

# Using Option Pricing Theory to Infer About Equity Premiums.

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## Abstract

In this paper we make use of option pricing theory to infer about historical equity premiums. This we do by comparing the prices of an American perpetual put option computed using two different models: The first is the standard one with continuous, zero expectation, Gaussian noise, the second is a strikingly similar model, except that the zero expectation noise is of Poissonian type.

The interesting fact that makes this comparison worthwhile, is that the probability distribution under the risk adjusted measure turns out to depend on the equity premium in the Poisson model, while this is not so for the standard, Brownian motion version. This difference is utilized to find the intertemporal, equilibrium equity premium.

We apply this technique to the US equity data of the last century and find that, if the risk free short rate was around one per cent, this corresponds to a risk premium on equity about two and a half per cent. On the other hand, if the risk free rate was about four per cent, we find that this corresponds to an equity premium of around four and a half per cent.

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The advantage with our approach is that we only need equity data and option pricing theory, no consumption data was necessary to arrive at these conclusions.

We round off the paper by investigating if the procedure also works for incomplete models.

*KEYWORDS: historical equity premiums, perpetual American put option, equity premium puzzle, risk free rate puzzle, geometric Brownian motion, geometric Poisson process, CCAPM.*

## Introduction.

The paper develops a technique for inferring historical risk premiums by the use of option pricing theory. This we do by comparing the prices of an American perpetual put option computed using two different models: The first is the standard one, where the accumulated return of a risky asset follows a continuous processes with zero expectation, Gaussian noise. In the second model the accumulated return also has a continuous component, but here the zero expectation noise is of Poissonian type, so this process has discontinuities in its paths.

It turns out that in the Poisson driven model the probability distribution under the risk adjusted measure depends on the equity premium, while this is not the case for the Brownian motion driven version. This difference we utilize to find equity premiums when the two different models are calibrated to yield the same average state prices, and have the same volatilities.

To see if this is reasonable or not, one must consider the question if a representative investor would be willing to pay the same price for an American perpetual put option when the price of the underlying follows a geometric Brownian motion, as in the case when it follows a geometric Poisson process. Both processes have the same volatilities, and the compared contracts have the same other characteristics, of course.

Since a Poisson random variable is *infinitely divisible* (e.g., Sato (1999)), meaning that it can be written as a sum of an arbitrary number of i.i.d. Poisson random variables, by the Central Limit Theorem we have that the accumulated returns process in the Poisson case is close to normal. Hence the probability distributions are approximately the same at the time when the representative investor may choose to exercise the option. Thus we should expect that the answer to the above question is yes.

Why do we use the perpetual American put option in this regard? First this instrument does not depend upon any time horizon. Second, by equating average state prices we get rid of the effects of different strike prices. Third

we know how to adjust for risk in both models. Fourth we have explicit, simple expressions for its market price in both cases.

An advantage with our approach is that we do not need consumption data to obtain equilibrium intertemporal equity premiums, as the quality of these data has been questioned.

Another candidate to produce intertemporal risk premiums without consumption data is the ICAPM of Merton (1973b). This model, on the other hand, requires certain state variables to be identifiable, which means that empirical testing of the ICAPM quickly becomes difficult.

Further attempts to overcome the inaccuracies in consumption data include Campbell (1993) and (1996). Briefly explained, a log-linear approximation to the representative agent's budget constraint is made and this is used to express unanticipated consumption as a function of current and future returns on wealth. This expression is then combined with the Euler equation resulting from the investor's utility maximization to substitute out consumption of the model. As is apparent, our approach is rather different from this line of research.

The paper is organized as follows: In Section 2 we present the two models. Here the solutions to the American perpetual option pricing problems are recalled for both models, where the relevant risk adjustments are emphasized. In Section 3 we discuss the relevant properties of these two models for our purposes and compare them further. In Section 4 we use these results to infer about historical equity premiums. In this section we also discuss the consequences of our findings for the equity premium puzzle and the risk-free rate puzzle. In Section 5 we replace the geometric Poisson process by geometric jump processes with continuously distributed jump sizes, and the last section concludes.

## **I The Problem**

### **I-A Introduction**

We utilize our methodology to the problem of estimating the equity premiums in the twentieth century. This has been a challenge in both finance and in macro economics for some time. The problem dates back to the paper by Mehra and Prescott (1985), introducing the celebrated "equity premium puzzle". Closely related there also exists a so called "risk-free rate puzzle", see e.g., Weil (1989), and both puzzles have been troublesome for the consumption-based asset pricing theory.

The problem has its root in the small estimate of the covariance between

equities and aggregate consumption, and the small estimate of the variance of aggregate consumption, combined with a large estimate of the equity premium. Using a representative agent equilibrium model of the Lucas (1978) type, the challenge has been to reconcile these values with a reasonable value for the relative risk aversion of the representative investor (the equity premium puzzle), and also with a reasonable value for his subjective interest rate (the risk free rate puzzle). Mehra and Prescott (1985) estimated the short term interest rate to one per cent, and the equity premium was estimated to around six per cent.

McGrattan and Prescott (2003) re-examine the equity premium puzzle, taking into account some factors ignored by the Mehra and Prescott: Taxes, regulatory constraints, and diversification costs - and focus on long-term rather than short-term savings instruments. Accounting for these factors, the authors find that the difference between average equity and debt returns during peacetime in the last century is less than one per cent, with the average real equity return somewhat under five per cent, and the average real debt return almost four per cent. If these values are correct, both puzzles are solved at one stroke (see e.g., Aase (2004)).

From these studies it follows that there is some confusion about the appropriate value of the equity premium of the last century, at least what numerical value to apply in models. It also seems troublesome to agree on the value of the short term interest rate for this period.

Our results for the US equity data of the last century indicate an equity premium of around 2.5 per cent if the risk free short rate has been about one per cent. If the latter rate has been around four per cent, on the other hand, we find that this corresponds to an equity premium of around 4.4 per cent. Both these values disagree somewhat with the two above studies. Our value of around 2.5 per cent equity premium yields a more reasonable coefficient of relative risk aversion than the one obtained by Mehra and Prescott (1985). If, on the other hand, the average real debt return was around 4 per cent during this time period, our 4.4 per cent risk premium differs somewhat from the 1 per cent estimate in McGrattan and Prescott (2003). See also Siegel (1992), who found that a risk free rate of one per cent was not really representative of the relevant period. We now turn to our methodology.

## I-B The two models

First we establish the dynamics of the assets in the underlying two models: In each case there is an underlying probability space  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$  satisfying the usual conditions, where  $\Omega$  is the set of states,  $\mathcal{F}$  is the set of events,  $\mathcal{F}_t$  is the set of events observable by time  $t$ , for any  $t \geq 0$ , and  $P$  is the

given probability measure, governing the probabilities of events related to the stochastic price processes in the market. In each case there is one locally riskless asset, thought as the evolution of a bank account with dynamics

$$d\beta_t = r\beta_t dt, \quad \beta_0 = 1,$$

and one risky asset. In the standard model the continuous price process  $S^c$  of the risky asset follows the dynamics

$$\frac{dS_t^c}{S_t^c} = \mu dt + \sigma dB_t, \quad (1)$$

which means that  $S_t^c$  is a geometric Brownian motion. Here  $\mu$  and  $\sigma$  are constants, and  $B$  is a standard Brownian motion, the zero mean Gaussian noise term. The accumulated return processes  $R_t^c := \mu t + \sigma B_t$  is seen to be a Brownian motion with drift, a Gaussian process.

In the Poissonian model the dynamic equation for the price process  $S^d$  of the discontinuous risky asset is given by

$$\frac{dS_t^d}{S_{t-}^d} = \mu dt + z_0 d\tilde{N}_t. \quad (2)$$

Here  $\tilde{N}_t := N_t - \lambda t$ , where  $N_t$  is a Poisson process with frequency parameter  $\lambda$ ,  $\tilde{N}_t$  is called the *compensated* Poisson process, and  $S_t^d$  is consequently a geometric Poisson process. The corresponding accumulated return processes  $R_t^d := \mu t + z_0 \tilde{N}_t$  is seen to be a Poisson process with drift. The parameter  $z_0$  signifies the jump sizes, and the compensated Poisson process  $\tilde{N}_t$  is a zero mean noise term, corresponding to the term  $B_t$  in the first model.

We see that the accumulated return processes  $R_t^d$  and  $R_t^c$  both have means  $\mu t$ , and their respective variances are  $(z_0^2 \lambda)t$  and  $\sigma^2 t$ . Suppose we calibrate the two processes  $S^c$  and  $S^d$  such that these latter quantities are equal, i.e.,  $\sigma = z_0 \sqrt{\lambda}$ . For any integer  $n$ , the Poisson random variable  $N_t$  with parameter  $\lambda t$  can be expressed as a sum of  $n$  independent, Poisson random variables  $N_t^i$  with parameter  $\lambda t/n$ . This property, called *infinite divisibility*, can be interpreted as saying that a Poisson random variable can be "divided" into an arbitrary number of i.i.d. random variables.

Recalling the key role the i.i.d. assumption plays in the Central Limit Theorem, it then follows from this theorem that, when  $\lambda t$  is sufficiently large, the probability distributions of the return process  $R_t^d$  will be well approximated by the normal distribution. Since  $N_t$  is infinitely divisible, this approximation can indeed be accurate even for moderate to small values of  $\lambda t$ . Since the return process  $R_t^c$  is normally distributed for any  $t$ , and since the normal distribution is completely characterized by its two first moments, the result is that  $R_t^c$  and  $R_t^d$  will have approximately the same probability distribution, for any  $t$ , under these conditions.

## I-C The price of a perpetual American put option

For the standard Gaussian model this problem has been solved by Merton (1973a). We recall his result: Let  $x$  be the price of the underlying asset at initiation of the contract. Then the market price  $\psi(x)$  of the perpetual put is given by

$$\psi(x) = \begin{cases} (K - c)\left(\frac{c}{x}\right)^\gamma, & \text{if } x \geq c; \\ (K - x), & \text{if } x < c, \end{cases} \quad (3)$$

where the continuation region  $\mathcal{C}$  is given by

$$\mathcal{C} = \{(x, t) : x > c\},$$

and the trigger price  $c$  is a constant. This constant is given by

$$c = \frac{\gamma K}{\gamma + 1}, \quad (4)$$

where the constant  $\gamma$  solves the following quadratic equation

$$-r - r\gamma + \frac{1}{2}\sigma^2\gamma(\gamma + 1) = 0 \quad (5)$$

For the Poisson model this problem has been solved by Aase (2005). The result is that the market price  $\psi(x)$  of the American perpetual put option is given by the same equation (3) as above, with the only exception that the parameter  $\gamma$  now solves the following nonlinear equation

$$-r - r\gamma + \lambda^Q(1 + \alpha z_0)^{-\gamma} + z_0\lambda^Q\gamma - \lambda^Q = 0 \quad (6)$$

Here  $\lambda^Q$  is the frequency of the Poisson process  $N$  under the risk adjusted measure  $Q$  in the jump model, to be explained in the next subsection.<sup>1</sup>

Let us consider the nonlinear equation (6) for  $\gamma$ , since the solution to (5) is well known. If  $z_0 > 0$  this equation is seen to have a unique positive solution in  $\gamma$  if  $r < \lambda^Q\alpha z_0$  for interest rate  $r > 0$ . If  $r \geq \lambda^Q\alpha z_0 > 0$  there is no solution. For  $z_0 < 0$  there is one positive and one negative solution for  $\gamma$  when  $r > 0$ , where only the positive one has economic meaning.

Note that we may interpret the term  $\left(\frac{x}{c}\right)^{-\gamma} I_{\{\tau(\omega) \in [t, t+dt)\}}(\omega)$  as the "state price" when  $x \geq c$ , where  $I$  indicates if exercise happens at time  $t$  or not: If exercise takes place at time  $t$ , then  $(K - c)$  units are paid out at a price  $(x/c)^{-\gamma}$  per unit when  $x \geq c$ , and  $(K - x)$  units are paid at price 1 per unit if  $x < c$ . Hence the term  $(x/c)^{-\gamma}$  in (3) can be interpreted as an "average state price" when  $x \geq c$ .

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<sup>1</sup>When the jump size parameter  $z_0 < 0$ , the above solution is only approximate. This will have no consequences in our treatment.

## I-D Risk adjustments

While the concept of an equivalent martingale measure is well known in the case of diffusion price processes with a finite time horizon  $T < \infty$ , the corresponding concept for jump price processes is lesser known. In addition we have an infinite time horizon, in which case it is not true that the "risk neutral" probability measure  $Q$  is equivalent to the given probability measure  $P$  (see e.g., Huang and Pagès (1992) or Revuz and Yor (1991)).

Suppose  $P$  and  $Q$  are two probability measures, and let  $P_t := P|_{\mathcal{F}_t}$  and  $Q_t := Q|_{\mathcal{F}_t}$  denote their restrictions to the information set  $\mathcal{F}_t$ . Then  $P_t$  and  $Q_t$  are equivalent for all  $t$  if and only if  $\sigma^P = \sigma^Q$  in the standard model, and if and only if  $z_0^P = z_0^Q$  in the Poisson model, i.e., the jump sizes  $z_0$  must be equal under the two different probability measures in the latter model.

In the standard model the market price of risk  $\theta_c$  solves the equation

$$S_t^c \sigma \theta_c = (\mu - r) S_t^c$$

which, since  $S_t^c > 0$  almost surely for all  $t$ , has the solution

$$\theta_c = \frac{\mu - r}{\sigma},$$

the Sharpe ratio. In this situation this connects to a unique risk adjusted measure  $Q$ , and the standard model is complete. The density process  $\xi_t$  is given by

$$\xi_t = \exp\{-\theta_c B_t - \frac{1}{2}\theta_c^2 t\}$$

where the connection between these two measures is given by

$$dQ_t(\omega) = \xi(t) dP_t(\omega) \tag{7}$$

and  $E(\xi(t)) = 1$  for all  $t$ . Under the probability measure  $Q$  the process  $B_t^Q := B_t + \theta_c t$  is a standard Brownian motion for all  $t \geq 0$ . This leads to the following dynamic for  $S^c$  under  $Q$ :

$$\frac{dS_t^c}{S_t^c} = r dt + \sigma dB_t^Q. \tag{8}$$

Notice that the only change from (1) is that the drift rate  $\mu$  has changed to  $r$ , while the noise terms have the same normal  $\mathcal{N}(0, \sigma^2 t)$ -distributions under their respective probability measures  $P$  and  $Q$ , since the volatility parameter  $\sigma$  has not changed. As a consequence of this, the second term in the equation (5) for  $\gamma$  is  $r\gamma$ , not  $\mu\gamma$  as would have been the case if we just solve an optimal stopping problem, and not a pricing problem.

Turning to the Poissonian case, we have a somewhat different, but in many respects also a similar, picture (see e.g., Aase (2005), or Øksendal and Sulem (2004) for details). Here we denote by  $\theta_d$  the quantity corresponding to the above market price of risk  $\theta_c$ . It is given by

$$\theta_d = \frac{\mu - r}{\lambda z_0}. \quad (9)$$

There is only one market price of risk parameter here as well, so this model is also complete. The density associated with the corresponding change of probability measure in (7) is now

$$\xi_t = \exp \{ \ln(1 - \theta_d)N_t + \theta_d z_0 \lambda t \} \quad (10)$$

for all  $t$ . From this expression we note that  $\theta_d \leq 1$ . Under the probability measure  $Q$  corresponding to this density process, the process  $\tilde{N}_t^Q$  is a compensated Poisson process, where

$$\tilde{N}_t^Q = N_t - (1 - \theta_d)\lambda t \quad (11)$$

for all  $t \geq 0$ , which follows from a version of Girsanov's theorem for jump processes. Comparing this to  $\tilde{N}_t = N_t - \lambda t$ , and recalling that the jump size parameter  $z_0$  has not changed under  $Q$ , we notice that the only parameter that has changed under  $Q$  is the frequency, which is now given by

$$\lambda^Q := \lambda(1 - \theta^d) = \lambda + \frac{r - \mu}{z_0}, \quad (12)$$

where we have also used the expression for the market price of risk  $\theta^d$  in equation (9). Note that the relation (11) can alternatively be written as  $\tilde{N}_t^Q = \tilde{N}_t + \theta_d \lambda t$ , which corresponds to  $B_t^Q = B_t + \theta_c t$  in the standard model. This leads to the following dynamic equation for  $S^d$  under  $Q$

$$\frac{dS_t^d}{S_{t-}^d} = r dt + z_0 d\tilde{N}_t^Q. \quad (13)$$

The drift rate has again changed to the risk-less spot rate  $r$  under  $Q$ . Also the noise term is of the same type under  $Q$  as under  $P$ , a compensated Poisson process times the constant  $z_0$ . However, here is one important difference from the standard model: The Poisson process  $N_t^Q$  has parameter  $\lambda^Q$  under  $Q$ , while the Poisson process  $N_t$  has parameter  $\lambda$  under  $P$ . Clearly  $\lambda^Q \neq \lambda$  if the equity premium  $e_p := (r - \mu) \neq 0$ , as is evident from equation (12). Thus the probability distributions of the zero mean noise terms are not the same under the respective probability distributions  $P$  and  $Q$ , as was the case



for the standard model. We shall see that this distinction becomes crucial in what follows.

As a consequence of (13), the term  $r\gamma$  appears in the equation (6) for  $\gamma$  instead of  $\mu\gamma$  as would have been the case if we just solved an optimal stopping problem, and the parameter  $\lambda^Q$  replaces  $\lambda$  in the latter case. Thus the market price of the perpetual American put option depends upon the risk adjusted frequency  $\lambda^Q$ , which in its turn is a function of the equity premium  $e_p$  through equation (12).

## II Discussion of the models

### II-A The equity premium

Let us start the discussion by briefly explaining why the jump parameter  $z_0$  does not change under  $Q$  in the jump model. Changing the frequency of jumps amounts to "reweighting" the probabilities on paths, and no new paths are generated by simply shifting the intensity. However, changing the jump sizes generates a different kind of paths. The frequency of a Poisson process can be modified without changing the "support" of the process, but changing the sizes of jumps generates a new measure which assigns nonzero probability to some events which were impossible under the old one. Thus  $z_0 := z_0^P = z_0^Q$  is the only possibility here when the probability measures  $P$  and  $Q$  are equivalent.

While it is a celebrated fact that the probability distribution of  $S_t^c$  under  $Q$ , in the standard model, does not depend on the drift parameter  $\mu$ , in the jump model it does. It is true that the drift term  $\mu$  was replaced by  $r$  in equation (13) for  $S^d$  under  $Q$ , but since  $\tilde{N}^Q$  depends upon  $\lambda^Q$ , the drift term, or the equity premium  $e_p$  rather, is reintroduced via the risk adjusted frequency  $\lambda^Q$  in equation (12). As a consequence of this the values of options must also depend on  $e_p$  in the latter model. For the American perpetual put option we see this from the equations (3), (4) and (6) for  $\gamma$ .

Let us briefly recall the argument why the drift parameter can not enter into the pricing formula for any contingent claim in the standard model: If two underlying assets existed with different drift terms  $\mu_1$  and  $\mu_2$  but with the same volatility parameter  $\sigma$ , there would simply be arbitrage. In the jump model different drift terms lead to different frequencies  $\lambda_1^Q$  and  $\lambda_2^Q$  through the equation (12), but this also leads to different volatilities of the two risky assets, since the volatility (under  $Q$ ) depends upon the jump frequency. Thus no inconsistency arises when the drift term enters the probability distribution under  $Q$  in the jump model.

We may solve the equations (6) and (12) in terms of  $e_p$ . This results in a linear equation for  $e_p$  with solution

$$e_p = z_0 \left\{ \frac{r(\gamma + 1)}{(1 + z_0)^{-\gamma} - (1 - z_0\gamma)} - \lambda \right\}. \quad (14)$$

Although this formula indicates a very simple connection between the equity premium and the parameters of the model, it is in some sense circular, since the parameter  $\gamma$  on the right hand side is not exogenous, but depends on all the parameters of the model. We will demonstrate below how this formula may be used to infer about historical risk premiums.

## II-B A calibration exercise: Two initial examples.

We would like to use the above two different models for the same phenomenon to infer about equity premiums in equilibrium. In order to do this, we calibrate the two models, which we propose to do in two steps. First we ensure that the martingale terms have the same variances in both models. Second, both models ought to yield the same option values. Let us argue why this should be the case: Recall the accumulated return processes for the two models. For the continuous standard model it is

$$R_t^c = \mu t + \sigma B_t$$

and for the geometric Poisson processes it is

$$R_t^d = \mu t + z_0 \tilde{N}_t.$$

In both cases  $E(R_t^c) = E(R_t^d) = \mu t$  and the variances are  $\sigma^2 t$  and  $z_0^2 \lambda t$  respectively. Furthermore the quantity  $\tilde{N}_t / \sqrt{\lambda t}$  converges in distribution to the standard normal  $\mathcal{N}(0, 1)$ -distribution as  $\lambda t$  increases, and of course,  $B_t / \sqrt{t}$  is  $\mathcal{N}(0, 1)$ -distributed for any value of  $t$ . As a consequence, when we calibrate the variances, these two models come across as almost identical, at least for large enough values of  $\lambda t$ . When a reasonably large value of  $\lambda$  is multiplied by the average time a typical investor would choose to hold this option, the normal approximation should be very appropriate for the Poisson process. Since the Poisson random variable is infinitely divisible, the normal approximation is particularly adequate, as explained earlier. Note that this argument does not depend upon the size of the jump parameter  $z_0$ . Notice also that the Poisson model has one more free parameter than its Gaussian counterpart, namely the frequency  $\lambda$ .

Also consider the solutions to the perpetual American put option valuation problem in these two cases. The value functions are in both cases given by

$$\psi(x) = \begin{cases} (K - c)\left(\frac{c}{x}\right)^\gamma, & \text{if } x \geq c; \\ (K - x), & \text{if } x < c, \end{cases} \quad (15)$$

where  $S_t^c = S_t^d = x$ , and where the trigger price  $c$  is

$$c = \frac{\gamma K}{\gamma + 1}. \quad (16)$$

If investors are convinced that the probability distributions are approximately the same, they would typically equate the average state prices in the two situations, for any value of  $x$ . These are both being given by  $(x/c)^{-\gamma}$  when  $x \geq c$ . Clearly for the same contracts both the prices  $S_t^c$  and  $S_t^d$  of the underlying security at initiation of the contract are the same and equal to  $x$ , and the exercise prices are the same constant  $K$ , which means that it suffices to equate the two  $\gamma_i$ -parameters and the trigger prices  $c_i$ ,  $i = \{c, d\}$ . From the equations (15) and (16) we see that it is enough to equate the  $\gamma$ -values, and this leads in its turn to the same values for the perpetual American put options in these two situations. Let us take an example:

Example 1. Choose  $\sigma = .165$  and  $r = .01$ . The significance of these particular values will be explained below.

Our calibration consists in the following two steps: (i) First, we match the volatilities. This gives the equation  $z_0^2 \lambda = \sigma^2 = .027225$ . We start with  $z_0 = .01$ , i.e., each jump size is positive and of size one per cent. This equation gives that  $\lambda = 272.27$ , which is roughly one jump each trading day on the average, where the time unit is one year. The compensated part of the noise term consists of a negative drift, precisely "compensating" for the situation that all the jumps are positive. Recall that the compensated part is the zero mean noise term.

(ii) Second, we calibrate the average state prices. From the discussion above, it follows from the equations (15) and (16) that this is equivalent to equating the values of  $\gamma$ . Thus we find the value of  $\lambda^Q$  that yields  $\gamma_d$  as a solution of equation (6) equal to the value  $\gamma_c$  resulting from solving the equation (5) for the standard, continuous model. For the volatility  $\sigma = .165$ , the latter value is  $\gamma_c = .73462$ . By trying different values of  $\lambda^Q$  in equation (6), we find that the equality in prices is obtained when  $\lambda^Q = 274.73$ .

Finally, by the equation (12) for the risk adjusted frequency  $\lambda^Q$  we can solve for the equity premium  $e_p = (r - \mu)$ , which is found to be .0248, or about 2.5 per cent.

Equivalent to the above is to use the formula for the equity premium  $e_p$  in equation (14) directly, using  $\gamma = \gamma_c = .73462$  in this equation and the above parameters values of  $z_0$ ,  $r$  and  $\lambda$ .

Is this value dependent of our choice for the jump size  $z_0$ ? Let us instead choose  $z_0 = .1$ . This choice gives the value of the frequency  $\lambda = 2.7225$  in step (i), the risk adjusted frequency  $\lambda^Q = 2.9700$  in step (ii) and the value for the equity premium is consequently  $e_p = .0248$ , or about 2.5 per cent again. Choosing the more extreme value  $z_0 = 1.0$ , i.e., the upward jump sizes are all 100 per cent of the current price, gives the values  $\lambda = .0272$  in step (i), the risk adjusted frequency  $\lambda^Q = .0515$  follows from step (ii), and finally the equity premium  $e_p = .0248$ , again exactly the same value for this quantity.  $\square$ .

In the above example the value of  $\sigma = .165$  originates from an estimate of the volatility for the Standard and Poor's composite stock price index during parts of the last century. Thus the value of the equity premium around 2.5 per cent has independent interest in financial and macro economics.

Notice that  $\mu < r$  in equilibrium. This is a consequence of the fact that we are analyzing a perpetual put option, which can be thought of as an insurance product. The equilibrium price of a put is larger than the expected pay-out, because of risk aversion in the market. For a call option we have just the opposite, i.e.,  $\mu > r$ , but the perpetual call option is of no use to us here, since its market value equals  $x$ , the initial stock price.

Notice that in the above example we have essentially two free parameters to choose, namely  $z_0$  and  $\lambda$ . The question remains how robust this procedure is regarding the choice of these parameters. The example indicates that our method is rather insensitive to the choice of these two parameters as long as the volatility  $z_0^2\lambda$  stays constant. In the next section, after we have studied the comparative statics for the new parameters  $\lambda^Q$  and  $z_0$ , we address this problem in a more systematic way, but let us just round off with the following example.

Example 2. Set  $\sigma = .165$  and  $r = .01$ , and consider the case when  $z_0 < 0$ . Using the equations (3), (4) and (6) to approximate the put price also for negative jumps, first choose  $z_0 = -.01$ , i.e., each jump size is negative and of size one per cent all the time. Then we get  $\lambda = 272.27$  in step (i) and the risk adjusted frequency is now  $\lambda^Q = 269.78$  in step (ii). This gives for the equity premium  $e_p = .0247$ , or again close to 2.5 per cent. The compensated part of the noise term will now consist of a positive drift, again compensating for the situation that all the jumps are negative in the mean zero noise term.

The value  $z_0 = -.1$  gives  $\lambda = .0273$  in step (i), the risk adjusted frequency  $\lambda^Q = 2.473$  in step (ii) and an estimate for the equity premium is  $e_p = .0251$ . The more extreme value of  $z_0 = -.5$ , i.e., each jump results in cutting the

price in half, provides us with the values  $\lambda = .1089$  in step (i),  $\lambda^Q = .0585$  in step (ii) and  $e_p = .0252$  follows. This indicates a form of robustness regarding the choice of the jump size parameter and the frequency.  $\square$

## II-C Comparative statics of the option price in the Poisson model

In this section we indicate some comparative statics results for price of the perpetual American put option in the Poisson model. We have here two new parameters  $\lambda^Q$  and  $z_0$  to concentrate on. In addition we have the drift rate  $\mu$ , or the equity premium  $e_p$ . Since we do not have a closed form solution of the equation for  $\gamma$ , we have to rely on numerical methods. Here the following result is useful. From the equations (3), (4) and (6) we deduce that

$$\frac{\partial \psi}{\partial \gamma} = \begin{cases} -\frac{c}{\gamma} \left(\frac{c}{x}\right)^\gamma \ln\left(\frac{x}{c}\right), & \text{if } x \geq c; \\ 0, & \text{if } x < c, \end{cases} \quad (17)$$

or, in other words, the put price is a decreasing function of  $\gamma$  when  $x := S_t \geq c$ . From this result we only have to find the effects on the parameter  $\gamma$  in order to obtain the conclusions regarding the option value itself.

First let us consider the jump size parameter  $z_0$ , and we start with negative jumps. The results are reported in tables 1 and 2.

$z_0$	-.999	-.99	-.90	-.80	-.70	-.60	-.50	-.40	-.30
$\gamma$	$9.9 \cdot 10^{-3}$	.016	.0412	.071	.113	.179	.290	.495	.941

Table 1: The parameter  $\gamma$  for different negative values of the jump sizes  $z_0$ :  $\lambda^Q = 1$ ,  $r = .06$

$z_0$	-.20	-.10	-.05	-.03	-.02	-.01	-.001	-.0001
$\gamma$	2.18	8.19	27.36	62.27	115.40	311.35	5882.94	85435.51

Table 2: The parameter  $\gamma$  for different negative values of the jump sizes  $z_0$ :  $\lambda^Q = 1$ ,  $r = .06$

We notice from these that when the jump sizes are large and negative, the parameter  $\gamma$  is small and close to zero, meaning that the corresponding option value is close to its upper value of  $K$ , regardless of the value of the underlying stock  $S_t$ . As the jump sizes become less negative, the parameter  $\gamma$  increases, ceteris paribus, meaning that the corresponding option values decrease. As the jump sizes become small in absolute value,  $\gamma$  grows large,

reflecting that the option value decreases (eventually towards its lowest possible value, which is  $\psi_l(x, K) = (K - x)^+$  when  $c = K$ ). Notice that there exists a solution  $\gamma$  to equation (6) across the whole range of  $z_0$ -values in  $(-1, 0)$ , which follows from our earlier observations. When  $z_0 < 0$  our put values are only approximations, which get better as  $x$  increases (see e.g. Aase (2005)).

For positive values of the jump size  $z_0$  parameter, the exact results are reported in tables 3 and 4.

$z_0$	.0600003	.060005	.061	.062	.065	.07	.10	.30	.50
$\gamma$	$.353 \cdot 10^7$	$.212 \cdot 10^6$	1060	530	212	106	24	1.86	.693

Table 3: The parameter  $\gamma$  for different positive values of the jump sizes  $z_0$ :  $\lambda^Q = 1$ ,  $r = .06$

$z_0$	.80	1	2	6	10	20	100	10 000
$\gamma$	.299	.204	.068	.015	.0079	.0035	$.63 \cdot 10^{-3}$	$.60 \cdot 10^{-5}$

Table 4: The parameter  $\gamma$  for different positive values of the jump sizes  $z_0$ :  $\lambda^Q = 1$ ,  $r = .06$

From these we see that small positive jump sizes have the same effect on  $\gamma$  as small negative jump sizes, giving low option values. As the jump size parameter  $z_0$  increases, the value of the option increases (eventually towards its upper value of  $K$ ).

These tables show that increasing the absolute value of the jump sizes, has the effect of decreasing the values of  $\gamma$ , which means that the values of the option increase. Here the jump size can not be decreased lower than  $-1$ , which is a singularity of the equation (6). Notice from Table 3 that equation (6) only has a solution when  $.06 < z_0$ , which is consistent with the requirement  $r < \lambda^Q \alpha z_0$  in this situation.

Turning to the risk adjusted frequency parameter  $\lambda^Q$  under  $Q$ , tables 5 and 6 show that as the risk adjusted frequency  $\lambda^Q$  increases, the parameter  $\gamma$  decreases. Increasing the frequency means increasing the "volatility" of the underlying stock (under  $Q$ ), and this should imply increasing option prices, which is also the conclusion here following from the result (17). Note from Table 5 how the requirement  $r < \lambda^Q \alpha z_0$  comes into play: There is no solution  $\gamma$  of the equation (6) for  $z_0 \leq .06$  for these parameter values, in agreement with our earlier remarks.

From these latter two tables we are also in position to analyze how the option price depends on the drift parameter  $\mu$ . Suppose  $\mu$  decreases. Then,

$\lambda^Q$	.0600003	.060005	.061	.062	.070	.10	.80	.90
$\gamma$	$4.0 \cdot 10^5$	$.2.4 \cdot 10^3$	121.00	61.00	13.00	3.82	.26	.23

Table 5: The parameter  $\gamma$  for different values of the risk adjusted jump frequency  $\lambda^Q$ :  $z_0 = 1$ ,  $r = .06$

$\lambda^Q$	1	1.5	2	3	10	100	1000	10000
$\gamma$	.2042	.1341	.1000	.0661	.0196	.0019	.0002	$2 \cdot 10^{-5}$

Table 6: The parameter  $\gamma$  for different values of the risk adjusted jump frequency  $\lambda^Q$ :  $z_0 = 1$ ,  $r = .06$

ceteris paribus,  $\lambda^Q$  increases, and the tables indicate that the parameter  $\gamma$  decreases and accordingly the put option value increases. Thus a decrease of the (objective) drift rate  $\mu$  makes the put option more valuable, which seems reasonable, since this makes it more likely that the option gets in the money.

Finally, suppose the equity premium  $e_p$  increases. Then, ceteris paribus,  $\lambda^Q$  increases, and again from the tables we see that the parameter  $\gamma$  decreases and accordingly the put option value increases. An increase in the risk premium could typically go along with an increase in the risk aversion in the market, in which case it seems natural that the put option price increases, since this product can be interpreted as an insurance product.

An increase in the risk premium could alternatively go along with an increase in the covariance rate between the underlying stock and aggregated consumption in the market, in which case the representative investor would typically value the underlying stock lower, and as a result the put value should again increase.

Notice that this kind of economic logic does not apply to the standard model, which may be considered a weakness of that model.

## II-D Related risk adjustments of frequency

Risk adjustments of the frequency has been discussed earlier in the academic literature, in particular in insurance, see e.g., Aase (1999). This type of adjustment is, however, often referred to as something else in most of the actuarial literature; typically it is called a "loading" on the frequency. The reason for this is that in part of this literature there is no underlying financial model, and prices of insurance products are exogenous. In life insurance, for instance, the mortality function used for pricing purposes is usually not the statistically correct one, i.e., an estimate  $\hat{\lambda}$  of  $\lambda$ , but a different one depending on the nature of the contract. For a whole life insurance product, where the

insurer takes on mortality risk, the employed frequency is typically larger than  $\hat{\lambda}$ , while for an endowment insurance, such as a pension or annuity, it is typically smaller.

One way to interpret this is as risk adjustments of the mortality function, increasing the likelihood of an early death for a whole life product, and increasing the likelihood of longevity for a pensioner. Both these adjustments are in favor of the insurer, making the contract premiums higher, but must at the same time also be accepted by the insured in order for these life insurance contracts to be traded. Thus one may loosely interpret these adjustments as market based risk adjustments, although this is not the interpretation allowed by most traditional actuarial models, for reasons explained above.

A different matter is hedging in the present model. This is not so transparent as in the standard model, and has been solved using Malliavin calculus, see Aase, Øksendal, Ubøe and Privault (2000) and Aase, Øksendal and Ubøe (2001), for details.

### III Implications for equity premiums

#### III-A Introduction

In this section we turn to the problem of estimating the premium on equity of the twentieth century mentioned in the introduction. As indicated in examples 1 and 2, we suggest to use the results of Section 2 to infer about the equity risk premium.

The situation is that we have two complete financial models of about the same level of simplicity, which are similar in all the important aspects. At the micro level there are, admittedly, some noticeable differences: One has continuous price paths of unbounded variation, a property that is, by the way, very hard to visualize. The other has cruder price paths, containing occasional jumps, but these price paths at least pass the test of a "magnifying glass": When inspecting the paths through a microscope, we do see more structure the more powerful our microscope is, which is not the case for the Brownian motion: If one attempts to follow the path of the Brownian motion with pencil, one would use all the lead in the world in even the tiniest fraction of a second. At the macro level, however, the probability distributions of these two price processes, at a reasonably distant point in the future, are approximately the same, namely normally distributed in both cases, and this is all that matters for pricing options, since the rational investor can only relate to probability distributions and not to microscopic price paths.

We adapt these two models to the Standard and Poor's composite stock



price index for the time period mentioned above, and compute the value of an American perpetual put option written on any risky asset having the same volatility as this index.

Since the two models are both complete, and at about the same level of sophistication regarding the matters that are important for option prices, we make the assumption that the theoretical option prices so obtained are approximately equal. The argument here has been relying on two facts: The probability distributions of the underlying price processes are approximately the same at the time when a representative investor will typically exercise this option. Given this, the representative investor would equate the average state prices, which implies that the put option prices are equalized.

Now we use the fact that the standard continuous model provides option prices that do not depend on the actual risk premium of the risky asset, whereas the geometric Poisson model does. Exactly this difference enables us to find an estimate, based on calibrations, of the relevant equity premium. This corresponds to a no-arbitrage value, and since both the financial models are complete, these values are also consistent with a financial equilibrium, and can alternatively be thought of as equilibrium risk premiums.

### III-B The calibration

We now perform the calibration indicated in examples 1 and 2. Starting with the two no-arbitrage models of the previous section, we recall that this is carried out in two stages. First we match the volatilities in the two models under the given probability measure  $P$ : This gives the equation  $\sigma^2 = z_0^2 \lambda$ . Notice that for a given value of  $\sigma$  there are infinitely many values of  $z_0$  and  $\lambda$  that fit this equation. This step is built on a presumption that there is a linear relationship between equity premiums and volatility in equilibrium.

The consumption based capital asset pricing model (CCAPM) is a general equilibrium model, different from the option pricing model that we consider, where aggregate consumption is the single state variable. As noted by several authors, there is consistency between the option pricing model and the general equilibrium framework (e.g., Bick (1987), Aase (2002)). Accepting this for the moment, a consequence of the CCAPM is that the instantaneous correlation between consumption and the stock index is equal to one for the continuous model, and this leads to a linear relationship between equity premiums and volatility. For the discontinuous model this linear relationship is not true in general (e.g., Aase (2004)), but holds with good approximation for the Poisson model we consider, a point we return to in the next subsection.

Second we calibrate the average state prices. From the discussion in the previous sections, we have argued that this is equivalent to equating the

values of  $\gamma$ , and thus the values of the perpetual American put options. The advantage of using this particular financial instrument in this manner, makes our comparisons independent of the strike price  $K$ , the initial price  $x$  of the underlying, as well as of the maturity of the option since we consider a perpetual. Thus we set out to find the value of  $\lambda^Q$  that yields  $\gamma_d$  as a solution to equation (6) equal to the value  $\gamma_c$  resulting from solving the equation (5) for the standard, continuous model.

Having completed the previous stage, we finally infer  $e_p$  from equation (12) for the risk adjusted frequency  $\lambda^Q$ . As noticed before, the required computations may be facilitated by using the formula (14) for the equity premium  $e_p$ , where we substitute the value  $\gamma = \gamma_c$ .

Since  $\lambda^Q = \lambda + (r - \mu)/z_0$ , this procedure presumably still depends on the various values of the jump size  $z_0$  and the frequency  $\lambda$  that are chosen, granting that  $z_0^2\lambda = \sigma^2$ . The following tables indicate, however, that the procedure is rather insensitive to the choices of  $z_0$  and  $\lambda$ , as long as  $\sigma^2 = \lambda z_0^2$ . We start with the short rate equal to one per cent.

$z_0$	0.001	0.01	0.1	0.5	1.0	10	100
$\lambda$	27225	272.25	2.7225	.10890	.02723	.00027	.27225 $\cdot 10^{-5}$
$\lambda^Q$	27249	274.73	2.9700	.15810	.05157	.00266	.23927 $\cdot 10^{-3}$
$e_p$	0.0240	0.0248	0.0248	0.0246	0.0245	0.0239	0.0237

Table 7: The equity premium  $e_p$ , the jump frequency  $\lambda$  and the risk adjusted jump frequency  $\lambda^Q$  for various values of the jump size parameter  $z_0$ . The short term interest rate  $r = .01$ , and  $\gamma_c = \gamma_d = .73462$ .

$z_0$	- 0.9	- 0.7	- 0.5	- 0.3	- 0.1	- 0.01	- 0.001
$\lambda$	.03361	.05556	.10890	.30250	2.7225	272.25	27225
$\lambda^Q$	.00461	.01912	.05847	.21911	2.4737	269.77	27200
$e_p$	0.0261	0.0255	0.0252	0.0250	0.0249	0.0248	0.0250

Table 8: The equity premium  $e_p$ , the jump frequency  $\lambda$  and the risk adjusted jump frequency  $\lambda^Q$  for various values of the jump size parameter  $z_0$ . The short term interest rate  $r = .01$ , and  $\gamma_c = \gamma_d = .73462$ .

From tables 7 and 8 we notice that the value of the equity premium is rather stable, and fluctuates very little around .025. Even the extreme values of  $z_0 = 1.0, 10$  and  $100$ , corresponding to a bonanza economy with sudden "upswings" of 100, 1000 and 10,000 per cent respectively (but increasingly rarely as  $\lambda$  becomes correspondingly small), provide values of the equity premium of around 2.4 per cent.

We also try negative values of the jump size parameter.<sup>2</sup> Using this, we find that also for the extreme values of  $z_0$  in the other end,  $-0.9$ ,  $-0.7$  and  $-0.5$ , the values of the equity premium is rather close to 2.5 per cent. These latter values of  $z_0$  correspond to a "crash economy", where a dramatic downward adjustment occurs very rarely.<sup>3</sup> For a related, but different, discrete time model of a crash economy, see e.g., Rietz (1988).

We conclude that the values of the equity premium found by this method is robust with respect to the jump size parameter  $z_0$  and frequency  $\lambda$  at the level of accuracy needed here. This holds for the short interest rate  $r = .01$ .

Tables 9 and 10 give a similar picture for the interest rate  $r = .04$ , but now the equity premium has changed to about 4.4 per cent.

$z_0$	0.001	0.01	0.1	0.5	1.0	10	100
$\lambda$	27225	272.25	2.7225	.10890	.02723	.00027	$.27225 \cdot 10^{-5}$
$\lambda^Q$	27270	276.75	3.1774	.20379	.07615	.00555	$.53796 \cdot 10^{-3}$
$e_p$	0.0452	0.0449	0.0455	0.0474	0.0489	0.0528	0.0535

Table 9: The equity premium  $e_p$ , the jump frequency  $\lambda$  and the risk adjusted jump frequency  $\lambda^Q$  for various values of the jump size parameter  $z_0$ . The short term interest rate  $r = .04$ , and  $\gamma_c = \gamma_d = 2.93848$ .

$z_0$	- 0.9	- 0.7	- 0.5	- 0.3	- 0.1	- 0.01	- 0.001
$\lambda$	.03361	.05556	.1089	.30250	2.7225	272.25	27225
$\lambda^Q$	.00018	.00503	.030321	.1623	2.2821	267.78	27181
$e_p$	0.0301	0.0354	0.0393	0.0421	0.0440	0.0447	0.0436

Table 10: The equity premium  $e_p$ , the jump frequency  $\lambda$  and the risk adjusted jump frequency  $\lambda^Q$  for various values of the jump size parameter  $z_0$ . The short term interest rate  $r = .04$ , and  $\gamma_c = \gamma_d = 2.93848$ .

Table 11 gives the connection between the short interest rate and the equity premium in our approach, when the volatility of the stock index is held constant. The computations are carried out for the jump size parameter  $z_0 = .01$ . The results are identical when  $z_0 = -.01$ .

<sup>2</sup>In this case the solution of the perpetual American put problem given above for the Poisson model can only be considered an approximation, which is better the larger  $x$  is. Since our arguments are independent of this quantity, we might as well just assume that  $x$  is large enough for this approximation to be accurate.

<sup>3</sup>That there is virtually no difference between the results in tables 7 and 8 we take an indication that the approximation to the put value is indeed accurate in this situation. Compare also tables 9 and 10.

$r$	0.0001	0.001	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08
$e_p$	.017	.019	.025	.032	.038	.045	.052	.058	.065	.072

Table 11: The equity premium  $e_p$ , as a function of the short term interest rate. The volatility of the stock index is fixed at 0.165,  $z_0 = .01$ .

### III-C The relation to the CCAPM

The presented tables are consistent with the CCAPM for this particular Poisson jump process, a fact we now demonstrate. To this end let us recall an expression for the CCAPM for jump-diffusions (eq. (29) in Aase (2004)) when there is one common source of jump risk. Adapted to the present model it looks like the following:

$$e_p = (RRA)(\sigma_c \cdot \sigma) - \lambda z_0 ((1 + z_{0,c})^{-RRA} - 1) = (RRA)(\sigma_c \cdot \sigma + \lambda z_0 z_{0,c}) - \frac{1}{2} \lambda (RRA)(RRA + 1) z_0 z_{0,c}^2 + \dots \quad (18)$$

Here  $(RRA)$  stands for the coefficient of relative risk aversion, assumed to be a constant,  $\sigma$  is the volatility of the continuous part of the stock index,  $\sigma_c$  the standard deviation of the continuous part of the aggregate consumption,  $z_0$  is the the jump size in the stock index and  $z_{0,c}$  is the corresponding jump size in the aggregate consumption.

If there are no jump terms, we notice that  $e_p = (RRA)\sigma_c \cdot \sigma$ , so the equity premium  $e_p$  is proportional to the volatility parameter  $\sigma$ . The next term inside the parenthesis in the second line of (18) is the jump analogue of the first term, and then higher order terms follow. Neglecting the latter for the moment, we notice that for the Poisson jump model of Section 2, where  $\sigma = \sigma_c = 0$ , equation (18) can be written, to the first order approximation

$$e_p = -\lambda z_0 ((1 + z_{0,c})^{-RRA} - 1) \approx (RRA) \left( \sqrt{\lambda z_0^2} \cdot \sqrt{\lambda z_{0,c}^2} \right). \quad (19)$$

We see that  $e_p$  is approximately proportional to the quantity  $\sigma_d := (\lambda z_0^2)^{1/2}$ , the volatility of the stock index, if we can neglect higher order terms. Tables 12 and 13 show the values of the equity premium  $e_p$  as the  $\sigma_d$  parameter varies. In Table 12 the short term interest rate  $r = 0.01$  and in Table 13  $r$  is four per cent. In addition the tables present values of the relative risk aversion  $(RRA)$  from the equality in equation (19), and its first order approximation  $(RRA)_0$  stemming from the approximation in this equation.

We have computed the values of  $e_p$  as in the tables 7-10, by calibrating to the continuous, standard model. When finding the values of the two relative

$\sigma_d$	0.05	0.08	0.10	0.165	0.18	0.20	0.22	0.25	0.30
$(RRA)_0$	4.68	3.83	3.74	4.21	4.40	4.67	4.98	5.41	6.22
$(RRA)$	4.70	3.85	3.76	4.23	4.43	4.69	5.01	5.45	6.27
$e_p$	.008	.011	.013	.025	.028	.033	.039	.048	.067

Table 12: The relative risk aversion ( $RRA$ ), its first order approximation  $(RRA)_0$ , and and the equity premium  $e_p$  for various values of the stock index volatility parameter  $\sigma_d$ . The short term interest rate  $r = .01$

$\sigma_d$	0.10	0.12	0.14	0.165	0.18	0.20	0.22	0.25	0.30
$(RRA)_0$	9.24	8.40	8.00	7.64	7.47	7.42	7.51	7.62	8.12
$(RRA)$	9.35	8.49	8.08	7.71	7.54	7.49	7.58	7.69	8.21
$e_p$	.033	.036	.040	.045	.048	.053	.059	.068	.087

Table 13: The the relative risk aversion ( $RRA$ ), its first order approximation  $(RRA)_0$ , and and the equity premium  $e_p$  for various values of the stock index volatility parameter  $\sigma_d$ . The short term interest rate  $r = .04$

risk aversion parameters, we have used the following procedure: The volatility is varied by changing only the jump size parameter  $z_0$ , keeping the frequency parameter  $\lambda$  fixed at the value 272.25. Then we have used the value of the standard deviation of the aggregate consumption  $\sigma_{d,c} := z_{0,c}\sqrt{\lambda} = 0.0357$ , as is the estimate of this quantity from the Mehra and Prescott (1985) study (see also Constantinides (1990)). This means that  $z_{0,c} = 2.1636 \cdot 10^{-3}$  for our choice of frequency  $\lambda$ . By changing the volatility of the stock index this way we managed to keep the standard deviation of the aggregate consumption  $\sigma_{d,c}$  constant at the value .0357 as  $\sigma_d$  varies.

If the second equality of (19) is a good approximation in the range of  $\sigma_d$ -values of interest, then we should obtain the same value of  $(RRA)_0$  throughout. We observe that there is some variation, although rather moderate in the range of volatilities close to the market estimate of 0.165. The values of  $(RRA)_0$  are close to those of  $(RRA)$ , an indication that the approximation in (19) is fairly accurate. When a broader range of  $\sigma_d$ -values are considered, there is some variation in the values of  $(RRA)$ , as is to be expected from the nonlinear relationship between  $e_p$  and this quantity. As noted above, however, the approximation is acceptable in the economically interesting range of values of the volatility parameter, at least at the level of accuracy needed here.

We conclude that in the geometric Poisson model the simple "volatility parameter"  $\sigma_d := (z_0^2\lambda)^{1/2}$  captures the individual asset risk, and hence the product of jump size squared and frequency is a "sufficient statistic" of this

type of risk, as a reasonable first order approximation.

Thus our assumption that  $e_p$  is proportional to the volatility of the stock holds approximately for the geometric Poisson model, and the above results are accordingly seen to be consistent with the CCAPM. This explains why  $e_p$  is invariant to changes in  $z_0$  and  $\lambda$  as long as  $z_0^2\lambda$  equals a constant, as seen in the tables 7-10. Finally, the comparisons in tables 12 and 13 also serve as a diagnostics check of our calibration procedure.

It follows from the general jump version of the CCAPM that we can not expect this kind of invariance results if our competing model to the geometric Brownian motion is more complex than the geometric Poisson process. For any jump-diffusion with a nonzero jump term, the probability distribution under  $Q$  does indeed depend on the equity premium, but the equity premium will no longer be proportional to the "volatility" of the underlying asset, in which case our premise in the first step of the calibration in Section 4.2 is violated. Computations confirm this. We have tried a large range of different jump-diffusion models, ranging from models including just a diffusion component in addition to the Poisson term, several different jump sizes, both positive and negative, and even continuously distributed jump sizes. In neither of these models we obtained the simple and transparent results of the above analysis.

However, our methodology will still work granted an appropriate empirical investigation, where option prices are observed in the market. It would then be a simple matter to calibrate a more complex jump-diffusion model to the standard model, and find *implied* equity premiums.

As an illustration, we present in the last section of the paper the relevant computations when the jump size distribution is continuous for (a) normal, and (b) exponential tails. Both these models are, of course, incomplete.

### III-D The relation to the classical puzzles.

From our results we can say something about the two puzzles mentioned in the introductory part of the paper. We reexamine the two puzzles using the above model, and find values for the parameters of the representative agent's utility function for different values of the equity premiums and short term interest rates, calibrated to the first two moments of the US consumption-equity data for the period 1889-1978.

Consider first the case where  $r = .01$  and the equity premium is 2.5 per cent. This is, as noted above, not consistent with the Mehra and Prescott (1985) study, where  $r = .01$  and the equity premium was 6 per cent. The jump model can explain a relative risk aversion coefficient in the range around 4.2, as seen from Table 12, which must be considered a plausible value for

this quantity. This follows as explained earlier from the CCAPM for jump processes, since the value used for the standard deviation  $\sigma_{d,c}$  of the aggregate consumption is the market estimate 0.0357 from the last century. Turning to the Mehra and Prescott (1985)-case, the value of the relative risk aversion is estimated to 10.2 using the standard, continuous model, which is simply the equity premium puzzle.

For the reexamined values presented by McGrattan and Prescott (2003), the short term interest rate was estimated to be four per cent, with an equity premium of only one per cent. This is not consistent with our approach, which gives the equity premium of about 4.4 per cent in this situation. Our case corresponds to a relative risk aversion of about 7.6, as seen from Table 13.

In both situations above we still get a (slightly) negative value for the subjective interest rate.

## IV The model with a continuous jump size distribution.

We round off this paper by considering the situation with a continuous distribution for the jump sizes. In this case the model is incomplete as long as there is a finite number of assets, since there is "too much uncertainty" compared to the number of assets.

The case with countably many jump sizes in the underlying asset could perhaps be approached by introducing more and more risky assets. In order for the market prices of risk  $\theta_1, \theta_2, \dots$  to be well defined, presumably only mild technical conditions need to be imposed. One line of attack is to weakly approximate any such distribution by a sequence of discrete distributions with finite supports. This would require more and more assets, and in the limit, an infinite number of primitive securities in order for the model to possibly be complete.

Here we will not elaborate further on this, but only make the assumption that the pricing rule is linear, which would be the case in a frictionless economy where it is possible to take any short or long position. This will ensure that there is *some* probability distribution and frequency for the jumps giving the appropriate value for  $\gamma$ , corresponding to a value for the perpetual American put option.

Below we limit ourselves to a discussion of the prices obtained this way for two particular choices of the jump distribution, where the risk adjustment is carried out mainly through the frequency of jumps.

The model is the same as in Section 2 with one risky security  $S$  and one locally riskless asset  $\beta$ . The risky asset has price process  $S$  satisfying

$$dS_t = S_{t-}[\mu dt + \alpha \int_{-1/\alpha}^{\infty} z \tilde{N}(dt, dz)]. \quad (20)$$

Here  $\tilde{N}(dt, dz) = N(dt, dz) - \lambda F(dz)dt$  is the compensated Poisson random measure, corresponding to our earlier  $\tilde{N}$  for the Poisson process, but now taking into account that there are several allowed jump sizes  $z \in [-1/\alpha, \infty)$ .  $N(t, U)$  is a random measure counting the number of times that jumps fall in the set  $U$  before time  $t$ . The function  $F(\cdot)$  is the probability distribution function of the jump sizes. The quantity  $\alpha z \geq -1$  for all values of  $z$ , and  $\alpha$  is just a constant. As before is  $\lambda$  the frequency of the jumps, and the integral in (20) represents the zero mean noise term, corresponding to our earlier  $z_0 d\tilde{N}_t$ -term.

The density process of  $S$  is associated to a change of probability measure is given by

$$\xi(t) = \exp\left\{\int_0^t \int_{-1/\alpha}^{\infty} \ln(1 - \theta(z)) N(ds, dz) + \int_0^t \int_{-1/\alpha}^{\infty} \theta(z) \lambda F(dz) ds\right\}. \quad (21)$$

Here  $\theta(z)$  is the market price of risk function and the distribution function of the jump sizes  $F(dz)$  is assumed absolutely continuous with a probability density  $f(z)$ . According to the results of Aase (2005), if the market price of risk satisfies the following equation

$$\int_{-1/\alpha}^{\infty} z \theta(z) f(z) dz = \frac{\mu - r}{\lambda \alpha}, \quad (22)$$

then the risk adjusted compensated jump process can be written

$$\tilde{N}^Q(dt, dz) = N(dt, dz) - (1 - \theta(z)) \lambda f(z) dz dt. \quad (23)$$

This means that the term

$$\lambda^Q f^Q(z) := \lambda(1 - \theta(z)) f(z) \quad (24)$$

determines the product of the risk adjusted frequency  $\lambda^Q$  and the risk adjusted density  $f^Q(z)$ , when  $\theta$  satisfies equation (22). If the market price of risk  $\theta$  is a constant, there is no risk adjustment of the density  $f(z)$ . The densities  $f(z)$  and  $f^Q(z)$  are mutually absolutely continuous with respect to each other, which means in particular that the domains where they are both positive must coincide.



Clearly the equation (22) has many solutions  $\theta$ , so the model is incomplete.

In solving the perpetual American put problem for this model, it follows from the results of Aase (2005) that the equation for  $\gamma$  is given by

$$-r - r\gamma + \int_{-1/\alpha}^{\infty} \{(1 + \alpha z)^{-\gamma} - 1 + \alpha\gamma z\} \lambda^Q f^Q(dz) = 0, \quad (25)$$

where we have carried out the relevant risk adjustments.

As before, the corresponding solution to the perpetual American put pricing problem is given by the equations (3) and (4), where  $\gamma$  is the solution to equation (25) above. This solution is an approximation when jumps can be negative, and exact when jumps can not take negative values. The problem arises if exercise can happen at a time of jump of the underlying price process  $S$ . For a given stock price  $S_t = x$  and jump size  $Z$  (a random variable), we are asking what is the probability that  $x(1 + \alpha Z) < c$ . This being the case, the term  $\psi(x + \alpha xz)$  appearing in the integro-differential-difference equation for the market price in the solution procedure of the optimal stopping problem must be replaced by the linear function  $(K - (x + \alpha xz))$ . The probability that this replacement should take place is

$$\int_{(-\frac{1}{\alpha})}^{(\frac{c}{\alpha x} - \frac{1}{\alpha})} f(z) dz$$

which is seen to become smaller the larger  $x$  is. Thus we conjecture that the error committed can not be large if we approximate the linear function by the nonlinear  $\psi$  in this situation, which is what our solution really does. Below we consider two situations where the jumps can be both negative and positive, and we use our presented solution, which is here an approximation, and a relatively more accurate one the larger the values of  $x$ .

We now turn to the illustrations by considering two special cases for the jump density  $f(z)$ .

#### IV-A The truncated normal case

Here we analyze normally distributed returns. In our model formulation we have chosen the stochastic exponential, and the domain of  $F$  is accordingly the interval  $[-1/\alpha, \infty)$ . In this case we choose to consider a truncated normal distribution at  $-1/\alpha$ . By and large we restrict our attention to risk adjustments associated with a constant  $\theta$  only. In the present case the most straightforward risk adjustments of the normal density  $f(z)$  having mean  $m$  and standard deviation  $s$  would be another normal density having mean

$m^Q$  and standard deviation  $s^Q$ , with a similar adjustment for the truncated normal distribution. Here we only notice that a joint risk adjustment of the jump distribution  $f$  to another truncated normal with parameters  $m^Q$  and  $s^Q$ , and of the frequency  $\lambda$  to  $\lambda^Q$ , means that the equity premium can be written

$$e_p = \alpha(\lambda^Q E^Q\{Z|m^Q, S^Q\} - \lambda E\{Z|m, s\}), \quad (26)$$

where the expectations are taken of the truncated normal random variable  $Z$  with respect to the parameters indicated. The above formula then follows from (22) and (24). Notice that  $\alpha$  does not change under the measure  $Q$ , since the supports of  $f$  and  $f^Q$  must coincide.

The exponential pricing model with normal jump sizes was considered by Merton (1976). In that case the probability density of the pricing model  $S_t$  is known explicitly. In contrast to Merton, who assumed that the jump size risk was not priced, or, he did not adjust for this type of risk, we will risk adjust precisely the jump risk, and our model is the stochastic exponential, not the exponential as he used.

Below we have calibrated this model to the standard continuous one using the same technique as outlined earlier. Since the equity premium is not proportional to the volatility of  $S$  in this model, we can not expect to confirm the simple results of Section 4. For  $Z$  a random variable with a truncated normal distribution at  $-1/\alpha$ , we first solve the equation  $\lambda\alpha^2 E(Z^2) := \sigma_d^2 = .027225$ , or

$$\lambda\alpha^2 \frac{\int_{-1/\alpha}^{\infty} z^2 \frac{1}{\sqrt{2\pi}s} e^{-\frac{1}{2}\left(\frac{z-m}{s}\right)^2} dz}{\int_{-1/\alpha}^{\infty} \frac{1}{\sqrt{2\pi}s} e^{-\frac{1}{2}\left(\frac{z-m}{s}\right)^2} dz} = \sigma_d^2$$

for various values of  $m$  and  $s$ , and find the frequency  $\lambda$ . Then we solve equation (25), using the relevant values for  $r$  and  $\gamma = \gamma_c(r)$ , to find the risk adjusted frequency  $\lambda^Q$ , and finally we use equation (22) to find the equity premium  $e_p = (r - \mu)$ , assuming  $\theta$  is a constant, so that  $\lambda^Q = \lambda(1 - \theta)$  and  $f = f^Q$ . Some results are summarized in tables 14 and 15.

$\alpha$	1	1	1	.01	.8	3
$(m, s)$	(.1, .1)	(.4, .7)	(.4, 2.0)	(10, 10)	(.01, .01)	(.01, .01)
$\lambda$	1.36	.042	0.0065	1.36	212.70	15.13
$\lambda^Q$	1.60	.079	.019	1.60	215.78	15.94
$e_p$	0.024	0.026	0.025	0.024	0.025	0.024

Table 14: The equity premium  $e_p$  when  $r = 0.01$  and  $\gamma = 0.73462$ , for various values of the parameters. The jumps are truncated normally distributed.

By decreasing the parameter  $\alpha$  we notice from the above equation that this has the effect of increasing the frequency of jumps  $\lambda$ . Alternatively this can be achieved by decreasing the values of  $m$  and  $s$ , as can be observed in Table 15, where the spot rate is equal to 4 per cent. A decrease in the standard deviation  $s$ , within certain limits, moves the present model closer to the one of Section 4.

$\alpha$	1	1	1	.9	2	10
$(m, s)$	(.1, .1)	(-.01, .01)	(1.0, 0.1)	(.01, .01)	(.011, .01)	(.01, .01)
$\lambda$	1.36	136.13	0.027	136.06	30.80	1.36
$\lambda^Q$	1.79	131.62	.076	173.01	32.91	1.79
$e_p$	0.043	0.045	0.049	0.045	0.046	0.043

Table 15: The equity premium  $e_p$  when  $r = 0.04$  and  $\gamma = 2.93848$ , for various values of the parameters. The jumps are truncated normally distributed.

## IV-B Exponential tails

In this model the distribution of the jump sizes is an asymmetric exponential with density of the form

$$f(z) = pae^{-a|z|}I_{[-\infty,0)}(z)/(1 - e^{-a/\alpha}) + (1 - p)be^{-bz}I_{[0,\infty)}(z)$$

with  $a > 0$  and  $b > 0$  governing the decay of the tails for the distribution of negative and positive jump sizes and  $p \in [0, 1]$  representing the probability of a negative jump. Here  $I_A(z)$  is the indicator function of the set  $A$ . The probability distribution of returns in this model has semi-heavy (exponential) tails. Notice that we have truncated the left tail at  $-1/\alpha$ . The exponential pricing version of this model, without truncation, has been considered by Kou (2002).

Below we calibrate this model along the lines of the previous section. Also here we restrict attention to risk adjusting the frequency only. We then have the following expression for the equity premium:

$$e_p = \alpha(\lambda^Q - \lambda) \left( p \left( \frac{e^{-a/\alpha}}{\alpha(1 - e^{-a/\alpha})} - \frac{1}{a} \right) + (1 - p) \frac{1}{b} \right), \quad (27)$$

where the frequency is risk adjusted, but not  $f$ . A formula similar to (26) can be obtained if also the density  $f$  is to be adjusted for risk. The simplest way to accomplish this here is to consider another probability density  $f^Q$  of

the same type as the above  $f$  with strictly positive parameters  $p^Q, a^Q$  and  $b^Q$ . This would constitute an absolutely continuous change of probability density, but there are of course very many other possible changes that are allowed. In finding the expression (27) we have first solved the equation (22) with a constant  $\theta$ , and then substituted for the market price of risk using the equation  $\lambda^Q = \lambda(1 - \theta)$ .

Proceeding as in the truncated normal case, we first solve the equation  $\lambda\alpha^2 E(Z^2) := \sigma_d^2 = 0.027225$ , which can be written

$$\lambda\alpha^2 \left( p(1 - e^{-a/\alpha})^{-1} \left( \frac{2}{a^2} - e^{-a/\alpha} \left( \frac{1}{\alpha^2} + \frac{2}{a\alpha} + \frac{2}{a^2} \right) \right) + (1 - p) \left( \frac{2}{b^2} \right) \right) = \sigma_d^2. \quad (28)$$

Then we determine reasonable parameters through the equation  $\alpha E(Z) = R_e$  for various values of  $R_e$ . This equation can be written:

$$R_e := \alpha \left( p \left( \frac{e^{-a/\alpha}}{\alpha(1 - e^{-a/\alpha})} - \frac{1}{a} \right) + (1 - p) \frac{1}{b} \right). \quad (29)$$

In order to arrive at reasonable values for the various parameters, we solve the two equations (28) and (29) in  $a$  and  $b$  for various values of the parameters  $\alpha, p$  and  $R$ , where we have fixed the value of  $\lambda = 250$ . Then for the spot rates  $r = 0.01$  and  $r = 0.04$  with corresponding values of  $\gamma = \gamma_c(r)$  respectively, we solve the equation (25) to find the value of  $\lambda^Q$ . Finally we compute the value of the equity premium from the formula (27).

$(\alpha, p)$	(1, .45)	(1, .55)	(1, .60)	(.01, .40)	(.01, .45)	(.01, .60)
$R_e$	.004	-.004	-.004	.0045	.004	.0035
$a$	350.23	104.07	110.34	3.76	3.50	5.54
$b$	140.07	350.23	278.21	1.08	1.04	.87
$\lambda^Q$	255.92	243.92	244.55	255.85	252.71	257.41
$e_p$	0.024	0.024	0.022	0.026	0.024	0.026

Table 16: The equity premium  $e_p$  when  $r = 0.01$  and  $\gamma = .73462$ , for various values of the parameters, where  $\lambda = 250$ . The jumps are truncated, asymmetric exponentials.

Under the same assumptions as in Section 4.3, the CCAPM in the present class of model is the following

$$e_p = -\lambda \int_{-1/\alpha}^{\infty} \int_{-1/\alpha}^{\infty} \left( (1 + z_c)^{(-RRA)} - 1 \right) z f(z, z_c) dz dz_c = \quad (30)$$

$$(RRA) \cdot \lambda \int_{-1/\alpha}^{\infty} \int_{-1/\alpha}^{\infty} z z_c f(z, z_c) dz dz_c - \dots$$

$(\alpha, p)$	(1, .40)	(1, .45)	(1, .60)	(.01, .40)	(.01, .45)	(.01, .60)
$R_e$	- .0035	-.0035	-.0035	.0045	.004	.0035
$a$	87.29	93.62	113.58	3.76	3.50	5.54
$b$	554.30	420.88	224.41	1.08	1.04	.87
$\lambda^Q$	236.24	237.53	241.29	260.54	260.67	263.38
$e_p$	0.048	0.044	0.048	0.047	0.043	0.047

Table 17: The equity premium  $e_p$  when  $r = 0.04$  and  $\gamma = 2.93848$ , for various values of the parameters, where  $\lambda = 250$ . The jumps are truncated, asymmetric exponentials.

Here  $f(z, z_c)$  is the joint probability density of the shocks in the stock return and the growth rate of consumption. Focusing on the first order approximation, the instantaneous correlation between the stock index and the consumption is equal to one only if both these jump sizes are deterministic, which follows from the Schwartz inequality. In the truncated normal case, for example, a good candidate for  $f(z, z_c)$  is the bivariate normal density.

It follows that in neither of these models is the equity premium proportional to the volatility of  $S$ , so we can not expect to obtain the simple and transparent results of Section 4.

The above tables all identify parameters that are consistent with the simple results obtained in Section 4, and are not meant to be representative of the variation one may obtain for  $e_p$ . These tables primarily illustrate numerical solutions of the basic equation (25) for  $\gamma$ , and how the calibration procedure works to infer about  $e_p$  in more complex models.

Notice that, as for the simple case of the geometric Poisson process, the probability distribution of returns under any risk adjusted probability measure  $Q$  depends on the equity premium here as well, as we have demonstrated in this section. This means that once we have estimated the various process parameters from, say, time series data, and observed option prices in the market, we may find implied equity premiums in much the same manner as implied volatility is found in various pricing models. This method does not require a comparison to the standard model.

## V Conclusions

We have used the price of a perpetual American put option to infer about equity premiums, when the underlying asset follows a geometric Poisson process. This was made possible since the probability distribution under the risk

adjusted measure depends on the equity premium for this type of model, which is not the case for the standard geometric Brownian motion process. Precisely this difference is utilized to find intertemporal, equilibrium equity premiums.

In the standard, continuous model the equity premium is proportional to the volatility of the underlying asset, while this is approximately the case for the geometric Poisson process. This fact implies that we obtain equity premiums that are almost invariant with respect to changes of its frequency or jump size parameter, as long as these parameters jointly determine a given volatility.

We applied this technique to the US equity data of the last century, and found an indication that a risk premium on equity about two and a half per cent is consistent with the observed historical volatility when we use the option approach, provided the risk free short rate was around one per cent. On the other hand, if the latter rate was about four per cent, we similarly find that this corresponds to an equity premium of around four and a half per cent.

This also indicates that the high estimate of 6 per cent equity premium in the last century is not consistent with the option model, nor with the CCAPM. If we believe in e.g., the option pricing model, such a premium is not likely to prevail into the future, unless the volatilities of stocks increase substantially.

The advantage with our approach is that we needed only equity data and option pricing theory, no consumption data was necessary to arrive at these conclusions.

In the last section of the paper we consecutively replaced the geometric Poisson process by two different processes, where the jump sizes are continuously distributed. The resulting financial models are thus incomplete. In these models the equity premium is no longer proportional to the "volatility" of the respective risky asset, as follows from the CCAPM for jump-diffusions. This is also confirmed by the computations in this section.

An econometric investigation, where option prices are observed in the market, would enable us to find implied equity premiums also for these more complex models, since the probability distribution under the risk adjusted measure still depends on the equity premium.

## References

- [1] Aase, K. K. (2005). "The perpetual American put option for jump-diffusions with applications". Working paper, Anderson Graduate

School of Management, UCLA.

- [2] Aase, K. K. (2005). "*The Continuous-Time Lucas Model with Jump Dynamics*" Working paper No. 15, Anderson School of Management, UCLA
- [3] Aase, K. K. (2002). "Equilibrium pricing in the presence of cumulative dividends following a diffusion." *Mathematical Finance* 12(3) 173-198.
- [4] Aase, K. K. (1999). "An Equilibrium Model of Catastrophe Insurance Futures and Spreads." *The Geneva Papers on Risk and Insurance Theory* 24, 69-96.
- [5] Aase, K. K., B. Øksendal, and J. Ubøe (2001). "Using the Donsker Delta Function to compute Hedging Strategies." *Potential Analysis* 14, 351-374.
- [6] Aase, K. K., B. Øksendal, J. Ubøe, and N. Privault (2000). "White noise generalizations of the Clark-Hausmann-Ocone Theorem with applications to Mathematical Finance." *Finance and Stochastics* 4, 4, 465-496.
- [7] Bick, A. (1987). "On the Consistency of the Black-Scholes Model with a General Equilibrium Framework." *J. Financial Quant. Anal.* 33, 259-275.
- [8] Campbell, J. Y. (1996). "Understanding Risk and Return." *The Journal of Political Economy* 104, 2, 298-345.
- [9] Campbell, J. Y. (1993). "Intertemporal Asset Pricing without Consumption Data." *The American Economic Review* 83, 3, 487-512.
- [10] Constantinides, G. M. (1990). "Habit formation: a resolution of the equity premium puzzle." *Journal of Political Economy* 98, 519-543.
- [11] Huang, C. -F., and H. Pagès (1992). "Optimal Consumption and Portfolio Policies with an Infinite Horizon: Existence and Convergence." *Annals of Applied Probability* 2; 36-64.
- [12] Kou, S. (2002). "A jump-diffusion model for option pricing." *Management Science* 48, 1086-1101.
- [13] Lucas, R. (1978). "Asset prices in an exchange economy." *Econometrica* 46, 1429-1445.

- [14] McGrattan, E. R. and E. C. Prescott (2003). “Average Debt and Equity Returns: Puzzling?” Federal Bank of Minneapolis, Research Department Staff Report 313.
- [15] Mehra, R. and E. C. Prescott (1985). “The Equity Premium: A puzzle.” *Journal of Monetary Economy*, 145-161.
- [16] Merton, R. (1976). ”Option pricing when the underlying stock returns are discontinuous.” *J. Financial Economics* 3, 125-144.
- [17] Merton, R. C. (1973a). ”Theory of Rational Option Pricing.” *Bell Journal of Economics and Management Science*, 141-183.
- [18] Merton, R. C. (1973b). ”An Intertemporal Capital Asset Pricing Model.” *Econometrica* 41, 5; 867-887.
- [19] Øksendal, B. and A. Sulem (2004). *Applied Stochastic Control of Jump Diffusions*. Springer Verlag; Berlin, Heidelberg, New York.
- [20] Revuz, D., and M. Yor (1991). *Continuous Martingales and Brownian Motion*. New York: Springer.
- [21] Rietz, T. A. (1988). “The equity risk premium: A solution.” *Journal of Monetary Economics* 22, 133-136.
- [22] Sato, K. (1999). *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press; Cambridge, U.K.
- [23] Siegel, J. J. (1992). “The real rate of interest from 1800-1900.” *J. Monetary Econ.* 29, 227-252.
- [24] Weil, P. (1989). “The equity premium puzzle and the risk-free rate puzzle.” *J. Monetary Econ.* 24, 501-421.