Trade and Communication Under Subjective Information

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Abstract

This paper models an economy where agents perceive the choices they face subjectively, and have subjective interpretations of the terminology they use in a shared business language. Preferences are defined on what an agent perceives, and not on what is objectively presented to an agent.

A business language enables agents to trade, provided the terminology in the language is sufficiently vague: once agents can express more detail than their trading partners can perceive, the language ceases to be useful. Under some regularity conditions on the language, an appropriately defined notion of competitive equilibrium exists. However, much less can be said about welfare than in the neoclassical case, as there are counter-examples to both welfare theorems.

Key Terms: Language, Perceptual Limits, Unawareness

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1 Introduction

For economic activity to occur, both parties in a trade need to have a shared understanding of what is being exchanged. The economic theorist's usual starting point is assuming that the commodity space is common knowledge and that agents have preference relations defined directly on this commodity space. While agents may fail to consider alternatives, as in the literature on unawareness (Fagin and Halpern 1988, Dekel, Lipman, and Rustichini 1998a, Modica and Rustichini 1999), they must still have preferences defined over commodity bundles described in arbitrarily fine detail. That is, agents must know in principle everything that could become available to them, in any state of the world, at any location or date.

In such a world, agents in principle can describe everything they might offer or desire in trade. There may be strategic reasons that reliable communication is restricted (Crawford and Sobel 1982), social benefits of restricted communication (Kanodia, Singh, and Spero 2005), or technical restrictions placed on the language (Rubinstein 2000), but there is no inherent reason why an agent cannot know infinitely fine details about what could be traded.

It is easy to imagine situations where subtle distinctions among possible choices are lost on an agent, and in fact where the same agent may be able to perceive the same object in multiple ways. For example, an agent might be unable to taste the difference between a medium-bodied wine and a light-bodied wine, if the two wines differ relatively little in their sugar and glycerin concentrations. On the other hand, the same agent may be unable to taste the difference between the same medium-bodied wine and a full-bodied wine, again for sufficiently small differences in the concentrations of these substances. Thus a fixed bottle of wine, which objectively may be medium-bodied, can be perceived as both a full-bodied wine and a light-bodied wine. Whether the agent knows the wine's objective sugar and glycerin concentrations is not what necessarily determines how well the agent enjoys the wine; instead, how the agent subjectively perceives the wine's body is what matters.

Conversely, an agent facing distinct real-world objects may perceive them in the same way: in the above example, both the full-bodied wine and the light-bodied wine taste the same, at least sometimes. These two phenomena underlie the notion of a just noticeable difference. It has been observed that non-degenerate just noticeable differences are problematic for the assumption of complete, transitive preferences on the objective commodity space (Mas-Colell, Whinston, and Green 1995, Knoblauch 1998, Dubra, Maccheroni, and Ok 2004).

As different agents can have different perceptions, a business language will typically enable agents to describe their proposed trades only approximately. The same perceived object might be reportable in multiple ways—a given bottle of wine might justifiably be called full-bodied or medium bodied—and conversely the same description might apply to multiple objects. These two phenomena are the basis of the accountant's notion of an immaterial difference. Unless everyone perceives the world in the same way, it turns out to be useful for a business language to have a non-trivial notion of immaterial differences: vague requests can be fulfilled more reliably than excessively detailed ones. In the above wine example, imagine that a connoisseur asks the advice of a novice wine steward. The connoisseur will discover that an overly precise request is one that the wine steward cannot reliably fill. After enough unpleasant surprises, the connoisseur presumably will learn that it would be better to requests that the steward cannot get wrong.

This paper formalizes these ideas, in order to study the economic consequences of subjective information. The next section presents an analytic model of an agent's subjective perceptions. Section three models the use of a shared business language among agents with private, subjective views of the world, and provides conditions under which agents can receive in trade what they believe they agreed upon. Section provides conditions for existence of an appropriately defined notion of equilibrium, and presents counter-examples to both welfare theorems. Section five concludes.

2 Perception

2.1 Basic Model of Perception

Throughout this paper, there is assumed to be a set I of agents. The model of individual perception presented here consists of two collections and a binary relation between them. The first collection consists of possible consumption choices ("real-world objects") labeled X. Agents do not observe X, and in fact need not be aware of X; one can think of X as chiefly being in the model for the researcher's convenience.

For agent $i \in I$, there is a collection S_i, known only to i, called $i's$ set of subjective *conceptions*. When no confusion can arise, I drop the subscript and write S for an arbitrary agent's subjective conceptions. Agent i's preferences are defined on S_i, i.e., on the world as the agent understands it.

A *perception* is some conception that the agent observes when facing a real-world object. That is, when an agent sees some $x \in X$, the agent perceives some subjectively meaningful $a \in S$. A given $x \in X$ need not have a unique conception $a \in S$ as the only way it can be perceived, but an agent is assumed to have coarse enough conceptions to enable every object to be perceived as something. Conversely, not every conception $a \in S$ is necessarily the perception of a unique $x \in X$. In terms of the wine example from the introduction, there are individual bottles of wine that the agent might in some contexts perceive as full-bodied and in others as light-bodied, and there are individual conceptions of wine such as "full-bodied" that many wines may be perceived as.

Because of this many-to-many relationship, perception is modeled as a binary relation, written as $\Vdash$, between X and S.[1] For $x \in X$ and $a \in S$, the relation $x \Vdash a$ is read as, "real-world object x can be perceived as subjective conception a." With some technical assumptions, it is shown that the agent's perceptions induce a topology on X, with S as a base of this topology. Intuitively, this means that the agent's preferences are defined on approximate consumption bundles, where the approximation is determined by the fineness of the agent's conceptual framework S and by the fineness of the agent's perceptual apparatus $\Vdash$. Thus the framework here is related to the approximate price sensitivity in Allen and Thisse (1992), the model of consumer choice over sets in Kreps (1979), and the approximate meanings of numerical stimuli in Dickhaut and Eggleton (1975)

There are two correspondences associated with the perception relation:

Definition 2.1. The correspondence $X \xrightarrow{\Vdash} S$ is given by

$$(\forall x \in X) \qquad \Vdash (x) \equiv \{a \in S \,|x \Vdash a\}.$$

The inverse correspondence $S \xrightarrow{\Vdash^{-1}} X$ is given by

$$(\forall a \in S) \qquad \Vdash^{-1} (a) \equiv \{x \in X \,|\, x \Vdash a\}.$$

[1]This symbol is the forcing relation introduced by Cohen. For background and historical discussion, see Avigard (2004).

There are two possible notions of the image a subset $D \subseteq X$ of the commodity space. One can take the *strong* image, giving the members of S that are in the image of every point in D, or one can take the *weak* image, giving the members of S that are in the image of some point in D.

Definition 2.2. The *strong* (or *universal*) *image* of $D \subseteq X$ under $X \xrightarrow{\Vdash} S$ is

$$\Box D \equiv \{a \in S \mid \Vdash^{-1}(a) \subseteq D\} = \{a \in S \,|(\forall x \in X)(x \Vdash a \to x \in D)\}.$$

The *weak* (or *existential*) *image* of D under $X \xrightarrow{\Vdash} S$ is

$$\Diamond D \equiv \{a \in S \mid \Vdash^{-1}(a) \between D\} = \{a \in S \,|(\exists x \in X)(x \Vdash a \text{ and } x \in D)\},$$

where $\Vdash^{-1}(a) \between D$ denotes that the two sets intersect, i.e., that $\exists x \in \Vdash^{-1}(a) \cap D$.

The right-hand sides in the above definitions show that the strong and weak images of a correspondence are logically dual. The former gives the subset of S whose members can only be the image of some point in D, whereas the latter gives the subset of S whose members can possibly be the image of some point in D. By analogy with alethic modal logic, the strong image is thus written as $\Box D$ (read "necessarily D"), while the weak image is written as $\Diamond D$ (read "possibly D").[2]

In an entirely analogous way, the inverse correspondence $S \xrightarrow{\Vdash^{-1}} X$ generates both a strong and weak inverse image of any $U \subseteq S$. In the literature, the strong image is called the *restriction* of U, while the weak image is the *extent* of U; see for example Johnstone (1977) or Negri (2002).

Definition 2.3. The *strong inverse image* (or *restriction*) of $U \subseteq S$ under $X \xrightarrow{\Vdash} S$ is

$$\text{rest } U \equiv \{x \in X \mid \Vdash(x) \subseteq U\} = \{x \in X \,|(\forall a \in S)(x \Vdash a \to a \in U)\}.$$

The *weak inverse image* (or *extent*) of U under $X \xrightarrow{\Vdash} S$ is

$$\text{ext } U \equiv \{x \in X \mid \Vdash(x) \between U\} = \{x \in X \,|(\exists a \in S)(x \Vdash a \text{ and } a \in U)\}.$$

In addition to the operators $\Box$, rest, $\Diamond$, and ext, the topological interpretation of perceptions requires two axioms. First, as indicated above, the agent must have some way, however coarse, of perceiving anything that might be traded:

[2]When more than one relation is being discussed, the symbols $\Box$ and $\Diamond$ are written with appropriate subscripts. The notation here follows Sambin and Gebellato (1998) and Sambin (2001).

Axiom 2.1. *For each $x \in X$, there is some $a \in S$ such that $x \Vdash a$.*

Second, the agent's perceptions must satisfy a consistency condition:

Axiom 2.2. *For each $a, a' \in S$, if there is some $x \in X$ such that*

$$x \Vdash a \quad \text{and} \quad x \Vdash a',$$

then there is some $a'' \in S$ such that, for all $x' \in X$,

$$x' \Vdash a'' \quad \text{iff} \quad x' \Vdash a \text{ and } x' \Vdash a'.$$

Thus, an agent who can perceive the same object in multiple ways must have a conception of objects that can be possibly be perceived in these ways.

2.2 The Topological Interpretation of Perception

Under the stated axioms, it is possible to define the topology induced by perceptions. The process is to compose the operations from $X \longrightarrow S$ with the operations $S \longrightarrow X$, in order to define operators from $X \longrightarrow X$, and then show that these operators are valid notions of interior and closure.

Intuitively, one can view the members of S as the names of open neighborhoods in the base of a topology on X. That is, each $a \in S$ is associated with

$$\Vdash^{-1}(a) = \{x \in X | x \Vdash a\},$$

which one can view as the neighborhood in X that a names. (Compare Valentini (2001).) Under this interpretation, all of S becomes associated with the family

$$\{\{x \in X | x \Vdash a\} | a \in S\}.$$

Definition 2.4 (Vickers (1988))**.** Let X be a topological space with base S. A subset $D \subseteq X$ is *open* iff every $x \in D$ has a neighborhood $N(x) \in S$ such that $N(x) \subseteq D$.

Associating S with neighborhoods in X modifies this definition as follows: $D \subseteq X$ is open if and only if, for every $x \in D$, there is some $a \in S$ such that $x \Vdash a$ and $\Vdash^{-1}(a) \subseteq D$.

Definition 2.5. The *interior* operator on X induced by $\Vdash$ is

$$\text{int} \equiv \text{ext}\,\Box$$

Thus, for $D \subseteq X$, $\text{int}\ D = \text{ext}\Box D$.

Expanding this definition shows its justification: for $D \subseteq X$,

$$\text{int}\ D \equiv \text{ext}\Box D$$

$$= \{x \in X | (\exists a \in S)(x \Vdash a \text{ and } a \in \Box D)\}$$

$$= \{x \in X | (\exists a \in S)(x \Vdash a \text{ and } (\forall x' \in x)(x' \Vdash a \to x' \in D))\}.$$

Noting that $x \Vdash a$ iff $x \in \Vdash^{-1}(a)$, the last expression above for int D says that it consists of those points belonging to a neighborhood (named by some $a \in S$) whose points are all in D. That is, the interior of an arbitrary $D \subseteq X$ matches Definition 2.4.

Conversely, the following holds:

Proposition 2.1 (Sambin (2001))**.** *For arbitrary* $D \subseteq X$*,* $\text{int}(\text{int}\ D) = \text{int}\ D$*.*

Proof. This will be shown by showing $\Box\text{ext}\Box = \Box$. The result then follows by composing both sides on the left with ext.

Expanding definitions gives, for arbitrary $D \subseteq X$,

$$\Box\text{ext}\Box D = \{a \in S | (\forall x \in X)(x \Vdash a \to x \in \text{ext}\Box D)\}$$

$$= \{a \in S | (\forall x \in X)(x \Vdash a \to (\exists a' \in S)(x \Vdash a' \text{ and } (\forall x' \in X)(x' \Vdash a' \to x' \in D)))\}.$$

The last expression says that $\Box\text{ext}\Box D$ names neighborhoods whose points have a neighborhood a' that is contained in D; thus, $\Box\text{ext}\Box D \subseteq \Box D$.

The reverse inclusion is immediate, by choosing $a' = a$. $\Box$

Given that the interior operator matches the classical definition and has the idempotent property in Proposition 2.2, I make the following definition:

Definition 2.6. A subset D of X is *open* in the topology induced by $\Vdash$ iff

$$D = \text{ int } D.$$

The standard definition of a closed set is one that contains all of its limit points. That is, the closure of a set is the collection of points for which every open neighborhood intersects the set.

This would suggest that the closure of an arbitrary $D \subseteq X$ should be defined as

$$\{x \in X | (\forall a \in S)(x \Vdash a \to (\exists x' \in X)(x' \Vdash a \text{ and } x' \in D))\}.$$

Thus the natural definition of closure in this context is the logical dual of interior:

Definition 2.7. The *closure* operator on X induced by $\Vdash$ is

$$\text{cl} \equiv \text{ rest}\Diamond$$

Thus, for $D \subseteq X$, cl $D =$ rest$\Diamond D$.

An analogous argument to that in the Proposition 2.2 shows that $\Diamond\text{ext}\Diamond = \Diamond$, and hence that closure is idempotent. This justifies the following definition:

Definition 2.8. A subset D of X is *closed* in the topology induced by $\Vdash$ iff

$$D = \text{ cl } D.$$

Expanding this definition for $D \subseteq X$ gives

$$\text{cl } D \equiv \text{ rest}\Diamond D$$

$$= \{x \in X | (\forall a \in S)(x \Vdash a \to a \in \Diamond D)\}$$

$$= \{x \in X | (\forall a \in S)(x \Vdash a \to (\exists x' \in X)(x' \Vdash a \text{ and } x' \in D))\}$$

as desired. Intuitively, a real-world object is in the perceptual closure of D iff every way of perceiving it is a way of perceiving something in D.

It can now be shown that an agent's subjective perceptions induce a topology on X.

Lemma 2.1. *In the perceptual topology, $\varnothing$ is clopen.*

Proof. By definition,

$$\text{int } \varnothing = \text{ ext}\Box\varnothing$$

$$= \{x \in X | (\exists a \in S)(x \Vdash a \text{ and } (\forall x' \in X)(x' \Vdash a \to x' \in \varnothing))\} = \varnothing.$$

Thus, $\varnothing$ is open.

Analogously,

$$\text{cl } \varnothing = \text{ rest}\Diamond\varnothing$$

$$= \{x \in X | (\forall a \in S)(x \Vdash a \to (\exists x' \in X)(x' \Vdash a \text{ and } x' \in \varnothing))\} = \varnothing.$$

Thus, $\varnothing$ is closed. □

Lemma 2.2. *In the perceptual topology, X is clopen.*

Proof. By definition,

$$\text{int } X = \text{ ext}\Box X$$

$$= \{x \in X | (\exists a \in S)(x \Vdash a \text{ and } (\forall x' \in X)(x' \Vdash a \to x' \in X))$$

$$= \{x \in X | (\exists a \in S) x \Vdash a\}.$$

By Axiom 2.1, this is all of X, so X is open.

Analogously,

$$\text{cl } X = \text{ rest}\Diamond X$$

$$= \{x \in X | (\forall a \in S)(x \Vdash a \to (\exists x' \in X)(x' \Vdash a \text{ and } x' \in X))\} = X.$$

Thus, X is closed. □

Lemma 2.3. *The union of open sets in the perceptual topology is open.*

Proof. By definition,

$$\bigcup_{\alpha} \text{int } D_\alpha = \{x \in X | (\exists \alpha)(\exists a \in S)(x \Vdash a \text{ and } (\forall x' \in X)(x' \Vdash a \to x' \in D_\alpha))\}$$

$$\subseteq \{x \in X | (\exists a \in S)(x \Vdash a \text{ and } (\forall x' \in X)(x' \Vdash a \to (\exists \alpha)(x' \in D_\alpha)))\}$$

$$= \text{int} \bigcup_\alpha \text{ int } D_\alpha.$$

To see the reverse inclusion, note that for any $D \subseteq X$, if $x \in$ int D, then

$$(\exists a \in S)(x \Vdash a \text{ and } (\forall x' \in S)(x' \Vdash a \to x' \in D)).$$

Picking $x' = x$ gives

$$x \in \text{ int } D \to x \in D,$$

i.e., int $D \subseteq D$. In particular,

$$\text{int} \bigcup_\alpha \text{ int } D_\alpha \subseteq \bigcup_\alpha \text{ int } D_\alpha.$$

Combining these gives the result. □

Lemma 2.4. *The intersection of finitely many open sets in the perceptual topology is open.*

Proof. It suffices to show that the intersection of two open sets is open, as the result then follows by induction. For $D, E \subseteq X$,

$$\text{int } D \bigcap \text{ int } E = \{x \in X | (\exists a, b \in S)(x \Vdash a \text{ and } x \Vdash b$$

$$\text{and } (\forall x' \in X)(x' \Vdash a \to x' \in D) \text{ and } (\forall x'' \in X)(x'' \Vdash b \to x'' \in E))\}.$$

By Axiom 2.2, if $x \Vdash a$ and $x \Vdash b$, then there is some $c \in S$ such that $x \Vdash c$ and

$$(\forall x' \in X)(x' \Vdash c \to x' \Vdash a \text{ and } x' \Vdash b),$$

which in turn implies,

$$(\forall x' \in X)(x' \Vdash c \to x' \in D \bigcap E).$$

Thus, int $D \bigcap$ int $E \subseteq$ int $(D \bigcap E)$.

Conversely, if $x \in$ int $(D \bigcap E)$, then there is a neighborhood $a \in S$ of x that is contained in $D \bigcap E$, which means that $\Vdash^{-1} (a) \subseteq D$ and $\Vdash^{-1} (a) \subseteq E$. This says that $x \in$ int $D \bigcap$ int E, which by the arbitrariness of x implies int$(D \bigcap E) \subseteq$ int $D \bigcap$ int E.

Combining these shows that the finite intersection property holds. □

The above lemmata are expressed in the following theorem:

Theorem 2.1 (Perceptual Topology)**.** *The open sets induced by perceptions, under Axioms 2.1 and 2.2, form a topology.*

2.3 Remarks on Complementation

Nothing has been asserted about the negation of the relation $\Vdash$. In particular, it has not been stipulated whether $(x, a) \notin \Vdash$ should be read as "x cannot be perceived as a," or whether this should merely indicate that the agent, faced with x, cannot say whether it is an instance of a.

Which reading one chooses determines the appropriate notion of complementation. Following Bridges, Schuster, and Vîţă (2002) and Vîţă and Bridges (2003), I define the *classical logical complement* of $D \subseteq X$ as the members of X that do not belong to D:

$$\neg D \equiv \{x \in X | x \notin D\}.$$

The classical logical complement of an open set is then those objects that are not perceived as something in the set. Depending on how one reads absence from from $\Vdash$, one might want a stronger notion of complementation. Specifically, the *apartness complement* of $D \subseteq X$, written D^c, is the set of objects that are always perceived as something outside of D:

$$D^c \equiv \{x \in X | (\forall a \in S) x \Vdash a \rightarrow a \notin D\}.$$

If the agent cannot say whether x is an instance of some $a \in D$, then x is in the classical logical complement $\neg D$, but is not in the apartness complement D^c. For the two notions of complementation to coincide, the agent must be able to perceive exactly when a given object belongs to a given set and when it does not. Since this reasoning applies in particular to every singleton $\{x\} \subset X$, $\neg D = D^c$ if and only if the agent has perfect perceptions.

From the agent's viewpoint, the apartness complement is the only notion of complementation that can be useful. Note, however, that the apartness complement of an open set is not closed. This feature, though somewhat strange to the uninitiated, is actually desirable: it reflects the fact that perceptual limits prevent the agent from distinguishing boundaries.

Because closed sets are not defined as the complement of open sets (assuming one views the apartness complement as the most useful notion), the topology induced by perceptions is intuitionistic rather than classical. (See Brouwer (1907), Heyting (1956), Troelstra and van Dalen (1988), or Dummett (2000) for details.) Thus existence proofs proofs must be constructive.

Note that this feature provides an alternative notion of unawareness to that in the literature (e.g., Fagin and Halpern (1988), Aumann and Brandenburger (1995), or Dekel, Lipman, and Rustichini (1998b)). If there exists classically a solution to an agent's problem, but no solution exists intuitionistically, the agent can be viewed as unaware of a solution.

3 Business Language

3.1 Basic Reporting Model and Topological Interpretation

For two agents to trade, they must be able to reach some sort of understanding about what they are offering or requesting in exchange. Trades cannot be stated in terms of X, as agents do not observe X directly, or even have mental conceptions of what is in X. Moreover, agents cannot offer what is in the individual sets of conceptions S_i, as these are private and subjective. I introduce a shared language as a way around this difficulty. The terminology in the shared language is public, so that the agents can use the language to try to reach some sort of consensual validation of what objects are under discussion.

A *business language* is modeled as a set T, interpreted as shared terminology, and, for each $i \in I$, a binary relation R_i between the agent's private conceptions S_i and T. The relation represents i's private semantics, i.e., how the agent may report a subjective conception in the business language.[3]

A report may be valid for more than one conception that the agent has in mind; in this case, the conceptions that the agent can report in the same way are said to be within an *immaterial* difference. Conversely, the same conception may have more than one way it can be reported.

[3] I do not discuss the development or evolution of the language, though there is a literature closely related to this issue. See for example Ahn (2000).

Recalling the wine example from the introduction, if the language includes only the terms "full-bodied" and "light-bodied," then a wine taster may be able to report a medium-bodied wine using either term, and conversely may have many wines in mind for which a given report is valid. This is similar to the vague language in Lipman (2001), but because of the perceptual limits here, the role of ambiguous language in this model is more beneficial than in Lipman's.

For $(a, t) \in S_i \times T$, $a\, R_i\, t$ is read as, "a can be reported as t by agent i." In parallel with the discussion on perceptions, there are two correspondences associated with the agent's reporting relation:

$$(\forall a \in S_i) \qquad R_i(a) \equiv \{t \in T | a\, R_i\, t\}$$

and

$$(\forall t \in S_i) \qquad R_i^{-1}(t) \equiv \{a \in S_i | a\, R_i\, t\}.$$

The inverse correspondence gives the agent's interpretation of what a report means. These two correspondences generate the operations $\Diamond$, ext, $\Box$, and rest. Thus, for $U \subseteq S_i$ and $W \subseteq T$,

$$\Diamond_{R_i}(U) \equiv \{t \in T | (\exists a \in S_i)(a\, R_i\, t \text{ and } a \in U)\},$$

$$\Box_{R_i}(U) \equiv \{t \in T | (\forall a \in S_i)(a\, R_i\, t \to a \in U)\},$$

$$\mathrm{ext}_{R_i}(W) \equiv \{a \in S_i | (\exists t \in T)(a\, R_i\, t \text{ and } t \in W)\},$$

and

$$\mathrm{rest}_{R_i}(W) \equiv \{a \in S_i | (\forall t \in T)(a\, R_i\, t \to t \in W)\}.$$

Reporting can be interpreted in terms of the topology (called the *reporting topology*) on an agent's conceptions. As in Theorem 2.1, the derivation requires two axioms:

Axiom 3.1. *For each $a \in S_i$, there is a $t \in T$ such that $a\, R_i\, t$.*

Axiom 3.2. *For each $t, t'' \in T$, if there is an $a \in S_i$ such that*

$$a\, R_i\, t \quad \text{and} \quad a\, R_i\, t',$$

then there is some $t'' \in T$ such that, for all $a' \in S_i$,

$$a'\, R_i\, t'' \quad \text{iff} \quad a'\, R_i\, t \quad \text{and} \quad a'\, R_i\, t'.$$

Axioms 3.1 is a non-degeneracy requirement. It says that there must be some way, however vague, of reporting anything the agent may want to report. That is, the language must have some sufficiently broad terms ("stuff," for example) to cover anything. Axiom 3.2 requires consistency of the language: if there are conceptions that can be reported more than one way, there must be a way to express that there are multiple possible reports. Thus, if a wine could be called "full-bodied" or "light-bodied," then there must be a term such as "medium-bodied" for wines that could be reported in both these ways.

The following definitions are analogous to those under perception:

Definition 3.1. The *reporting interior* operator is

$$\mathrm{int}_{R_i} \equiv \ \mathrm{ext}_{R_i} \Box_{R_i}.$$

The *reporting closure* operator is

$$\mathrm{cl}_{R_i} \equiv \ \mathrm{rest}_{R_i} \Diamond_{R_i}.$$

A subset U of S_i is *open* in the reporting topology if and only if $U = \mathrm{int}_{R_i}(U)$, and is *closed* in the reporting topology if and only if $U = \mathrm{cl}_{R_i}(U)$.

Theorem 3.1 (Reporting Topology). *The open sets induced by the agent's reporting relation, under Axioms 3.1 and 3.2, form a topology.*

Proof. Entirely analogous to the proof of Theorem 2.1. □

3.2 Communication

The topological interpretations of perceptions and semantics make it possible to address how agents with fundamentally different worldviews can nevertheless find a reliable way to trade. The idea is that what one agent offers in the business language must be something that a trading partner interprets as a faithful representation of what is actually traded, as the trading partner perceives it. Because this subjective notion of a faithful representation differs from the accountant's objective notion (FASB 1980), I refer to the idea studied here as *heterogeneous faithfulness.*

As neither reports nor perceptions are in general unique, the business language cannot guarantee that the same agent *necessarily* issues the same report when faced with the same object. The most that can be required is that one agent reports what he or she sees in a way that a trading partner would agree is a valid possible report.

Definition 3.2. Let $i, j \in I$ be two agents, with conceptions S_i, S_j, perception relations $\Vdash_i, \Vdash_j$, and reporting relations R_i, R_j for a set of common terminology T. The business language is *heterogeneously faithful* between i and j if and only if the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, "\Vdash_i"] \arrow[d, "\Vdash_j"'] & S_i \arrow[d, "R_i"] \\
S_j \arrow[r, "R_j"'] & T
\end{tikzcd}$$

That is, the business language is heterogeneously faithful if and only if

$$R_i \circ \Vdash_i = R_j \circ \Vdash_j \; .$$

If this holds for every $i, j \in I$, then the language is said to be *heterogeneously faithful*.

The following proposition shows that the direction of the definition could be reversed; that is, an equivalent requirement is that two agents interpreting the same report have the same collection of real-world objects in mind.

Proposition 3.1. *Suppose a business language is heterogeneously faithful between two agents, $i, j \in I$. Then the interpretation of the reports is also heterogeneously faithful, i.e.,*

$$\Vdash_i^{-1} \circ R_i^{-1} = \Vdash_j^{-1} \circ R_j^{-1}.$$

Proof. The heterogeneous faithfulness between i and j means that, given $x \in X$ and $t \in T$, agent i can report x as t iff agent j can do so also. For i to be able to report x as t, there must be some $a \in S_i$ that is a way i can perceive x which i can report as t:

$$(\exists a \in S_i)(x \Vdash_i a \text{ and } a \, R_i \, t),$$

which also says that i can interpret t as x, i.e., that

$$(\exists a \in S_i)(a \in R_i^{-1}(t) \text{ and } x \in \Vdash_i^{-1}(a)).$$

By an identical argument, if j can report x as t, then j can interpret t as x. Thus, heterogeneous faithfulness between i and j means that the following square also commutes:

$$\begin{tikzcd}
X & S_i \arrow[l, "\Vdash_i^{-1}"'] \\
S_j \arrow[u, "\Vdash_j^{-1}"] & T \arrow[l, "R_j^{-1}"] \arrow[u, "R_i^{-1}"']
\end{tikzcd}$$

□

Proposition 3.1 thus says that a language is useful for reporting entities if and only if it is useful for end users. This feature depends on the fact that reporting and perception are defined by binary relations and not necessarily by functions.

Theorems 2.1 and 3.1 show that $\langle X, S_i \rangle$ and $\langle S_j, T \rangle$ are topological spaces. Heterogeneous faithfulness thus requires each agent's perceptions to induce a correspondence that carries collections of open neighborhoods to collections of open neighborhoods, i.e., that takes open sets to open sets. Thus, the condition here is related to lower hemi-continuity (Berge 1963):

Definition 3.3. A correspondence $X \xrightarrow{\Vdash} S$ is *lower hemi-continuous* if, for every $U \subseteq S$,

$$\{x \in X \mid \Vdash(x) \between \operatorname{int} U\}$$

is open (where $\between$ denotes occupied intersection).

Theorem 3.2 (Heterogeneous Faithfulness and Lower Hemi-Continuity)**.** *If a business language is heterogeneously faithful between two agents $i, j \in I$, then j's perception correspondence $\Vdash_j(\cdot)$ is a lower hemi-continuous correspondence between X, endowed with i's perceptual topology, and S_j, endowed with j's reporting topology.*

Conversely, suppose $\Vdash_j(\cdot)$ is a lower hemi-continuous correspondence between the topological spaces $\langle X, S_i \rangle$ and $\langle S_j, T \rangle$. Then there is in principle a reporting relation $R_i(\cdot)$ for i that makes the language heterogeneously faithful.

Proof. By the proof of Proposition 2.2, $\Box\ \mathrm{ext}\ \Box\ =\ \Box$, and an analogous argument shows that $\mathrm{ext}\ \Box\ \mathrm{ext}\ =\ \mathrm{ext}$.

If $U \subseteq S_j$ is the extent of some $W \subseteq T$, then $\mathrm{int}\ U =\ \mathrm{int}(\mathrm{ext}(W)) =\ \mathrm{ext}\ \Box\ \mathrm{ext}\ W =\ \mathrm{ext}\ W = U$. Conversely, if $U =\ \mathrm{int}\ U$, then automatically U is the extent of some $W \subseteq T$, namely $\Box U$. Thus $U \subseteq S_j$ is open in the reporting topology if and only if it is the extent of some $W \subseteq T$, i.e., iff it is the inverse image of some subset of T along R_j. A similar argument holds for an open $D \subseteq X$ in i's perceptual topology.

The definition of lower hemi-continuity thus says that the inverse image of any $W \subseteq T$ along $\Vdash_j^{-1} \circ R_j^{-1}$ is the extent of some subset $U' \subset S_i$. So if $W =\ \mathrm{ext}_{R_i} U'$ for some $U' \subseteq S_i$, then the relation $\Vdash_j$ is lower hemi-continuous. But this just says that the square below commutes:

$$\begin{tikzcd}
X & S_i \arrow[l, "\Vdash_i^{-1}"'] \\
S_j \arrow[u, "\Vdash_j^{-1}"] & T \arrow[l, "R_j^{-1}"] \arrow[u, "R_i^{-1}"']
\end{tikzcd}$$

By Proposition 3.1, this is equivalent to heterogeneous faithfulness. Therefore, heterogeneous faithfulness implies lower hemi-continuity.

For the second part of the theorem, define the reporting relation for i by $R_i \equiv R_j \circ \Vdash_j \circ \Vdash_i^{-1}$. The continuity of $\Vdash_j$ means that R_i takes open neighborhoods in i's perceptual topology to open neighborhoods in j's reporting topology, which is just the definition of heterogeneous faithfulness. $\Box$

Remark. The phrase "in principle" in the second part of Theorem 3.2 reflects that the existence is non-constructive. An agent using a language that is heterogeneously faithful can observe what others say and can introspect; by so doing, the agent will fail to refute the claim that the language has the desired property. Thus while the first portion of the theorem could not be established, it could in practice at least withstand scrutiny.

The second part, however, requires more. To construct the desired reporting relation for agent i, one would need access to S_i and S_j (along with the various relations that are composed). This would imply knowledge of others' perceptions and of X, but the phenomena being studied is that no one has such knowledge.

Accordingly, what the latter part of Theorem 3.2 establishes is that the non-existence of the desired relation is contradictory. To one who is omniscient, this is equivalent to existence, but it is clear that the relation used in the proof could not in practice be constructed. In light of the first part of Theorem 3.2, one might view heterogeneous faithfulness as a form of continuity, classically equivalent to lower hemi-continuity, but constructively stronger.[4]

There are in general many business languages that enable heterogeneously faithful reporting. Two degenerate cases are the autarkic language, where $T = \varnothing$, and the universal language, where T is a singleton (e.g, {"stuff"}). In both cases, the reporting relations are trivally faithful. In the autarkic language, contracting is perfectly incomplete, and the only possible allocation is autarky. In the universal language, any agents who agree to trade have no information on what they are bargaining for, and provide no information on what they are offering, so that any economy is a grab bag. In any other case, it is natural to ask what sort of allocations are attainable as equilibria. I turn to this question in the next section.

4 Equilibrium and Welfare

4.1 Syntactical Requirements on the Language

Since all trade occurs in the shared language T, equilibrium must be stated in terms of T. For market clearing to be a meaningful concept, T cannot be an arbitrary language, or even an arbitrary heterogeneously faithful language. There is a syntactical requirement that an operation of addition is defined on T, that T be closed under addition, that there is an additive identity (so that excess demands can be said to sum to zero), and that every $t \in T$ have an additive inverse (so that it is possible to say how markets could clear from an arbitrary set of endowments). Moreover,

[4]Related ideas on continuity are discussed in Grandis (1997) and Gebellato and Sambin (2001). The notion of heterogeneous faithfulness as a stability property of a language seems related to the strategic stability of Kohlberg and Mertens (1986), as strategic stability is likewise a form of lower hemi-continuity.

the clearing of markets must be consistent with conservation of the physical and perceived flow of goods. These requirements are summarized as follows:

Axiom 4.1. *The shared language T is an Abelian (i.e., commutative) additive group, as are X and the S_i. The addition operation on T is common knowledge.*

The perception and reporting relations cannot be assumed to be group homomorphisms. For example, one cannot assume that $\Vdash (x+y) = \Vdash (x) + \Vdash (y)$: both x and y may differ imperceptibly from 0, but their sum may be large enough to be distinct from 0. Similarly, a material quantity can be split up into multiple immaterial quantities. It seems reasonable instead to require the following:

Axiom 4.2. *For all $i \in I$, the correspondences induced by $\Vdash_i$ and R_i are consistent with group homomorphisms. That is, for $x, y \in X$, $\Vdash_i (x+y) \subseteq \Vdash_i (x) + \Vdash_i (y)$, and analogously for the R_i.*

4.2 Optimality and Equilibrium

Notions of equilibrium and optimality depend on an agent's preference relation, which for agent i is assumed to be defined on S_i. While there are philosophical reasons that one might want to permit preferences to be incomplete (e.g., one might argue that conceptions that could be different ways of perceiving the same object should not have a preference defined between them), a minimal requirement would seem to be that strict preferences are irreflexive and transitive. Because the absence of a strict preference might or might not be interpreted as a weak preference, I write $a \not\succ_i b$ as a shorthand for $\neg(a \succ_i b)$. I then have the following notion of Pareto dominance:

Definition 4.1. Let $(a_i), (a_i') \in \prod_{i \in I} S_i$ be perceived allocations. Allocation (a_i) *Pareto dominates* (a_i') iff, for every $i \in I$, $a_i' \not\succ_i a_i$ and, for some $j \in I$, $a_j \succ_j a_j'$.

The notion of an agent's lower contour set is likewise defined in terms of perceived consumption:

Definition 4.2. For $i \in I$ and $U \subseteq S_i$, the conceptions *not better than* U are

$$\{a_i' \in S_i | (\exists a_i \in S_i)(a_i' \not\succ_i a_i \text{ and } a_i \in U)\} \equiv \mathrm{ext}_{\not\succ_i} U.$$

Given price functional $T \xrightarrow{p} \mathbb{R}$, agent i can afford to purchase $t' \in T$ if i's perceived allocation can be reported (under R_i) as a member of the following set:

$$I(p, t') \equiv \{t \in T | p \cdot t' \leq p \cdot t\},$$

where, in keeping with convention, the value of $t \in T$ under p is written $p \cdot t$.

Definition 4.3. For agent $i \in I$ with perceived endowment $a_i \in S_i$, some $t' \in T$ is *budget feasible* for i if and only if

$$(\exists t \in T)(a_i \, R_i \, t \text{ and } t \in I(p, t')),$$

i.e., iff $R_i(a_i) \between I(p, t')$.

An optimal choice is now defined as follows:

Definition 4.4. For agent $i \in I$ with perceived endowment $a_i \in S_i$, a choice of $t' \in T$ is *optimal* given prices $T \xrightarrow{p} \mathbb{R}$ iff

1. t' is budget feasible, and
2. For every budget feasbile $t'' \in T$, if there are $b, c \in S_i$ with b reportable as t', c reportable as t'', and $c \succ_i b$, then $R_i^{-1}(t'') \between \mathrm{ext}_{\not\succ_i} R_i^{-1}(t')$.

Thus, the agent must choose something that has at least one interpretation that is no worse than any budget feasible alternative. Diagramatically, Definitions 4.3 and 4.4 say that optimality is a commutative square:

$$\begin{tikzcd}
S_i \arrow[r, "R_i"] \arrow[d, "\mathrm{ext}_{\not\succ_i}"'] & T \arrow[d, "I(p,\cdot)"] \\
S_i \arrow[r, "R_i"'] & T
\end{tikzcd}$$

This leads to the following result:

Lemma 4.1. *A consumer's budget problem has an optimal choice if and only if the agent's preferences are lower hemi-continuous in the reporting topology.*

Proof. Immediate consequence of Theorem 3.2 and Definition 4.4. □

The definition of equilibrium is now straightforward: an equilibrium is budget feasible and optimal for each agent, with markets clearing in the group operation defined in Axiom 4.1.

Definition 4.5. A *competitive equilibrium* is a price mapping $T \xrightarrow{p} \mathbb{R}$ and a collection $\{(t_i, t'_i)_{i \in I}\}$ of pairs in $T \times T$ such that, if the initial perceived endowments are $(a_i \in S_i)_{i \in I}$, then for each $i \in I$, the following conditions hold:

Feasibility $a_i \, R_i \, t_i$ and $p \cdot t'_i \leq p \cdot t$.

Optimality For feasible $t''_i \in T$, if $(\exists b_i \in R_i^{-1}(t'_i))$ and $(\exists c_i \in R_i^{-1}(t''_i))$ such that $c_i \succ_i b_i$, then there are $b'_i \in R_i^{-1}(t'_i)$ and $c'_i \in R_i^{-1}(t''_i)$ such that $c'_i \not\succ_i b'_i$.

Market Clearing $\sum_{i \in I} t_i = \sum_{i \in I} t'_i$, where the summations are in the group operation defined on T by Axiom 4.1.

Since T is the base of a topology on S_i, then if T is finite, every open cover of every subset of S_i (for every agent $i \in I$) necessarily has a finite subcover. In other words, T is finite, then every set is compact. This would seem to be a powerful result for assuring the existence of equilibrium. However, Axiom 4.1 restricts the cases where T can be finite. In particular, the following holds:

Proposition 4.1. *If T is finite and addition is not isomorphic to addition mod n for some $n \in \mathbb{Z}_{++}$, then T is a singleton.*

Proof. See Lang (2000), pages 23–5. An intuitive argument is as follows: by Axiom 4.1, T is an additive group. If $t \in T$ and $t \neq 0$, then $t + t \in T$. If $t + t = t$, then $t + (t + (-t)) = t + 0 = t$, violating the hypothesis. But then $t + t \neq t$ for every $t \neq 0$. If T is not a singleton, this means that either addition must be isomorphic to addition mod n, which is ruled out by hypothesis, or T must be infinite. Thus, either $T = \{0\}$ with the only possible notion of addition, or T is infinite. □

Nevertheless, it is always possible to coarsen T (even when T is infinite) enough in order to make every set compact:

Definition 4.6. T is a *Stone-compact* space iff $(\forall W \subseteq T)(\exists W_0 \subseteq W)$ such that

1. W_0 is finite, and
2. $\text{ext } W_0 = \text{ext } W$.

T is a *Scott-compact* space iff $(\forall W \subseteq T)(\forall t \in T)$

$$\text{if ext } \{t\} \subseteq \text{ext } W, \text{ then } (\exists t' \in W)(\text{ext } \{t\} \subseteq \text{ext } \{t'\}).$$

Lemma 4.2. *If T is Stone-compact, or if T is Scott-compact and $I(p, \cdot)$ is finite-set valued for every p, then a competitive equilibrium exists provided the language is heterogeneously faithful and each agent's preferences are lower hemi-continuous (in the sense of Lemma 4.1).*

Proof. From heterogeneous faithfulness, preferences are consistent with continuous functions, and in particular with continuous group homomorphisms (because of Axiom 4.2). Thus, each agent's optimization problem includes a continuous group homomorphism over a compact set, so every agent has a budget feasible optimal choice, given a price functional, and in particular when prices are market clearing. □

Lemma 4.3 (Negri 1996)**.** *R_i and T always can be coarsened so that $\langle S_i, R_i, T \rangle$ is Stone-compact.*

Proof. See Negri (1996), Proposition 2.8 and Theorem 2.10. □

Theorem 4.1. *A business language can always be coarsened so that competitive equilibrium exists.*

Proof. Immediate corollary of Lemma 4.2 and Lemma 4.3. □

Axioms 4.1 and 4.2 guarantee that equilibrium is consistent with conservation of the flow of goods; this is the most that can be hoped for.

Proposition 4.2. *Suppose that T is heterogeneously faithful and that individual preferences are lower hemi-continuous in the sense of Lemma 4.1. Let $\omega_i \in X$ be the initial real-world allocation* *to agent $i \in I$, and let $x_i \in X$ be what agent i receives in equilibrium. If*

$$\sum_{i \in I} \omega_i = \sum_{i \in I} x_i,$$

then, for each $i \in I$, there are reports $t_i \in R_i \circ \Vdash_i (\omega_i)$ and $t'_i \in R_i \circ \Vdash_i (x_i)$ such that markets clear. Conversely, if markets clear in T, then there are (ω_i) and (x_i) satisfying the above equation, i.e., such that the flow of goods is conserved.

Proof. From Axiom 4.1, the above expressions make sense. The correspondences from $X \longrightarrow S_i$ and from $S_i \longrightarrow T$ contain continuous group homomorphisms because of Axiom 4.2, so their composition contains a continuous group homomorphism. $\square$

4.3 Failure of the Welfare Theorems

The results above are promising, in that they show that competitive equilibrium exists for any sufficiently coarsened business language. Unfortunately, much less can be said about the welfare properties of equilibrium than in the standard neoclassical case. There can be competitive equilibria that are Pareto dominated by feasible allocations, and there can be Pareto optimal allocations that cannot be attained as a competitive equilibrium. Thus, neither welfare theorem holds. Because of the topological interpretations, one might expect competitive equilibrium to be approximately Pareto optimal, but even this is untrue. Thus, the welfare theorems do not seem robust to any ambiguities in perception or language, or to any possible miscommunication.

In the non-heterogeneously faithful case, the failure of the welfare theorems is seen in the following:

Example 4.1. Let $X = \{x, y\}$, $I = \{1, 2\}$, $S_1 = \{a_1, b_1\}$, $S_2 = \{a_2, b_2\}$. Assume $x \Vdash_1 a_1$, $x \Vdash_2 a_2$, $y \Vdash_1 b_1$, and $y \Vdash_2 b_2$. Thus each agent perceives X perfectly.

Let T have two terms, t_1 and t_2, and assume that agents 1 and 2 report differently, with $a_1 \, R_1 \, t_1$, $b_1 \, R_1 \, t_2$, $b_2 \, R_2 \, t_1$, and $a_2 \, R_2 \, t_2$. That is, agent 1 uses t_1 to mean x and t_2 to mean y, and agent 2 reverses these. Since T is an additive group, assume T is given by all integer multiples of t_1 and t_2, with addition defined componentwise.

Let preferences be given by

$$b_1 \succ_1 a_1 \qquad a_2 \succ_2 b_2,$$

and assume that initially agent 1 is endowed with x and agent 2 with y. Thus both agents can offer t_1 in trade and are willing to trade it for t_2. If the price of t_2 exceeds that of t_1, then autarky is an equilibrium. However, if both agents were to swap their endowments, they would both be better off. That is, the allocation (b_1, a_2) is feasible and Pareto dominates the equilibrium (a_1, b_2).

In fact, the allocation (a_1, b_2) is Pareto optimal. However, if it is to emerge as equilibrium, then both agents must offer to sell t_1 at the equilibrium prices, and then purchase t_1 at the equilibrium prices. Since both agents will purchase t_2 if it is affordable, in equilibrium the price of t_2 must exceed that of t_1. If both agents sell their t_1 and buy the other agent's t_1, then the Pareto optimum is achieved. However, these equilibrium prices and reports are the same as those for which autarky is an equilibrium. Therefore, no equilibrium prices assure trading from the Pareto dominated endowments to the Pareto optimal allocation.

The above example depends on the lack of heterogeneous faithfulness. However, it is easy to see that the welfare theorems fail even in the heterogeneously faithful case. The universal language provides the simplest example:

Example 4.2. Let $T = \{0\}$ be a singleton, with the only possible definition of addition. Then, irrespective of X and the S_i, every agent always issues the same report, and markets always clear. Thus, the market is always in equilibrium. If there is *any* possible Pareto-dominated allocation, it can arise as a result of the equilibrium, and nothing can assure a move to a Pareto-superior allocation. That is, competitive equilibrium need not be optimal, and optimal allocations need not be attainable as an equilibrium.

This example can be extended to any setting where an agent cannot distinguish in the reporting language between conceptions over which the agent has a strict preference.

Examples 4.1 and 4.2 show how limits on the business language, in either the form of misunderstandings or ambiguities in interpretation, can make competitive equilibrium suboptimal. The counter-examples do not depend on perceptual limits; indeed, Example 4.1 used perfect perceptions.

It is straightforward to see that, when agents have perceptual limits, the language will necessarily either have ambiguous semantics or fail to be heterogeneously faithful. Thus subjective perceptions or subjective interpretations of the language conflict with the welfare theorems.

A natural way of ranking business languages is by their informativeness. Let T, T' be two sets of shared terminologies, with reporting relations R_i, R_i' for each agent $i \in I$. If, for each $t \in T$ and each $i \in I$, there is some $t' \in T'$ such that $\text{ext}_{R_i'}\{t'\} \subset \text{ext}_{R_i}\{t\}$, then T' is a (weakly) finer information structure than T. This is closely related to Blackwell's informativeness.

The results in this section show that ranking business languages by their informativeness is unlikely to be a reasonable social preference. Once a language allows distinctions to become overly precise, it ceases to be heterogeneously faithful; i.e., the agents can no longer use it and know what they are trading. Thus, as in Dubra and Echenique ((2001) and (2004)), more detailed information can be less desirable than coarser information.

5 Concluding Remarks

This paper studies the economic consequences of subjective information, by modeling how an agent subjectively perceives real-world objects, and by modeling how such an agent can nevertheless use a shared business language. The framework here is then used to study the interaction between subjective perceptions and subjective semantics, and how these affect the way an economy functions.

The models of perceptions and of use of a language have useful topological interpretations. The connection between what an agent perceives and what an agent reports is shown to be a form of continuity in these topological spaces. Intuitively, the continuity condition says that an agent reports a real-world object in a way that a trading partner, viewing the same object, could see as justified. Because preferences are defined on the agents' subjective worlds, the topological interpretation of perceptions can be thought of as saying that agents' preferences are defined on open neighborhoods of consumption choices. Thus the model here might be viewed as an argument in favor of Kreps's (1979) definition of utilities on sets rather than on points, and with Jeffrey's (1983) definition of utility on propositions (hence on subsets). With some regularity conditions on agents' preferences, the topological interpretation is used again to provide conditions for the existence of equilibrium.

Much less can be said about the welfare properties of competitive equilibria under subjective information than in the usual case. Neither welfare theorem necessarily holds, even though preferences are continuous and there are no externalities. The reason is that each agent does not see exactly what is being traded, and cannot convey exactly what is desired. The invisible hand is therefore not always able to allocate resources optimally, largely because the agents are unable to convey precisely what they want it to do.

The notion of agents' beliefs thus arises in a different sense from the usual probabilistic one. One agent may observe another agent's use of a language, and infer the distinctions that the other agent is capable of making. Thus an extension of the current model to a dynamic model would enable one to discuss beliefs about another agent's subjective understanding of the world.

References

Ahn, R. M. C. (2000): "Agents, Objects, and Events: A Computational Approach to Knowledge, Observation, and Communication," Ph.D. thesis, Technische Universitiet Eindhoven.

Allen, B., and J. Thisse (1992): "Price Equilibria in Pure Strategies for Homogeneous Oligopoly," *Journal of Economics and Management Strategy*, 1(1), 63–81.

Aumann, R. J., and A. Brandenburger (1995): "Epistemic Conditions for Nash Equilibrium," *Econometrica*, 63(5), 1161–80.

Avigard, J. (2004): "Forcing in Proof Theory," *Bulletin of Symbolic Logic*, 10(3), 305–33.

Berge, C. (1963): *Topological Spaces*. Oliver and Boyd.

Bridges, D. S., P. Schuster, and L. S. Vîţă (2002): "Apartness, Topology, and Uniformity: a Constructive View," *Mathematical Logic Quarterly*, 48(Supplement 1), 16–28.

Brouwer, L. E. J. (1907): "On the Foundations of Mathematics," Ph.D. thesis, University of Amsterdam, Translated by Arend Heyting and reprinted in *L. E. J. Brouwer Collected Works I*, North-Holland 1975.

Crawford, V. P., and J. Sobel (1982): "Strategic Information Transmission," *Econometrica*, 50(6), 1431–51.

Dekel, E., B. L. Lipman, and A. Rustichini (1998a): "Recent Developments in Modeling Unforeseen Contingencies," *European Economic Review*, 42(3–5), 523–42.

——— (1998b): "Standard State-Space Models Preclude Unawareness," *Econometrica*, 66(1), 159–73.

Dickhaut, J. W., and I. R. C. Eggleton (1975): "An Examination of the Process Underlying Comparative Judgements of Numerical Stimuli," *Journal of Accounting Research*, 13(4), 38–72.

Dubra, J., and F. Echenique (2001): "Monotone Preferences over Information," *Topics in Theoretical Economics*, 1(1), 1–15.

——— (2004): "Information is not about Measurability," *Mathematical Social Sciences*, 47(2), 177–85.

Dubra, J., F. Maccheroni, and E. A. Ok (2004): "Expected Utility Theory without the Completeness Axiom," *Journal of Economic Theory*, 115(4), 118–33.

Dummett, M. A. E. (2000): *Elements of Intuitionism*, no. 39 in Oxford Logic Guides. Oxford University Press, second edn.

Fagin, R., and J. Y. Halpern (1988): "Belief, Awareness, and Limited Reasoning," *Artificial Intelligence*, 34, 39–76.

FASB (1980): "Qualitative Characteristics of Accounting Information," Statement of Financial Accounting Concepts No. 2, Financial Accounting Standards Board.

Gebellato, S., and G. Sambin (2001): "The Essence of Continuity (the Basic Picture, II)," Preprint 27, University of Padua Department of Pure and Applied Mathematics.

Grandis, M. (1997): "Weak Subobjects and Weak Limits in Categories and Homotopy Categories," *Cahiers de Topologie et Géometrie Différentielle Catégoriques*, 38, 301–26.

Heyting, A. (1956): *Intuitionism: An Introduction*. North-Holland.

Jeffrey, R. C. (1983): *The Logic of Decision*. University of Chicago Press, second edn.

Johnstone, P. T. (1977): *Topos Theory*. Academic Press.

Kanodia, C., R. Singh, and A. Spero (2005): "Imprecision in Accounting Measurement: Can it be Value Enhancing?," *Journal of Accounting Research*, 43(2), 487–519.

Knoblauch, V. (1998): "Order Isomorphism for Preferences with Intransitive Indifference," *Journal of Mathematical Economics*, 30(4), 421–31.

Kohlberg, E., and J.-F. Mertens (1986): "On the Strategic Stability of Equilibria," *Econometrica*, 54(5), 1003–38.

Kreps, D. M. (1979): "A Representation Theorem for 'Preference for Flexibility'," *Econometrica*, 47(3), 565–78.

LANG, S. (2000): *Algebra*. Springer-Verlag, 3rd edn.

LIPMAN, B. L. (2001): "Why Is Language Vague?," Discussion paper, Boston University.

MAS-COLELL, A., M. D. WHINSTON, AND J. R. GREEN (1995): *Microeconomic Theory*. Oxford University Press.

MODICA, S., AND A. RUSTICHINI (1999): "Unawareness and Partitional Information Structures," *Games and Economic Behavior*, 27(2), 265–98.

NEGRI, S. (1996): "Stone Bases, Alias the Constructive Content of Stone Representation," in *Logic and Algebra*, ed. by A. Ursini, and P. Aglianó, pp. 617–36. Dekker.

——— (2002): "Continuous Domains as Formal Spaces," *Mathematical Structures in Computer Science*, 12, 19–52.

RUBINSTEIN, A. (2000): *Economics and Language*. Cambridge University Press.

SAMBIN, G. (2001): "The Basic Picture, a Structure for Topology (the Basic Picture, I)," Discussion paper, University of Padua Department of Pure and Applied Mathematics.

SAMBIN, G., AND S. GEBELLATO (1998): "A Preview of the Basic Picture: A New Perspective on Formal Topology," in *Types and Proofs for Programs*, vol. 1657 of *Lecture Notes in Computer Science*, pp. 194–207. Springer-Verlag.

TROELSTRA, A. S., AND D. VAN DALEN (1988): *Constructivism in Mathematics: An Introduction*, no. 121 and 123 in Studies in Logic and the Foundations of Mathematics. Elsevier, Two volumes.

VALENTINI, S. (2001): "Fixed Points of Continuous Functions Between Formal Spaces," Discussion paper, Department of Pure and Applied Mathematics, University of Padua.

VICKERS, S. J. (1988): *Topology via Logic*, no. 5 in Cambridge Tracts in Theoretical Computer Science. Cambridge University Press.

VÎŢĂ, L. S., AND D. S. BRIDGES (2003): "A Constructive Theory of Point-Set Nearness," *Theoretical Computer Science*, 503(1–3), 473–89.