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Discussion paper

Attitudes towards income risk in the presence of quantity constraints

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Abstract:

Considering a consumer with standard preferences, I trace out the consequences for risk aversion and prudence of quantity constraints on markets. I first show how the effect can be decomposed into a price risk effect and an endogenously changing risk aversion/prudence effect. Next, I calibrate locally both effects on relative risk aversion and prudence, using estimates on household demand for durables and labour supply. Finally, I perform a global numerical analysis of these effects. I conclude that quantity constraints have counter-intuitive and pronounced non-linear effects on risk attitudes.

Keywords: household demand, income risk aversion, prudence, quantity constraints, labour supply.

JEL Classification: D11; D81.

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1 Introduction

How are a consumer's attitudes towards income risk affected when her trading opportunities get restricted because of quantity constraints, such as having to work full-time while wishing a part-time job (or vice versa), or being stuck with a small car, when in need for a large one? I consider two types of attitude towards risk: (i) risk aversion and (ii) prudence or downside risk aversion. The Arrow-Pratt coefficient of risk aversion measures a consumer's willingness to pay for disposing of any zero mean risk. Likewise, Kimball's coefficient of prudence indicates the certain reduction in income required to bring the *marginal* utility of consumption in line with the expected marginal utility of consumption when a zero mean risk is added.[1]

First intuition suggests that quantity constraints make a consumer both more risk averse and more prudent, since they reduce the opportunity set and thus allow for smaller adjustments of the consumption bundle after the income risk has realised. Consider, e.g., the case where the utility function over consumption (c) and leisure (ℓ) is $u(c, \ell) = v(c) + \ell$, with $v', v''' > 0$ and $v'' < 0$. Because *preferences* are quasi-linear, all exogenous income risk of a worker (with spare time) is absorbed by leisure. Since also the *utility function* is linear in leisure, the consumer is risk neutral with respect to this income risk and exhibits zero prudence. But if she faces a binding quantity constraint on her labour supply, the exogenous income risk is absorbed by the consumption of other goods, whose marginal utility is strictly falling and convex. Hence, the quantity constraint turns the consumer into a strictly risk averse and prudent person with respect to income risk.

This intuition, however, tells only part of the story. When it comes to risk aversion, the effect of a quantity constraint can be decomposed in two effects. The first is what I call the price risk effect (PRE): had the constraint on labour supply been only weakly binding (i.e., under certainty, notional labour supply coincides with the quantity constraint), then risk aversion goes up because small income shocks now have to be absorbed by consumption alone, any adjustment through leisure being ruled out. I will propose to think of a quantity constraint as turning income shocks into (compensated)

[1] Since the solution to the consumer's decision problem under uncertainty will satisfy a condition on the expected marginal utility of consumption, the optimal response to changes in the zero mean risk will depend on the sign and size of the prudence coefficient. Hence, this coefficient measures 'the propensity to prepare or forearm oneself in the face of uncertainty' (Kimball, 1990, p 54). While *prudence* was originally defined in a temporal context (savings decision), the notion of *downside risk aversion* is often used in the atemporal context. For brevity, I use *prudence* thoughout in the paper.

price shocks: when a consumer experiences a positive income shock but cannot expand consumption of a particular good, then it is *as if* she is suddenly facing a higher price for that good (a virtual price) while at the same time her income is increased to make the new bundle (but with the same quantity of the constrained good) fit into the budget (a virtual income). Hence, the presence of a constraint means that the nominal income risk is (i) turned into a virtual income risk and (ii) amplified by an virtual price risk. Below, I show that the PRE adds a positive ordinal term to the coefficient of relative risk aversion (RR) that depends positively on the income elasticity of the constrained good and its importance in the consumer's budget, and negatively on its compensated price elasticity. Graphically, the PRE mark up is due to the fact that consumer's indirect utility function in terms of income is the upper envelope of the constrained indirect utility function. If the former is concave, the latter must be even more concave. This part of the story squares with intuition.

But what happens if the quantity constraint gets tighter, e.g., when the quota of what you are allowed to purchase of a good or the amount of labour you manage to supply gets striclty smaller than the optimal amount under certainty? I show that four effects will take place: (i) an income effect on RR (because a tighter constraint makes a person worse off), (ii) a relative price effect on RR (because a tighter constraint affects the virtual price), (iii) a scale effect on RR (because we are interested in the risk premium as a fraction of mean *nominal* income, not virtual income), and finally (iv) an effect on the size of the PRE (mainly because the budget share is affected). I call the sum of these four effects the endogenously changing risk aversion effect (ECRAE) because it picks up what happens when the consumer is forced to move along the budget line. Its sign and size is an empirical issue. Hence, if a consumer is initially constrained at the optimal demand under certainty, the ECRAE following a change in the quota may enforce the PRE or it may go in opposite direction. In the last case, the consumer may even turn less risk averse than when unconstrained. A similar decomposition applies for the effect on relative prudence. The PRE now pertains to the change in curvature of the marginal utility of income function. But since this function when unconstrained is not a maximum value function, it is not the envelope of the corresponding function when constrained, and therefore the PRE need not be positive.

Neary and Roberts (1980) introduced the concepts of virtual price and income in modern microeconomics to analyse the effects of quantity constraints on consumer behaviour under certanty. I show these concepts are also useful to trace out the effects on attitudes towards risk and result in expressions that can easily be calibrated using information on income and compensated price elasticities. To illustrate, I carry out such a calibration for two sets of empirical studies: household demand for durables and labour supply. In the case of constraints on labour supply, I find small price risk effects that are offset by endogenously changing risk aversion effects due to a 10% underemployment constraint. The small size of these effects is due to the low income elasticity of labour supply. Also the PRE on relative prudence is small and positive. In case of underemployment, it is enhanced by the endogenously changing prudence effect. For durables, the PRE for risk aversion is strong, and enforced by the ECRAE if a household is prevented from expanding the durable good to its optimal level–*vice versa* the ECRAE will mitigate the PRE when the household cannot downscale the durable good to its optimal level. The evidence on the PRE for prudence is mixed (i.e., both positive or negative). The endogenously changing prudence effect is negative: being prevented from expanding the durable stock raises relative prudence.

In addition to calibrating local effects of quantity constraints on attitudes towards risk, I explore the global effects by restricting preferences to the CES-CRRA class. I show that relative risk aversion and prudence, when constrained, are weighted averages of the corresponding measures when unconstrained and the elasticity of substitution. I then provide conditions under which relative risk aversion when constrained can be lower than when unconstrained, and when a constraint may turn a prudent consumer into an imprudent one. I illustrate these results using numerical examples. These show that constraints have non–monotone and pronounced non-linear effects on attitudes towards risk.

The subject of the present paper is related to recent work on how frictions and constraints affect risk taking behaviour or the willingness to take risk and the normative implications this may have for contract design. For example, Chetty and Szeidl (2007) show within an expected utility model how the presence of consumption commitments may make the indirect utility function more concave in some income regions, but convex in others. They explore how this may help to reconcile a number of empirical puzzles, such as the simultaneous purchasing of insurance and lottery tickets, or the presence of substantial aversion towards moderate gambles without implying unrealistically high aversion towards large gambles. Drawing an a similar observation as Chetty and Szeidl, Postlewaite *et al.* (2008) show that efficient employment contracts should allow for layoffs when consumer/workers make consumption commitments. A contract that allows for layoffs in case of a negative productivity shock balances the desire for wage smoothing of committed workers with the moral hazard constraint that in bad states the wage cannot exceed marginal productivity. Moreover, because consumption commitments introduce a non-concavity in the indirect utility function, the consumer prefers to bear the ensuing employment risk (and a smooth 'high' wage if employed throughout) to a smooth but low wage under a tenure contract. The strength of this argument, and the optimal degree of wage rigidity depends–among other thing–on the effect of consumption commitments on risk aversion. It is an empirical question how strong the effects are and the expressions that I derive in this paper will allow to assess their size without requiring particular assumptions on preferences. Gollier (2009) considers a general dynamic choice problem and asks whether an agent who can choose a lottery and take some action after observing the outcome of the lottery, has a larger willingness to bear risks than an agent who has to commit to an action before observing the lottery outcome. Gollier derives a set of sufficient conditions for the flexible context to lead to a higher risk tolerance. He then examines how rigidities may induce a household to more risk-prone behaviour in portfolio allocation and/or savings decisions. While the present paper addresses a similar question, its focus is very different. Gollier's focus is on decision taking under risk: does the ability to postpone an action until the uncertainty is resolved always lead to more risk taking? In the present paper, I examine the effect of one particular set of constraints–quantity constraints on purchased levels of goods and services–on the willingness to accept small income risks, and decompose it in terms of consumer preferences. To the best of my knowledge, this paper is also the first to examine the consequences of constraints for the decision maker's rate of prudence and thus her willingness to change precautionary behaviour when background risk increases.

Section 2 provides a mean variance analysis of the PRE on risk aversion, and indicates why this effect may be counteracted by the ECRAE. Section 3 gives a reminder of the consumer's decision problem, its properties, and formulates the coefficients of risk aversion and prudence with respect to income risk in terms of the direct utility function. In section 4, I introduce quantity constraints and derive their effect on the consumer's aversion with respect to income risks using the virtual price approach. Section 5 uses the same approach to look at the effect of a quantity constraint on the degree of relative prudence. In Section 6, I illustrate these effects for CRRA-CES preferences. Concluding remarks are presented in Section 7.

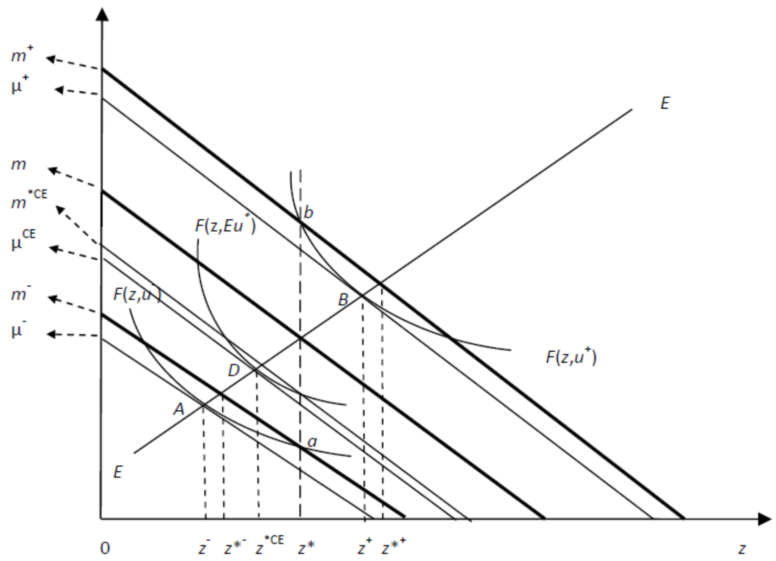

Figure 1: Equivalent income prospects (μ^+, μ^-), certainty equivalent income (μ^{CE}), and implementing CE income (m^{CE}) with a weakly binding quantity constraint ($z = z^*$).

2 A mean-variance argument

In this section, I will explain the effect of a constraint in a simple model where the consumer cares about two goods (z, x) where the first is subjected to a quantity constraint. The price of the z-good is p_z, that of the x-good is normalised to 1. In Figure 1, the Engel curve is drawn as EE (straight for simplicity). Hence, with an income m, the consumer purchases the bundle $(z^*, m - p_z z^*)$. Suppose now that income is uncertain, and takes the values m^+ and m^- with equal probability. Without any constraint, the optimal amounts for the z-good are z^{*+} and z^{*-}, respectively. But if she is constrained at z^*, any income shock must be absorbed by the x-good. Thus with a negative shock, the consumer ends up at a and with a positive shock at b. The corresponding utility levels are $u^-$and u^+, respectively.

I am now interested in computing the certainty equivalent income in the presence of this constraint. For this purpose, I draw the indifference curves through a and b and ask which income levels would make the consumer equally well off when not facing any constraint. The answer is μ^- and μ^+; I call these the *equivalent incomes.*

Let $F(z,u)$ be the *numéraire* function, i.e., the amount of the x-good the consumer requires to achieve utility level u when given z. Then clearly $m^+ = F(z^*, u^+) + p_z z^*$. Taking a second order Taylor approximation of the right-hand side around z^+ gives

$$\begin{aligned} m^+ &\simeq \mu^+ + \left[F_z(z^+,u^+) + p_z\right](z^* - z^+) + \frac{1}{2}F_{zz}(z^+,u^+)(z^* - z^+)^2 \\ &= \mu^+ - \frac{1}{2}\frac{1}{k_{zz}^B}(z^* - z^+)^2, \end{aligned} \tag{1}$$

where the equality sign comes from the fact that the slope of the indifference curve at B, $-F_z(z^+,u^+)$, equals the price p_z, and that F_{zz} is (minus) the inverse of the own Hicksian price effect on the z-good, k_{zz}. Since z^* and z^+ are the optimal amounts for incomes m and μ^+, respectively, we have $z^* - z^+ = z_m^B \cdot (m - \mu^+) + O((m-\mu^+)^2)$ where z_m denotes the income effect for z. Therefore, (1) may also be written as

$$m^+ = \mu^+ - \frac{1}{2}\frac{\left(z_m^B\right)^2}{k_{zz}^B}(m-\mu^+)^2 + O((m-\mu^+)^3).$$

Likewise,

$$m^- = \mu^- - \frac{1}{2}\frac{\left(z_m^A\right)^2}{k_{zz}^A}(m-\mu^-)^2 + O((m-\mu^-)^3).$$

The mean and variance of the prospect $(\mu^+, \mu^-; \frac{1}{2}, \frac{1}{2})$ are then given by[2]

$$E\mu \simeq m + \frac{1}{2}\frac{(z_m^C)^2}{k_{zz}^C}\varepsilon^2 + O(\varepsilon^3), \text{ and } var\mu = \varepsilon^2 + O(\varepsilon^3),$$

where superscript C indicates evaluation at a bundle on the Engel curve somewhere between A and B.

Suppose the consumer's degree of absolute risk aversion at expected income m is AR. Then the certainty equivalent income, μ^{CE}, is approximately

$$\mu^{CE} \simeq E\mu - \frac{AR}{2}var\mu = m + \frac{1}{2}\frac{\left(z_m^C\right)^2}{k_{zz}^C}\varepsilon^2 - \frac{AR(m)}{2}\varepsilon^2.$$

[2]Note that $m - \mu^- \simeq \varepsilon + m^+ - \mu^+ \simeq \varepsilon - \frac{1}{2}\frac{\left(z_m^B\right)^2}{k_{zz}^B}(m-\mu^+)^2$ where the second approximation follows from (1) and $z^* - z^+ \simeq z_m^B(m - \mu^+)$. Because $\mu^+ - m = \varepsilon + O(\varepsilon^2)$, it follows that $(m-\mu^+)^2 = \varepsilon^2 + O(\varepsilon^3)$, and $(m-\mu^-)^2 = \varepsilon^2 + O(\varepsilon^3)$.

μ^{CE} is the certainty equivalent income that *in the absence of any constraint* yields Eu^*. But since the consumer is constrained, what is needed is the certain income level that *in the presence of the constraint* z^* yields Eu^*. I call this the implementing certainty equivalent income and denote it as m^{*CE}. Transforming μ^{CE} into m^{*CE} can be done in a similar way as in (1):

$$m^{*CE} \simeq \mu^{CE} - \frac{1}{2}\frac{1}{k_{zz}^D}(z^* - z^{*CE})^2.$$

But since $z^* - z^{*CE} \simeq z_m^D(m - \mu^{CE})$ and $m - \mu^{CE} = O(\varepsilon^2)$, it follows that $m^{*CE} - \mu^{CE} = O(\varepsilon^4)$, and thus can be ignored.

I can now ask what is the implied degree of absolute risk aversion when having expected income m and being constrained at z^*, AR^*. Since absolute risk aversion is approximately twice the risk premium per unit of variance, the answer is

$$AR^* \simeq \frac{2}{\varepsilon^2}\left(m - \mu^{CE}\right) = \frac{2}{\varepsilon^2}\left(\frac{AR}{2}\varepsilon^2 - \frac{1}{2}\frac{\left(z_m^C\right)^2}{k_{zz}^C}\varepsilon^2\right) = AR - \frac{\left(z_m^C\right)^2}{k_{zz}^C}.$$

I summarise this as

Claim 1 *When the constraint is weakly binding (i.e., coinciding with the optimal demand for the z-good at the expected income level) the variance of equivalent income is (almost) the same, but the expected value is lower. Hence, certainty equivalent income is lower and risk aversion has a mark up of* $-\frac{\left(z_m^C\right)^2}{k_{zz}^C}$ *to risk aversion in the absence of a constraint.*

Suppose next that the quantity constraint is slightly increased: from z^* to $\overline{z}$. As Figure 2 shows, the equivalent income in the high income state increases, while that in the low income state falls by approximately the same amount. Hence the variance of μ increases, while the mean is almost constant. Consequently, expected utility, and certainty equivalent income fall: $\frac{\partial \mu^{CE}}{\partial \overline{z}} < 0$.

Again, what is of interest is how the implementing certainty equivalent income is affected. The answer is[3]

$$\frac{\partial m^{CE}}{\partial \overline{z}} = \frac{\partial \mu^{CE}}{\partial \overline{z}} - \frac{1}{k_{zz}}(\overline{z} - \overline{z}^{CE})\left(1 - z_m\frac{\partial \mu^{CE}}{\partial \overline{z}}\right).$$

[3] Since $m^{CE}(\overline{z}) \simeq \mu^{CE} - \frac{1}{2}\frac{1}{k_{zz}}(\overline{z} - \overline{z}^{CE})^2 = \mu^{CE} - \frac{1}{2}\frac{1}{k_{zz}}(z_m(m - \mu^{CE}) + \overline{z} - z^*)^2$, differentiating with respect to $\overline{z}$ then gives the result.

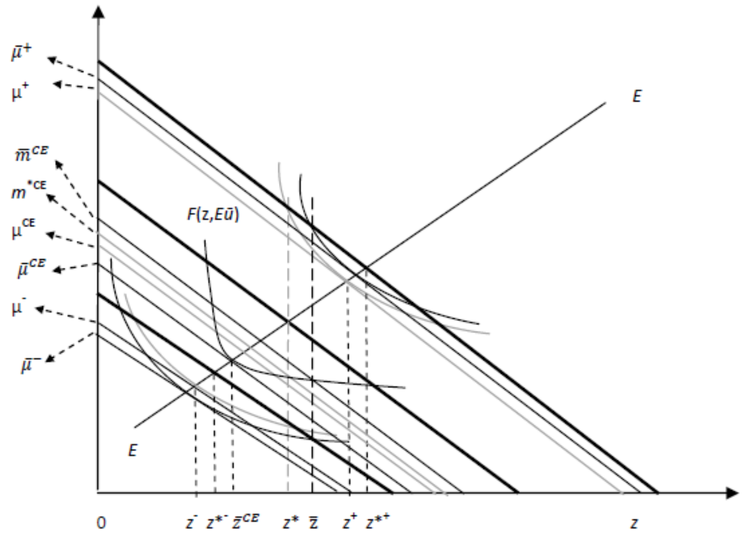

Figure 2: Equivalent income prospect $(\overline{\mu}^+, \overline{\mu}^-)$, certainty equivalent income $(\overline{\mu}^{CE})$, and implementing CE income $(\overline{m}^{CE})$ with a strictly binding quantity constraint $\overline{z}$.

If z is a normal good, the large round bracket term takes a positive value. Since $k_{zz} < 0$, the second right-hand side term may offset the first term and increase the implementing certainty equivalent income (as it does in Figure 2: $\overline{m}^{CE} > m^{*CE}$). In particular, this will happen when the elasticity of substitution between the two goods is very small. The opposite is true for a quantity constraint $\overline{z}$ slightly below z^*. Now the variance of equivalent income falls (the mean is approx. unaffected), hence μ^{CE} rises. But the implementing certainty equivalent income m^{CE} may fall if the substitution effect is small. I therefore make

Claim 2 *When the Hicksian substitution effect is small enough, forced consumption will lower risk aversion, while rationing will increase it.*

Since the compensated wage effect on labour supply is typically small, Claim 2 means that when a worker is underemployed (forced consumption of leisure) she may become less risk averse than when constrained at the optimal number of hours, or even when not constrained at all, as I illustrate in the calibration exercise in Section 4.

The claims in this section relied on a mean-variance argument. I considered the consequences for absolute risk aversion, but not for relative risk aversion, nor for relative prudence, and I assumed absolute risk aversion to be independent of income. The remainder of the paper will take up these issues in a rigorous way.

3 Income risk aversion and prudence without quantity constraints

A consumer cares about n commodities whose quantities are given by the bundle $q \in R^n_+$. Let the price vector be certain and given by $p \in R^n_+$. The consumer's income $\widetilde{m}$, however, is random with expectation m and variance σ^2_m. Her preferences are represented by a cardinal Bernoulli utility function $u(\cdot)$ which is monotone and strongly concave.

Suppose that the consumer is informed about the income draw before she makes her consumption decision. Suppose as well that the income draw coincides with the expected income m.[4] Her problem is then to solve

$$\max_q u(q) \text{ s.t. } p'q = m \quad (\lambda).$$

Let the unique solution be given by the bundle $q(p,m)$ satisfying the first order conditions[5]

$$u_q(q(p,m)) = \lambda(p,m)\ p, \tag{2}$$

where $\lambda(p,m)$ is the equilibrium value of the Lagrange multiplier.

The local properties of $q(p,m)$ are well known but repeated here for future reference. Defining K as the matrix of Slutsky substitution effects and q_m as the vector of income effects, we have:

$$\begin{aligned} \text{(i) } p'q_m &= 1, \text{ (ii) } \frac{\partial q}{\partial p'} = K - q_m q', \text{ (iii) } K = K', \\ \text{(iv) } Kp &= 0, \text{ and (v) } y'Ky < 0 \text{ for } y \neq \alpha p \text{ (α real scalar).} \end{aligned} \tag{3}$$

Expression (3-ii) is the Slutsky decomposition. A similar decomposition of the price effect on the marginal utility of income, λ, is

$$\frac{\partial \lambda}{\partial p} = -\lambda_m q - \lambda q_m. \tag{4}$$

[4] This is for notational convenience, since I will later evaluate the risk aversion measures at $m = \mathrm{E}\widetilde{m}$.

[5] Subscripts with u (and with λ, q and v below) denote derivatives.

The first *right-hand side* term is a real income effect that can be neutralized by an appropriate change in income. The second *right-hand side* term is a substitution effect: the change in the marginal utility of income when the consumer is compensated so as to remain at the same utility level.

The indirect utility function is defined as $v(p,m) \stackrel{\text{def}}{=} u(q(p,m))$ and satisfies $v_m = \lambda(p,m)$. Differentiating both sides of (2) with respect to m, and making use of the adding-up property (3-(i)) gives $v_{mm} = \lambda_m = q'_m u_{qq} q_m$. Since $\lambda = q'_m u_q$, the Arrow-Pratt coefficient of absolute risk aversion, measuring twice the risk premium the consumer is willing to pay (per unit of variance) to get rid of the income risk, is given by

$$AR(p,m) \stackrel{\text{def}}{=} -\frac{v_{mm}}{v_m} = -\frac{q'_m u_{qq} q_m}{q'_m u_q}. \tag{5}$$

This expression may be added to Hanoch's list of alternative representations of relative risk aversion (Hanoch, 1977, Theorem 1).

When the consumer faces an uninsurable income risk but can take actions to mitigate this risk, Kimball's (1990) coefficient of absolute prudence measures the sensitivity of these actions to the risk. When the action and the income risk enter the utility on equal terms (as in the case of future uncertain income and savings), this coefficient of absolute prudence is defined as

$$AP(p,m) \stackrel{\text{def}}{=} -\frac{v_{mmm}}{v_{mm}}, \tag{6}$$

and an increase in risk is said to trigger prudent behaviour if $AP(p,m) > 0$. Eeckhoudt and Schlesinger (2006) have shown more generally that a decision maker is prudent if and only if she prefers to subject her income to the lottery $(-k, \widetilde{\varepsilon}; \frac{1}{2}, \frac{1}{2})$ rather than to the lottery $(0, -k+\widetilde{\varepsilon}; \frac{1}{2}, \frac{1}{2})$, for any loss k and any a zero mean risk $\widetilde{\varepsilon}$. Thus she prefers to disaggregate the two 'pains' rather than to face them both in the same state of the world.

For a consumer who cares about many goods, one would expect that $AP(p,m)$ depends on the set of third (cross) derivatives of the utility function $u(\cdot)$. This is indeed the case. In the appendix, I show that

$$AP(p,m) = -\frac{\frac{\partial q'_m u_{qq} q_m}{\partial q'} q_m}{q'_m u_{qq} q_m}, \tag{7}$$

where $\frac{\partial q'_m u_{qq} q_m}{\partial q'}$ is the effect on the quadratic form $q'_m u_{qq} q_m$ because of a perturbation in the Hessian following dq.[6] Expression (7) thus reveals that the coefficient of absolute prudence for income risk can be expressed as the ratio of a cubic form in the (three dimensional) array of third derivatives of $u(\cdot)$ to a quadratic form in the Hessian of $u(\cdot)$.

Proposition 1 *When a consumer has a utility function $u(\cdot)$ defined over n commodities, the coefficients of absolute risk aversion and prudence are given by (5) and (7), respectively.*

Later in the paper, my main concern will be with the effect of quantity constraints on the degree of relative risk aversion and prudence, which are defined as $RR(p,m) \stackrel{\text{def}}{=} AR(p,m)m$ and $PR(p,m) \stackrel{\text{def}}{=} AP(p,m)m$, respectively. In this respect, it is useful to know how these measures are affected by an m-compensated increase in the price of good i. In the appendix, I prove the following theorem:

Theorem 1 *The m-compensated effects of p_i on RR and RP are*

$$\frac{\partial RR(p,m)}{\partial \log p_i}|_{dv=0} = \frac{p_i q_i}{m}\left(RR(p,m)\left(1-\frac{\partial q_i}{\partial m}\frac{m}{q_i}\right)+\frac{\partial^2 q_i}{\partial m^2}\frac{m^2}{q_i}\right), \tag{8}$$

$$\frac{\partial RP(p,m)}{\partial \log p_i}|_{dv=0} = \frac{p_i q_i}{m}\left(RP(p,m)\left(1-\frac{\partial q_i}{\partial m}\frac{m}{q_i}\right)\right. \tag{9}$$

$$\left.+\left(3-\frac{RP}{RR}\right)\frac{\partial^2 q_i}{\partial m^2}\frac{m^2}{q_i}-\frac{1}{RR}\frac{\partial^3 q_i}{\partial m^3}\frac{m^3}{q_i}\right).$$

The term $\frac{\partial^2 q_i}{\partial m^2}\frac{m^2}{q_i}$ measures the curvature of the Engel curve for good i; $\frac{\partial^3 q_i}{\partial m^3}\frac{m^3}{q_i}$ is the third order counterpart. If (8) [(9)] is zero for all i then relative risk aversion [prudence] is constant along the indifference curve (but can vary along an Engel curve).[7] Clearly, this will be the case with homothetic preferences (when all Engel curves are straight lines through the origin).

[6] With two goods, we have $\frac{\partial u_{qq}}{\partial q'} = \begin{pmatrix} u_{111} & u_{121} & \vdots & u_{112} & u_{122} \\ u_{211} & u_{221} & \vdots & u_{212} & u_{222} \end{pmatrix}$, and the numerator of (7) is the binary cubic form $q_{m1}^3 u_{111} + 3q_{m1}^2 q_{m2} u_{112} + 3q_{m1} q_{m2}^2 u_{122} + q_{m2}^3 u_{222}$.

[7] The case of constant RR along the indifference curve was first studied by Deschamps (1973, section 3). Hanoch (1977, section 3) completed the analysis, by deriving the indirect utility function that corresponds to this assumption.

4 Effects of quantity constraints on risk aversion

4.1 Virtual coefficient of risk aversion

Suppose now that $q = \binom{x}{z}$, $p = \binom{p_x}{p_z}$ and that the consumer can no longer choose the sub-bundle z which is fixed at $\overline{z}$. Her problem then turns into

$$\max_x u(x, \overline{z}) \text{ s.t. } p_x' x + p_z' \overline{z} = m \quad (\lambda^r).$$

Let the solution be given by $x^r(p, m, \overline{z})$, satisfying the first order condition $u_x = \lambda^r p_x$. The indirect utility function is now $v^r(p, m, \overline{z}) \stackrel{\text{def}}{=} u(x^r(p, m, \overline{z}), \overline{z})$. Repeating the procedure of section 2, the coefficient of absolute risk aversion for income risk is given by

$$AR(p, m|\overline{z}) \stackrel{\text{def}}{=} -\frac{v_{mm}^r}{v_m^r} = -\frac{x_m^{r\prime} u_{xx} x_m^r}{x_m^{r\prime} u_x}. \tag{10}$$

In order to relate $AR(p, m|\overline{z})$ to $AR(p, m)$, I will use the 'virtual price' approach of Neary and Roberts (1980). This consists in defining a virtual price vector π_z for the sub-bundle z, and adjusting the consumer's income to the virtual income level $m^v \stackrel{\text{def}}{=} m + (\pi_z - p_z)'\overline{z}$ such that the consumer's notional demand for that bundle coincides with the imposed quantities. That is,

$$\begin{aligned} \overline{z} &\equiv z(p_x, \pi_z, m + (\pi_z - p_z)'\overline{z}), & (11)\\ x^r(p_x, p_z, m, \overline{z}) &\equiv x(p_x, \pi_z, m + (\pi_z - p_z)'\overline{z}), & (12)\\ v^r(p_x, p_z, m, \overline{z}) &\equiv v(p_x, \pi_z, m + (\pi_z - p_z)'\overline{z}). & (13) \end{aligned}$$

Implicitly differentiating (11) and using the Slutsky equation (3-ii) shows that

$$\frac{\partial \pi_z}{\partial m} = -K_{zz}^{-1} z_m, \tag{14}$$

where z_m is the vector of income effects for sub-bundle z and K_{zz} is the block in K related to z, i.e., $K_{zz} = \frac{\partial z}{\partial p_z'} + z_m z'$. Intuitively, the consumer would like to respond to a marginal income increase by $\mathrm{d}z = z_m \mathrm{d}m$. However, the quantity constraints prevents her from doing so, and therefore the virtual prices of that bundle have to go up with $-K_{zz}^{-1}\mathrm{d}z = -K_{zz}^{-1} z_m \mathrm{d}m$. *The constraint translates the income risk into price risks.*

The marginal utility of income is then

$$v_m^r \equiv (v'_{\pi_z} + v_m \overline{z}')\frac{\partial \pi_z}{\partial m} + v_m = v_m,$$

where the equality sign follows from Roy's identity. Differentiating one more time with respect to m yields

$$\begin{aligned} v_{mm}^r &\equiv v'_{m\pi_z}\frac{\partial \pi_z}{\partial m} + v_{mm}(1 + \overline{z}'\frac{\partial \pi_z}{\partial m}) \\ &= -v_{mm}\overline{z}'\frac{\partial \pi_z}{\partial m} - v_m z'_m \frac{\partial \pi_z}{\partial m} + v_{mm}(1 + \overline{z}'\frac{\partial \pi_z}{\partial m}) \\ &= v_{mm} - v_m z'_m \frac{\partial \pi_z}{\partial m}, \end{aligned} \tag{15}$$

where the second equality follows upon using (4). Use of (14) the leads to:

Theorem 2 *When facing the quantity constraints $\overline{z}$, the absolute degree of risk aversion can be decomposed into a virtual absolute degree risk aversion and a positive ordinal term:*

$$AR(p, m|\overline{z}) = AR(p_x, \pi_z, m^v) - z'_m K_{zz}^{-1} z_m. \tag{16}$$

The first term on the right hand side of (16) can be coined the virtual coefficient of absolute risk aversion. It corresponds to twice the risk premium per unit of variance in case the consumer is facing a small risk around the virtual income m^v, that can be traded for commodities at the price vector (p_x, π_z). Since K_{zz} is a negative definite matrix, so is its inverse. Therefore the quadratic form $z'_m K_{zz}^{-1} z_m$ is strictly negative (and entirely ordinal).

The result that absolute risk aversion under quantity constraints exceeds virtual absolute risk aversion can be explained as follows. Ideally, the consumer would like to respond to a small deviation in income, $\mathrm{d}m$, from its expected value, by increasing the demand for z commodities with $\mathrm{d}z = z_m \mathrm{d}m$. Since this is not feasible, the virtual price vector of z-goods increases with $\mathrm{d}\pi_z = -K_{zz}^{-1} z_m \mathrm{d}m$. This price increase has a double effect on the marginal utility of income: $\mathrm{d}\lambda = -\lambda_m z' \mathrm{d}\pi_z - \lambda z'_m \mathrm{d}\pi_z$. The first effect is the change in marginal utility because real income falls, while the second effect is the compensated price effect on marginal utility. The first effect is eliminated, however, because the consumer's virtual income, $m + (\pi_z - p_z)'\overline{z}$, is by definition adjusted with exactly $z\mathrm{d}\pi_z$. Hence, the change in marginal utility due to the virtual price change is $\lambda z'_m K_{zz}^{-1} z_m \mathrm{d}m$, and the relative change in marginal utility is $z'_m K_{zz}^{-1} z_m \mathrm{d}m$.

Assume first that the quantity constraints $\overline{z}$ are weakly binding, i.e., that they exactly coincide with $z^* \stackrel{\text{def}}{=} z(p, m)$, the levels the consumer would have chosen if her income takes the expected value. Then $\pi_z = p_z$, $m^v = m$ and the virtual degree of absolute risk aversion reduces to $AR(p, m)$. Proposition 2 then confirms Claim 1.[8]

Proposition 2 *If quantity constraints are weakly binding,*

$$AR(p, m|z^*) = AR(p, m) - z_m' K_{zz}^{-1} z_m.$$

This proposition is a generalization of a result by Drèze and Modigliani (1972). They considered a consumer deciding about the amount to save while facing an uncertain future income. They compared the attitudes towards income risk under two settings: (i) a timeless income risk where the consumer is informed about her income draw before making her savings decision, and (ii) a temporal income risk where the savings decision is made before the income draw is known. Drèze and Modigliani (1972, eq 2.9) showed that the risk aversion for temporal income risks exceeds that for timeless income risks by an ordinal term positively related to the (squared) income effect on current consumption and reciprocally related to the degree of substitution between current and future consumption. The constraint arises in the temporal context because savings decision can not respond to income shocks. Nevertheless, the constraint is weakly binding because the decision has been made optimally.

I now give a similar decomposition of the coefficient of relative risk aversion under quantity constraints. For this purpose, let Π_z denote the diagonal matrix with the virtual price vector π_z on its main diagonal. Eq (16) may now be rewritten as:

$$AR(p, m|\overline{z})\ m = AR(p_x, \pi_z, m^v)\ m^v \frac{m}{m^v} - z_m' \Pi_z \left(\Pi_z K_{zz} \Pi_z\right)^{-1} \Pi_z z_m\ m^v \frac{m}{m^v}.$$

The left-hand side is the degree of relative risk aversion under rationing, $RR(p, m|\overline{z})$. Define $b_z \stackrel{\text{def}}{=} \Pi_z z_m$, and $S_{zz} \stackrel{\text{def}}{=} \frac{1}{m^v} \Pi_z K_{zz} \Pi_z$. These are the Rotterdam parameterisations of the income and substitution effects, evaluated at virtual prices and income (cf Theil, 1976). Since my main focus will be on the relative risk aversion, Theorem 3 is useful:

[8] Alternatively, Proposition 2 may be seen as an application of the second-order envelope property of maximum value functions. Cf Dixit (1990, p 113-4).

Theorem 3 *The coefficient of relative risk aversion under quantity constraints may be decomposed as*

$$RR(p,m|\overline{z}) = \left\{RR(p_x,\pi_z,m^v) - b_z' S_{zz}^{-1} b_z\right\} \cdot \frac{m}{m^v}. \tag{17}$$

If the quantity constraints are weakly binding, this relationship reduces to $RR(p,m|\overline{z}) = RR(p,m) - b_z' S_{zz}^{-1} b_z$. In the next section, I will assess the difference between $RR(p,m|\overline{z})$ and $RR(p,m)$ when these constraints become strictly binding.

4.2 Strictly binding quantity constraints

I will now identify the four effects that in the introduction were claimed to make up the ECRA effect. For simplicity, I focus in the remainder on the case where z is a scalar and use z^* as a shorthand for $z(p_x,p_z,m)$. If the consumer is *rationed* (i.e., $z^* > \overline{z}$) π_z will exceed p_z and $m^v > m$, and *mutatis mutandis* with *forced consumption* (i.e., $z^* < \overline{z}$).

I start from the following identity:

$$RR(p,m|\overline{z}) - RR(p,m) \equiv [RR(p,m|z^*) - RR(p,m)] + [RR(p,m|\overline{z}) - RR(p,m|z^*)]$$

The first square bracket term is what I called in the introduction the price risk effect: the effect of being constrained at the optimal level z^*. The second one is the endogenously changing risk aversion effect: the effect of having to consume $\overline{z}$ rather than z^* and having to move along the budget line.

Theorems 1-3 provide the tools necessary to quantify both effects. Note that the 'Rotterdam' responses b_z and s_{zz} may also be written as $w_z^v \eta_z$ and $w_z^v \widehat{\varepsilon}_{zz}$ where w_z^v is the (virtual) budget share of good z, $w_z^v \overset{\text{def}}{=} \frac{\pi_z \overline{z}}{m^v}$, and η_z and $\widehat{\varepsilon}_{zz}$ denote the income and compensated own price elasticity, respectively. Therefore, (17) can be rewritten as

$$RR(p,m|\overline{z}) = \left(RR(p_x,\pi_z,m^v) - w_z^v \frac{\eta_z^2}{\widehat{\varepsilon}_{zz}}\right) \frac{m}{m^v}, \tag{18}$$

where it should be kept in mind that the behavioural responses (like η_z and $\widehat{\varepsilon}_{zz}$) are evaluated at the triple (p_x,π_z,m^v). The price risk effect is then obtained by evaluating (18) at (p_x,p_z,m) and subtracting $RR(p,m)$: $-w_z \frac{\eta_z^2}{\widehat{\varepsilon}_{zz}}$.

On the other hand, the effect of endogenously changing risk aversion may be approximated as:

$$RR(p,m|\overline{z}) - RR(p,m|z^*) \simeq \frac{\mathrm{d}RR(p,m|\overline{z})}{\mathrm{d}\overline{z}}|_{\overline{z}=z^*} (\overline{z} - z^*). \tag{19}$$

Thus I can write

$$RR(p,m|\overline{z}) - RR(p,m) \simeq \underbrace{-w_z\frac{\eta_z^2}{\widehat{\varepsilon}_{zz}}}_{\text{PRE}} + \underbrace{\frac{\mathrm{d}RR(p,m|\overline{z})}{\mathrm{d}\log\overline{z}}|_{\overline{z}=z^*}}_{\text{ECRAE}}\frac{\overline{z}-z^*}{z^*}.$$

I will now develop an expression for the endogenously changing risk aversion effect in the neighbourhood of the notional demand, $\frac{\mathrm{d}RR(p,m|\overline{z})}{\mathrm{d}\log\overline{z}}|_{\overline{z}=z^*}$. I will do so under

Assumption C *In the neighbourhood of the notional demand z^*, income and compensated price elasticities are constant.*

Expression (18) reveals three channels through which $\overline{z}$ affects $RR(p,m|\overline{z})$. First, its virtual counterpart, $RR(p_x,\pi_z,m^v)$, changes because π_z and m^v change:

$$\frac{\mathrm{d}RR(p_x,\pi_z,m^v)}{\mathrm{d}\log\overline{z}} = \left(\frac{\partial RR}{\partial\pi_z} + \frac{\partial RR}{\partial m^v}\overline{z}\right)\pi_z\frac{\partial\log\pi_z}{\partial\log\overline{z}} + \frac{\partial RR}{\partial m^v}(\pi_z - p_z)\,\overline{z}. \tag{20}$$

Since I look at the neighbourhood of z^*, $\pi_z = p_z$ and the income effect vanishes. Thus we are left with a compensated price effect. In the appendix, I show how Theorem 1 and Assumption C allow me to write this effect as

$$\frac{\mathrm{d}RR(p_x,\pi_z,m^v)}{\mathrm{d}\log\overline{z}}|_{\overline{z}=z^*} = w_z(1-\eta_z)(\eta_z - RR^*)\frac{w_z}{\widehat{\varepsilon}_{zz}},$$

where $RP^* = RP(p,m)$.

Second, the ratio $w_z^v\frac{\eta_z^2}{\widehat{\varepsilon}_{zz}}$ will change. Under Assumption C, this will happen in proportion with w_z^v. I show in the appendix that

$$\frac{\partial\log w_z^v}{\partial\log\overline{z}}|_{\overline{z}=z^*} = 1 + \frac{1-w_z}{\widehat{\varepsilon}_{zz}}. \tag{21}$$

Finally, there is the scaling factor $\frac{m}{m^v}$. I show in the appendix that

$$\frac{\partial\log m^v}{\partial\log\overline{z}}|_{\overline{z}=z^*} = \frac{w_z}{\widehat{\varepsilon}_{zz}} < 0. \tag{22}$$

Collecting results then gives the following operational expression for the ECRA-effect:

$$\begin{aligned}\frac{\mathrm{d}RR(p,m|\overline{z})}{\mathrm{d}\log\overline{z}}|_{\overline{z}=z^*} &= w_z(1-\eta_z)(\eta_z - RR^*)\frac{w_z}{\widehat{\varepsilon}_{zz}} \\ &+ \left(-w_z\frac{\eta_z^2}{\widehat{\varepsilon}_{zz}}\right)\left(1+\frac{1-w_z}{\widehat{\varepsilon}_{zz}}\right) - RR(p,m|z^*)\frac{w_z}{\widehat{\varepsilon}_{zz}}.\end{aligned} \tag{23}$$

In Table 1, I present for some recent empirical studies reporting on elasticities and budget shares for durable goods, the calibration of the price risk effect (column 5) and–under the assumption that $RR^* = 2$–the endogenously changing risk aversion effect (column 6).

Table 1. Calibration of the price risk effect (PRE) and the endogenously changing risk aversion/prudence effect (ECRAE/ECPE) for some recent empirical studies on durable goods demand.

	η_z	$\widehat{\varepsilon}_{zz}$	w_z	PRE_{RA}	$\text{ECRAE}^{b,c}$	τ^b	$\text{PRE}_P^{b,c,d}$	$\text{ECPE}^{b,c,d}$
1. Olney (1990) US (1920-83)	1.28	−.138	.10	1.22	−4.16	5.39	−0.54	−2.58
2. Pakoš (forthcoming) US (1951:Q1-2001:Q4)	1.50	−.247	.13	1.18	−1.18	2.14	0.44	−1.29
3. Deschamps (1993) UK (1955:Q1-1983:Q2)	2.98	−2.51	.05	0.18	0.11	0.15	0.17	.17
4. Deschamps (2003) UK (1955:Q1-1997:Q4)[a]	3.43	−.291	.15	6.05	−9.27	15.85	−.29	−16.30
average				2.16	−3.65		0.09	−4.96
av. excl. 3				2.82	−4.87		0.06	−6.68
av. excl. 3[e]					−6.04		−1.23	−10.16
av. excl. 3[f]					−3.70		0.86	−4.67

[a] The elasticities were calculated on the basis of the reported average budget share and the posterior medians of the coefficient distributions in Table VI. [b] Assuming $\widehat{\varepsilon}_{zz}$ and η_z to be locally constant. [c] Assuming $RR^*= 2$. [d] Assuming $RP^*= 3$. [e] Assuming $RR^*= 1$ and $RP^*= 2$. [f] Asuming $RR^*= 3$ and $RP^*= 4$.

Olney (1990) estimates a single equation of per capita net investment in durables in terms of a price index, per capita disposable income and other variables. Pakoš (2009) estimates the Euler equation corresponding to an intertemporal utility maximisation problem where the period utility index is a generalised CES function (allowing for non-homotheticity) defined over the service flow of durables and the consumption of non-durables. The price risk effects for these studies are around 1.2, while they differ in terms of the endogenously changing risk aversion effect, due to a larger value for $|\widehat{\varepsilon}_{zz}|$ in Pakoš (2009). Deschamps (1993, 2003) estimates a dynamic demand system of (6 and 9, resp.) commodity groups, one of them being durable household goods. The budget share of durables in the former study, however, is suspiciously low (Power (2004: 22), e.g., reports a share around 25% for the UK). The large income elasticity (3.43) for the second UK study produces very large values for the two effects. The unweighted table averages for the two effects are 2.2 and −3.7. Disregarding the 3th row, the averages are 2.8 and −4.9. Being stuck at the optimal service flow thus raises RR on average with 2.8, while a 10% reduction of the service flow below its optimal value further increases relative risk aversion with about .5.

Next, I look at the local effects of a quantity constraint on labour supply. Like in Section 2, I consider a consumer/worker caring about consumption and leisure. Labour time is sold at a wage rate a. Non-labour income is y. The indirect utility function defined on a and y is $v(a, y)$, and the analysis above can then be repeated to find that

$$RR(a, y|\overline{L}) = \left(RR(a^v, y^v) + \frac{a^v \overline{L}}{y^v} \frac{\eta_L^2}{\widehat{\varepsilon}_{LL}} \right) \frac{y}{y^v}, \tag{24}$$

where a^v and y^v are the virtual wage rate and non-labour income supporting $\overline{L}$ as the solution to the standard utility maximisation problem, η_L is the elasticity of labour supply w.r.t. non-labour income, $\widehat{\varepsilon}_{LL}$ is the compensated wage elasticity, and $RR(a, y) \stackrel{\text{def}}{=} -\frac{v_{yy}(a,y)y}{v_y}$.

Under Assumption C, I show in the appendix that

$$\frac{\mathrm{d}\left(RR(a, y|\overline{L})\right)}{\mathrm{d}\log \overline{L}}|_{\overline{L}=L^*} = \frac{\frac{aL^*}{y}}{\widehat{\varepsilon}_{LL}}(\eta_L - 1)\left(RR^* + \eta_L\right) \tag{25}$$

$$+\frac{\eta_L^2}{\widehat{\varepsilon}_{LL}} \frac{aL^*}{y} \left(1 + \frac{1 + \frac{aL^*}{y}}{\widehat{\varepsilon}_{LL}} \right) + \left(RR^* + \frac{aL^*}{y} \frac{\eta_L^2}{\widehat{\varepsilon}_{LL}} \right) \frac{\frac{aL^*}{y}}{\widehat{\varepsilon}_{LL}},$$

where RR^* is a shorthand for $RR(a, y)$ and L^* is the otpimal labour supply under certainty. Again, the first term is the compensated effect on $RR(a, y)$, the second term is the effect on the PRE-term, and the third term is the effect through the scaling factor $\frac{y}{y^v}$.

Note that when $\eta_L \to 0$ both the PRE ($\frac{aL^*}{y} \frac{\eta_L^2}{\widehat{\varepsilon}_{LL}}$) the ECRAE ($\frac{\mathrm{d}\left(RR(a,y|\overline{L})\right)}{\mathrm{d}\log \overline{L}}|_{\overline{L}=L^*}$) vanish, though the latter at a smaller rate (since it is $O(\eta_L)$). If labour supply is perfectly income inelastic, preferences are quasi-linear in consumption. Then all income shocks are ideally absorbed by consumption and restrictions on labour supply do not prevent that. Hence there is no effect on risk aversion.

Chetty (2006, Table 1) collects the values for $\frac{aL^*}{y}$, $\widehat{\varepsilon}_{LL}$, and η_L from 14 empirical studies on labour supply in US/Europe. On the basis of these data, I calculate in Table 2 $\frac{aL^*}{y} \frac{\eta_L^2}{\widehat{\varepsilon}_{LL}}$ (column 5), as well as the right-hand side of (25) under the assumption that $RR^* = 2$ (column 6) in Table 2.

Table 2. Calibration of the price risk effect (PRE) and the endogenously changing risk aversion/prudence effect (ECRAE/ECPE) using Chetty's (2006) collection of labour supply elasticities.

Row in Chetty (2006, Table 1)	η_L	$\widehat{\varepsilon}_{LL}$	$\frac{aL^*}{y}$ [a]	PRE_{RA}	$\text{ECRAE}^{b,c}$	τ^b	PRE_P [b,c,d]	$\text{ECPE}^{b,c,d}$
1	−.020	.130	1.99	.006	−.06	.014	.016	−1.01
2	−.120	.567	1.977	.050	.12	.107	.125	−1.15
3	−.010	.035	4.145	.012	1.98	.004	.020	−3.07
4	−.030	.192	4.632	.022	.48	.031	.048	−1.52
5	−.040	.088	.408	.007	−.02	.016	.019	−.61
6	−.297	.545	.707	.114	.35	.240	.276	−.94
7	−.185	.301	.513	.058	.19	.117	.142	−.80
8	−.008	.033	.815	.002	−.07	.004	.004	−.72
9	−.038	.288	.137	.001	−.01	.002	.002	−.090
10	−.110	1.040	2.025	.024	−.05	.061	.065	−.82
11	−.110	.660	2	.037	.02	.085	.096	−1.04
12	−.278	.646	.394	.047	.06	.121	.128	−.57
13	−.251	.432	2	.291	2.78	.187	.463	−3.11
14	−.222	.375	2.007	.264	2.87	.137	.410	−3.35
av. (st. dev.)	−.12 (11)			.067 (.094)	.62 (1.07)		.13 (.15)	−1.34 (1.04)
av. (st. dev.)[e]					1.10 (1.46)		.13 (.13)	−1.53 (1.94)
av. (st. dev.)[f]					.13 (.70)		.13 (.16)	−1.57 (.97)

[a] The value of $\frac{aL^*}{y}$ is implicitly available from Chetty's (2006) Table 1 as $1-\frac{\widehat{\varepsilon}_{LL}}{\eta_L}\times$ value in his column (6). [b] Assuming $\widehat{\varepsilon}_{LL}$ and η_L to be constant. [c] Assuming $RR^*= 2$. [d] Assuming $RP^*= 3$. [e] Assuming $RR^*= 1$ and $PR^*= 2$. [f] Assuming $RR^*= 3$ and $PR^*= 4$.

Compared with Table 1, PRE and ECRAE have a smaller order of magnitude. This is due to the fact that the typical income elasticity is small (the average is $-.12$). For 9 out of 14 studies, the ECRA effect is positive, meaning that an underemployment constraint makes the worker less risk averse. The average positive ECRAE is around 1, i.e., being underemployed for 10% reduces RR with 0.1. For the whole 'sample', the average ECRAE is 10 times larger than the average PRE. Taken at face value, this means that the average worker, when being underemployed for 10% is not more risk averse than when she can choose hours of work freely. While this conclusion rests on two assumptions–a base rate relative risk aversion of 2 and locally constant income and Hicksian wage elasticities of labour supply–it shows one cannot take for granted that people in an underemployment status are less willing to take risks. In fact, if $RR = 1$, then ECRAE equals 1.1 (last row), and a 10% underemployment constraint lowers RR (with $.11 - .07 = .04$).

The analysis in this section has shown that the effect of a local quantity constraint on relative income risk aversion is intricate. *A fortiori*, this will be the case when leaving the neighbourhood of the notional demand/supply. In section 5, I will illustrate the global behaviour of $RR(p, m|\overline{z})$ when the elasticity of substitution is constant.

5 Prudence with a quantity constraint

To investigate the effect of quantity constraints on the coefficients of prudence, I continue to assume that such constraint only applies to a single good (z). In the appendix, it is shown that

$$v_{mmm}^{r} = v_{mmm} - 3v_{mm}z_{m}\frac{\partial \pi_{z}}{\partial m} - v_{m}\frac{\tau}{m^{2}}, \tag{26}$$

where τ is a dimensionless term collecting all 'second order' ordinal responses:

$$\tau \stackrel{\text{def}}{=} \left(2z_{mm}\frac{\partial \pi_{z}}{\partial m} - \left(z_{m}^{2} - z_{m\pi_{z}} - z_{mm}\overline{z}\right)\left(\frac{\partial \pi_{z}}{\partial m}\right)^{2} + z_{m}\frac{\partial^{2}\pi_{z}}{\partial m^{2}}\right)m^{2}. \tag{27}$$

Dividing (26) through by (15) leads to

Theorem 4 *When facing a quantity constraint, the coefficient of absolute prudence is given by*

$$AP(p, m|\overline{z}) \stackrel{def}{=} -\frac{v_{mmm}^{r}}{v_{mm}^{r}} = \frac{AR \cdot \left(AP - 3\frac{z_{m}^{2}}{k_{zz}}\right)}{AR - \frac{z_{m}^{2}}{k_{zz}}} - \frac{\tau}{AR - \frac{z_{m}^{2}}{k_{zz}}}\frac{1}{(m^{v})^{2}}, \tag{28}$$

where all right-hand side terms are evaluated at (p_{x}, π_{z}, m^{v}).

Multiplying through by m gives the corresponding expression for the coefficient of relative prudence under a quantity constraint:

Corollary 1 *Under a quantity constraint, the coefficient of relative prudence is given by*

$$RP(p, m|\overline{z}) \stackrel{def}{=} -\frac{v_{mmm}^{r}}{v_{mm}^{r}}m = \frac{RR \cdot \left(RP - 3\frac{b_{z}^{2}}{s_{zz}}\right)}{RR - \frac{b_{z}^{2}}{s_{zz}}}\frac{m}{m^{v}} - \frac{\tau}{RR - \frac{b_{z}^{2}}{s_{zz}}}\frac{m}{m^{v}}. \tag{29}$$

As with relative risk aversion, I decompose the effect of a constraint as

$$RP(p,m|\overline{z}) - RP(p,m) \simeq [RP(p,m|z^*) - RP(p,m)] + \frac{\mathrm{d}RP(p,m|\overline{z})}{\mathrm{d}\log\overline{z}}|_{\overline{z}=z^*}\frac{\overline{z}-z^*}{z^*}.$$

The square bracket term is a price risk effect (but now of the marginal utility of income function), while the next term is an endogenously changing prudence effect. The former effect is no longer unambiguously positive. This can be seen by evaluating (29) at (p_x, p_z, m) and subtracting $RP^* = RP(p,m)$:

$$RP(p,m|z^*) - RP^* = \frac{w_z\frac{\eta_z^2}{\widehat{\varepsilon}_{zz}}(RP^* - 3RR^*) - \tau}{RR^* - w_z\frac{\eta_z^2}{\widehat{\varepsilon}_{zz}}}. \tag{30}$$

Making use of Assumption C, I show in the appendix that

$$\tau = w_z\frac{\eta_z^2}{\widehat{\varepsilon}_{zz}^2}\left(\widehat{\varepsilon}_{zz}(3-\eta_z) - \eta_z w_z(2+\eta_z) + \eta_z\right). \tag{31}$$

Table 1, column 7, gives the corresponding values for τ, all of which are positive. This tends to make the PRE negative. But unless PR^* is large relative to RR^*, this tendency is reversed. Indeed, with CRRA preferences, $RP^* = RR^* + 1$, so that $RP^* > 3RR^*$ iff $RR^* < \frac{1}{2}$, which is empirically not very likely (e.g., Barsky *et al.*, 1997). Column 8 of Table 1 gives the price risk effect for prudence on the assumption that $RP^* = 3$. In the appendix, I develop an expression for $\frac{\mathrm{d}RP(p,m|\overline{z})}{\mathrm{d}\log\overline{z}}|_{\overline{z}=z^*}$; its evaluation is presented in column 9. On average, the PRE is small, but this is due to the diverging numbers for the individual studies. The Olney figures suggest that short run constraints on the service flows of durables make the degree of relative prudence fall with around .5, while the Pakoš figures predict an increase with around .4. The ECPE figures for the US are negative (i.e., forced consumption reduces prudence) but small. Deschamps' (2003) elasticities for the UK imply a PRE on prudence of $-.3$, but a strong ECPE such that a 10% forced consumption reduces RP from an assumed value of 3 down to with 1.9.

In Table 2 (columns 8 and 9), I have reported on the corresponding measures for the 14 labour supply studies. All price risk effects for prudence (column 8) are positive, while all endogenously changing prudence effects (column 9) are negative.[9] On average, and in absolute value, the latter is

[9] PRE is now given by $RP(a,y|L^*) - RP^* = \frac{-sh_L\frac{\eta_L^2}{\widehat{\varepsilon}_{LL}}(RP^*-3RR^*)+\tau}{RR^*+sh_L\frac{\eta_L^2}{\widehat{\varepsilon}_{LL}}}$, where (assuming constant $\widehat{\varepsilon}_{LL}$ and η_L) $\tau = sh_L\frac{\eta_L^2}{\widehat{\varepsilon}_{LL}^2}\left(\widehat{\varepsilon}_{LL}(3-\eta_L) + sh_L\eta_L(2+\eta_L) + \eta_L\right)$. The expression for $\frac{\mathrm{d}RP(a,y|\overline{L})}{\mathrm{d}\log\overline{L}}|_{\overline{L}=L^*}$ is developed in the appendix.

about 10 times as large as the former, just like in the case of relative risk aversion. All else equal, the degree of relative prudence for a worker who is underemployed for 10% exceeds the degree for an unconstrained worker with about $\frac{1}{4}$. This conclusion is insensitive to the assumption on RR^* and RP^* (cf. last two rows).

6 CRRA-CES preferences and examples

In this section, I explore the global effects of quantity constraints on $RR(p, m|\overline{z})$ and $RP(p, m|\overline{z})$ by imposing more structure on preferences. In particular, I will assume a CES utility function homogenous of degree $1-\gamma$:

$$u(x, z) = \frac{1}{1-\gamma}\left[\alpha x^{\rho} + (1-\alpha) z^{\rho}\right]^{\frac{1-\gamma}{\rho}},$$

($-\infty < \rho \leq 1$, $0 < \alpha < 1$, and $\gamma > 0$). Denote $\sigma \stackrel{\text{def}}{=} \frac{1}{1-\rho}$ as the elasticity of substitution and

$$\delta_x(x, z) \stackrel{\text{def}}{=} \frac{\alpha x^{\rho}}{\alpha x^{\rho} + (1-\alpha) z^{\rho}}$$

as the intensity of consumption of the x-good (with a similar definition for δ_z). Then it is well known that in equilibrium the marginal budget share b_x equals the average budget share w_x, which in turn equals δ_x.

By construction, this utility function has $RR = \gamma$ and $RP = \gamma + 1$. I show in the appendix that

$$RR(p, m|\overline{z}) \cdot w_x = \left(\delta_x \gamma + \delta_z \frac{1}{\sigma}\right), \text{ and} \tag{32}$$

$$PR(p, m|\overline{z}) \cdot w_x = \begin{pmatrix} \delta_x & \delta_z \end{pmatrix} \begin{pmatrix} a_{xx} & a_{xz} \\ a_{xz} & a_{zz} \end{pmatrix} \begin{pmatrix} \delta_x \\ \delta_z \end{pmatrix}, \tag{33}$$

where $a_{xx} = \gamma(1+\gamma)$, $a_{xz} = \frac{1}{2}\frac{1}{\sigma}(2 - \frac{1}{\sigma} + 3\gamma)$ and $a_{zz} = \frac{1}{\sigma}(1 + \frac{1}{\sigma})$.

Consider first (32). The right-hand side is a weighted average of the relative risk aversion when unconstrained, and the inverse of the elasticity of substitution. Since $\frac{\delta_x}{w_x} = \frac{m}{m^v}$, (32) can be rewritten as

$$RR(p, m|\overline{z}) = \left(\gamma + \frac{1}{\sigma}\frac{w_z^v}{1 - w_z^v}\right)\frac{m}{m^v}. \tag{34}$$

In terms of the previous discussion, the ordinal term $\frac{1}{\sigma}\frac{w_z^v}{1-w_z^v}$ corresponds to the PRE. The ECRA effect of a change in $\overline{z}$ consists of two components:

the fact that the PRE is variable and depends on $\overline{z}$,[10] and the scaling factor $\frac{m}{m^v}$. The CRRA assumption effectively shuts down the income and relative price effects of a change in the quota on RR. Thus, two factors regulate the relationship between $RR(p,m|\overline{z})$ and γ. One is the degree of substitutability between the z-good and the other commodity. The lower this degree, the higher is the second, ordinal, term. Intuitively, a high substitution elasticity allows one to "work around" the constraint easily by consuming other goods. The other is the relationship between nominal income m and virtual income m^v. If the consumer is forced to consume more than her ideal demand, then $\pi_z < p_z$ and $m^v < m$. In this case, $RR(p,m|\overline{z})$ will exceed γ both because of a low degree of substitutability and because of forced consumption. On the other hand, if the consumer is rationed in the sense that her notional demand exceeds $\overline{z}$, then $\pi_z > p_z$ and $m^v > m$. The coefficient $RR(p,m|\overline{z})$ then falls below γ whenever

$$\frac{1}{\sigma}\frac{\pi_z\overline{z}}{p_x x} < \gamma\frac{m^v - m}{m^v}.$$

Since $\frac{m^v-m}{m^v} = \frac{\pi_z\overline{z}-p_z\overline{z}}{p_x x+\pi_z\overline{z}} < \frac{\pi_z\overline{z}}{p_x x}$, a necessary condition for this to happen is $\frac{1}{\sigma} < \gamma$.

The right-hand side of (33) is a weighted average of the elements of the symmetric matrix (a_{ij}). The terms a_{xx} and a_{zz} measure the 'base' rate of prudence and the difficulty to substitute, respectively; they are clearly non-negative. If $\sigma \geq \frac{1}{2+3\gamma}$, then $a_{xz} \geq 0$ and the quadratic form is positive for any $\delta_x \in [0,1]$. On the other hand, if $\sigma < \frac{1}{2+3\gamma}$, then $a_{xz} < 0$ and the quadratic form will become negative for some $\delta_x \in [0,1]$ if the corresponding determinant is negative. The solid lines in figure 3 delineate the regions where $\det(a_{ij})$ takes a positive or negative sign. Pairs of (γ,σ) below the dashed line result in $a_{xz} < 0$. Hence, combinations of (γ,σ) below the lower solid line will for some $\delta_x > 0$ result in a negative value for the quadratic form $\delta' A\delta$.[11] This means that for some $\overline{z}$, the quantity constrained consumer, while still risk averse, has become imprudent.

Proposition 3 *With CRRA-CES preferences, for every value of $RR > 0$, (i) there exists a sufficiently high elasticity of substitution σ, such that for some level of the quantity constraint $\overline{z}$ ($< z^*$), $RR(p,m|\overline{z}) < RR$, and (ii) there exists a sufficiently low σ, such that for some level of the quantity constraint $\overline{z}$, $PR(p,m|\overline{z}) < 0$.*

[10] Unless $\sigma = 1$ (Cobb-Douglas) in which case $w_z^v \equiv 1-\alpha$.

[11] The expression for the boundary line is $\sigma = \frac{1}{8}\frac{4+5\gamma-\gamma\sqrt{24+25\gamma}}{1+\gamma}$.

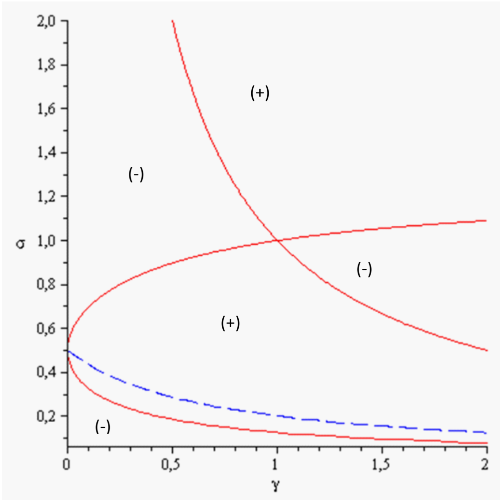

Figure 3: Regions for positive and negative definitness of the quadratic form (33).

I conclude this section with two examples. In both examples, $\alpha = \frac{1}{2}$, $m = 10$ and $p_x = p_z = 1$, so that the notional demand for each good is 5 units, and the budget shares when unconstrained equal $\frac{1}{2}$. In the first example, illustrated in figure 4, $\sigma = 2$ and $\gamma = 2$. The solid bold lines show $RR(p, m|\overline{z})$ (left) and $PR(p, m|\overline{z})$ (right). The horizontal dotted lines represent γ (left) and $\gamma + 1$ (right). The solid tin line and the dashed tin line correspond to the first and second right-hand side terms in (18) and (29). If the quantity constraint on z is less than 2.95 units, the consumer turns less risk averse than without facing any constraint at all: the endogenous diminishing risk aversion effect has overtaken the price risk effect. And once it falls below 2 units, she also becomes less prudent.

The next example, shown in figure 5, is for $\sigma = \frac{1}{5}$ and $\gamma = \frac{1}{2}$. Again, the ideal amount of the z-good is 5 units. Now, both $RR(p, m|\overline{z})$ and $RP(p, m|\overline{z})$ display a pronounced non-monotone behaviour in $\overline{z}$. A comparison with figure 4 shows that this is due to the behaviour of the response term. That $RR(p, m|\overline{z})$ is falling in the neighbourhood of 5 units squares with Claim 2 made in section 2. Relative prudence falls below the 'base' rate for $\overline{z}$-values above the notional demand, i.e., with forced consumption.

Figures 4 and 5 may give the impression that when the constraint is weakly binding, relative prudence always exceeds its base rate. This impression is incorrect. Indeed, if $m = 10$, $p_z = .55334$, $\alpha = .75$, $\sigma = .2$,

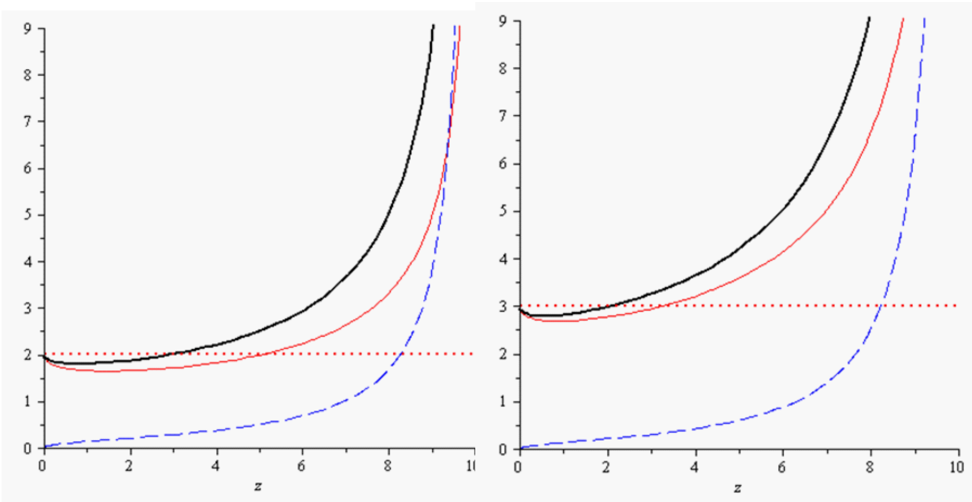

Figure 4: Relative risk aversion (left) and relative prudence (right): $\gamma = 2, \sigma = 2$. Notional demand for z is 5.

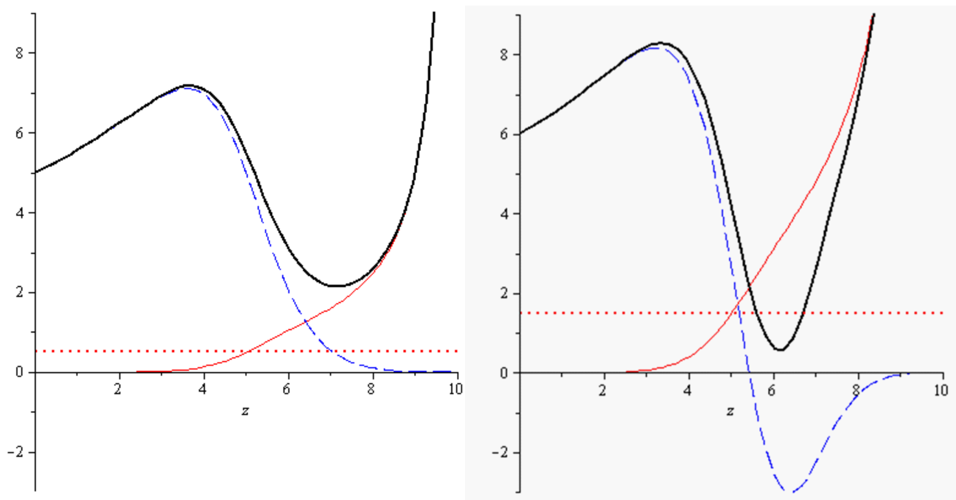

Figure 5: Relative risk aversion (left) and relative prudence (right): $\gamma = \frac{1}{2}, \sigma = \frac{1}{5}$. Notional demand for z is 5.

and $\gamma = .3$, the optimal demands are $x^* = 6.667$ and $z^* = 6.024$. When $\overline{z} = z^*$, the response term τ vanishes and $PR(p, m|\overline{z}) = .9428 + 0 < 1 + .3$. Furthermore, if $\overline{z} = 7$ (i.e., mild forced consumption), then $PR(p, m|\overline{z}) = 1.357 - 1.908 = -.551$.

These examples, and the discussion in the preceding section, illustrate that even with very 'regular' preferences, quantity constraints have intricate effects on the degrees of relative risk aversion and prudence. Stated differently, (income) insurance and the z-good can be both complements, as well as substitutes, depending on the level of the constraint. And the presence of a quantity constraint may both enhance and lower the precautionary savings motive.

7 Conclusion

I have traced out how quantity constraints on one or more goods or services affect the consumer's attitude towards income risk. Using the virtual price approach, I have decomposed the effect into a price risk effect and an endogenously changing risk aversion/prudence effect, and shown how these can be measured using information on price and income elasticities and relative degrees of risk aversion/prudence. Calibrations using empirical studies on the demand for durables and labour supply have illustrated the sign and order of magnitude of the two effects.

In addition, for CRRA-CES preferences, I identified conditions under which a quantity constraint may turn a consumer less risk averse or imprudent, and showed that the relationship between the degree of relative risk aversion/relative prudence, and the quantity constraint can easily become very non-monotonic.

In recent years, a literature has developed to explain the empirical variation in measures of risk aversion (e.g., Barsky *et al.* 1997, Guiso and Paiella, 2008, Aarbu and Schroyen, 2009) and prudence (e.g., Deck and Schlesinger, 2010) by regressing these measures on socioeconomic characteristics of the decision maker. I believe the findings in this paper show that the employment status of a worker/consumer, the imperfect malleability of durables, and transaction costs more generally, all may contribute to a person's attitude towards risk, and not necessarily in a uniform manner. This suggests one should account for these effects as much as possible to avoid imprecise and even inconsistent estimates of the parameters of interest.

The effects of employment constraints on attitudes towards risk obviously have macroeconomic applications. If the anticipation of future quantity constraints impacts on the degrees of risk aversion and prudence, this will trigger changes in the optimal amount of savings (see, e.g., Bauer and Buchholz, 2008) of which the macroeconomic effects may confirm the expectation. Therefore, a better understanding of these impacts is also relevant for studying the nature of underemployment equilibria.

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Appendix

Derivation of (7)

Since $\lambda_m = q_m' u_{qq} q_m$, differentiating both sides with respect to m yields

$$\lambda_{mm} = 2q_m' u_{qq} q_{mm} + \frac{\partial q_m' u_{qq} q_m}{\partial q'} q_m$$

where $\frac{\partial q_m' u_{qq} q_m}{\partial q'}$ is the effect on the quadratic form $q_m' u_{qq} q_m$ because of a perturbation in the Hessian following dq. Differentiating the first order condition (2) with respect to m yields

$$\lambda_m p = u_{qq} q_m.$$

Doing the same with the adding up condition (3-i) gives $p' q_{mm} = 0$. Therefore $q_m' u_{qq} q_{mm} = 0$ and $\lambda_{mm} = \frac{\partial q_m' u_{qq} q_m}{\partial q'} q_m$. The coefficient of absolute prudence with respect to income risk is then given by

$$AP(p,m) = -\frac{\lambda_{mm}}{\lambda_m} = -\frac{\frac{\partial q_m' u_{qq} q_m}{\partial q'} q_m}{q_m' u_{qq} q_m}.$$

Since $\frac{\partial q_m' u_{qq} q_m}{\partial q'}$ is the effect on the quadratic form $q_m' u_{qq} q_m$ because of a perturbation in the Hessian following dq, I can write it as

$$\begin{aligned}
\frac{\partial q_m' u_{qq} q_m}{\partial q'} &= [q_m' \frac{\partial u_{qq}}{\partial q_1} q_m, ..., q_m' \frac{\partial u_{qq}}{\partial q_n} q_m] \\
&= q_m' [\frac{\partial u_{qq}}{\partial q_1} q_m, ..., \frac{\partial u_{qq}}{\partial q_n} q_m] \\
&= \underbrace{q_m'}_{1\times n} [\underbrace{\frac{\partial u_{qq}}{\partial q_1}}_{n\times n}, ..., \underbrace{\frac{\partial u_{qq}}{\partial q_n}}_{n\times n}] \underbrace{(I_n \otimes q_m)}_{n^2\times n},
\end{aligned}$$

where I_n is the $(n \times n)$ identity matrix and $\otimes$ is the Kronecker product operator. Still another way of writing $\frac{\partial q_m' u_{qq} q_m}{\partial q'}$ is

$$\text{vec}(q_m q_m')' \frac{\partial \text{vec}(u_{qq})}{\partial q'} q_m.$$

Proof of Theorem 1: expression (8)

$$\begin{aligned}
\frac{\partial RR}{\partial p_i} &= -\frac{m}{v_m}\left(v_{mmp_i} - \frac{v_{mm}}{v_m} v_{mp_i}\right), \\
\frac{\partial RR}{\partial m} &= -\frac{v_{mm}}{v_m} - \frac{m}{v_m}\left(v_{mmm} - \frac{v_{mm}}{v_m} v_{mm}\right).
\end{aligned}$$

To keep utility constant, $\mathrm{d}p_i$ should be accompanied by $\mathrm{d}m = q_i \mathrm{d}p_i$. Therefore

$$\begin{aligned}\frac{\partial RR}{\partial p_i}|_{\mathrm{d}v=0} &= \frac{\partial RR}{\partial p_i} + \frac{\partial RR}{\partial m} q_i \\ &= -\frac{m}{v_m}\left(v_{mmp_i} + v_{mmm}q_i - \frac{v_{mm}}{v_m}(v_{mp_i} + v_{mm}q_i\right) - \frac{v_{mm}}{v_m}q_i.\end{aligned}$$

Since

$$\begin{aligned}v_{mp_i} &= v_{p_i m} = -v_{mm}q_i - v_m\frac{\partial q_i}{\partial m}, \text{ and} \\ v_{mmp_i} &= v_{p_i mm} = -v_{mmm}q_i - 2v_{mm}\frac{\partial q_i}{\partial m} - v_m\frac{\partial^2 q_i}{\partial m^2},\end{aligned}$$

the previous expression can be written as

$$\frac{\partial RR}{\partial p_i}|_{\mathrm{d}v=0} = \frac{v_{mm}m}{v_m}\frac{\partial q_i}{\partial m} + m\frac{\partial^2 q_i}{\partial m^2} - \frac{v_{mm}m}{v_m}\frac{q_i}{m}$$

Multiplying through by p_i and making use of the definition of RR and the fact that $\frac{\partial p_i q_i}{\partial m} = \frac{p_i q_i}{m}\frac{\partial q_i}{\partial m}\frac{m}{q_i}$ then gives (8).

Proof of Theorem 1, expression (9)

$$\begin{aligned}\frac{\partial RP}{\partial p_i} &= -\frac{m}{v_{mm}}\left(v_{mmmp_i} - \frac{v_{mmm}}{v_{mm}}v_{mmp_i}\right) \\ \frac{\partial RP}{\partial m} &= \frac{v_{mmm}}{v_{mm}} - \frac{m}{v_{mm}}\left(v_{mmmm} - \frac{v_{mmm}}{v_{mm}}v_{mmm}\right)\end{aligned}$$

Therefore

$$\begin{aligned}\frac{\partial RP}{\partial p_i}|_{\mathrm{d}v=0} &= \frac{\partial RP}{\partial p_i} + \frac{\partial RP}{\partial m} q_i \\ &= -\frac{m}{v_m}\left(v_{mmmp_i} + v_{mmmm}q_i - \frac{v_{mmm}}{v_{mm}}(v_{mmp_i} + v_{mmm}q_i\right) - \frac{v_{mmm}}{v_{mm}}q_i\end{aligned}$$

Since

$$v_{mmmp_i} = v_{p_i mmm} = -v_{mmmm}q_i - 3v_{mmm}\frac{\partial q_i}{\partial m} - 3v_{mm}\frac{\partial^2 q_i}{\partial m^2} - v_m\frac{\partial^3 q_i}{\partial m^3},$$

the previous expression can be written as

$$\frac{\partial RP}{\partial p_i}|_{\mathrm{d}v=0} = -\frac{v_{mmm}}{v_{mm}}(\frac{q_i}{m} - \frac{\partial q_i}{\partial m}) + (3 - \frac{v_{mmm}m}{v_{mm}}\frac{v_m}{v_{mm}m})\frac{\partial^2 q_i}{\partial m^2}m + \frac{v_m m}{v_{mm}}\frac{\partial^3 q_i}{\partial m^3}.$$

Multiplying through by p_i and making use of the definitions of RR and RP then gives (9).

Derivation of (23)

Totally differentiate (11) with respect to. $\overline{z}$ to get

$$\frac{\mathrm{d}\pi_z}{\mathrm{d}\overline{z}} = k_{zz}^{-1}\left[1 - z_m(\pi_z - p_z)\right],$$

so that

$$\frac{\mathrm{d}\log \pi_z}{\mathrm{d}\log \overline{z}} = \frac{1 - \frac{\partial \log z}{\partial \log m} w_z^v \frac{\pi_z - p_z}{\pi_z}}{\frac{\partial \log z}{\partial \log p_z}|_{\mathrm{d}u=0}}. \tag{35}$$

Therefore

$$\frac{\mathrm{d}\log \pi_z}{\mathrm{d}\log \overline{z}}|_{z^*} = \frac{1}{\frac{\partial \log z}{\partial \log p_z}|_{\mathrm{d}u=0}} = \frac{1}{\widehat{\varepsilon}_{zz}}. \tag{36}$$

The effect of a marginal change in $\overline{z}$ on virtual income $m^v = m + (\pi_z - p_z)\overline{z}$ is given by

$$\begin{aligned} \frac{\partial \log m^v}{\partial \log \overline{z}} &= \frac{\overline{z}}{m^v}\left(\frac{\partial \pi_z}{\partial \overline{z}}\overline{z} + (\pi_z - p_z)\right) \\ &= w_z^v \frac{\partial \log \pi_z}{\partial \log \overline{z}} + \frac{\overline{z}}{m^v}(\pi_z - p_z), \end{aligned} \tag{37}$$

Evaluating at $\overline{z} = z^*$ and making use of (36) then results in

$$\frac{\mathrm{d}\log m^v}{\mathrm{d}\log \overline{z}}|_{z^*} = \frac{w_z}{\widehat{\varepsilon}_{zz}}. \tag{38}$$

By definition, $w_z^v = \frac{\pi_z \overline{z}}{m^v} = \frac{\pi_z \overline{z}}{m + (\pi_z - p_z)\overline{z}}$. Then

$$\frac{\partial \log w_z^v}{\partial \log \overline{z}} = \frac{\partial \log \pi_z}{\partial \log \overline{z}} + 1 - \frac{\partial \log m^v}{\partial \log \overline{z}}. \tag{39}$$

Evaluating at $\overline{z} = z^*$ and making use of (36) and (38) then results in

$$\frac{\partial \log w_z^v}{\partial \log \overline{z}} = 1 + \frac{1 - w_z}{\widehat{\varepsilon}_{zz}}. \tag{40}$$

When income elasticities are constant,

$$z_m = \frac{z}{m}\eta_z, \text{ and } z_{mm} = \frac{z}{m^2}\eta_z(\eta_z - 1).$$

Then for $i = z$, (8) becomes

$$\frac{\partial RR}{\partial \log p_z}|_{\mathrm{d}v=0} = w_z(1-\eta_z)(RR^* - \eta_z). \tag{41}$$

When income and compensated price elasticities are constant,

$$\begin{aligned}\frac{\mathrm{d}RR(p, m|\overline{z})}{\mathrm{d}\log \overline{z}}|_{\overline{z}=z^*} &= (1-\eta_z)(RR^* - \eta_z)\frac{w_z}{\widehat{\varepsilon}_{zz}} \\ &\quad + \left(RR^* - w_z\frac{\eta_z^2}{\widehat{\varepsilon}_{zz}}\right)\left(-\frac{w_z}{\widehat{\varepsilon}_{zz}}\right) - w_z\frac{\eta_z^2}{\widehat{\varepsilon}_{zz}}\left(1 + \frac{1-w_z}{\widehat{\varepsilon}_{zz}}\right),\end{aligned}$$

which rearranged gives (23) in the text.

Derivation of (25)

Let sh_L^v be a shorthand for $\frac{a^v\overline{L}}{y^v}$. Differentiating (24) w.r.t. $\overline{L}$ gives

$$\begin{aligned}\frac{\mathrm{d}R(a, y|\overline{L})}{\mathrm{d}\log \overline{L}} &= \left(R(a,y) + sh_L^v\frac{\eta_L^2}{\widehat{\varepsilon}_{LL}}\right)(-\frac{y}{y^v})\frac{\partial \log y^v}{\partial \log \overline{L}} \\ &\quad + \frac{\eta_L^2}{\widehat{\varepsilon}_{LL}} sh_L^v \frac{\partial \log sh_L^v}{\partial \log \overline{L}}.\end{aligned}$$

Since $y^v = y + (a - a^v)\overline{L}$,

$$\frac{\partial \log y^v}{\partial \log \overline{L}} = (a - a^v)\frac{\overline{L}}{y^v} - sh_L^v \frac{\partial \log a^v}{\partial \log \overline{L}},$$

and

$$\frac{\partial \log sh_L^v}{\partial \log \overline{L}} = \left(\frac{\partial \log a^v}{\partial \log \overline{L}} + 1 - sh_L^v\left(\frac{a-a^v}{a^v} - \frac{\partial \log a^v}{\partial \log \overline{L}}\right)\right).$$

From implicit differentiation of $\overline{L} = L(a^v, y + (a - a^v)\overline{L})$, one gets that

$$\frac{\partial \log a^v}{\partial \log \overline{L}} = \frac{1 - sh_L^v\eta_L\frac{a-a^v}{a^v}}{\widehat{\varepsilon}_{LL}}.$$

It may be verified that with constant income elasticity $\frac{\partial R(a,y)}{\partial \log a}|_{\mathrm{d}v=0}$ takes the form

$$\frac{\partial R(a, y)}{\partial \log a}|_{\mathrm{d}v=0} = sh_L(\eta_L - 1)(\eta_L + RR(a, y)). \tag{42}$$

Collecting results and evaluating at $\overline{L} = L^*(= L(a,y))$ then gives

$$\begin{aligned}\frac{\mathrm{d}R(a,y|\overline{L})}{\mathrm{d}\log\overline{L}}|_{\overline{L}=L^*} &= (\eta_L - 1)(\eta_L + RR(a,y))\frac{sh_L}{\widehat{\varepsilon}_{LL}} \\ & \left(RR(a,y) + sh_L\frac{\eta_L^2}{\widehat{\varepsilon}_{LL}}\right)\frac{sh_L}{\widehat{\varepsilon}_{LL}} + \frac{\eta_L^2}{\widehat{\varepsilon}_{LL}}sh_L\left(1 + \frac{1+sh_L}{\widehat{\varepsilon}_{LL}}\right).\end{aligned}$$

Rearranging then gives (25) in the text.

Derivation of (26)

Differentiating (15) with respect to m, gives

$$\begin{aligned} v_{mmm}^r &= v_{mm\pi_z}\frac{\partial\pi_z}{\partial m} + v_{mmm}\left(1 + \overline{z}\frac{\partial\pi_z}{\partial m}\right) \\ & -(v_{mm} - v_m z_m\frac{\partial\pi_z}{\partial m})z_m\frac{\partial\pi_z}{\partial m} \\ & -v_m\left[z_{m\pi_z}\frac{\partial\pi_z}{\partial m} + z_{mm}\left(1 + \overline{z}\frac{\partial\pi_z}{\partial m}\right)\right] \\ & -v_m z_m\frac{\partial^2\pi_z}{\partial m^2}. \end{aligned} \tag{43}$$

Totally differentiating (4) for π_z with respect to m, gives

$$v_{mm\pi_z} = -2v_{mm}z_m - v_m z_{mm} - v_{mmm}\overline{z}.$$

Then (43) can be written as (26) in the text.

Derivation of (31).

By definition, $z_m(p_x, \pi_z, m^v) = \eta_z\frac{z(p_x,\pi_z,m^v)}{m^v}$, where η_z is assumed to be constant. Since z_{mm} and $z_{m\pi_z}$ are partials of z_m (i.e., derivatives w.r.t. 3th and 2nd argument, respectively), we get

$$\begin{aligned} z_{mm} &= \frac{\eta_z}{m^v}\left(z_m - \frac{\overline{z}}{m^v}\right) = \frac{\overline{z}\eta_z}{(m^v)^2}(\eta_z - 1), \\ z_{m\pi_z} &= \frac{\eta_z}{m^v}z_{\pi_z} = \frac{\overline{z}\eta_z}{\pi_z m^v}\varepsilon_{zz}, \end{aligned}$$

where ε_{zz} is the uncompensated own price elasticity.

Recall that $\frac{\partial\pi_z}{\partial m} = -\frac{z_m}{k_{zz}} = -\frac{\pi_z}{m^v}\frac{\eta_z}{\widehat{\varepsilon}_{zz}}$. Differentiating w.r.t. m and assuming $\widehat{\varepsilon}_{zz}$ and η_z constant gives

$$\begin{aligned} \frac{\partial^2\pi_z}{\partial m^2} &= -\frac{1}{m^v}\frac{\eta_z}{\widehat{\varepsilon}_{zz}}\left(\frac{\partial\pi_z}{\partial m} - \frac{\pi_z}{m^v}\left(1 + \frac{\partial\pi_z}{\partial m}\overline{z}\right)\right) \\ &= \frac{\pi_z}{(m^v)^2}\frac{\eta_z}{\widehat{\varepsilon}_{zz}^2}\left((1 - w_z)\eta_z + \widehat{\varepsilon}_{zz}\right). \end{aligned}$$

Therefore

$$\begin{aligned} 2z_{mm}\frac{\partial \pi_z}{\partial m} &= -2\frac{\overline{z}\eta_z}{(m^v)^2}(\eta_z - 1)\frac{\pi_z}{m^v}\frac{\eta_z}{\widehat{\varepsilon}_{zz}} \\ &= 2\frac{w_z^v}{(m^v)^2}\frac{\eta_z^2}{\widehat{\varepsilon}_{zz}}(1-\eta_z) \end{aligned}$$

and

$$\begin{aligned} z_m^2 - z_{m\pi_z} - z_{mm}\overline{z} &= \frac{\eta_z^2\overline{z}^2}{(m^v)^2} - \frac{\overline{z}\eta_z}{\pi_z m^v}\varepsilon_{zz} - \frac{\overline{z}^2\eta_z}{(m^v)^2}(\eta_z - 1) \\ &= \frac{\eta_z^2\overline{z}^2}{(m^v)^2} - \frac{\overline{z}\eta_z}{\pi_z m^v}(\varepsilon_{zz} + w_z\eta_z - w_z) \\ &= \frac{\eta_z^2\overline{z}^2}{(m^v)^2} - \frac{\overline{z}\eta_z}{\pi_z m^v}(\widehat{\varepsilon}_{zz} - w_z) \\ &= [\eta_z(w_z^v)^2 - \eta_z w_z^v(\widehat{\varepsilon}_{zz} - w_z)]\frac{1}{\pi_z^2}. \end{aligned}$$

Hence

$$(z_m^2 - z_{m\pi_z} - z_{mm}\overline{z})\left(\frac{\partial \pi_z}{\partial m}\right)^2 = [\eta_z(w_z^v)^2 - \eta_z w_z^v(\widehat{\varepsilon}_{zz} - w_z)]\frac{1}{(m^v)^2}\frac{\eta_z^2}{\widehat{\varepsilon}_{zz}^2}.$$

And finally,

$$\begin{aligned} z_m\frac{\partial^2 \pi_z}{\partial m^2} &= \frac{\eta_z\overline{z}}{m^v}\frac{\pi_z}{(m^v)^2}\frac{\eta_z}{\widehat{\varepsilon}_{zz}^2}((1-w_z)\eta_z - \widehat{\varepsilon}_{zz}) \\ &= \frac{w_z^v}{(m^v)^2}\frac{\eta_z^2}{\widehat{\varepsilon}_{zz}^2}((1-w_z)\eta_z + \widehat{\varepsilon}_{zz}). \end{aligned}$$

Then (27) can be written as

$$\begin{aligned} \tau &= w_z^v\frac{\eta_z^2}{\widehat{\varepsilon}_{zz}^2}\left(2\widehat{\varepsilon}_{zz}(1-\eta_z) - \eta_z^2 w_z^v + \eta_z\widehat{\varepsilon}_{zz} - 2\eta_z w_z^v + \eta_z + \widehat{\varepsilon}_{zz}\right) \\ &= w_z^v\frac{\eta_z^2}{\widehat{\varepsilon}_{zz}^2}(\widehat{\varepsilon}_{zz}(3-\eta_z) - \eta_z w_z^v(2+\eta_z) + \eta_z), \end{aligned}$$

which is (31) in the text.

Derivation of the ECP-effect used to compute column 9 in Table 1.

Assuming constant income and price elasticities. The effect of a small change in $\overline{z}$ in the neighbourhood of z^* on $RP(p, m|\overline{z})$ is the sum of two effects. The first effect traces the effects because of changes in w_z^v and m_v; it is given by

$$RP(p,m|\overline{z}) = \frac{RR \cdot \left(RP - 3w_z\frac{\eta_z^2}{\widehat{\varepsilon}_{zz}}\right)}{RR - w_z\frac{\eta_z^2}{\widehat{\varepsilon}_{zz}}}\frac{m}{m^v} - \frac{\tau}{RR - w_z\frac{\eta_z^2}{\widehat{\varepsilon}_{zz}}}\frac{m}{m^v}.$$

$$\begin{aligned}\frac{\mathrm{d}RP(p,m|\overline{z})}{\mathrm{d}\log\overline{z}}|^{\text{part 1}}_{\overline{z}=z^*} &= \frac{-3RRw_z\frac{\eta_z^2}{\widehat{\varepsilon}_{zz}} - \frac{\partial\tau}{\partial\log w_z} + RP(p,m|\overline{z})w_z\frac{\eta_z^2}{\widehat{\varepsilon}_{zz}}}{RR - w_z\frac{\eta_z^2}{\widehat{\varepsilon}_{zz}}}\frac{\partial\log w_z^v}{\partial\log\overline{z}}|_{\overline{z}=z^*}\\ &\quad -RP(p,m|\overline{z})\frac{\partial\log m^v}{\partial\log\overline{z}}|_{\overline{z}=z^*}\end{aligned}$$

where

$$\frac{\partial\tau}{\partial\log w_z} = \tau - w_z^2\frac{\eta_z^2}{\widehat{\varepsilon}_{zz}^2}\eta_z(2+\eta_z)$$

The second effect is the direct effect on $RR(p_x,\pi_z,m^v)$ and $RP(p_x,\pi_z,m^v)$– cf (20) and the corresponding expression for RP.

$$\begin{aligned}\frac{\mathrm{d}RP(p,m|\overline{z})}{\mathrm{d}\log\overline{z}}|^{\text{part 2}}_{\overline{z}=z^*} &= \frac{1}{RR - w_z\frac{\eta_z^2}{\widehat{\varepsilon}_{zz}}}\left(\frac{\partial RR}{\partial\log\pi_z}|_{\mathrm{d}v=0}\left(RP - 3w_z\frac{\eta_z^2}{\widehat{\varepsilon}_{zz}}\right) + RR\frac{\partial RP}{\partial\log\pi_z}|_{\mathrm{d}v=0}\right.\\ &\quad \left. -RP(p,m|\overline{z})\frac{\partial RR}{\partial\log\pi_z}|_{\mathrm{d}v=0}\right)|_{\overline{z}=z^*}\frac{1}{\widehat{\varepsilon}_{zz}},\end{aligned}$$

where $\frac{\partial RR}{\partial\log\pi_z}|_{\mathrm{d}v=0}$ is given by (41) and the corresponding expression for $\frac{\partial RP}{\partial\log\pi_z}|_{\mathrm{d}v=0}$ is obtained from (9) and the fact that when income elasticity is constant then $z_{mm}\frac{m^2}{z} = \eta_z(\eta_z-1)$ and $z_{mmm}\frac{m^3}{z} = \eta_z(\eta_z-1)(\eta_z-2)$:

$$\frac{\partial RP}{\partial\log\pi_z}|_{\mathrm{d}v=0} = w_z(\eta_z-1)\left(3\eta_z - \frac{RP}{RR}\eta_z - \frac{\eta_z(\eta_z-2)}{RR} - PR\right).$$

The figures in Table 1, column 9, are then obtained as $\frac{\mathrm{d}RP(p,m|\overline{z})}{\mathrm{d}\log\overline{z}}|^{\text{part 1}}_{\overline{z}=z^*} + \frac{\mathrm{d}RP(p,m|\overline{z})}{\mathrm{d}\log\overline{z}}|^{\text{part 2}}_{\overline{z}=z^*}$.

Derivation of the expressions in footnote 7

First, note that $v_{yy}^r(a,y|\overline{L})$ can be obtained in the same way as (15) and $v_{yyy}^r(a,y|\overline{L})$ in the same way as (26):

$$\begin{aligned}v_{yy}^r &= v_{yy} + v_yL_y\frac{\partial a^v}{\partial y},\\ v_{yyy}^r &= v_{yyy} + 3v_{yy}L_y\frac{\partial a^v}{\partial y} + v_y\left(2L_{yy}\frac{\partial a^v}{\partial y} + (L_y^2 + L_{ya} - L_{yy}\overline{L})\left(\frac{\partial a^v}{\partial y}\right)^2 + L_y\frac{\partial^2 a^v}{\partial y^2}\right).\end{aligned}$$

Since $L_y = \eta_L \frac{L}{y}$, we have that

$$\begin{aligned} L_{yy} &= \eta_L \frac{L}{(y^v)^2}(\eta_L - 1) \\ L_{ya} &= \frac{L}{a^v}\frac{\eta_L}{y^v}\varepsilon_{LL}, \end{aligned}$$

where $\varepsilon_{LL} = \widehat{\varepsilon}_{LL} + sh_L\eta_L$, the uncompensated wage elasticity.

Furthermore, since $\frac{\partial a^v}{\partial y} = -\frac{L_y}{\widehat{L}_a} = -\frac{a^v}{y^v}\frac{\eta_L}{\widehat{\varepsilon}_{LL}}$, we have that

$$\frac{\partial^2 a^v}{\partial y^2} = \frac{a^v}{(y^v)^2}\frac{\eta_L}{\widehat{\varepsilon}^2_{LL}}\left((1 + sh_L)\eta_L + \widehat{\varepsilon}_{LL}\right).$$

Combining results, we obtain:

$$-\frac{v^r_{yyy}}{v^r_{yy}} y = \frac{RR^*\left(PR^* + 3sh^v_L\frac{\eta^2_L}{\widehat{\varepsilon}_{LL}}\right) + \tau}{\left(RR^* + sh^v_L\frac{\eta^2_L}{\widehat{\varepsilon}_{LL}}\right)}\frac{y}{y^v},$$

where

$$\tau = sh^v_L\frac{\eta^2_L}{\widehat{\varepsilon}^2_{LL}}\left(\widehat{\varepsilon}_{LL}(3 - \eta_L) + sh_L\eta_L(2 + \eta_L) + \eta_L\right).$$

The PRE is then found by evaluating this expression at (a, y) in stead of (a^v, y^v) and subtracting PR^*.

The figures obtained in Table 2, column 9 are obtained as the sum of

$$\begin{aligned} \frac{\mathrm{d}RP(a, y|\overline{L})}{\mathrm{d}\log\overline{L}}\Big|^{\text{part 1}}_{\overline{L}=L^*} &= \frac{3\ RR\ sh_L\frac{\eta^2_L}{\widehat{\varepsilon}_{LL}} + \frac{\partial\tau}{\partial\log sh_L} - RP(p, m|\overline{z})\ sh_L\frac{\eta^2_L}{\widehat{\varepsilon}_{LL}}}{RR + sh_L\frac{\eta^2_L}{\widehat{\varepsilon}_{LL}}}\frac{\partial\log sh^v_L}{\partial\log\overline{L}}\Big|_{\overline{L}=L^*} \\ &\quad -RP(a, y|\overline{L})\frac{\partial\log y^v}{\partial\log\overline{L}}\Big|_{\overline{L}=L^*} \end{aligned}$$

where

$$\frac{\partial\tau}{\partial\log sh_L} = \tau + sh^2_L\frac{\eta^2_L}{\widehat{\varepsilon}^2_{LL}}\eta_L(2 + 2\eta_L)$$

and

$$\begin{aligned} \frac{\mathrm{d}RP(a, y|\overline{L})}{\mathrm{d}\log\overline{L}}\Big|^{\text{part 2}}_{\overline{L}=L^*} &= \frac{1}{RR + sh_L\frac{\eta^2_L}{\widehat{\varepsilon}_{LL}}}\left(\frac{\partial RR}{\partial\log a^v}\Big|_{\mathrm{d}v=0}\left(RP + 3sh_L\frac{\eta^2_L}{\widehat{\varepsilon}_{LL}}\right) + RR\frac{\partial RP}{\partial\log a^v}\Big|_{\mathrm{d}v=0}\right. \\ &\quad \left. -RP(a, y|\overline{L})\frac{\partial RR}{\partial\log a^v}\Big|_{\mathrm{d}v=0}\right)\Big|_{\overline{L}=L^*}\frac{1}{\widehat{\varepsilon}_{LL}}. \end{aligned}$$

where $\frac{\partial RR}{\partial \log a^v}|_{\mathrm{d}v=0}$ is given by (42) and the corresponding expression for $\frac{\partial RP}{\partial \log a^v}|_{\mathrm{d}v=0}$ is

$$\frac{\partial RP}{\partial \log a^v}|_{\mathrm{d}v=0} = sh_L(\eta_L - 1)\left(-3\eta_L + \frac{RP}{RR}\eta_L + \frac{\eta_L(\eta_L - 2)}{RR} + PR\right).$$

Example with CRRA-CES preferences.

Solving

$$\begin{aligned} \max_{x,z} u(x, z) &= \frac{1}{1-\gamma}\left[\alpha x^\rho + (1-\alpha)z^\rho\right]^{\frac{1-\gamma}{\rho}} \\ \text{s.t.}\ \ p_x x + p_z z &= m, \end{aligned}$$

yields the notional demands

$$\begin{aligned} x(p_x, p_z, m) &= \left(\frac{\alpha}{p_x}\right)^\sigma \left[\alpha^\sigma p_x^{1-\sigma} + (1-\alpha)^\sigma p_z^{1-\sigma}\right]^{-1} m, \\ z(p_x, p_z, m) &= \left(\frac{1-\alpha}{p_z}\right)^\sigma \left[\alpha^\sigma p_x^{1-\sigma} + (1-\alpha)^\sigma p_z^{1-\sigma}\right]^{-1} m. \end{aligned}$$

The compensated price elasticity for good z is then

$$\frac{\partial \log z}{\partial \log p_z}|_{\mathrm{d}u=0} = -\sigma(1 - w_z),$$

where the budget share w_z is given by

$$w_z = \frac{p_z z(p_x, p_z, m)}{m} = (1-\alpha)^\sigma p_z^{1-\sigma}\left[\alpha^\sigma p_x^{1-\sigma} + (1-\alpha)^\sigma p_z^{1-\sigma}\right]^{-1}.$$

Solving $\overline{z} = z(p_x, \pi_z, m + (\pi_z - p_z)\overline{z})$ for π_z gives

$$\pi_z = \frac{1-\alpha}{\alpha} p_x^{1-\frac{1}{\sigma}} \left(\frac{m - p_z\overline{z}}{\overline{z}}\right)^{\frac{1}{\sigma}}.$$

This gives a virtual income

$$m^v = m + \left[\frac{1-\alpha}{\alpha} p_x^{1-\frac{1}{\sigma}} \left(\frac{m - p_z\overline{z}}{\overline{z}}\right)^{\frac{1}{\sigma}} - p_z\right]\overline{z}.$$

The indirect utility function is

$$v(p, m, \overline{z}) = \frac{1}{1-\gamma}\left[\alpha x(\overline{z})^\rho + (1-\alpha)\overline{z}^\rho\right]^{\frac{1-\gamma}{\rho}},$$

where $x(\overline{z}) \stackrel{\text{def}}{=} m - p_z\overline{z}$. Denote $c(\overline{z}) = [\alpha x(\overline{z})^\rho + (1-\alpha)\overline{z}^\rho]$.

Then

$$\begin{aligned}
v_m &= \alpha c(\overline{z})^{\frac{1-\gamma}{\rho}-1} x(\overline{z})^{\rho-1} \\
v_{mm} &= \alpha^2 (1-\gamma-\rho)\, c(\overline{z})^{\frac{1-\gamma}{\rho}-2} x(\overline{z})^{2(\rho-1)} \\
&\quad + \alpha(\rho-1) c(\overline{z})^{\frac{1-\gamma}{\rho}-1} x(\overline{z})^{\rho-2} \\
v_{mmm} &= \alpha^3 (1-\gamma-\rho)(1-\gamma-2\rho)\, c(\overline{z})^{\frac{1-\gamma}{\rho}-3} x(\overline{z})^{2(\rho-1)} \\
&\quad + 3\alpha^2 (1-\gamma-\rho)(\rho-1) c(\overline{z})^{\frac{1-\gamma}{\rho}-2} x(\overline{z})^{2\rho-3} \\
&\quad + \alpha(\rho-1)(\rho-2) c(\overline{z})^{\frac{1-\gamma}{\rho}-1} x(\overline{z})^{\rho-3}.
\end{aligned}$$

v_{mm} can be factored as

$$\begin{aligned}
v_{mm} &= \alpha c(\overline{z})^{\frac{1-\gamma}{\rho}-1} x(\overline{z})^{\rho-2} \left\{ \gamma \frac{\alpha x(\overline{z})^\rho}{c(\overline{z})} + (1-\rho) \frac{(1-\alpha)\overline{z}^\rho}{c(\overline{z})} \right\} \\
&= \alpha c(\overline{z})^{\frac{1-\gamma}{\rho}-1} x(\overline{z})^{\rho-2} \left\{ \gamma\delta_x + (1-\rho)\delta_z \right\}
\end{aligned}$$

Likewise, v_{mmm} may be factored as

$$\begin{aligned}
v_{mmm} &= \alpha c(\overline{z})^{\frac{1-\gamma}{\rho}-1} x(\overline{z})^{\rho-3} \Bigg\{ (1-\gamma-\rho)(1-\gamma-2\rho) \left(\frac{\alpha x(\overline{z})^\rho}{c(\overline{z})} \right)^2 \\
&\quad + 3(1-\gamma-\rho)(\rho-1) \frac{\alpha x(\overline{z})^\rho}{c(\overline{z})} + (\rho-1)(\rho-2) \Bigg\} \\
&= \alpha c(\overline{z})^{\frac{1-\gamma}{\rho}-1} x(\overline{z})^{\rho-3} \left\{ (1-\gamma-\rho)(1-\gamma-2\rho){\delta_x}^2 \right. \\
&\quad \left. + 3(1-\gamma-\rho)(\rho-1)\delta_x + (\rho-1)(\rho-2) \right\}.
\end{aligned}$$

The curly bracket term is a 2$^{\text{nd}}$ degree polynomial in δ_x.

Since $\delta_z = 1 - \delta_x$, we may ask for which coefficients a_{xx}, a_{zz}, a_{xz} the quadratic form $(\delta_x \;\; \delta_z)\left(\begin{smallmatrix} a_{xx} & a_{xz} \\ a_{xz} & a_{zz} \end{smallmatrix}\right)\left(\begin{smallmatrix} \delta_x \\ \delta_z \end{smallmatrix}\right)$ results in the above polynomial. The answer is $a_{xx} = \gamma(1+\gamma)$, $a_{xz} = \frac{1}{2}\frac{1}{\sigma}(2 - \frac{1}{\sigma} + 3\gamma)$ and $a_{zz} = \frac{1}{\sigma}(1 + \frac{1}{\sigma})$.

Hence,

$$\begin{aligned}
-\frac{v_{mm}}{v_m} &= \frac{1}{x(\overline{z})} \left\{ \gamma\delta_x + (1-\rho)\delta_z \right\}, \\
-\frac{v_{mmm}}{v_{mm}} &= \frac{1}{x(\overline{z})} (\delta_x \;\; \delta_z) \begin{pmatrix} a_{xx} & a_{xz} \\ a_{xz} & a_{zz} \end{pmatrix} \begin{pmatrix} \delta_x \\ \delta_z \end{pmatrix}.
\end{aligned}$$

Multiplying through by m and noting that $w_x = \frac{x(\overline{z})}{m}$ since $p_x = 1$ gives (32) and (33) in the text.

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