Compound Contingent Claims

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Abstract

This paper explores similarities and differences between a compound option and a two-period guarantee. A generalised *compound contingent claim* that captures these two claims as special cases is constructed. The underlying asset of the compound contingent claim is a generalised *simple contingent claim*. Similar parities as the put-call parity are derived for both these claims. Also several other claims captured by the two general claims are revealed. We also show that the derivation of a closed form solution for the market value of a compound option under stochastic interest rates is likely to be non-trivial, if possible at all.

Keywords and phrases: Compound option, multi-period guarantee, Heath, Jarrow, and Morton term structure model of interest rates.

JEL Classification: C63, G12, G13.

1 Introduction

Many seemingly different assets may in fact be more similar than they first appear. In this paper our main goal is to point out similarities between a compound option and a multi-period guarantee. Once the similarities are pointed out, also some of the differences will be displayed.

Compound options were first analysed by Geske (1977) and Geske (1979). A compound option is an option with another option as the underlying asset. We limit our analysis to a call option written on a call option. The underlying option is assumed written on a stock.

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A multi-period guarantee is an asset that secures that the holder gets the maximum of the return on the underlying asset and some minimum guaranteed rate of return within each period. In this paper we focus, for simplicity, on a two-period guarantee, see e.g., Miltersen and Persson (1999). We assume that the underlying return of the guarantee is the return on the stock in which the call option above is written on. It is straightforward to generalise to a compound option that is written on another compound option and so on. Also, generalising to guarantees lasting for more than two periods is straightforward. However, these generalisations will make the intuition harder to grasp and will not be necessary for our purposes.

To explore the similarities between these two claims, a general compound contingent claim capturing both claims as special cases is constructed. To this end we start by constructing a generalised *simple* contingent claim, i.e., a claim that is written on primary traded assets such as stocks and bonds, not other contingent claims. This asset has the necessary generality to capture both a call option and a maturity guarantee[1] as special cases. To construct the generalised compound contingent claim, we assume that there exists a contingent claim written on the simple contingent claim described above. This asset captures both the compound option and the two-period guarantee as special cases. It puts us in a position where we can easily see similarities between these two claims. It is our hope, since we have not found any connections in the literature between the compound option, which was first analysed in the literature some 25 years ago, and the relatively newly analysed two-period guarantee, that this will shed some new light into these two claims. Our analysis may also give an alternative introduction to the theory of multi-period guarantees for the reader familiar to compound options and vice versa.

Using different specifications for the two claims we construct, we find that the claims also capture several other claims as special cases, not just the call option, the maturity guarantee, the compound option, and the two-period guarantee. Several of these are trivial in the sense that their payoffs do not represent real-world contingent claims and can even be constants. Some of the possible specifications lead to claims where we are not able to derive closed form solutions for the market value. However, based on more or less well-known results relevant for option pricing, we have pointed out for what specifications we have been able to obtain closed form solutions.

An important difference between our framework and that of Geske (1977) and Geske (1979) is that we work under stochastic interest rates. Although this is in principle a trivial extension, it is interesting to notice that a closed form solution for the market value of a compound option as analysed by Geske (1979), i.e., a call option on a standard Black and Scholes call option, is not trivially obtainable, if obtainable at all. This is caused by difficulties

[1] A maturity guarantee is effectively the same as a *one*-period guarantee.

concerning the exercise probability for the compound option.

From the put-call parity we know that there is a close relationship between a call option and a put option. The put option has a "mirror imaged" payoff structure of what the call option has and vice versa. We therefore denote the put option the *mirror claim* for the call option. By defining the mirror claims for the two generalised claims, we show how to derive parities for these claims. This is an issue also addressed in Haug (2002).

We have also picked five specifications of the generalised compound contingent claim and given them a more thorough analysis.

The paper is organised as follows: In section 2 we give a description of our economic model and some preliminaries. In section 3 a short comparison of a call option and a maturity guarantee is given. In section 4 we construct a generalised contingent claim. In section 5 a short comparison of a compound option and a two-period guarantee is given. In section 6 we construct a generalised compound contingent claim that is written on the general contingent claim constructed in section 4. In section 7 some claims that are special cases of the general compound contingent claim are given a thorough analysis. The paper is ended in section 8 with some concluding remarks.

2 The Economic Model and Preliminaries

We assume a continuous trading economy on the time interval $[0, \mathcal{T}]$, for some fixed horizon $\mathcal{T} > 0$, and with no transaction costs. A filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ is fixed, where Ω is the state space, $\mathcal{F}$ is a σ-algebra, $\mathbb{F} = \{\mathcal{F}_t, 0 \leq t \leq \mathcal{T}\}$ is a filtration where $\mathcal{F}_\mathcal{T} = \mathcal{F}$ and $\mathcal{F}_0 = \{\emptyset, \Omega\}$, where $\emptyset$ is the empty set, and P is a probability measure. The σ-algebra is generated by a d-dimensional, $d \geq 1$, Brownian motion, W_t. We further assume a complete market, i.e., there exists one unique equivalent martingale measure Q, see e.g., Harrison and Kreps (1979).

Following the model of Heath, Jarrow, and Morton (1992), the instantaneous continuously compounded forward rate at time s as seen from time t, $t \leq s \leq \mathcal{T}$, under the equivalent martingale measure Q, is given by

$$f(t,s) = f(0,s) + \int_0^t \sigma_f(v,s) \int_v^s \sigma_f(v,u)dudv + \int_0^t \sigma_f(v,s)dW_v,$$

where $\sigma_f(t,s)$ is the volatility function for the instantaneous continuously compounded forward rate at time s as seen from time t, satisfying some technical regularity conditions, see Heath et al. (1992). The short-term interest rate is obtained by setting s equal to t, i.e., $r_t = f(t,t)$. The volatility function is assumed deterministic, implying Gaussian interest rates. Under deterministic interest rates we formally set $\sigma_f(v,u) = 0$. We also assume that there is a continuum of bonds that trade in the market.

We let the market value of the non-dividend paying primary traded securities i, S_t^i, be given under the equivalent martingale measure Q by the equation[2]

$$S_t^i = S_0^i + \int_0^t r_v S_v^i dv + \int_0^t \sigma_{S^i}(v) S_v^i dW_v,$$

where $r_t S_t^i$ satisfies the integrability condition $\int_0^t |r_v S_v^i| dv < \infty$ almost surely for all t. Here $\sigma_{S^i}(t)$ is the volatility function for the return on asset i and satisfies the square integrability condition $E\Big[\int_0^t (\sigma_{S^i}(v) S_v^i)^2 dv\Big] < \infty$ (for further details on integrability conditions, see e.g., Duffie (1996)). Also this volatility function is assumed to be a deterministic function of time. This class of assets will be referred to as stocks. For simplicity, when only one stock is present, we write $S_t^1 = S_t$.

We also assume that there exists an instantaneously risk-free asset, a money market account, that accrues interest according to the short-term interest rate, yielding a time t market value of

$$M_t = M_0 + \int_0^t r_v M_v dv, \qquad M_0 = 1, \tag{1}$$

where $r_t M_t$ satisfies the integrability condition $\int_0^t |r_v M_v| dv < \infty$ almost surely for all t. The return on the money market account, under the equivalent martingale measure Q, over the time period from time T_1 to T_2 is given by (see e.g., Miltersen and Persson (1999))

$$\begin{aligned} \beta_{T_2-T_1} &= \int_{T_1}^{T_2} r_v dv = -\ln F(0,T_1,T_2) + \frac{1}{2}\sigma^2_{\beta_{T_2-T_1}} + c_{T_2-T_1,T_1} \\ &\quad + \int_0^{T_1}\int_{T_1}^{T_2} \sigma_f(v,u) du dW_v + \int_{T_1}^{T_2}\int_v^{T_2} \sigma_f(v,u) du dW_v, \end{aligned}$$

where $F(0,T_1,T_2)$ is the time 0 forward price for delivery at time T_1 of a zero-coupon bond maturing at time T_2 and is given by

$$F(0,T_1,T_2) = \frac{P(0,T_2)}{P(0,T_1)},$$

where $P(0,t)$ is the time zero market value of a zero-coupon bond maturing at time $t \geq 0$. Here $\sigma^2_{\beta_{T_2-T_1}}$ is the variance of the return on the money market account over the time period from time T_1 to T_2 and is given by

$$\sigma^2_{\beta_{T_2-T_1}} = \int_0^{T_1} (\int_{T_1}^{T_2} \sigma_f(v,u) du)^2 dv + \int_{T_1}^{T_2} (\int_v^{T_2} \sigma_f(v,u) du)^2 dv$$

[2]In this paper it is sufficient that $i \in \{1, 2, \ldots, 6\}$.

and $c_{T_2-T_1,T_1}$ is the covariance between the return on the money market account over the time period from time 0 to T_1 and from time T_1 to T_2 and is given by

$$c_{T_2-T_1,T_1} = \int_0^{T_1} \Big(\int_v^{T_1} \sigma_f(v,u)du \Big) \Big(\int_{T_1}^{T_2} \sigma_f(v,u)du \Big) dv.$$

The return on the stock under the equivalent martingale measure Q over the same time interval is given by

$$\delta_{T_2-T_1} = \int_{T_1}^{T_2} (r_v - \frac{1}{2}\sigma_S(v)^2)dv + \int_{T_1}^{T_2} \sigma_S(v)dW_v,$$

with variance

$$\sigma^2_{\delta_{T_2-T_1}} = \sigma^2_{\beta_{T_2-T_1}} + 2\int_{T_1}^{T_2} \sigma_S(v) \int_v^{T_2} \sigma_f(v,u)dudv + \int_{T_1}^{T_2} \sigma^2_S(v)dv. \qquad (2)$$

3 Options and Guarantees

Let us start by considering a standard call option and a maturity guarantee. The terminal time T payoff for the call option is given by $\max(S_T - X, 0)$ for some exercise price $X \in (0, \infty)$, while the terminal payoff for the maturity guarantee is given by $\max(S_T, X)$, or, equivalently, $\max(S_T - X, 0) + X$. As we can see, there is a close relationship between these two claims.

The call option gives the owner the right to receive one unit of the stock by at the same time delivering X units of account, or, since the face value of a zero-coupon bond is equal to one, X units of the face value of a zero-coupon bond. From Merton (1973) we know that the market value of the call option at time $t < T$ is given by

$$\pi_t^c = S_t\Phi(d_1) - P(t,T)X\Phi(d_2), \qquad (3)$$

where

$$d_1 = \frac{\ln(\frac{S_t}{P(t,T)X}) + \frac{1}{2}\sigma^2_{\delta_{T-t}}}{\sigma_{\delta_{T-t}}},$$
$$d_2 = d_1 - \sigma_{\delta_{T-t}},$$

$\Phi(\cdot)$ is the cumulative normal probability distribution, and $\sigma^2_{\delta_{T-t}}$ follows from (2).

First we notice that the option only will be exercised if the condition $S_T > X$ is satisfied. The market value at time t can be interpreted as consisting of two parts; the first, $S_t\Phi(d_1)$, is the time t market value of the stock multiplied by the probability of receiving the stock at time T. This probability is under the equivalent probability measure where the stock price is used as numeraire. The second, $P(t,T)X\Phi(d_2)$, is the time t market value of delivering X units of the face value of a zero-coupon bond multiplied by the probability (under the equivalent probability measure where the bond price, $P(t,T)$, is used as a numeraire, i.e., the forward probability measure, see e.g., Jamshidian (1989)) that the face valued has to be delivered.

Using the symmetry properties of the normal probability distribution, it follows from (3) that the time t market value of the maturity guarantee is given by

$$\pi_t^g = S_t\Phi(d_1) + P(t,T)X\Phi(-d_2).$$

From the above we conclude that the main difference between a call option and a maturity guarantee is that the call option gives the holder the choice between receiving one unit of the stock by delivering X units of the face value of a zero-coupon bond or nothing, while the maturity guarantee gives the holder the right to choose between receiving one unit of the stock or X units of the face value of a zero-coupon bond at no cost. Intuitively, we can think of it as being free to "exercise" the maturity guarantee while it is costly to exercise the call option. However, this is paid for up front since the maturity guarantee has a higher initial market value than the call option.

4 A Generalised Simple Contingent Claim

Let us now construct a generalised contingent claim that captures the two claims analysed above as special cases. We denote this a *simple* contingent claim. By a simple contingent claim we mean a contingent claim that is only a function of primary traded assets such as stocks and bonds, not other contingent claims.

There are many different ways in which such a simple contingent claim can be constructed. We let the final time T payoff be given by

$$g_T = \max(A_T - B_T, C_T). \tag{4}$$

We further let each of A_T, B_T, and C_T be equal to one of the following:

1. zero,
2. a strictly positive constant, or
3. a positive valued random variable.

By a "positive valued random variable" we mean a linear[3] function of the market value of a primary traded asset.

Though the claim in (4) may seem somewhat ad-hoc, it does in fact do the job of describing a call option and a maturity guarantee. To obtain a call option, let $A_T = S_T$, $B_T = X$, and $C_T = 0$, i.e.,

$$g_T = \max(S_T - X, 0).$$

If instead $B_T = 0$ and $C_T = X$ we have that

$$g_T = \max(S_T, X),$$

and the maturity guarantee is obtained as a special case.

In general, the time 0 market value of the simple claim can be calculated in the following way

$$\begin{aligned} g_0 &= E_Q\Big[e^{-\beta_T}\max(A_T - B_T, C_T)\Big] \\ &= A_0 Q_{\mathbf{1}}(\mathcal{A}) - B_0 Q_{\mathbf{2}}(\mathcal{A}) + C_0 Q_{\mathbf{3}}(\bar{\mathcal{A}}), \end{aligned} \tag{5}$$

where $A_0 \equiv E_Q\Big[e^{-\beta_T} A_T\Big]$, $B_0 \equiv E_Q\Big[e^{-\beta_T} B_T\Big]$, and $C_0 \equiv E_Q\Big[e^{-\beta_T} C_T\Big]$.[4] We define $Q_{\mathbf{1}}$, $Q_{\mathbf{2}}$, and $Q_{\mathbf{3}}$ by

$$\begin{aligned} \frac{dQ_{\mathbf{1}}}{dQ} &= \frac{e^{-\beta_T} A_T}{E_Q\Big[e^{-\beta_T} A_T\Big]}, \\ \frac{dQ_{\mathbf{2}}}{dQ} &= \frac{e^{-\beta_T} B_T}{E_Q\Big[e^{-\beta_T} B_T\Big]}, \end{aligned}$$

and

$$\frac{dQ_{\mathbf{3}}}{dQ} = \frac{e^{-\beta_T} C_T}{E_Q\Big[e^{-\beta_T} C_T\Big]}.$$

Here $\mathcal{A} = \{A_T - B_T > C_T\}$ and $\bar{\mathcal{A}}$ is the complement to $\mathcal{A}$.

For a constant A_T we define $Q_{\mathbf{1}} = Q_T$, for B_T constant $Q_{\mathbf{2}} = Q_T$, and finally for C_T constant $Q_{\mathbf{3}} = Q_T$, where Q_T is the forward probability measure. Similarly, we define $\frac{e^{-\beta_T} A_T}{E_Q[e^{-\beta_T} A_T]} \equiv 0$ for $A_T = 0$, $\frac{e^{-\beta_T} B_T}{E_Q[e^{-\beta_T} B_T]} \equiv 0$ for $B_T = 0$, and $\frac{e^{-\beta_T} C_T}{E_Q[e^{-\beta_T} C_T]} \equiv 0$ for $C_T = 0$. As an example, assume that $B_T = 0$. (5) would then be reduced to $A_0 Q_{\mathbf{1}}(\mathcal{A}) + C_0 Q_{\mathbf{3}}(\bar{\mathcal{A}})$.

[3] A linear function is a function on the form $y = ax$ for some non-zero constant a.

[4] Notice that these definitions are only used for notational simplicity and do not necessarily mean that $e^{-\beta_T} A_T$, $e^{-\beta_T} B_T$, or $e^{-\beta_T} C_T$ are Q-martingales. For instance, if A_T is a constant, say, A, it follows trivially that $A_0 \neq A$.

So far we have considered two possible specifications of the claim in (4); a call option and a maturity guarantee. However, also several other claims can be constructed by choosing other specifications. A natural question that then arises is the following: For what specifications of the claim in (4) do there exist a closed form solution for the market value?

The usual definition of a closed form solution is that it is a (deterministic) function that takes its arguments from a set of known parameter values and returns a scalar; the market value. This means that there can be no unknown parameters in the pricing formula such as future stock prices or level of interest rates. All the arguments used at time t have to be $\mathcal{F}_t$-measurable. Even though, in a Gaussian setting, the cumulative normal probability distribution has to be approximated by some numerical integration routine, we follow tradition and also denote an expression for the market value of a claim containing a cumulative normal probability distribution a closed form solution.

In total, it is possible to construct $3^3 = 27$ different combinations for the claim in (4), not all of which are equally interesting. In Table 1 - 3 we have showed the possible specifications. ($A_T = \bar{A}$ means that A_T is a constant and $A_T = \tilde{A}$ that A is a random variable. The same also applies for B_T and C_T, with the obvious change of notation. "*" indicates no obtainable closed form solution.)

The abbreviations in Table 1 - 3 define what the market value of the different specifications of the general claim are equal to. They are defined as follows:

a) = 0.

b) = a constant.

c) = a positive valued random variable.

d) = a call option.

e) = a put option.

f) = an exchange option.

g) = a maturity guarantee.

h) = e) + b).

i) = d) + b).

j) = a spread option + b).

k) = the maximum of two assets.

l) = a spread option + b) − c).

Table 1: Specifications for the simple claim g for $C_T = 0$.

	$A_T = 0$	$A_T = \bar{A}$	$A_T = \tilde{A}$
$B_T = 0$	a) $\max(0, 0)$	b) $\max(\bar{A}, 0)$	c) $\max(\tilde{A}, 0)$
$B_T = \bar{B}$	a) $\max(-\bar{B}, 0)$	a) or b) $\max(\bar{A} - \bar{B}, 0)$	d) $\max(\tilde{A} - \bar{B}, 0)$
$B_T = \tilde{B}$	a) $\max(-\tilde{B}, 0)$	e) $\max(\bar{A} - \tilde{B}, 0)$	f) $\max(\tilde{A} - \tilde{B}, 0)$

Table 2: Specifications for the simple claim g for $C_T = \bar{C}$.

	$A_T = 0$	$A_T = \bar{A}$	$A_T = \tilde{A}$
$B_T = 0$	b) $\max(0, \bar{C})$	b) $\max(\bar{A}, \bar{C})$	f) $\max(\tilde{A}, \bar{C})$
$B_T = \bar{B}$	b) $\max(-\bar{B}, \bar{C})$	b) $\max(\bar{A} - \bar{B}, \bar{C})$	i) $\max(\tilde{A} - \bar{B}, \bar{C})$
$B_T = \tilde{B}$	b) $\max(-\tilde{B}, \bar{C})$	b) or h) $\max(\bar{A} - \tilde{B}, \bar{C})$	j)* $\max(\tilde{A} - \tilde{B}, \bar{C})$

Table 3: Specifications for the simple claim g for $C_T = \tilde{C}$.

	$A_T = 0$	$A_T = \bar{A}$	$A_T = \tilde{A}$
$B_T = 0$	c) $\max(0, \tilde{C})$	g) $\max(\bar{A}, \tilde{C})$	f) $\max(\tilde{A}, \tilde{C})$
$B_T = \bar{B}$	c) $\max(-\bar{B}, \tilde{C})$	c) or g) $\max(\bar{A} - \bar{B}, \tilde{C})$	m)* $\max(\tilde{A} - \bar{B}, \tilde{C})$
$B_T = \tilde{B}$	c) $\max(-\tilde{B}, \tilde{C})$	l)* $\max(\bar{A} - \tilde{B}, \tilde{C})$	n)* $\max(\tilde{A} - \tilde{B}, \tilde{C})$

m) = a spread option + c).

n) = an exchange option to deliver $B_T + C_T$ to receive A_T + c).

If two (or three) of A_T, B_T, and C_T are equal (or are linear functions of the same random variable), the definitions above may not apply because the claim degenerates to another claim. Notice also that the spread option is defined as a call on the spread.

4.1 A Parity for the Simple Contingent Claim

Using the put-call parity, the market value of a call option can be expressed in terms of the market value of a put option, the underlying asset, and the present value of the strike price. In this subsection we find a parity for the simple contingent claim given in (4).

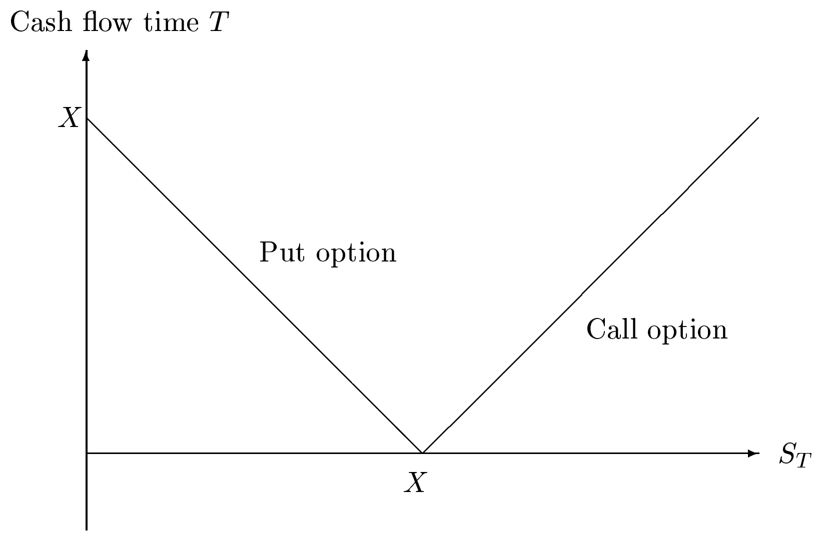

Figure 1: The terminal payoff for a call and a put option with exercise price X.

Consider the call and the put option in Figure 1 with the market value of the underlying asset on the x-axis and the terminal payoff on the y-axis. When the market value of the underlying asset is greater than X, the payoff of the call option is given by a 45°-line. Otherwise, the market value is given by a horizontal line at $y = 0$. Now, consider placing a vertical two-sided mirror at $x = X$. Looking in the mirror from right to left, we see a 45°-line rising away from us, i.e., the payoff of a put option when the market value of the underlying asset is less than X. On the other hand, looking in the mirror from left to right, we see a horizontal line at $y = 0$ going away from us, i.e., the payoff of a put option when the market value of the underlying asset is greater than X. Because the put option has this "mirror imaged" payoff structure of the call option, we will in the following refer to the put option as the *mirror* claim for the call option and vice versa.

Definition 1. *For a claim with terminal payoff*[5] $\max(Z_1, Z_2) = (Z_1 - Z_2)^+ + Z_2$, *we define the mirror claim as the claim with terminal payoff* $-\min(Z_1, Z_2) = -(Z_1 - Z_2)^- - Z_2 = \max(-Z_1, -Z_2)$.

The terminal time T market value of a call option written on a stock with market value S_T is given by $\max(S_T - X, 0)$. From Definition 1 we have that the market value of the corresponding put option is given by $-\min(S_T - X, 0)$. Alternatively, the terminal market value of the put option can be found by changing signs (i.e., by multiplying by minus one) *inside* the max-operator in the expression for the terminal market value of the

[5] Let $(Z)^+ = \max(Z, 0)$ and $(Z)^- = \min(Z, 0)$, for some $Z \in \mathbb{R}$.

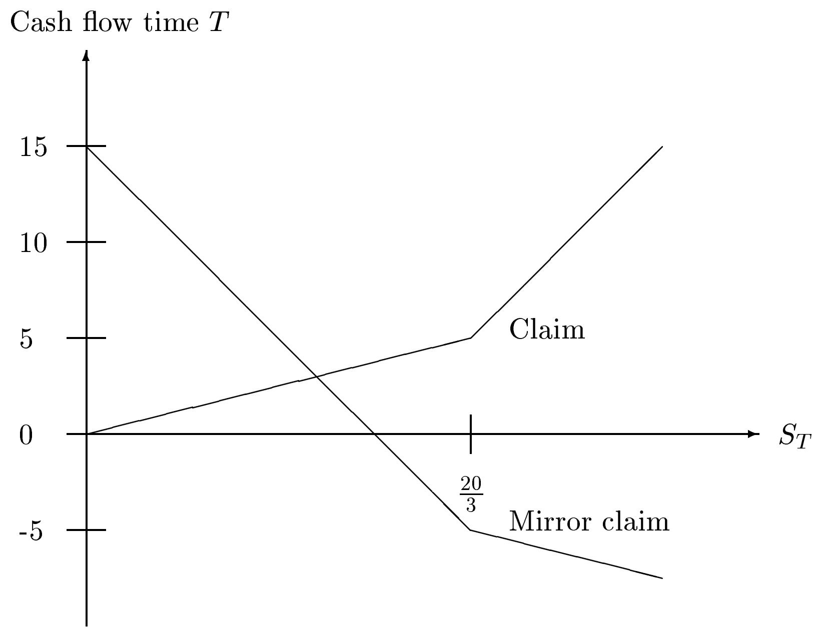

Figure 2: Illustration of the terminal cash flow for the claim $\max(3S_T - 15, 0.75S_T)$ and for the mirror claim $\max(15 - 3S_T, -0.75S_T)$.

call option. This gives the more familiar expression for the terminal market value of the put option, i.e., $\max(-1 \cdot S_T - (-1)X, -1 \cdot 0) = \max(X - S_T, 0)$.

Using Definition 1 on the simple claim, we find that the terminal market value of the mirror claim is given by $g_T^m = \max(B_T - A_T, -C_T)$. This is illustrated in Figure 2 for the simple claim and the mirror claim for $A_T = 3S_T$, $B_T = 15$, and $C_T = 0.75S_T$.

Let g_t and g_t^m be the time t market value of the simple claim and the mirror claim, respectively. Further, define $A_t \equiv E_Q\left[e^{-\int_t^T r_v dv} A_T\right]$, $B_t \equiv E_Q\left[e^{-\int_t^T r_v dv} B_T\right]$, and $C_t \equiv E_Q\left[e^{-\int_t^T r_v dv} C_T\right]$.

Theorem 1. *For the simple contingent claim, we have the following parity*

$$g_t = g_t^m + A_t - B_t + C_t.$$

Proof. In the absence of arbitrage, this follows since both the left and the right-hand side of the parity have the same terminal payoff. □

Another way to justify this interpretation of the mirror claim is the following rewriting (using the terminal market values)

$$\begin{aligned} g_T &= \max(A_T - B_T, C_T) \\ &= \max(A_T - B_T - C_T, 0) + C_T \\ &= \max(B_T - A_T, -C_T) + A_T - B_T + C_T. \end{aligned}$$

5 Compound Option and Two-period Guarantee

Let us now consider two somewhat more complicated claims. First we consider a compound option (see e.g., Geske (1979)), i.e., a call option with another call option as the underlying asset. We assume that the compound option can be exercised at time T_1 at a cost of X_1 and that the underlying option is written on a stock and can be exercised at time $T_2 \geq T_1$ at a cost of X_2. Let π_t^1 be the time $t \leq T_1$ market value of the compound option. We then have that

$$\pi_{T_1}^1(\pi^c) = \max(\pi_{T_1}^c - X_1, 0),$$

where π^c is the underlying call option with time T_1 market value $\pi_{T_1}^c$. Thus, the compound option can be interpreted in the same way as the call option; it gives the holder the right to acquire one unit of the underlying asset by delivering X_1 units of the face value of a zero-coupon bond.

A two-period guarantee secures that the holder receives the maximum of the return on some underlying asset and some minimum guaranteed rate of return in each of the two periods. Assume that the minimum guaranteed rate of return in period i, $i \in \{1, 2\}$, is given by g_i. If the guarantee is written on the return on the stock, the terminal payoff is given by

$$\pi_{T_2}^{mg} = \max(\frac{S_{T_1}}{S_0}, e^{g_1}) \cdot \max(\frac{S_{T_2}}{S_{T_1}}, e^{g_2}).$$

The expression $\max(\frac{S_{T_2}}{S_{T_1}}, e^{g_2})$ is the same payoff as that of a maturity guarantee over the time period from time T_1 to T_2 and where the initial amount to accrue interest is normalised to one. The time T_1 market value of the two-period guarantee is therefore equal to

$$\pi_{T_1}^{mg}(\pi^g) = \max(\frac{S_{T_1}}{S_0}, e^{g_1}) \cdot \pi_{T_1}^g,$$

where π^g is the maturity guarantee and $\pi_{T_1}^g$ is the time T_1 market value of the maturity guarantee.

The interpretation of the two-period guarantee is somewhat different than the interpretation of the maturity guarantee. The two-period guarantee gives the holder the opportunity to choose between two different quantities (one of them $\mathcal{F}_{T_1}$-measurable) of the underlying asset (i.e., the maturity guarantee), whereas the maturity guarantee gave the holder the choice between one unit of the underlying asset and X. This choice can be made at time T_1 at no cost. Comparing this to the compound option, we see that also the holder of the compound option can choose between two different quantities of the underlying asset (i.e., the call option); one or zero units, and if the holder chooses to receive one unit, it comes at a cost.

If we instead think of the maturity guarantee as offering the holder the choice between a stochastic ($\frac{S_T}{S_0}$) and a deterministic (e^g) number of units of account, where one unit of account is equal to 1, the two-period guarantee is almost identical to the maturity guarantee. The main difference is that for the two-period guarantee one unit of account is equal to $\pi^g_{T_1}$.

The above shows that also the two-period guarantee can be interpreted as a compound contingent claim, just as the compound option can. This feature does not seem to have been recognised in the existing literature on multi-period guarantees. In the next section we construct a generalised compound contingent claim that captures these two claims as special cases.

6 A Generalised Compound Contingent Claim

We will now, as for the simple contingent claim in section 4, construct a generalised *compound contingent claim* that captures the compound option and the two-period guarantee as special cases. By a compound contingent claim we mean a contingent claim that is written on some other contingent claim. In fact, we let the simple contingent claim in section 4 be the underlying asset.

Consider now a claim with the following time T_1 market value

$$f_{T_1}(g) = \max(\alpha g_{T_1} - K, \gamma g_{T_1}), \tag{6}$$

where each of α, γ, and K is equal to either zero, a strictly positive constant, or a positive valued random variable (i.e., the same possibilities as for A_T, B_T, and C_T in section 4). Again, the claim is somewhat ad-hoc; though it has the necessary generality to capture the compound call option and the two-period guarantee as special cases. To show this, let $\alpha = 1$, $K = X_1$, $\gamma = 0$, and $g_{T_1} = \pi^c_{T_1}$. This gives

$$f_{T_1}(g) = \max(\pi^c_{T_1} - X_1, 0),$$

and is equal to the time T_1 market value of a compound call option. If instead $\alpha = \frac{S_{T_1}}{S_0}$, $K = 0$, $\gamma = e^{g_1}$, and $g_{T_1} = \pi^g_{T_1}$, we get

$$f_{T_1}(g) = \max(\frac{S_{T_1}}{S_0}\pi^g_{T_1}, e^{g_1}\pi^g_{T_1}) = \max(\frac{S_{T_1}}{S_0}, e^{g_1}) \cdot \pi^g_{T_1},$$

and this is equal to the time T_1 market value of a two-period guarantee.

Using the results in section 4, changing the maturity date for the simple claim from time T to T_2, and valuing the claim at time T_1 instead of at time 0, the market value can be written as

$$g_{T_1} = A_{T_1} E_{Q_1}\left[1_{\mathcal{A}_2}\middle|\mathcal{F}_{T_1}\right] - B_{T_1} E_{Q_2}\left[1_{\mathcal{A}_2}\middle|\mathcal{F}_{T_1}\right] + C_{T_1} E_{Q_3}\left[1_{\bar{\mathcal{A}}_2}\middle|\mathcal{F}_{T_1}\right], \tag{7}$$

where $\mathcal{A}_2 = \mathcal{A}$ and $\bar{\mathcal{A}}_2$ is the complement to $\mathcal{A}_2$. The time 0 market value of the compound contingent claim can be written as

$$f_0(g) = E_Q\Big[e^{-\beta_{T_1}} \max(\alpha g_{T_1} - K, \gamma g_{T_1})\Big].$$

Define

$$\begin{aligned}
\alpha A_0 &\equiv E_Q\Big[e^{-\beta_{T_1}} \alpha A_{T_1}\Big], \\
\alpha B_0 &\equiv E_Q\Big[e^{-\beta_{T_1}} \alpha B_{T_1}\Big], \\
\alpha C_0 &\equiv E_Q\Big[e^{-\beta_{T_1}} \alpha C_{T_1}\Big], \\
K_0 &\equiv E_Q\Big[e^{-\beta_{T_1}} K\Big], \\
\gamma A_0 &\equiv E_Q\Big[e^{-\beta_{T_1}} \gamma A_{T_1}\Big], \\
\gamma B_0 &\equiv E_Q\Big[e^{-\beta_{T_1}} \gamma B_{T_1}\Big],
\end{aligned}$$

and

$$\gamma C_0 \equiv E_Q\Big[e^{-\beta_{T_1}} \gamma C_{T_1}\Big].$$

Define further the following Radon-Nikodym derivatives

$$\begin{aligned}
\frac{dQ_{\mathbf{4}}}{dQ} &= \frac{e^{-\beta_{T_1}} \alpha A_{T_1}}{E_Q\Big[e^{-\beta_{T_1}} \alpha A_{T_1}\Big]}, \\
\frac{dQ_{\mathbf{5}}}{dQ} &= \frac{e^{-\beta_{T_1}} \alpha B_{T_1}}{E_Q\Big[e^{-\beta_{T_1}} \alpha B_{T_1}\Big]}, \\
\frac{dQ_{\mathbf{6}}}{dQ} &= \frac{e^{-\beta_{T_1}} \alpha C_{T_1}}{E_Q\Big[e^{-\beta_{T_1}} \alpha C_{T_1}\Big]}, \\
\frac{dQ_{\mathbf{7}}}{dQ} &= \frac{e^{-\beta_{T_1}} K}{E_Q\Big[e^{-\beta_{T_1}} K\Big]}, \\
\frac{dQ_{\mathbf{8}}}{dQ} &= \frac{e^{-\beta_{T_1}} \gamma A_{T_1}}{E_Q\Big[e^{-\beta_{T_1}} \gamma A_{T_1}\Big]}, \\
\frac{dQ_{\mathbf{9}}}{dQ} &= \frac{e^{-\beta_{T_1}} \gamma B_{T_1}}{E_Q\Big[e^{-\beta_{T_1}} \gamma B_{T_1}\Big]},
\end{aligned}$$

and

$$\frac{dQ_{\mathbf{10}}}{dQ} = \frac{e^{-\beta_{T_1}}\gamma C_{T_1}}{E_Q\left[e^{-\beta_{T_1}}\gamma C_{T_1}\right]}.$$

In any of the cases where the denominator in the expressions for the Radon-Nikodym derivatives equals zero, we define, as in section 4, the Radon-Nikodym derivative to be equal to zero.

Combining the above, the time zero market value of the compound contingent claim can be written as

$$\begin{aligned} f_0(g) &= \alpha A_0 Q_{\mathbf{4}}(\mathcal{A}_1 \cap \mathcal{A}_2) - \alpha B_0 Q_{\mathbf{5}}(\mathcal{A}_1 \cap \mathcal{A}_2) + \alpha C_0 Q_{\mathbf{6}}(\mathcal{A}_1 \cap \bar{\mathcal{A}}_2) \\ &\quad - K_0 Q_{\mathbf{7}}(\mathcal{A}_1) + \gamma A_0 Q_{\mathbf{8}}(\bar{\mathcal{A}}_1 \cap \mathcal{A}_2) - \gamma B_0 Q_{\mathbf{9}}(\bar{\mathcal{A}}_1 \cap \mathcal{A}_2) \\ &\quad + \gamma C_0 Q_{\mathbf{10}}(\bar{\mathcal{A}}_1 \cap \bar{\mathcal{A}}_2), \end{aligned} \tag{8}$$

where $\mathcal{A}_1 = \{\alpha g_{T_1} - K \geq \gamma g_{T_1}\}$ and $\bar{\mathcal{A}}_1$ is the complement to $\mathcal{A}_1$.

To determine the market value of the compound contingent claim we need to be able to determine the exercise probabilities, under the appropriate probability measures, for the claim under consideration. This is the same as saying that we need to determine for what values of the underlying asset(s) the claim will be exercised. For the compound contingent claim this means that we must be able to determine for what value(s) of the underlying asset(s) the following inequality holds with equality

$$\alpha g_{T_1} - K \geq \gamma g_{T_1}. \tag{9}$$

We know from the discussion on page 8 that we must be able to determine when (9) holds with equality based on the information available at time zero.

As a first example, consider the compound option analysed by Geske (1979), i.e., a call option on a call option under deterministic interest rates. (9) then becomes (where d_1 and d_2 are "adjusted" to time T_1)

$$S_{T_1}\Phi(d_1) - P(T_1, T_2)X_2\Phi(d_2) \geq X_1, \tag{10}$$

where the left-hand side of the inequality in (10) now is the time T_1 market value of a call option maturing at time $T_2 > T_1$. Since the call option is strictly increasing in the market value of the underlying stock, it follows by the intermediate value property[6] that there exists a stock price s^* that makes (10) hold with equality for all $X_1 \in (0, \infty)$, and the probabilities for the compound option being exercised can then be calculated.

Consider now the setting in this paper, i.e., stochastic interest rates. Then there is no longer one unique s^* for each X_1, but several, each as a

[6]See e.g., Rudin (1976) Theorem 4.23.

function of the $\mathcal{F}_{T_1}$-measurable random variable $P(T_1, T_2)$. This complicates matters quite considerably since there does not seem to exist any trivial relationship between the stock price and the bond price that can be used to determine the exercise probabilities for the compound option. Hence, a closed form solution for the market value of a compound option in a stochastic interest rate framework does not seem to be easily obtainable. Searching the literature, the only work on compound options and stochastic interest rates that we have found is in Geman, El Karoui, and Rochet (1995), but their analysis seems flawed in that they assume that there exists a unique $\mathcal{F}_0$-measurable s^*.

If the holder of the compound option instead of delivering X_1 units of the face value of a zero-coupon bond for exercising it at time T_1 could deliver X_1 units of the zero-coupon bond maturing at time T_2, i.e., $X_1 P(T_1, T_2)$, (10) could be simplified to (the only difference is that the maturity date for the bond delivered is changed from time T_1 to T_2)

$$R_{T_1}\Phi(d_1) - X_2\Phi(d_2) \geq X_1, \tag{11}$$

where $R_{T_1} = \frac{S_{T_1}}{P(T_1,T_2)}$ can be interpreted as the market value of the underlying asset of a call option with zero interest rates (see e.g., Carr (1988)). Again using the fact that a call option is strictly increasing in the market value of the underlying asset, it follows that there exists a unique R^* that makes (11) hold with equality. Hence, the probabilities for the compound option being exercised can then be calculated.

It seems like if the rewriting above (and similar ones) is possible, it will also be *sufficient* for the derivation of a closed form solution, i.e., the rewriting that makes it possible to calculate the exercise probabilities for the compound option. However, since we have not tried every possible approach, we cannot claim that it is *necessary* to be able to perform such a rewriting for there to exist a closed form solution.

For what specifications of the compound contingent claim do there exist a closed form solution? First, for $g \in \{$a),b),c)$\}$ (see Table 1 - 3) the claim $f(g)$ is not a compound contingent claim, but at best a contingent claim, and we will therefore not give any attention to these specifications in this section. Since there does not exist a closed form solution for the simple claim when $g \in \{$j),l),m),n)$\}$, we will not be able determine when (9) holds with equality, hence, we are not able to find a closed form solution for the market value of the compound contingent claim. It turns out that $g \in \{$d),e),f),g),h),i),k)$\}$ are quite similar.

When the simple claim falls into the categories d), e), and g), the time T_1 market value can be written on the form

$$g_{T_1} = \pm \mathcal{S}_{T_1}\Phi(\varphi_1) \pm P(T_1, T_2)X\Phi(\varphi_2),$$

for the categories f) and k)

$$g_{T_1} = \pm \mathcal{S}_{T_1} \Phi(\varphi_1) \pm \mathcal{V}_{T_1} \Phi(\varphi_2),$$

and, finally, for the categories h) and i) as

$$g_{T_1} = \pm \mathcal{S}_{T_1} \Phi(\varphi_1) \pm P(T_1, T_2) X \Phi(\varphi_2) + P(T_1, T_2) \bar{\mathcal{K}}.$$

From now on we define $\mathcal{S}_{T_1}$ as the market value of the *first* asset and $P(T_1, T_2)$ and $\mathcal{V}_{T_1}$ as the market value of the *second* asset. Here $\bar{\mathcal{K}}$ is a constant. φ_1 and φ_2 will typically not be the same across the different specifications, but it will not be necessary to specify them any closer here. Using the definitions and descriptions below, we have in Table 4 - 6 showed for what specifications of the compound contingent claim in (6) the market value can be obtained in closed form solution.

The abbreviations in Table 4 - 6 are defined as follows:

a') = 0.

b') = a constant number of g.

c') = a constant number of call options on g.

d') = exchange K to receive a constant number of g - solvable if K is a function of the second asset.

e') = a random number of g.

f') = a call option on a random number of g.

g') = exchange K to receive a random number of g. Solvable if α is a function of the first asset and K is a function of the second asset.

h') = $\alpha > \gamma \Rightarrow$ a given number of the Geske (1979)-option + b'), otherwise b').

i') = $\alpha \leq \gamma \Rightarrow$ b'), $\alpha > \gamma \Rightarrow$ d') + b') if K a function of the second asset, otherwise not solvable.

j') = the maximum of a random and a constant number of g.

k') = the maximum of a random number of g subtracted a constant and a constant number of g.

l') = the maximum of a random number of g subtracted a random variable and a constant number of g.

m') = the maximum of a constant number of g subtracted a constant and a random number of g.

Table 4: Specifications for the claim $f(g)$ for $\gamma = 0$.

	$\alpha = 0$	$\alpha = \bar{\alpha}$	$\alpha = \tilde{\alpha}$
$K = 0$	a') $\max(0, 0)$	b') $\max(\bar{\alpha} g_{T_1}, 0)$	e') $\max(\tilde{\alpha} g_{T_1}, 0)$
$K = \bar{K}$	a') $\max(-\bar{K}, 0)$	c')* $\max(\bar{\alpha} g_{T_1} - \bar{K}, 0)$	f')* $\max(\tilde{\alpha} g_{T_1} - \bar{K}, 0)$
$K = \tilde{K}$	a') $\max(-\tilde{K}, 0)$	d') $\max(\bar{\alpha} g_{T_1} - \tilde{K}, 0)$	g') $\max(\tilde{\alpha} g_{T_1} - \tilde{K}, 0)$

Table 5: Specifications for the claim $f(g)$ for $\gamma = \bar{\gamma}$.

	$\alpha = 0$	$\alpha = \bar{\alpha}$	$\alpha = \tilde{\alpha}$
$K = 0$	b') $\max(0, \bar{\gamma} g_{T_1})$	b') $\max(\bar{\alpha} g_{T_1}, \bar{\gamma} g_{T_1})$	j') $\max(\tilde{\alpha} g_{T_1}, \bar{\gamma} g_{T_1})$
$K = \bar{K}$	b') $\max(-\bar{K}, \bar{\gamma} g_{T_1})$	h') $\max(\bar{\alpha} g_{T_1} - \bar{K}, \bar{\gamma} g_{T_1})$	k')* $\max(\tilde{\alpha} g_{T_1} - \bar{K}, \bar{\gamma} g_{T_1})$
$K = \tilde{K}$	b') $\max(-\tilde{K}, \bar{\gamma} g_{T_1})$	i') $\max(\bar{\alpha} g_{T_1} - \tilde{K}, \bar{\gamma} g_{T_1})$	l')* $\max(\tilde{\alpha} g_{T_1} - \tilde{K}, \bar{\gamma} g_{T_1})$

Table 6: Specifications for the claim $f(g)$ for $\gamma = \tilde{\gamma}$.

	$\alpha = 0$	$\alpha = \bar{\alpha}$	$\alpha = \tilde{\alpha}$
$K = 0$	e') $\max(0, \tilde{\gamma} g_{T_1})$	j') $\max(\bar{\alpha} g_{T_1}, \tilde{\gamma} g_{T_1})$	o') $\max(\tilde{\alpha} g_{T_1}, \tilde{\gamma} g_{T_1})$
$K = \bar{K}$	e') $\max(-\bar{K}, \tilde{\gamma} g_{T_1})$	m')* $\max(\bar{\alpha} g_{T_1} - \bar{K}, \tilde{\gamma} g_{T_1})$	p')* $\max(\tilde{\alpha} g_{T_1} - \bar{K}, \tilde{\gamma} g_{T_1})$
$K = \tilde{K}$	e') $\max(-\tilde{K}, \tilde{\gamma} g_{T_1})$	n')* $\max(\bar{\alpha} g_{T_1} - \tilde{K}, \tilde{\gamma} g_{T_1})$	q')* $\max(\tilde{\alpha} g_{T_1} - \tilde{K}, \tilde{\gamma} g_{T_1})$

n') = the maximum of a constant number of g subtracted a random variable and a random number of g.

o') = the maximum of two random numbers of g.

p') = the maximum of a random number of g subtracted a constant and a random number of g.

q') = the maximum of a random number of g subtracted a random variable and a random number of g.

As in section 4, the above may not apply if two or more of the variables coincide or are linear functions of the market value of the same asset.

6.1 A Parity for the Compound Contingent Claim

We will in this subsection derive a parity for the compound contingent claim.

The mirror claim for the compound contingent claim has the following time T_1 market value

$$f_{T_1}^m(g) = \max(K - \alpha g_{T_1}, -\gamma g_{T_1}).$$

We now define the following for $t < T_1$, $K_t \equiv E_Q\left[e^{-\int_t^{T_1} r_v dv} K\right]$, $\alpha g_t \equiv E_Q\left[e^{-\int_t^{T_1} r_v dv} \alpha g_{T_1}\right]$, and $\gamma g_t \equiv E_Q\left[e^{-\int_t^{T_1} r_v dv} \gamma g_{T_1}\right]$.

Theorem 2. *For the compound contingent claim, we have the following parity for* $t \leq T_1$

$$f_t(g) = f_t^m(g) + \alpha g_t - K_t + \gamma g_t.$$

Proof. The left and the right-hand side of the parity have the same time T_1 market value, and the result follows therefore in the absence of arbitrage. □

7 Other Claims Captured by (6)

In this section we give a closer analysis of some of the claims captured by the general claim in (6). The market values are found using the general formula in (8). In the proofs we have for simplicity only taken into account the terms in (8) that are non-zero.

7.1 A Compound Exchange Option

An exchange option seems first to have been analysed by Fischer (1978) and Margrabe (1978). This is a contingent claim that gives the holder the option to exchange a given number of units of one assets in return for one unit of another asset, say, deliver X units of an asset with market value S_T^2 to receive one unit of an asset with market value S_T^1. Carr (1988) analysed a compound exchange option, i.e., an option to exchange a given number of units of an asset to receive one unit of an exchange option.

Consider the following specification of (6): $A_{T_2} = S_{T_2}^1$, $B_{T_2} = X_2 S_{T_2}^2$, $C_{T_2} = 0$, $\alpha = 1$, $K = X_1 S_{T_1}^2$, and $\gamma = 0$. This gives the same payoff as the compound exchange option.

Proposition 1. (Carr (1988)) *The time 0 market value of an exchange option on an exchange option is given by*

$$\begin{aligned} f_0(g) &= S_0^1 \Phi(d_3, d_4, \rho) - X_2 S_0^2 \Phi(d_3 - v(T_1), d_4 - v(T_2), \rho) \\ &\quad - X_1 S_0^2 \Phi(d_3 - v(T_1)), \end{aligned}$$

where

$$d_3 = \frac{\ln(\frac{R_0}{R^*}) + \frac{1}{2}v^2(T_1)}{v(T_1)},$$

$$d_4 = \frac{\ln(\frac{S_0^1}{X_2 S_0^2}) + \frac{1}{2}v^2(T_2)}{v(T_2)},$$

$$\rho = \frac{v(T_1)}{v(T_2)},$$

$$R_0 = \frac{S_0^1}{S_0^2},$$

$$v^2(T_i) = \int_0^{T_i} \Big(\sigma_{S^1}^2(v) - 2\sigma_{S^1}(v)\sigma_{S^2}(v) + \sigma_{S^2}^2(v)\Big)\, dv,$$

$\Phi(a, b, p)$ is the cumulative bivariate normal probability distribution evaluated at the points a and b with correlation p, and R^* is the critical ratio of $\frac{S_{T_1}^1}{S_{T_1}^2}$ that makes the time T_1 market value of the underlying exchange option equal to $X_1 S_{T_1}^2$.

Proof. The market value can be found using (8). For the compound exchange option it follows that $\alpha A_0 = S_0^1$, $\alpha B_0 = X_2 S_0^2$, and $K_0 = X_1 S_0^2$. The three probability measures $Q_{\mathbf{4}}$, $Q_{\mathbf{5}}$, and $Q_{\mathbf{7}}$ are defined by the Radon-Nikodym derivatives

$$\frac{dQ_{\mathbf{4}}}{dQ} = e^{-\frac{1}{2}\int_0^t \sigma_{S^1}^2(v)dv + \int_0^t \sigma_{S^1}(v)dW_v}$$

and

$$\frac{dQ_{\mathbf{5}}}{dQ} = \frac{dQ_{\mathbf{7}}}{dQ} = e^{-\frac{1}{2}\int_0^t \sigma_{S^2}^2(v)dv + \int_0^t \sigma_{S^2}(v)dW_v}.$$

From this we get that

$$f_0(g) = S_0^1 Q_{\mathbf{4}}(\mathcal{A}_1 \cap \mathcal{A}_2) - X_2 S_0^2 Q_{\mathbf{5}}(\mathcal{A}_1 \cap \mathcal{A}_2) - X_1 S_0^2 Q_{\mathbf{7}}(\mathcal{A}_1),$$

where $\mathcal{A}_1 = \{\pi_{T_1}^{eo} > X_1 S_{T_1}^2\}$, $\mathcal{A}_2 = \{S_{T_2} > X_2 S_{T_2}^2\}$, and $\pi_{T_1}^{eo}$ is the time T_1 market value of the underlying exchange option. The result then follows. □

It is interesting to notice that the result in Proposition 1 that is derived under stochastic interest rates is (if $\sigma_{S^i}(v)$ is time independent) identical to the result in Carr (1988) where the result is derived under deterministic interest rates. This is in line with the comment in Carr (1988) that "... there is no presumption that the term structure of interest rates be flat or even known."

Carr (1988) analysed several claims that can be shown to be special cases of his formula and different interpretations of the compound exchange option. All these claims and interpretations are of course also captured by the claim in (6).

7.2 An Option on a Maturity Guarantee

Another version of a compound contingent claim is the following (this is, to the best of our knowledge, a claim that has not previously been analysed). Assume that one at time T_1 has the right to exchange X_1 units of a zero-coupon bond maturing at time T_2 for one unit of a maturity guarantee maturing at time T_2. The compound contingent claim in (6) and this claim are seen to coincide when using the following specification: $A_{T_2} = S_{T_2}$, $B_{T_2} = 0$, $C_{T_2} = X_2$, $\alpha = 1$, $K = X_1 P(T_1, T_2)$, and $\gamma = 0$.

Proposition 2. *The time 0 market value of an option to exchange X_1 units of a zero-coupon bond maturing at time T_2 for one unit of a maturity guarantee maturing at time T_2 is given by*

$$\begin{aligned} f_0(g) &= S_0 \Phi(d_5, d_6, \rho) + X_2 P(0, T_2) \Phi(d_5 - \sigma_{R_{T_1}}, -d_6 + \sigma_{\delta_{T_2}}, -\rho) \\ &\quad - X_1 P(0, T_2) \Phi(d_5 - \sigma_{R_{T_1}}), \end{aligned}$$

where

$$\begin{aligned} d_5 &= \frac{\ln(\frac{R_0}{R^*}) + \frac{1}{2}\sigma^2_{R_{T_1}}}{\sigma_{R_{T_1}}}, \\ d_6 &= \frac{\ln(\frac{S_0}{X_2 P(0,T_2)}) + \frac{1}{2}\sigma^2_{\delta_{T_2}}}{\sigma_{\delta_{T_2}}}, \\ \sigma^2_{R_{T_1}} &= \int_0^{T_1} (\int_v^{T_2} \sigma_f(v,u) du)^2 dv + 2 \int_0^{T_1} \sigma_S(v) \int_v^{T_2} \sigma_f(v,u) du dv \\ &\quad + \int_0^{T_1} \sigma^2_S(v) dv, \\ \rho &= \frac{\operatorname{cov}(\ln(R_{T_1}), \delta_{T_2})}{\sigma_{R_{T_1}} \sigma_{\delta_{T_2}}} = \frac{\sigma_{R_{T_1}}}{\sigma_{\delta_{T_2}}}, \end{aligned}$$

$R_0 = \frac{S_0}{P(0,T_2)}$, and R^* is the critical ratio $\frac{S_{T_1}}{P(T_1,T_2)}$ that makes the time T_1 market value of the maturity guarantee equal to $X_1 P(T_1, T_2)$.

Proof. The time 0 market value can be found using (8). For the exchange option on the maturity guarantee it follows that $\alpha A_0 = S_0$, $\alpha C_0 = X_2 P(0, T_2)$, and $K_0 = X_1 P(0, T_2)$. The probability measures $Q_{\mathbf{4}}$, $Q_{\mathbf{6}}$, and $Q_{\mathbf{7}}$ are defined by the Radon-Nikodym derivatives

$$\frac{dQ_{\mathbf{4}}}{dQ} = e^{-\frac{1}{2}\int_0^t \sigma^2_S(v) dv + \int_0^t \sigma_S(v) dW_v}$$

and

$$\frac{dQ_{\mathbf{6}}}{dQ} = \frac{dQ_{\mathbf{7}}}{dQ} = e^{-\frac{1}{2}\int_0^t (\int_v^{T_2} \sigma_f(v,u) du)^2 dv - \int_0^t \int_v^{T_2} \sigma_f(v,u) du dW_v},$$

respectively. It then follows that

$$f_0(g) \quad = \quad S_0 Q_{\mathbf{4}}(1_{\mathcal{A}_1 \cap \mathcal{A}_2}) + X_2 P(0, T_2) Q_{\mathbf{6}}(1_{\mathcal{A}_1 \cap \bar{\mathcal{A}}_2}) - X_1 P(0, T_2) Q_{\mathbf{7}}(1_{\mathcal{A}_1}),$$

where $\mathcal{A}_1 = \{R_{T_1} > R^*\}$, $\mathcal{A}_2 = \{S_{T_2} > X_2\}$, and $\bar{\mathcal{A}}_2$ is the complement to $\mathcal{A}_2$.

Consider now the inequality (where d_1 and d_2 are "adjusted" to time T_1)

$$S_{T_1}\Phi(d_1) + X_2 P(T_1, T_2)\Phi(-d_2) \geq X_1 P(T_1, T_2). \tag{12}$$

The left-hand side of (12) is the time T_1 market value of the underlying maturity guarantee and the right-hand side is the time T_1 exercise price for the compound contingent claim. Dividing through by $P(T_1, T_2)$, we get

$$R_{T_1}\Phi(d_1) + X_2\Phi(d_2) \geq X_1. \tag{13}$$

That there exists an R^* that makes (13) hold with equality follows since the left-hand side of (13) can be thought of as the time T_1 market value of a maturity guarantee with $R_{T_1} = \frac{S_{T_1}}{P(T_1,T_2)}$ being the market value of the underlying asset and with zero interest rates. The market value of this claim is strictly increasing in R_{T_1} and there does therefore exist a solution to (13), i.e., a parameter R^*.

The result then follows. □

7.3 Instantaneous Compound Contingent Claims

We now analyse a type of contingent claims that we have not found previously to been treated as compound contingent claims. For the assets we have in mind here, the two exercise dates, T_1 and T_2, coincide and are termed T. These claims do not exactly fit into our general claims. However, replacing the max-operator in the expression for the simple claim by a min-operator, things work out fine.

Consider first a capped call option, i.e., a contingent claim that gives the final time T payoff

$$\begin{aligned} f_T(g) &= \max(\min(S_T, X_2) - X_1, 0) \\ &= \max(-\max(-S_T, -X_2) - X_1, 0), \end{aligned} \tag{14}$$

where we assume that $X_2 \geq X_1 > 0$. The expression in (14) can be rewritten as

$$\begin{aligned} f_T(g) &= \max(S_T - X_1, \max(S_T - X_2, 0)) - \max(S_T - X_2, 0) \\ &= \max(S_T - X_1, 0) - \max(S_T - X_2, 0), \end{aligned}$$

since $X_2 \geq X_1$. This is the difference between two call options, and from section 4 we know that the market value is easily obtainable in closed form solution (corresponds to the case denoted d) in Table 1 - 3).

This compound contingent claim can be obtained as a special case of (6) by using the following specification: $A_T = S_T$, $B_T = 0$, $C_T = X_2$, $\alpha = 1$, $K = X_1$, and $\gamma = 0$.

Another compound contingent claim, though somewhat similar as the one in (14), is a call option on the minimum of two assets and has been analysed by Stulz (1982) and Johnson (1987). This claim has the terminal payoff

$$f_T(g) = \max(\min(S_T^1, S_T^2) - X, 0).$$

The specification for the claim in (6) that corresponds to a call option on the minimum of two assets is as follows: $A_T = S_T^1$, $B_T = 0$, $C_T = S_T^2$, $\alpha = 1$, $K = X$, and $\gamma = 0$.

7.4 A Random Number of Call Options

We end this section by considering a claim that is captured by the general claim in (6) but that is not a compound contingent claim. Assume that we at time T_1 will receive a random number of call options, more precisely S_{T_1} units. This could for instance be some sort of a bonus mechanism for the employees. Instead of using more traditional stock options as an incentive, we could strengthen the incentive by also making the number of call options depend on the development in the stock price. This is a sort of a *quanto* option, see e.g., Reiner (1992).

This claim is obtained by the following specification: $A_{T_2} = S_{T_2}$, $B_{T_2} = X$, $C_{T_2} = 0$, $\alpha = S_{T_1}$, $K = 0$, and $\gamma = 0$. What is the value of such a claim?

Proposition 3. *The time 0 market value of the claim with time T_2 payoff $S_{T_1}\max(S_{T_2} - X, 0)$ is given by*

$$f_0(g) = \frac{(S_0)^2}{P(0,T_1)} e^{\sigma_{\delta_{T_1}}} \Phi(d_7) - S_0 F(0,T_1,T_2) X e^{-\operatorname{cov}(\delta_{T_2-T_1}, \delta_{T_1})} \Phi(d_8),$$

where

$$d_7 = \frac{\ln(\frac{S_0}{XP(0,T_2)}) + \frac{1}{2}\sigma^2_{\delta_{T_2}} + \sigma_{\delta_{T_1}} + \operatorname{cov}(\delta_{T_2-T_1}, \delta_{T_1})}{\sigma_{\delta_{T_2}}},$$

$$d_8 = \frac{\ln(\frac{S_0}{XP(0,T_2)}) + \frac{1}{2}\sigma^2_{\delta_{T_2}} - \sigma_{\delta_{T_2}-\delta_{T_1}} - \operatorname{cov}(\delta_{T_2-T_1}, \delta_{T_1})}{\sigma_{\delta_{T_2}}},$$

and

$$\operatorname{cov}(\delta_{T_2-T_1}, \delta_{T_1}) = c_{T_2-T_1,T_1} + \int_0^{T_1} \sigma_S(v) \int_{T_1}^{T_2} \sigma_f(v,u)\,du\,dv.$$

Proof. From (8) it follows that

$$\alpha A_0 = \frac{(S_0)^2}{P(0,T_1)} e^{\sigma_{\delta_{T_1}}}$$

and

$$\alpha B_0 = S_0 F(0,T_1,T_2) X e^{-\operatorname{cov}(\delta_{T_2-T_1},\delta_{T_1})}.$$

The exercise set for this claim is given by $\mathcal{A} = \{S_{T_2} > X\}$. Using the Radon-Nikodym derivatives

$$\frac{dQ_{\mathbf{4}}}{dQ} = \frac{S_{T_1} S_{T_2}/M_{T_2}}{E_Q\left[S_{T_1} S_{T_2}/M_{T_2}\right]}$$

and

$$\frac{dQ_{\mathbf{5}}}{dQ} = \frac{S_{T_1}/M_{T_2}}{E_Q\left[S_{T_1}/M_{T_2}\right]},$$

it follows that the market value can be written as

$$f_0(g) = \frac{(S_0)^2}{P(0,T_1)} e^{\sigma_{\delta_{T_1}}} Q_{\mathbf{4}}(\mathcal{A}) - S_0 F(0,T_1,T_2) X e^{-\operatorname{cov}(\delta_{T_2-T_1},\delta_{T_1})} Q_{\mathbf{5}}(\mathcal{A}).$$

The result then follows. □

Another interpretation of this claim can be obtained by replacing $\alpha = S_{T_1}$ with a time T_2 currency exchange rate, say, Y_{T_2}, and then by interpreting the call option as an option on a stock in a foreign economy. By arbitrage arguments, it is easily seen that the time 0 market value of such a claim is equal to $Y_0 g_0$, where g_0 now is the time 0 market value of the call option denoted in the foreign economy's currency.

8 Conclusions

We have in this paper constructed two general contingent claims. The first a simple claim that is written on primary traded assets. Among the claims that were captured by this claim, special attention was given on a call option and a maturity guarantee. The second was a compound contingent claim that was written on the simple claim. First the focus was on the similarities between a compound option and a two-period guarantee. The analysis also showed that the market value of a compound option under stochastic interest rates is not easily obtainable. In addition, also a few of the other claims captured by the general compound contingent claim were given a deeper analysis. Among these, the compound exchange option analysed by Carr (1988) was rediscovered, but this time under stochastic interest rates.

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