

# Negative volatility and the Survival of the Western Financial Markets

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## Abstract

The paper discusses situations where certain parameters are given values that are outside their natural ranges. One case is obtained when plugging in a negative value for the volatility parameter  $\sigma$  in the Black and Scholes formula. This leads to seemingly “new” results.

A different setting is considered related to the developments in time of biological populations. Here deterministic models lead to chaotically fluctuating population sizes, which came as a surprise to workers with population data.

It is argued that the origins for the seemingly new and original results may be related.

*KEYWORDS: The Black and Scholes model, negative volatility, population models, chaotic fluctuations, bifurcation*

## Introduction

The motivation for this article is an entertaining paper by E.G. Haug (2002), including the comic strip “The Collector” in “From Russia with vol”. The Collector had a nightmare one night, dreaming that the volatility had gone *negative*. An internet search for negative volatility came up with a single link - to Albert Shiryaev at the Steklov Mathematical institute, Moscow, Russia. After kicking in Shiryaev’s door, the Collector brings forth his mission - to terminate the Russian secret weapon known as negative volatility. The Collector feared that this weapon might destroy the western capitalist system,

and in particular ruin his own option strategy, buying cheap options for very low volatility. But Shiryaev reassures The Collector that negative volatility is “just a mathematical concept”. The Collector was first astonished, then relieved, and finally enthusiastic about the prospects of negative volatility.

## Negative volatility - a direct approach

Instead of starting, as Haug (2002) does, with the classical Merton, Black and Scholes formula, and plugging in a negative volatility directly in this formula, I suggest we take a look at the dynamic equation for the risky asset. It is

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dB(t), \quad (1)$$

where  $S(t)$  is the spot price of the risky asset at time  $t$ ,  $\mu$  is the drift parameter and  $\sigma$  is the volatility parameter.  $B(t)$  is the Brownian motion stochastic process at time  $t$  under the given probability measure  $P$ .  $B(t)$  is normally distributed with mean zero and variance  $t$  (under  $P$ ). Informally one can think of the increment  $dB(t)$  as normally distributed with mean zero and variance  $dt$ . The term  $\sigma dB(t)$  is thus normally distributed with mean zero and variance  $\sigma^2 dt$ . A representation for the price process  $S$  is as follows:

$$S(t) = S(0)e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma B(t)}. \quad (2)$$

The only stochastic component in this relation is the term  $\sigma B(t)$ , which has a normal distribution with mean zero and variance  $\sigma^2 t$ .

When deriving the formula for a call option written on a stock with price process  $S$ , Black and Scholes (1973) made the implicit assumption that the volatility parameter is positive, i.e.,  $\sigma > 0$ . Let us assume the opposite, namely that  $\sigma < 0$ , i.e., a negative volatility. Going back to equation (1), the term  $\sigma dB(t)$  is still normally distributed with mean zero and variance  $\sigma^2 dt$ . Positive and negative increments in the Brownian motion are equally likely, and multiplying these increments by a constant, it being negative or positive, yields a random increment with the same distribution, regardless of the sign of  $\sigma$ . Similarly the term  $\sigma B(t)$  in equation (2) has still a normal distribution with mean zero and variance  $\sigma^2 t$ , so there are in fact no changes in this regard from the situation case with positive volatility.

This does not mean that the sample paths are the same; the stretch of data  $\{S(t); 0 \leq t \leq T\}$  generated from the equation (1) with positive  $\sigma$  is different from the same stretch generated from the same equation but only

with  $\sigma$  being negative (and with the same absolute value). However, these two stretches would have the same finite dimensional distributions, and that is all what matters when deriving the Black and Scholes formula. Let us quickly see how this can be done.

## The Value of a European Call Option for any Value - Positive or Negative - of the Volatility

Under the risk adjusted probability measure  $Q$ , the equations (1) and (2) can be written

$$\frac{dS(t)}{S(t)} = r dt + \sigma dB^Q(t), \quad (3)$$

and

$$S(t) = S(0)e^{(r-\frac{1}{2}\sigma^2)t+\sigma B^Q(t)}, \quad (4)$$

respectively, where  $r$  is the risk free interest rate, and where  $B^Q(t)$  is a Brownian motion process under the probability measure  $Q$ . From the similarity with the equations (1) and (2), we notice that the same story as above can be told under the risk adjusted measure  $Q$ , since, by Girsanov's theorem, only the drift term has changed.

Using the above representation (4), we now demonstrate how the Black and Scholes formula looks like in the case of a negative volatility parameter  $\sigma$ . The price  $c(S, K, T, r, \sigma)$  of a European call option having strike price  $K$  and time to maturity  $T$  is given as

$$c(S, K, T, r, \sigma) = E^Q\{e^{-rT}(S(T) - K)^+\}. \quad (5)$$

This can be written

$$c(S, K, T, r, \sigma) = e^{-rT} \int_{-\infty}^{\infty} (S(0)e^{(r-\frac{1}{2}\sigma^2)T+x} - K)^+ \frac{1}{\sqrt{2\pi\sigma^2T}} e^{-\frac{x^2}{2\sigma^2T}} dx. \quad (6)$$

Notice that we have here integrated with respect to the probability density of the random variable  $X =: \sigma B^Q(T)$ , which, as observed above, has a normal distribution with mean zero and variance  $\sigma^2T$ . The result is, of course, the following formula

$$\begin{aligned} c(S, K, T, r, \sigma) = S(0)\Phi\left(\frac{\ln(S(0)/K) + (r + \sigma^2/2)T}{|\sigma|\sqrt{T}}\right) \\ - e^{-rT}K\Phi\left(\frac{\ln(S(0)/K) + (r - \sigma^2/2)T}{|\sigma|\sqrt{T}}\right), \end{aligned} \quad (7)$$

where the function  $\Phi(\cdot)$  is the standard normal cumulative probability distribution function. The only difference between this formula and the standard Black and Scholes one is the absolute value sign of the volatility, the term  $|\sigma|$  instead of only  $\sigma$ , in the denominator of the distribution function  $\Phi(\cdot)$ .<sup>1</sup>

When the volatility is positive, the formulae are the same. When the volatility is negative, the price of a call option written on the stock has the same value as a call option written on a stock having positive volatility of the same magnitude (absolute value). In this case, for  $\sigma < 0$ , we could have written  $-\sigma$  instead of  $|\sigma|$  in the above formula.

A related point of view is that volatility should *never* be negative, based on the convention  $\sqrt{\sigma^2} = |\sigma|$ . If we assume  $\sigma > 0$  and write the term  $\sigma(-B(T))$ , applying the minus sign to the Brownian motion directly and not via the term  $\sigma$ , we would obtain the same formula, since  $-B(T)$  has the same probability distribution as  $B(T)$ . In this case we would not even need the absolute value sign  $|\cdot|$  in the formula. A statistician would, typically, prefer this interpretation.

The Collector may thus rest in peace. His hectic trip to Moscow was not really warranted, and quite likely, the western capitalist system will survive even this crisis of negative volatility.

## Negative volatility - the Haug interpretation

In the article by Haug (2002), and with reference to a paper by Peskir and Shiryaev (2001), a negative volatility is plugged directly into the standard Black and Scholes (1973) - formula without making the adjustment leading to the formula (7). Recall that the original formula is derived under the implicit assumption that the volatility parameter is positive. From the properties of the normal cumulative probability distribution function  $\Phi(\cdot)$  it is straightforward to show that this leads to the following relation:

$$c(S, K, T, r, -\sigma) = -c(S, K, T, r, \sigma) + S(0) - e^{rT}K = -p(S, K, T, r, \sigma), \quad (8)$$

where the parameter  $\sigma$  is now positive, and  $p(S, K, T, r, \sigma)$  is the price of a European put option. The last equality above follows from the put-call parity. The first equality follows from plugging in  $-\sigma$ ,  $\sigma > 0$ , in formula (7) instead of the term  $|\sigma|$ , and using the antisymmetry of the function  $\Phi(\cdot)$ .

So, just by this kind of formulashopping, a seemingly new and interesting relationship is discovered. This is of course not incorrect, but should be considered in the light of the previous section. The left hand side of the

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<sup>1</sup>This type of derivation can be found in e.g., Aase (2002)

formula in equation (8) has of course little to do with the price of a European call option any more.

At this stage Haug (2002) enters into the field of particle physics in search of a deeper truth in mathematical finance. To this I would only like to point out that related to particle physics is of course the *uncertainty principle*, strongly supported by the Danish physicist and Noble prize winner Niels Bohr,<sup>2</sup> via “The Copenhagen School”. He formulated, among other things, the correspondence principle,<sup>3</sup> which postulates a detailed analogy between the quantum theory and the classical theory appropriate to the mental picture employed. This analogy does not merely serve as a guide to the discovery of formal laws; its special value is that it furnishes the interpretation of the laws that are found in terms of the mental picture used.

Bohr’s student, Werner Heisenberg (1930), developed the uncertainty principle.<sup>4</sup> The principle is known to have altered our whole philosophy of science.

Albert Einstein on the other hand, never really accepted this view of the world, and believed that a system of deterministic differential equations would suffice to describe the world.<sup>5</sup> There is a rumor that the two professors, Bohr and Einstein, had long walks together in the streets of Copenhagen, sometimes also taking the bus, but they typically forgot to get off at the right stop, then took another bus in the opposite direction, and again forgot to get off at the intended destination, and so on. Despite these apparent random oscillations, Einstein was not convinced about the existence of genuine uncertainty. Einstein’s God was a God of order and there was no place in his scheme of things for a God who, in his famous phrase, “plays dice with the world”.<sup>6</sup>

One may conjecture that if this Albert had ever turned to finance, he would either have been in for a big surprise, or become a very popular consultant.<sup>7</sup>

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<sup>2</sup>He won a *real* Noble prize, the one in Physics in 1922, unlike “The Sveriges Riksbank (Bank of Sweden) Price in Economic Sciences in Memory of Alfred Nobel”, which is the one granted to economists and people in related fields. As we know, Merton and Scholes were both sharing this price in 1997. Fisher Black had, unfortunately, passed away some time before this event took place. Likewise, of the three creators of the CAPM only Sharpe received this prize in 1990, Lindtner and Mossin were both dead when this prize was awarded (Jan Mossin, e.g., died in 1987 at age 50).

<sup>3</sup>See Bohr (1923)

<sup>4</sup>Werner Heisenberg is also a Nobel Laureate in Physics in 1932.

<sup>5</sup>Einstein got the Nobel prize in Physics in 1921 for his discoveries regarding the photoelectric effect.

<sup>6</sup>Clark, op. cit., p. 327. See e.g., Bartholomew for a treatise on God and Uncertainty.

<sup>7</sup>There is a rumor indicating that Einstein was not terribly interested in money. This

# Chaotic Behavior from Deterministic Dynamics

The following deterministic model has several applications in population dynamics:

$$x_{t+1} = \theta x_t(1 - x_t), \quad (9)$$

$x_0 \in [0, 1]$ ,  $0 \leq \theta \leq 4$ . The process  $x$  evolves in the interval  $[0, 1]$  for the given range of  $\theta$ , and displays a rather complex path behavior as a function of the this population parameter (see, for example, Lie and York (1975) or May (1976)). Just to illustrate, for  $\theta = 0.8$  the path shows only one fixed point  $x^* = 0$ , which is locally monotone and stable. For  $\theta = 2.0$  two fixed points appear,  $x_1^* = 0$  and  $x_2^* = 1/2$ . The point  $x_1^*$  is not stable (repelling), but  $x_2^*$  is stable (attracting). Increasing the parameter the paths display ever more complex behavior, where pitchfork curves and bifurcation appears.

One can even move the parameter outside its natural range  $[0, 4]$  and obtain really chaotic results from this deterministic model. It is not likely that the resulting behavior has much to do with the evolution of biological populations anymore. Still the mathematical behavior can be explored, and even statistical estimation is possible based on observed sample paths, introducing stochastic shocks in the model (9) (see e.g, Aase (1983)). It was surprising that introducing additive noise in this type of models presented no problems when the goal was to estimate the unknown parameters.

## Conclusions

The starting point of this paper was the negative volatility “discovered” by “The Collector”. We have pointed out that a statistician would simply not accept the notion of negative volatility. For the geometric Brownian motion process it is, nevertheless, possible to explore this subject a bit further, and we have presented the price of an European call option when the volatility is negative. This we have contrasted with the development in Haug (2002) and Peskir and Shiryaev (2001). While this latter interpretation is “just a mathematical concept”, its direct application leads to seemingly new formulae, without any real theoretical underpinning.

Instead of connecting this to the theory of antimatter in Physics as is elegantly pursued by Haug (2002), we indicated a connection to this discipline 

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rumor says that he promised his first wife Mileva Maric, who had allegedly assisted him a great deal on parts of his work, the future Nobel Prize money as alimony (as long as he got the honor).

via the uncertainty principle. This leads us to wonder who would have been the best portfolio manager, Bohr or Einstein? We rounded off the paper by a linkage to deterministic chaos.

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