Discussion paper

Regulation of pollution in a Cournot equilibrium

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Abstract

In the traditional Cournot model producers influence each other's decisions through revenue earned in the product market. Rosen (1965) introduced the notion of a *coupled constraints* equilibrium which allowed players to affect their rivals' strategy space as well. Krawczyk (2005) applied this idea to the regulation of environmental pollution where a cap on aggregate emissions implies a constraint across firms' activity levels. He solved the model with a diagonalization algorithm obtaining linear convergence. We formulate this problem as a complementarity problem and apply a Newton algorithm obtaining a quadratic convergence. We show that the conditions under which this model has a unique solution and the algorithm computes the solution are analogous to those for the traditional Cournot model.

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1. **Introduction**

The complementarity problem (CP) and variational inequalities (VI) have long been applied for modeling economic equilibria. A variety of algorithmic methods have been suggested and employed for computation. Eaves (1978) and Josephy (1979) suggested a Newton process which was subsequently used by among others Friesz *et.al.* (1983), Pang (1984), and Jones *et.al.* (1985). Mathiesen (1985) demonstrated the efficiency of SLCP[1] in solving both partial and general equilibrium models.

Several researchers have studied the oligopoly of *N* rivals following a Cournot-Nash behavior from a computational perspective (Murphy *et.al.* 1982, Sherali *et.al.* 1983, Harker 1984, Dafermos and Naguerney 1987, Mathiesen *et.al.* 1987). The problem involves a collection of *N* interrelated optimization models where profit for agent *j* depends on his as well as rivals' strategies. In this respect, it is a non-standard mathematical programming problem. Various approaches have been suggested for model (re)formulation and computation.

The CP/VI-approach is to solve the combined set of *N* first order conditions for individual profit maximizing volumes. Kolstad and Mathiesen (1991) examined convergence criteria for SLCP applied to the Cournot-Nash problem and provided conditions under which the problem has a unique solution and SLCP converges locally and globally.

Rosen (1965) contributed to the questions of existence, uniqueness and computation of Nash equilibrium for *N*-person games by introducing more general *coupling constraints*. An economic interpretation is that while Cournot only considered interactions between rivals through revenue earned in the product market and thus profits, Rosen allowed players to affect each other's strategy space as well. His model is an instance of what later has become known as quasi-variational inequalities (QVI). (See e.g. Harker (1991).)

Krawczyk (2005) applied Rosen's idea to the regulation of environmental pollution which implies constraints on the sum of individual emissions and hence production levels. He used a diagonalization algorithm to solve the model.[2]

The interpretation of the application is that there is an authority with the capability to enforce regulation by some means, e.g., issuing individual firm non-tradable or tradable quotas, or levying differentiated or uniform taxes. We consider a situation where emitters face a uniform price, which may be obtained either through a uniform tax or a market for tradable quotas.

We model the regulatory problem as a CP (VI), show the conditions under which it has a unique solution, and show that a Newton algorithm converges locally and at a quadratic rate. Finally, we apply these ideas to Krawczyk's 3-firm example and the non-linear 5-firm model of Murphy *et.al.* (1982) augmented with regulatory constraints. The observation that both examples are solved with considerably less effort than Krawczyk reports is unsurprising in view of the different convergence rates that these algorithms enjoy.

In the next sections we review the traditional Cournot model and the model extended with coupling constraints. Numerical experiments are reported in Section 4.

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[1] SLCP is an acronym for a Newton process solving a *S*equence of *L*inear *C*omplementarity *P*roblems where Lemke's algorithm solves each linear approximation of the typically non-linear models. Rutherford (1995) implemented this approach as two modules MCP and MPSGE in the commercial software package GAMS.
[2] Harker (1984) showed how diagonalization decomposed a Cournot model into *N* one-variable models laying the ground for an efficient algorithm. In Harker (1987) he demonstrated how this kind of algorithm with linear convergence sometimes in practice suffers from non-convergence.

2. The Cournot model as a complementarity problem

Let there be N firms, each with cost $C_j(x_j)$ for producing output $x_j \geq 0$ of a homogeneous product. x_j is a component of the vector $x = (x_1,..., x_N)^T$. The firms face an inverse demand function $P(X)$, $(X = \sum_j x_j)$. Define $X_j = X - x_j$. Profit for the j'th firm is given by

(1) $\pi_j(x) = x_j P(x_j + X_j) - C_j(x_j)$, $j = 1,...,N$.

Traditionally, producer j's strategy is constrained to $x_j \geq 0$. Feasible strategies x in (1) are thus contained in the set $C = \{x \mid x \geq 0\}$ which is non-empty, closed and convex.

A Cournot-Nash equilibrium is an output-vector x^* such that for any firm j,

(2) $\pi_j(x^*) = \max_{x_j} \{ \pi_j(x_j + X_j^*) \mid X_j^* = \sum_{k \neq j} x_k^*, \ (x_1^*,..., x_j, ..., x_N^*)^T \text{ in } C\}$, $j = 1,...,N$.

Producer j does his best given the actions (x_k^*) of his N-1 rivals, $k \neq j$. As is well known, (2) is not a standard optimization problem. Rather it is a collection of N dependent optimization problems[3]. Because of this interrelationship, several suggestions have been offered for (re)formulation and computation of equilibrium.[4]

Given the Cournot-behavior[5], first-order conditions for individual profit maximum are: Find x_j^*, $j = 1,...,N$, such that

(3) $\partial\pi_j/\partial x_j = P(x_j^* + X_j^*) + x_j^* P'(x_j^* + X_j^*) - C'_j(x_j^*) \leq 0$, $x_j^* \geq 0$, $x_j^*[\partial\pi_j/\partial x_j] = 0$.

The first condition states that at equilibrium marginal profits cannot be positive. The last implies that for production to occur, output must be a stationary point of the profit function. (3) is an instance of a complementarity problem (CP) which can be written

CP(f).   Find x^* such that $f(x^*) \geq 0$, $x^* \geq 0$, $x^T f(x^*) = 0$.

The following definition provides the bridge between (3) and CP(f).

(4) $f_j(x) \equiv -\partial\pi_j/\partial x_j$, $j = 1,...,N$.

Lemma 1. Assume that costs and inverse demand functions are twice continuously differentiable. Then any Cournot equilibrium solves CP(f). Further, if profits are pseudo-concave with respect to own output, then x^* is a Cournot equilibrium if and only if it solves CP(f).

Karamardian (1972) showed that every CP is a variational inequality (VI). Let Y denote a non-empty, closed and convex set in R^n and let φ be a mapping from R^n into itself. The VI is

VI(Y, φ)   Find y* in Y such that $(y^* - y)^T \varphi(y^*) \geq 0$, for all y in Y.

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[3] Cf. Scarf (1973). "The determination of prices that simultaneously clear all markets cannot, in general, be formulated as a maximization problem in a useful way. Rather than being a single maximization problem, the competitive model involves the interaction and mutual consistency of a number of maximization problems separately pursued by a variety of economic agents. The problem involves, in a fundamental way, the reconciliation of distinct objectives and not the maximization of a single indicator of social preference."

[4] Samuelson (1952) showed that the fictitious objective of maximizing the sum of consumers' and producers' surpluses over the feasible production set generated the same (first order) conditions as those for an equilibrium, whereby the competitive equilibrium was converted into a (non)linear programming model. In a Cournot model the gradient $g(x)$ of *individual* profit functions may be non-integrable. For the particular case of linear demand Slade (1988) constructed a function Γ with the property that $\partial\Gamma(x)/\partial x = g(x)$, whereby the N individual, but interrelated optimization problems are consistently described by one optimization model.

[5] Producer j *conjectures* that rival i does not change volume in reaction to j's adjustment, i.e., $\partial x_i/\partial x_j = 0$, $i \neq j$.

$VI(R_+^N, f)$ has the same solutions as CP(f), if any. Hence, we could as well analyze VI. However, because of our focus is on solving a sequence of LCPs we stick to the CP.

CP(f) is about finding a solution on a non-empty, convex and closed set $C = \{x \mid x \geq 0\}$. As C is unbounded, continuity of f is insufficient in order to establish existence. Additional conditions on (the growth of) f have been stated, e.g. f being convex or coercive, e.g. Moré (1974a), all essentially aimed at securing $f > 0$ for some x in C. Kolstad and Mathiesen (1991) introduced the economically motivated notion of a bound on industry output.

Definition. Industry output is said to be *bounded* by $Q > 0$ if output in excess of Q from any producer implies negative marginal profits to all firms, and thus $f > 0$.

Let Π denote the Jacobian matrix of marginal profits with respect to x, i.e., $\Pi = [\partial^2 \pi_i / \partial x_i \, \partial x_j]$. Based on results by Karamardian (1972) and Moré (1974b), Kolstad and Mathiesen (1991) established the existence of a unique Cournot equilibrium assuming that

      i)   each firm's profit $\pi_j(x)$ is twice continuously differentiable and concave in x_j,
      ii) industry output is bounded, and
      iii) Π has a negative dominant diagonal[6] for all $x \geq 0$.

When Π has a negative dominant diagonal, $F = [\partial f_i / \partial x_j]$ has a positive dominant diagonal and is positive definite (not necessarily symmetric).[7]

The SLCP algorithm applied to the function f is to linearize f at some initial point x^0 and solve the resulting linear complementarity problem (LCP). f is relinearized at this solution and the process is continued until convergence is obtained, i.e., $\mid x_j^t f_j(x^t) \mid < \delta$ for all j and a tolerance δ. The linearization of f at y is a first-order Taylor expansion: $Lf(x|y) = f(y) + F(y)(x-y)$, and the LCP($f|y$) is as follows

LCP(f|y): Find x such that $Lf(x|y) = q + Mx \geq 0$, $\;\; x \geq 0$, $\;\; x^T(q + Mx) = 0$,
where $q = [f(y) - F(y)y]$ and $M = F(y)$.

Let M be a positive definite matrix. Then LCP has a unique complementary solution and Lemke's algorithm will compute it (Cottle and Dantzig, 1968). So, when F has a dominant diagonal it is positive definite and we conclude that iterates of SLCP are well-defined.

Based on Pang and Chan (1982), Kolstad and Mathiesen (1991) established local and global convergence of SLCP applied to the Cournot model. Local convergence follows by norm-contraction when the Jacobian F is positive definite. When F is Lipschitz continuous the convergence rate is quadratic.

The proof for global convergence was based on the monotone approach and requires the following assumptions: Industry output is bounded, marginal profit (-f_j) is concave, the Jacobian F has a positive, dominant diagonal with unitary scales, and the process is initiated at a feasible solution (i.e., $x \geq 0$, $f(x) \geq 0$).

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[6] See the Appendix for notation and definitions.
[7] A positive definite and non-symmetric matrix is often called positive quasi-definite. This language becomes too cumbersome when we below will study positive semi-definite matrices.

3. The Cournot model with coupling constraints

In the Cournot model the strategy space $C = \{x \mid x \geq 0\}$ is the full Cartesian product of the individual strategy sets $\{x_j \geq 0\}$ and reflects that player j cannot affect rivals' strategy sets. Rosen (1965) admitted the more general constraint set $\{x \mid h(x) \geq 0\}$, where $h(x) = (h_1(x), ..., h_K(x))$ with each $h_i(x)$, $i=1,\ldots,K$, concave in x. Such a constraint set may generate what has become known as quasi-variational inequalities. It is critical to the outcome of the game how such interaction is played out. Who, if any, has the authority to enforce these constraints? And which incentives do players have to comply? See Harker (1991).

A model of regulation
We follow Krawczyk's (2005) application of Rosen's idea where producer j emits pollution (a_{ij}) and there is a cap (b_i) on aggregate pollution

(5) $\qquad \sum_j a_{ij} x_j \leq b_i$, $i=1,\ldots,K$.

We will assume $a_{ij} > 0$ and $b_i > 0$ for all i and j. Hence the set

(5') $\qquad \boldsymbol{D} = \{x \mid Ax \leq b, x \geq 0\}$,

is non-empty, compact, and convex. What kind of game is this pollution regulation? Several reasonable assumptions may be made.[8] We will assume that there is some agency with the undisputed authority to enforce regulation and furthermore that such regulation, e.g. popular policies as tradable quotas or uniform taxes, implies that all players face uniform prices related to these constraints. Taking quota prices (or taxes) as given, there is no game between the Cournot players related to the coupling constraints, and the extended Cournot problem can be described as

VI(D, f) $\quad$ Find x^* in D such that $\quad (x^* - x)^T f(x^*) \geq 0$ for all x in $\boldsymbol{D}$.

When the feasible set is non-empty, compact, and convex, continuity of the mapping f gives existence of a solution (Hartman and Stampacchia 1966). When in addition f is strictly monotone uniqueness follows. (See below.) Thus, VI($\boldsymbol{D}, f$) has a unique solution x^*.

A complementarity formulation
VI($\boldsymbol{D}, f$) does not immediately correspond to a CP because $\boldsymbol{D} \neq R_+^N$. There is, however, a closely related CP. Let $\lambda = (\lambda_1, \ldots, \lambda_K)$ denote shadow prices corresponding to the constraints (5). The Cournot model with coupled constraints can be cast as the following CP:

Find volumes x^* in R^N and shadow prices λ^* in R^K such that

(6.1) $\quad g_j(x^*, \lambda^*) = -\partial \pi_j / \partial x_j + \sum_i a_{ij} \lambda_i^* \geq 0$, $\quad x_j^* \geq 0$, $\quad x_j^*[g_j(x^*, \lambda^*)] = 0$, $\quad j = 1,\ldots,N$,
(6.2) $\quad g_{i+N}(x^*, \lambda^*) = b_i - \sum_j a_{ij} x_j^* \quad \geq 0$, $\quad \lambda_i^* \geq 0$, $\quad \lambda_i^*[g_{i+N}(x^*, \lambda^*)] = 0$, $\quad i = 1,\ldots,K$.

Let $z = (x, \lambda)^T$. (6) can then be more compactly written as

CP(g): $\quad$ Find z^* in R^{N+K} such that $g(z^*) \geq 0$, $\quad z^* \geq 0$, $\quad z^{*T} g(z^*) = 0$.

Let $n = N+K$, $\boldsymbol{Y} = R_+^n$, and $\varphi=g$, whereby CP(g) is equivalent to VI(R_+^n, g).

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[8] A regulation with firm specific, non-tradable quotas could easily initiate a lobbying game between firms in order to secure for one-self the largest amount of quotas. Furthermore, an allocation of non-tradable quotas would in general imply that firms faced different shadow prices for one and the same pollutant.

Economic interpretation

Corresponding to each of the two policy instruments uniform taxes and tradable quotas, there is a nice economic interpretation of the term $\sum_i a_{ij} \lambda_i^*$ in (6.1). First, let λ_i denote a tax rate per unit pollution of type i. Then the term represents the total taxes producer j has to pay per unit of his product. As usual the shadow price λ_i rations the scarce cap b_i in (6.2). When aggregate pollution from unregulated production threatens to exceed cap i, the tax λ_i is levied, thereby reducing the profitability of producers so that each of them has an incentive to reduce production individually, whereby they all reduce emissions and collectively meet the cap.

Alternatively, assume that regulation takes place by means of tradable quotas issued in amounts b_i, $i=1,...,K$. Then λ_i has the interpretation of the price of a quota of type i, i.e., a permit to pollute one unit of i, and $(\sum_i a_{ij} \lambda_i^*)$ is the expenses of producer j for buying necessary permits in order to produce one unit. In both interpretations, the term is a cost to producer j and with λ_i taken as given the term is independent of what rival k does. Hence there is no interaction between producers; they play against each other only in the product market and the coupling constraints function like they would do in a competitive model.

Existence of a unique solution to (6).

To learn more about the LCP we analyze (6). Let G denote the Jacobian matrix of $g(z)$. It has the following structure

$$G = \begin{bmatrix} F(x) & A^T \\ -A & 0 \end{bmatrix}.$$

F(x) is the Jacobian of the traditional Cournot model, *A* originates with coupling constraints, and *0* is a *KxK* matrix of zeroes. *G* is clearly not positive definite as *F* is. Hence, the results of the previous section on existence and uniqueness of solution, on solving each LCP, and on the convergence of SLCP do not apply to CP*(g)*. The constraints and their dual variables converts the problem into a CP with an unbounded feasible set (R^{N+K}) whereby we have to conclude as above that continuity of *g* is insufficient for establishing existence. In the traditional Cournot model the bound Q on industry output ensured negative marginal profits, that is $f > 0$. The coupling constraints (5) may rule out such a large output, but the dual variables associated with these constraints do the job.

Lemma 2. CP*(g)* as defined as in (6) has interior, feasible solutions, i.e., $z \geq 0$ and $g(z) > 0$.

The proof is obvious and has the following nice economic interpretation. At non-zero production x' with $(b - Ax') > 0$ (and zero taxes) marginal profit ($\partial \pi_j / \partial x_j, j=1,..., N$) may be negative or positive. If marginal profit is positive for some *j* stipulate some tax-rates (λ_i) sufficiently high such that marginal profit after tax is negative for all *j*, whereby $g > 0$.

Now we can apply the following result on existence by Moré (1974b; Theorem 3.2).

Lemma 3. Let *g* be continuous and monotone. If there is an $z \geq 0$ with $g(z) > 0$[9], then CP*(g)* has a complementary solution *z**.

Monotonicity of *g* follows from monotonicity of *f* as the skew-symmetric parts (see G) cancel.

Lemma 4. Assume *f* is strictly monotone. Then CP*(g)* as defined in (6) has at most one complementary solution.

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[9] Megiddo (1977) demonstrated the necessity of a strict inequality ($g(z) > 0$), i.e., a constraint qualification.

Assume $z \neq w$ are solutions to (6). g strictly monotone means $0 < (z-w)^T(g(z)-g(w)) = z^Tg(z) - z^Tg(w) - w^Tg(z) - w^Tg(w) = -z^Tg(w) - w^Tg(z) \leq 0$, a contradiction.

Lemmas 1-4 provide existence of a unique solution to the CP-formulation of the Cournot model with coupling constraints under largely the same assumptions as the traditional model.

Theorem. Let CP*(g)* be defined as in (6). Assume that
> i)   each firm's profit is twice continuously differentiable and concave in x_j,
> ii) the constraint set D is non-empty, convex, and compact, and
> iii) the Jacobian Π of marginal profits has a negative dominant diagonal.
> Then CP*(g)* has a unique complementary solution.

When Π has a negative dominant diagonal F is positive definite whereby f is strictly monotone.

The LCP
Turning now to the LCP, we observe the possibility to build matrices being copositive plus from smaller matrices having this property (Cottle and Dantzig, 1968). Let the square matrices M_1 and M_2 be copositive plus. Then so is the matrix

$$(7) \qquad M = \begin{pmatrix} M_1 & A^T \\ -A & M_2 \end{pmatrix},$$

where A is a matrix of appropriate dimensions. We observe that

$$(7') \qquad u^T M u = u_1^T M_1 u_1 + u_2^T M_2 u_2 + u_2^T A u_1 - u_1^T A^T u_2 = u_1^T M_1 u_1 + u_2^T M_2 u_2,$$

for all u, where $u = (u_1, u_2)^T$ is partitioned according to the dimensions of M_1 and M_2. Thus, M is positive (semi)definite if both M_1 and M_2 are positive (semi)definite.

G is structured like (7). So when F is positive definite, G is positive semi-definite. Hence, the following results from the literature on LCP are relevant. (See the appendix.)

Lemma 5. Let M be positive semi-definite. The LCP has a complementary solution if it has feasible solutions, and Lemke's algorithm finds the solution, which is unique if it is non-degenerate.

Degeneracy of LCP-solutions might imply non-uniqueness of iterates and thus jeopardize the convergence of the iterative process. Kolstad and Mathiesen (1987) pointed out that the Cournot model is regular meaning that it almost surely has non-degenerate solution, and if there is a Cournot model with a degenerate solution, the parameters of inverse demand or costs may be perturbed without changing the model's regularity properties (Kehoe, 1985). Regularity is thus a generic property, and each LCP will be uniquely solved.

An iterative process.
Finally, consider the Newton process. Results in Pang and Chan (1982), as applied above, are essentially based on the assumption that Jacobian of the mapping is positive definite. The sub-matrix *0* of G invalidates this assumption for (6). Here is where VI(D, f) comes into play.

Motivated by the PIES energy model Eaves (1978) considered the application of a Newton process to economic equilibrium models. Framed as a CP these models are:

Find $z^* = (x^*, u^*, v^*)^T$ such that

$$H(z^*) = \begin{pmatrix} c & + A^T u^* - B^T v^* \\ a - Ax^* & \\ Bx^* & - \zeta(v^*) \end{pmatrix} \geq 0, \quad z^* \geq 0, \quad z^{*T} H(z^*) = 0.$$

Here c and a are vectors, A and B are matrices, and $\zeta(v)$ is a function. The LCP is

$$q = \begin{bmatrix} c \\ a \\ -e \end{bmatrix}, \quad M = \begin{pmatrix} 0 & A^T & -B^T \\ -A & 0 & 0 \\ B & 0 & -E \end{pmatrix},$$

$e + Ev$ is the linearization (of demand) where $E = [\partial\zeta/\partial v]$ is the Jacobian assumed to be negative definite.[10]

Recall the definition of a VI. Let Y denote a non-empty, closed and convex set in R^n and let φ be a function from R^n into itself. A point y^* in Y is defined to be stationary point or a solution to the VI(Y, φ) if

VI(Y, φ) $(y^* - y)^T \varphi(y^*) \geq 0$, for all y in Y.

Eaves considered a special structure where $y = (u, v)^T$, $\varphi(u, v) = (-a, -\zeta(v))^T$, i.e., φ is composed of a linear (a) and a non-linear part ($\zeta(v)$), and $Y = \{(u, v)^T \mid -A^T u + B^T v \leq c\}$. For our purposes his results can be summarized as follows.

Lemma 6. Let $y^* = (u^*, v^*)^T$ be a solution to VI(Y, φ) with Y and φ defined as above. Assume that ζ has negative definite derivatives ζ' (not necessarily symmetric). If v^0 is chosen close to v^*, the iterates v^t generated by the algorithm converge to v^* at a super-linear rate. If in addition ζ' is Lipschitz continuous at v^* the convergence rate is quadratic.

In our model (6) there are no u's. (Compare G and H.) The correspondences between VI(Y, φ) and VI(D, f) are given by $v = x$, $\zeta = -f$, and $Y = D = \{x \mid Ax \leq b, x \geq 0\}$.

VI(D, f) is stated above with D defined in (5'). We have essentially assumed that f (defined by (4)) is a mapping with positive definite derivatives. Because VI(D, f) has a unique solution x^*, lemma 6 applies and we conclude that the Newton process convergences to x^* when initiated at an x^0 close to x^*. If the derivatives $f' = F$ are Lipschitz continuous a quadratic convergence rate follows.

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[10] Mathiesen (1977) formulated this LCP and observed that when the Jacobian of demand is negative definite, M is positive semi-definite and Lemke's algorithm will compute the equilibrium or show that none exists.

4. Numerical examples

4.1 The River Basin Pollution game. (Haurie and Krawczyk 1997.)
Three firms produce a homogeneous product at levels $x_j, j=1, 2, 3$. Cost functions are:

$$C_i = (c_{1j} + c_{2i} x_j) x_j, \quad j=1, 2, 3.$$

Inverse market demand is

$$P(X) = d_1 - d_2 X = 3 - 0.01X, \quad \text{where} \ \ X = \textstyle\sum_j x_j,$$

Firm *j*'s profit from operating in this market is

(8) $\qquad \pi_j(x) = x_j P(X) - C_j(x_j) = (d_1 - d_2(x_1+x_2+x_3)) x_j - (c_{1j} + c_{2j} x_j) x_j, \ j=1, 2, 3.$

The firms are located along a river and pollutants may be expelled into to the river, where they disperse. Emission (e_j) is assumed to be proportional to production level, i.e., $e_j x_j$. A regulatory authority monitors concentration of pollution at two downstream stations. Decay and transportation coefficient of pollution from firm *j* to control station *i* is denoted u_{ij}. The regulator's constraints are

(9) $\qquad \sum_j u_{ij} e_j x_j \le b_i, \quad i=1, 2.$

Admissible pollution levels $b_1 = b_2 = 100$. Parameters c_{1j}, c_{2j}, u_{ij}, and e_j are shown in Table 1. Observe from (9) that only products $u_{ij} e_j$ matter. Define $a_{ij} = u_{ij} e_j$, for all *i* and *j*, and write

(9') $\qquad \sum_j a_{ij} x_j \le b_i, \quad i=1, 2.$[11]

Table 1. Marginal cost, emission, transportation and decay data.

Firm	c_{1j}	c_{2j}	e_j	u_{j1}	u_{j2}	a_{1j}	a_{2j}
1	0.10	0.01	0.50	6.5	4.583	3.25	2.2915
2	0.12	0.05	0.25	5.0	6.250	1.25	1.5625
3	0.15	0.01	0.75	5.5	3.750	4.125	2.8125

The Cournot model without regulation
Absent taxes or tradable quotas for pollution, profits are as shown in (8), which is quadratic in decision variables. Hence the first order conditions are linear

$$-\partial \pi_j(x)/\partial x_j = (c_{1j} - d_1) + [2(d_2+c_{2j})x_j + d_2\textstyle\sum_{k\ne j} x_j] \ge 0, \quad x_j \ge 0, \quad x_j [-\partial \pi_j(x)/\partial x_j] = 0,$$

whereby a model of the unregulated Cournot equilibrium is an LCP with (q, M)

(10) $\quad \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} c_{11} - d_1 \\ c_{12} - d_1 \\ c_{13} - d_1 \end{bmatrix}, \quad M = \begin{bmatrix} 2(d_2+c_{21}) & d_2 & d_2 \\ d_2 & 2(d_2+c_{22}) & d_2 \\ d_2 & d_2 & 2(d_2+c_{23}) \end{bmatrix} = d_2 \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} + 2 \begin{bmatrix} c_{21} & 0 & 0 \\ 0 & c_{22} & 0 \\ 0 & 0 & c_{23} \end{bmatrix}.$

When either $d_2 > 0$ or $c_{2j} > 0$ (or both), for all *j*, i.e., downward sloping demand and increasing marginal costs, *M* in (10) is positive definite whereby (10) has a unique[12] solution that Lemke's algorithm computes (Cottle and Dantzig, 1968).

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[11] Pollution is in this example thought of as one pollutant measured at two control stations. (9') can alternatively be interpreted as constraints across emissions of two pollutants.

If $q_j > 0$ for all j, $x_j^* = 0$ for all j. This is the case when willingness to pay is insufficient to cover the marginal cost of even the most efficient producer, e.g. in a not yet opened market. If however, $q_j < 0$ for at least one j, the solution involves positive output.

With the data of Table 1 (and absent any regulation), the activity levels and individual profits for the unconstrained case are as shown in the second and third columns of Table 2. Pollutions as measured at the stations exceed targets. The solution was obtained by the MCP-solver in GAMS in 4 pivots. (Because this model is linear, the solution is found in one iteration.)

Table 2. Solutions to the River Basin Pollution game[#].

Firm	Unconstrained emissions Cournot-Nash Act.levels (x)	Profit	Constrained emissions Rosen-Nash Act.levels (x)	Profit	
1	67.577	45.666	21.145	8.942	
2	22.192	4.925	16.028	15.414	
3	65.077	42.350	2.726	0.149	
Emiss.1	515.81		100.00	0.574	
Emiss.2	372.56		81.16	0.000	

[#] For all three solutions $\max_j (|f_j(x^*)|)$ is on the order of 10^{-7} or less.

The Cournot model with regulation

Assume that the regulator can levy a tax λ_i based on pollution as measured at station i, and that the regulator knows the individual coefficients (a_{ij} in our notation). Hence, he can infer each firm's contributed pollution and tax accordingly. Finally, it is assumed that each firm regards the tax rate λ_i as exogenous. Profits in (8) are modified as follows

$$(8') \qquad \psi_j(x) = \pi_j(x) - \sum_i \lambda_i a_{ij} x_j, \quad j=1, 2, 3.$$

$(8')$ is also quadratic in volumes x_j, whereby the first order conditions for profit maximum within regulatory constraints are linear in decision variables (x_j) and dual variables (λ_i)

$$(11.1) \quad -\partial\psi_j(x)/\partial x_j = -\partial\pi_j(x)/\partial x_j + \sum_i \lambda_i a_{ij} \geq 0, \quad x_j \geq 0, \quad x_j[-\partial\psi_j(x)/\partial x_j] = 0, \quad j=1,2,3,$$
$$(11.2) \qquad\qquad b_i - \sum_j a_{ij} x_j \geq 0, \quad \lambda_i \geq 0, \quad \lambda_i[b_i - \sum_j a_{ij} x_j] = 0, \quad i=1,2.$$

Compared to the non-regulated case (10), the LCP in (11), including regulation, has two added constraints and corresponding (dual) variables λ_i, $i=1,2$.

$$(12) \quad \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \end{pmatrix} = \begin{pmatrix} c_{11} - d_1 \\ c_{12} - d_1 \\ c_{13} - d_1 \\ b_1 \\ b_2 \end{pmatrix}, \quad M = \begin{pmatrix} 2(d_2+c_{21}) & d_2 & d_2 & a_{11} & a_{21} \\ d_2 & 2(d_2+c_{22}) & d_2 & a_{12} & a_{22} \\ d_2 & d_2 & 2(d_2+c_{23}) & a_{13} & a_{23} \\ -a_{11} & -a_{12} & -a_{13} & 0 & 0 \\ -a_{21} & -a_{22} & -a_{23} & 0 & 0 \end{pmatrix}$$

M in (12) is partitioned as in (7) i.e., $M = \begin{pmatrix} F & A^T \\ -A & 0 \end{pmatrix}$.

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[12] Observe that M is positive definite even if some or all c_{2j} are slightly negative. Kolstad and Mathiesen (1987) proved uniqueness of the equilibrium if the determinant of F is positive at all equilibria.

F is assumed to have a positive dominant diagonal implying it is positive definite; the zero matrix is positive semi-definite, and by (7') *M* in (12) is positive semi-definite. Thus, if we can show that (12) has feasible solutions, it follows from lemma 5 that (12) has a complementary solution which Lemke's algorithm will compute, and which is unique if it is non-degenerate.

Above we argued that (6) has (interior) feasible solutions. Similarly, it is obvious that each LCP of the iterative process has feasible solutions. Hence, Lemke's algorithm computes a complementary solution. In fact, GAMS computed the solution in 5 pivots. See the Rosen-Nash solution of Table 2. By inspection, it is non-degenerate and therefore unique.

Comparison of computational effort.
Krawczyk (2005) programmed his algorithm in MATLAB and reported 20 iterations to solve for an approximate Nash-Cournot equilibrium from an initial guess $x^0 = (0, 0, 0)$. In each iteration a constrained optimization problem of three variables and two constraints is solved. The computational effort to solve each such sub-problem amounts to that of solving our model (11), which, however, is solved once.

Being a LCP this example may not provide the best benchmark for comparison of the two algorithms. Therefore, we construct and solve a non-linear CP that necessarily involves several iterations by SLCP. As is well known, a Newton process is the most efficient in the neighborhood of the solution as it may converge at a quadratic rate. In the first stages of the process it may be less efficient than other iterative schemes, like e.g. the NIRA algorithm.

4.2 An oligopoly model with 5 firms. (Murphy *et.al.* 1982.)
Consider an oligopoly with five firms producing and selling a homogeneous product following the Cournot behavior. The inverse demand for the product is given by

$$P(X) = (5000/X)^e,$$

where $X = \sum_j x_j$ and $e = 1/1.1$. Firm *j* has a marginal cost function

$$C''_j = c_j + (x_j/L_j)^{bj}, \ j=1,...,5.$$

Table 3 lists the parameters of these functions.[13]

Murphy *et.al.* (1982), Harker (1984), and Kolstad and Mathiesen (1991) initiated algorithms at $x_j^0 = 10, j=1,...,N$.[14] Convergence with SLCP was obtained in 6 iterations to the solution *x** in Table 3. Because of the observation of an indefinite Jacobian at low *x*-values and in order

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[13] With non-linear demand and marginal cost functions, the Jacobian *F* of *f* has non-constant entries. Does *F* have a dominant digonal as assumed? Let f_{jk} denote the general entry. Differentiating f_j (using (3) and (4)), we obtain diagonal entries $f_{jj} = C''_j - 2P' - P''x_j, j=1,...,N$, and off-diagonal entries $f_{jk} = -P' - P''x_j, k \neq j$.[13] We find that $P' < 0$ and $P'' > 0$. As $P''x_j = ((e-1)(x_j/X))P'$ off-diagonal entries (f_{jk}) are positive. Because C''_j is positive, also diagonal entries (f_{jj}) are positive. Using these formulas to check for a dominant diagonal seems less helpful. But numerical tests at various points *x* are simple. Stipulate $x_j = y, j=1,...,N$, at values *y* = 3, 5, 10, and 30, in addition to equilibrium values (see Table 3). When *y* = 3, there is non-dominance and *F* is indefinite. When *y* = 5, the diagonal is dominant. In fact, *F* is almost a Hadamard matrix (equal weights). For all larger y-values *F* is a Hadamard matrix and positive definite.
[14] Kolstad and Mathiesen (1991) noted that for *j*=1, 2, 3, f_j are concave (not convex as assumed for establishing global convergence). Furthermore x^0 is not feasible; i.e., $f_j(x^0) < 0$. Hence global convergence of SLCP is not assured for this problem. Our experiments suggest convergence from any $x^0 > 0$; observe that *P*(X) is not defined for $x = (0,..., 0)^T$.

to check the robustness of SLCP, the process was also initiated $x_j^0 = 1$, from where it converged in 9 iterations.

Table 3. Parameters and (approximate[#]) solutions of a five-firm Cournot model.

Firm j	c_j	L_j	b_j	x_j^*	a_{1j}	a_{2j}	x_j^{**}
1	10	5	1/1.2	36.9325	1.5	0.6	27.445
2	8	5	1/1.1	41.8182	1.25	0.75	30.805
3	6	5	1	43.7066	1	1	31.031
4	4	5	1/0.9	42.6593	0.75	1.25	30.142
5	2	5	1/0.8	39.1190	0.6	1.5	27.814

[#] $\text{Max}_j (|f_j(x^*)|)$ is on the order of 10^{-11}. The tolerance in GAMS is set to 10^{-6}.

A non-linear model with coupling constraints.
We now combine the previous models by adding emissions and coupling constraints to the Murphy model. Let emission (a_{ij}) of pollutant i, $i=1,2$, per unit of activity of firm j be as in columns six and seven of Table 3, and let total admissible pollution be 150 of each type.

The solution x_j^{**} is shown in the last column of Table 3. At x^{**} both constraints bind and the values of the corresponding dual variables are 1.896 respectively 5.823. The solution was obtained from $x_j^0 = 10$ (1), $j=1,..., N,$ in 4 (8) iterations.

In private communication, Krawczyk informs that his NIRA algorithm spent 24-26 iterations to obtain similar accuracy from the same initial points. It seems fair to state that SLCP outperforms NIRA. The reason is obvious. A diagonalization algorithm like NIRA obtains a linear convergence rate[15] while a Newton process enjoys a quadratic convergence rate when functions are Lipschitz continuous, and functions that are used in economic equilibrium applications typically have this property.

Initiating the Newton process at x^0 far from the equilibrium x^* increases the number of iterations. In practical use doing analyses of 'real' problems, this may imply that it takes some effort to obtain a solution at the first run with a model. In subsequent runs, however, the effort to perform (sensitivity) analyses, i.e., by changing parameters of demand and cost, will be minimal. Thus, it is important to be able to employ a Newton process for solving economic equilibrium models.

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[15] Harker (1987) illustrates the potential drawback of the diagonalization algorithm. Extending the Murphy model from five to ten firms causes the standard implementation to spend almost 1000 iterations. This motivates an acceleration step that reduces the number of iterations by two orders of magnitude.

5. Conclusion

Rosen (1965) introduced the notion of *coupled constraint* equilibrium of the Cournot model. Krawczyk (2005) applied this idea to the regulation of environmental pollution, where a cap on aggregate emissions implies a constraint across agents' activity levels. Considering regulatory mechanisms like tradable quotas or uniform taxes where all agents face the same price for a given constraint, this paper formulates the model as a complementarity problem and demonstrates that conditions for existence and uniqueness of solution are similar to those of the traditional Cournot model (without such constraints).

The Newton-process SLCP implemented as module MCP in GAMS outperforms Krawczyk's diagonalization algorithm NIRA when applied to his numerical example and a non-linear 5-firm model. This follows as a diagonalization algorithm only obtains a linear convergence rate while a Newton process enjoys a quadratic convergence rate when functions are Lipschitz continuous.

Functions that are employed for the computation of economic equilibria typically have this property, and there is a large body of experience of excellent convergence applying Newton processes to such models. Fewer results are established in terms of proofs for existence of unique solution and (local) convergence of a Newton process applied to these models.

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Appendix A. Notation and LCP-results.

Notation.
Let $A = [a_{ij}]$ be a real $N \times N$ matrix. A is said to have a row dominant diagonal or row dominance if there exists N positive scalars d_i, such that

$$d_i \, |a_{ii}| > \sum_{j \neq i} d_j \, |a_{ij}| \quad \text{for } i=1,\ldots,N,$$

with a similar definition for column dominance. If $d_i = 1$ for all i, then A is a Hadamard matrix. A matrix with column dominance has row dominance although the vectors of scalars may differ between the two. If all diagonal entries are positive (negative) A has positive (negative) diagonal dominance. If A has positive row or column diagonal dominance, then it is a P-matrix. A is said to be positive quasi-definite if $u^T A u > 0$ for all $x \neq 0$. Definiteness is a special case when A is symmetric. In fact, the non-symmetric matrix A is positive quasi-definite when the symmetric matrix $(A+A^T)$ is positive definite.[16] If a positive scaling vector d exists, for which A has both row and column dominance, we say A has a dominant diagonal. If A has a positive dominant diagonal, then A is positive definite.

A is called copositive plus when $u^T A u \geq 0$ for all $u \geq 0$ and $(A+A^T)u = 0$ if $u^T A u = 0$ and $u \geq 0$. (Cottle and Dantzig 1968.) The class of such matrices includes
i) all strictly copositive matrices, i.e., those for which $u^T A u > 0$ when $0 \neq u \geq 0$,
ii) all positive semi-definite matrices, i.e., those for which $u^T A u \geq 0$ for all u.

Let f be a mapping of a subset D in R^N, *i.e.*, $f: \mathbf{D}$ in $R^N \to R^N$.

f is called *monotone* on the set $\mathbf{D}$ if $(x^1 - x^2)^T(f(x^1) - f(x^2)) \geq 0$, for each x^1, x^2 in $\mathbf{D}$.
f is said to be *strictly* monotone if the inequality is strict.

Let f be defined by $f = Mx+q$ for some matrix M and vector q. Then f is monotone (resp. strictly monotone) on $\mathbf{D}$ if and only if M is positive semi-definite (resp. positive definite).

f is *coercive* when $\|x\| \to \infty$ implies that $[f(x) - f(x^0)] / \|x - x^0\| \to \infty$ for some x^0 in $\mathbf{D}$.

f is *Lipschitz continuous* when there exists a constant $K > 0$ such that for all x^1, x^2 in $\mathbf{D}$,
$|f(x^1) - f(x^2)| < K \, |x^1 - x^2|$.

LCP-results.
LCP Find x in R^N such that $q + Mx \geq 0$, $x \geq 0$, $x^T(q + Mx) = 0$.

An x that satisfy $q + Mx \geq 0$ and $x \geq 0$ is called feasible.

Lemma 1. Let M be positive definite. Then the LCP has an optimal solution.[17]

Lemma 2. Let M be positive semi-definite. If LCP has feasible solutions, it has a complementary solution. [Cottle 1964].

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[16] In economic equilibrium applications, like the Cournot model, the Jacobian matrix is typically non-symmetric. The notation with a quasi-label becomes too cumbersome when we consider semi-definite matrices and will be dropped.

[17] Cottle (1964) attributes this result to Dorn.

Remark. The relaxation to a *semi*-definite M requires some additional condition, a constraint qualification, i.e., the existence of interior solutions to the constraints.

Lemma 3. Let M be a P-matrix. Then LCP has an optimal solution for all q. [Cottle 1966].

Lemma 4. If M has positive principal minors Lemke's algorithm computes the complementary solution for any q. [Lemke 1964; Cottle and Dantzig 1968].

Remark. A positive definite matrix has positive principal minors and is thus a P-matrix. The converse may not be true when M is non-symmetric.

Lemma 5. If M is copositive plus Lemke's algorithm terminates with a complementary solution or shows that no feasible solution exists. [Cottle and Dantzig 1968].

Remark. A positive semi-definite matrix is copositive plus. Hence, a complementary solution is not guaranteed when M is positive semi-definite. If, however, it can be shown that the LCP has a feasible solution, lemma 2 says it has a complementary solution and lemma 5 then implies that Lemke's algorithm finds it.

Lemma 6. If M is positive semi-definite and the LCP has a non-degenerate[18] complementary solution, then this solution is unique.[19]

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[18] If a solution of LCP uses at least N linearly independent columns it is said to be non-degenerate.
[19] Eaves (1971, p. 626) attributes this result to an unpublished paper by Lemke.