

Benefit efficient statistical distributions on patient lists

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ABSTRACT. In this paper we consider statistical distributions of different types of patients on the patient lists of doctors. In our framework different types of patients have different preferences regarding their preferred choice of doctor. Assuming that the system is benefit efficient in the sense that distributions with larger total utility have higher probability, we can construct unique probability measures describing the statistical distribution of the different types of patients.

Keywords: Patient lists, efficient welfare, statistical distributions

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1. Introduction

The norwegian medical system is based on a construction where all people living in Norway are given the option to be a part of the “patient list-system in general practice”. A person can choose not to be part of the system, but a large majority (99.5%) have chosen to participate.

The system works in the following way: In every community there is a pool of doctors having agreed to take care of certain numbers of patients. In the following we will refer to these numbers as the list lengths of the doctors. The list lengths may vary from doctor to doctor, and there are usually between 500 and 2 000 patients on every list. The doctors receive a fixed annual income for each patient on their lists. In return the doctors agree to be medically responsible for the patients on their lists.

It goes without saying that some doctors are more popular than others. Hence some patient lists are full, i.e., the doctor cannot undertake responsibility of more patients. A list of all doctors with vacancies is made public every month. Newcomers to the system can apply for vacancies, and those that are already members can apply to be transferred to a new doctor.

To obtain a statistical model which takes into account that different people have different preferences, we will consider a situation where there are S groups of patients and T types of doctors. The groups differ in their preferences for the different types of doctors. In particular we pay attention to a case where we divide the patients into male and female doctors, i.e., $T = 2$, and where the patients are divided into $S = 4$ groups:

- MM - men who want a male doctor
- MF - men who want a female doctor
- WM - women who want a male doctor
- WF - women who want a female doctor

The challenge is then to formulate a model expressing that different groups may differ in the strength of their preferences. In such a model we should allow constructions where some patients do not stay on any list, and allow for cases where we have vacancies.

The approach chosen in this paper is based on a newly developed theory of efficient systems, see Jörnsten & Ubøe (2005). The basic idea in this theory is to assume that each individual

has a certain utility/disutility attached to his or hers allocation in the system. Given a specific allocation of all the individuals, we can describe the total welfare of that particular allocation by the sum of the utility of all the individuals. The individuals cannot freely choose their allocation; they can only choose allocations that are compatible with the various list lengths. In a well functioning society, authorities introduce incentives/legislation to avoid suboptimal allocations. In such cases we would then expect that the so-called efficiency principle holds:

Let A_1 and A_2 denote any two allocations of patients to lists. If the total utility of A_1 is larger than of A_2 , the probability of A_1 should be larger than the probability of A_2 .

Assuming that our system is efficient, we should then seek to find all distributions that are compatible with the efficiency principle. It is surprising to observe that there are very few probability measures of this sort. Following the construction used in Jörnsten & Ubøe (2005), the only such probability measures can be described by explicit one-parameter formulas, see (2.2) below. The parameter quantifies the strength of peoples preferences, and can be interpreted as a choice of unit for utility. Once a choice of unit has been made, we are left with a unique probability measure, and we refer to this measure as the benefit efficient probability measure. Assuming that there are many patients of each type, we then expect to observe allocations that are compatible with this measure.

The main theory in Jörnsten & Ubøe (2005) is based on well known principles from gravity modeling. Since the first major results in the late 60s, e.g., Wilson (1967), gravity models has been a topic of intensive study. Models of this kind have found widespread use in several different areas, e.g., road planning and computer tomography. For a seminal textbook on gravity models, see Sen & Smith (1995). Gravity models can be derived in many different ways. Anas (1983) was the first to show that gravity models can be derived from random utility theory, and we refer to Erlander and Stewart (1990) for a number of different derivations of these models. Of special importance is the derivation from cost efficiency principles, see Erlander and Smith (1990). Jörnsten & Ubøe (2005) use the core arguments in Erlander and Smith (1990) and Jörnsten et al (2004) to consider new applications of the cost efficiency principle. The basic construction we will use in our paper is based on the same underlying principles.

Our paper is organized as follows: In Section 2 we describe the general framework. Using the results from Jörnsten & Ubøe (2005), we obtain explicit formulas for the statistical distributions. As some proofs are rather technical, they are placed in the appendix. In Section 3 we consider

a series of numerical experiments to show how the distributions may look like in special cases. In particular we observe that all doctors of the same type get the same distribution of patients. In Section 4 we proceed to cover aggregate states, i.e., we pool all doctors of the same type together and observe that this does not change the final distributions. Numerical simulation of microsystems with a large number of individual doctors are difficult/impossible. The aggregation principles in Section 4 show that such subdivisions can be avoided without loss of generality, i.e., that microsimulations will eventually lead to the same distributions that we obtain using macrosimulation. Finally in Section 5 we offer some concluding remarks.

2. The framework

In this section we will consider a general version where we have D doctors, T types of doctors and S types of patients. A patient is either on the the list of a doctor or on a waiting list for a doctor. For simplicity we will ignore the case where a patient prefer to have no formal relation with any of the doctors. Assumptions:

- We order the doctors $i = 1, \dots, D$, assuming that doctor i has a patient list of length L_i .
- We group the patients of each doctor into $2S$ different categories.

The first S groups consist of patients of type s that is on the list of the doctor, and the remaining groups consist of patients of type s that is on a waiting list for the doctor. We let N_s denote the total number of patients of type s .

- Each doctor has a list of vacant entries which is empty if his or hers list is full.

Utilities

From the above we see that the listings related to any of the doctors can be divided into a total of $2S + 1$ groups. The utility of the various combinations are defined as follows:

$$U_{st} = \begin{cases} \text{Utility of a patient of group } s \text{ having a doctor of type } t & \text{if } s = 1, \dots, S \\ \text{Utility of a patient of group } s \text{ waiting for a doctor of type } t & \text{if } s = S + 1, \dots, 2S \\ \text{(Dis)utility per vacant entry of a doctor of type } t & \text{if } s = 2S + 1 \end{cases}$$

Remarks

We do not put any constraints on the length of the waiting lists. Moreover, every patient is member of one and only one group. That excludes scenarios where a patient is on several waiting lists, or is on the list of one doctor while waiting for another.

Choose and fix one arbitrary distribution of all the patients in the system. We will now need to order the vacancies and patients into a single vector \mathbf{f} of length $D(2S + 1)$. The ordering is as follows:

f_1, \dots, f_S are the numbers of the S different types of patients on the list of doctor 1, f_{S+1}, \dots, f_{2S} are the corresponding numbers on the waiting list for doctor 1, and f_{2S+1} is the number of vacant entries for doctor 1. Then we continue with the corresponding numbers for doctor 2, and so on. We may order the utilities in the same way, i.e., we consider a utility vector \mathbf{U} of length $D(2S + 1)$, such U_i is the utility of the patients/vacancies in f_i .

Note that we have two essentially different notations for the utilities, and we will need to switch between the two notations according to the context. We hope that this is not too confusing for the reader.

The total utility $B[\mathbf{f}]$ of the distribution \mathbf{f} is a sum of two terms. The first is the sum of the utility of all the patients and the second is the sum of the (dis)utilities of the vacant entries of all the doctors.

We will assume that the following efficiency principle holds:

If $B[\mathbf{f}] \geq B[\mathbf{g}]$, then the probability of \mathbf{f} should be at least as high as the probability of \mathbf{g} .

It is surprising to observe that this simple behavioral principle singles out a unique probability measure. Assuming that there is a large number of patients in each category, the patients will be distributed according to this measure. This measure is defined in terms of the restrictions on the system. Before we can state a formal result, we have to define these.

There are altogether $D + S$ restrictions on the system. The first D restrictions express that doctor i has a patient list of length L_i , and the last S restrictions express that there is a total number E_s of patients of type s . Since these are linear restrictions, it is hence possible to find an $(D + S) \times D(2S + 1)$ matrix M such that these restrictions are satisfied if and only if

$$M\mathbf{f}^\perp = (L_1, \dots, L_D, E_1, \dots, E_S)^\perp \quad (1)$$

where \perp signifies transposition. The final result can then be formulated as follows:

THEOREM 2.1

Assume that a system of patient allocation is benefit efficient. Then given $\beta \geq 0$ there exist unique real numbers u_1, \dots, u_{D+S} such that

$$\mathbf{f} = \exp[(u_1, \dots, u_{D+S})M + \beta \mathbf{U}] \quad (2)$$

satisfies the constraints in (1). These distributions are the only ones that are compatible with the efficiency principle.

PROOF

See the appendix. □

The problem with this relatively abstract result is that it is not very transparent. In our particular case, however, the result can be given in more transparent form.

THEOREM 2.2

Assume that the system is efficient, and let P_{ist} denote the expected number of patients/-vacancies in group s belonging to doctor i , and assume that doctor i has type t . Then we can find real numbers A_i , $i = 1, \dots, D$ and B_s , $s = 1, \dots, S$, such that

$$P_{ist} = \begin{cases} A_i B_s \exp[\beta U_{st}] & \text{if } s = 1, \dots, S \\ B_{s-S} \exp[\beta U_{st}] & \text{if } s = S + 1, \dots, 2S \\ A_i \exp[\beta U_{st}] & \text{if } s = 2S + 1 \end{cases} \quad (3)$$

PROOF

See the appendix. □

As we will see in the next section, the formulas in (3) admit quite explicit interpretations, see Corollary 3.1 and also Theorem 4.1. Moreover, Theorem 2.2 offers a clear link to the classical theory of gravity models, see (17) and Theorem 4.2.

As we mentioned in the introduction gravity models have been used extensively in many connections for a long period of time. Very efficient software has been developed to study

such system, and they have found widespread use. In particular we refer to Herman et al (1978) where these methods have been used for medical image reconstruction, i.e., computer tomography. In such systems extreme numbers of constraints (millions!) can be handled without problems. The general procedure to find numerical solutions of such systems has been based on the Bregman balancing algorithm, see Bregman (1967). The Bregman algorithm can easily be modified to cover the cases we study in this paper. We refer to Jörnsten & Ubøe (2005) for a general discussion of numerical solutions of extensions to the gravity model.

3. Examples and interpretations

We will now consider a few examples to see how this works. We will take a look at a case where there are $D = 7$ doctors categorized by their sex, i.e., $T = 2$. There are 4 male and 3 female doctors, and there are $S = 4$ different types of patients. The different types of patients can be described as follows:

- MM - men who want a male doctor
- MF - men who want a female doctor
- WM - women who want a male doctor
- WF - women who want a female doctor

That splits the record of each doctor into 9 different categories

$$(MMp, MFp, WMp, Wfp, MMw, MFw, WMw, Wfw, e) \quad (4)$$

where p signifies patients on the list, w signifies patients waiting for a place on the list, and e signifies the vacant entries on the list. It follows from Theorem 2.2 that β can be interpreted as a numeraire for utility. With properly chosen units for U, we can hence assume that $\beta = 1$ without loss of generality. This is done in all the cases below.

Case 1

Consider a case where all 7 doctors have a list of length 2 000, i.e.

$$(l_1, \dots, l_7) = (2\,000, 2\,000, 2\,000, 2\,000, 2\,000, 2\,000, 2\,000) \quad (5)$$

and where the total number of each type is as follows

MM - 5 000 MF - 5 000 WM - 5 000 WF - 5 000

Hence a total number of 20 000 patients are “competing” for the 14 000 entries. Utilities are given in the table below:

Table 1: Utilities, column 1 shows the sex of the doctor									
D	MMp	MFp	WMp	WFp	MMw	MFw	WMw	WFw	e
M	1	0	0	-1	-5	-5	-5	-5	0
F	-1	0	0	1	-5	-5	-5	-5	0

These utilities correspond to a situation where MM and WF have a moderate preference for a doctor of their own sex, MW and WM are indifferent, all patients strongly prefer to be on a list, and doctors suffer no loss in utility for vacancies. A numerical computation of the model in Theorem 2.2 gives the results below.

$$\begin{bmatrix}
 881 & 486 & 486 & 147 & 164 & 246 & 246 & 202 & 0 \\
 881 & 486 & 486 & 147 & 164 & 246 & 246 & 202 & 0 \\
 881 & 486 & 486 & 147 & 164 & 246 & 246 & 202 & 0 \\
 881 & 486 & 486 & 147 & 164 & 246 & 246 & 202 & 0 \\
 109 & 446 & 446 & 999 & 164 & 246 & 246 & 202 & 0 \\
 109 & 446 & 446 & 999 & 164 & 246 & 246 & 202 & 0 \\
 109 & 446 & 446 & 999 & 164 & 246 & 246 & 202 & 0
 \end{bmatrix} \quad (6)$$

The first 4 rows show the distribution of patients belonging to the 4 male doctors, and the following 3 rows show the corresponding results for the female doctors. We notice that male and female doctors have a different distribution of patients on their lists, while the waiting lists are uniformly distributed among all the doctors.

Case 2

Keeping the utilities in Table 1, we change the length of the lists of some of the doctors, and consider a case where

$$(l_1, \dots, l_7) = (1\,000, 2\,000, 2\,000, 3\,000, 1\,000, 2\,000, 3\,000) \quad (7)$$

The total capacity of the male doctors and the female doctors are fixed, and that leads to the following distribution

$$\begin{bmatrix} 441 & 243 & 243 & 74 & 164 & 246 & 246 & 202 & 0 \\ 881 & 486 & 486 & 147 & 164 & 246 & 246 & 202 & 0 \\ 881 & 486 & 486 & 147 & 164 & 246 & 246 & 202 & 0 \\ 1322 & 729 & 729 & 221 & 164 & 246 & 246 & 202 & 0 \\ 55 & 223 & 223 & 499 & 164 & 246 & 246 & 202 & 0 \\ 109 & 446 & 446 & 999 & 164 & 246 & 246 & 202 & 0 \\ 164 & 669 & 669 & 1498 & 164 & 246 & 246 & 202 & 0 \end{bmatrix} \quad (8)$$

We notice that the all male doctors and all female doctors have the same number of patients on their waiting lists as before. The number of patients on the individual lists are different, but they have all been changed in proportion to the length of the lists. Hence all doctors of the same type has the same distribution of patients on their patient lists.

Case 3

In case 1 and 2 we have a considerable shortage of doctors. To examine the case with a surplus of doctors we change the numbers of patients to

$$\text{MM} - 2\,500 \quad \text{MF} - 2\,500 \quad \text{WM} - 2\,500 \quad \text{WF} - 2\,500$$

otherwise we fix everything in case 2. That leads to the following distribution

$$\begin{bmatrix} 286 & 184 & 184 & 51 & 0 & 0 & 0 & 0 & 295 \\ 571 & 369 & 369 & 102 & 0 & 0 & 0 & 0 & 590 \\ 571 & 369 & 369 & 102 & 0 & 0 & 0 & 0 & 590 \\ 857 & 553 & 553 & 153 & 0 & 0 & 0 & 0 & 885 \\ 36 & 171 & 171 & 349 & 0 & 0 & 0 & 0 & 273 \\ 72 & 342 & 342 & 698 & 0 & 0 & 0 & 0 & 547 \\ 108 & 513 & 513 & 1\,046 & 0 & 0 & 0 & 0 & 820 \end{bmatrix} \quad (9)$$

We notice that all male doctors have the same share of vacancies, but that the share is different from the share of the female doctors. The reason is of course that there is less surplus of female doctors.

The features we have seen in cases 1-3 are all true in general. They are all easy consequences of Theorem 2.2, and the results can be stated as follows:

COROLLARY 3.1

Given a set of preferences U , then:

- All doctors of the same type have the same distribution of patients and vacancies, i.e., these distributions do not depend on the length of their lists.
- All doctors of the same type has the same number of people on their waiting lists, i.e., these numbers do not depend on the length of their lists.

PROOF

The first statement follows from the first and the third line in (3), and the second follows from the second line in (3).

□

Case 4

To proceed, we now wish to see what happens when we change preferences. In the cases above, doctors suffer no loss in utility if they have vacancies. We consider a case where all doctors suffer a utility loss of 1 unit for each vacancy. Otherwise everything is as in case 3. That leads to the following distribution:

$$\begin{bmatrix} 286 & 184 & 184 & 51 & 0 & 0 & 0 & 0 & 295 \\ 571 & 369 & 369 & 102 & 0 & 0 & 0 & 0 & 590 \\ 571 & 369 & 369 & 102 & 0 & 0 & 0 & 0 & 590 \\ 857 & 553 & 553 & 153 & 0 & 0 & 0 & 0 & 885 \\ 36 & 171 & 171 & 349 & 0 & 0 & 0 & 0 & 273 \\ 72 & 342 & 342 & 698 & 0 & 0 & 0 & 0 & 547 \\ 108 & 513 & 513 & 1046 & 0 & 0 & 0 & 0 & 820 \end{bmatrix} \quad (10)$$

We observe that there is no change! To create a difference, one type of doctors must be more sensitive to utility loss than the other.

Case 5

We now assume that female doctors suffer a loss of 5 units of utility while male doctors suffer a loss of 1 unit. That leads to the following distribution:

$$\begin{bmatrix} 255 & 118 & 118 & 24 & 0 & 0 & 0 & 0 & 485 \\ 511 & 236 & 236 & 47 & 0 & 0 & 0 & 0 & 971 \\ 511 & 236 & 236 & 47 & 0 & 0 & 0 & 0 & 971 \\ 766 & 353 & 353 & 71 & 0 & 0 & 0 & 0 & 1456 \\ 76 & 260 & 260 & 385 & 0 & 0 & 0 & 0 & 20 \\ 152 & 519 & 519 & 770 & 0 & 0 & 0 & 0 & 39 \\ 228 & 779 & 779 & 1155 & 0 & 0 & 0 & 0 & 59 \end{bmatrix} \quad (11)$$

We see that in this case it is more efficient if female doctors have fewer vacancies. It is an appropriate discussion if such a transfer of welfare is indeed possible, and what administrative measures that must be taken to ensure such a policy, but that is outside the scope of this paper.

Case 6

One question of interest is the following: Is it possible to find a set of preferences such that some doctors have vacancies even when there is a surplus of doctors in the system? The answer is yes, and one possibility scenario is the following system:

- 4 male and 3 female doctors, all with a list length of 2 000.

MM - 2 000 MF - 2 000 WM - 2 000 WF - 7 000

Table 2: Utilities, column 1 shows the sex of the doctor									
D	MMp	MFp	WMp	WFp	MMw	MFw	WMw	WFw	e
M	1	0	0	-10	-10	-10	-10	-10	0
F	-1	0	0	5	-10	-10	-10	-1	0

This corresponds to a situation where the people within the WF-group prefer to stay on the waiting list for a female doctor, rather than be accepted on the list of a male doctor. The distribution in that case is as follows:

$$\begin{bmatrix} 500 & 500 & 500 & 18 & 0 & 0 & 0 & 0 & 482 \\ 500 & 500 & 500 & 18 & 0 & 0 & 0 & 0 & 482 \\ 500 & 500 & 500 & 18 & 0 & 0 & 0 & 0 & 482 \\ 500 & 500 & 500 & 18 & 0 & 0 & 0 & 0 & 482 \\ 0 & 0 & 0 & 2000 & 0 & 0 & 0 & 309 & 0 \\ 0 & 0 & 0 & 2000 & 0 & 0 & 0 & 309 & 0 \\ 0 & 0 & 0 & 2000 & 0 & 0 & 0 & 309 & 0 \end{bmatrix} \quad (12)$$

As we can see from the last two columns there are plenty of vacancies and people on waiting lists in this case.

4. Aggregation of states

We have seen that all doctors of the same type has the same distribution of patients on their lists. Hence if we consider the sum of the patient lists for doctors of each type, we will get the same distribution. Formally we can prove the following result where a $\tilde{\cdot}$ on top of a letter signifies aggregation (within types) of that quantity:

THEOREM 4.1

If the system is efficient, we can find the aggregate distribution of patients with doctors of type t solving the aggregate system

$$\tilde{P}_{st} = \begin{cases} \tilde{A}_t B_s \exp[\beta U_{st}] & \text{if } s = 1, \dots, S \\ D_t B_{S-s} \exp[\beta U_{st}] & \text{if } s = S + 1, \dots, 2S \\ \tilde{A}_t \exp[\beta U_{st}] & \text{if } s = 2S + 1 \end{cases} \quad (13)$$

together with the restrictions

$$\sum_{s=1}^S \tilde{P}_{st} + \tilde{P}_{(2S+1)t} = \tilde{L}_t \quad t = 1, \dots, T \quad \sum_{t=1}^T (\tilde{P}_{st} + \tilde{P}_{(s+S)t}) = E_s \quad s = 1, \dots, S \quad (14)$$

PROOF

See the appendix. □

Theorem 4.1 says that we will obtain the same distributions if we model the system in terms of macrostates where all doctors of the same type are pooled together. If the number of doctors are large, we need to solve large non-linear systems to find the corresponding microstates. Alternatively we can use the approach in Theorem 4.1 and find the same solutions solving much smaller systems. That greatly facilitates numerical computations.

EXAMPLE 4.2

To explain how this works, we return to case 6 in the previous section. In that case the male doctors have a total capacity of 8 000 entries, and the corresponding number for the female doctors is 6 000. Assuming that all preferences are as in case 6, we can use Theorem 4.1 with $D_1 = 4$ and $D_2 = 3$ to compute the aggregate distributions. The result is as follows:

$$\begin{bmatrix} 2000 & 2000 & 2000 & 73 & 0 & 0 & 0 & 0 & 1927 \\ 0 & 0 & 0 & 6000 & 0 & 0 & 0 & 927 & 0 \end{bmatrix} \quad (15)$$

The first line shows the expected distribution of the patients belonging to male doctors, and the second line the corresponding numbers for female doctors. As we could expect from Theorem 4.1, these distributions are exactly the same as the distributions shown in (12). In fact Theorem 4.1 says that we will obtain the results in (12) if we split the first line in (15) into 4 equal parts, and the second line into 3 equal parts. The reader can easily check that this is correct (except for decimal rounding errors).

Remark

Note that if we simply assume that there is one male doctor with a total list length of 8 000 and one female doctor with a total list length of 6 000, we obtain a different (incorrect) result:

$$\begin{bmatrix} 2\,000 & 2\,000 & 2\,000 & 183 & 0 & 0 & 0 & 0 & 1\,817 \\ 0 & 0 & 0 & 6\,000 & 0 & 0 & 0 & 817 & 0 \end{bmatrix} \quad (16)$$

We hence see that aggregation must be handled with some care, i.e., the microstate result in Theorem 2.2 is essential to obtain the correct macrostate given by (13).

Gravity models

In the theory of spatial economics much attention has been put on so-called gravity models. These are models where the expected numbers can be expressed on the form

$$P_{st} = A_t B_s \exp[\beta U_{st}] \quad (17)$$

As we can see from (13), our model is not of this type. In one particular case, however, our aggregate system can be reduced to a system of gravity type. The result reads as follows:

THEOREM 4.3

Assume that the system is efficient, that there is no shortage of doctors, and that all groups have very strong disutilities of staying on waiting lists. Then the system in (13) can be reduced to a model of gravity type.

PROOF

See the appendix.

□

5. Concluding remarks

In this paper we have built a theoretical model for statistical distributions of patient lists. The model takes into account that different groups of patients have different preferences for different types of doctors. We have modeled this from a setting with microstates, i.e., a setting where we compute the distributions of patients belonging each individual doctor. Our model has been built from the efficiency principle, i.e., a framework where the basic assumption is that allocations with larger total utility will be more probable. This is the same basic core underlying the models of gravity type; a type of well established models with widespread use.

In the paper we have shown how our models are simplified if we pool all doctors of the same type together. This means that we might just as well model the system using macrostates, i.e., frameworks that only make use of the total number of doctors of each type. Aggregation of doctors of the same type does not change the final distributions in the model.

If we assume that all patients have a strong disutility of being on a waiting list and we have a surplus of doctors, we have shown that the pooled distributions are of gravity type. This makes a nice link to the extensive literature on such models. Our general case, however, is not of gravity type. Still we are in a position where the basic numerical tools in gravity theory can be used with only minor modifications.

In the paper we have only studied examples with two different types of doctors. Numerical methods will in general admit extensions to system with very large numbers of different types and/or different groups. In that respect there are virtually no restrictions on the size of the systems we can handle.

Our model may be helpful for providing qualitative insights on how the parameters involved affect the allocation. In particular this may tell politicians and regulators something about “the limits to change”. One example is how the share of female patients allocated to female doctors is expected to change when the fraction of female doctors increases. For quantitative applications we are left with the problem of elicitation of the preference structure: Here two venues are available: Devise a scheme for questioning patients or infer parameters from observed allocations. These issues are, however, outside the scope of this paper.

As is quite clear from the discussion in Jörnsten and Ubøe (2005), the theory that we use in this paper can be applied in much more general settings than the one we study in this paper. Loosely

speaking our theory can be used in all situations where we have some types of agents that have some utilities from doing some specified actions, and where the allocations of the agents are constrained by linear constraints. Assuming that we have an efficient distribution of welfare, we can apply the methods in Jörnsten and Ubøe (2005) to compute explicit statistical distributions for the agents. It is our hope that the present paper may inspire the reader to apply the same theory to any kind of setting concerning distribution of welfare in general. Viewed as such our paper might have implications far beyond the actual application we have studied in this paper.

6. Appendix

Proof of Theorem 2.1

The proofs in Jörnsten & Ubøe (2005) can be carried out with no essential changes, and we refer to that paper for the complete details. In Jörnsten & Ubøe (2005), however, uniqueness can only be proved in non-degenerate cases. Here we can obtain a slightly stronger result, and this can be seen as follows:

From the construction in Jörnsten & Ubøe (2005) u_1, \dots, u_{D+S} always exist and \mathbf{f} given $\beta \geq 0$ is always unique. In degenerate cases with strong linear dependence between the columns in M , then u_1, \dots, u_{D+S} may not be unique. When M is defined as in (1), e.g. (18), we can easily see that given \mathbf{f} , β , and \mathbf{U} , there can be at most one solution of (2). Hence in our case u_1, \dots, u_{D+S} are always unique.

□

Proof of Theorem 2.2

For simplicity we only consider the case $D = 2, D_1 = 1, D_2 = 1$ and $S = 4$, i.e., a case with one male and one female doctor. In that particular case we get

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (18)$$

From Theorem 2.1, we get that there exist real numbers $u_1, u_2, u_3, u_4, u_5, u_6$ and $\beta \geq 0$ such that

$$\mathbf{f} = \exp[(u_1, u_2, u_3, u_4, u_5, u_6)M + \mathbf{U}] \quad (19)$$

Hence

$$\begin{aligned}
f_1 &= e^{u_1+u_3+\beta U_1} & f_2 &= e^{u_1+u_4+\beta U_2} & f_3 &= e^{u_1+u_5+\beta U_3} & f_4 &= e^{u_1+u_6+\beta U_4} \\
f_5 &= e^{0+u_3+\beta U_5} & f_6 &= e^{0+u_4+\beta U_6} & f_7 &= e^{0+u_5+\beta U_7} & f_8 &= e^{0+u_6+\beta U_8} \\
f_9 &= e^{u_1+0+\beta U_9} & & & & & & \\
f_{10} &= e^{u_2+u_3+\beta U_{10}} & f_{11} &= e^{u_2+u_4+\beta U_{11}} & f_{12} &= e^{u_2+u_5+\beta U_{12}} & f_{13} &= e^{u_2+u_6+\beta U_{13}} \\
f_{14} &= e^{0+u_3+\beta U_{14}} & f_{15} &= e^{0+u_4+\beta U_{15}} & f_{16} &= e^{0+u_5+\beta U_{16}} & f_{17} &= e^{0+u_6+\beta U_{17}} \\
f_{18} &= e^{u_2+0+\beta U_{18}} & & & & & &
\end{aligned} \tag{20}$$

If we put $A_1 = e^{u_1}, A_2 = e^{u_2}$ and $B_1 = e^{u_3}, B_2 = e^{u_4}, B_3 = e^{u_5}, B_4 = e^{u_6}$, we can write the expressions in (20) as follows:

$$\begin{aligned}
f_1 &= A_1 B_1 e^{\beta U_1} & f_2 &= A_1 B_2 e^{\beta U_2} & f_3 &= A_1 B_3 e^{\beta U_3} & f_4 &= A_1 B_4 e^{\beta U_4} \\
f_5 &= B_1 e^{\beta U_5} & f_6 &= B_2 e^{\beta U_6} & f_7 &= B_3 e^{\beta U_7} & f_8 &= B_4 e^{\beta U_8} \\
f_9 &= A_1 e^{u_1+0+\beta U_9} & & & & & & \\
f_{10} &= A_2 B_1 e^{\beta U_{10}} & f_{11} &= A_2 B_2 e^{\beta U_{11}} & f_{12} &= A_2 B_3 e^{\beta U_{12}} & f_{13} &= A_2 B_4 e^{\beta U_{13}} \\
f_{14} &= B_1 e^{\beta U_{14}} & f_{15} &= B_2 e^{\beta U_{15}} & f_{16} &= B_3 e^{\beta U_{16}} & f_{17} &= B_4 e^{\beta U_{17}} \\
f_{18} &= A_2 e^{\beta U_{18}} & & & & & &
\end{aligned} \tag{21}$$

This proves (3) in this particular case. Clearly we can obtain the general case by the same type of argument if we put

$$A_i = e^{u_i}, i = 1, \dots, D \quad B_j = e^{u_{D+j}}, j = 1, \dots, S \tag{22}$$

□

Proof of Theorem 4.1

Assume that there are D_t doctors of type t , let $I_t = \{i | \text{doctor } i \text{ is of type } t\}$ and define

$$\tilde{P}_{st} = \sum_{i \in I_t} P_{ist} \tag{23}$$

i.e., the total number of patients belonging to doctors of type t . Let $A_i, i = 1, \dots, D, B_s, s = 1, \dots, S$, and β be the numbers found in Theorem 2.2. If we define

$$\tilde{A}_t = \sum_{i \in I_t} A_i \quad t = 1, \dots, T \tag{24}$$

it follows from Theorem 2.2, that

$$\tilde{P}_{st} = \begin{cases} \tilde{A}_t B_s \exp[\beta U_{st}] & \text{if } s = 1, \dots, S \\ D_t B_{S-s} \exp[\beta U_{st}] & \text{if } s = S + 1, \dots, 2S \\ \tilde{A}_t \exp[\beta U_{st}] & \text{if } s = 2S + 1 \end{cases} \tag{25}$$

The numbers \tilde{A}_t , $t = 1, \dots, T$, B_s , $s = 1, \dots, S$ can be found directly using (25) together with the constraints

$$\sum_{s=1}^S \tilde{P}_{st} + \tilde{P}_{(2S+1)t} = \sum_{i \in I_t} L_i \quad t = 1, \dots, T \quad \sum_{t=1}^T (\tilde{P}_{st} + \tilde{P}_{(s+S)t}) = E_s \quad s = 1, \dots, S \quad (26)$$

If we define $\tilde{L}_t = \sum_{i \in I_t} L_i$, i.e., the total capacity of the doctors of type t , we get the constraints

$$\sum_{s=1}^S \tilde{P}_{st} + \tilde{P}_{(2S+1)t} = \tilde{L}_t \quad t = 1, \dots, T \quad \sum_{t=1}^T (\tilde{P}_{st} + \tilde{P}_{(s+S)t}) = E_s \quad s = 1, \dots, S \quad (27)$$

□

Proof of Theorem 4.2.

With the assumptions above, the numbers of patients on waiting lists are negligible and can be ignored. Consider a gravity model on the form

$$\tilde{P}_{st} = C_t D_s \exp[\beta U_{st}] \quad s = 1, \dots, S, s = 2S + 1, t = 1, \dots, T \quad (28)$$

with the constraints

$$\sum_{s=1}^S \tilde{P}_{st} + \tilde{P}_{(2S+1)t} = \tilde{L}_t \quad t = 1, \dots, T \quad \sum_{t=1}^T \tilde{P}_{st} = \begin{cases} E_s & \text{if } s = 1, \dots, S \\ K & \text{if } s = 2S + 1 \end{cases} \quad (29)$$

and where K is the total surplus in the system, i.e., $K = \sum_{t=1}^T \tilde{L}_t - \sum_{s=1}^S E_s$. Assume that (28) and (29) hold. Put

$$\tilde{A}_t = C_t \cdot D_{2S+1} \quad t = 1, \dots, T \quad (30)$$

$$B_s = D_s / D_{2S+1} \quad s = 1, \dots, S \quad (31)$$

If we define \tilde{P}_{st} by (13), it is then straightforward to verify that (14) holds. To prove the converse result, assume that (13) and (14) hold. Put

$$C_t = \tilde{A}_t \quad t = 1, \dots, T \quad (32)$$

$$D_s = \begin{cases} B_s & \text{if } s = 1, \dots, S \\ 1 & \text{if } s = 2S + 1 \end{cases} \quad (33)$$

If we define \tilde{P}_{st} by (28), and $s = 1, \dots, S$ (29) follows directly from (14). It remains to consider (29) in the case $s = 2S + 1$. In that case we get

$$\begin{aligned}
\sum_{t=1}^T C_t D_s \exp[\beta U_{st}] &= \sum_{t=1}^T A_t \exp[\beta U_{st}] \\
&= \sum_{t=1}^T \left(\tilde{L}_t - \sum_{s=1}^S \tilde{A}_t B_s \exp[\beta U_{st}] \right) \\
&= \sum_{t=1}^T \tilde{L}_t - \sum_{s=1}^S \left(\sum_{t=1}^T \tilde{A}_t B_s \exp[\beta U_{st}] \right) \\
&= \sum_{t=1}^T \tilde{L}_t - \sum_{s=1}^S E_s = K
\end{aligned} \tag{34}$$

□

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