Poverty Measurement:

The Critical Comparison Value

by

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Abstract

The basic problem in poverty measurement is how to weigh the income of different groups. This is a normative problem on which people differ in opinion, and hence we should seek a way of dealing with the issue that takes into account this plurality. In the paper, we suggest an approach to poverty measurement which avoids incorporating any particular normative position on how to weigh the interests of various poor groups, but rather reports on changes in poverty by making explicit the link between various normative positions and ordinal conclusions in poverty measurement. Within this framework, by applying a generalized version of Decartes' Rule of Signs, we present results that should provide useful guidance in a poverty comparison.

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1. Introduction

Suppose the population size in a society is constant and we agree on an income poverty line.[2] In this case, the remaining problem in poverty measurement is how to weigh the income of different groups. Conventionally, we neglect the income of non-poor people, and hence the focus is on income differences in the poor group. Obviously, from an ordinal perspective, measurement of poverty is trivial if the change in income has the same direction for everyone in the poor group. However, in some situations, there is a conflict of interest within this group, and then in most cases we have to specify a weight function (or a class of functions) in order to solve the measurement problem. This is a normative problem on which people differ in opinion, and hence we should seek a way of dealing with the issue that takes into account this plurality.

In the poverty measurement literature, there are two kinds of responses to this problem. On the one hand, poverty is frequently measured by applying a few specific weight functions (which also make available cardinal information). However, the choice of weight functions can be criticized for being rather arbitrary, and consequently it is of considerable interest to notice the framework of partial poverty orderings. Within this framework, general dominance results are established, which provide useful guidelines for when we can attain unambiguous ordinal conclusions in poverty measurement.[3]

In this paper, we shall suggest a third possibility, which can provide some additional insight to this question. In the suggested approach, we will avoid incorporating any particular normative position on how to weigh the interests of various poor groups, but rather report on changes in poverty by making explicit the link between various normative positions and ordinal conclusions in poverty measurement.[4] We motivate this approach further in section 2, and elaborate on the formal framework in sections 3 and 4. The main results are presented in section 5.

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[2] Hence, the focus of this paper is on economic poverty. See, among others, Sen (1992) for a broader approach to the problem of poverty.

[3] Ravallion (1994) and Zheng (1997) provide useful surveys of these approaches.

[4] Formally, this framework is closely related to the literature on internal rates of return in corporate finance, but with some important differences. In particular, see Pratt and Hammond (1979).

2. Motivation

Let us consider two examples, which will highlight some difficulties with the conventional approaches. In this discussion, assume that the poverty line z=100, the population size n=9, and q is the number of poor people.

First, consider two discrete income distributions $x = \{30,40,40,90,90,90,90,90,90\}$ and $y = \{20,50,50,85,85,85,85,85,85\}$, where we assume that each person obtains the same position in the alternatives. In this case, there is a conflict of interest within the poor group when we compare x and y (which, by way of illustration, could be the income distribution before and after some policy reform), and we need a weight function in order to compare the level of poverty in these two situations. Frequently, members of the Foster-Greer-Thorbecke class of poverty measures are applied. If we assume that the elements of the income distributions are pre-sorted in nondecreasing order, then this class is defined as follows:[5]

(1) $$P^F(x,a) = \frac{1}{n}\sum_{i=1}^{q}\left(\frac{z-x_i}{z}\right)^a ;$$

where a is a parameter which represents a normative position on how to weigh interests within the poor group.

But which members of this class should we apply? Usually, when the focus is on strictly convex poverty measures, i.e. measures that are distribution sensitive, we consider the values $a = 2$ and $a = 3$. In this case, we find that the level of poverty is higher in y than x. However, this result is not very robust, because for every value of a that belongs to

[5] See Foster et.al. (1984).

the interval $[2.1,\ 2.9]$ we attain the opposite conclusion.[6] Thus, this approach, which reports conclusions on the level of poverty on the basis of a few weight functions, can be very misleading. In some cases, it might give an impression of robustness that is not well founded.

Second, compare *x* to *w*, where $w = \{29.999,89,89,89,89,89,89,89,89\}$. Neither in this case is it possible to attain any unambiguous result in poverty measurement, unless we restrict ourselves to $a \in [1,\ 51]$.[7] Therefore, if general dominance (giving an unambiguous conclusion for every $a \in [1, \infty)$) is the underlying criterion, we would not be able to rank *x* and *w* in a poverty comparison. But is general dominance necessarily an appropriate approach to this problem? If there is overlapping consensus in a society about the relevant range of values of a, say the interval $[1,\ 51]$, then we might consider it as sufficient to establish a restricted dominance result for this range. In this case, the framework of general dominance could give the impression of ambiguity that is not well-founded in the public opinion.

The conventional approaches face opposite problems. If we content ourselves with applying a few specific weight functions, then we might demand to little in order to attain a conclusion in a poverty comparison; general dominance, on the other hand, might demand too much. In the following, we will outline an approach that in some sense avoids these problems, because the aim of the approach is slightly different from the standard in the literature. The aim is not to attain a specific ranking, but to elucidate in a comprehensive manner the link between various normative positions and ordinal conclusions in poverty comparisons.

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6 $a \approx 2.088$ or $a \approx 2.959$ implies that the level of poverty is the same in *x* and *y*.

7 $a \approx 51{,}32$ implies that the level of poverty is the same in *x* and *z*.

3. Preliminaries

In the following, we will work with cumulative distribution functions drawn from the set $\mathcal{F} := \{F: [0,\infty) \to [0,1] \mid F \text{ is nondecreasing and right continuous; } F(0)=0\}$. Given $F \in \mathcal{F}$, set $F_1 := F$ and define inductively $F_t(y) := \int_0^y F_{t-1}(u)du$, for each positive integer *t*.

In general, let $a \in [1,\infty)$ be a parameter which value represents a normative position on how to weigh interests within the poor group. A class of poverty measures can now be defined as a function *P:* $\mathcal{F} \times [1,\infty) \to [0,1]$, where we assume that $z > 0$. Hence, by way of illustration, $P(F,\bar{a})$ is the degree of poverty associated with the cumulative income distribution *F* when the normative parameter value is $\bar{a}$. Let $\mathcal{P}$ be the set of all continuous functions of this kind.

In the analysis, we will narrow the discussion to the Foster et.al. class of poverty measures, which is a prominent poverty measure in the poverty measurement literature.[8] By integrating by parts, the class can be represented as follows when we work with continuous income distributions:

(2) $$P^F(F,a) = \frac{a}{z}\int_0^z F(y)\left(\frac{z-y}{z}\right)^{a-1}dy.$$

4. The critical comparison value

An interesting strategy in ordinal poverty measurement might be to report on the normative parameter values that imply that the extent of poverty is considered to be the

[8] In a companion paper, Manne and Tungodden (1998) discuss some extensions.

same in two income distributions. This approach will avoid the problems illustrated in section 2, and make the link between poverty measurement and normative reasoning more transparent.

In the following, we will refer to these normative values as critical comparison values, which can be defined formally as follows:

Critical Comparison Values: *For any* $F, G \in$ *F*, $P^I \in$ *P, and* $k, l \geq 1$, *the set of critical comparison values in the interval (k,l) is defined as* $C^I_{k,l}(F,G) := \{\alpha \in (k,l) \mid P^I(F,\alpha) = P^I(G,\alpha)\}$.

Obviously, the set of critical comparison values may in some cases be empty, as illustrated by the various dominance results in the literature for $\alpha \in [1,\infty)$. In the following discussion, however, we will aim at providing a more general understanding of what determines the cardinality of C^I.

Before we outline the analysis, we should comment on the possibility of providing a normative interpretation of the critical comparison values. If this were impossible, then we could seriously question the relevance of the suggested approach. But it turns out to be rather easy to get a good understanding of the normative position implied by any particular parameter value. Let me illustrate this by considering the problem within the framework of the Foster et.al. class of poverty measures. Consider the cases where $\alpha = 2$ and $\alpha = 3$. Both measures assign weight 1 to people without any income. Let this group be the reference poor. Now, choose another income level below the poverty line, say the income level $z/2$. In this case, $\alpha = 2$ ($\alpha = 3$) implies that we consider a situation where one reference poor is lifted out of poverty as equivalent to a situation where four (eight) people with income level *z/2* is lifted out of poverty. Surely, this line of reasoning only provides a rough evaluation of these normative positions, but, in my view, it illustrates the possibility of reflecting on the appropriate range of values. For example, my guess is that most people would reject the claim that to lift one reference poor out of poverty is equivalent to lifting one million people with income *z/2* out of poverty, which is the implication of $a \approx 20$ (if we stay within the Foster et.al. class of poverty measures).

5. Analysis

At the outset, it is of interest to notice that there is no upper limit on the cardinality of the set of critical comparison values.

***Observation**: For any positive integer t, there are distribution functions* $F, G \in F$ *such that the cardinality of* $C^F_{1,\infty}(F,G)$ *is t.*

Proof. See Appendix.

Thus, according to this observation, it might be quite difficult to single out the relevant critical comparison values in ordinal poverty measurement. But the picture becomes much more structured if we take into account the following proposition.

Proposition: *For any* $F, G \in F$ *and* $k \geq 1$, *the number of changes in the sign of* $F_k - G_k$ *in the interval* $[0, z]$ *determines an upper limit on the cardinality of* $C^F_{k-1,\infty}(F,G)$.

Proof. Suppose that for some $F, G \in F$ and $k \geq 1$, there are q changes in the sign of $F_k - G_k$ in the interval $[0, z]$.We will prove that this implies that q is the upper limit of $C^F_{k-1}(F,G)$.

(a) By integrating by parts, it follows that for $a > k$-1:

(4) $$\Delta(F,G,a) = \frac{a \cdot (a-1)\cdots(a-k+1)}{z^{k-1}} \int_0^z [F_k(y) - G_k(y)](\frac{z-y}{z})^{a-k} dy \; .$$

(4) implies that $\Delta(F,G,a)$=0 when $\overline{\Delta} \equiv \int_0^z [F_k(y) - G_k(y)](\frac{z-y}{z})^{a-k} dy = 0$.

(b) (4) can be rewritten as follows:

(5) $$\overline{\Delta} \equiv \int_0^z [F_k(y) - G_k(y)] e^{(a-k)\ln(\frac{z-y}{z})} dy \; .$$

$\overline{\Delta}$ is a Riemann integral, and then in the usual manner we can write (5) as follows:

(6) $$\overline{\Delta} = \lim \sum_{j=1}^{m} \left[F_k(y_j) - G_k(y_j)\right] e^{(a-k)\ln(\frac{z-y_j}{z})} \Delta y_j \; .$$

$(e^{(a-k)\ln(\frac{z-y_1}{z})}, \ldots, e^{(a-k)\ln(\frac{z-y_m}{z})})$ is a Decartes system on $a \in \langle k-1, \infty]$, and thus, by choosing a sufficiently fine partition, the result follows from the generalized version of Decartes' Rule of Signs (see Borwein and Erdélyi (1995, p. 100-104)). ð

Two immediate corollaries of this proposition should be noticed.

Corollary 1: *For any* $F, G \in F$ *and* $k \geq 2$, *if* $F_k - G_k > 0$ *for every income level in the interval* $[0, z]$, *then* $P^F(F, a) - P^G(G, a) > 0$ *for every parameter value in the interval* $a \in (k-1, \infty)$.

Corollary 2: *For any* $F, G \in F$ *and* $k \geq 2$, *there is at most a unique critical comparison value in the interval* $a \in (k-1, \infty)$ *if there is only one change in the sign of* $F_k - G_k$ *in the interval* $[0, z]$.

Corollary 1 reports a restricted dominance result. Corollary 2 highlights the fact that if the cumulative distribution functions cross only once, there is at most a unique critical comparison value for the whole range of plausible normative positions. In this case, if the unique critical comparison value exists, a poverty comparison is a two-edged story, where the conclusion depends on whether you defend a normative position defined by a value below or above the critical comparison value.

6. Concluding remarks

In poverty measurement, disagreement about how to weigh interests within the poor population sometimes makes it difficult to reach a conclusion that everyone can support. In these cases, it can be of importance to make clear the implications of the relevant normative positions, and then leave it to the public debate (eventually) to settle the dispute. In this paper, we have introduced a framework that follows this route, and makes explicit the link between ordinal poverty measurement and various normative positions.

Two objections to this approach should be considered. First, it is clear that presently most individuals do not know their preferred value (or range) of the parameter, and thus we might question the relevance of making explicit the critical comparison values.

However, as illustrated in section 4 of this paper, it is trivial to provide a rough evaluation of these parameter values, and hence this objection should be rather easy to overcome in public debates. Second, this approach is narrower than the general literature on dominance results, because the critical comparison values are reported for a specific class of poverty measures. Consequently, those who prefer, let us say, the Sen poverty measure might not be convinced by the reported values. Surely, this is a reasonable objection, but still we will argue that the suggested approach should be of interest in many cases. If we are unable to attain general dominance results, then we should aim at clarifying whether this incompleteness reflects disagreement among normative positions well founded in the public opinion. One way of doing this is the chose a class of poverty measures and report on the critical comparison values. It is an imperfect approach, but still it might enhance our understanding of the underlying problem.

Appendix

Proof of Observation

The proof goes by induction, but it will be useful first to notice the following lemma.

Lemma: *For any* $F,G\in$ *F, if there is some* $\bar{y}<z$ *such that:* $F(\bar{y})>G(\bar{y})$ *and* $F(y)\geq G(y),\ \forall\ y<\bar{y}$, *then* $\lim_{a\to\infty}\Delta(F,G,a)\equiv P^F(F,a)-P^F(G,a)>0$.

(a) The lemma is trivial in cases where $F(y)\geq G(y),\forall\ y\in[0,z]$. Otherwise, it can be proved by establishing the following inequality:

(3) $$\lim_{a\to\infty}\left(\frac{z}{z-\bar{y}}\right)^{a-1}\int_0^z[F(y)-G(y)]\left(\frac{z-y}{z}\right)^{a-1}dy>0.$$

Divide the integral into two parts, integrating first from 0 to $\bar{y}$ and then from $\bar{y}$ to *z*. It is now easily seen that the left hand side of (3) is at least as large as $\left[F(\bar{y})-G(\bar{y})\right]$. The result follows.

We are now ready to prove the Observation.

(b) Consider two distribution functions $F^1,G^1\in$ *F* , where for some interval $(a,b)\subset[0,\bar{y}]\subset[0,z]$:

$$F^1(y)=G^1(y),\ \forall\ y\in[0,\bar{y}]\setminus(a,b),$$

$$F^1(y) > G^1(y), \forall y \in (a,b)$$
$$F_2^1(z) < G_2^1(z).$$

In this case, it follows straightforwardly that $P^F(F^1) < P^F(G^1)$ when $a = 1$. Moreover, by taking into account the lemma, we see that $\lim_{a\to\infty} \Delta(F,G,a) = P^F(F^1,a) - P^F(G^1,a) > 0$. Δ is continuous in a, and thus there exists at least one critical comparison value. In addition, it is easily seen that Δ changes sign for one such value.

(c) Consider two distribution functions $F^t, G^t \in$ *F*, where for some interval $(a,b) \subset [0,\bar{y}] \subset [0,z]$:

$$F^t(y) = G^t(y),\ \forall\, y \in [0,\bar{y}] \setminus (a,b),$$
$$F^t(y) > G^t(y),\ \forall\, y \in (a,b),$$

and there exist at least *t* critical comparison values where Δ changes sign. It remains to prove that in this case we can find distribution functions in *F* which imply at least *t+1* critical comparison values. For this purpose, consider the two distribution functions $F^{t+1}, G^{t+1} \in$ *F*, where for some interval $(c,d) \subset (0,a)$:

$$0 < G^{t+1}(y) - F^{t+1}(y) \le e,\ \forall\, y \in (c,d),$$
$$F^{t+1}(y) = F^t(y)\ \&\ G^{t+1}(y) = G^t(y),\ \forall\, y \notin (c,d).$$

The lemma implies that for any e>0, $\lim_{a \to \infty} \Delta(F^{t+1}, G^{t+1}, a) = P^F(F^{t+1}, a) - P^F(G^{t+1}, a) < 0$. However, since for F^t, G^t there exist at least t critical comparison values where Δ changes sign, it follows that by choosing a sufficiently small e, we can construct F^{t+1}, G^{t+1} such that there exist at least t+1 critical comparison values in this case. QED.

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