An Analysis of a Combinatorial Auction

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Abstract. Our objective is to find prices on individual items in a combinatorial auction that support the optimal allocation of bundles of items, i.e. the solution to the winner determination problem of the combinatorial auction. The item-prices should price the winning bundles according to the corresponding winning bids, whereas the bundles that do not belong to the winning set should have strictly positive reduced cost. I.e. the bid on a non-winning bundle is strictly less than the sum of prices of the individual items that belong to the bundle, thus providing information to the bidders why they are not in the winning set. Since the winner determination problem is an integer program, in general we cannot find a linear price-structure with these characteristics. However, in this article we make use of sensitivity analysis and duality in linear programming to obtain this kind of price-information. The ideas are illustrated by means of numerical examples.

1 Introduction

In some auctions/markets, a participant's valuation of an object depends significantly on which other objects the participant acquires. Objects can be substitutes or complements, and the valuation of a particular bundle of items may not be equal to the sum of the valuations of the individual items, i.e. valuations are not additive. This may be represented by letting bidders of the auction have preferences not just for particular items, but for sets or bundles of items as well. In this setting, economic efficiency is increased by allowing bidders to bid on combinations of objects, which is exactly what a combinatorial auction does.

A recent survey of combinatorial auctions is provided by de Vries and Vohra (2000). In the literature, there are a number of examples of combinatorial auctions, ranging from the allocation of rights to radio frequencies (FCC (1994)), auctions for airport time slots (Rassenti et al. (1982)), railroad segments (Brewer (1999)) and delivery routes (Caplice (1996)). Bundle pricing (Hanson and Martin (1990)) and the effects of discounts on vendor selection (Moore et al. (1991)) can also be analyzed within this framework.

2 The Winner Determination Problem

Given a set of bids for subsets of objects, selecting the winning set of bids is denoted "the winner determination problem". This problem can be formulated as an integer programming problem. Let N be the set of bidders, M the set of m distinct objects, and S a subset of M. Agent j's ($j \in N$) bid for bundle S is denoted by $b^j(S)$, and we let

$$b(S) = \max_{j \in N} b^j(S)$$

The binary variable x_S is equal to 1 if the highest bid on S is accepted, and 0 otherwise. The winner determination problem can then be formulated as

$$\begin{array}{lll} \max & \sum_{S \subset M} b(S) \cdot x_S & \\ \text{s.t.} & \sum_{S \ni i} x_S \leq 1 & \forall i \in M \\ & x_S = 0/1 & \forall S \subset M \end{array}$$

The objective function maximizes the "revenue", i.e. the value of the bids, whereas the first set of constraints requires that no object can be assigned to more than one bidder. An alternative interpretation is the following: If bidders submit their true values, i.e. bid their reservation prices on different bundles, implying that $b^j(S) = v^j(S)$, for all $j \in N$ and $S \subset M$, the solution to the winner determination problem represents the efficient allocation of indivisible objects in an exchange economy.

The formulation above is valid for the winner determination problem in case of superadditive bids, i.e. if

$$b^j(A) + b^j(B) \leq b^j(A \cup B) \qquad \forall j \in N,\ A, B \subset M \text{ and } A \cap B = \emptyset$$

In case of substitutes, dummy variables can be introduced to make the formulation valid, or a more general formulation can be used as shown in de Vries and Vohra (2000). In any case, the formulation of the winner determination problem is an instance of the set packing problem (SPP). The linear programming relaxation of the SPP produces integer solutions in a number of cases (ref. de Vries and Vohra (2000)), we will however focus on instances where the LP-relaxation gives fractional solutions. In general, the SPP belongs to the class of NP-hard problems, and is closely related to set partitioning and set covering problems.

The objective of this paper however, is not to focus on solution methods for the winner determination problem, but rather to find prices on individual items that support the optimal allocation of bundles of items. By "support", we mean that the prices on the individual objects should price the winning bundles according to the winner bids, whereas the bundles that do not belong to the winning set should have strictly positive reduced cost, i.e.

bid on non-winning bundle	<	Σ prices of individual objects that belong to bundle

Prices with these characteristics will provide information to the bidders why they are not in the winning set, and this information may be used in a specific market design. Since the winner determination problem is an integer problem, in general, we will have to consider non-linear price structures.

3 Incentive Issues

Solving a combinatorial auction consists of two parts:

1) The Allocation Problem: “Who gets what?” and
2) The Pricing Problem: “How much do winners pay?”

Bidders would prefer to pay less than their reservation price on their winning bids, i.e. depending on how bids determine what a winner should pay, there may be incentives to bid less than the reservation price, $b^j(S) < v^j(S)$ for some j and S, in which case the auction is not incentive compatible.

In the literature, a number of incentive problems are discussed, we will briefly mention a few:

“Exposure Problem”:

Refers to the problem of bidders paying too much for individual items, or in order to avoid this happen, bidders with preferences for certain bundles dropping out early in the bidding process in order to limit losses.

“Threshold Problem” / “Free Rider Problem”:

Bidders may bid less than their reservation price in order to pay less, but risk losing the auction. In the following example, 3 bidders bid on 2 objects, X and Y, and the true valuations of the bidders are given in the table below:

	X	**Y**	**XY**
1	0	0	100
2	75	0	0
3	0	40	0

If the bidders bid their reservation prices, 2 should get X and 3 Y. However, under the assumption that bidder 3 bids truthfully, bidder 2 has an incentive to reduce his bid, for instance to 65. In that case, bidders 2 and 3 still win the auction, but bidder 2 pays less, he is “free riding”. The same argument holds for bidder 3, but if they both reduce their bids, they may lose the auction to bidder 1. In other words, the “threshold problem” refers to the difficulty of a collection of bidders, whose combined valuation for a subset of items exceeds the bid submitted by some other bidder, to coordinate their bids in order to outbid the larger bidder.

“Strategic Bids”:

A bidder may alter his bid on some object or bundle in order to increase his chances of obtaining a bundle that he prefers. The following example has 2 bidders and 2 objects, A and B, and reservation prices are given in the table:

	A	B	AB
1	3	3	7
2	5	0	0

If bids are equal to reservation prices, bidder 2 gets A and 1 gets B. However, bidder 1 may reduce his bid on B to 1 in order to obtain bundle AB with a value of 7.

4 Suggested Method

As stated before, our objective is to find prices on individual items in a combinatorial auction that support the optimal allocation of bundles of items, i.e. the solution to the winner determination problem of the combinatorial auction. The prices should price the winning bundles according to the winner bids, whereas the bundles that do not belong to the winning set of bids should have strictly positive reduced cost, implying that the bid of a non-winning bundle is strictly less than the sum of prices of the individual items that belong to the bundle.

It is only possible to find a single price-vector that excludes all non-winning bids if

- the LP-relaxation of the winner determination problem has an integer solution, and
- the LP-relaxation has a unique dual solution such that every non-winning bundle has reduced cost (RC) > 0.

As will be illustrated in the next section by means of a simple example, it seems to be difficult to find a unique price-vector with the characteristics searched for, therefore in this article, we suggest making use of sensitivity analysis to obtain this kind of price-information. The results of the sensitivity analysis are used to reduce the size of the (primal) winner determination problem and obtain a degenerate dual of the linear programming relaxation of the reduced primal. This generates a convex set of price-vectors such that RC > 0 for at least one price-vector for all non-winning bids. When reducing the primal, we search for a minimal reduction of the problem in order to retain as much information as possible in the problem.

5 Example

In this section, we will use a small constructed example to illustrate the ideas. We assume that the following 9 bids have been handed in for different combinations of 7 objects, A-G:

Bid	17	10	10	9	20	12	4	15	26
Object									
A	1	1	1		1		1	1	
B		1	1					1	
C		1			1	1		1	1
D				1					
E	1		1		1	1			1
F		1	1				1		1
G	1			1	1			1	1

The interpretation of the table is as follows: the bid of 17 includes objects A, E and G, the next bid of 10 is on objects A, B, C and F, etc. The winner determination problem of the combinatorial auction can be formulated as the following set packing problem:

$$
\begin{array}{lcccccccccl}
\max & 17x_1 + & 10x_2 + & 10x_3 + & 9x_4 + & 20x_5 + & 12x_6 + & 4x_7 + & 15x_8 + & 26x_9 & \\
\text{s.t.} & x_1 + & x_2 + & x_3 + & & x_5 + & & x_7 + & x_8 & & \le 1 \\
 & & x_2 + & x_3 + & & & & & x_8 & & \le 1 \\
 & & x_2 + & & & x_5 + & x_6 + & & x_8 + & x_9 & \le 1 \\
 & & & & x_4 & & & & & & \le 1 \\
 & x_1 + & & x_3 + & & x_5 + & x_6 + & & & x_9 & \le 1 \\
 & & x_2 + & x_3 + & & & & x_7 + & & x_9 & \le 1 \\
 & x_1 + & & & x_4 + & x_5 + & & & x_8 + & x_9 & \le 1 \\
 & x_i & \text{binary} & & & & & & & &
\end{array}
$$

The optimal integer solution has a value of 26, $x_9 = 1$ and all other variables are zero. In the following, we will consider various potential price-structures, based on 1) the LP-relaxation, 2) exact IP marginal values, 3) the dual function and 4) using sensitivity analysis together with linear programming.

1) LP-relaxation

If we relax the integer restrictions on the variables and solve the corresponding linear program, we obtain a fractional solution with value 26.5, where $x_1 = x_2 = x_9 = 0.5$ and all other variables are equal to zero. The shadow prices for the seven constraints are given by the vector (0.5, 0, 6, 0, 7.5, 3.5, 9) implying reduced costs for the 9 bundles that have been bid on equal to (0, 0, 1.5, 0, 3, 1.5, 0, 0.5, 0). However, this dual solution is not very useful in combinatorial auction terms, since it produces reduced cost equal to zero for a number of inferior bids. This is so for bids 1 and 2, that is part of the fractional solution, but it is also so for bids 4 and 7, which are inferior even in the LP-relaxation.

Note that there exist multiple dual solutions to the linear program. The alternative dual solutions are (0.5, 0, 5, 0, 7, 4.5, 9.5) and (0.5, 0, 6, 0, 6, 3.5, 10.5), with reduced costs (0, 0, 2, 0.5, 2, 0, 1, 0, 0) and (0, 0, 0, 1.5, 3, 0, 0, 2, 0), respectively. We notice that for all the alternative dual solutions, several inferior bids have reduced cost equal to zero, but not necessarily in all the alternative solutions. Only the inferior alternatives consisting of bids 1 and 2 do not get any indications of the inferiority of the value of their bids.

2) Exact IP Marginal Values

An alternative way to generate price-information, which requires a substantial amount of computation, is to calculate exact marginal values by solving a number of integer programming problems apart from the original integer program. These other integer programs to be solved, are generated by in turn adding one more object, or alternatively deleting one object, in each new problem to be solved. This will give the following price information:

- (0, 0, 7, 0, 7, 5, 10), where each price is calculated by *deleting* the corresponding object (setting the right hand side equal to 0) and comparing the IP solution generated with the original problem's IP solution. The reduced costs are given by (0, 2, 2, 1, 4, 2, 1, 2, 3), so this price-structure prices out x_2, but not x_1. Moreover, x_9 is not priced according to the winning bid, since the reduced cost for bundle 9 is different from 0.
- (0, 0, 5, 0, 5, 4, 9), where each price is calculated by *adding* one more object (setting the right hand side equal to 2) and comparing the IP solution value with the IP solution value of the original problem. In this case neither x_1 nor x_2 are priced out, and the price-vector does not price bundle 9 to 26 either.

Consequently, these two price-vectors indicate by how much each bidder might lower or rise their price, but do not give enough information to the bidders. Moreover, the procedure is computationally burdensome, as the number of IP solutions needed to generate the prices is 2 times *m*, the number of objects in the auction.

3) Cutting Planes and Dual Functions

We know that a price-system that works for a combinatorial auction must in general be non-linear. Such a non-linear price-system can be derived using a cutting plane or branch-and-bound technique when solving the winner determination problem. For our example we will use a cutting plane approach to generate a non-linear price-system. By adding constraints 1, 3 and 5, dividing by two and rounding down, the following cutting plane is derived:

$$x_1 + x_2 + x_3 + x_5 + x_6 + x_8 + x_9 \leq 1$$

If we append this cutting plane to the winner determination problem and solve the new linear programming relaxation, we get the solution $x_9 = 1$ with value 26, shadow prices (0, 0, 0, 0, 0, 4, 9, 13), and reduced costs (5, 7, 7, 0, 2, 1, 0, 7, 0). I.e. bundles 1 and 2 are priced out, but not bundles 4 and 7.

There are however, multiple dual solutions, as shown in the table below:

Dual Solutions ($\boldsymbol{\pi}$,μ)

Constraint	1	2	3	4	5	6	7
A	0	0	0	0	0	0	0
B	0	0	0	0	0	0	0
C	0	5	0	4	5	0	4
D	0	0	0	0	0	0	0
E	0	7	0	7	6	0	6
F	4	4	4	4	4	5	5
G	9	9	10	10	10	9	9
Cut	13	1	12	1	1	12	2

Each of these dual solutions gives rise to a nonlinear price-function of the form

$$\mathbf{F}(\mathbf{a}_j) = \boldsymbol{\pi}\mathbf{a}_j + \mu \cdot \lfloor (a_{1j}+a_{3j}+a_{5j})/2 \rfloor$$

where $\mathbf{a}_j$ is the coefficient vector of the jth bundle, $\boldsymbol{\pi}$ is the vector of shadow prices for the original constraints A-G, μ is the shadow price of the cut, a_{ij} is the coefficient in the ith row of bundle j, and $\lfloor u \rfloor$ represents the greatest integer less than or equal to u. However, none of the above dual non-linear price-functions alone will produce a positive reduced cost for all the unsuccessful bids. Nevertheless, a convex combination of the dual solutions yields such a price-system. Take for instance the convex combination consisting of 0.98 of dual solution 2 and 0.01 of dual solutions 3 and 6, giving prices (0, 0, 4.9, 0, 6.86, 4.01, 9.01, 1.22) and reduced costs (0.09, 0.13, 2.09, 0.01, 1.99, 0.98, 0.01, 0.13, 0).

This price-vector has the characteristics searched for, and this gives rise to the following questions:

Question 1: Does there always exist a set of cutting planes that produces prices such that all unsuccessful bids have RC > 0?

Note that from a pure integer programming point of view, this property is uninteresting, but when using integer programming in a combinatorial-auction-setting, this property is necessary in order to achieve decentrability.

Question 2: If such a set of cutting planes exists, how difficult is it to derive it, as compared with deriving only a set of cutting planes that yields the desired integer solution?

The two questions raised above are of theoretical interest. However, when it comes to implementing a multi-unit auction in practice, we need more easily accessible information to be distributed to the bidders. Thus, we are interested in finding out whether such information can be derived, and if so, in which form it should be distributed to the bidders in order to create a market mechanism for a general combinatorial auction.

4) Using Sensitivity Analysis and Linear Programming

Let us use the example as we try to derive a system of prices that gives valuable information to the bidders by just solving a number of linear programming problems. One alternative that first comes to mind, and that can generate valuable information to the bidders, is to perform a sensitivity analysis of each bid in turn, based on the assumption that the other bidders do not change their bids.

The information we are looking for is, *by how much must bidder i rise his bid in order to be guaranteed to get his bid accepted, given that the other bidders do not change their bids?* One way to get that information is to solve the parametric linear program until the variable corresponding to bid i takes the value 1 in the linear program. Note that the corresponding solutions for the other variables need not be integer, hence this value is just an indication of the necessary rise for bid i.

In our example we get the following results:

Bidder	1	2	3	4	5	6	7	8	9
Original bid	17	10	10	9	20	12	4	15	26
New bid	36	25	23	11	30	14	6	36	27

It is noteworthy to see that the competing coalitions consisting of bidders 4, 6 and 7 and bidder 9 get low rises, whereas the stand-alone bidders get high rises. However, this information is far from fair since bidder 3, by rising his bid to 18, while all other bidders keep their bids constant, would be in a winning combination together with bidder 4.

Question: Is there a way to generate a non-linear price-system by solving only linear programs?

What we are looking for is a price-system that yields reduced costs that are positive for all bids that are not present in the optimal solution to the winner determination problem.

One way of finding such a system of prices is to reduce the winner determination problem by deleting bids such that the reduced winner determination problem, when solved as a linear program, yields the integer programming solution. That this can always be achieved is obvious, since we can reduce the winner determination problem to only include the winning bids. However, doing such a radical reduction will give us a price-system with very little information. Performing the maximal reduction in the example, with only the winning bid left, gives alternative optimal dual price-vectors equal to (0,0,26,0,0,0,0), (0,0,0,0,26,0,0), (0,0,0,0,0,26,0) and (0,0,0,0,0,0,26). All deleted bids will have strictly positive reduced cost for at least one of the price-vectors. However, the price-structure indicates for the bidders that all of them need to rise their bids to 26, which is clearly a fraud.

In the following, we find the minimal reduction of the winner determination problem that yields the result sought for, i.e. we delete as few as possible of the most probably losing bids, such that the reduced problem has an integer solution to the LP-relaxation. In the example, we achieve this by deleting bids 1, 2, 3 and 8. Solving the restricted LP-problem gives the winning bid, bid 9, and a set of extreme dual solutions. The extreme dual solutions are

Dual Solutions $\boldsymbol{\pi}$

Constraint	1	2	3	4	5	6
A	0	0	0	0	0	0
B	0	13	12	0	12	0
C	0	0	0	0	0	0
D	0	0	0	0	0	0
E	13	0	0	12	0	12
F	4	4	4	4	5	5
G	9	9	10	10	9	9

If we require that all bids should be able to tackle each of these prices and all the convex combinations of them, the (maximal) reduced costs for the 9 bids are as follows:

Bid	1	2	3	4	5	6	7	8	9
Reduced cost	5	7	7	1	2	1	1	7	0

As is evident from the reduced costs, the suggested price-structure prices out all non-winning bids, and it also gives a realistic indication of the necessary rise for each bidder, in order to be in the winning set.

Assuming that the bidders rise their prices accordingly, i.e. no bidder withdraws because his reservation price has been exceeded, we get a revised winner determination problem. Only the objective function has changed compared to the old problem, as the bids have been increased by the maximal reduced cost. The new objective is

$$\max \quad 22x_1 + 17x_2 + 17x_3 + 10x_4 + 22x_5 + 13x_6 + 5x_7 + 22x_8 + 26x_9$$

Now the linear programming relaxation of the winner determination problem has objective function value 33.5 with $x_2 = x_3 = x_6 = 0.5$ and $x_4 = 1$, whereas the integer programming solution has value 28 with $x_4 = x_6 = x_7 = 1$.

Constructing once more the minimal reduced problem, requires that all bids except the winning bids must be deleted from the problem in order to have an integer solution to the LP-relaxation of the reduced winner determination problem. Solving the reduced problem and calculating the corresponding price-structure yields

Dual Solutions $\boldsymbol{\pi}$

Constraint	1	2	3	4	5	6	7	8
A	5	5	0	0	5	5	0	0
B	0	0	0	0	0	0	0	0
C	0	13	13	0	0	13	13	0
D	0	0	0	0	10	10	10	10
E	13	0	0	13	13	0	0	13
F	0	0	5	5	0	0	5	5
G	10	10	10	10	0	0	0	0

and

Bid	1	2	3	4	5	6	7	8	9
Reduced cost	6	1	1	0	6	0	0	6	2

Again assuming that bidders rise bids accordingly, the revised objective function is

$$\max \quad 28x_1 + 18x_2 + 18x_3 + 10x_4 + 28x_5 + 13x_6 + 5x_7 + 28x_8 + 28x_9$$

This winner determination problem has an LP-relaxation with value 37.5, whereas the integer solution has value 28. In fact, there exist alternative integer solutions, all with value 28, these are

Solutions:

1: $x_1 = 1$	3: $x_3 = x_4 = 1$	5: $x_5 = 1$	7: $x_9 = 1$
2: $x_2 = x_4 = 1$	4: $x_4 = x_6 = x_7 = 1$	6: $x_8 = 1$	

Since all variables appear in at least one of the alternative solutions, there does not exist a price-structure with the desired property. However, each bidder has now got the information on the relative value of their original bid.

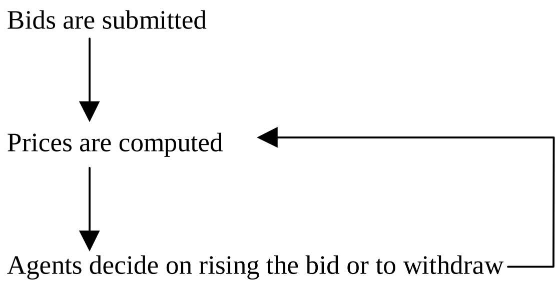
If we instead use the information from the LP-relaxation of the original winner determination problem, i.e. base the reduced costs on the three dual solutions given in part 1) of this section, our changed objective function will be

$$\max \quad 17x_1 + 10x_2 + 11.5x_3 + 10.5x_4 + 22x_5 + 13.5x_6 + 5x_7 + 17x_8 + 26x_9$$

which of course does not alter the bids on bundles 1 and 2. The corresponding solution to the linear programming relaxation of this perturbed winner determination problem is integer valued, with value 29. Hence, the LP-prices make the procedure converge with a higher value paid to the auctioneer.

6 Outline of Market Design and Suggestions for Future Research

The suggested price-structure can be used in a sequential combinatorial auction. The process could be the following:



When prices converge, in the sense that there is no more information to withdraw from the price-structures, the auction could be closed by picking the previous integer solution as the "best" allocation. The incentive-properties of an auction design like this is a topic for future research, as is also more extensive testing of the pricing-algorithm. However, the suggested method seems promising when it comes to providing information supporting decentralized decision making in a market setting.

References

Brewer, P. J. (1999): Decentralized Computation Procurement and Computational Robustness in a Smart Market. *Economic Theory*. 13;41-92.

Caplice, C. G. (1996): *An Optimization Based Bidding Process: A New Framework for Shipper-Carrier Relationships*. Thesis, Department of Civil and Environmental Engineering, School of Engineering, MIT.

de Vries, Sven / Rakesh Vohra (2000): *Combinatorial Auctions: A Survey*. Manuscript.

FCC Auction, Broadband Personal Communication Services, Bidder Information Package (1994): Washington DC.

Hanson, W. / R. K. Martin (1990): Optimal Bundle Pricing. *Management Science*, 36;155-174.

Moore, E. W. / J. M. Warmke / L. R. Gorban (1991): The Indispensable Role of Management Science in Centralizing Freight Operations at Reynolds Metals Company. *Interfaces*, 21;107-129.

Rassenti, S. J. / V. J. Smith / R. L. Bulfin (1982): A Combinatorial Auction Mechanism for Airport Time Slot Allocation. *Bell Journal of Economics*, 13;402-417.