

# Do Koopmans' postulates lead to discounted utilitarianism?\*

Geir B. Asheim<sup>†</sup> and Bertil Tungodden<sup>‡</sup>

December 9, 2004

Discussion Paper 32/04

## Abstract

In this paper we consider variations of Koopmans' (1960) postulates and demonstrate that these lead to a class of social preferences that is wider than discounted utilitarianism. We formulate a utilitarian condition (**PFL**), and introduce a one-sided equity condition (**HEF**) stating that a sacrifice of the present generation leading to an equal gain for all future generations is weakly desirable if the present remains better off than the future. We investigate the consequences of imposing **HEF**, and obtain a new axiomatization of discounted utilitarianism by assuming that **PFL** holds and **HEF** does not hold.

**Keywords and Phrases:** Intergenerational justice, Discounted utilitarianism, Sustainability.

**JEL Classification Numbers:** D63, D71, Q01.

---

\*We are grateful for many helpful discussions with Wolfgang Buchholz on the topic of this paper, comments by Walter Bossert, Graciela Chichilnisky, Larry Karp, Michael Mandler, Marco Mariotti, Tapan Mitra, Tomoichi Shinotsuka, Kotaro Suzumura, and seminar participants in Tokyo, Ascona, Osaka and Madrid, and financial support from the Research Council of Norway (Ruhrgas grant).

<sup>†</sup>Department of Economics, University of Oslo, P.O. Box 1095 Blindern, NO-0317 Oslo, Norway (e-mail: g.b.asheim@econ.uio.no) Corresponding author.

<sup>‡</sup>Norwegian School of Economics and Business Administration, Helleveien 30, NO-5045 Bergen, Norway (e-mail: bertil.tungodden@nhh.no)

# 1 Introduction

This paper considers conditions for social preferences over infinite utility streams and explores a middle ground between discounted utilitarianism and maximin. Maximin makes no use of interpersonal unit comparability (even if utilities are unit comparable), while discounted utilitarianism makes no use of interpersonal level comparability (even if utilities are level comparable). One can, however, argue that intuitive notions of justice, also in the case of social preferences over infinite intergenerational utility streams, make non-trivial use of both unit comparability and level comparability.

The framework in the main parts of this paper (Sections 2–5) follows the approach of Koopmans (1960)<sup>1</sup> by assuming completeness, transitivity, and continuity (in the sup topology), entailing numerical representability. However, we open up for a class of social preferences that is considerably wider than the class of discounted utilitarian social preferences. In particular, we adapt variations of Koopmans’ Postulates 1–5, while not considering his additional Postulate 3’. It is only by means of the latter postulate—which in the words of Heal (2001) is “restrictive” and “surely not innocent”—that Koopmans moves beyond a recursive form (also obtained in Proposition 1 below) to arrive at discounted utilitarianism.

We show that by introducing a one-sided equity condition (“Hammond Equity for the future”), stating that a sacrifice of the present generation leading to an equal gain for all future generations is weakly desirable if the present remains better off than the future, it is possible to make use of interpersonal level comparability of (at least) ordinally measurable utility within the recursive form, leading to a sustainability constraint. On the other hand, by making use of interpersonal unit comparability of (at least) cardinally measurable utility through a utilitarian condition (“Present-future linearity”) and imposing that “Hammond Equity for the future” does not

---

<sup>1</sup>See also Koopmans (1986a,b).

hold, we obtain a new characterization of discounted utilitarianism. We use these results to shed light on Chichilnisky’s (1996) ‘sustainable’ social preferences.

In the context of the Dasgupta-Heal-Solow model of capital accumulation and resource depletion, it has been argued that both discounted utilitarianism and maximin lead to problematic outcomes: Discounted utilitarianism undermines the livelihood of generations in the distant future (Dasgupta and Heal, 1974, 1979), while maximin may perpetuate poverty (Solow, 1974). We indicate how a middle ground between discounted utilitarianism and maximin—by accepting trade-off between the present and the future if and only if the present is worse off than the future—yields interesting, and possibly appealing outcomes, in the Dasgupta-Heal-Solow model.

Koopmans (1960) has often been interpreted as presenting the definite case for discounted utilitarianism. Sections 2–5 of this paper seek to weaken this impression, by exploring other avenues within the general setting of his approach.

In the final Section 6 we leave Koopman’s framework and explore the consequences of relaxing continuity. It is well-known (see, e.g., Diamond, 1965; Basu and Mitra, 2003a; Suzumura and Shinotsuka, 2003; Bossert et al., 2004) that continuity (and even an assumption of numerical representability) is not without normative significance when combined with sensitivity conditions. In line with these contributions, we note that our condition “Hammond Equity for the future” cannot be combined with both continuity and Strong Pareto. Building on Asheim and Tunngodden (2004), Basu and Mitra (2003b), and Bossert et al. (2004), we show that, if we drop continuity, there exist social preferences that satisfy our remaining conditions, including Strong Pareto and “Hammond Equity for the future”. However, if we drop “Hammond Equity for the future” while keeping continuity, Strong Pareto, and the other conditions we consider, we obtain an alternative characterization of discounted utilitarianism.

## 2 Basic characterization result

Consider a discrete time setting, with an infinite but countable number of generations, where the instantaneous well-being of generation  $t$  is represented by utility  $u_t$  that can take on values in the unit interval  $[0, 1]$ .<sup>2</sup> Assume that, at each time  $t$ , there are social preferences  $\succeq_t$  over the utility streams  ${}_t\mathbf{u} = (u_t, u_{t+1}, \dots)$  in  $[0, 1]^\infty$  starting at time  $t$ , where  $\infty = |\mathbb{N}|$  and  $\mathbb{N}$  denotes the set of natural numbers.

Throughout this paper we assume at least ordinally measurable level comparable utilities; i.e. what Blackorby et al. (1984) refer to as “level-plus comparability”. Write  ${}_{\text{con}}w := (w, w, \dots)$  for a stream with a constant level of utility equal to  $w \in [0, 1]$ .

Consider the following conditions on  $\succeq_t$ .

**Condition O** (*Order*) For each  $t \geq 1$ ,  $\succeq_t$  is complete and transitive.

**Condition C** (*Continuity*) For each  $t \geq 1$ , if  $\lim_{n \rightarrow \infty} \sup_{s \geq t} |u_s^n - u_s'| = 0$  and, for all  $n$ ,  ${}_t\mathbf{u}^n \succeq_t {}_t\mathbf{u}''$  (resp.  ${}_t\mathbf{u}'' \succeq_t {}_t\mathbf{u}^n$ ), then  ${}_t\mathbf{u}' \succeq_t {}_t\mathbf{u}''$  (resp.  ${}_t\mathbf{u}'' \succeq_t {}_t\mathbf{u}'$ ).

**Condition TI** (*Time invariance*) For each  $t \geq 1$ ,  ${}_t\mathbf{u}' \succeq_t {}_t\mathbf{u}''$  if and only if  ${}_1\mathbf{v}' \succeq_1 {}_1\mathbf{v}''$ , where, for each  $s \geq 1$ ,  $v_s' = u_{t+s-1}'$  and  $v_s'' = u_{t+s-1}''$ .

**Condition S** (*Sensitivity*) For each  $t \geq 1$ , if for all  $s \geq t$ ,  $u_s' > u_s''$ , and there exists  $T \geq t$  such that for all  $s \geq T$ ,  $u_s' = w'$  and  $u_s'' = w''$ , then  ${}_t\mathbf{u}' \succ_t {}_t\mathbf{u}''$ .

**Condition IF** (*Independent future*) For each  $t \geq 1$ ,  $(u_t, {}_{t+1}\mathbf{u}')$   $\succeq_t (u_t, {}_{t+1}\mathbf{u}'')$  if and only if  ${}_{t+1}\mathbf{u}' \succeq_{t+1} {}_{t+1}\mathbf{u}''$ .

---

<sup>2</sup>A more general formulation is, as used by Koopmans (1960), to assume that the well-being of generation  $t$  depends on a  $n$ -dimensional vector  $\mathbf{c}_t$  that takes on values in a connected set  $C$ . However, by representing the well-being of generation  $t$  by a scalar  $u_t$ , we can do without his Postulate 3(a) and focus on intergenerational issues. In doing so, we follow, e.g., Chichilnisky (1996), Suzumura and Shinotsuka (2003) and Bossert et al. (2004).

**Condition EP** (*Extreme streams*) For each  $t \geq 1$  and for all  ${}_t\mathbf{u} \in [0, 1]^\infty$ ,

$${}_{\text{con}}0 \preceq_t {}_t\mathbf{u} \preceq_t {}_{\text{con}}1.$$

Conditions **O**, **C** and **TI** correspond to Koopmans' (1960) Postulate 1, condition **S** substitutes a weak form of Weak Pareto (comparing only streams where the tails have constant well-being) for his Postulate 2, condition **IF** is equivalent to his Postulates 3b and 4, while condition **EP** coincides with his Postulate 5.

If **TI** is invoked, then  $\succeq_t$  is independent of  $t$ , and we may write  $\succeq$  for the common preferences (i.e., for each  $t$ ,  ${}_t\mathbf{u}' \succeq_t {}_t\mathbf{u}''$  if and only if  ${}_t\mathbf{u}' \succeq {}_t\mathbf{u}''$ ). Provided that **TI** is satisfied, say that a social welfare function  $W : [0, 1]^\infty \rightarrow [0, 1]$  represents the social preferences,  $\succeq$ , if the following two statements are equivalent for each  $t$ :

- (1)  ${}_t\mathbf{u}' \succeq {}_t\mathbf{u}''$ .
- (2)  $W({}_t\mathbf{u}') \geq W({}_t\mathbf{u}'')$ .

**Proposition 1** Assume that **O**, **C**, **TI**, **S**, **IF**, and **EP** hold. Then the social preferences,  $\succeq$ , are represented by a social welfare function  $W : [0, 1]^\infty \rightarrow [0, 1]$  satisfying

- (a) if  $\lim_{n \rightarrow \infty} \sup_{s \geq t} |u_s^n - u_s'| = 0$  and, for all  $n$ ,  $W({}_t\mathbf{u}^n) \geq W({}_t\mathbf{u}'')$  (resp.  $W({}_t\mathbf{u}'') \geq W({}_t\mathbf{u}^n)$ ), then  $W({}_t\mathbf{u}') \geq W({}_t\mathbf{u}'')$  (resp.  $W({}_t\mathbf{u}'') \geq W({}_t\mathbf{u}')$ ), and
- (b) for all  ${}_t\mathbf{u} \in [0, 1]^\infty$ ,  ${}_{\text{con}}W({}_t\mathbf{u}) \sim_t {}_t\mathbf{u}$  and  $W({}_t\mathbf{u}) = \Upsilon(u_t, W({}_{t+1}\mathbf{u}))$ , where  $\Upsilon(u, w)$  is continuous, non-decreasing in  $u$ , and increasing in  $w$ .

**Proof.** By **TI**, we may let the streams start at time 1. Consider any  ${}_1\mathbf{u} \in [0, 1]^\infty$ . By **O**, **C**, **S**, and **EP** there is a unique  $w \in [0, 1]$  such that  ${}_{\text{con}}w \sim {}_1\mathbf{u}$ ; i.e.,  ${}_{\text{con}}w$  is the stationary equivalent of  ${}_1\mathbf{u}$ . Let  $W({}_1\mathbf{u}) = w$ . It follows that  $W : [0, 1]^\infty \rightarrow [0, 1]$  represents  $\succeq$  and is continuous in the sup topology.

It now follows from **IF** that, for any  $u_1 \in [0, 1]$ , there exists a positive monotone transformation  $\Upsilon(u_1, \cdot)$  such that, for all  ${}_2\mathbf{u}$ ,  $W(u_1, {}_2\mathbf{u}) = \Upsilon(u_1, W({}_2\mathbf{u}))$ . This determines  $\Upsilon$ , where  $\Upsilon(u, w)$  is increasing in  $w$ . Since  $W : [0, 1]^\infty \rightarrow [0, 1]$  is continuous in the sup topology, we have that  $\Upsilon$  is continuous.

Suppose  $u'_1 > u''_1$  and  $\Upsilon(u'_1, W(2\mathbf{u})) < \Upsilon(u''_1, W(2\mathbf{u}))$ . Write  $W(2\mathbf{u}) = w''$  and keep in mind that  $2\mathbf{u} \sim_{\text{con}} w''$ . By continuity, there exists  $w' > w''$  such that

$$\Upsilon(u'_1, W(2\mathbf{u})) < \Upsilon(u'_1, W(\text{con}w')) < \Upsilon(u''_1, W(2\mathbf{u})) = \Upsilon(u''_1, W(\text{con}w'')),$$

contradicting **S**. Hence  $\Upsilon(u, w)$  is non-decreasing in  $u$ . ■

### 3 Hammond Equity for the future

Consider a stream  $(u, \text{con}w)$  having the property that well-being is constant from the second period on. For such a stream we may unequivocally say that, if  $u < w$ , then the present is worse off than the future. Likewise, if  $u > w$ , then the present is better off than the future. This lays a foundation for introducing the following condition.

**Condition HEF** (*Hammond Equity for the future*) For each  $t \geq 1$ ,  $u'' > u' > w' > w''$  implies  $(u', \text{con}w') \succeq_t (u'', \text{con}w'')$ .

For streams where well-being is constant from the second period on, Condition **HEF** states the following: If the present is better off than the future and a sacrifice now leads to an equal gain for all future generations, then such a transfer from the present to the future is weakly desirable in social evaluation, as long as the present remains better off than the future. To appreciate the weakness of condition **HEF**, consider first the standard Hammond Equity condition (Hammond, 1976) and a weak version of Lauwers' (1998) non-substitution condition.

**Condition HE** (*Hammond Equity*) For each  $t \geq 1$ ,  ${}_t\mathbf{u}' \succeq_t {}_t\mathbf{u}''$  whenever  ${}_t\mathbf{u}'$  and  ${}_t\mathbf{u}''$  satisfy that there exists a pair  $q, r$  such that  $u''_q > u'_q > u'_r > u''_r$  and  $u'_s = u''_s$  for all  $s \geq t$  with  $s \neq q, r$ .

**Condition WNS** (*Weak non-substitution*) For each  $t \geq 1$ ,  $w' > w''$  implies  $(u', \text{con}w') \succeq_t (u'', \text{con}w'')$ .

By assuming, in addition, that utilities are at least cardinally measurable and fully comparable, we may also consider weak versions of the Pigou-Dalton and Lorenz domination principles.

**Condition WPD** (*Weak Pigou-Dalton*) For each  $t \geq 1$ ,  ${}_t\mathbf{u}' \succeq_t {}_t\mathbf{u}''$  whenever  ${}_t\mathbf{u}'$  and  ${}_t\mathbf{u}''$  satisfy that there exist a positive number  $\varepsilon$  and a pair  $q, r$  such that  $u''_q - \varepsilon = u'_q \geq u'_r = u''_r + \varepsilon$  and  $u'_s = u''_s$  for all  $s \geq t$  with  $s \neq q, r$ .

**Condition WLD** (*Weak Lorenz domination*) For each  $t \geq 1$ ,  ${}_t\mathbf{u}' \succeq_t {}_t\mathbf{u}''$  whenever  ${}_t\mathbf{u}'$  and  ${}_t\mathbf{u}''$  satisfy that there exists  $T > t$  such that  ${}_t\mathbf{u}'_T$  Lorenz dominates  ${}_t\mathbf{u}''_T$  and  ${}_{T+1}\mathbf{u}' = {}_{T+1}\mathbf{u}''$ .<sup>3</sup>

While it is clear that condition **HEF** is implied by **WNS**—as **HEF** in contrast to **WNS** does not preclude that a finite improvement for the first generation can compensate for a uniform loss for all future generations, provided that the present is worse off than the future—it is perhaps less obvious that, under transitivity and sensitivity, **HEF** is weaker than *each* of **HE**, **WPD**, and **WLD**.

**Proposition 2** *Assume that **O** and **S** hold. Then each of **HE**, **WPD**, and **WLD** implies **HEF**.*

**Proof.** Assume  $u'' > u' > w' > w''$ . We must show under **O** and **S** that each of **HE**, **WPD**, and **WLD** implies, for each  $t$ ,  $(u', \text{con}w') \succeq_t (u'', \text{con}w'')$ .

Since  $u'' > u' > w' > w''$ , there exist an integer  $n$  and utilities  $v, x \in [0, 1]$  satisfying  $u' > v \geq w' > x > w''$  and  $u'' - v = n(x - w'')$ .

If **HE** holds, then  $(v, x, \text{con}w'') \succeq_t (u'', \text{con}w'')$ , and by **S**,  $(u', \text{con}w') \succ_t (v, x, \text{con}w'')$ . By transitivity,  $(u', \text{con}w') \succ_t (u'', \text{con}w'')$ .

---

<sup>3</sup>For any utility stream starting at time  $t$   ${}_t\mathbf{u} = (u_t, u_{t+1}, \dots) \in [0, 1]^\infty$  and any time  $T \geq t$ , let  ${}_t\mathbf{u}_T = (u_t, u_{t+1}, \dots, u_T) \in [0, 1]^{T-t+1}$  denote the truncation of  ${}_t\mathbf{u}$  at  $T$ . Recall that  ${}_t\mathbf{u}'_T$  Lorenz dominates  ${}_t\mathbf{u}''_T$  if  $\sum_{s=t}^T u'_s = \sum_{s=t}^T u''_s$  and the Lorenz curve of  ${}_t\mathbf{u}'_T$  lies uniformly above  ${}_t\mathbf{u}''_T$ .

Consider next **WPD** and **WLD**. Let  ${}_t\mathbf{u}^0 = (u'', \text{con}w'')$ , and define,  $\forall i \in \{1, \dots, n\}$ ,  ${}_t\mathbf{u}^i$  inductively as follows:

$$\begin{aligned} u_s^i &= u_s^{i-1} - (x - w'') && \text{for } s = t \\ u_s^i &= x && \text{for } s = t + i \\ u_s^i &= u_s^{i-1} && \text{for } s \neq t, t + i. \end{aligned}$$

If **WPD** holds, then,  $\forall i \in \{1, \dots, n\}$ ,  ${}_t\mathbf{u}^i \succeq_t {}_t\mathbf{u}^{i-1}$ , and by **S**,  $(u', \text{con}w') \succ_t {}_t\mathbf{u}^n$ . By transitivity,  $(u', \text{con}w') \succ_t (u'', \text{con}w'')$  since  ${}_t\mathbf{u}^0 = (u'', \text{con}w'')$ .

If **WLD** holds, then  ${}_t\mathbf{u}^n \succeq_t {}_t\mathbf{u}^0$ , and by **S**,  $(u', \text{con}w') \succ_t {}_t\mathbf{u}^n$ . By transitivity,  $(u', \text{con}w') \succ_t (u'', \text{con}w'')$  since  ${}_t\mathbf{u}^0 = (u'', \text{con}w'')$ . ■

Note that condition **HEF** involves a comparison between a sacrifice by a single generation and an equal gain for each member of an infinite set of generations that are worse off. Hence, contrary to the standard Hammond Equity condition, the weakly welfare increasing transfer from the better-off present to the worse-off future specified in condition **HEF** always increases the total amount of utility along a stream, if utilities are made (at least) cardinally measurable and fully comparable. This entails that condition **HEF** is implied by both the Pigou-Dalton principle of transfers and the Lorenz domination principle, independently of what specific cardinal utility scale is imposed. Hence, the condition can be endorsed both from an egalitarian and utilitarian point of view. In particular, condition **HEF** is much weaker and more compelling than the standard Hammond Equity condition.

In an environment where **O**, **C**, **TI**, **S**, **IF**, and **EP** hold, the addition of **HEF** leads to the following condition.

**Lemma 1** *Assume that **O**, **C**, **TI**, **S**, **IF**, **EP**, and **HEF** hold. Then, for each  $t \geq 1$ ,  $u_t'' > u_t' > W({}_{t+1}\mathbf{u}') > W({}_{t+1}\mathbf{u}'')$  implies  $W({}_t\mathbf{u}') \geq W({}_t\mathbf{u}'')$ .*

**Proof.** Write  $W({}_{t+1}\mathbf{u}') = w'$  and  $W({}_{t+1}\mathbf{u}'') = w''$ . By Proposition 1,  ${}_t\mathbf{u}' \sim_t$



$(u'_t, \text{con}w')$  and  ${}_t\mathbf{u}'' \sim_t (u''_t, \text{con}w'')$ , while, by **HEF**,  $(u'_t, \text{con}w') \succeq (u''_t, \text{con}w'')$ . Transitivity yields the result. ■

If we say that the present is worse (better) off than the future whenever  $u_t < W_{(t+1)\mathbf{u}}$  ( $u_t > W_{(t+1)\mathbf{u}}$ ), also for streams where well-being is *not* constant from the second period on, then Lemma 1 means the following: If the present is better off than the future, and a sacrifice now leads to a gain for the future that is equivalent to an increase in their stationary equivalent, then such a transfer from the present to the future is weakly desirable in social evaluation, as long as the present remains better off than the future.

For each  $t \geq 1$ , the social preferences at time  $t$ ,  $\succeq_t$ , evaluate utility streams starting at time  $t$ . Since the cardinality of infinite streams is the same independently of when they start, one may, however, pose the following question: Do generations  $\{t, t+1, \dots\}$  *envy* (cf. Varian, 1974) the situation that generations  $\{s, s+1, \dots\}$  will be in, if the *social preferences* at time  $t$ ,  $\succeq_t$ , are used to compare  ${}_t\mathbf{u}$  and  ${}_s\mathbf{u}$ ? I.e., does  ${}_t\mathbf{u} \succeq_t {}_s\mathbf{u}$  hold? In the same way, generations  $\{s, s+1, \dots\}$  may use the social preferences at time  $s$ ,  $\succeq_s$ , to compare their utility stream with the stream of generations  $\{t, t+1, \dots\}$ ; i.e., does  ${}_t\mathbf{u} \succeq_s {}_s\mathbf{u}$  hold? By condition **TI**, such a comparison by means of social preferences across time based on the concept of ‘envy’ does not depend on whether one applies the social preferences at time  $t$  or at time  $s$ . Hence, we may unequivocally determine whether, along  ${}_1\mathbf{u}$ , the stream starting at time  $t$  is socially preferred to the stream starting at time  $s$ .

By invoking **HEF** in addition to the conditions of Proposition 1, it turns out that, along any  ${}_1\mathbf{u}$ , the stream starting at an earlier time  $t$  will never be socially preferred to the stream starting at a later time  $s$ .

**Proposition 3** *There exist social preferences,  $\succeq$ , satisfying **O**, **C**, **TI**, **S**, **IF**, **EP**, and **HEF**. For any such social preferences  $\succeq$ , the following holds: For any  ${}_1\mathbf{u} \in [0, 1]^\infty$ ,  ${}_t\mathbf{u} \preceq {}_s\mathbf{u}$  whenever  $t < s$ .*

**Proof.** Consider the social preferences represented by the following social welfare function:

$$W({}_1\mathbf{u}) = \lambda \limsup_{t \rightarrow \infty} u_t + (1 - \lambda) \liminf_{t \rightarrow \infty} u_t, \quad \text{where } 0 \leq \lambda \leq 1.$$

It can be verified by inspection that these social preferences satisfy **O**, **C**, **TI**, **S**, **IF**, **EP**, and **HEF**, establishing the first part of the proposition.

For the second part, it is by Proposition 1 sufficient to show that we will arrive at a contradiction if  $W({}_1\mathbf{u}) > W({}_2\mathbf{u})$ . Therefore, suppose  $W({}_1\mathbf{u}) > W({}_2\mathbf{u})$ . Write  $W({}_1\mathbf{u}) = w$  and keep in mind that  ${}_1\mathbf{u} \sim_{\text{con}} w$ .

If  $u_1 \leq w$ , then, by the properties of  $\Upsilon$ ,

$$W({}_1\mathbf{u}) = \Upsilon(u_1, W({}_2\mathbf{u})) < \Upsilon(w, W({}_{\text{con}}w)) = W({}_1\mathbf{u}),$$

ruling out this case.

If  $u_1 > w$ , select some  $w' \in (W({}_2\mathbf{u}), w)$ . By Lemma 1,

$$W({}_1\mathbf{u}) = \Upsilon(u_1, W({}_2\mathbf{u})) \leq \Upsilon(w, W({}_{\text{con}}w')) < \Upsilon(w, W({}_{\text{con}}w)) = W({}_1\mathbf{u}),$$

since  $u_1 > w > W({}_{\text{con}}w') > W({}_2\mathbf{u})$ , ruling out also this case. ■

This result can be interpreted as follows: In any stream where the present is better off than the future, i.e.,  $u_t > W({}_{t+1}\mathbf{u})$ , then reducing  $u_t$  to the stationary equivalent of  $W({}_{t+1}\mathbf{u})$  without changing  ${}_{t+1}\mathbf{u}$ , i.e., letting  $u_t = W({}_{t+1}\mathbf{u})$ , does not affect the social evaluation of the stream. This in turn means that, in social evaluation of streams across time, a stream starting at  $t + 1$  cannot be worse than a stream starting at  $t$ .

The result means that it might be unwise to define the concept of a ‘sustainable development’ in terms of non-decreasing intergenerational social welfare  $W$ , rather than in terms of the feasibility of sharing  $u_t$  with future generations: the condition of non-decreasing intergenerational social welfare  $W$  is a vacuous restriction under the conditions of Proposition 3, while requiring generation  $t$  to be able potentially to share  $u_t$  with future generations is a non-vacuous restriction.

One might consider “Hammond equity for the present” by requiring, for each  $t \geq 1$ ,  $u'' < u' < w' < w''$  implies  $(u', \text{con}w') \succeq_t (u'', \text{con}w'')$ . However, this contradicts a finding included in Proposition 1, namely that  $\Upsilon(u, w)$  is increasing in  $w$ . Moreover, some would question the ethical appeal of requiring *any* sacrifice from an infinite number of generations to help the worst-off first generation. Maximin implies that only the present matters in comparisons where the first generation is worst-off, and this would be inconsistent with the framework of Section 2. Consequently, maximin cannot be characterized within this framework.

## 4 Present-future linearity

Consider again utility streams  $(u', \text{con}w')$  and  $(u'', \text{con}w'')$  with well-being constant from the second period on. Assume that utilities are at least cardinally measurable and fully comparable. Introduce the following utilitarian condition.

**Condition PFL** (*Present-future linearity*) For each  $t \geq 1$ ,  $(u', \text{con}w') \succeq_t (u'', \text{con}w'')$  implies  $(v', \text{con}x') \succeq_t (v'', \text{con}x'')$  whenever  $v' - u' = v'' - u''$  and  $x' - w' = x'' - w''$ .

By combining condition **PFL** with the conditions of Proposition 1, we show through the following two lemmas that the social preferences can be represented by a social welfare function that is linear in present utility and future utility. This in turn means that we can obtain a new characterization of discounted utilitarianism by imposing that **PFL** holds and **HEF** does not hold.

**Lemma 2** *Assume that **O**, **C**, **TI**, **S**, **IF**, **EP**, and **PFL** hold. Then*

$$W(u', \text{con}w') \geq W(u'', \text{con}w'') \text{ implies } W(u', \text{con}w') \geq W(u, \text{con}w) \geq W(u'', \text{con}w'')$$

$$W(u', \text{con}w') > W(u'', \text{con}w'') \text{ implies } W(u', \text{con}w') > W(u, \text{con}w) > W(u'', \text{con}w'')$$

*whenever  $\alpha \in (0, 1)$ ,  $u = (1 - \alpha)u' + \alpha u''$  and  $w = (1 - \alpha)w' + \alpha w''$ .*

**Proof.** By Proposition 1, the social preferences,  $\succeq$ , are represented by  $W$  having the property that  $W(u, \text{con}w) = \Upsilon(u, w)$  for a stream where well-being is constant from the second period on.

Assume that  $W(u', \text{con}w') \geq W(u'', \text{con}w'')$ . Since, by Proposition 1,  $\Upsilon(u, w)$  is increasing in  $w$ , it follows that the lemma is trivial if  $u' = u''$ . Therefore, assume  $u' \neq u''$ .

Suppose that  $(u, \text{con}w)$  satisfies  $u = (1 - \alpha)u' + \alpha u''$  and  $w < (1 - \alpha)w' + \alpha w''$  for some  $\alpha \in (0, 1)$ . Then there exists a rational number  $\tilde{\alpha} = m/n$ , where  $m$  and  $n$  are two integers satisfying  $0 < m < n$ , such that  $\tilde{u} := (1 - \tilde{\alpha})u' + \tilde{\alpha}u'' \geq u$  and  $\tilde{w} := (1 - \tilde{\alpha})w' + \tilde{\alpha}w'' > w$ . Let,  $\forall i \in \{0, 1, \dots, n\}$ ,  $u_i = u' + i(u'' - u')/n$  and  $w_i = w' + i(w'' - w')/n$ . Hence,  $(u', \text{con}w') = (u_0, \text{con}w_0)$ ,  $(\tilde{u}, \text{con}\tilde{w}) = (u_m, \text{con}w_m)$ , and  $(u'', \text{con}w'') = (u_n, \text{con}w_n)$ . Note that,  $\forall i \in \{2, \dots, n\}$ ,  $u_i - u_1 = u_{i-1} - u_0 = (i-1)(u'' - u')/n$  and  $w_i - w_1 = w_{i-1} - w_0 = (i-1)(w'' - w')/n$ . If  $W(u_0, \text{con}w_0) < W(u_1, \text{con}w_1)$ , then by **PFL**,  $W(u_{i-1}, \text{con}w_{i-1}) < W(u_i, \text{con}w_i)$  for all  $i \in \{2, \dots, n\}$ , leading by transitivity to the contradiction that  $W(u', \text{con}w') < W(u'', \text{con}w'')$ . Therefore,  $\forall i \in \{1, \dots, n\}$ ,  $W(u_{i-1}, \text{con}w_{i-1}) \geq W(u_i, \text{con}w_i)$ . By transitivity,

$$W(u', \text{con}w') = W(u_0, \text{con}w_0) \geq W(u_m, \text{con}w_m) = W(\tilde{u}, \text{con}\tilde{w}) > W(u, \text{con}w)$$

since  $\tilde{u} \geq u$ ,  $\tilde{w} > w$ , and  $\Upsilon(u, w)$  is non-decreasing in  $u$  and increasing in  $w$ .

Since  $\Upsilon(u, w)$  is continuous, it now follows that  $W(u', \text{con}w') \geq W(u, \text{con}w)$  whenever  $\alpha \in (0, 1)$ ,  $u = (1 - \alpha)u' + \alpha u''$  and  $w = (1 - \alpha)w' + \alpha w''$ .

An analogous argument yields  $W(u, \text{con}w) \geq W(u'', \text{con}w'')$  whenever  $\alpha \in (0, 1)$ ,  $u = (1 - \alpha)u' + \alpha u''$  and  $w = (1 - \alpha)w' + \alpha w''$ .

If  $W(u', \text{con}w') > W(u'', \text{con}w'')$ , then it can likewise be shown that  $W(u', \text{con}w') > W(u, \text{con}w)$  and  $W(u, \text{con}w) > W(u'', \text{con}w'')$  whenever  $\alpha \in (0, 1)$ ,  $u = (1 - \alpha)u' + \alpha u''$  and  $w = (1 - \alpha)w' + \alpha w''$ . ■

**Lemma 3** *Assume that **O**, **C**, **TI**, **S**, **IF**, **EP**, and **PFL** hold. Then there exists  $\delta \in (0, 1]$  such that the social preferences,  $\succeq$ , are represented by a social welfare*

function  $W : [0, 1]^\infty \rightarrow [0, 1]$  satisfying, for all  $t\mathbf{u} \in [0, 1]^\infty$ ,

$$W(t\mathbf{u}) = (1 - \delta)u_t + \delta W(t_{+1}\mathbf{u}).$$

**Proof.** Since, by Proposition 1, the social preferences,  $\succeq$ , are represented by  $W$  having the property that  $W(t\mathbf{u}) = W(u_t, \text{con}w)$ , where  $W(t_{+1}\mathbf{u}) = w$ , it is sufficient to consider streams  $(u', \text{con}w')$  and  $(u'', \text{con}w'')$  where well-being is constant from the second period on. Moreover, since by Proposition 1,  $W(u, \text{con}w) = \Upsilon(u, w)$  for any such stream and  $\Upsilon(u, w)$  is increasing in  $w$ , it is sufficient to show that there exists  $\delta \in (0, 1]$  such that  $W(u', \text{con}w') = W(u'', \text{con}w'')$  whenever  $(1 - \delta)u' + \delta w' = (1 - \delta)u'' + \delta w''$ .

Since  $\Upsilon(u, w)$  is continuous, non-decreasing in  $u$  and increasing in  $w$ , there exist  $(v', \text{con}x')$  and  $(v'', \text{con}x'')$  with  $v' < v''$  and  $x' \geq x''$  such that  $W(v', \text{con}x') = \Upsilon(v', x') = \Upsilon(v'', x'') = W(v'', \text{con}x'')$ . Define  $\delta \in (0, 1]$  by

$$\delta := \frac{v'' - v'}{v'' - v' + x' - x''},$$

so that  $(1 - \delta)v' + \delta x' = (1 - \delta)v'' + \delta x''$ .

Consider any  $(u', \text{con}w')$  and  $(u'', \text{con}w'')$  satisfying  $(1 - \delta)u' + \delta w' = (1 - \delta)u'' + \delta w''$ . If  $u' = u''$ , then  $w' = w''$  as  $\delta > 0$ , and  $W(u', \text{con}w') = W(u'', \text{con}w'')$  follows trivially. Therefore, assume w.l.o.g. that  $u' < u''$ . Choose some  $d \in (0, \min\{u'' - u', v'' - v'\})$  and determine  $\alpha \in (0, 1)$  and  $\beta \in (0, 1)$  by  $(1 - \alpha)u' + \alpha u'' = u' + d$  and  $(1 - \beta)v' + \beta v'' = v' + d$ . Define  $(u, \text{con}w)$  and  $(v, \text{con}x)$  by

$$\begin{aligned} u &:= (1 - \alpha)u' + \alpha u'' & w &:= (1 - \alpha)w' + \alpha w'' \\ v &:= (1 - \beta)v' + \beta v'' & x &:= (1 - \beta)x' + \beta x'' \end{aligned}$$

By Lemma 2,  $W(v', \text{con}x') = W(v, \text{con}x)$ . Furthermore, since  $u - u' = d = v - v'$  and  $w' - w = \frac{1-\delta}{\delta} \cdot d = x' - x$ , so that  $u' - v' = u - v$  and  $w' - x' = w - x$ , **PFL** now implies  $W(u', \text{con}w') = W(u, \text{con}w)$ . By applying Lemma 2 once more, we obtain  $W(u', \text{con}w') = W(u'', \text{con}w'')$ . ■

**Proposition 4** *There exist social preferences,  $\succeq$ , satisfying **O**, **C**, **TI**, **S**, **IF**, **EP**, **PFL**, and **HEF**. For any such social preferences  $\succeq$ , the following holds:  ${}_1\mathbf{u}' \succeq {}_1\mathbf{u}''$  implies  $({}_1\mathbf{v}'_T, {}_{T+1}\mathbf{u}')$   $\succeq$   $({}_1\mathbf{v}''_T, {}_{T+1}\mathbf{u}'')$  for any  $T \geq 1$  and  ${}_1\mathbf{v}'_T, {}_1\mathbf{v}''_T \in [0, 1]^T$ .*

**Proof.** Consider the social preferences represented by the following social welfare function:

$$W({}_1\mathbf{u}) = \lambda \limsup_{t \rightarrow \infty} u_t + (1 - \lambda) \liminf_{t \rightarrow \infty} u_t, \quad \text{where } 0 \leq \lambda \leq 1.$$

It can be verified by inspection that these social preferences satisfy **O**, **C**, **TI**, **S**, **IF**, **EP**, **PFL**, and **HEF**, establishing the first part of the proposition.

Assume that the social preferences,  $\succeq$ , satisfy **O**, **C**, **TI**, **S**, **IF**, **EP**, and **PFL**. By Lemma 3, there exists  $\delta \in (0, 1]$  such that  $\succeq$  is represented by a social welfare function  $W : [0, 1]^\infty \rightarrow [0, 1]$  satisfying, for all  ${}_t\mathbf{u} \in [0, 1]^\infty$ ,  $W({}_t\mathbf{u}) = (1 - \delta)u_t + \delta W({}_{t+1}\mathbf{u})$ . Suppose  $\delta < 1$ . Construct  $(u', {}_{\text{con}}w')$  and  $(u'', {}_{\text{con}}w'')$  where

$$u'' > u' > w' > w'' \quad \text{and} \quad u'' - u' > \frac{\delta}{1 - \delta}(w' - w'')$$

implying that  $W(u', {}_{\text{con}}w') = (1 - \delta)u' + \delta w' < (1 - \delta)u'' + \delta w'' = W(u'', {}_{\text{con}}w'')$ . This precludes that **HEF** holds. Hence, if the social preferences  $\succeq$  satisfy **O**, **C**, **TI**, **S**, **IF**, **EP**, **PFL**, and **HEF**, then  $\succeq$  is represented by a social welfare function  $W : [0, 1]^\infty \rightarrow [0, 1]$  satisfying, for all  ${}_t\mathbf{u} \in [0, 1]^\infty$ ,  $W({}_t\mathbf{u}) = W({}_{t+1}\mathbf{u})$ . Then

$$\begin{aligned} W({}_1\mathbf{u}') &= W({}_{T+1}\mathbf{u}') = W({}_1\mathbf{v}'_T, {}_{T+1}\mathbf{u}') \\ W({}_1\mathbf{u}'') &= W({}_{T+1}\mathbf{u}'') = W({}_1\mathbf{v}''_T, {}_{T+1}\mathbf{u}'') \end{aligned}$$

follows by transitivity, implying that the social evaluation of two streams does not depend on the utilities at times  $1, 2, \dots, T$ , for any  $T \geq 1$ . ■

Proposition 4 means that, if both **PFL** and **HEF** are added to the conditions of Proposition 1, then the social preferences entail invariance for the well-being during any finite part of the stream; only the limiting behavior of the utility stream, as

time goes to infinity, matters. Hence, with **PFL** as an additional condition it is not only the case that  $u_t'' > u_t' > W_{(t+1)\mathbf{u}'} > W_{(t+1)\mathbf{u}''}$  implies  $W_{(t)\mathbf{u}'} \geq W_{(t)\mathbf{u}''}$  as reported in Lemma 1 under **HEF** and the conditions of Proposition 1. Rather,  $W_{(t+1)\mathbf{u}'} > W_{(t+1)\mathbf{u}''}$  implies  $W_{(t)\mathbf{u}'} > W_{(t)\mathbf{u}''}$  even if  $u_t'' > u_t' > W_{(t+1)\mathbf{u}'}$  does not hold. If this were not the case—i.e., that a gain to the first generation outweighed a loss to future generations when the first generation is worst off—then we would immediately get a conflict with **HEF** by applying **PFL** and considering a similar case where the first generation is better off.

This motivates looking at the consequences of adding **PFL** to the conditions of Proposition 1 in a setting where **HEF** does *not* hold.<sup>4</sup> As the following result establishes, this leads to a new axiomatization of discounted utilitarianism.

**Proposition 5 (Discounted utilitarianism)** *Assume that **O**, **C**, **TI**, **S**, **IF**, **EP**, and **PFL** hold and that **HEF** does not hold. Then there exists  $\delta \in (0, 1)$  such that the social preferences,  $\succeq$ , are represented by the social welfare function  $W : [0, 1]^\infty \rightarrow [0, 1]$  defined by, for all  ${}_t\mathbf{u} \in [0, 1]^\infty$ ,*

$$W({}_t\mathbf{u}) = (1 - \delta) \cdot \sum_{s=t}^{\infty} \delta^{s-t} u_s. \quad (\text{DU})$$

**Proof.** Assume that the social preferences,  $\succeq$ , satisfy **O**, **C**, **TI**, **S**, **IF**, **EP**, and **PFL**. By Lemma 3, there exists  $\delta \in (0, 1]$  such that  $\succeq$  is represented by a social welfare function  $W : [0, 1]^\infty \rightarrow [0, 1]$  satisfying, for all  ${}_t\mathbf{u} \in [0, 1]^\infty$ ,  $W({}_t\mathbf{u}) = (1 - \delta)u_t + \delta W_{(t+1)\mathbf{u}}$ .

Since **HEF** does not hold, there exist  $(u', {}_{\text{con}}w')$  and  $(u'', {}_{\text{con}}w'')$  such that

$$u'' > u' > w' > w'' \quad \text{and} \quad (u', {}_{\text{con}}w') \prec (u'', {}_{\text{con}}w''),$$

precluding that  $\delta = 1$ . Thus, since  $\delta \in (0, 1)$ ,  $W({}_t\mathbf{u})$  can be written as in (DU).

---

<sup>4</sup>It follows from the definition of condition **HEF** that **HEF** does not hold if there exist  $t \geq 1$  and  $u'' > u' > w' > w''$  such that  $(u', {}_{\text{con}}w') \prec_t (u'', {}_{\text{con}}w'')$ .

It can be verified by inspection that, under the social preferences determined by (DU) for some  $\delta \in (0, 1)$ , **O**, **C**, **TI**, **S**, **IF**, **EP**, and **PFL** hold, while **HEF** does not hold. ■

Proposition 5 means that, under the conditions of Proposition 1, assuming that **PFL** holds and **HEF** does not hold precludes the use of interpersonal level comparability in social evaluation, since discounted utilitarianism does not rely on such information (even if utilities are level comparable).

Chichilnisky (1996) presents an axiomatic approach to ‘sustainable’ social preferences. ‘Sustainable’ social preferences satisfy our conditions **O**, **C**, **S**, and **EP**,<sup>5</sup> as well as Chichilnisky’s Axiom 1 (“No dictatorship of the present”) and Axiom 2 (“No dictatorship of the future”). She shows existence of ‘sustainable’ social preferences that satisfy the additional property of ‘independence’, which is related to our condition **PFL**. However, our Propositions 4 and 5 imply that there exists no ‘sustainable’ social preferences that satisfy both **TI** and **IF** in addition to **PFL**:

- On the one hand, if **HEF** holds, then it follows from Proposition 4 that **O**, **C**, **TI**, **S**, **IF**, **EP**, and **PFL** are in direct conflict with Chichilnisky’s Axiom 2 (“No dictatorship of the future”), which rules out all social welfare functions depending solely on the limiting behavior of the utility streams.
- On the other hand, if **HEF** does not hold, then it follows from Proposition 5 that **O**, **C**, **TI**, **S**, **IF**, **EP**, and **PFL** are in direct conflict with Chichilnisky’s Axiom 1 (“No dictatorship of the present”), which rules out all forms of discounted sums of utilities.

It appears to be an open question whether there exist ‘sustainable’ social preferences satisfying both **TI** and **IF**.

---

<sup>5</sup>I.e., ‘sustainable’ social preferences satisfy all conditions of Proposition 1 except conditions **TI** and **IF**. Chichilnisky (1996) actually imposes sensitivity in the sense of Strong Pareto. This condition implies **S** and **EP**, as discussed in Section 6 below.



## 5 Combining HEF with a restricted form of PFL

Consider a setting where the conditions of Proposition 1 hold. Proposition 4 shows that the combination of **PFL** and **HEF** leads to a complete disregard for any finite portion of the utility stream. Hence, if we want to combine **PFL** with sensitivity for the interests of a finite number of generations, we must rule out **HEF**.

Since **HEF** may be thought of as a weak and compelling equity condition (cf. Proposition 2), it is of interest to investigate whether the condition can be combined with a weakened form of the utilitarian condition **PFL**, while keeping sensitivity for the interests of the present in certain situations. Hence, consider **PFL** restricted to situations where the present is worse off than the future.

**Condition RPFL** (*Restricted present-future linearity*) For each  $t \geq 1$ ,  $(u', \text{con}w')$   $\succeq_t (u'', \text{con}w'')$  implies  $(v', \text{con}x')$   $\succeq_t (v'', \text{con}x'')$  whenever  $v' - u' = v'' - u''$ ,  $x' - w' = x'' - w''$ ,  $u' \leq w'$ ,  $u'' \leq w''$ ,  $v' \leq x'$ , and  $v'' \leq x''$ .

The following result is obtained from the analysis of Section 4 by substituting **RPFL** for **PFL**.

**Proposition 6** *Assume that **O**, **C**, **TI**, **S**, **IF**, **EP**, **RPFL**, and **HEF** hold. Then there exists  $\delta \in (0, 1]$  such that the social preferences,  $\succeq$ , are represented by a social welfare function  $W : [0, 1]^\infty \rightarrow [0, 1]$  satisfying, for all  ${}_t\mathbf{u} \in [0, 1]^\infty$ ,*

$$W({}_t\mathbf{u}) = \begin{cases} (1 - \delta)u_t + \delta W({}_{t+1}\mathbf{u}) & \text{if } u_t < W({}_{t+1}\mathbf{u}) \\ u_t = W({}_{t+1}\mathbf{u}) & \text{if } u_t = W({}_{t+1}\mathbf{u}) \\ W({}_{t+1}\mathbf{u}) & \text{if } u_t > W({}_{t+1}\mathbf{u}). \end{cases}$$

Hence, with  $\delta < 1$ , social preferences satisfying **RPFL** and **HEF** in addition to the conditions of Proposition 1 allow the present to matter if it is worse off than the future, while only the future matters if the present is better off. Thereby, a sustainability constraint is imposed on discounted utilitarianism. It also follows that

Chichilnisky’s (1996) Axioms 1 and 2 are both satisfied.<sup>6</sup> Note that if  $\delta < 1$ , then social preferences satisfying **RPFL** and **HEF** in addition to the conditions of Proposition 1 make non-trivial use of both unit comparability and level comparability.

It has been shown by Asheim (1988) and others that discounted utilitarianism combined with a sustainability constraint in the Dasgupta-Heal-Solow model of capital accumulation and resource depletion leads to streams that may appeal to our ethical intuition. The reason is that this allows for development in an initial phase when a small capital stock and a large resource stock lead to high capital productivity, while protecting generations in the distant future against the grave consequences of discounting when resource exhaustion leads to a vanishing flow of resource extraction and low and diminishing capital productivity. Asheim et al. (2004) apply the social preferences that are partially characterized by Proposition 6 to this model and show that they lead to the streams investigated in Asheim (1988, Section 3).

## 6 Relaxing continuity

The axiomatization of discounted utilitarianism given above in Proposition 5 differs from the one established by Koopmans (1960) in several respects:

- Koopmans’ period independence conditions Postulate 3 and Postulate 3’ have been replaced by our condition **PFL**.
- Koopmans’ sensitivity condition Postulate 2 has been replaced by our condition **S** and an assumption that condition **HEF** does not hold.

Let us investigate the latter of these differences closer.

Koopmans’ Postulate 2 requires, within the current setting where  $u_t$  is one-dimensional, that there exist  $u'_t$ ,  $u''_t$ , and  ${}_{t+1}\mathbf{u}$  such that  $(u'_t, {}_{t+1}\mathbf{u}) \succ (u''_t, {}_{t+1}\mathbf{u})$ .

---

<sup>6</sup>Such social preferences are still not ‘sustainable’ in the sense of Chichilnisky (1996) since Strong Pareto does not hold; cf. footnote 5.

Thus, this condition imposes sensitivity for the interests of the first generation. It therefore rules out the kind of social preferences that satisfy **O**, **C**, **TI**, **S**, **IF**, **EP**, **PFL**, and **HEF** (cf. Proposition 4), as these entail invariance for the well-being during any finite part of the stream.

Since sensitivity for the interests of each generation is an appealing ethical intuition, it is of interest to consider the following strengthening of our condition **S**.

**Condition SP** (*Strong Pareto*) For each  $t \geq 1$ , if for all  $s \geq t$ ,  $u'_s \geq u''_s$ , and for some  $r \geq t$ ,  $u'_r > u''_r$ , then  ${}_t\mathbf{u}' \succ_t {}_t\mathbf{u}''$ .

It is clear that condition **SP** implies both Koopmans' Postulate 2 and our condition **S**. It also entails our condition **EP**, which coincides with Koopmans' Postulate 5.

Since **SP** implies conditions **S** and **EP** and leads to sensitivity for the interests of the first generation, we obtain the following version of Proposition 1.

**Proposition 1\*** *Assume that **O**, **C**, **TI**, **SP**, and **IF** hold. Then the social preferences are represented by a social welfare function  $W : [0, 1]^\infty \rightarrow [0, 1]$  satisfying*

- (a) *if  $\lim_{n \rightarrow \infty} \sup_{s \geq t} |u'_s - u''_s| = 0$  and, for all  $n$ ,  $W({}_t\mathbf{u}^n) \geq W({}_t\mathbf{u}'')$  (resp.  $W({}_t\mathbf{u}'') \geq W({}_t\mathbf{u}^n)$ ), then  $W({}_t\mathbf{u}') \geq W({}_t\mathbf{u}'')$  (resp.  $W({}_t\mathbf{u}'') \geq W({}_t\mathbf{u}')$ ), and*
- (b) *for all  ${}_t\mathbf{u} \in [0, 1]^\infty$ ,  ${}_{\text{con}}W({}_t\mathbf{u}) \sim_t {}_t\mathbf{u}$  and  $W({}_t\mathbf{u}) = \Upsilon(u_t, W({}_{t+1}\mathbf{u}))$ , where  $\Upsilon(u, w)$  is continuous, and increasing in  $u$  and  $w$ .*

However, it turns out that **SP** is in direct conflict with **C** and **HEF**.

**Proposition 7** *There exist no social preferences,  $\succeq_t$ , satisfying **C**, **SP**, and **HEF**.*

**Proof.** Let

$$u'' > u^1 = u^2 = \dots = u^n = \dots = u' > w^1 > w^2 > \dots > w^n > \dots > w' = w''$$

and assume that  $\lim_{n \rightarrow \infty} w^n = w'$ . Then, for all  $n$ ,  $(u^n, {}_{\text{con}}w^n) \succeq_t (u'', {}_{\text{con}}w'')$  by **HEF** and  $(u', {}_{\text{con}}w') \prec_t (u'', {}_{\text{con}}w'')$  by **SP**. This contradicts **C** since  $(u^n, {}_{\text{con}}w^n)$  converges to  $(u', {}_{\text{con}}w')$  in the sup topology. ■

Proposition 7 is related to a well-known result established by Diamond (1965), in the sense that it shows another questionable implication of combining **SP** and **C**. Diamond establishes that **Weak Anonymity**—meaning that two utility streams are indifferent in social evaluation when the one is derived from the other through a finite permutation—is inconsistent with **C** and **SP**. In a recent paper, Basu and Mitra (2003a) extend Diamond’s (1965) result by showing that **SP** even in combination with an assumption of numerical representability is in conflict with **Weak Anonymity**.

Equally worrying, we will claim, is the fact that **C** and **SP** are inconsistent with assigning priority to an infinite number of worst off generations in comparisons where the assignment of such priority only reduces the well-being of the better-off present generation, as expressed by **HEF**. In this respect, note that **HEF** neither implies nor is implied by **Weak Anonymity**, and thus the results are also different from a formal point of view.

Since **HEF** is implied by each of the Pigou-Dalton principle of transfers **WPD** and the Lorenz domination principle **WLD** under **O** and **SP** (cf. Proposition 2 and recall that **SP** implies **S**), it follows as a corollary to Proposition 7 that **O**, **C**, and **SP** are inconsistent with each of **WPD** and **WLD**, i.e, with assigning priority to the worse off in comparisons where the assignment of such priority does not reduce total utility. In this respect, Proposition 7 is closely related to the investigation by Suzumura and Shinotsuka (2003), who however show that **SP** are inconsistent with Pigou-Dalton and Lorenz domination principles under weaker versions of **O** and **C**.

Proposition 7 implies that **C** must be excluded from the list of conditions if we want to impose both **SP** and **HEF**. This leads to the following versions of Propositions 3 and 4.

**Proposition 3\*** *There exist social preferences,  $\succeq$ , satisfying **O**, **TI**, **SP**, **IF**, and **HEF**.*

**Proof.** Consider the orderings **Infinite-horizon leximin** and **Infinite-horizon utilitarianism**, as defined by Bossert et al. (2004). Alternatively, consider the **Utilitarian** quasi-ordering, as defined by Basu and Mitra (2003b), or the quasi-orderings **S-Leximin**, **W-Leximin**, **Catching Up**, and **Overtaking**, as defined by Asheim and Tungodden (2004), and complete these by invoking Szpilrajn’s (1930) Lemma (cf. Svensson, 1980). It can be verified that each of these social preferences satisfies **O**, **TI**, **SP**, **IF**, and **HEF**. ■

**Proposition 4\*** *There exist social preferences,  $\succeq$ , satisfying **O**, **TI**, **SP**, **IF**, **PFL**, and **HEF**.*

**Proof.** Consider the ordering **Infinite-horizon utilitarianism**, as defined by Bossert et al. (2004). Alternatively, consider the **Utilitarian** quasi-ordering, as defined by Basu and Mitra (2003b), or the quasi-orderings **Catching Up** and **Overtaking**, as defined by Asheim and Tungodden (2004), and complete these by invoking Szpilrajn’s (1930) Lemma (cf. Svensson, 1980). It can be verified that each of these social preferences satisfies **O**, **TI**, **SP**, **IF**, **PFL**, and **HEF**. ■

Moreover, Proposition 7 entails that we can characterize discounted utilitarianism without assuming that **HEF** does not hold, if we strengthen **S** to **SP**.

**Proposition 5\*** (**Discounted utilitarianism**) *Assume that **O**, **C**, **TI**, **SP**, **IF**, and **PFL** hold. Then there exists  $\delta \in (0, 1)$  such that the social preferences,  $\succeq$ , are represented by the social welfare function  $W : [0, 1]^\infty \rightarrow [0, 1]$  defined by, for all  ${}_t\mathbf{u} \in [0, 1]^\infty$ ,*

$$W({}_t\mathbf{u}) = (1 - \delta) \cdot \sum_{s=t}^{\infty} \delta^{s-t} u_s. \quad (\text{DU})$$

**Proof.** Assume that the social preferences,  $\succeq$ , satisfy **O**, **C**, **TI**, **SP**, **IF**, and **PFL**. Since **SP** implies that **S** and **EP** hold, and by Proposition 7, that **HEF** does not hold, it follows by Proposition 5 that  $\succeq$  is represented by a social welfare

function  $W : [0, 1]^\infty \rightarrow [0, 1]$  determined by (DU) for some  $\delta \in (0, 1)$ .

It can be verified by inspection that the social preferences determined by (DU) for some  $\delta \in (0, 1)$  satisfy **O**, **C**, **TI**, **SP**, **IF**, and **PFL**. ■

In line with earlier literature, the analysis of this section indicates that **C** is not an innocent technical assumption; rather, the condition has significant normative implications in the social evaluation of infinite horizon utility streams. Perhaps it is informative that such normative implications be conveyed explicitly, as in Proposition 5, rather than function in a implicit way, as in the alternative characterization of discounted utilitarianism given in Proposition 5\*.

Moreover, even though continuity conditions can be useful from a pragmatic point of view, as such conditions allow for the application of Weierstrass' Theorem, they do not by themselves provide a normative justification for ruling out the kind of social preferences considered in the proofs of Propositions 3\* and 4\*. Such social preferences satisfy **O**, **TI**, **SP**, **IF**, **EP**, **HEF**. In the case of **Infinite-horizon utilitarianism** (Bossert et al., 2004) and the completed **Utilitarian** (Basu and Mitra, 2003b), **Catching Up** and **Overtaking** criteria (Asheim and Tungodden, 2004, inspired by Atsumi, 1965, and von Weizsäcker, 1965) they even satisfy **PFL**. However, in comparisons where the interests of the future are infinitely more important, such preferences contradict **C** in order to allow for sensitivity for the interests of the present generation.

## References

- Asheim, G.B. (1988), Rawlsian intergenerational justice as a Markov-perfect equilibrium in a resource technology. *Review of Economic Studies* **55**, 469–483.
- Asheim, G.B., Buchholz, W, and Mitra, T. (2004), On sustainability and discounted utilitarianism in an exhaustible resource model, mimeo (under preparation), University of Oslo, University of Regensburg, and Cornell University.

- Asheim, G.B. and Tungodden, B. (2004), Resolving distributional conflicts between generations. *Economic Theory* **24**, 221–230.
- Atsumi, H. (1965), Neoclassical growth and the efficient program of capital accumulation. *Review of Economic Studies* **32**, 127–136.
- Basu, K. and Mitra, T. (2003a), Aggregating infinite utility streams with intergenerational equity: the impossibility of being Paretian, *Econometrica* **32**, 1557–1563.
- Basu, K. and Mitra, T. (2003b), Utilitarianism for infinite utility streams: A new welfare criterion and its axiomatic characterization, CAE Working Paper #03-05, Cornell University.
- Blackorby, C., Donaldson, D., and Weymark, J.A. (1984), Social choice with interpersonal utility comparisons: A diagrammatic introduction, *International Economic Review* **25**, 327–356.
- Bossert, W., Sprumont, Y., and Suzumura, K. (2004), The possibility of ordering infinite utility streams, mimeo, Département de Sciences Economiques and CIREQ, Université de Montréal, and Institute of Economic Research, Hitotsubashi University.
- Chichilnisky, G. (1996), An axiomatic approach to sustainable development. *Social Choice and Welfare* **13**, 231–257.
- Dasgupta, P.S. and Heal, G.M. (1974), The optimal depletion of exhaustible resources. *Review of Economic Studies* (Symposium), 3–28.
- Dasgupta, P.S. and Heal, G.M. (1979), *Economic Theory and Exhaustible Resources*. Cambridge University Press, Cambridge, UK.
- Diamond, P. (1965), The evaluation of infinite utility streams. *Econometrica* **33**, 170–177.
- Heal, G.M. (2001), Intergenerational welfare economics and the environment. Forthcoming in Mäler, K.-G. and Vincent, J. (eds.), *Handbook of Environmental Economics*, North-Holland, Amsterdam.
- Hammond, P.J. (1976), Equity, Arrow's conditions and Rawls' difference principle. *Econometrica* **44**, 793–804.

- Koopmans, T.C. (1960), Stationary ordinal utility and impatience. *Econometrica* **28**, 287–309.
- Koopmans, T.C. (1986a), Representation of preference orderings with independent components of consumption. In McGuire, C.B. and Radner, R. (eds.), *Decision and Organization: A Volume in Honor of Jacob Marschak*, 2nd edition, University of Minnesota Press, Minneapolis, pp. 57–78. Previously published in 1972.
- Koopmans, T.C. (1986b), Representation of preference orderings over time. In McGuire, C.B. and Radner, R. (eds.), *Decision and Organization: A Volume in Honor of Jacob Marschak*, 2nd edition, University of Minnesota Press, Minneapolis, pp. 79–100. Previously published in 1972.
- Lauwers, L. (1998), Intertemporal objective functions: Strong Pareto versus anonymity. *Mathematical Social Sciences* **35**, 37–55.
- Solow, R.M. (1974), Intergenerational equity and exhaustible resources. *Review of Economic Studies* (Symposium), 29–45.
- Suzumura, K. and Shinotsuka, T. (2003), On the possibility of continuous, Paretian, and egalitarian evaluation of infinite utility streams, mimeo, Project on intergenerational equity, Institute of Economic Research, Hitotsubashi University.
- Svensson, L.-G. (1980), Equity among generations. *Econometrica* **48**, 1251–1256.
- Szpilrajn, E. (1930), Sur l'extension de l'ordre partiel. *Fundamenta Mathematicae* **16**, 386–389.
- Varian, H.R. (1974), Equity, envy, and efficiency. *Journal of Economic Theory* **9**, 63–91.
- von Weizsäcker, C. C. (1965), Existence of optimal program of accumulation for an infinite time horizon. *Review of Economic Studies* **32**, 85–104.