Tax-adjusted discount rates with investor taxes and risky debt

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October 2005, first version October 2004

Abstract

This paper derives tax-adjusted discount rate formulas with Miles-Ezzell leverage policy, investor taxes, and risky debt in the context of a standard tax system. This expands on other formulas that are commonly used and that, for example, assume riskless debt or make different tax assumptions. The paper shows that the errors from using these other formulas are material at reasonable parameter values. Expressions are also given for the asset beta and implementation using the CAPM is discussed.

JEL codes: G31, G32

Keywords: Capital structure, valuation, risky debt, cost of capital.

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1 Introduction

The treatment of the tax saving from debt in valuation has recently become a subject of renewed interest (Fernandez (2004), Cooper and Nyborg (2005)). However, there is still no standard approach with respect to calculating tax-adjusted discount rates in the presence of risky debt. Taggart (1991) extends the Miles-Ezzell (1980) formula for tax-adjusted discount rates by including the effect of personal taxes, but assumes that corporate debt is riskless. In practice, the cost of corporate debt includes a default spread. This can be a significant part of the cost of capital for some firms, especially with low interest rates, high leverage, and the lower equity risk premia that are now often used. Therefore, the correct treatment of the debt risk premium is necessary for realistic valuation. This paper extends Taggart's analysis by deriving the tax-adjusted discount rate formula for companies that follow the Miles-Ezzell (ME) leverage policy when both investor taxes and risky debt are included.

An alternative treatment is provided by Sick (1990), who derives a formula which he contrasts with an expression given in Brealey and Myers (1988).[1] According to Sick, the Brealey and Myers formula differs from his "..by the incorrect treatment of risky debt, as well as the failure to recognize that tax shields should be discounted at a cost of equity..." (Sick, 1990, p.1441). In this paper, we unify the treatments of Sick and Brealey and Myers. We clarify the source of the difference between their formulas for tax-adjusted discount rates, which turns out to depend on the way that companies are taxed when they are in default. Sick makes a particular assumption about this, which leads to a simplification of the way that interest tax shields are valued. We then derive valuation and cost of capital procedures with an assumption about taxes which we believe to be more realistic than that used by Sick. This leads to an extended version of the Miles-Ezzell formula that is similar, but not identical, to that given by Brealey and Myers. We show that the different approaches can lead to economically significantly different discount rates and values.

We include the possibility that the effect of investor taxes may make the tax benefit to borrowing less than the full corporate tax rate. For the US, there is evidence that this may be the case. Fama and French (1998) fail to find any increase in firm value associated with debt tax savings. On the other hand, Kemsley and Nissim (2002) use a different estimation technique and find that the net tax advantage to debt similar to the corporate tax rate. Thus, while the US empirical evidence is inconclusive, it certainly admits the possibility that investor taxes make the net tax advantage to

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[1]This expression is still in Brealey and Myers (2003),

debt less than the full corporate tax rate.[2] The changes to the US tax code introduced in 2003 increase any advantage of dividend income over interest income for most investors, which will tend to reduce the tax benefit of debt. In other countries, there are even stronger reasons to believe that investor taxes offset part of the corporate tax advantage to debt. In particular, where countries have imputation taxes, such an effect is built directly into their tax systems (Rajan and Zingales (1995)).

2   The tax-adjusted discount rate

We operate under the Miles-Ezzell (1980) leverage assumption, that leverage is maintained at a constant proportion of the market value of the firm. This is the most realistic simple leverage policy, and also the one that is consistent with the use of the weighted average cost of capital (WACC). The Miles-Ezzell formula applies to any profile of cash flows as long as the company maintains constant market value leverage. It gives a relationship between the leveraged discount rate, R_L, and the unleveraged rate, R_U. We analyze a firm with expected pre-tax cash flows C_t, at dates $t = 1, \ldots, T$. Between these dates, leverage remains fixed. After each cash flow, leverage is reset to be a constant proportion, L, of the leveraged value of the firm. The two rates R_U and R_L are defined implicitly as the discount rates that give the correct unleveraged and leveraged values when the after-tax operating cash flows are discounted:[3]

$$V_{Ut} = \Sigma_{i=t+1}^{T} C_i (1 - T_C)/(1 + R_U)^i \qquad\qquad t = 1, ...T \qquad\qquad (1)$$

$$V_{Lt} = \Sigma_{i=t+1}^{T} C_i (1 - T_C)/(1 + R_L)^i \qquad\qquad t = 1, ...T \qquad\qquad (2)$$

where V_{Ut} is the unleveraged value of the firm and V_{Lt} its leveraged value.

We assume that the representative investor has tax rates of T_{PD} on debt returns and T_{PE} on equity income and capital gains. The increase in the after-tax cash flow to this investor resulting from an incremental dollar of corporate interest is:

$$T_S = (1 - T_{PD}) - (1 - T_C)(1 - T_{PE}) \qquad\qquad (3)$$

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[2]Graham (2000) finds that the net tax advantage to debt is less than the full corporate tax rate because of non-debt tax shields. The valuation consequences of this are more complicated, because it implies that the tax rate that should be used in valuation is state and time-dependent.

[3]We assume that tax is levied on the operating cash flows. A more complex treatment does not alter the results.

This result is standard. We define the related variable, T^*, by:

$$T^* = T_S/(1 - T_{PD}) \tag{4}$$

Thus:

$$1 - T^* = \frac{(1 - T_C)(1 - T_{PE})}{1 - T_{PD}} \tag{5}$$

Following Taggart (1991), we define the required return on riskless equity as:

$$R_{FE} = \frac{R_F(1 - T_{PD})}{(1 - T_{PE})} = \frac{R_F(1 - T_C)}{(1 - T^*)} \tag{6}$$

The first equality results from setting the after-investor-tax returns on riskless debt and riskless equity equal to each other. The second equality follows from (5). Note that if T_{PD} and T_{PE} are equal then $R_{FE} = R_F$.

The equation that relates R_U and R_L is one of the most important in valuation. It is used whenever discount rates are adjusted to a different amount of leverage. Even when adjusted present value methods are used to take account of the tax saving from debt, this relationship may first need to be used to derive the unlevered required return. The original Miles-Ezzell formula that relates R_U and R_L was derived with an informal treatment of risky debt and no investor taxes. It depends on the 'cost of debt' where this could be interpreted either as the yield or the expected return. For clarity, we define the yield on the debt as Y_D, and the expected return on the debt as R_D.[4] The difference is the effect of expected default.[5] Miles-Ezzell give the relationship between R_U and R_L as:

$$R_L = R_U - \frac{L R_D T_C (1 + R_U)}{1 + R_D} \tag{7}$$

where their 'cost of debt' has been interpreted here as its expected return. Brealey and Myers (2003, page 542), generalize this to the case of investor taxes by using T^* rather than T_C:

$$R_L = R_U - \frac{L R_D T^* (1 + R_U)}{1 + R_D} \tag{8}$$

Sick (1990) derives a different relationship:[6]

$$R_L = R_U - \frac{L R_{FE} T^* (1 + R_U)}{1 + R_{FE}} \tag{9}$$

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[4]In particular, Y_D is the coupon on debt issued at par.

[5]See Cooper and Davydenko (2001) and (2004) for a more extensive discussion of this point.

[6]The notation in Sick (1990) is slightly different, but the result can be derived by simple substitution.

The key difference, as stated by Sick, is that his result involves the use of the riskless interest rate rather than the cost of debt, and adjusts the interest rate for the relative tax treatment of debt and equity by using R_{FE}.

3   The assumptions underlying the different approaches

The difference between the assumptions underlying the Sick and 'Brealey and Myers' approaches concerns whether a tax payment will be made when the company is insolvent. Sick assumes that an insolvent firm will make a tax payment equal to the gain made from writing off the debt relative to its face value multiplied by the tax rate [see Sick (1990), p.1437]. If the write-off is total then the entire principal of the debt will be taxed when the firm is in default. The assumption implicit in the 'Brealey and Myers' approach is that no such payment will be made. The issue is, therefore, whether insolvent firms can be expected to make such tax payments. The evidence is that insolvent companies do not pay tax (Gilson (1997)). Therefore, the tax payment in the insolvent state that Sick assumes will not, in general, be made.
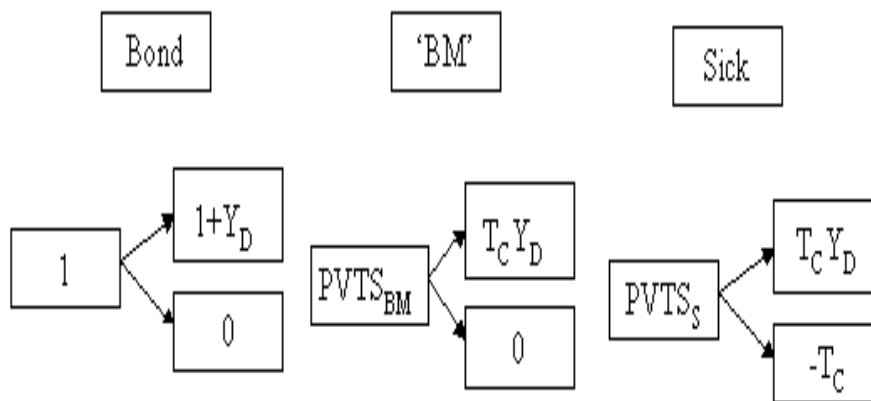
We illustrate the implications of the assumption made by Sick and that made by Brealey and Myers using a binomial tree. Figure 1 panel 1A shows two alternative assumptions about the tax saving from interest. The first tree, headed, 'Bond' shows a bond that pays a yield of Y_D if it does not default. Default is shown by the lower branch of the tree, which results in a total loss. The second panel, headed 'BM' shows a tax saving that occurs only if the firm is solvent. If the company is solvent, it saves tax of an amount $T_C Y_D$ per dollar of face value of the bond. If the company is insolvent, there is no tax impact of having the debt, as the company is assumed to pay no tax in this state. In this case, the tax saving has the same level of risk as the bond. In contrast, the third tree shows the assumption made by Sick (1990). Here the tax saving in the solvent state is the same. However, when the company is insolvent, the existence of the debt results in an extra tax payment of T_C. This is the extra tax resulting from the gain made by writing off the initial debt of $1.

It is straightforward to show, by no-arbitrage, that the value of the tax saving per dollar of debt in the 'BM' case is:
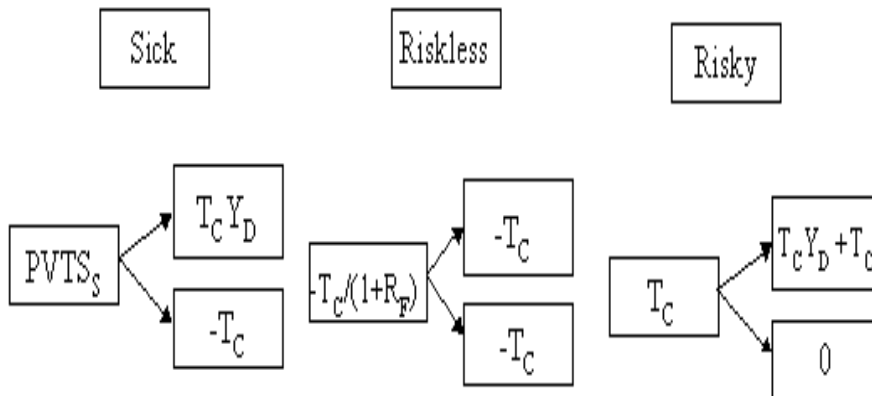
$$PVTS_{BM} = T_C Y_D / (1 + Y_D) \tag{10}$$

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Figure 1: The risk of the tax saving from interest

1A: Alternative assumptions of Brealey and Myers and Sick

Bond	'BM'	Sick

1 → $1+Y_D$ / 0

$PVTS_{BM}$ → $T_C Y_D$ / 0

$PVTS_S$ → $T_C Y_D$ / $-T_C$

1B: Replication of the tax saving in Sick

Sick	Riskless	Risky

$PVTS_S$ → $T_C Y_D$ / $-T_C$

$-T_C/(1+R_F)$ → $-T_C$ / $-T_C$

T_C → $T_C Y_D + T_C$ / 0

and the result shown by Sick (1990) is:

$$PVTS_S = T_C R_F / (1 + R_F) \qquad (11)$$

Because they are based on no-arbitrage, these values are independent of the probability of the firm remaining solvent. As with all no-arbitrage values, however, there is an alternative interpretation in terms of expected cash flows discounted at expected rates of return. We assume that the true probability of the firm remaining solvent is p. In the Brealey and Myers case, the expected tax saving is $p T_C Y_D$, and the appropriate required return is the expected return on the bond $[p(1 + Y_D) - 1]$. Therefore:

$$PVTS_{BM} = p T_C Y_D / [p(1 + Y_D)] = T_C Y_D / (1 + Y_D) \qquad (12)$$

Discounting the expected tax saving at the expected return on the bond is shown by the second term in this equation. This gives the correct value of the tax saving, because it is equal to the no-arbitrage value, as shown by the second equality.

With the assumptions made by Sick, the valuation of the tax saving using expected returns is more complex. This can be seen by decomposing the binomial representation of the tax saving into two parts, shown in Figure 1B. One part is riskless, and the other has a level of risk equal to that of the bond. The first part can be valued by discounting at the riskless rate and the second by replication with the bond. Thus:

$$PVTS_S = -T_C / (1 + R_F) + p(T_C Y_D + T_C) / [p(1 + Y_D)] = T_C R_F / (1 + R_F) \qquad (13)$$

The second term in the equation is the two components of the expected payoff, each discounted at an appropriate expected return. This is equal to the value based on no-arbitrage, shown by the final term. Sick states the result in terms of this no-arbitrage value.

As we have discussed above, the assumption made by the 'Brealey and Myers' approach appears to be more consistent with the actual tax treatment of firms in default. Therefore, we make their assumption throughout the rest of the paper. In particular, we assume that the tax saving from risky debt should be valued by discounting the expected tax saving at the expected return on the debt, as in the first equality in (12).

4 The relationship between leveraged and unleveraged rates

Appendix 1 shows that, under the standard Miles-Ezzell assumptions with the inclusion of risky debt and investor taxes, the relationship between R_L and R_U is:

$$R_L = R_U - \frac{[LR_DT^*(1+R_U)/(1+R_{FE})](R_{FE}/R_F)[1+R_F(1-T_{PD})]}{1+R_D(1-T_{PD})} \tag{14}$$

which is equal to:

$$R_L = R_U - \frac{[LR_DT^*(1+R_U)/(1+R_{FE})][(1-T_C)/(1-T^*)][1+R_F(1-T_{PD})]}{1+R_D(1-T_{PD})} \tag{15}$$

The difference from Brealey and Myers' expression lies mainly in the second term in square brackets. They do not include this term in their intuitively derived expression. The difference from Sick's expression can be seen by making the debt riskless. Then $R_D = R_F$, giving:

$$R_L = R_U - \frac{LR_{FE}T^*(1+R_U)}{1+R_{FE}} \tag{16}$$

This is the expression given in Sick (1990) for risky debt and in Taggart (1991) for riskless debt. If the debt is riskless and there are also no investor taxes, then $R_D = R_F$ and $T^* = T_C$, giving:

$$R_L = R_U - \frac{LR_FT_C(1+R_U)}{1+R_F} \tag{17}$$

This is the standard Miles-Ezzell result with riskless debt. As we have shown above, the general form of the relationship between R_L and R_U is (14), rather than this simpler expression. We discuss below the size of the errors that result from using various approximations.

If the period between rebalancing the leverage becomes short, (14) converges to:

$$R_L = R_U - LR_DT^*\frac{1-T_C}{1-T^*} \tag{18}$$

Both this and the more complex version of the ME formula given by (14) are approximations. Neither policy exactly reflects the actual leverage policies that firms follow. It is not clear which expression more accurately reflects the way that companies actually determine their leverage over time. Apart from its simplicity, (18) has one significant advantage. As we show below, when $T_C = T^*$ it is consistent with the standard formula for asset betas.

4.1 Errors arising from using different formulas

To evaluate the size of possible errors arising from using approximations to the correct formulas, we examine three types of firm. These are shown in Table 1 Panel A. The first firm, in Cases 1

and 4, has a typical amount of leverage and a debt spread of 1%. The second firm, shown in Cases 2 and 5, has higher leverage and a debt spread of 2%. The third firm, in cases 3 and 6, has very high leverage and a debt spread of 3%. All firms have an unleveraged cost of capital of 8%. The general parameters are a riskless rate of 4%, a corporate tax rate of 40%, and a marginal investor tax rate on debt of 40%. In cases 1-3 the tax saving from debt, T^*, is the full corporate tax rate, whereas in cases 3-6 it is half the corporate tax rate. This implicitly varies the marginal investor's tax rate on equity returns from 20% in cases 1-3 to 40% in cases 4-6. We compute R_L from (14) and compare the values given by four other formulas.

Table 1: Errors resulting from different formulas for R_L.

This table shows the errors arising from using different formulas to calculate the leveraged cost of capital, R_L. The unlevered cost of capital, R_U, is 8 % and other variables are : $R_F = 4\%, T_C = 40\%, T_{PD} = 40\%$. Panel A shows three different firms, with leverage of 30%, 60% and 80% and debt costs of 5%, 6% and 7% respectively. The tax saving from interest, T^*, is assumed to be either 40% or 20%, which is equivalent to setting $T_{PE} = 40\%$ or 20%, respectively. The (benchmark) values of R_L in Panel A are calculated using (14). Panel B rows (a), (b), (c), and (d) show the errors in R_L (alternative formula less benchmark) resulting from using the alternatives (8), (9), (18), and 16 respectively.

	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6
(A) Inputs						
L	30%	60%	80%	30%	60%	80%
R_D	5%	6%	7%	5%	6%	7%
T^*	40%	40%	40%	20%	20%	20%
R_L	7.38%	6.52%	5.71%	7.77%	7.44%	7.13%
(B) Errors in estimates of R_L from using different formulas						
a. BM	0.00%	0.01%	0.03%	-0.07%	-0.172%	-0.26%
b. Sick	0.10%	0.44%	0.91%	0.04%	0.16%	0.34%
c. Continuous Approximation	0.02%	0.04%	0.05%	0.01%	0.02%	0.03%
d. Taggart	0.12%	0.48%	0.96%	0.05%	0.18%	0.36%

Panel B shows the errors resulting from using three other formulas. The first is the formula given in Brealey and Myers (8) shown in row (a). When the tax saving is equal to the full corporate

tax rate, this is very accurate, as one would expect. However, the accuracy depends on interpreting the cost of debt correctly as the expected return on debt, rather than its yield. If the yield is used, significant errors can result, as shown in Cooper and Davydenko (2001). The main error in the formula arises because it does not treat investor taxes properly. This error shows up when the leverage is high, and the tax saving is less than the full corporate tax saving, in case 6. The error here is a quarter of one percent, enough to cause significant errors in company valuation. Row (b) shows the error arising from using the Sick formula (9). Here the errors are larger, especially when the debt spread is high. The difference here occurs because of the different treatment of the impact of default on the tax saving from interest, discussed above. Row (c) shows the error from using the continuous approximation (18) rather than the discrete formula (14). This error is small, indicating that the extra complexity of the formulas that assume discrete adjustment of leverage is probably unnecessary. The expression (18) appears to be the best way to adjust discount rates for the tax effect of leverage. It has the merits of being simple, very close to the discrete adjustment formula (14), and easy to use to convert R_L to R_U as well as vice versa. Finally, row (d) shows the errors from using Taggart's formula (16), which assumes riskfree debt. Not surprisingly, the errors here become quite large as leverage and the debt spread grow. For example, in Case 3 where the tax advantage to debt is at its maximum, the leverage ratio is 80%, and the debt spread is 2%, the error is almost a full percentage point.

5   Implementation using the CAPM

A common approach is to use formulas for tax adjusted discount rates in conjunction with the CAPM. For example, one first estimates the equity beta and unlevers it to get the asset beta, then one calculates R_U using the CAPM, and finally calculates R_L using one of the formulas above. Alternatively, one calculates R_L as the weighted average cost of capital by plugging equity and debt betas into the CAPM. From that one can infer R_U and perhaps calculate new R_L's for a different leverage ratios. In this section, we derive formulas for the implementation using the CAPM that are consistent with (18).

5.1   The CAPM with personal taxes

The consensus investor will set returns so that the risk-return ratio from assets is equal on an after-investor-tax basis. This means that the standard version of the CAPM, where returns and

betas are measured before investor taxes, will be affected by investor tax rates. Appendix 2 shows that the version of the CAPM that is consistent with the assumptions about tax that determine T^* is:

$$R_E = R_{FE} + \beta_E P \tag{19}$$

where R_E is the expected rate of equity return before investor taxes, β_E is the beta of the equity, and

$$P = R_M - R_{FE} \tag{20}$$

R_M and R_F are measured in the standard way, using returns before investor taxes. Betas are also measured in the standard way, using pre-tax returns.

Note that only if $T^* = T_C$ (or, equivalently, $T_{PE} = T_{PD}$) is the standard version of the CAPM, with an intercept equal to R_F, valid. In particular, this means that the assumption in, for instance, Modigliani and Miller (1963) that $T^* = T_C$ corresponds to the normal CAPM. In contrast, the Miller (1977) view that $T^* = 0$ corresponds to a CAPM where the intercept is $R_F(1 - T_C)$. A similar effect can be seen in the formula for the required return on unlevered assets:[7]

$$R_U = R_{FE} + \beta_U P \tag{21}$$

As previously pointed out by Sick (1990) [see also Benninga and Sarig (2003)], the required return on debt follows a different version of the CAPM, because the tax treatment of debt and equity differ in all cases other than the standard MM case:

$$R_D = R_F + \beta_D P \tag{22}$$

Regardless of the assumption about taxes, the pre-tax CAPM holds for debt, because all debt is taxed in the same way.[8]

5.2 The WACC and the asset beta

The standard WACC relationship is:

$$R_D(1 - T_C)L + R_E(1 - L) = WACC \tag{23}$$

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[7]Care must be taken to distinguish between the unlevered beta, β_U, and the beta of the firm value given by the sum of debt and equity. The latter is affected by the beta of the tax saving, as we discuss below.

[8]There is another complexity with risky debt. It is not clear that the entire premium over the riskless rate is due to beta risk. We do not deal with this issue here. See Cooper and Davydenko (2004) for a discussion.

Under the ME assumptions, the WACC is equal to R_L, so, using the continuous rebalancing assumption [see (18)]:

$$R_D(1 - T_C)L + R_E(1 - L) = R_U - LT^*R_D\frac{1 - T_C}{1 - T^*} \tag{24}$$

Substituting in R_E, R_U, and R_D from (19), (21), and (22), respectively, we have that the unlevered asset beta is

$$\beta_U = \beta_D[(1 - T_C)/(1 - T^*)]L + \beta_E(1 - L) \tag{25}$$

If $T^* = T_C$ [or, equivalently $T_{PE} = T_{PD}$, see (5)], this reduces to the standard asset beta equation:

$$\beta_U = \beta_D L + \beta_E(1 - L) \tag{26}$$

We can intuitively understand the relationships between betas in the following way. The leveraged firm's operating assets are the same as those for the all-equity firm. But the leveraged firm generates extra value through the tax saving from interest and changes the after-tax risk of the cash flow by channeling some of it to debtholders rather than equityholders, which changes the associated tax treatment. The weighted average of the equity beta and the tax-adjusted debt beta for the leveraged firm must equal the asset beta adjusted for the effect of the tax saving:

$$E\beta_E + D\frac{1 - T_{PD}}{1 - T_{PE}}\beta_D = \beta_U(V_L - V_{TS}) + V_{TS}\beta_{TS} \tag{27}$$

where E is the value of the equity, D the value of the debt, $V_L = E + D$, V_{TS} is the value of the tax shield and β_{TS} is its beta. The value $(V_L - V_{TS})$ is the all-equity value of the firm, which has beta equal to β_U. The adjustment $\frac{1 - T_{PD}}{1 - T_{PE}}$ to the debt beta reflects the fact, shown in (39), that the differential tax treatment of debt and equity results in a change in beta when cash flow is switched from equity to debt, even apart from the effect on the value of the firm. With the ME assumptions and continuous debt rebalancing, $\beta_{TS} = \beta_U$, giving (25).[9]

6 Conclusions

We have presented tax-adjusted discount rate and asset beta formulas with Miles-Ezzell constant debt to value leverage policy, investor taxes, and risky debt. Formulas assuming riskless debt have

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[9]In general, with the ME debt policy β_{TS} and β_U are not equal, because of discrete rebalancing and also the effect of debt default (Sick (1990)). However, with continuous rebalancing, β_{TS} and β_U are equal because the effect of discrete rebalancing disappears and the difference cause by debt default becomes negligible.

previously been derived by Taggart (1991). We also have compared and contrasted our formula for tax-adjusted discount rates with those of (1990) and Brealey and Myers (1988), who also allow for risky debt. Differences arise because of nonstandard assumptions about the tax system (Sick) or missing tax effects at the personal level (Brealey and Myers). Using realistic parameter values, we have shown that errors from these formulas can be almost a full percentage point at high leverage levels. The correct formula, assuming discrete rebalancing of debt, is complex. However, the approximation based on continuous rebalancing is very accurate:

$$R_L = R_U - LR_D T^* \frac{(1 - T_C)}{(1 - T^*)} \tag{28}$$

This approximation has the merits of being simple, very close to the discrete adjustment formula, and easy to use to convert R_L to R_U as well as vice versa. We have also presented the formulas for asset betas and implementation using the CAPM that are consistent with this.

Appendix 1: Proof of the relationship between R_L and R_U

We derive relationship between R_L and R_U by induction, starting at time $T-1$. At that time, the only cash flow remaining is C_T. The unleveraged value is:

$$V_{UT-1} = C_T(1 - T_C)/(1 + R_U) \tag{29}$$

This is the unleveraged value of the last cash flow. It includes implicitly the value to the representative investor of the tax deduction associated with the purchase price, V_{UT-1}.

From the leveraged firm, the representative shareholder will receive, at date T, a cash flow after personal taxes of $C_T(1 - T_C)(1 - T_{PE}) + I_T T_S$. The first part of this cash flow is identical to that from the unleveraged firm and so has value equal to V_{UT-1}, if it comes with a tax deduction of V_{UT-1}. The second flow has risk equal to the debt of the firm, and should be discounted at the after-tax rate appropriate to the debt of the firm, $R_D(1 - T_{PD})$. There is a third component of value. Relative to an investment in the unleveraged firm, the representative investor also gets an extra tax deduction equal to $(V_{LT-1} - V_{UT-1})$. This is riskless, and is discounted at his after-tax riskless rate. Using $I_T = LV_{LT-1}R_D$, the resulting value of the leveraged firm is:

$$V_{LT-1} = V_{UT-1} + \frac{LV_{LT-1}R_D T_S}{1 + R_D(1 - T_{PD})} + \frac{(V_{LT-1} - V_{UT-1})T_{PE}}{1 + R_F(1 - T_{PD})} \tag{30}$$

The last term in this expression is the incremental saving in capital gains taxes, which are assumed

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to be paid every year.[10] The tax basis is higher for the leveraged firm by the amount $(V_{LT-1} - V_{UT-1})$, and capital gains taxes are consequently reduced.

Using (6), we can rearrange (30)as:

$$V_{LT-1} = V_{UT-1} + \frac{LV_{LT-1}T^*R_{FE}R_D(1 + R_F(1 - T_{PD}))}{(1 + R_{FE})R_F(1 + R_D(1 - T_{PD}))} \tag{31}$$

At time T-1, R_L and R_U are defined by:

$$(1 + R_L) = C_T(1 - T_C)/V_{LT-1} \tag{32}$$

$$(1 + R_U) = C_T(1 - T_C)/V_{UT-1} \tag{33}$$

Thus $V_{UT-1} = V_{LR-1}(1 + R_L)/(1 + R_U)$. Substituting this into (31), we get

$$R_L = R_U - \frac{LR_DT^*(1 + R_U)R_{FE}(1 + R_F(1 - T_{PD}))}{(1 + R_{FE})R_F(1 + R_D(1 - T_{PD}))} \tag{34}$$

We will establish the generality of this formula by induction. Consider first time $T - 2$. Consider the expected cash flow at $T - 1$, C_{T-1}, and the continuation value, V_{LT-1}, as two separate but equally levered flows. Denote their values at $T - 2$ by w_{T-2} and W_{T-2}, respectively. The same argument as above establishes that $w_{T-2} = C_{T-1}(1 - T_C)/(1 + R_L)$, with R_L given by (34). Now, we have established above that V_{LT-1} is proportional to V_{UT-1}. Therefore, its unleveraged value at $T - 2$ is $V_{LT-1}/(1 + R_U)$. Thus, using the same argument as above, the leveraged value of V_{LT-1} at $T - 2$ is $W_{T-2} = V_{LT-1}/(1 + R_L)$, with R_L given by (34). This establishes that

$$V_{T-2} = \frac{C_{T-1}(1 - T_C)}{1 + R_L} + \frac{C_T(1 - T_C)}{(1 + R_L)^2} \tag{35}$$

This argument can be repeated for arbitrary T, which establishes the generality of (34).

Appendix 2: Relationships between betas and returns

The representative investor sets returns so that after-tax returns are in equilibrium. However, the CAPM is usually stated in terms of pre-tax betas and risk premia. This Appendix uses the after-tax CAPM to derive the pre-tax version that is consistent with the assumptions about the tax saving on debt.

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[10]There is an emerging literature that introduces realistic treatment of capital gains taxes into the capital structure literature (see Lewellen and Lewellen (2004)). The implications of their results for practical valuation are not yet clear.

Assuming that the market portfolio consists of only equities and not risky debt, and denoting betas that are after personal tax by primes, and pre-tax betas with no primes:

$$\beta_E' = \frac{Cov(\tilde{R}_E(1-T_{PE}), \tilde{R}_M(1-T_{PE}))}{Var(\tilde{R}_M(1-T_{PE}))} = \frac{Cov(\tilde{R}_E, \tilde{R}_M)}{Var(\tilde{R}_M)} = \beta_E \tag{36}$$

similarly:

$$\beta_A' = \beta_U \tag{37}$$

and:

$$\beta_D' = \frac{Cov(\tilde{R}_D(1-T_{PD}), \tilde{R}_M(1-T_{PE}))}{Var(\tilde{R}_M(1-T_{PE}))} = \frac{(1-T_{PD})}{(1-T_{PE})}\beta_D. \tag{38}$$

Expected returns are set by the consensus investor to give equal after-tax risk premia per unit of after-tax beta:

$$
\begin{aligned}
R_E(1-T_{PE}) &= R_F(1-T_{PD}) + \beta_E P' \\
R_U(1-T_{PE}) &= R_F(1-T_{PD}) + \beta_U P' \\
R_D(1-T_{PD}) &= R_F(1-T_{PD}) + \beta_D \frac{(1-T_{PD})}{(1-T_{PE})} P'
\end{aligned}
\tag{39}
$$

where P' is the post-tax market risk premium; $P' = R_M(1-T_{PE}) - R_F(1-T_{PD}) = R_M(1-T_{PE}) - R_{FE}(1-T_{PE})$. Substituting $P = P'/(1-T_{PE})$ for P' gives the expressions for the pre-tax returns on equity and debt:

$$R_E = R_{FE} + \beta_E P \tag{40}$$

$$R_U = R_{FE} + \beta_U P \tag{41}$$

$$R_D = R_F + \beta_D P. \tag{42}$$

Most empirical observations about risk premia are made in terms of pre-tax returns, so these are the expressions that should be used to interpret data on risk premia. Note, however, that the equity risk premia should be measured relative to the tax-adjusted riskless rate, R_{FE}, rather than the pre-tax rate R_F.

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