was near 5% or slightly lower in most of the periods. $\hat{b}_1$ may be taken as the long term rate of interest on Government bonds, and the fact that it was close to the coupon of the 5% bonds means that there is little spread in the set of possible present values of these bonds, which gives small computed variances. The 5% coupon bonds may therefore be the ones which are most affected by the use of the BTP yield curves, and the high $\hat{\gamma}_{5\%}$ should be interpreted with this in mind.

TABLE 1

$$\pi_{jd}^{6\%} = a_d + \gamma_d \sigma_{jd}^2 + h_d sd_{jd}$$

Number	1	8	14	25	36
Date d	11.5.59	19.5.60	19.5.61	11.5.62	3.6.63
$\hat{a}_d$	2·36* (0·446)	0·88* (0·1181)	1·55* (0·271)	3·82* (0·263)	1·81 (0·635)
$\hat{\gamma}_d$	0·78* (0·112)	0·59* (0·023)	0·57* (0·045)	0·58* (0·052)	0·47* (0·063)
$\hat{h}_d$	−0·01 (0·480)	0·20 (0·158)	−0·08 (0·524)	0·33 (0·550)	0·63 (0·661)
R^2	0·93	0·99	0·98	0·98	0·98
ESS	1·3270	0·1217	0·7592	0·8415	0·7418
Number of observations	7	7	7	7	5

The 6% coupons dominate in terms of the number of observations, and it is for this group that one probably has the most reliable estimate of the market price of risk, $\hat{\gamma}_{6\%} = 0{\cdot}5$. It is a fair difference between the BTP yields and the 6% coupon, which gives variation in the computed σ^2. The number of 6% bonds on the market at any time varied between 4 and 7.

The above suggests that a market price of risk exists, and we shall test whether it was stable over time. Using (20), (20.1), (20.2) and (20.3) to test the homogeneity across coupons of the coefficients in $\pi = a + \gamma\sigma^2 + h \cdot sd$, the statistic $F_{4 \cdot 352} = 103{\cdot}29$ clearly rejects this. The stability test is, therefore, limited to the subsample of bonds carrying a 6% coupon. One observation date was chosen for each year, with the date closest to the 15th May being picked arbitrarily.

Significant estimates are marked with asterisks. Testing the stability of the market price of risk over yearly intervals, the null hypothesis is $\pi_{jd}^{6\%} = a_d + \gamma\sigma_{jd}^2 + h_d sd_{jd}$, against the alternative hypothesis above.

Introducing the dummy variables:

$$td_1 = \begin{cases} 1 \text{ if the observation is from the first date} \\ 0 \text{ if the observation is from the second date} \end{cases}$$

$$td_2 = 1 - td_1,$$

we have the regressions in Table 2:

TABLE 2

$$\pi_{jt}^{6}\% = \gamma\sigma_{jt}^{2}+a_1td_1+a_2td_2+a_3td_1sd_{jt}+a_4td_2sd_{jt}$$

Observation dates	1 and 8	8 and 14	14 and 25	25 and 36
$\hat{\gamma}$	0·65 (0·053)	0·58 (0·026)	0·57 (0·032)	0·52 (0·042)
$\hat{a}_1$	2·76 (0·291)	0·92 (0·171)	1·54 (0·226)	4·04 (0·269)
$\hat{a}_2$	0·65 (0·297)	1·52 (0·174)	3·84 (0·207)	1·39 (0·473)
$\hat{a}_3$	0·21 (0·381)	0·23 (0·272)	−0·09 (0·485)	0·61 (0·623)
$\hat{a}_4$	0·04 (0·428)	−0·11 (0·365)	0·34 (0·479)	0·34 (0·554)
R^2	0·96	0·99	0·98	0·97
ESS	2·1070	0·8921	1·6030	2·1060
Number of observations	14	14	14	12
F Statistic	$F_{1 \cdot 8}$ = 3·64	$F_{1 \cdot 8}$ = 0·10	$F_{1 \cdot 8}$ = 0·01	$F_{1 \cdot 6}$ = 1·98

As $F_{1 \cdot 8}(0{\cdot}05) = 5{\cdot}32$ and $F_{1 \cdot 6}(0{\cdot}05) = 5{\cdot}99$, the market price of risk was stable over each one year period. A similar test for the five years gave $\text{ESS})_0 = 5{\cdot}9982$ with 33 observations, for the null hypothesis. With $\text{ESS})_a = 3{\cdot}7912$ for the alternative, the statistic is $F_{4 \cdot 18} = 2{\cdot}62$, against $F_{4 \cdot 18}(0{\cdot}05) = 2{\cdot}93$. So that the market price of risk was also stable over the five years studied. These tests, therefore, indicate both short and more long run stability of the market price of risk.

IV. CONCLUSION

This paper has studied the formation of market prices on contracts in uncertain future payments. For the case of certainty, the well known criterion is to choose investments so as to maximize present wealth. Given a market for certainty claims, the mean/variance equilibrium model under uncertainty then gives a testable hypothesis on observable transaction prices for securities.

The mean/variance equilibrium model was tested against data from the Italian bond market. The equilibrium model was supported. The estimated market price of risk appears to be close to £0·5 per unit of variance. The market price of risk was stable over the five-year period studied, and also over one-year subintervals.

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Chapter V

Conclusion.

The intention of the papers has been to obtain information on individual behaviour under risk and to study the market prices for contracts in uncertain future payments. Bearer's bonds in loans where the repayment is made through a series of lotteries offer a godd opportunity for such studies. The interesting characteristic of these bonds is that the lotteries provide a set of probabilities, which may be considered as objectively given data in the valuation of the bonds. All investors may further be said to have identical information on the probabilities.

The intention of chapter I was to classify the various types of bonds and to derive the probabilities. Some commonly observable rules of thumb which are used by investors, give an indication that lottery bonds offer interesting possibilities for empirical testing of uncertainty theories. The natural next step was then to design a more extensive experiment, which is reported in chapter II. In this experiment the main result was that typical investor behaviour is in accordance with the von Neumann and Morgenstern theory. The typical investors are risk averse, and their Arrow-Pratt absolute risk aversion functions increase with increasing wealth. This admits for example the cummulated normal distribution and the quadratic utility function for money.

Emphasis has been laid on the treatment of the time dimension. The various cash flows which a lottery bond may offer span different lengths of time into the future. As the utility functions were defined for money, the parallell consept for ranking of investments is present value. For each

Bueno del Tesoro Poliennali - BTP bonds.

Coupon for all bonds: 5% per year. Fixed redemption.
Maturity dates of issues:

Loan no.		Maturity
1	-	1. 4. 1960
2	-	1. 1. 1961
3	-	1. 1. 1962
4	-	1. 1. 1963
5	-	1. 4. 1964
6	-	1. 4. 1965
7	-	1. 4. 1966
8	-	1. 1. 1968
9	-	1.10. 1966
10	-	1. 4. 1969
11	-	1. 1. 1970
12	-	1. 1. 1971

Semi-annual coupon payments.
Simple interest is not included in the observed market prices.

B T P PRICES

DA.MO.YE.	1	2	3	4	5	6	7	8	9	10	11	12
11. 5.59.	101.775	101.300	101.525	101.425	101.375	101.475	101.550	101.475	0.000	0.000	0.000	0.000
22. 5.59.	101.450	101.300	101.300	101.275	101.100	101.325	101.325	101.250	0.000	0.000	0.000	0.000
10. 6.59.	101.100	100.975	100.825	100.825	100.750	100.825	101.000	101.050	0.000	0.000	0.000	0.000
17. 6.59.	101.200	101.100	101.000	101.075	101.000	101.125	101.175	101.275	0.000	0.000	0.000	0.000
28. 9.59.	100.700	100.300	100.125	99.950	99.900	99.875	99.925	99.950	0.000	0.000	0.000	0.000
3. 5.60.	0.000	101.225	101.250	101.275	101.300	101.325	101.525	101.575	101.250	0.000	0.000	0.000
10. 5.60.	0.000	101.275	101.275	101.350	101.325	101.325	101.525	101.525	101.375	0.000	0.000	0.000
19. 5.60.	0.000	101.725	101.725	101.725	101.875	101.775	101.950	101.825	101.700	0.000	0.000	0.000
27. 5.60.	0.000	101.800	101.925	101.875	101.875	102.125	102.350	102.550	101.950	0.000	0.000	0.000
20. 9.60.	0.000	101.400	101.375	101.450	101.500	101.575	101.700	101.900	101.575	101.825	0.000	0.000
23. 9.60.	0.000	101.350	101.275	101.400	101.400	101.475	101.575	101.675	101.275	101.650	0.000	0.000
26. 9.60.	0.000	101.300	101.150	101.275	101.275	101.300	101.375	101.500	101.100	101.450	0.000	0.000
5. 5.61.	0.000	0.000	102.225	102.525	102.850	102.825	103.025	103.275	102.900	103.225	0.000	0.000
19. 5.61.	0.000	0.000	102.075	102.375	102.525	102.625	102.875	102.800	102.625	102.875	0.000	0.000
24. 5.61.	0.000	0.000	102.025	102.275	102.375	102.525	102.550	102.475	102.425	102.675	102.675	0.000
25. 5.61.	0.000	0.000	102.000	102.200	102.275	102.425	102.475	102.550	102.250	102.675	102.700	0.000
26. 5.61.	0.000	0.000	102.000	102.250	102.275	102.400	102.500	102.625	102.325	102.750	102.700	0.000
21. 6.61.	0.000	0.000	102.450	102.600	102.675	102.825	102.875	102.950	102.600	103.150	103.150	0.000
4. 9.61.	0.000	0.000	101.875	102.300	102.450	102.700	103.050	103.000	102.475	102.975	103.225	0.000
13. 9.61.	0.000	0.000	101.650	102.100	102.175	102.600	102.775	102.675	102.450	102.950	103.200	0.000
18. 9.61.	0.000	0.000	101.675	102.150	102.400	102.675	102.850	102.775	102.450	102.925	102.950	0.000
25. 9.61.	0.000	0.000	101.850	102.325	102.525	102.575	102.825	102.800	102.325	102.775	102.900	0.000
28. 9.61.	0.000	0.000	101.775	102.350	102.500	102.500	102.700	102.725	102.275	102.675	102.825	0.000
29. 9.61.	0.000	0.000	101.650	102.325	102.500	102.525	102.725	102.725	102.325	102.725	102.850	0.000
11. 5.62.	0.000	0.000	0.000	103.050	103.090	103.250	103.440	103.500	103.475	103.700	104.425	0.000
4. 6.62.	0.000	0.000	0.000	101.200	100.950	101.875	101.650	101.600	101.125	101.400	103.425	0.000
20. 6.62.	0.000	0.000	0.000	100.125	100.125	100.275	100.325	100.350	100.050	100.300	100.450	100.250
22. 6.62.	0.000	0.000	0.000	100.050	100.000	100.200	100.225	100.275	100.000	100.175	100.475	100.200
27. 6.62.	0.000	0.000	0.000	100.075	99.900	99.950	100.000	100.050	100.200	99.975	100.550	99.900
6. 9.62.	0.000	0.000	0.000	100.725	100.975	101.200	101.475	101.575	101.000	102.150	102.575	102.100
14. 9.62.	0.000	0.000	0.000	100.525	100.725	100.750	100.975	101.350	100.900	101.250	102.100	101.925
18. 9.62.	0.000	0.000	0.000	100.300	100.250	100.325	100.600	100.625	100.250	100.725	101.400	101.375
19. 9.62.	0.000	0.000	0.000	100.300	100.250	100.400	100.100	100.775	100.475	100.625	101.475	101.100
7. 5.63.	0.000	0.000	0.000	0.000	100.425	101.500	101.975	101.825	101.725	101.925	102.825	102.925
29. 5.63.	0.000	0.000	0.000	0.000	100.375	100.750	101.600	102.025	101.225	101.750	102.675	102.475
3. 6.63.	0.000	0.000	0.000	0.000	100.225	100.500	101.350	101.450	101.075	101.575	102.425	102.525
11. 6.63.	0.000	0.000	0.000	0.000	100.325	100.575	101.600	101.775	101.250	102.300	102.750	102.825
3. 9.63.	0.000	0.000	0.000	0.000	100.225	100.800	101.150	101.600	101.975	101.650	102.975	102.825
4. 9.63.	0.000	0.000	0.000	0.000	100.300	100.850	101.200	101.575	100.875	101.900	102.950	102.675
10. 9.63.	0.000	0.000	0.000	0.000	100.275	100.575	101.200	101.400	100.475	101.650	102.675	102.350

B T P MATURITIES

DA.MO.YE.	1	2	3	4	5	6	7	8	9	10	11	12
11. 5.59.	0.88	1.63	2.63	3.63	4.88	5.88	6.88	8.63	0.00	0.00	0.00	0.00
22. 5.59.	0.85	1.60	2.60	3.60	4.85	5.85	6.85	8.60	0.00	0.00	0.00	0.00
10. 6.59.	0.80	1.55	2.55	3.55	4.80	5.80	6.80	8.55	0.00	0.00	0.00	0.00
17. 6.59.	0.78	1.53	2.53	3.53	4.78	5.78	6.78	8.53	0.00	0.00	0.00	0.00
28. 9.59.	0.50	1.25	2.25	3.25	4.50	5.50	6.50	8.25	0.00	0.00	0.00	0.00
3. 5.60.	0.00	0.66	1.66	2.66	3.91	4.91	5.91	7.66	6.41	0.00	0.00	0.00
10. 5.60.	0.00	0.64	1.64	2.64	3.89	4.89	5.89	7.64	6.39	0.00	0.00	0.00
19. 5.60.	0.00	0.61	1.61	2.61	3.86	4.86	5.86	7.61	6.36	0.00	0.00	0.00
27. 5.60.	0.00	0.59	1.59	2.59	3.84	4.84	5.84	7.59	6.34	0.00	0.00	0.00
20. 9.60.	0.00	0.28	1.28	2.28	3.53	4.53	5.53	7.28	6.03	8.53	0.00	0.00
23. 9.60.	0.00	0.27	1.27	2.27	3.52	4.52	5.52	7.27	6.02	8.52	0.00	0.00
26. 9.60.	0.00	0.26	1.26	2.26	3.51	4.51	5.51	7.26	6.01	8.51	0.00	0.00
5. 5.61.	0.00	0.00	0.65	1.65	2.90	3.90	4.90	6.65	5.40	7.90	0.00	0.00
19. 5.61.	0.00	0.00	0.61	1.61	2.86	3.86	4.86	6.61	5.36	7.86	0.00	0.00
24. 5.61.	0.00	0.00	0.60	1.60	2.85	3.85	4.85	6.60	5.35	7.85	8.60	0.00
25. 5.61.	0.00	0.00	0.60	1.60	2.85	3.85	4.85	6.60	5.35	7.85	8.60	0.00
26. 5.61.	0.00	0.00	0.59	1.59	2.84	3.84	4.84	6.59	5.34	7.84	8.59	0.00
21. 6.61.	0.00	0.00	0.52	1.52	2.77	3.77	4.77	6.52	5.27	7.77	8.52	0.00
4. 9.61.	0.00	0.00	0.32	1.32	2.57	3.57	4.57	6.32	5.07	7.57	8.32	0.00
13. 9.61.	0.00	0.00	0.30	1.30	2.55	3.55	4.55	6.30	5.05	7.55	8.30	0.00
18. 9.61.	0.00	0.00	0.28	1.28	2.53	3.53	4.53	6.28	5.03	7.53	8.28	0.00
25. 9.61.	0.00	0.00	0.26	1.26	2.51	3.51	4.51	6.26	5.01	7.51	8.26	0.00
28. 9.61.	0.00	0.00	0.25	1.25	2.50	3.50	4.50	6.25	5.00	7.50	8.25	0.00
29. 9.61.	0.00	0.00	0.25	1.25	2.50	3.50	4.50	6.25	5.00	7.50	8.25	0.00
11. 5.62.	0.00	0.00	0.00	0.63	1.88	2.88	3.88	5.63	4.38	6.88	7.63	0.00
4. 6.62.	0.00	0.00	0.00	0.57	1.82	2.82	3.82	5.57	4.32	6.82	7.57	0.00
20. 6.62.	0.00	0.00	0.00	0.53	1.78	2.78	3.78	5.53	4.28	6.78	7.53	8.53
22. 6.62.	0.00	0.00	0.00	0.52	1.77	2.77	3.77	5.52	4.27	6.77	7.52	8.52
27. 6.62.	0.00	0.00	0.00	0.51	1.76	2.76	3.76	5.51	4.26	6.76	7.51	8.51
6. 9.62.	0.00	0.00	0.00	0.31	1.56	2.56	3.56	5.31	4.06	6.56	7.31	8.31
14. 9.62.	0.00	0.00	0.00	0.29	1.54	2.54	3.54	5.29	4.04	6.54	7.29	8.29
18. 9.62.	0.00	0.00	0.00	0.28	1.53	2.53	3.53	5.28	4.03	6.53	7.28	8.28
19. 9.62.	0.00	0.00	0.00	0.28	1.53	2.53	3.53	5.28	4.03	6.53	7.28	8.28
7. 5.63.	0.00	0.00	0.00	0.00	0.90	1.90	2.90	4.65	3.40	5.90	6.65	7.65
29. 5.63.	0.00	0.00	0.00	0.00	0.83	1.83	2.83	4.58	3.33	5.83	6.58	7.58
3. 6.63.	0.00	0.00	0.00	0.00	0.82	1.82	2.82	4.57	3.32	5.82	6.57	7.57
11. 6.63.	0.00	0.00	0.00	0.00	0.80	1.80	2.80	4.55	3.30	5.80	6.55	7.55
3. 9.63.	0.00	0.00	0.00	0.00	0.57	1.57	2.57	4.32	3.07	5.57	6.32	7.32
4. 9.63.	0.00	0.00	0.00	0.00	0.57	1.57	2.57	4.32	3.07	5.57	6.32	7.32
10. 9.63.	0.00	0.00	0.00	0.00	0.55	1.55	2.55	4.30	3.05	5.55	6.30	7.30

B T P YIELDS

DA.MO.YE.	1	2	3	4	5	6	7	8	9	10	11	12
11. 5.59.	2.953	4.174	4.385	4.570	4.680	4.710	4.735	4.790	0.000	0.000	0.000	0.000
22. 5.59.	3.273	4.154	4.465	4.615	4.745	4.737	4.770	4.820	0.000	0.000	0.000	0.000
10. 6.59.	3.614	4.344	4.655	4.745	4.825	4.835	4.825	4.847	0.000	0.000	0.000	0.000
17. 6.59.	3.454	4.254	4.575	4.665	4.765	4.775	4.795	4.815	0.000	0.000	0.000	0.000
28. 9.59.	3.614	4.755	4.940	5.015	5.025	5.025	5.013	5.008	0.000	0.000	0.000	0.000
3. 5.60.	0.000	3.113	4.214	4.485	4.635	4.695	4.700	4.752	4.770	0.000	0.000	0.000
10. 5.60.	0.000	2.973	4.184	4.455	4.625	4.695	4.700	4.760	4.747	0.000	0.000	0.000
19. 5.60.	0.000	2.172	3.894	4.294	4.465	4.590	4.615	4.712	4.687	0.000	0.000	0.000
27. 5.60.	0.000	1.932	3.744	4.229	4.465	4.505	4.537	4.597	4.640	0.000	0.000	0.000
20. 9.60.	0.000	0.010	3.894	4.324	4.535	4.610	4.647	4.690	4.695	4.737	0.000	0.000
23. 9.60.	0.000	0.050	3.954	4.344	4.565	4.635	4.672	4.725	4.755	4.762	0.000	0.000
26. 9.60.	0.000	0.050	4.054	4.405	4.605	4.675	4.715	4.752	4.787	4.790	0.000	0.000
5. 5.61.	0.000	0.000	1.572	3.414	3.954	4.209	4.309	4.425	4.390	4.510	0.000	0.000
19. 5.61.	0.000	0.000	1.612	3.474	4.054	4.254	4.339	4.505	4.445	4.560	0.000	0.000
24. 5.61.	0.000	0.000	1.612	3.524	4.109	4.279	4.410	4.560	4.485	4.590	4.620	0.000
25. 5.61.	0.000	0.000	1.652	3.574	4.144	4.309	4.427	4.547	4.520	4.590	4.615	0.000
26. 5.61.	0.000	0.000	1.612	3.534	4.144	4.314	4.420	4.535	4.505	4.580	4.615	0.000
21. 6.61.	0.000	0.000	0.331	3.243	3.974	4.184	4.327	4.475	4.440	4.515	4.550	0.000
4. 9.61.	0.000	0.000	-0.750	3.213	3.989	4.179	4.259	4.450	4.450	4.530	4.530	0.000
13. 9.61.	0.000	0.000	-0.509	3.333	4.094	4.204	4.319	4.507	4.455	4.535	4.532	0.000
18. 9.61.	0.000	0.000	-0.830	3.273	3.994	4.179	4.299	4.490	4.450	4.537	4.567	0.000
25. 9.61.	0.000	0.000	-1.871	3.113	3.934	4.204	4.304	4.482	4.477	4.560	4.575	0.000
28. 9.61.	0.000	0.000	-1.791	3.073	3.944	4.224	4.334	4.495	4.487	4.575	4.585	0.000
29. 9.61.	0.000	0.000	-1.390	3.093	3.944	4.214	4.329	4.495	4.475	4.567	4.580	0.000
11. 5.62.	0.000	0.000	0.000	0.210	3.293	3.804	4.034	4.294	4.124	4.372	4.314	0.000
4. 6.62.	0.000	0.000	0.000	2.893	4.455	4.284	4.525	4.670	4.710	4.757	4.462	0.000
20. 6.62.	0.000	0.000	0.000	4.775	4.925	4.895	4.905	4.925	4.985	4.948	4.928	4.963
22. 6.62.	0.000	0.000	0.000	4.895	4.995	4.925	4.935	4.940	5.000	4.970	4.923	4.970
27. 6.62.	0.000	0.000	0.000	4.855	5.055	5.020	5.000	4.990	4.945	5.005	4.910	5.015
6. 9.62.	0.000	0.000	0.000	2.693	4.354	4.500	4.545	4.660	4.725	4.615	4.582	4.692
14. 9.62.	0.000	0.000	0.000	3.213	4.515	4.685	4.700	4.710	4.755	4.775	4.657	4.717
18. 9.62.	0.000	0.000	0.000	3.934	4.835	4.865	4.815	4.865	4.930	4.870	4.770	4.797
19. 9.62.	0.000	0.000	0.000	3.894	4.835	4.830	4.970	4.830	4.870	4.885	4.757	4.837
7. 5.63.	0.000	0.000	0.000	0.000	4.515	4.174	4.269	4.560	4.445	4.625	4.505	4.542
29. 5.63.	0.000	0.000	0.000	0.000	4.535	4.575	4.395	4.505	4.600	4.655	4.525	4.610
3. 6.63.	0.000	0.000	0.000	0.000	4.715	4.715	4.485	4.645	4.645	4.687	4.570	4.600
11. 6.63.	0.000	0.000	0.000	0.000	4.575	4.665	4.385	4.565	4.585	4.545	4.510	4.555
3. 9.63.	0.000	0.000	0.000	0.000	4.615	4.475	4.525	4.590	4.304	4.660	4.455	4.542
4. 9.63.	0.000	0.000	0.000	0.000	4.455	4.435	4.505	4.595	4.690	4.610	4.460	4.565
10. 9.63.	0.000	0.000	0.000	0.000	4.495	4.615	4.495	4.635	4.830	4.660	4.507	4.617

I M I - B O N D S

Loan No.	Coupon	Coupon payment		Lottery		Instalment payment		Period of bond sale to market
		Dates	No. per yr.	Dates	No. per yr.	First	Last	
1	5.00	1.4. 1.10.	2	15.1. 15. 7.	2	1.10.55.	1.10.60.	1. 1.45. - 31.12.45.
2	5.50	1.1. 1. 7.	2	15.4. 15.10.	2	1. 1.48.	1. 1.61	1. 1.47. - 31.12.48.
3	6.00	1.4. 1.10.	2	15.1. 15. 7.	2	1. 4.53.	1. 4.65.	1. 1.50. - 31.12.55.
4	6.00	1.4. 1.10.	2	0.0. 15. 1.	1	1. 4.52.	1. 4.66.	1. 1.51. - 31.12.54.
5	6.00	1.4. 1.10.	2	0.0. 15. 1.	1	1. 4.55.	1. 4,64.	1. 1.54. - 31.12.54.
6	6.00	1.4. 1.00.	2	15.1. 15. 7.	2	1.10.49.	1. 4.69.	1. 1.49. - 31.12.59.
7	6.00	1.4. 1.10.	2	0.0. 15. 1.	1	1. 4.55.	1. 4.74.	1. 1.54. - 31.12.64.
8	6.00	1.4. 1.10.	2	0.0. 15. 1	1	1. 4.61.	1. 4.70.	1. 1.55. - 31.12.60.
9	6.00	1.4. 1.10.	2	0.0. 15. 1.	1	1. 4.61.	1. 4.77.	1. 1.57. - 31.12.57.
10	5.00	1.4. 1.10.	2	0.0. 15. 1.	1	1. 4.60.	1. 4.74.	1. 1.59. - 31.12.64.
11	5.00	1.4. 1.10.	2	0.0. 15. 7.	1	1.10.64.	1.10.78.	1. 1.61. - 31.12.68.
12	5.00	1.1. 1. 7.	2	0.0. 15.10.	1	1. 1.63.	1. 1.82.	1. 1.62. - 31.12.62.
13	5.50	1.1. 1. 7.	2	0.0. 15.10.	1	1. 1.64.	1. 1.83.	28. 1.63. - 14.10.72.
14	6.00	1.1. 1. 7.	2	0.0. 14.10.	1	1. 1.65.	1. 1.80.	3. 4.63. - 14.10.67.

IMI - prices

BOND NO.	1	2	3	4	5	6	7	8	9	10	11	12	13	14
COUPON	5.00	5.50	5.00	6.00	6.00	6.00	6.00	6.00	6.00	5.00	5.00	5.00	5.50	6.00
DATE OF REDEMPTION	10.60.	1.61.	4.65.	4.66.	4.64.	4.69.	4.74.	4.70.	4.77.	4.74.	10.78.	1.82.	1.83.	1.80.
OBS. DATE														
1 11. 5.59.	100.80	100.50	101.05	100.90	101.00	101.00	100.95	100.95	101.20	0.0	0.0	0.0	0.0	0.0
2 22. 5.59.	100.60	100.40	101.10	101.05	101.20	101.00	101.05	101.05	101.25	0.0	0.0	0.0	0.0	0.0
3 10. 6.59.	101.30	100.40	101.95	101.85	101.85	101.90	102.20	101.90	102.60	0.0	0.0	0.0	0.0	0.0
4 17. 6.59.	101.50	100.50	102.10	102.05	102.00	102.05	102.10	102.40	102.50	0.0	0.0	0.0	0.0	0.0
5 28. 9.59.	101.00	99.85	100.80	101.05	101.05	100.95	101.40	101.00	101.05	96.40	0.0	0.0	0.0	0.0
6 3. 5.60.	101.20	101.20	102.50	102.60	103.20	102.50	102.55	102.75	102.85	97.60	0.0	0.0	0.0	0.0
7 10. 5.60.	100.40	100.50	102.50	102.45	102.60	102.40	102.55	102.55	103.05	97.80	0.0	0.0	0.0	0.0
8 19. 5.60.	100.30	101.55	102.50	102.65	102.70	102.45	102.60	102.80	103.00	98.15	0.0	0.0	0.0	0.0
9 27. 5.60.	100.30	101.45	102.55	102.60	102.60	102.60	102.65	102.75	102.80	98.30	0.0	0.0	0.0	0.0
10 20. 9.60.	100.00	101.00	102.75	102.40	102.60	102.50	103.00	102.50	102.90	98.70	0.0	0.0	0.0	0.0
11 23. 9.60.	100.00	102.00	102.75	102.30	102.40	102.50	102.60	102.50	102.80	98.50	0.0	0.0	0.0	0.0
12 26. 9.60.	100.00	102.00	103.10	102.50	102.60	102.60	102.75	102.65	102.90	98.50	0.0	0.0	0.0	0.0
13 5. 5.61.	0.0	0.0	103.90	102.65	102.60	102.90	102.70	102.20	102.85	98.40	0.0	0.0	0.0	0.0
14 19. 5.61.	0.0	0.0	102.30	102.00	102.60	102.00	102.25	102.15	102.20	97.80	0.0	0.0	0.0	0.0
15 24. 5.61.	0.0	0.0	102.30	102.30	102.40	102.30	102.30	102.30	102.30	97.50	0.0	0.0	0.0	0.0
16 25. 5.61.	0.0	0.0	102.30	102.30	102.40	102.30	102.30	102.30	102.40	97.60	0.0	0.0	0.0	0.0
17 26. 5.61.	0.0	0.0	102.30	102.30	102.40	102.40	102.30	102.30	102.40	97.60	0.0	0.0	0.0	0.0
18 21. 6.61.	0.0	0.0	101.90	102.70	102.45	102.10	102.45	102.25	102.60	97.50	0.0	0.0	0.0	0.0
19 4. 9.61.	0.0	0.0	101.90	102.00	101.90	101.80	102.00	102.00	101.90	97.20	0.0	0.0	0.0	0.0
20 13. 9.61.	0.0	0.0	101.10	101.80	101.85	101.50	101.55	101.55	101.70	96.70	0.0	0.0	0.0	0.0
21 18. 9.61.	0.0	0.0	102.00	101.70	101.75	101.20	101.50	101.50	102.00	97.00	0.0	0.0	0.0	0.0
22 25. 9.61.	0.0	0.0	102.00	101.70	101.70	102.00	101.80	101.90	102.20	97.00	96.70	0.0	0.0	0.0
23 28. 9.61.	0.0	0.0	102.45	101.90	101.80	102.05	101.95	101.95	102.35	96.90	96.75	0.0	0.0	0.0
24 29. 9.61.	0.0	0.0	102.45	101.80	101.90	102.05	101.75	102.15	102.35	96.95	96.70	0.0	0.0	0.0
25 11. 5.62.	0.0	0.0	100.90	100.60	100.90	100.75	100.70	100.75	101.10	96.30	95.10	0.0	0.0	0.0
26 4. 6.62.	0.0	0.0	100.10	100.60	100.30	100.25	100.30	100.30	100.40	94.60	94.10	0.0	0.0	0.0
27 20. 6.62.	0.0	0.0	100.30	99.60	100.25	99.65	99.70	99.75	99.70	93.60	93.50	0.0	0.0	0.0
28 22. 6.62.	0.0	0.0	99.90	99.85	100.30	99.80	99.80	99.85	100.10	93.70	93.60	0.0	0.0	0.0
29 27. 6.62.	0.0	0.0	99.85	99.90	100.60	100.30	99.90	99.90	99.90	93.60	93.00	0.0	0.0	0.0
30 6. 9.62.	0.0	0.0	100.10	100.10	100.60	100.05	100.15	100.20	100.10	93.00	92.90	92.40	0.0	0.0
31 14. 9.62.	0.0	0.0	100.10	100.10	100.30	100.05	99.90	99.90	100.10	92.30	91.80	91.90	0.0	0.0
32 18. 9.62.	0.0	0.0	100.10	100.10	100.10	100.00	100.10	99.90	100.20	92.30	91.90	91.80	0.0	0.0
33 19. 9.62.	0.0	0.0	100.10	100.00	100.10	100.20	100.15	99.95	100.15	92.30	92.10	92.00	0.0	0.0
34 7. 5.63.	0.0	0.0	0.0	0.0	0.0	100.50	100.65	100.40	101.00	93.15	92.70	92.70	96.35	0.0
35 29. 5.63.	0.0	0.0	0.0	0.0	0.0	100.10	100.30	100.15	100.50	92.55	92.00	91.60	95.85	0.0
36 3. 6.63.	0.0	0.0	0.0	0.0	0.0	100.20	100.30	100.30	100.40	92.60	92.30	92.10	96.20	100.00
37 11. 6.63.	0.0	0.0	0.0	0.0	0.0	100.25	100.20	100.30	100.20	93.00	92.10	92.10	96.30	100.00
38 3. 9.63.	0.0	0.0	0.0	0.0	0.0	99.90	99.90	100.00	100.20	91.75	91.20	91.05	94.55	99.85
39 4. 9.63.	0.0	0.0	0.0	0.0	0.0	99.80	99.85	100.15	100.30	92.00	90.90	90.95	94.50	99.80
40 10. 9.63.	0.0	0.0	0.0	0.0	0.0	99.80	99.50	99.60	100.00	91.80	90.60	90.60	94.45	99.60

Computation of mean and variance for a bond.

It will be shown by way of an example how the mean and variance are computed for bond on an observation date. The bond chosen is from loan no. 8 and the observation date Sept. 10, 1963. Bonds in loan no. 8 carry 6% coupon, payable semiannually. Lotteries are arranged every Jan. 15, and the subsequent repayments of the loan are made every April 1. On Sept. 10 the first forthcoming repayment in the loan was to be made on April 1, 1964. And the final repayment on April 1, 1970.

Yield on alternative, safe investment is given by the estimated yield curve for BTP bonds on the observation date: $i = 4{,}645 - 0{,}077/t$. The set of certainty alternative values for the lottery bond is obtained by computing the value for i from the regression above for each possible time to maturity for the bond. The obtained value for i together with the t-value are then used in the formula for the transaction price in chapter III:

$$(2) \qquad B_t = (1+i)^{\alpha} \left\{\frac{r}{i} + \frac{i-r}{i} v^n\right\}$$

to get the set of alternative bond values.

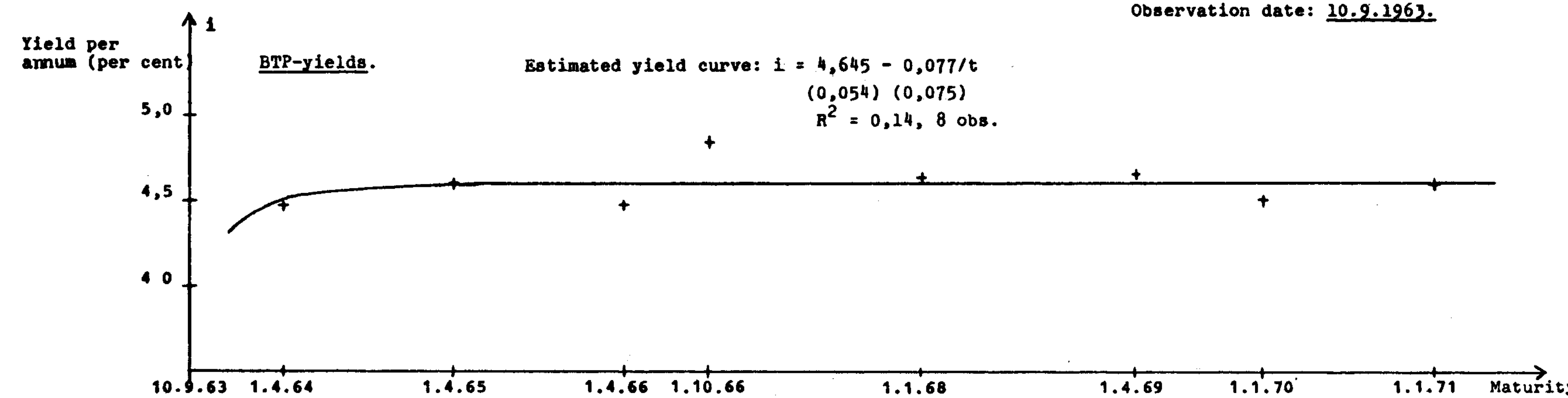
In the following figure is given the estimated yield curve for BTP bonds on the observation date Sept. 10, 1963, together with the computed yields. In the diagram below are the repayment dates of loan no. 8 and the set of certainty alternative values for an IMI bond in loan no. 8. From the set of certainty alternative values and the lottery probabilities may be computed the mean $\mu = 107{,}28$ and variance $\sigma^2 = 5{,}1143$.

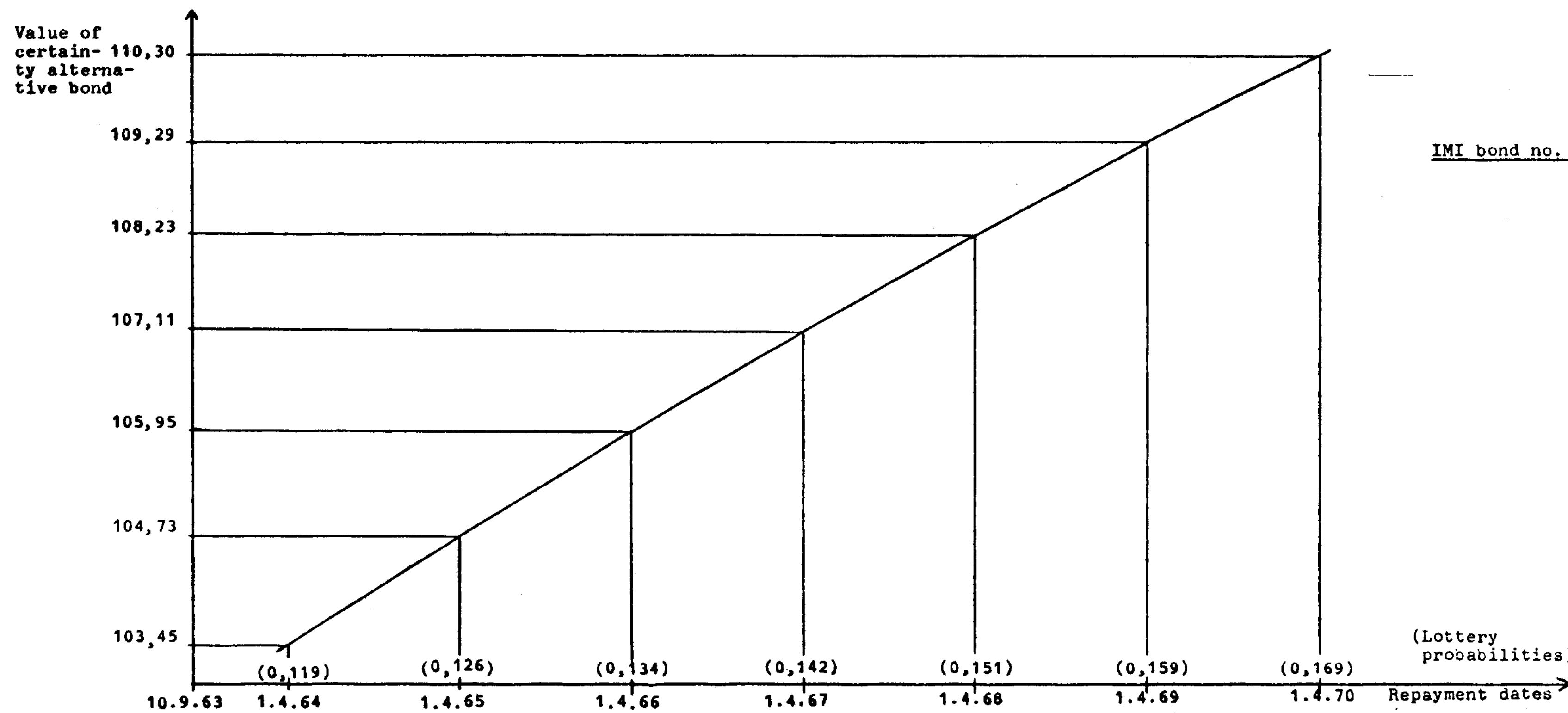
The observed market price of the bond is L99,60. The first forthcoming coupon payment is due on Oct. 1, and on that day a bond holder will receive a coupon payment of 6%/2 = 3%.

The remaining time to coupon payment is 20 days, or 20/180 = 0,11 of a coupon period. The accrued simple interest is then L3(1-0,11) = L2,66, and the transaction price is p = L99,60 + L2,66 = L102,26. The observed risk premium is then $\Pi = \mu - p = 107{,}28 - 102{,}26 =$ L5,02.

From the enclosed set of data may be seen that the computed risk premium for a bond is loan no. 8 on Sept. 10, 1963 was L5,0264. The risk premiums are thus represented with more decimal points than the observed prices on IMI bonds. The alternative would be to truncate or to round the computed figures to an accuracy of two decimal points. It was chosen not to round or truncate figures in any intermediate step of the data processing and regression analysis.

Observation date: 10.9.1963.





Istituto Mobiliare Italiano - IMI

Variables

LO	=	Loan number
RPR	=	Risk premium
VAR	=	Variance
SD	=	Sales dummy
		SD = 1 - bonds on sale from IMI
		SD = 0 - bonds not on sale
6-C	=	6 - coupon
C1 and		
C2	=	coupon dummies
		C1 = 1 if coupon is equal to 6%
		C1 = 0 otherwise
		C2 = 1 if coupon is equal to 5%
		C2 = 0 otherwise

OBS.NO. 1 DATE 11. 5.59.

LO.	1	2	3	4	5	6	7	8	9
RPR	0.8864	1.6386	3.4347	4.1756	3.2603	5.0830	7.1093	5.9652	8.0793
VAR	0.0016	0.0397	1.7855	2.2245	1.2347	4.0278	6.6932	3.5312	6.9382
SD	0.0	0.0	0.0	0.0	0.0	1.0000	1.0000	1.0000	0.0
6-C	1.0000	0.5000	0.0	0.0	0.0	0.0	0.0	0.0	0.0
C1	0.0	0.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
C2	1.0000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

OBS.NO. 2 DATE 22. 5.59.

LO.	1	2	3	4	5	6	7	8	9
RPR	0.9232	1.4610	3.1644	3.8282	2.8315	4.9250	6.9311	5.7401	8.0015
VAR	0.0007	0.0480	1.9251	2.3985	1.3306	4.3474	7.2392	3.8142	7.5169
SD	0.0	0.0	0.0	0.0	0.0	1.0000	1.0000	1.0000	0.0
6-C	1.0000	0.5000	0.0	0.0	0.0	0.0	0.0	0.0	0.0
C1	0.0	0.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
C2	1.0000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

OBS.NO. 3 DATE 10. 6.59.

LO.	1	2	3	4	5	6	7	8	9
RPR	-0.2267	1.0909	1.8849	2.5894	1.7557	3.5699	5.2939	4.4216	6.1444
VAR	0.0009	0.0439	1.8679	2.3258	1.2910	4.2149	7.0153	3.6963	7.2827
SD	0.0	0.0	0.0	0.0	0.0	1.0000	1.0000	1.0000	0.0
6-C	1.0000	0.5000	0.0	0.0	0.0	0.0	0.0	0.0	0.0
C1	0.0	0.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
C2	1.0000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

OBS.NO. 4 DATE 17. 6.59.

.O.	1	2	3	4	5	6	7	8	9
.PR	-0.3242	1.0957	1.8993	2.5787	1.7605	3.6527	5.7136	4.1902	6.6189
'AR	0.0003	0.0463	2.0235	2.5219	1.3987	4.5736	7.6276	4.0153	7.9297
.D	0.0	0.0	0.0	0.0	0.0	1.0000	1.0000	1.0000	0.0
-C	1.0000	0.5000	0.0	0.0	0.0	0.0	0.0	0.0	0.0
.1	0.0	0.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.2	1.0000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

OBS.NO. 5 DATE 28. 9.59.

.O.	1	2	3	4	5	6	7	8	9	10
PR	-0.2502	1.1277	2.0336	2.2690	1.5740	3.2355	4.4524	3.8840	5.8352	2.9060
'AR	0.0431	0.0107	1.2337	1.6194	0.9019	2.8596	4.8254	2.5529	4.9794	0.4311
D	0.0	0.0	0.0	0.0	0.0	1.0000	1.0000	1.0000	0.0	1.0000
-C	1.0000	0.5000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000
:1	0.0	0.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0
:2	1.0000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000

OBS.NO. 6 DATE 3. 5.60.

LO.	1	2	3	4	5	6	7	8	9	10
RPR	0.4160	0.2302	1.3345	1.9203	0.3795	3.1825	5.4354	3.5633	6.3083	3.6278
VAR	0.0000	0.0140	1.6765	2.2019	1.0446	4.4479	8.0175	5.0522	10.0271	0.0094
SD	0.0	0.0	0.0	0.0	0.0	0.0	1.0000	1.0000	0.0	1.0000
6-C	1.0000	0.5000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000
C1	0.0	0.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0
C2	1.0000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000

OBS.NO. 7 DATE 10. 5.60.

LO.	1	2	3	4	5	6	7	8	9	10
RPR	0.8706	0.9783	1.3568	2.0886	1.0030	3.2943	5.4337	3.7715	6.0995	3.6398
VAR	0.0000	0.0130	1.6574	2.1776	1.0329	4.3977	7.9248	4.9954	9.9089	0.0061
SD	0.0	0.0	0.0	0.0	0.0	0.0	1.0000	1.0000	0.0	1.0000
6-C	1.0000	0.5000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000
C1	0.0	0.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0
C2	1.0000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000

OBS.NO. 8 DATE 19. 5.60.

LO.	1	2	3	4	5	6	7	8	9	10
RPR	1.4212	0.5835	1.7713	2.2999	1.3180	3.6513	5.7805	3.9256	6.5406	3.6891
VAR	0.0000	0.0091	1.6410	2.1598	1.0238	4.3602	7.8570	4.9549	9.8194	0.0032
SD	0.0	0.0	0.0	0.0	0.0	0.0	1.0000	1.0000	0.0	1.0000
6-C	1.0000	0.5000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000
C1	0.0	0.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0
C2	1.0000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000

OBS.NO. 9 DATE 27. 5.60.

LO.	1	2	3	4	5	6	7	8	9	10
RPR	1.4955	0.5551	1.8983	2.5672	1.5798	3.7883	6.1589	4.3006	7.2430	3.9573
VAR	0.0000	0.0097	1.8390	2.4232	1.1471	4.8988	8.8564	5.5724	11.0870	0.0465
SD	0.0	0.0	0.0	0.0	0.0	0.0	1.0000	1.0000	0.0	1.0000
6-C	1.0000	0.5000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000
C1	0.0	0.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0
C2	1.0000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000

OBS.NO. 10 DATE 20. 9.60.

LO.	1	2	3	4	5	6	7	8	9	10
RPR	1.5613	0.5352	0.9639	2.0064	0.8347	3.1148	4.9872	3.7589	6.2997	2.976[illegible]
VAR	0.0000	0.0000	1.7162	2.3472	1.1119	4.6818	8.5617	5.3921	10.7111	0.0182
SD	0.0	0.0	0.0	0.0	0.0	0.0	1.0000	1.0000	0.0	1.0000
6-C	1.0000	0.5000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000
C1	0.0	0.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0
C2	1.0000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000

OBS.NO. 11 DATE 23. 9.60.

.O.	1	2	3	4	5	6	7	8	9	10
RPR	1.5806	-0.5176	0.8628	1.9832	0.9340	2.9662	5.1784	3.5919	6.1610	2.9838
VAR	0.0000	0.0000	1.6137	2.2404	1.0618	4.4425	8.1564	5.1414	10.1979	0.0047
SD	0.0	0.0	0.0	0.0	0.0	0.0	1.0000	1.0000	0.0	1.0000
6-C	1.0000	0.5000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000
C1	0.0	0.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0
C2	1.0000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000

OBS.NO. 12 DATE 26. 9.60.

.O.	1	2	3	4	5	6	7	8	9	10
RPR	1.7285	-0.5734	0.4088	1.6397	0.6219	2.6915	4.7656	3.2375	5.7563	2.7411
VAR	0.0000	0.0000	1.4464	2.0952	0.9936	4.0898	7.6076	4.8018	9.5026	0.0006
SD	0.0	0.0	0.0	0.0	0.0	0.0	1.0000	1.0000	0.0	1.0000
6-C	1.0000	0.5000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000
C1	0.0	0.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0
C2	1.0000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000

OBS.NO. 13 DATE 5. 5.61.

.O.	3	4	5	6	7	8	9	10
RPR	0.7366	2.8103	1.7336	3.9553	6.9440	4.8189	8.2223	4.9872
VAR	1.5492	2.2086	0.8092	5.1667	10.2882	6.0307	13.3012	0.3100
SD	0.0	0.0	0.0	0.0	1.0000	0.0	0.0	1.0000
6-C	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000
C1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0
C2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000

OBS.NO. 14 DATE 19. 5.61.

.O.	3	4	5	6	7	8	9	10
RPR	2.0964	3.1977	1.5011	4.5547	7.0149	5.1471	8.4519	5.2524
VAR	1.4670	2.0917	0.7667	4.8895	9.7205	5.7050	12.5564	0.2204
SD	0.0	0.0	0.0	0.0	1.0000	0.0	0.0	1.0000
6-C	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000
C1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0
C2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000

OBS.NO. 15 DATE 24. 5.61.

.O.	3	4	5	6	7	8	9	10
RPR	1.9633	2.7301	1.5804	4.0281	6.6173	4.7378	7.9404	5.2329
VAR	1.3445	1.9169	0.7031	4.4747	8.8749	5.2186	11.4479	0.1136
SD	0.0	0.0	0.0	0.0	1.0000	0.0	0.0	1.0000
6-C	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000
C1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0
C2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000

OBS.NO. 16 DATE 25. 5.61.

LO.	3	4	5	6	7	8	9	10
RPR	1.9150	2.6736	1.5352	3.9576	6.5181	4.6596	7.7260	5.0407
VAR	1.3163	1.8764	0.6885	4.3791	8.6798	5.1061	11.1921	0.0936
SD	0.0	0.0	0.0	0.0	1.0000	0.0	0.0	1.0000
6-C	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000
C1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0
C2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000

OBS.NO. 17 DATE 26. 5.61.

LO.	3	4	5	6	7	8	9	10
RPR	1.9277	2.6935	1.5453	3.8895	6.5748	4.6982	7.7959	5.0960
VAR	1.3406	1.9116	0.7012	4.4616	8.8483	5.2029	11.4132	0.1106
SD	0.0	0.0	0.0	0.0	1.0000	0.0	0.0	1.0000
6-C	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000
C1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0
C2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000

OBS.NO. 18 DATE 21. 6.61.

LO.	3	4	5	6	7	8	9	10
RPR	2.6788	2.6318	1.8476	4.5123	6.7115	5.0604	7.8638	5.5225
VAR	1.2936	1.8599	0.6804	4.3360	8.6119	5.0671	11.1003	0.0783
SD	0.0	0.0	0.0	0.0	1.0000	0.0	0.0	1.0000
6-C	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000
C1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0
C2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000

OBS.NO. 19 DATE 4. 9.61.

LO.	3	4	5	6	7	8	9	10
RPR	2.1399	2.8905	1.8112	4.5567	7.2759	5.1518	8.8758	6.0515
VAR	1.6464	2.4180	0.8838	5.6201	11.3085	6.6164	14.6408	0.4484
SD	0.0	0.0	0.0	0.0	1.0000	0.0	0.0	1.0000
6-C	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000
C1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0
C2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000

OBS.NO. 20 DATE 13. 9.61.

LO.	3	4	5	6	7	8	9	10
RPR	2.7178	2.8770	1.6323	4.6638	7.5717	5.4185	8.9426	6.4173
VAR	1.6806	2.4824	0.9081	5.7605	11.6179	6.7941	15.0483	0.5087
SD	0.0	0.0	0.0	0.0	1.0000	0.0	0.0	1.0000
6-C	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000
C1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0
C2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000

OBS.NO. 21 DATE 18. 9.61.

LO.	3	4	5	6	7	8	9	10
RPR	1.8556	3.0164	1.7590	5.0227	7.7101	5.5328	8.7485	6.2105
VAR	1.6853	2.5351	0.9271	5.8556	11.8769	6.9422	15.3885	0.5580
SD	0.0	0.0	0.0	0.0	1.0000	0.0	0.0	1.0000
6-C	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000
C1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0
C2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000

OBS.NO. 22 DATE 25. 9.61.

LO.	3	4	5	6	7	8	9	10	11
RPR	2.0043	3.0750	1.9079	4.2573	7.3134	5.1126	8.3960	6.1362	6.9874
VAR	1.3680	2.3687	0.8661	5.2531	11.0651	6.4784	14.3181	0.3950	0.3703
SD	0.0	0.0	0.0	0.0	1.0000	0.0	0.0	1.0000	1.0000
6-C	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000	1.0000
C1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0	0.0
C2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000	1.0000

OBS.NO. 23 DATE 28. 9.61.

LO.	3	4	5	6	7	8	9	10	11
RPR	2.4845	2.8185	1.7438	4.5980	7.1357	5.0207	8.2287	6.2141	6.9341
VAR	6.2641	2.3995	0.8774	5.7321	11.2135	6.5632	14.5139	0.4218	0.3960
SD	0.0	0.0	0.0	0.0	1.0000	0.0	0.0	1.0000	1.0000
6-C	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000	1.0000
C1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0	0.0
C2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000	1.0000

OBS.NO. 24 DATE 29. 9.61.

LO.	3	4	5	6	7	8	9	10	11
RPR	1.1013	2.8892	1.5730	4.0371	7.4665	4.8727	8.4168	6.2864	7.2123
VAR	3.3781	2.5724	0.9410	7.2658	12.0559	7.0450	15.6240	0.5899	0.5573
SD	0.0	0.0	0.0	0.0	1.0000	0.0	0.0	1.0000	1.0000
6-C	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000	1.0000
C1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0	0.0
C2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000	1.0000

OBS.NO. 25 DATE 11. 5.62.

LO.	3	4	5	6	7	8	9	10	11
RPR	3.8753	4.9629	3.5869	6.1455	8.8671	6.8787	9.8333	7.4278	8.9150
VAR	0.7901	1.2869	0.2801	3.6867	8.1866	4.4420	10.9125	0.1202	0.1349
SD	0.0	0.0	0.0	0.0	1.0000	0.0	0.0	1.0000	1.0000
6-C	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000	1.0000
C1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0	0.0
C2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000	1.0000

OBS.NO. 26 DATE 4. 6.62.

LO.	3	4	5	6	7	8	9	10	11
RPR	2.7359	3.0180	2.2540	4.6812	7.2679	5.3560	8.5161	7.2400	8.0150
VAR	0.7864	1.2595	0.2773	3.6030	7.9667	4.3253	10.6229	0.1077	0.1230
SD	0.0	0.0	0.0	0.0	1.0000	0.0	0.0	1.0000	1.0000
6-C	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000	1.0000
C1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0	0.0
C2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000	1.0000

OBS.NO. 27 DATE 20. 6.62.

LO.	3	4	5	6	7	8	9	10	11
RPR	1.3345	2.7478	1.1286	3.8912	6.2338	4.4500	7.4545	6.7016	6.8770
VAR	0.6567	1.0450	0.2310	2.9834	6.5607	3.5732	8.7297	0.0080	0.0092
SD	0.0	0.0	0.0	0.0	1.0000	0.0	0.0	1.0000	1.0000
6-C	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000	1.0000
C1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0	0.0
C2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000	1.0000

OBS.NO. 28 DATE 22. 6.62.

LO.	3	4	5	6	7	8	9	10	11
RPR	1.6506	2.4130	0.9951	3.6547	6.0442	4.2626	6.9633	6.5189	6.6921
VAR	0.6551	1.0423	0.2305	2.9753	6.5430	3.5633	8.7067	0.0075	0.0087
SD	0.0	0.0	0.0	0.0	1.0000	0.0	0.0	1.0000	1.0000
6-C	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000	1.0000
C1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0	0.0
C2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000	1.0000

OBS.NO. 29 DATE 27. 6.62.

LO.	3	4	5	6	7	8	9	10	11
RPR	1.6565	2.2887	0.6618	3.0291	5.7133	4.0584	6.8773	6.4102	6.9917
VAR	0.6012	0.9558	0.2115	2.7256	5.9803	3.2623	7.9481	0.0005	0.0006
SD	0.0	0.0	0.0	0.0	1.0000	0.0	0.0	1.0000	1.0000
6-C	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000	1.0000
C1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0	0.0
C2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000	1.0000

OBS.NO. 30 DATE 6. 9.62.

.O.	3	4	5	6	7	8	9	10	11	12
;PR	2.1388	2.9914	1.3288	4.4784	7.2683	5.1212	8.8004	8.8622	9.4382	10.0846
/AR	0.9325	1.5036	0.3307	4.3028	9.5718	5.1782	12.7983	0.3171	0.3682	0.5965
;D	0.0	0.0	0.0	0.0	1.0000	0.0	0.0	1.0000	1.0000	1.0000
;-C	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000	1.0000	1.0000
:1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0	0.0	0.0
:2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000	1.0000	1.000C

OBS.NO. 31 DATE 14. 9.62.

.O.	3	4	5	6	7	8	9	10	11	12
RPR	1.8631	2.6811	1.3664	4.1076	7.0231	5.0170	8.2395	9.1041	9.9728	9.9839
VAR	0.8597	1.3824	0.3047	3.9524	8.7661	4.7513	11.7057	0.1908	0.2210	0.3579
SD	0.0	0.0	0.0	0.0	1.0000	0.0	0.0	1.0000	1.0000	1.0000
6-C	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000	1.0000	1.0000
C1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0	0.0	0.0
C2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000	1.0000	1.0000

OBS.NO. 32 DATE 18. 9.62.

.O.	3	4	5	6	7	8	9	10	11	12
RPR	1.5301	2.2820	1.2588	3.6433	6.0729	4.4390	7.2651	8.3965	8.9636	9.1100
VAR	0.7285	1.1648	0.2573	3.3257	7.3347	3.9890	9.7703	0.0394	0.0453	0.0733
SD	0.0	0.0	0.0	0.0	1.0000	0.0	0.0	1.0000	1.0000	1.0000
6-C	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000	1.0000	1.0000
C1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0	0.0	0.0
C2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000	1.0000	1.0000

OBS.NO. 33 DATE 19. 9.62.

.O.	3	4	5	6	7	8	9	10	11	12
RPR	1.5154	2.3601	1.2465	3.4095	5.9645	4.3486	7.2438	8.3431	8.6888	8.8282
VAR	0.7147	1.1431	0.2525	3.2629	7.1931	3.9137	9.5790	0.0300	0.0344	0.0557
SD	0.0	0.0	0.0	0.0	1.0000	0.0	0.0	1.0000	1.0000	1.0000
6-C	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000	1.0000	1.0000
C1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0	0.0	0.0
C2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000	1.0000	1.0000

OBS.NO. 34 DATE 7. 5.63.

.O.	6	7	8	9	10	11	12	13
RPR	4.1213	7.5904	5.2135	9.1123	9.6493	11.0773	11.5801	12.5183
VAR	4.9617	12.8177	6.2175	17.9457	1.4609	2.1536	3.0424	12.4981
SD	0.0	1.0000	0.0	0.0	1.0000	1.0000	0.0	1.0000
6-C	0.0	0.0	0.0	0.0	1.0000	1.0000	1.0000	0.5000
C1	1.0000	1.0000	1.0000	1.0000	0.0	0.0	0.0	0.0
C2	0.0	0.0	0.0	0.0	1.0000	1.0000	1.0000	0.0

OBS.NO. 35 DATE 29. 5.63.

.O.	6	7	8	9	10	11	12	13
RPR	4.1585	7.4011	5.0533	8.9783	9.7778	11.1657	11.9985	12.2641
VAR	4.5061	11.5983	5.6408	16.2066	1.0765	1.5819	2.2318	10.7037
SD	0.0	1.0000	0.0	0.0	1.0000	1.0000	0.0	1.0000
6-C	0.0	0.0	0.0	0.0	1.0000	1.0000	1.0000	0.5000
C1	1.0000	1.0000	1.0000	1.0000	0.0	0.0	0.0	0.0
C2	0.0	0.0	0.0	0.0	1.0000	1.0000	1.0000	0.0

OBS.NO. 36 DATE 3. 6.63.

LO.	6	7	8	9	10	11	12	13	14
RPR	3.8241	7.0990	4.6507	8.7400	9.4456	10.5321	11.1390	11.5284	10.9370
VAR	4.3359	11.1459	5.4258	15.5640	0.9495	1.3944	1.9668	10.0696	17.1514
SD	0.0	1.0000	0.0	0.0	1.0000	1.0000	0.0	1.0000	1.0000
6-C	0.0	0.0	0.0	0.0	1.0000	1.0000	1.0000	0.5000	0.0
C1	1.0000	1.0000	1.0000	1.0000	0.0	0.0	0.0	0.0	1.0000
C2	0.0	0.0	0.0	0.0	1.0000	1.0000	1.0000	0.0	0.0

OBS.NO. 37 DATE 11. 6.63.

LO.	6	7	8	9	10	11	12	13	14
RPR	3.9876	7.5876	4.9114	9.4223	9.4314	11.2505	11.7278	12.0611	11.5188
VAR	4.7814	12.3321	5.9887	17.2525	1.2914	1.9020	2.6861	11.7516	19.0533
SD	0.0	1.0000	0.0	0.0	1.0000	1.0000	0.0	1.0000	1.0000
6-C	0.0	0.0	0.0	0.0	1.0000	1.0000	1.0000	0.5000	0.0
C1	1.0000	1.0000	1.0000	1.0000	0.0	0.0	0.0	0.0	1.0000
C2	0.0	0.0	0.0	0.0	1.0000	1.0000	1.0000	0.0	0.0

OBS.NO. 38 DATE 3. 9.63.

LO.	6	7	8	9	10	11	12	13	14
RPR	4.1472	7.8011	5.0491	9.3909	10.7583	12.2888	12.9416	13.9277	11.6870
VAR	5.0594	13.0651	6.3392	18.2905	1.4676	2.1640	3.0576	12.6921	20.2176
SD	0.0	1.0000	0.0	0.0	1.0000	1.0000	0.0	1.0000	1.0000
6-C	0.0	0.0	0.0	0.0	1.0000	1.0000	1.0000	0.5000	0.0
C1	1.0000	1.0000	1.0000	1.0000	0.0	0.0	0.0	0.0	1.0000
C2	0.0	0.0	0.0	0.0	1.0000	1.0000	1.0000	0.0	0.0

OBS.NO. 39 DATE 4. 9.63.

LO.	6	7	8	9	10	11	12	13	14
RPR	4.1384	7.5409	4.7360	8.8726	10.2174	12.1408	12.5111	13.3911	11.2053
VAR	4.5347	11.6652	5.6762	16.2944	1.0405	1.5280	2.1550	10.6566	17.9641
SD	0.0	1.0000	0.0	0.0	1.0000	1.0000	0.0	1.0000	1.0000
6-C	0.0	0.0	0.0	0.0	1.0000	1.0000	1.0000	0.5000	0.0
C1	1.0000	1.0000	1.0000	1.0000	0.0	0.0	0.0	0.0	1.0000
C2	0.0	0.0	0.0	0.0	1.0000	1.0000	1.0000	0.0	0.0

OBS.NO. 40 DATE 10. 9.63.

LO.	6	7	8	9	10	11	12	13	14
RPR	3.9273	7.4996	5.0264	8.6852	10.0569	11.9404	12.2868	12.8102	10.8161
VAR	4.0887	10.4793	5.1128	14.6103	0.7199	1.0539	1.4839	8.9957	16.0695
SD	0.0	1.0000	0.0	0.0	1.0000	1.0000	0.0	1.0000	1.0000
6-C	0.0	0.0	0.0	0.0	1.0000	1.0000	1.0000	0.5000	0.0
C1	1.0000	1.0000	1.0000	1.0000	0.0	0.0	0.0	0.0	1.0000
C2	0.0	0.0	0.0	0.0	1.0000	1.0000	1.0000	0.0	0.0

APPENDIX II

"Le prix du risque",
Annales de l'INSEE,
no. 9, 1972, pp. 89-118.

OBS.NO. 36 DATE 3. 6.63.

LO.	6	7	8	9	10	11	12	13	14
RPR	3.8241	7.0990	4.6507	8.7400	9.4456	10.5321	11.1390	11.5284	10.9370
VAR	4.3359	11.1459	5.4258	15.5640	0.9495	1.3944	1.9668	10.0696	17.1514
SD	0.0	1.0000	0.0	0.0	1.0000	1.0000	0.0	1.0000	1.0000
6-C	0.0	0.0	0.0	0.0	1.0000	1.0000	1.0000	0.5000	0.0
C1	1.0000	1.0000	1.0000	1.0000	0.0	0.0	0.0	0.0	1.0000
C2	0.0	0.0	0.0	0.0	1.0000	1.0000	1.0000	0.0	0.0

OBS.NO. 37 DATE 11. 6.63.

LO.	6	7	8	9	10	11	12	13	14
RPR	3.9876	7.5876	4.9114	9.4223	9.4314	11.2505	11.7278	12.0611	11.5188
VAR	4.7814	12.3321	5.9887	17.2525	1.2914	1.9020	2.6861	11.7516	19.0533
SD	0.0	1.0000	0.0	0.0	1.0000	1.0000	0.0	1.0000	1.0000
6-C	0.0	0.0	0.0	0.0	1.0000	1.0000	1.0000	0.5000	0.0
C1	1.0000	1.0000	1.0000	1.0000	0.0	0.0	0.0	0.0	1.0000
C2	0.0	0.0	0.0	0.0	1.0000	1.0000	1.0000	0.0	0.0

OBS.NO. 38 DATE 3. 9.63.

LO.	6	7	8	9	10	11	12	13	14
RPR	4.1472	7.8011	5.0491	9.3909	10.7583	12.2888	12.9416	13.9277	11.6870
VAR	5.0594	13.0651	6.3392	18.2905	1.4676	2.1640	3.0576	12.6921	20.2176
SD	0.0	1.0000	0.0	0.0	1.0000	1.0000	0.0	1.0000	1.0000
6-C	0.0	0.0	0.0	0.0	1.0000	1.0000	1.0000	0.5000	0.0
C1	1.0000	1.0000	1.0000	1.0000	0.0	0.0	0.0	0.0	1.0000
C2	0.0	0.0	0.0	0.0	1.0000	1.0000	1.0000	0.0	0.0

OBS.NO. 39 DATE 4. 9.63.

LO.	6	7	8	9	10	11	12	13	14
RPR	4.1384	7.5409	4.7360	8.8726	10.2174	12.1408	12.5111	13.3911	11.2053
VAR	4.5347	11.6652	5.6762	16.2944	1.0405	1.5280	2.1550	10.6566	17.9641
SD	0.0	1.0000	0.0	0.0	1.0000	1.0000	0.0	1.0000	1.0000
6-C	0.0	0.0	0.0	0.0	1.0000	1.0000	1.0000	0.5000	0.0
C1	1.0000	1.0000	1.0000	1.0000	0.0	0.0	0.0	0.0	1.0000
C2	0.0	0.0	0.0	0.0	1.0000	1.0000	1.0000	0.0	0.0

OBS.NO. 40 DATE 10. 9.63.

LO.	6	7	8	9	10	11	12	13	14
RPR	3.9273	7.4996	5.0264	8.6852	10.0569	11.9404	12.2868	12.8102	10.8161
VAR	4.0887	10.4793	5.1128	14.6103	0.7199	1.0539	1.4839	8.9957	16.0695
SD	0.0	1.0000	0.0	0.0	1.0000	1.0000	0.0	1.0000	1.0000
6-C	0.0	0.0	0.0	0.0	1.0000	1.0000	1.0000	0.5000	0.0
C1	1.0000	1.0000	1.0000	1.0000	0.0	0.0	0.0	0.0	1.0000
C2	0.0	0.0	0.0	0.0	1.0000	1.0000	1.0000	0.0	0.0

APPENDIX II

"Le prix du risque",
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Le prix du risque*

par **Cornelius M. SCHILBRED** **

Cet article expose les résultats d'une étude appliquée portant sur le prix de marché du risque. Les titres utilisés sont les obligations émises en Italie par l'Istituto mobiliare italiano (I.M.I.) ainsi que les obligations de l'Etat italien.

L'I.M.I. emprunte de l'argent sur le marché financier italien en émettant des obligations dont le remboursement s'effectue à l'aide de loteries qui déterminent quelles obligations vont être remboursées à chaque échéance. Comme les montants de ces remboursements sont fixés à l'avance dans les conditions de l'emprunt, on peut considérer que les bons de l'I.M.I. ont, en ce qui concerne les paiements futurs, une distribution de probabilité qui est une donnée pour les détenteurs de ces bons.

Afin de tester l'hypothèse selon laquelle il existe un prix de marché du risque, l'article débute en spécifiant un modèle d'équilibre instantané espérance-variance. On suppose que l'équivalent certain qu'il convient de considérer comme une alternative aux obligations de l'I.M.I. est constitué par les obligations émises par l'Etat italien, de sorte que le taux de ces

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dernières peut être utilisé pour tenir compte du fait que les paiements relatifs à différents emprunts de l'I.M.I. ne sont pas simultanés.

On a alors sélectionné au hasard quarante jours ouvrables pour la période 1958-1963, et estimé la courbe de rendement pour les bons d'Etat italien à chacune de ces dates. Puis on a utilisé les probabilités associées aux obligations de l'I.M.I. en circulation pour calculer les espérances et les variances de leurs valeurs actualisées.

A l'aide de ces espérances et variances ainsi calculées, on a pu effectuer différents tests de l'hypothèse selon laquelle il existe un prix d'équilibre du risque. Les tests tendent à confirmer l'existence d'un tel prix du risque, qui apparait voisin de 0,5 lire par unité de variance. On a trouvé que ce prix du risque était stable sur l'ensemble de la période (cinq ans) ainsi que sur des sous-intervalles d'une année.

Pour terminer, on a montré au moyen d'exemples simples comment ce modèle d'équilibre espérance-variance peut être utilisé dans les décisions relatives aux investissements et à leur financement, tout en soulignant certaines caractéristiques générales de ces modèles.

Introduction

Il est bien connu qu'en avenir certain une entreprise doit choisir ses investissements de manière à maximiser la valeur actualisée de ses actifs [6, 11]. *En ce qui concerne le meilleur critère d'investissement en avenir incertain, l'approche la plus prometteuse semble être dans l'utilisation de modèles d'équilibre général en incertitude. Les premiers modèles de ce type ont été construits par* ALLAIS [1] *et* ARROW [2]. BORCH [3] *a étudié un tel modèle pour le marché de la réassurance, et a utilisé le critère espérance-variance dû à* MARKOVITZ *et* TOBIN [12, 26] *pour établir, à partir d'une fonction d'utilité quadratique, une formule explicite du prix d'équilibre des titres en avenir incertain. Le modèle de* BORCH *est atemporel, et l'on peut donc considérer que les investisseurs expriment leurs préférences sur les distributions de probabilité de leur richesse actualisée.*

Des développements plus récents du modèle d'équilibre espérance-variance ont été obtenus par SHARPE [25], LINTNER [9] *et* MOSSIN [15]. *Leurs modèles sont des modèles à une période. Les investisseurs font leur choix parmi les distributions de probabilité de leur richesse finale de façon à minimiser la variance pour une espérance mathématique fixée de leur richesse finale, ou bien ils maximisent l'espérance d'une fonction d'utilité quadratique. On suppose que tous les investisseurs ont les mêmes distributions de probabilité, et que toute activité économique cesse à la fin de la période.*

Le modèle espérance-variance à une seule période est satisfaisant pour l'analyse de certains problèmes économiques, tels que la composition du portefeuille-titres d'un investisseur. Mais généralement, dans une opération d'investissement, une entreprise engage ses ressources pour plus d'une période, et les investissements qui s'offrent à elle ont des durées de vie différentes. Il est alors possible de considérer des modèles à plusieurs périodes, tels que ceux de MOSSIN [16] *et de* HAKANSON [7], *tout en supposant l'existence de préférences sur la richesse finale. Mais ces modèles deviennent vite fort compliqués et semblent impossibles à tester empiriquement.*

1 Le modèle d'équilibre espérance-variance

Le modèle espérance-variance suppose que les investisseurs choisissent entre des distributions de probabilité en ne tenant compte que de leurs espérances et de leurs variances. BORCH [4] a montré alors que, si les investisseurs se comportent en accord avec la théorie de VON NEUMANN et MORGENSTERN, ils possèdent des fonctions d'utilité quadratiques en ce qui concerne la monnaie ou la richesse actualisée :

$$u_i(x) = x - c_i x^2$$

où x est la monnaie et c_i une constante pour chaque investisseur i. De plus, SAMUELSON [21] a montré qu'une fonction d'utilité quadratique donne une approximation convenable pour les deux premiers moments de toute fonction d'utilité. L'espérance de l'utilité s'écrit alors :

$$\text{(1)} \qquad \mathrm{E}\,[u_i(x)] = \mathrm{E}_i - c_i\,(\mathrm{E}_i^2 + \mathrm{S}_i^2)$$

où E_i est l'espérance mathématique et S_i^2 la variance du portefeuille du $i^{\text{ième}}$ investisseur.

L'économie est Walrassienne, en ce sens que les investisseurs considèrent les prix comme donnés et que l'ensemble des prix d'équilibre est atteint par un processus de tâtonnement. Les n investisseurs arrivent sur le marché avec une richesse initiale donnée W_i ($i = 1, \ldots, n$). Il y a m titres différents, reprérés par l'indice $j = 1, \ldots, m$. A l'équilibre, l'investisseur i détient une fraction z_{ij} du titre j. Les conditions d'équilibre du marché sont, pour un nombre fixé de titres :

$$\text{(2)} \qquad \sum_i z_{ij} = 1 \qquad (j = 1, \ldots, m)$$

Lorsque les titres ont pour prix p_j, les investisseurs ont pour contrainte budgétaire :

$$\text{(3)} \qquad \sum_j z_{ij}p_j - \mathrm{W}_i = 0 \qquad (i = 1, \ldots, n)$$

Les investisseurs choisissent alors un portefeuille de titres de façon à maximiser l'espérance mathématique de leurs fonctions d'utilité quadratiques. L'espérance mathématique d'un portefeuille est :

$$\text{(4)} \qquad \mathrm{E}_i = \sum_j z_{ij}\,\mu_j \qquad (i = 1, \ldots, n)$$

où μ_j est l'espérance de la valeur actualisée du titre j. La variance totale d'un portefeuille est :

(5)
$$S_i^2 = \sum_j \sum_k z_{ij} z_{ik} \sigma_{jk} \qquad (i = 1, \ldots, n)$$

où σ_{jk} est la covariance entre les valeurs actualisées des titres j et k.

Pour maximiser l'espérance mathématique de l'utilité sous la contrainte budgétaire (3), formons les Lagrangiens :

$$L_i = \sum_j z_{ij}\mu_j - c_i \left[\sum_j \sum_k z_{ij} z_{ik} \sigma_{jk} + \left(\sum_j z_{ij} \mu_j \right)^2 \right] + \lambda_i \left(\sum_j z_{ij} p_j - W_i \right)$$
$$i = 1, \ldots, n$$

Annulant les données partielles, il vient :

$$\frac{\partial L_i}{\partial z_{ij}} = \mu_j - 2\, c_i \left[\sum_k z_{ik}\, \sigma_{jk} + \mu_j \sum_j z_{ij} \mu_j \right] + \lambda_i p_j = 0$$
$$i = 1, \ldots, n \qquad j = 1, \ldots, m$$

Divisant par c_i, sommant par rapport à i et utilisant les conditions d'équilibre de marché (2), on obtient :

(6)
$$\mu_j \sum_i \frac{1}{c_i} - 2 \sum_k \sigma_{jk} - 2\, \mu_j \sum_j \mu_j + p_j \sum_i \frac{\lambda_i}{c_i} = 0 \qquad j = 1, \ldots, m$$

Faisons alors l'hypothèse qu'il existe un titre dont la valeur est connue avec certitude, par exemple la monnaie, ou un numéraire. Pour un tel titre, que nous supposerons être le m^{ieme}, $p_m = \mu_m$ et $\sigma_{mk} = 0$ pour tout k, de sorte que l'équation (6) pour ce titre m s'écrit :

(7)
$$\sum_i \frac{1}{c_i} - 2 \sum_j \mu_j + \sum_i \frac{\lambda_i}{c_i} = 0$$

Reportant dans (6) la valeur de $\sum_i \frac{\lambda_i}{c_i}$ tirée de (7), on obtient alors les $m - 1$ équations :

$$\mu_j \sum_i \frac{1}{c_i} - 2 \sum_k \sigma_{jk} - 2\, \mu_j \sum_j \mu_j + p_j \left[2 \sum_j \mu_j - \sum_i \frac{1}{c_i} \right] = 0$$

ce qui donne le prix d'équilibre pour les m titres :

$$p_j = \mu_j - \frac{1}{\sum_i \frac{1}{2\, c_i} - \sum_j \mu_j} \sum_k \sigma_{jk} \qquad j = 1, \ldots, m$$

ou encore :

(8)
$$p_j = \mu_j - \gamma \sum_k \sigma_{jk} \qquad j = 1, \ldots, m$$

avec :

$$\gamma = \frac{1}{\sum_i \frac{1}{2 c_i} - \sum_j \mu_j}$$

γ est le prix du risque à l'équilibre. Pour des fonctions d'utilité et des distributions de probabilité données, γ est une quantité fixée, la même pour tous les titres.

La formulation (8) en termes de réassurance se trouve dans [3, p. 433]. Dans le modèle à une période de SHARPE-LINTNER-MOSSIN, les quantités du membre de droite dans (8) correspondent à des distributions de probabilité et des fonctions d'utilité pour la période terminale. Dans leur modèle, le membre de droite de (8) est divisé par un facteur $(1 + R)$ où R est le taux d'intérêt certain. Cela revient à prendre la valeur actualisée à la date présente de l'équivalent-certain de la distribution de probabilité en fin de période. On voit ainsi clairement le lien qui existe entre les deux formulations du modèle d'équilibre espérance-variance.

2 Les distributions de probabilité et les obligations italiennes

Nous allons tester le modèle d'équilibre espérance-variance en utilisant les prix auxquels sont échangés les obligations émises par l'Istituto mobiliare italiano (I.M.I.) et les bons du Trésor italien, ou bueno del Tesoro poliennali (bons B.T.P.).

L'I.M.I. émet des obligations garanties par l'État; son président est nommé par le Président de la République, et 7 des 19 membres de son conseil d'administration sont nommés par le Trésor italien [19]. Dans toutes ses opérations, l'I.M.I. semble agir en contact étroit avec le Gouvernement italien. L'I.M.I. finance ses opérations en vendant des obligations sur le marché italien. Lorsqu'il émet un nouvel emprunt, les obligations sont mises en vente pendant une période spécifiée à l'avance à l'issue de laquelle l'émission est close. Le remboursement de l'emprunt se fait selon un échéancier tel qu'à chaque échéance, la somme des intérêts versés et des remboursements de titres reste constante. Chaque obligation émise par l'I.M.I. porte un numéro, et tous les deux mois et demi environ, on organise une loterie dans cette tranche de façon à déterminer quelles obligations rembourser.

Du fait que les obligations sont remboursées selon des loteries où le nombre de titres à tirer est fixé à l'avance, chaque obligation est affectée d'une probabilité objective concernant sa date future de remboursement. Si une obligation rapporte un intérêt r par période, et si l'émission doit être remboursée en q périodes futures, la probabilité qu'une obligation donnée soit remboursée par tirage après la s^{ieme} loterie est [22, p. 47] :

$$(9) \qquad g_s(r, q) = \frac{r(1+r)^{s-1}}{(1+r)^q - 1} \qquad s = 1, 2, \ldots, q$$

Ce que nous allons chercher à estimer, c'est le prix de marché de ce risque qui tient au remboursement par loterie. La situation idéale aurait été que l'I.M.I. ait aussi émis suffisamment d'obligations à échéance fixée pour que l'une de ces obligations vienne à échéance à chaque date de remboursement des obligations à échéance aléatoire tout en rapportant un taux d'intérêt d'un même montant.

L'ensemble des prix de transaction pour chaque obligation à échéance aléatoire s'obtiendrait alors directement à partir des prix de transaction des obligations à échéance fixe (équivalents certains); les probabilités (9) permettraient alors de calculer les deux premier moments nécessaires pour l'estimation de γ dans (8) [1].

Mais il est de fait que l'I.M.I. n'émet pas d'obligations à échéance fixe. L'alternative à ces obligations de l'I.M.I. qui puisse jouer le rôle d'alternative certaine pour les investisseurs italiens consiste en bons B.T.P. Le Trésor italien émet ces bons B.T.P. à peu près tous les ans, et il n'émet que des bons à échéance fixe.

La date d'échéance des bons B.T.P. ne coïncide généralement pas avec les dates de mise en paiement des obligations I.M.I. Afin d'obtenir les prix de transaction pour d'hypothétiques bons B.T.P. qui auraient l'échéance convenable, des courbes de rendement ont été estimées sur la base des prix observées de ces bons B.T.P.

Pour un bon à échéance fixe qui vient à échéance à la date t, $[t]$ paiements de coupons seront faits, $[t]$ étant le nombre entier immédiatement supérieur à t. A la date t, le détenteur de ce bon reçoit en paiement la valeur nominale du bon augmentée du coupon. La valeur actualisée d'un bon est donc égale à la valeur actualisée de la valeur nominale et des coupons payés :

$$(10) \qquad B(r, i, t) = (1+i)^{\alpha} \left[\sum_{\theta=1}^{[t]} r \left(\frac{1}{1+i} \right)^{\theta} + 1 \left(\frac{1}{1+i} \right)^{[t]} \right] 100$$

$$= (1+i)^{\alpha} \left[\frac{r}{i} + \frac{i-r}{i} \left(\frac{1}{1+i} \right)^{[t]} \right] 100$$

où :

- 100 = valeur nominale de bon;
- t = durée de vie du bon mesurée en unités de périodes de mise en payement du coupon (nombre réel);
- α = $[t] - t$, de sorte que $1 - \alpha$ = fraction de la période de mise en paiement du coupon qui reste à courir jusqu'au paiement suivant (voir note 1);
- i = rendement par période du bon qui vient à échéance fixe à la date t;
- $B(r, i, t)$ = valeur actualisée du bon.

---

1. Le prix de transaction est égal au prix de marché affiché augmenté de l'intérêt simple au taux de l'obligation considérée. Si l'on appelle α la fraction de période écoulée depuis le paiement du dossier coupon, l'intérêt simple est αr. En ajoutant l'intérêt simple au prix du marché, les courtiers en valeurs ne font que corriger l'erreur mathématique que l'on commettrait en utilisant les tables donnant la valeur des obligations. Le prix de transaction entre l'acheteur et le vendeur qui résulte de cette correction est une bonne approximation de la valeur de l'obligation actualisée au taux d'intérêt constant i.

Un algorithme a été utilisé pour calculer le rendement d'un bon B.T.P. à une date donnée d'observation. On obtient ainsi une valeur de i qui donne une valeur actualisée du bon (10) assez précise pour se situer à l'intérieur de la fourchette des prix observés pour ce bon [2]. Le nombre des bons B.T.P. sur le marché à une date quelconque était de 8 ou 9, et la durée restant à courir avant échéance allait de 0 à 9 ans. Une courbe de rendement en forme d'hyperbole équilatère a été ajustée à cet ensemble de rendements calculés :

$$i = b_1 + \frac{b_2}{t} + \varepsilon \tag{11}$$

où ε est un terme résiduel distribué normalement avec une espérance nulle et une variance constante. Cette courbe de rendement pour les bons du Trésor représente le taux de rendement du marché à une date donnée sur des payements futurs qu'on peut considérer comme certains. A partir de cette courbe du rendement estimé, on peut alors déterminer d'après (10) ce qu'aurait été le prix du marché pour une obligation à échéance aléatoire si l'on avait connu avec certitude sa future date d'échéance. Compte tenu des probabilités d'échéance et de cet ensemble de valeurs actualisées possibles à une date donnée, l'espérance de la valeur actualisée d'un bon à échéance aléatoire est :

$$\mu_B = \sum_{s=1}^{q} B(r,\ i,\ t) \cdot g_s(r,\ q) \tag{12}$$

où s est la loterie qui prend place avant la date de remboursement t. La variance due à ces loteries est alors :

$$\sigma_B^2 = \sum_{s=1}^{q} [B(r,\ i,\ t) - \mu_B]^2\, g_s(r,\ q) \tag{13}$$

Les obligations de l'I.M.I. viennent à échéance entre 0 et 19 ans, l'espérance mathématique de l'échéance étant de 12 ans pour l'obligation ayant l'échéance observée la plus lointaine. Cela implique que les courbes de rendement des bons B.T.P. ont été utilisées bien au-delà des valeurs observées en ce qui concerne le temps qui reste à courir jusqu'à l'échéance de ces bons B.T.P. L'estimation $\hat{b}_1$ dans (11) peut être considérée comme le rendement à long terme des bons du Trésor; les écarts-types estimés des $\hat{b}_1$ sont faibles, la statistique t de Student la plus faible étant $t = 49{,}63$, alors que le seuil théorique correspondant $t_6(0{,}05) = 2{,}45$. Les $\hat{b}_1$ doivent donc donner une indication assez sûre de ce que seraient les rendements de marché à long terme pour des payements futurs que l'on peut raisonnablement tenir pour certains.

Il faut d'autre part tenir compte des covariances entre les résultats des loteries relatives à différentes émissions. Comme il n'y a aucun lien entre les loteries successives, toutes les covariances dues aux seules loteries sont nulles. Il existe cependant une source de covariation entre les payements des différents emprunts, c'est celle qui provient de la possibilité que l'I.M.I. soit incapable de faire place à ces remboursements futurs. On doit tenir compte de cette source d'incertitude parce que les prix de marché observés pour des titres alternatifs qu'on peut raisonnablement tenir pour certains sont ceux des bons B.T.P. du Gouvernement à échéance fixe, et non des bons de l'I.M.I.

Le risque tenant à la seule loterie peut donc ne pas être la seule source d'incertitude : il peut également y avoir une différence dans les risques de cessation de payement des bons par l'I.M.I. et par le Trésor. En émettant et en vendant des bons, l'I.M.I. distribue des créances sur la distribution de probabilité de son flux de recettes (cash flow). Si cette distribution de probabilité est donnée, alors, quel que soit l'état de la nature qui se réalise, le montant total payé aux détenteurs de titres émis par l'I.M.I. doit être égal au montant que l'I.M.I. est en mesure de payer :

$$a_{jtw} = \sum_{l} a_{jltw} \tag{14}$$

où :

a_{jtw} = montant que peut payer la firme j (en l'occurence, l'I.M.I.) à l'époque t lorsque l'état de la nature w se réalise;

a_{jltw} = montant que doit payer la firme j au détenteur de titre l à 'lépoque t lorsque l'état de la nature w se réalise.

Lorsque le marché des créances dont le payement est certain est en équilibre, le prix de marché v_t d'une lire payable à la date t devient une donnée. Un ensemble de payements a dès lors une valeur de marché actualisée

$$a_{jlw} = \sum_{t} a_{jltw}\, v_t$$

si l'on omet l'indice temporel pour les valeurs actualisées, et de même :

$$a_{jw} = \sum_{t} a_{jtw}\, v_t$$

on déduit alors de (14).

$$a_{jw} = \sum_{l} a_{jlw} \tag{15}$$

c'est-à-dire que la valeur actualisée de la firme j lorsque l'état de la nature w se réalise (ici, il s'agit d'un flux de recettes) est égal à la somme des valeurs actualisées, dans ce même état de la nature, des l titres émis par l'entreprise j.

L'espérance mathématique de la valeur actualisée au titre l est :

$$\mu_{jl} = \sum_{w} a_{jlw}\, q_w$$

où q_w est la probabilité que la firme perçoive un flux de recettes w. La somme des espérances mathématiques des valeurs actualisées des titres l est :

$$\sum_{l} \mu_{jl} = \sum_{l} \sum_{w} a_{jlw}\, q_w$$

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2. Les procédures de calcul et l'estimation des courbes de rendement sont discutées plus en détail dans [24]. La construction des courbes de rendement pour les bons B.T.P. est étudiée aussi dans [10] et [13].

et, d'après (15) :

(16) $$\sum_{l} \mu_{jl} = \sum_{w} a_{jw} q_w = \mu_j$$

où μ_j est l'espérance mathématique de la valeur actualisée de la firme j. La variance de la distribution de probabilité des valeurs actualisées de la firme j est :

$$\sigma_j^2 = \sum_{w} (a_{jw} - \mu_j)^2 q_w \qquad (j = k)$$

Les variances-covariances entre les valeurs actualisées des titres l et o émis par la firme j sont :

(17') $${}_j\sigma_{l0} = \sum_{w} (a_{jlw} - \mu_{js})(a_{j0w} - \mu_{j0})\, q_w \qquad (j = k)$$

En sommant (17') par rapport à tous les titres émis par la firme j, puis en utilisant (15) et (16), on obtient :

$$\sum_{l} \sum_{0} {}_j\sigma_{l0} = \sum_{l} \sum_{0} \sum_{w} (a_{jlw} - \mu_{jl})(a_{j0w} - \mu_{j0})\, q_w$$

(17) $$\sum_{l} \sum_{0} {}_j\sigma_{l0} = \sum_{w} (a_{jw} - \mu_j)(a_{jw} - \mu_j)\, q_w = \sigma_j^2 \qquad (j = k)$$

cela signifie que la somme des variances et covariances entre les distributions de probabilité des payements effectués aux détenteurs de titres émis par la firme j est égale à la variance totale de la distribution des flux de recettes de la firme j. Par conséquent, en émettant des titres, la firme reporte sur ses titres la variance de son flux de recettes. Si ce flux de recettes est incertain, alors certaines variances et covariances entre ses titres seront non nulles.

Lorsque l'I.M.I. émet des bons, les variances et covariances entre tous les bons seront égales s'il existe un danger immédiat d'illiquidité pour l'I.M.I. Une telle situation implique que le même pourcentage de la valeur nominale des bons soit payé aux détenteurs de bons, étant donné que tous les bons ont même priorité vis-à-vis des actifs de l'I.M.I. Une possibilité de cessation de paiement à plus longue échéance implique, cependant, que la variance totale de la distribution du flux de recettes de l'I.M.I. soit reportée de façon inégale sur ses titres, les prêts à long terme supportant alors une plus grande proportion du risque de cessation de paiement. En ce cas, il est probable que tous les flux de recettes possibles envisagés par l'I.M.I. sont tels qu'ils permettent à cette institution de faire face à ses paiements d'obligations à court terme, et il n'y a pas alors de risque de cessation de paiement pour les prêts venant à maturité dans des délais assez courts.

Si les paiements futurs par l'I.M.I. sont incertains, et s'il y a aussi sur le marché des titres émis par d'autres firmes dont les flux de recettes futurs sont incertains, il se peut alors que les covariances entre les distributions de ces flux de recettes ne soient pas toutes nulles. La covariance entre les valeurs actualisées des firmes j et k est :

$$\sigma_{jk} = \sum_{w} \sum_{w'} (a_{jw} - \mu_j)(a_{kw} - \mu_k)\, q_{ww} \qquad (j \neq k)$$

où :

$q_{ww'}$ = probabilité que la firme j obtienne un flux de recettes w et le firme k un flux de recettes w' :

$$\sum_{w} \sum_{w'} q_{ww'} = 1 \qquad \sum_{w} q_{ww'} = q_w$$

la covariance de titre l de l'entreprise j et de celui de la firme k est :

$$_j\sigma_{lk} = \sum_{w} \sum_{w'} (a_{jlw} - \mu_{jl})(a_{kw} - \mu_k)\, q_{ww'} \qquad (j \neq k)$$

En faisant la somme des covariances des titres de la firme j et en utilisant (15) et (16), il vient :

(18) $$\sum_{l} {}_j\sigma_{lk} = \sigma_{jk} \qquad (j \neq k)$$

ce qui exprime que la somme des covariances des titres l de la firme j avec la firme k est égale à la covariance entre les firmes j et k.

En cas de risque de cessation de paiement par l'I.M.I., c'est la redistribution du risque total σ_{jk} (pour tout j et pour tout k) parmi ses titres $_j\sigma_{lk}$ qui intervient dans la détermination des prix relatifs des titres à l'équilibre. Ce risque de cessation de paiement intervient concurremment avec le risque dû aux loteries, de sorte que l'incertitude mesurée en termes de variance des loteries peut ne pas constituer une mesure de l'incertitude totale telle que l'exige l'hypothèse d'un système de prix à l'équilibre (8).

On peut soutenir cependant qu'un risque de loterie élevé (réduit) est associé à un risque de cessation du paiement réduit (élevé). Cela dépend de l'étendue des valeurs de marché des bons équivalents en avenir certain, de la durée qui reste à courir jusqu'à l'échéance du prêt et du caractère plus ou moins immédiat du risque de cessation de paiement. Faute de connaître la façon dont, à chaque date, les investisseurs perçoivent l'avenir du Gouvernement italien et celui de l'I.M.I., il n'est pas possible de dire a priori si le risque de cessation de paiements affecte l'estimation du prix du risque sur le marché italien.

3 Résultats empiriques

On a choisi comme période d'observation les années 1958 à 1963 [3], qui devraient constituer une période assez longue pour pouvoir effectuer un test de la stabilité intertemporelle de l'hypothèse d'existence d'un prix d'équilibre. Étant donné que les mois les plus actifs au point de vue financier sont ceux de mai, juin et septembre c'est, pour chaque année, parmi ces mois-là qu'a été tiré un échantillon aléatoire de quarante jours ouvrables.

En ce qui concerne les titres de l'I.M.I. négociables ces jours-là, on a utilisé les brochures d'information publiées par l'I.M.I. [8] en même temps que les courbes de rendement estimé pour les bons B.T.P. afin de calculer l'espérance mathématique μ_{jd} et la variance σ^2_{jd} des valeurs actualisées de bon j le jour d'observation d. L'intérêt simple au taux du coupon a été ajouté au prix de marché observé pour obtenir le prix de transaction p_{jd}.

Le modèle d'équilibre ayant été spécifié pour un montant donné de titres, une variable indicatrice a été utilisée pour tenir compte de l'effet possible de la vente de titres nouveaux sur les prix de marché observés :

$$sd = \begin{cases} 1 & \text{si de nouveaux bons sont émis par l'I.M.I.} \\ 0 & \text{sinon} \end{cases}$$

Pour tester l'hypothèse qu'il existe un prix de marché du risque γ_d, il est commode d'introduire le concept de prime de risque :

$$\Pi_{jd} = \mu_{jd} - p_{jd}$$

et de réécrire (8) sous la forme :

$$\vec{\Pi}_{jd} = \gamma_d \sigma^2_{jd}$$

Comme les loteries peuvent ne pas être la seule source d'incertitude, nous supposerons que la différence dans les risques de non-paiement entre les bons B.T.P. et les bons de l'I.M.I. affecte tous ces deniers de façon identique, de sorte qu'une telle différence dans le risque de non-paiement se traduit par un terme constant [4] non nul dans la régression :

$$\Pi_{jd} = a_d + \gamma_d \, \sigma^2_{jd} + h_d \, sd_{jd} + \varepsilon$$

A partir de toutes les observations on a obtenu [5] :

$$\Pi = 2{,}47 + 0{,}34\,\sigma^2 + 3{,}07\,sd \qquad (19)$$

$$(0{,}184) \quad (0{,}027) \quad (0{,}252)$$

$R^2 = 0{,}50$;
361 observations;
Somme des carrés des résidus (S.C.R.) = 1 857,89.

Ce résultat tend à confirmer le modèle d'équilibre espérance-variance. Le prix de marché du risque ainsi estimé est significativement différent de zéro. La significativité de la variable indicatrice *sd* suggère que dans l'ensemble, l'I.M.I. a subi une baisse de 3 % sur les prix des bons dont l'émission n'était pas encore close par rapport aux bons appartenant à des émissions déjà closes. Le fait que le terme constant soit significativement différent de zéro signifie que les bons de l'I.M.I. se négociaient à des prix qui étaient inférieurs d'environ 2,5 % aux prix qui auraient été payés pour des bons similaires qu'aurait émis le Trésor italien.

La régression (19) explique environ la moitié de la variance totale, si bien qu'il peut aussi exister d'autres facteurs affectant les primes de risque observées. La positivité du terme constant conduit à s'interroger sur la possibilité qu'il ait existé un risque de non-paiement dont il n'a pas été tenu compte en calculant les espérances et variances des bons de l'I.M.I.

Il se peut donc que l'estimation du prix de marché du risque soit affectée par le fait que les courbes de rendement ont été établies pour des bons B.T.P. et non pour des bons de l'I.M.I. à échéance fixe. La forme de la courbe de rendement des bons B.T.P. est censée traduire les anticipations relatives aux changements futurs de taux d'intérêt, aux prix des biens, aux risques de non paiement par le Trésor, ainsi qu'aux autres facteurs économiques pertinents. La raison qui aurait pû conduire l'I.M.I. à avoir une courbe de rendement pour les bons à échéance fixe différente de celle pour les B.T.P. serait la possibilité qu'il ait existé une différence dans les risques de non-paiements entre l'I.M.I. et le Trésor italien.

Une telle différence affecterait le calcul des espérances des valeurs actualisées et des variances dans (10). Cet effet dépendrait du coupon et du temps restant jusqu'à l'échéance du titre *t*. Étant donné que *t* et *r* ont déjà été utilisés tous les deux pour les calculs de μ_{jd} et de $\sigma^2{}_{jd}$, il n'existe pas de moyen direct d'étudier la différence possible dans les risques de non-paiement. L'introduction de *t* et *r* comme variables explicatives à côté de $\sigma^2{}_{jd}$ conduit à un problème d'identification. Cette question de risque de non-paiement étant cependant importante, on a choisi de l'étudier en introduisant des variables indicatrices pour les coupons des différents bons de l'I.M.I. Au cours de cette période, les bons de l'I.M.I. négociables sur le marché ont été des bons à 5 %, 5,5 % et 6 %. Les deux variables indicatrices sont :

$$C_1 = \begin{cases} 1 & \text{si le coupon est de } 6\,\% \\ 0 & \text{sinon} \end{cases}$$

$$C_2 = \begin{cases} 1 & \text{si le coupon est de } 5\,\% \\ 0 & \text{sinon} \end{cases}$$

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3. Les lois fiscales italiennes ayant été modifiées en 1971, la période d'observation a dû être choisie bien avant cette date pour que les prix relatifs des bons ne soient pas affectés par les effets attendus des changements futurs dans la législation fiscale.
4. Les Italiens ont eu depuis longtemps la possibilité d'utiliser des bons B.T.P. pour régler à un taux réduit le montant des droits de succession. Toutefois, aucune modification n'ayant été apportée à cette réglementation au cours de la période étudiée, cette possibilité s'est tout au plus répercutée sur le terme constant.
5. On a choisi un niveau de signification de 5 %. Les écarts-types estimés sont donnés entre parenthèses au-dessous de chaque estimation.

on obtient ainsi la régression :

(20) $$\Pi = 2{,}06 + 0{,}48\,\sigma^2 - 0{,}87\,C_1 + 2{,}38\,C_2 + 2{,}95\,sd$$
$$(0{,}429)\ (0{,}023)\ \ (0{,}409)\ \ \ (0{,}447)\ \ (0{,}192)$$

$R^2 = 0{,}72$;
361 observations;
S.C.R. = 1 048,51.

Il est clair d'après (20) que l'usage des courbes de rendement des B.T.P. soulève un problème, puisque les deux variables indicatrices de coupon sont significativement différentes de zéro et que le coefficient de détermination a considérablement augmenté. L'introduction de ces variables indicatrices de coupon augmente l'estimation du prix du risque. Si l'on étudie séparément ces trois sous-échantillons, on obtient :

(20.1) coupon 6 % $$\Pi = 1{,}65 + 0{,}50\,\sigma^2 + 0{,}38\,sd$$
$$(0{,}085)\ (0{,}013)\ \ (0{,}137)$$

$R^2 = 0{,}88$;
264 observations;
S.C.R. = 199,90.

(20.2) coupon 5,5 % $$\Pi = 0{,}68 + 0{,}21\,\sigma^2 + 9{,}68\,sd$$
$$(0{,}216)\ (0{,}228)\ \ (2{,}541)$$

$R^2 = 0{,}99$;
19 observations;
S.C.R. = 8,94.

(20.3) coupon 5 % $$\Pi = 1{,}65 + 3{,}78\,\sigma^2 + 4{,}12\,sd$$
$$(0{,}502)\ (0{,}278)\ \ (0{,}516)$$

$R^2 = 0{,}74$;
78 observations;
S.C.R. = 273,50.

Les estimations du prix du risque diffèrent fortement d'un groupe à l'autre, et pour les coupons à 5,5 % le coefficient γ n'est plus significatif. Ces bons à 5,5 % n'ont été négociables sur le marché que pendant 19 des 40 jours observés. Parmi les deux émissions à 5,5 %, l'une est venue à échéance en 1961 et une autre a été ouverte en 1963. Le placement de cette nouvelle émission a affaibli considérablement son prix, tandis que le risque de loterie n'a joué, semble-t-il, qu'un rôle négligeable dans la détermination du prix.

En ce qui concerne les bons à 5 % le nombre de telles émissions sur le marché a varié de 1 à 3 durant la période d'observation. Trois nouveaux emprunts à 5 % ont été émis, et le placement de ceux-ci a affecté significativement les prix du marché : le γ 5 % est particulièrement élevé. 11 fois plus grand que le prix du risque estimé à partir de l'ensemble de l'échantillon.

A partir de la colonne de droite du tableau 1, on voit que b_1 pour les courbes de rendement estimées pour les bons B.T.P. est voisin de 5 % (ou légèrement inférieur) pour la plupart des dates d'observation. On peut considérer b_1 comme le taux d'intérêt à long terme sur les bons d'État, et le fait qu'il soit proche du coupon des bons à 5 % indique une faible dispersion dans l'ensemble des valeurs actualisées possibles de ces bons et, partant, de faibles valeurs pour les variances calculées. Les bons à 5 % sont donc peut-être ceux qui sont le plus affectés par l'utilisation des courbes de rendement des B.T.P., et il convient de garder ce fait présent à l'esprit pour interpréter la valeur élevée de $\hat{\gamma}_{5\,\%}$.

Les coupons à 6 % dominent l'échantillon pour ce qui est du nombre d'observations, et c'est pour ce groupe que l'on a probablement l'estimation la plus digne de confiance du prix de marché du risque, $\hat{\gamma}_{6\,\%} = 0{,}5$. C'est une différence marquée entre les rendements du B.T.P. et le coupon à 6 % qui cause la variation du σ^2 estimé. Le nombre de différents bons à 6 % négociés sur le marché au cours de la période a varié entre 4 et 7.

Les résultats ci-dessus suggèrent qu'il existe bien un prix de marché du risque, et on va maintenant tester si ce prix de marché est resté stable au cours du temps. Le test est limité au sous-échantillon des bons à 6 %. Pour chaque année, on a choisi une seule date d'observation, celle qui est la plus proche du 15 mai. Cependant, en 1963, l'I.M.I. a émis un nouvel emprunt à 6 % le 3 avril. Cet emprunt n'a pas été coté en Bourse avant le 3 juin, et il est probable que le placement de ce nouvel emprunt a affecté les prix de marché des autres titres de l'I.M.I. pendant les deux mois où aucune cotation officielle n'a été faite sur le nouvel emprunt. Aussi, bien que la date d'observation la plus proche de la mi-mai 1963 ait été le 7 mai, la date retenue pour le test de stabilité a été celle du 3 juin : tout en étant l'une des dates d'observations, cette date est aussi celle où les bons du nouvel emprunt ont commencé à être cotés :

$$\Pi^{6\,\%}_{jd} = a_d + \gamma_d\, \sigma^2_{jd} + h_d\, sd_{jd}$$

Numéro de l'observation	1	8	14	25	36
Date	11-5-59	19-5-60	19-5-61	11-5-62	3-6-63
a_d	2,36* (0,446)	0,88* (0,1181)	1,55* (0,271)	3,82* (0,263)	1,81 (0,635)
γ_d	0,78* (0,112)	0,59* (0,023)	0,57* (0,045)	0,58* (0,052)	0,47* (0,063)
h_d	−0,01 (0,480)	0,20 (0,158)	−0,08 (0,524)	0,33 (0,550)	0,63 (0,661)
R^2	0,93	0,99	0,98	0,98	0,98
S.C.R	1,3270	0,1217	0,7592	0,8415	0,7418
Nombre d'observations à la date d.	7	7	7	7	5

Les coefficients estimés qui sont signicatifs sont marqués par une astérisque.

Pour tester la stabilité du prix de marché du risque sur des intervalles annuels (c'est-à-dire en comparant les dates deux à deux, on a choisi pour hypothèse nulle :

$$\Pi_{jd}^{0}\% = a_d + \gamma\,\sigma_{jd}^2 + h_d\,sd_{jd}$$

contre l'hypothèse alternative précédente. Si l'on introduit alors des variables indicatrices :

$$t\,d_1 = \begin{cases} 1 \text{ si l'observation a été faite à la première date} \\ 0 \text{ si l'observation a été faite à la deuxième date} \end{cases}$$

$$t\,d_2 = 1 - t\,d_1$$

on obtient les régressions :

$$\Pi_{jd}^{2}\% = \gamma\,\sigma_{jd}^2 + a_1 t\,d_1 + a_2 t\,d_2 + a_3 t\,d_1 sd_{jd} + a_4 t\,d_2 sd_{jd}$$

Numéros des observations	1 et 8	8 et 14	14 et 25	25 et 36
$\hat{\gamma}$	0,65 (0,053)	0,58 (0,026)	0,57 (0,032)	0,52 (0,042)
$\hat{a}_1$	2,76 (0,291)	0,92 (0,171)	1,54 (0,226)	4,04 (0,269)
$\hat{a}_2$	0,65 (0,297)	1,52 (0.174)	3,84 (0,207)	1.39 (0,473)
$\hat{a}_3$	0,21 (0,381)	0,23 (0,272)	—0,09 0,485)	0,61 (0,623)
$\hat{a}_4$	0,04 (0,428)	—0,11 (0,365)	0.34 (0,479)	0.34 (0,554)
R^2	0,96	0,99	0,98	0,97
S.C.R.	2,1070	0,8921	1,6030	2,1060
Nombre d'observations	14	14	14	12
Statistique F	$F_{1,8} = 3,64$	$F_{1,8} = 0,10$	$F_{1,8} = 0,01$	$F_{1,6} = 1,98$

Étant donné que :

$$F_{1,8}\,(0,05) = 5,32 \qquad F_{1,6}\,(0,05) = 5,99$$

on conclut que le prix de marché du risque est resté stable au cours de chaque période d'une année.

Un test analogue de stabilité portant sur l'ensemble des cinq années de la période étudiée a donné une somme de carrés des résidus valant 5,9982 dans l'hypothèse nulle et 3,7912 alternative (pour 33 observations). La statistique F correspondante est donc $F_{4,18} = 2{,}62$, qu'il faut comparer à la valeur théorique $F_{4,18}\ (0{,}05) = 2{,}93$. On en déduit que le prix de marché du risque est resté stable sur cette période de cinq ans.

L'ensemble des tests effectués sur cette période 1958-1963 permet donc de conclure à la stabilité du prix du risque, aussi bien à court terme (un an) qu'à moyen terme (cinq ans).

Il apparaît d'après (20), (20.1), (20.2) et (20.3) que les coefficients de $\Pi = a + \gamma\,\sigma^2 + h\,sd$ ne sont pas homogènes pour les différents coupons, toutes observations réunies, la statistique F correspondante étant alors :

$$F_{4,352} = 103{,}29$$

par conséquent, lorsque l'on étudie l'ensemble des prix d'équilibre pour tous les bons de l'I.M.I. négociés sur le marché à chaque date d'observation, le coupon devrait être utilisé pour distinguer les différents bons. Le nombre de bons de l'I.M.I. sur le marché à une date quelconque variant de 8 à 10, il ne resterait plus qu'un seul titre sur le marché lorsque des bons à 5,5 % sont négociés. Cela interdit d'utiliser des variables indicatrices de coupon comme dans (20). Alternativement, on peut considérer le coupon à 6 % comme le coupon normal pour les bons émis par l'I.M.I. (puisque les coupons à 6 % ont toujours constitué la majorité des titres échangés sur le marché) et introduire comme variable explicative la différence 6 — *r* entre le coupon normal et le coupon *r* du bon considéré.

Cette suggestion n'implique pas que le coupon ait un effet direct sur la prime de risque. Mais il peut sembler raisonnable, étant donné le faible nombre d'observations pour chaque date, de repérer les bons par leurs coupons afin de tenir compte de l'utilisation des courbes de rendement B.T.P. La variable indicatrice des ventes n'a pas été utilisée dans les régressions journalières parce que cette variable n'est pas significative pour l'une quelconque des 5 dates sélectionnées dans l'étude des bons à 6 %. Il serait sans doute trop grossier de repérer les efforts journaliers de vente par l'I.M.I. pour les bons des différents emprunts, et il ne semble pas qu'un gain possible en explication compense la perte d'un degré de liberté.

Le tableau 1 donne les régressions journalières pour les marchés à l'équilibre. Les prix du risque estimés sont tous significativement différents de zéro. Les coefficients de détermination sont élevés, ce qui fournit une indication supplémentaire de la signification économique de modèle d'équilibre espérance-variance. Toutefois, comme la variable coupon a été utilisée pour distinguer entre les bons, ces régressions ne constituent pas à proprement parler un test du modèle espérance-variance.

Ces régressions du tableau 1 devraient être de quelque intérêt pour l'I.M.I. dans sa politique d'émission. Compte tenu des courbes de rendement pour les bons B.T.P., l'I.M.I. peut obtenir certaines indications concernant le prix auquel il est possible de placer une nouvelle émission de bons annuels compte tenu du coupon et de l'échéance de ces bons. l'I.M.I. pourrait avoir intérêt à mettre au point une variable indépendante qui tienne compte de ses efforts journaliers de vente, à l'aide de laquelle il pourrait alors faire des prévisions très précises.

Tableau 1

$\Pi = a + \gamma \sigma^2 + h(6 - r)$

Date d'observation	$\hat{a}$	Écart-type estimé de $\hat{a}$	$\hat{\gamma}$	Écart-type estimé de $\hat{\gamma}$	$\hat{h}$	Écart-type estimé de $\hat{h}$	R^2	Nombre d'observations	Courbes de rendement des BTP $\hat{b}_1$	$\hat{b}_2$
1. 11 mai 1959	2,3580*	0,3439	0,7791*	0,0805	— 1,4784*	0,5885	0,97	9	5,04	— 1,75
2. 22 mai 1959	1,9555*	0,3487	0,7605*	0,0755	— 1,0386	0,5972	0,97	9	5,01	— 1,46
3. 10 juin 1959	1,1208*	0,3855	0,6521*	0,0861	— 1,1019	0,6603	0,95	9	5,04	— 1,13
4. 17 juin 1959	0,9361*	0,3413	0,6701*	0,0701	— 0,9569	0,5851	0,96	9	5,00	— 1,20
5. 28 septembre 1959	0,9054	0,6266	0,8957*	0,2045	0,2341	0,8795	0,80	10	5,21	— 0,78
6. 3 mai 1960	0,0571	0,6103	0,6579*	0,1109	1,7801	0,9258	0,84	10	4,93	— 1,20
7. 10 mai 1960	0,5261	0,5074	0,5975*	0,0932	1,6341	0,7697	0,86	10	4,94	— 1,26
8. 19 mai 1960	0,6748	0,4747	0,6318*	0,0880	1,6489	0,7202	0,89	10	4,93	— 1,70
9. 27 mai 1960	0,7548	0,5119	0,6065*	0,0842	1,6944	0,7766	0,89	10	4,87	— 1,75
10. 20 septembre 1960	0,2289	0,3369	0,5807*	0,0574	1,8765*	0,5123	0,94	10	4,91	— 1,37
11. 23 septembre 1960	0,0091	0,4880	0,6354*	0,0873	1,9022*	0,7426	0,89	10	4,94	— 1,33
12. 26 septembre 1960	— 0,2928	0,4389	0,6708*	0,0844	2,1842*	0,6691	0,90	10	4,97	— 1,29
13. 5 mai 1961	1,0022*	0,3869	0,5643*	0,0544	3,8101*	0,7296	0,95	8	4,74	— 2,09
14. 19 mai 1961	1,5501*	0,2415	0,5676*	0,0360	3,5772*	0,4561	0,98	8	4,77	— 1,97
15. 24 mai 1961	1,3736*	0,1388	0,5880*	0,0226	3,7925*	0,2626	0,99	8	4,82	— 1,95
16. 25 mai 1961	1,3484*	0,1470	0,5881*	0,0245	3,6373*	0,2784	0,99	8	4,83	— 1,98
17. 26 mai 1961	1,3409*	0,1489	0,5826*	0,0244	3,6907*	0,2818	0,99	8	4,82	— 1,93
18. 21 juin 1961	1,7940*	0,1862	0,5690*	0,0313	3,6840*	0,3529	0,98	8	4,83	— 2,37
19. 4 septembre 1961	1,5159*	0,1228	0,5121*	0,0157	4,3060*	0,2314	0,99	8	4,71	— 1,78
20. 13 septembre 1961	1,6597*	0,1932	0,5013*	0,0241	4,5025*	0,3641	0,98	8	4,70	— 1,56
21. 18 septembre 1961	1,5813*	0,3057	0,4994*	0,0373	4,3506*	0,5763	0,97	8	4,68	— 1,59
22. 25 septembre 1961	1,6439*	0,2197	0,4928*	0,0289	4,7294*	0,3314	0,98	9	4,72	— 1,77
23. 28 septembre 1961	1,3037	0,6067	0,4815*	0,0750	5,0735*	0,8469	0,89	9	4,72	— 1,70
24. 29 septembre 1961	0,8000	0,5352	0,5064*	0,0626	5,6588*	0,7673	0,92	9	4,68	— 1,58
25. 11 mai 1962	3,8180*	0,3312	0,5889*	0,0588	4,2783*	0,5229	0,95	9	4,78	— 2,90
26. 4 juin 1962	2,3278*	0,1857	0,6080*	0,0339	5,2295*	0,2931	0,98	9	4,84	— 1,11
27. 20 juin 1962	1,4170*	0,2614	0,7284*	0,0579	5,3661*	0,4144	0,97	9	4,96	— 0,11
28. 22 juin 1962	1,3921*	0,2189	0,6847*	0,0486	5,2078*	0,3470	0,97	9	4,96	— 0,08
29. 27 juin 1962	1,1427*	0,2427	0,7510*	0,0590	5,5578*	0,3847	0,97	9	5,01	— 0,06
30. 6 septembre 1962	1,7521*	0,2572	0,5738*	0,0390	7,4644*	0,3570	0,98	10	4,76	— 0,66
31. 14 septembre 1962	1,6122*	0,2428	0,5993*	0,0402	7,9210*	0,3394	0,98	10	4,82	— 0,47
32. 18 septembre 1962	1,3841*	0,2032	0,6324*	0,0403	7,4060*	0,2871	0,98	10	4,91	— 0,27
33. 19 septembre 1962	1,3522*	0,1750	0,6379*	0,0354	7,2423*	0,2475	0,99	10	4,92	— 0,27
34. 7 mai 1963	2,0560	0,8047	0,4371*	0,0639	7,9165*	0,7789	0,95	8	4,49	— 0,10
35. 29 mai 1963	2,0645	0,8503	0,4700*	0,0752	8,3182*	0,8399	0,95	8	4,56	— 0,04
36. 3 juin 1963	1,5295	0,6328	0,5208*	0,0515	8,2031*	0,6523	0,96	9	4,59	0,10
37. 11 juin 1963	1,6035	0,6780	0,4988*	0,0495	8,2966*	0,6865	0,96	9	4,52	0,06
38. 3 septembre 1963	1,5679	0,8792	0,4909*	0,0605	9,5589*	0,8842	0,95	9	4,50	0,04

4 Critères d'investissement

Les résultats empiriques ci-dessus devraient être utiles pour les entreprises en ce qui concerne leurs décisions d'investissement. Étant donné le rendement des bons d'État et ce prix de marché du risque, une entreprise peut se faire une idée approximative du prix auquel elle sera à même de vendre de nouveaux titres sur le marché. Elle peut aussi déterminer les effets d'investissements nouveaux sur le prix de marché de ses titres émis antérieurement.

A titre d'exemple simple, on peut considérer une entreprise qui lance un programme d'investissement étalé sur deux ans, après quoi l'entreprise cesse toute activité. La distribution de probabilité du flux de recettes résultant de cet investissement est :

Flux de recettes	Année		Probabilité du flux de recettes
	1	2	
1	102	100	1/4
2	100	100	1/2
3	98	100	1/4

Le rendement des titres d'état est supposé tel qu'un titre au nominal d'1 L à échéance d'une année se négocie à 0,9 L, tandis qu'un titre d'1 L à échéance de deux ans se négocie à 0,8 L. On en déduit que, pour l'investissement considéré ci-dessus, l'espérance de la valeur actualisée est $\mu = 170$ et la variance $\sigma^2 = 1{,}62$.

Si l'entreprise décide de financer ses opérations en vendant des actions sur le marché, la valeur totale de ses titres sur le marché sera (en supposant que les covariances avec les flux de recette des autres titres sur le marché sont nulles et que $\gamma = 1/2$) :

$$P_E = \mu - \gamma \sigma^2 = 169{,}19$$

Alternativement, l'entreprise peut envisager la possibilité de vendre un bon et de financer le solde par la vente d'une action. Le bon consiste en une promesse de payer 100 L au bout d'un an, avec des garanties suffisantes pour que l'on puisse considérer qu'un tel paiement sera effectué avec certitude.

Ce bon constitue donc un investissement sûr et l'entreprise pourra le vendre pour 100 × 0,9 = 90 L. Quant aux actions, leur distribution de probabilité est alors :

Flux de recettes	Année		Probabilité du flux de recettes
	1	2	
1	2	100	1/4
2	0	100	1/2
3	– 2	100	1/4

L'espérance de la valeur actualisée de cette action est $\mu_E = 80$ et sa variance $\sigma_E^2 = 1{,}62$, de sorte que le prix de marché de cette action sera de 79,19; la valeur totale sur le marché des titres de cette entreprise sera

$$90 + 79{,}19 = 169{,}19.$$

On voit ainsi que la valeur totale sur le marché des titres de l'entreprise ne dépend pas de la façon dont l'investissement est financé, ce qui confirme le théorème de MODIGLIANI-MILLER [14]. MOSSIN [17] a démontré que cette proposition est valide pour les modèles d'équilibre espérance-variance; il a également montré que le risque de non-paiement ne modifie pas la valeur totale de l'entreprise sur le marché.

Dans le cas de risque de non-paiement (ce qui revient à dire qu'il n'existe pas de garanties suffisantes), la distribution de probabilité pour le bon de 100 L est :

Flux de recettes	Année 1	Probabilité
1	100	1/4
2	100	1/2
3	98	1/4

$\mu_B = 89{,}55$;
$\sigma_B^2 = 0{,}6075$;

et la distribution de probabilité pour l'action :

Flux de recettes	Année		Probabilité
	1	2	
1	2	100	1/4
2	0	100	1/2
3	0	100	1/4

$\mu_E = 80{,}45;$
$\sigma_E^2 = 0{,}6075;$

comme le bon ne peut plus maintenant être considéré comme un investissement sûr, il y a une covariance non nulle $\sigma_{BE} = 0{,}2025$ entre les distributions de probabilité des valeurs actualisées du bon et de l'action. La valeur de marché du bon sera alors $\mu_B - \gamma\ (\sigma_B^2 + \sigma_{BE}) = 89{,}145$ et celle de l'action sera $\mu_E - \gamma(\sigma_E^2 + \sigma_{BE}) = 80{,}045$, de sorte que la valeur totale des titres de l'entreprise sur le marché sera 169,19, ce qui est la même valeur que pour les deux autres modes de financement. Le fait que l'entreprise n'offre pas de garantie pour le bon a abaissé de 90 L à 89,145 L le prix auquel le marché accepte ce bon. En ce qui concerne l'entreprise, cela peut s'interpréter de la façon suivante : lorsque ce paiement du bon est garanti, le taux d'intérêt sur ce bon est de $\frac{100}{90} - 1 \neq 0{,}11$, ou 11 % par an; sinon, il est de $\frac{100}{89{,}145} - 1 \neq 0{,}12$, ou 12 % par an. Ainsi, l'existence d'une garantie de paiement sur ce bon implique un abaissement de 1 % sur le taux d'intérêt auquel l'entreprise peut emprunter.

Ces exemples montrent que c'est en général la distribution de probabilité des flux futurs de recettes que l'entreprise peut engendrer qui détermine sa valeur totale sur le marché. Les valeurs sur le marché des titres de l'entreprise dépendent à leur tour des règles de partage concernant les flux totaux de recettes. Mais ces règles ne changent pas la valeur totale de ces titres sur le marché.

Pour voir qu'il s'agit là d'une propriété générale du modèle d'équilibre espérance-variance, on peut écrire le prix de marché du titre l émis par l'entreprise j :

(21) $$P_{jl} = \mu_{jl} - \gamma\left(\sum_{\substack{k \\ (k \neq j)}} {}_j\sigma_{lk} + \sum_{0} {}_j\sigma_{l_0}\right)$$

où $\sum_k {}_j\sigma_{lk}$ est la somme des covariances des titres de l'entreprise j avec les autres entreprises ($k \neq j$) et $\sum_0 {}_j\sigma_{l_0}$ est la somme des variance et covariances du titre l avec les autres titres émis par l'entreprise j. En sommant (21) pour tous les titres l, et en utilisant (16), (17), (18), il vient :

$$\sum_{l} p_{jl} = p_j$$

Cela signifie que, pour une distribution donnée des flux de recettes d'une entreprise, la valeur totale de cette entreprise sur le marché ne dépend pas de la structure de son capital. Il est alors intéressant de se demander si différentes structures de capital permettent à une entreprise de choisir entre différents ensembles d'activités économiques ou entre différents investissements. Lorsque c'est le cas, l'entreprise devrait choisir, parmi les possibilités qui lui sont ouvertes, les investissements et la structure de capital qui lui permettent de maximiser la valeur actualisée totale de ses titres sur le marché. C'est là un objectif qui se rencontre communément et qui correspond à la maximisation de la richesse actualisée dans le cas certain [6].

Conclusion

On a étudié dans cet article la formation des prix de marché pour des contrats portant sur des paiements futurs aléatoires. Dans le cas certain, il est bien connu que les investissements doivent être choisis de manière à maximiser la valeur actualisée de la richesse. En incertitude, le modèle d'équilibre espérance-variance fournit une hypothèse testable sur les prix observés des titres, pourvu qu'il existe un marché pour les droits certains.

On a testé ce modèle espérance-variance en utilisant les données du marché italien des obligations. Le modèle s'en trouve confirmé. L'estimation du prix du risque est voisine de 0,5 L par unité de variance; elle est stable sur un intervalle de cinq ans, et aussi sur des sous-intervalles d'une année.

Pour une entreprise qui assure son financement en vendant des titres sur le marché, on a montré à l'aide d'exemples comment elle peut évaluer les prix de marché de ses titres lorsqu'elle connaît le rendement des bons d'État et le prix de marché du risque.

6. Voir Diamond [5] et Sandmo [20], par exemple.

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Summary

The market price of risk

by **Cornelius Schilbred**

This article reports an empirical study of the market price of risk. The securities used are bonds issued by Istituto Mobiliare Italiano (I.M.I.) in Italy and also Italian Government bonds.

I.M.I. borrows money through the Italian securities market by issuing bonds. The repayment of its loans is then arranged by series of lotteries, in order to determine which bonds are to be redeemed at each particular installment date. As the amounts to be paid in installments are fixed in advance in the loan agreements, I.M.I.'s bonds may be considered to have their probability distributions of future payments to the bond holders given.

In order to test the hypothesis of a market price of risk, the article sets out by specifying an atemporal mean-variance equilibrium model. It is assumed that the relevant certainty alternative to investment in I.M.I. bonds, is investment in bonds issued by the Italian Government, so that their yield may be used to take care of the fact that payments fall due at different times in the various I.M.I. bond issues.

A random selection of 40 trading days during the years 1958-1963 was then made. The yield curves for Italian Government bonds on these dates were estimated. Then the probabilities associated with the outstanding I.M.I. bonds were used to compute their means and variances of present values.

With the computed means and variances, various tests of the equilibrium price hypothesis was carried out. The tests tend to confirm that there exists a market price of risk. The price of risk appears to be close to £ 0.5 per unit of variance. This price was found to be stable over the five year period, and also over sub-intervals of one year.

At the end, the article shows through simple examples how the mean-variance equilibrium model may be applied in investment and financing decisions. It also points to some general characteristics of such models.

Reseña

El precio del riesgo

por **Cornelius Schilbred**

Este artículo relata los resultados de un estudio aplicado relativo al precio del mercado del riesgo. Se utilizaron, para dicho estudio, las obligaciones emitidas en Italia por el Istituto mobiliare italiano (I.M.I.) así como las obligaciones del propio Estado italiano.

El I.M.I. pide prestado dinero en el mercado financiero italiano emitiendo obligaciones cuyo reembolso se efectúa mediante loterías que determinan cuales son las obligaciones que van a ser reembolsadas a cada vencimiento. Como los importes de estos reembolsos se determinan por antelación según modalidades del empréstito, puede considerarse que los bonos del I.M.I. tienen, por lo que se refiere a futuros pagos, una distribución de probabilidad, la cual constituye una información para los portadores de dichos bonos.

Con objeto de comprobar la hipótesis según la cual existe un precio del mercado del riesgo, el artículo comienza por la especificación de un modelo atemporal de equilibrio esperanza-variación. Se supone que el equivalente cierto que conviene considerar como una alternativa de las obligaciones del I.M.I. está constituído por las obligaciones emitidas por el Estado italiano, de manera que el tipo de estas últimas puede utilizarse para tener en cuenta el hecho de que los pagos relativos a diversos empréstitos del I.M.I. no son simutáneos.

Se ha elegido entonces, a ventura, cuarenta días hábiles para el período de 1958 a 1963 y se ha estimado la curva de rendimiento para los bonos del Estado italiano en cada una de estas fechas. Luego, se utilizaron las probabilidades asociadas con las obligaciones del I.M.I. en circulación para calcular las esperanzas y las variancias de sus valores actualizados.

Con la ayuda de estas esperanzas y variancias calculadas de este modo, pudieron llevarse a cabo diferentes pruebas de la hipótesis según la cual existe un precio de equilibrio del riesgo. Las pruebas tienden a confirmar la existencia de semejante precio del riesgo, el cual al parecer es de 0,5 lira aproximadamente por unidad de variancia. Apareció que este precio del riesgo era estable para el conjunto del período, o sea un quinquenio, así como para sub-intervalos de un año.

Por último, se demostró mediante ejemplos simples en qué forma puede utilizarse este modelo de equilibrio esperanza-variancia en las decisiones relativas a las inversiones y a su financiamiento, a la par que se subrayaba algunas características generales de dichos modelos.

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