Lower and Upper Bounds for Linear Production Games

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Abstract

We study a model of a production economy in which every set of agents owns a set of resources, and where they all have access to the same technology. The agents can cooperate by pooling their resources, and the total profit from the joint venture is given by the optimal value to a linear program. The problem of allocating the total profit among the participants of such a joint venture can be formulated as a cooperative game, as in Owen (1975), and it is well-known that some core points can be obtained from optimal solutions to the dual of the LP-problem corresponding to the grand coalition. We provide lower (upper) bounds on the values of the game by aggregating over columns (rows) of the LP-problem. By choosing aggregation weights corresponding to optimal solutions of the primal (dual) LP-problem, we can create new games whose core form a superset (subset) of the original core. An estimate of the resulting error, in terms of an ϵ-core, is obtained by solving a mixed-integer programming problem, and we also suggest an iterative procedure for improving the bounds. Using a set of numerical examples, we investigate how the performance of the aggregation approach depends on the structure of the problem data.

Keywords: linear programming, cooperative game theory

1 Introduction

We study a model of a production economy, in which the production technology is given by linear relationships, and where every group of agents have access to the same technology. There is a set of resources R that can be

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used to produce a set of products P. The production technology is given by a matrix A, where a_{ij} is the amount of resource i needed to produce one unit of product j. It is assumed that an infinite amount of product j can be sold at the price c_j, giving the price vector $c = \{c_1, \ldots, c_p\}$. The resources available is given by a vector $b = \{b_1, \ldots, b_r\}$, where b_i is the amount available of resource i. The maximal profit that can be made from the resource bundle b is given by

$$\max\left\{c^T x : Ax \leq b,\ x \in \mathbb{R}_+^p\right\}, \tag{1.1}$$

where x_j denotes the amount of product j that is produced.

The resources are owned by a set N of agents, and ownership of the resources is dispersed among the agents. The agents may operate on their own, or they may combine their resources in order to increase the total profit. Before they agree to cooperate, they will typically decide how to allocate the total profit among themselves. The resulting allocation will, among other things, depend on the outside options available to the agents, i.e., the profits that can be earned by sub-groups of agents if they should decide to establish their own production facilities. The problem of finding an allocation of the profit can be modeled as a TU-game, such as in Owen (1975), providing us with solution concepts such as the core. Generalizations, with respect to how resources are controlled by various subsets (coalitions) of agents, have been studied by Granot (1986) and Curiel et al. (1989).

To describe a solution to a TU-game, we need to know not only the profit that can be made by N, but also the corresponding values for some or all of the subsets $S \subset N$. Since there are $2^n - 1$ such subsets, the amount of computational work involved can be prohibitive. In this paper we present a method that provides us with lower and upper bounds on $v(S)$ for any $S \subseteq N$, while requiring less computational effort than actually computing $v(S)$. Our method is related to aggregation of columns and rows in linear programming problems, as in Zipkin (1980b) and Zipkin (1980a), respectively.

In Section 2 we define linear production processes and linear production games, as well as some concepts related to cooperative game theory. Section 3 describes how lower and upper bounds for linear production games can be found by aggregating columns and rows, respectively, and in Section 4 we give a method to find bounds on the error resulting from the aggregation. The method involves solving a mixed integer programming problem, and the solution from this problem also suggests how the weight matrix of the aggregated game may be updated in order to improve the bound. Finally, in Section 5, we investigate, using numerical examples, how the performance of the aggregation approach depends on the structure of the problem data.

2 Linear production games

The set of *agents* (players) is denoted by N, the set of *resources* by R, and the set of *products* by P, where $n := |N|$, $r := |R|$, and $p := |P|$. The production technology is described by the matrix $A \in \mathbb{R}^{r \times p}$, where a_{ij} is the amount of resource i needed to produce one unit of product j. The profit per unit sold of product j is c_j, making up the column[1] vector $c \in \mathbb{R}^p$. Each coalition $S \subseteq N$ owns the resources given by $b(S) = \{b_1(S), \dots, b_r(S)\}$, where $b_i(S)$ is the amount of resource i that the subset S controls.

Definition 2.1 The triple (A, b, c) is a *linear production process* if

(i) $a_{ij} \geq 0$ for all $i \in R$ and $j \in P$,

(ii) $b_i(S) \geq 0$ for all $i \in R$ and $S \subseteq N$,

(iii) if $c_j > 0$, then there exists some resource i such that $a_{ij} > 0$.

The above assumptions ensures that that the linear programs that we will define below have finite optimal solutions. For a linear production process (A, b, c), and for every $S \subseteq N$, the maximal profit that the agents in S can obtain by pooling their resources is given by

$$v^{(A,b,c)}(S) := \max \left\{ c^T x : Ax \leq b(S),\ x \in \mathbb{R}^p_+ \right\}. \tag{2.1}$$

We will refer to the LP-problem given by (2.1) as $LP(A, b, c, S)$, or, if this is unambiguous, just $LP(S)$. From the Duality Theorem of Linear Program-

[1]If nothing else is stated, a vector is assumed to consist of one column.

ming follows that we can also compute the value of $LP(S)$ from

$$v^{(A,b,c)}(S) = \min\left\{u^T b(S) : A^T u \geq c,\ u \in \mathbb{R}^r_+\right\}. \tag{2.2}$$

For every linear production process (A, b, c) we define a *linear production game* $(N, v^{(A,b,c)})$, where N is the set of *players*, and $v^{(A,b,c)} : 2^N \to \mathbb{R}$ is the *characteristic function*[2]. We will mostly skip the superscript and just write v for the characteristic function.

A solution of the game v is an allocation vector $z = \{z_1, \ldots, z_n\}$, where z_i specifies the amount of the total profit awarded to player i. A *core* solution satisfies *Pareto-efficiency*, i.e., $z(N) = v(N)$, as well as the participation constraints $z(N) \geq v(S)\ \forall S \subset N$. The core of the game v will be denoted $C(v)$. For some allocation vector $z \in \mathbb{R}^n$, let $e(v, S, z) := v(S) - z(S)$ denote the *excess value* of coalition $S \subseteq N$. The *strong ϵ-core*[3] is defined as

$$C_\epsilon(v) := \{z \in \mathbb{R}^n : z(N) = v(N) \text{ and } e(v, S, z) \leq \epsilon\ \forall S\}. \tag{2.3}$$

If $\epsilon = 0$, we have $C_\epsilon(v) = C(v)$, and for $\epsilon \leq 0$ we have $C_\epsilon(v) \subseteq C(v)$.

Several variations on linear production games, with respect to how the function b is defined, exist in the literature. Owen (1975) studies the situation where the resources are controlled by individual players, where b_{ik} denotes the amount of resource i controlled by player k. Owen assumes that a group of players can pool their resources by simply adding the individual amounts, i.e., $b_i(S) = \sum_{k \in S} b_{ik}$. In this case, an allocation in the core can be deduced from an optimal solution to the dual of $LP(N)$. If u is such an optimal dual solution then y is in the core of v, where $y_k := \sum_{i \in R} b_{ik} u_i$ for every $k \in N$. Gellekom et al. (1999) provide alternative characterizations of this allocation rule.

Granot (1986) generalizes this model, and studies the core of the linear production game $(N, v^{(A,b,c)})$ by looking at the *resource games* (N, b_i), $i \in R$. If the cores of all the resource games are nonempty, then the core of $(N, v^{(A,b,c)})$ is also nonempty. Moreover, if t^i is a core allocation for the resource game (N, b_i) for every $i \in R$, and u is an optimal dual solution

[2] Let 2^N denote the set of all subsets of N.

[3] See Maschler et al. (1979).

to $LP(N)$, then a core allocation for the game $(N, v^{(A,b,c)})$ is given by the vector y, where the amount allocated to player k is $y_k := \sum_{i \in R} t_k^i u_i$.

Curiel et al. (1989) assumes that each resource $i \in R$ is divided into d_i portions. The amount of resource i belonging to portion q, $1 \leq q \leq d_i$, is b_i^q. Portion q of resource i is controlled by a committee $Q \subseteq N$, meaning that a coalition $S \subseteq N$ can only use this portion if it contains Q. Formally, this is modeled using a simple game[4] (N, w_i^q), where $w_i^q(S) = 1$ only if $Q \subseteq S$. The amount of resource $i \in R$ controlled by coalition S is given by $b_i(S) := \sum_{q=1}^{d_i} b_i^q w_i^q(S)$. Curiel et al. show that the core of a linear production game is nonempty if all the games w_i^q, where $i \in R$ and $q \in \{1, \ldots, d_i\}$, have nonempty cores. Moreover, if z_i^q is in the core of the game (N, w_i^q) for every $i \in R$ and $q \in \{1, \ldots, d_i\}$, and if u is an optimal dual solution to $LP(N)$, then y is a core allocation for $(N, v^{(A,b,c)})$, where $y_k := \sum_{i \in R} u_i \sum_{q=1}^{d_i} b_i^q (z_i^q)_k$ for every $k \in N$.

3 Aggregation of columns and rows

Reducing the size of (each of) the linear programs that must be solved in order to compute v can be done by aggregating over columns or rows (or both), as in Zipkin (1980b) and Zipkin (1980a), respectively.

In Zipkin (1980b), column aggregation is performed by specifying a partition of the set of columns. The columns belonging to each partition member are combined using a pre-specified weight vector. After the aggregated problem has been solved, a feasible solution to the original problem can be obtained by disaggregating using the same weight vectors. Our approach is a generalization[5] of that of Zipkin, and the aggregation is performed by multiplying

[4] A game (N, g) is *simple* if $g(N) = 1$ and $g(S) \in \{0, 1\}$ for every $S \subseteq N$.

[5] In Zipkin (1980b), column aggregation is performed by specifying a partition $\sigma = \{P_k : k = 1, \ldots, K\}$ of P, and a weight vector g^k for each member of this partition. To illustrate how our approach relates to that of Zipkin, consider an example with four products, where $\sigma = \{\{1, 2\}, \{3, 4\}\}$, and where $g_1^1 = g_2^1 = g_1^2 = g_2^2 = 0.5$. In our case this corresponds to the matrix

$$G = \begin{bmatrix} 0.5 & 0 \\ 0.5 & 0 \\ 0 & 0.5 \\ 0 & 0.5 \end{bmatrix}$$

Note that our approach is more general than that of Zipkin, in that aggregation is done

A and c with the matrix $G \in \mathbb{R}_+^{p \times \bar{p}}$, where $\bar{p}$ is the number of "products" of the resulting linear production process $(AG, b, G^T c)$. Our purpose is to reduce the size of the LP-problems to be solved when computing the values of the linear production game, so we will typically have $\bar{p} < p$. The values of the resulting linear production game, which we label v^G, is given by, for every $S \subseteq N$,

$$v^G(S) := v^{(AG,b,G^T c)}(S) = \max \left\{ c^T G X : AGX \leq b(S),\ X \in \mathbb{R}_+^{\bar{p}} \right\}. \quad (3.1)$$

The linear program to be solved by coalition S will be denoted $LP^G(S) = LP(AG, b, G^T c, S)$. In order to distinguish between the solutions of the original and the aggregated LP-problem, we will use uppercase letters to denote solutions to the latter problem. In order to illustrate how v^G is constructed, we provide an example.

Example 3.1 [Figures 3.1 and 3.2] There are four products ($p = 4$) and two resources ($r = 2$), and the production technology and the profits that can be made are given by

$$A = \begin{bmatrix} 2 & 1 & 3 & 1 \\ 1 & 2 & 2 & 1 \end{bmatrix}, \text{ and } c^T = \begin{bmatrix} 6 & 6 & 8 & 5 \end{bmatrix}.$$

The resources are controlled by three players ($n = 3$), and, as in Owen (1975), we assume that $b(S) := Be_S^N$ for every $S \subseteq N$, where

$$B = \begin{bmatrix} 9 & 0 & 6 \\ 1 & 8 & 3 \end{bmatrix}.$$

The value of coalition S is computed as

$$\begin{aligned} v(S) = \quad & \max \quad 6x_1 + 6x_2 + 8x_3 + 5x_4 \\ & \text{s.t.} \quad 2x_1 + 1x_2 + 3x_3 + 1x_4 + s_1 = b_1(S) \\ & \qquad\ \ 1x_1 + 2x_2 + 2x_3 + 1x_4 + s_2 = b_2(S) \\ & \qquad\ \ x_j \geq 0 \text{ for } j = 1,2,3,4 \\ & \qquad\ \ s_i \geq 0 \text{ for } i = 1,2 \end{aligned}$$

and the (unique) optimal solutions of the primal problems are shown in Figure 3.1.

with respect to *coverings* of the set of columns, since each row of G can have more than one nonzero element (see Section 4 of Zipkin (1980b)).

S			x_1	x_2	x_3	x_4	s_1	s_2	$v(S)$	$v^G(S)$	$v^{G'}(S)$
1			1	0	0	0	7	0	6	5.25	6
	2		0	0	0	0	0	8	0	0	0
		3	3	0	0	0	0	0	18	15.75	18
1	2		0	0	0	9	0	0	45	37.8	45
1		3	4	0	0	0	7	0	24	21	24
	2	3	0	5	0	1	0	0	35	25.2	30
1	2	3	3	0	0	9	0	0	63	63	63

Figure 3.1: Optimal primal solutions for Example 3.1

Suppose now that we combine the columns of A using one of the solutions shown in Figure 3.1. Choosing the solution corresponding to the grand coalition, i.e.,

$$G = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 9 \end{bmatrix},$$

gives the new linear production process $(AG, b, G^T c)$, where

$$AG = \begin{bmatrix} 15 \\ 12 \end{bmatrix} \text{ and } c^T G = \begin{bmatrix} 63 \end{bmatrix}.$$

Since the aggregated game has a single column, its value for a particular coalition can be computed by solving a continuous knapsack problem, e.g., for the grand coalition the value is

$$\begin{aligned} v^G(N) &= \max\{63X : 15X \leq 15,\ 12X \leq 12,\ X \in \mathbb{R}^1_+\} \\ &= 63 \times \min\left\{\frac{15}{15}, \frac{12}{12}\right\} = 63 = v(N). \end{aligned}$$

Not surprisingly, for the grand coalition, from which we obtained the aggregation weights, the game v^G coincides with v. For the other coalitions, having smaller amounts of resources than N, the value of the aggregated game is obtained by scaling down the value of the grand coalition. E.g., for coalition $\{1, 3\}$,

$$\begin{aligned} v^G(1,3) &= \max\{63X : 15X \leq 15,\ 12X \leq 4,\ X \in \mathbb{R}^1_+\} \\ &= 63 \times \min\left\{\frac{15}{15}, \frac{4}{12}\right\} = 21 < v(1,3) = 24. \end{aligned}$$

We note that for all coalitions, the game v^G forms a lower bound for v.

Had we instead chosen the weight matrix

$$G' = \begin{bmatrix} 0 & 4 \\ 0 & 0 \\ 0 & 0 \\ 9 & 0 \end{bmatrix},$$

i.e., the columns of G' correspond to the optimal solutions of $LP(1,2)$ and $LP(1,3)$, the game $v^{G'}$, also shown in Figure 3.1, would result. The games v and $v^{G'}$ coincide for all but one coalition, namely $\{2,3\}$. An interesting point is that coincidence occurs even for coalitions for which we did not include the optimal solution in G'. We will show, in Proposition 3.2(iii), that coincidence will occur for a coalition S if and only if the optimal solution for $LP(S)$ can be obtained as a linear combination of the columns of G'. In the example, the optimal solution for the grand coalition can be obtained[6] by combining the solutions for $\{1,2\}$ and $\{1,3\}$ as

$$\left[\begin{array}{cccc} 3 & 0 & 0 & 9 \end{array}\right] = 1 \cdot \left[\begin{array}{cccc} 0 & 0 & 0 & 9 \end{array}\right] + \frac{3}{4}\left[\begin{array}{cccc} 4 & 0 & 0 & 0 \end{array}\right],$$

hence we will have $v^{G'}(N) = v(N)$. The weights in this expression correspond to the optimal primal solution of $LP^{G'}(N)$, i.e., $X_1^* = 1$ and $X_2^* = \frac{3}{4}$. $\triangleleft$

Proposition 3.2 *Let (A, b, c) be a linear production process, and $G \in \mathbb{R}_+^{p\times\bar{p}}$. Then the following statements are true:*

(i) $(AG, b, G^T c)$ is a linear production process.

[6]In fact, the two problems $LP(N)$ and $LP(1,2)$ have the same optimal *basis*, hence the solution of both problems could have been obtained using the corresponding basis matrix. For any coalition S we can write the primal of $LP(S)$ as

$$\begin{array}{rll} v(S) = & \max & c^T x \\ & \text{s.t} & Ax + Is = b(S) \\ & & x \geq 0,\ s \geq 0, \end{array} \tag{3.2}$$

where s is a vector of slack variables. Letting

$$d^T := [c^T\ 0^T],\ y := \left[\begin{array}{c} x \\ s \end{array}\right],\ \text{and } C := [A\ I],$$

we can rewrite (3.2) as

$$\begin{array}{ll} \max & d^T y \\ \text{s.t} & Cy = b(S) \\ & y \geq 0 \end{array} \tag{3.3}$$

The optimal basis matrix $\mathbf{B} \in \mathbb{R}^{r\times r}$, not to be confused with the matrix describing ownership of the resources, determines the solutions of the primal and dual, respectively, as

$$y_{\mathbf{B}}^* = \mathbf{B}^{-1} b(S) \text{ and } u^* = (d_{\mathbf{B}}^T \mathbf{B}^{-1})^T.$$

Hence, if $\mathbf{B}$ is an optimal basis also for some other coalition $R \neq S$, then $\mathbf{B}^{-1}b(R)$ is an optimal primal solution to $LP(R)$.

In the example, an optimal basis for coalitions N and $\{1,2\}$ corresponds to columns 1 and 4 of the matrix A, i.e., the basis matrix

$$\mathbf{B} = \left[\begin{array}{cc} 2 & 1 \\ 1 & 2 \end{array}\right] \text{ and its inverse } \mathbf{B}^{-1} = \left[\begin{array}{cc} 1 & -1 \\ -1 & 2 \end{array}\right].$$

The optimal solution for $LP(N)$ and $LP(1,2)$ are, respectively,

$$\mathbf{B}^{-1} b(N) = \mathbf{B}^{-1} \left[\begin{array}{c} 15 \\ 12 \end{array}\right] = \left[\begin{array}{c} 3 \\ 9 \end{array}\right] \text{ and } \mathbf{B}^{-1} b(1,2) = \mathbf{B}^{-1} \left[\begin{array}{c} 9 \\ 9 \end{array}\right] = \left[\begin{array}{c} 0 \\ 9 \end{array}\right].$$

(ii) $v^G(S) \leq v(S)$ *for every* $S \subseteq N$.

(iii) $v^G(S) = v(S)$ *if and only if there exists* $X \in \mathbb{R}^{\bar{p}}$ *such that* GX *is an optimal primal solution of* $LP(S)$.

Proof. (i) Since A and G have non-negative elements, the elements of AG must also be non-negative. Also, if $(G^T c)_j = \sum_{k \in P} c_k g_{kj} > 0$ for some $j \in \bar{P}$, then there must exist some $k \in P$ such that $c_k > 0$ and $g_{kj} > 0$. Then, since (A, b, c) is a linear production process, there must exist some $i \in R$ such that $a_{ik} > 0$, and hence $(AG)_{ij} = \sum_{l \in P} a_{il} g_{lj} \geq a_{ik} g_{kj} > 0$.

(ii) For $S \subseteq N$ and an optimal solution X to the primal of $LP^G(S)$, we have $AGX \leq b(S)$, implying that GX is a feasible solution to the primal of $LP(S)$, hence we must have $v(S) \geq c^T GX = v^G(S)$.

(iii) If GX is optimal in $LP(S)$, then

$$v(S) = c^T GX \leq v^G(S) \leq v(S) \Rightarrow v^G(S) = v(S).$$

The optimality of GX in $LP(S)$ implies $AGX \leq b(S)$, i.e., X is feasible in the primal of $LP^G(S)$, hence the first inequality. The second inequality follows from (ii).

Suppose $v^G(S) = c^T GX = v(S)$, where $X \in \mathbb{R}^{\bar{p}}$ is an optimal primal solution to $LP^G(S)$. Then clearly, $GX \in \mathbb{R}^p$ is feasible in $LP(S)$, since $GX \geq 0$ and $AGX \leq b(S)$. Then, since $v(S) = c^T GX$, the solution GX must be optimal in $LP(S)$. □

From Proposition 3.2(iii), we know that by including in G an optimal solution for the grand coalition, we can make $v^G(N) = v(N)$. Also, since $v^G \leq v$, by Proposition 3.2(ii), the core of v^G will contain the core of v. This is illustrated by Figure 3.2, where the solid lines represent the game v, and the dashed lines the game v^G.

We may also aggregate over the rows (resource constraints) of the LP-problem, as in Zipkin (1980a). Let $\bar{R}$ be the set of "resources" in the aggregated problem. Then, take some $H \in \mathbb{R}_+^{\bar{r} \times r}$ and define, for every

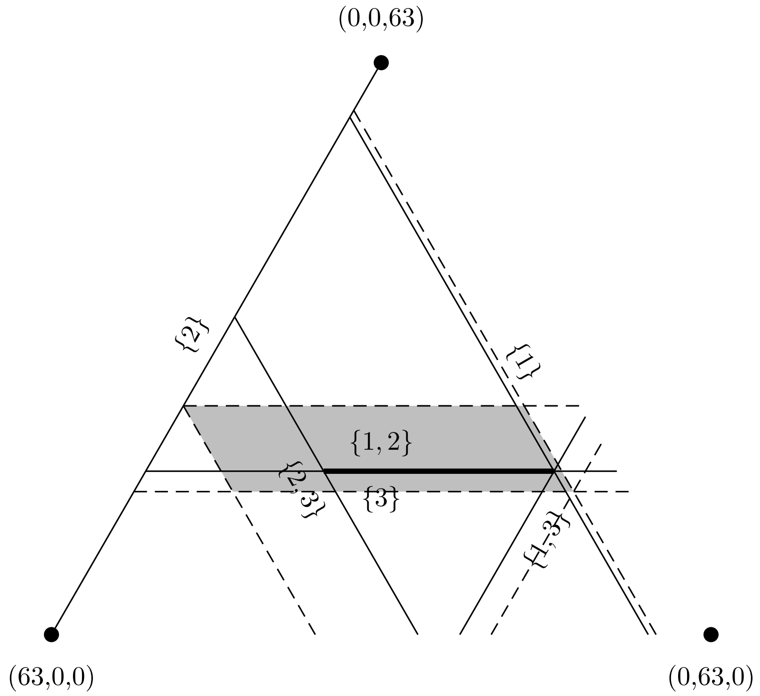


Figure 3.2: Core of v and v^G in Example 3.1

$S \subseteq N$,

$$\begin{aligned} v^H(S) &:= v^{(HA,b^H,c)}(S) = \max\left\{c^T x : HAx \leq b^H(S),\ x \in \mathbb{R}_+^p\right\} \\ &= \min\left\{U^T Hb(S) : U^T HA \geq c^T,\ U \in \mathbb{R}_+^{\bar{r}}\right\}, \end{aligned}$$

where $b^H(S) := Hb(S)$ for every $S \subseteq N$. The linear program to be solved by coalition S will be denoted $LP^H(S) = LP(HA, b^H, c, S)$.

Example 3.3 [Figures 3.3 and 3.4] There are two products ($p = 2$) and four resources ($r = 4$), and the production technology and the profits that can be made are given by

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 3 & 2 \\ 1 & 1 \end{bmatrix}, \text{ and } c^T = \begin{bmatrix} 6 & 6 \end{bmatrix}.$$

The resources are controlled by three players ($n = 3$), and $b(S) = Be_S^N$ for every $S \subseteq N$, where

$$B = \begin{bmatrix} 9 & 0 & 3 \\ 1 & 8 & 3 \\ 3 & 4 & 7 \\ 3 & 3 & 3 \end{bmatrix}.$$

The value of coalition S can be obtained as

$$\begin{aligned} v(S) = \quad \min \quad & u_1b_1(S) + u_2b_2(S) + u_3b_3(S) + u_4b_4(S) \\ \text{s.t.} \quad & 2u_2 + 1u_2 + 3u_3 + 1u_4 - s_1 = 6 \\ & 1u_1 + 2u_2 + 2u_3 + 1u_4 - s_2 = 6 \\ & u_i \geq 0 \text{ for } i = 1,2,3,4 \\ & s_j \geq 0 \text{ for } j = 1,2 \end{aligned}$$

and the optimal solutions of the dual problems are shown in Figure 3.3.

S			u_1	u_2	u_3	u_4	s_1	s_2	$v(S)$	$v^H(S)$	$v^{H'}(S)$	$v^{H''}(S)$
1			0	1.5	1.5	0	0	0	6	6	6	6
			0	6	0	0	0	6				
	2		6	0	0	0	6	0	0	18	12	0
		3	2	2	0	0	0	0	12	15	15	15
1	2		0	0	3	0	3	0	21	24	21	24
1		3	0	1.5	1.5	0	0	0	21	21	21	21
	2	3	6	0	0	0	6	0	18	33	33	18
1	2	3	0	1.5	1.5	0	0	0	39	39	39	39

Figure 3.3: Optimal dual solutions for Example 3.3

The weight matrix could e.g. be constructed from the dual solution for the grand coalition, i.e.,

$$H = \begin{bmatrix} 0 & 1.5 & 1.5 & 0 \end{bmatrix},$$

which produces the new linear production process[7] (HA, Hb, c), where

$$HA = \begin{bmatrix} 6 & 6 \end{bmatrix} \text{ and } HB = \begin{bmatrix} 6 & 18 & 15 \end{bmatrix}.$$

The dual of $LP^H(N)$ can now easily be solved as a continuous knapsack

[7]It is not obvious that the aggregation actually yields a linear production process, since we may use H such that for a product j for which $c_j > 0$, we have $(HA)_{ij} = 0$ for all $i \in \bar{R}$.

problem

$$\begin{aligned} v^H(N) &= \min\{39U : 6U \geq 6,\ 6U \geq 6,\ U \in \mathbb{R}^1_+\} \\ &= 39 \times \max\left\{\frac{6}{6}, \frac{6}{6}\right\} = 39 = v(N). \end{aligned}$$

Again, as for column-aggregation, the value of v^H for the grand coalition, which we used to generate H, coincides with the value of the original game. For other coalitions we get an upper bound on v, e.g.,

$$\begin{aligned} v^H(2) &= \min\{18U : 6U \geq 6,\ 6U \geq 6,\ U \in \mathbb{R}^1_+\} \\ &= 18 \times \max\left\{\frac{6}{6}, \frac{6}{6}\right\} = 18 \geq v(2). \end{aligned}$$

Note also that, since the dual constraints are the same for all coalitions, and they all have positive amounts of the single resource, $U^* = 1$ will be the optimal solution for all of them, and we have the additive structure given by $v^H(S) = \sum_{k \in S}(HB)_k$, i.e., the value for a coalition S is given by the total value of the resources owned by S, where the value is computed using the price vector included in H.

A slightly better bound is obtained by using

$$H' = \begin{bmatrix} 0 & 6 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix},$$

i.e., we use the dual solutions corresponding to the coalitions $\{1\}$ and $\{1, 2\}$. Note that we have $v(N) = v^{H'}(N)$, even though the optimal dual solution for the grand coalition is not included in H'. However, as we shall prove in Proposition 3.4(iii) below, coincidence follows from the fact that the optimal dual solution of $LP(N)$ can be written as a linear combination of the two row vectors of H', i.e.,

$$\begin{bmatrix} 0 & 1.5 & 1.5 & 0 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 0 & 6 & 0 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0 & 3 & 0 \end{bmatrix}.$$

The weights correspond to the optimal dual solution of $LP^{H'}(N)$, i.e., $U_1^* = 1/4$ and $U_2^* = 1/2$. $\triangleleft$

Proposition 3.4 *Let (A, b, c) be a linear production process, and $H \in \mathbb{R}^{\bar{r} \times r}_+$.*

(i) If (HA, b^H, c) is a linear production process, then $v^H(S) \geq v(S)$ for every $S \subseteq N$.

(ii) If, for some $S \subseteq N$, there exists $U \in \mathbb{R}^{\bar{r}}$ such that $H^T U$ is optimal in the dual of $LP(S)$, then (HA, b^H, c) is a linear production process.

(iii) $v^H(S) = v(S)$ if and only if there exists $U \in \mathbb{R}^{\bar{r}}$ such that $H^T U$ is optimal in the dual of $LP(S)$.

Proof. (i) Take $S \subseteq N$ and an optimal solution U to the dual of $LP^H(S)$. Then, since the optimality of U implies $U^T HA \geq c^T$, $U^T H$ must be feasible in the dual of $LP(S)$, which implies $v^H(S) = U^T Hb(S) \geq v(S)$.

(ii) Since the elements of A and H, as well as the values returned by the function b, are non-negative, this must also be the case for the elements of HA, as well as the values returned by b^H. Also, since $H^T U$ is optimal in the dual of $LP(S)$, we must have $\sum_{i \in \bar{R}} U_i (HA)_{ij} \geq c_j$ for all $j \in P$. So if $c_j > 0$, there must exist some $i \in \bar{R}$ such that $(HA)_{ij} > 0$.

(iii) If $H^T U$ is optimal in the dual of $LP(S)$, then

$$v(S) = U^T Hb(S) \geq v^H(S) \geq v(S) \Rightarrow v^H(S) = v(S).$$

The optimality of $H^T U$ implies $U^T HA \geq c^T$ and $H^T U \geq 0$, hence U must be feasible in the dual of $LP^H(S)$, which implies the first inequality. The second inequality follows from (i).

Suppose $v^H(S) = U^T Hb(S) = v(S)$, where $U \in \mathbb{R}^{\bar{r}}$ is optimal in the dual of $LP^H(S)$. Then $U^T H \geq 0$ and $U^T HA \geq c^T$ implies that $H^T U$ is feasible in the dual of $LP(S)$, and optimality follows from $v(S) = U^T Hb(S)$. □

The cores of v and v^H in Example 3.3 are illustrated in Figure 3.4, where the solid (dashed) lines are hyperplanes corresponding to v (v^H). Since v^H is an upper bound for v, the core of v^H is contained in the core of v. Note that the core of v^H consists of the single point

$$\begin{bmatrix} 6 & 18 & 15 \end{bmatrix} = HB,$$

i.e., the allocation where the resources of the players are valued at the price vector corresponding to the dual solution of $LP(N)$.

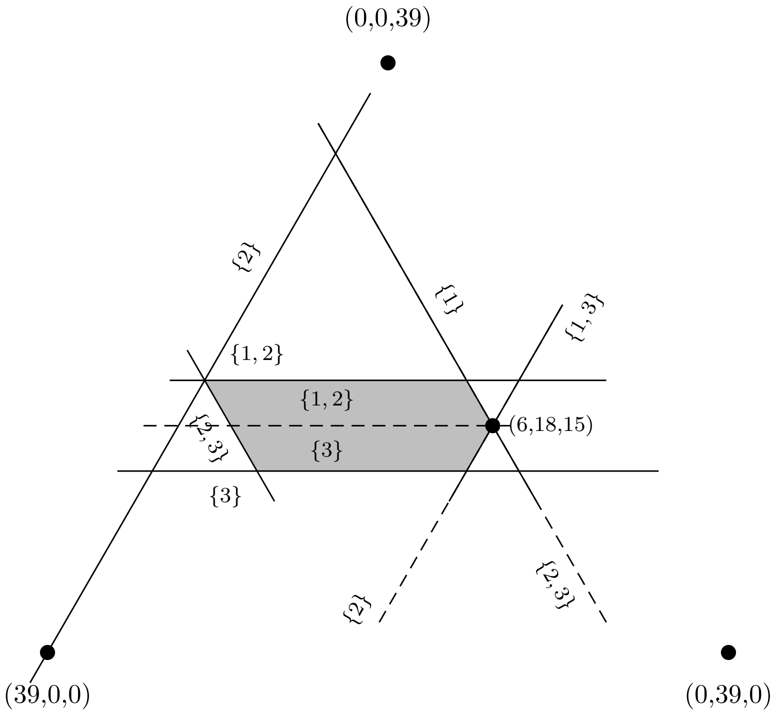


Figure 3.4: Core of v and v^H in Example 3.3

Proposition 3.5 *Let u be an optimal dual solution to $LP(Q)$ for some $Q \subseteq N$ such that $v(Q) > 0$. Then, if $H = u^T$, we have $v^H(S) = u^T b(S)$ for every $S \subseteq N$.*

Proof. Since the aggregated problem contains only one row, i.e., $\bar{r} = 1$, the value of the game can be computed, for any S, as

$$\begin{aligned} v^H(S) &:= \min \left\{ U u^T b(S) : U u^T A \geq c^T,\ U \in \mathbb{R}^1_+ \right\} \\ &= u^T b(S) \cdot \max_{\substack{j \in P \\ u^T A^j > 0}} \frac{c_j}{u^T A^j} \end{aligned} \tag{3.4}$$

Note that the feasibility of u, for any $Q \subseteq N$, implies that $u^T A^j \geq c_j$ holds for every $j \in P$. Moreover, since $v(Q) > 0$, it must be optimal for the coalition Q to produce at least one product, hence we must have $u^T A^j = c_j$ for at least one $j \in P$. Then $v^H(S) = u^T b(S)$ follows from (3.4). $\square$

4 Error bounds and ϵ-cores

The aggregated games presented in Section 3 enable us to analyze the original game with less computational effort. However, aggregation introduces a possible error, and the purpose of this section is to give an estimate of this error.

First, we need to make clear what we mean by "error". Since the core is one of the most widely used solution concepts for TU-games, it is natural to discuss error bounds relative to it. Suppose we use the game v^G as an approximation to the game v, where we have chosen G, according to Proposition 3.2, such that the core of v is contained in the core of v^G. Knowing that an allocation vector z belongs to $C(v^G)$ thus does not guarantee that it also belongs to $C(v)$, hence there might exist some coalition S that receives less than its stand-alone value, i.e., $z(S) < v(S)$. We shall use as a "distance measure" the excess $e(v, S, z) = v(S) - z(S)$. Suppose we know that

$$C(v^G) \subseteq C_\epsilon(v)$$

for some ϵ. Since $z \in C_\epsilon(v)$, we know that no coalition has an excess of more than ϵ, hence no coalition receives less than $v(S) - \epsilon$. We would like to find the smallest ϵ-core containing $C(v^G)$, i.e., we need to solve

$$\min\{\epsilon : C(v^G) \subseteq C_\epsilon(v)\}. \tag{4.1}$$

Since making ϵ sufficiently high always makes $C_\epsilon(v)$ nonempty, (4.1) always has a solution.

Proposition 4.1 *Let (A, b, c) be a linear production process, and v be the corresponding linear production game. Let G be a matrix constructed according to Proposition 3.2 such that $C(v) \subseteq C(v^G)$, and let*

$$\epsilon := \max_{S \in 2^N \setminus \{N, \emptyset\}} \{v(S) - v^G(S)\}. \tag{4.2}$$

Then $C(v^G) \subseteq C_\epsilon(v)$.

Proof. If $z \in C(v^G)$, and $Q \subset N$, then

$$z(Q) \geq v^G(Q) \Rightarrow v(Q) - z(Q) \leq v(Q) - v^G(Q) \leq \epsilon.$$

□

In Example 3.1, (4.2) gives $\epsilon = 9.8 = v(2,3) - v^G(\{2,3\})$, and Figure 4.1 illustrates[8] that $C_\epsilon(v) \supseteq C(v^G)$.

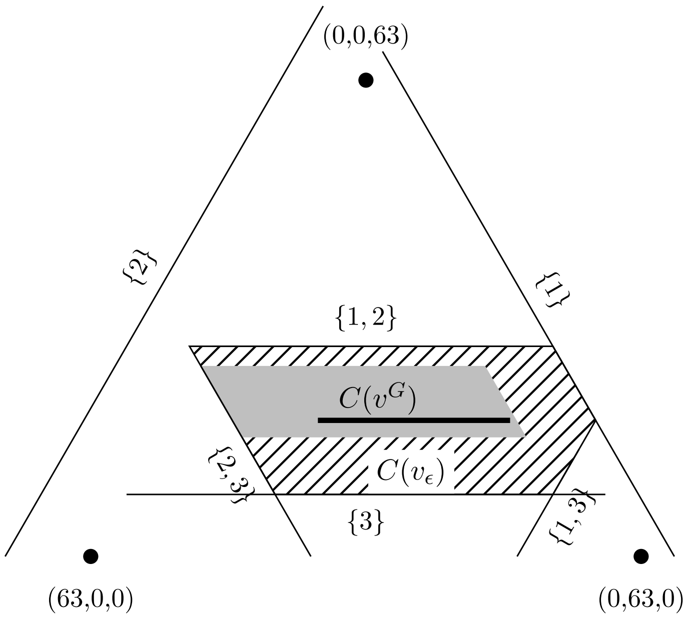


Figure 4.1: Core of v^G (shaded) and ϵ-core (hatched) in Example 3.1

Likewise, consider the games v and v^H, and suppose we know that

$$C(v^H) \supseteq C_\epsilon(v)$$

for some ϵ. By using v^H instead of v, we may exclude from consideration some elements of the core of v. However, we are certain to include all the points in $C_\epsilon(v)$, i.e., those with an excess less than or equal to ϵ. Of course, for $\epsilon = 0$, we exclude no core elements, and in this case it follows that the cores of v and v^H coincide. We would like to find the largest ϵ-core that is

[8]The solid lines correspond to the sets

$$H_S^\epsilon(v) := \{z \in \mathbb{R}^n : z(N) = v(N) \text{ and } z(S) = v(S) - \epsilon\}.$$

contained in $C(v^H)$, i.e., we solve

$$\max\{\epsilon : C_\epsilon(v) \subseteq C(v^H)\}. \tag{4.3}$$

Whereas (4.1) always has a solution, (4.3) does not, since $C_\epsilon(v)$ is empty for small enough values of ϵ.

Proposition 4.2 *Let (A, b, c) be a linear production process, and v the corresponding linear production game. Let H be a matrix constructed according to Proposition 3.4 such that $C(v^H) \subseteq C(v)$, and let*

$$\epsilon := \min_{S \in 2^N \setminus \{N, \emptyset\}} \{v(S) - v^H(S)\}. \tag{4.4}$$

Then, if $C_\epsilon(v) \neq \emptyset$, we have $C_\epsilon(v) \subseteq C(v^H)$.

Proof. If $z \in C_\epsilon(v)$, and $Q \subset N$, then

$$v(Q) - z(Q) \leq \epsilon \leq v(Q) - v^H(Q) \Rightarrow z(Q) \geq v^H(Q).$$

□

In Example 3.3, if the weight matrix

$$H'' = \begin{bmatrix} 0 & 1.5 & 1.5 & 0 \\ 6 & 0 & 0 & 0 \end{bmatrix}$$

is used, then (4.4) gives $\epsilon = -3 = v(1, 2) - v^{H'}(1, 2) = v(3) - v^{H'}(3)$. In Figure 4.2 the cores of v and $v^{H''}$ are given by the shaded area and the thick solid line, respectively. The ϵ-core of v is represented by the white solid line, and we see that $C_\epsilon(v) \subseteq C(v^{H''})$. Note that we deliberately chose H'' here in order to make the ϵ-core of nonempty, given that ϵ satisfies (4.4).

How can we find the error bounds given by (4.2) and (4.4) in practice? In addressing this question, we will limit our attention to the special case considered by Owen (1975). Here, player k controls b_{ik} units of resource i, where b_{ik} corresponds to row i and column k of the matrix $B \in \mathbb{R}_+^{r \times n}$. The coalition S pool their resources by simply summing them, i.e., they control the resource vector $b(S) := Be_S^N$. In what follows, we will let (A, B, c) denote a linear production process, where the matrix B has replaced the function b.

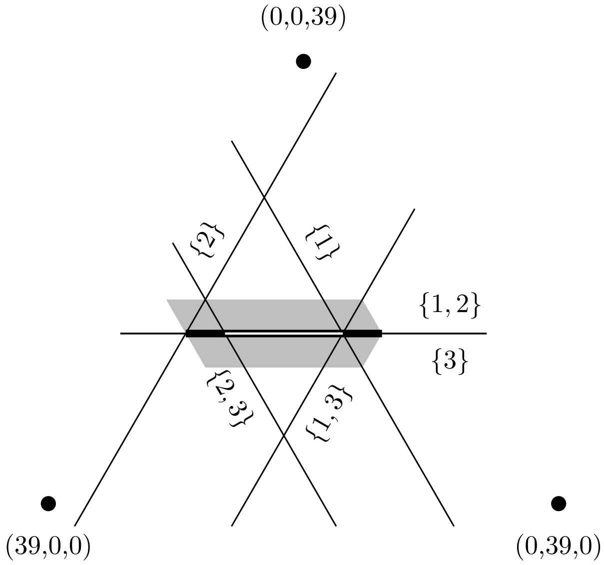


Figure 4.2: Core of $v^{H''}$ (black line) and ϵ-core (white line) in Example 3.3

Problem (4.2) may be formulated as

$$\epsilon^G := \max_{x,u,s} c^T x - u^T Bs \tag{4.5}$$

$$\text{subject to } Ax \leq Bs \tag{4.6}$$

$$u^T AG \geq c^T G \tag{4.7}$$

$$x \geq 0 \tag{4.8}$$

$$u \geq 0 \tag{4.9}$$

$$0 \leq s \leq 1 \tag{4.10}$$

$$s \text{ integer} \tag{4.11}$$

In a solution (x, u, s) to (4.5)-(4.11), $x \in \mathbb{R}^p$ is a solution to the primal of $LP(S)$, and $u \in \mathbb{R}^r$ is a solution to the dual of $LP^G(S)$. The coalition S corresponding to the solution is given by $S = \{k \in N : s_k = 1\}$. The objective function (4.5) maximizes the difference between the optimal values of the two problems. Primal feasibility of $LP(S)$ is ensured by (4.6) and (4.8), and dual feasibility of $LP^G(S)$ by (4.7) and (4.9). Problem (4.5)-

(4.11) may be rewritten as:

$$\max \sum_{j\in P} c_j x_j - \sum_{i\in R}\sum_{k\in N} b_{ik} u_i s_k \tag{4.12}$$

$$\text{subject to} \sum_{j\in P} a_{ij} x_j \leq \sum_{k\in N} b_{ik} s_k \qquad \forall i \in R \tag{4.13}$$

$$\sum_{i\in R} u_i \sum_{j\in P} a_{ij} g_{j\ell} \geq \sum_{j\in P} c_j g_{\ell j} \qquad \forall \ell \in \bar{P} \tag{4.14}$$

$$x_j \geq 0 \qquad \forall j \in P \tag{4.15}$$

$$u_i \geq 0 \qquad \forall i \in R \tag{4.16}$$

$$0 \leq s_k \leq 1 \qquad \forall k \in N \tag{4.17}$$

$$s \text{ integer} \tag{4.18}$$

Finding a solution to (4.12)-(4.18) is made more difficult by the fact that (4.12) is non-concave, and because of the integrality condition (4.18). Methods to linearize such problems are given by Petersen (1971), Glover (1975), and Adams and Sherali (1990). In Petersen (1971) the product term $u_i s_k$ is replaced by the variable w_{ik}, and the following constraints are added:

$$u_i - u_i^+(1 - s_k) \leq w_{ik} \leq u_i^+ s_k \qquad \forall i \in R,\ k \in N \tag{4.19}$$

$$w_{ik} \geq 0 \qquad \forall i \in R,\ k \in N \tag{4.20}$$

$$w_{ik} \leq u_i \qquad \forall i \in R,\ k \in N \tag{4.21}$$

The constant u_i^+ is an upper bound on the value of the variable u_i. If $s_k = 1$, then the first inequality of (4.19), together with (4.21) imply $u_i \leq w_{ik} \leq u_i$. In the case where $s_k = 0$, the second inequality of (4.19) together with (4.20) imply $0 \leq w_{ik} \leq 0$. Hence the equality $w_{ik} = u_i s_k$ always holds, and we may replace the objective function (4.12) by

$$\max \sum_{j\in P} c_j x_j - \sum_{i\in R}\sum_{k\in N} b_{ik} w_{ik}. \tag{4.22}$$

Problem (4.12)-(4.18) is equivalent to the mixed-integer programming problem given by (4.13)-(4.22), hereafter referred to as MIP^G. Note that in an optimal solution, either we have $u_i = 0$ for all $i \in R$, or at least one of the constraints (4.14) is binding. Hence the upper bounds for the variable u can be set to

$$u_i^+ := \max\left\{\max_{\substack{\ell\in\bar{P}\\ \sum_{j\in P} g_{j\ell}a_{ij}\neq 0}} \frac{\sum_{j\in P} g_{j\ell}c_j}{\sum_{j\in P} g_{j\ell}a_{ij}}, 0\right\} \qquad \forall i \in R.$$

Problem (4.4) may be formulated as

$$\begin{aligned}
\epsilon^H := \max_{x,u,s}\ & c^T x - u^T Bs && (4.23)\\
\text{subject to } & HAx \leq HBs && (4.24)\\
& u^T A \geq c^T && (4.25)\\
& x \geq 0 && (4.26)\\
& u \geq 0 && (4.27)\\
& 0 \leq s \leq 1 && (4.28)\\
& s \text{ integer}, && (4.29)
\end{aligned}$$

which, in a manner similar to that applied to (4.5)-(4.11), can be rewritten as

$$\begin{aligned}
\max\ & \sum_{j\in P} c_j x_j - \sum_{i\in R}\sum_{k\in N} b_{ik} w_{ik} && && (4.30)\\
\text{subject to } & \sum_{j\in P} x_j \sum_{i\in R} h_{\ell i} a_{ij} \leq \sum_{k\in N} s_k \sum_{i\in R} h_{\ell i} b_{ik} && \forall \ell \in \bar{R} && (4.31)\\
& \sum_{i\in R} u_i a_{ij} \geq c_j && \forall j \in P && (4.32)\\
& x_j \geq 0 && \forall j \in P && (4.33)\\
& u_i \geq 0 && \forall i \in R && (4.34)\\
& 0 \leq s_k \leq 1 && \forall k \in N && (4.35)\\
& s \text{ integer} && && (4.36)\\
& u_i - u_i^+(1-s_k) \leq w_{ik} \leq u_i^+ s_k && \forall i \in R,\ k \in N && (4.37)\\
& w_{ik} \geq 0 && \forall i \in R,\ k \in N && (4.38)\\
& w_{ik} \leq u_i && \forall i \in R,\ k \in N, && (4.39)
\end{aligned}$$

where

$$u_i^+ := \max\left\{\max_{\substack{j\in P\\ a_{ij}\neq 0}} \frac{c_j}{a_{ij}}, 0\right\} \qquad \forall i \in R.$$

The mixed integer programming problem given by (4.30)-(4.39) will hereafter be referred to as MIP^H.

Example 4.3 [Figure 4.3] The data of this example is given by $n = 5$, $p = 5$, $r = 10$,

$$A = \begin{bmatrix} 7 & 3 & 5 & 2 & 1 \\ 6 & 9 & 9 & 5 & 10 \\ 6 & 3 & 3 & 4 & 3 \\ 9 & 5 & 4 & 2 & 1 \\ 3 & 6 & 10 & 2 & 4 \\ 4 & 5 & 1 & 3 & 8 \\ 4 & 3 & 4 & 2 & 3 \\ 7 & 9 & 1 & 1 & 7 \\ 5 & 8 & 9 & 3 & 2 \\ 2 & 6 & 3 & 10 & 2 \end{bmatrix}, \; c = \begin{bmatrix} 53 \\ 57 \\ 49 \\ 34 \\ 41 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 4 & 0 & 15 & 0 & 0 \\ 0 & 22 & 18 & 0 & 0 \\ 9 & 0 & 11 & 0 & 0 \\ 0 & 17 & 0 & 5 & 0 \\ 19 & 0 & 0 & 7 & 0 \\ 0 & 13 & 0 & 9 & 0 \\ 2 & 0 & 0 & 0 & 15 \\ 0 & 22 & 0 & 0 & 4 \\ 12 & 0 & 0 & 0 & 16 \\ 0 & 23 & 0 & 0 & 0 \end{bmatrix}.$$

The value of the grand coalition is $v(N) = 241.046$. We aggregate rows using the matrix

$$H := \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 19.0 & 0.0 & 0.0 & 0.0 \end{bmatrix},$$

corresponding to the optimal dual solution of $LP(N)$. Solving MIP^H yields $\epsilon^H = 226.576$, corresponding to the coalition $\{2, 3, 4\}$. We add a new row corresponding to u^*, the optimal dual solution to $LP(2, 3, 4)$, to the weight matrix and obtain

$$H := \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 19.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.000 & 17.7 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix},$$

and by solving MIP^H again we obtain $\epsilon^H = 220.549$, corresponding to the coalition $\{2, 3, 5\}$. Continuing in this manner, new rows can be added to H until the value of ϵ^H is small enough. In Figure 4.3, the solutions of MIP^H, as new rows are added, are shown. After nine rows have been added, we have $\epsilon^H = 0$, implying that $v^H = v$. $\triangleleft$

5 Numerical results

The purpose of this section is to investigate how the performance of the aggregation approach introduced in Section 3 depends on properties of the

$\bar{r}$	ϵ^H	s^*					u^*									
1	226.6	0	1	1	1	0	0.0	0.0	0.0	0.0	17.7	0.0	0.0	0.0	0.0	0.0
2	220.5	0	1	1	0	1	0.0	0.0	19.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
3	123.7	0	1	0	1	1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	26.5
4	88.5	1	0	1	1	1	13.2	0.0	0.0	0.0	0.0	3.5	0.0	0.0	0.0	0.0
5	69.0	1	1	0	0	1	0.0	0.0	0.0	3.2	2.9	3.2	0.0	0.0	0.0	1.2
6	46.3	1	1	1	0	1	9.6	3.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
7	21.8	1	1	0	1	1	0.0	3.3	0.0	3.0	0.0	0.0	0.0	0.3	0.5	1.0
8	0.0															

Figure 4.3: Solutions of MIP^H for Example 4.3

problem data. The analysis will be based on Owen's (1975) model, where the ownership of the resources is given by the matrix B. In our analysis, we will especially focus on the density of A, and the degree to which ownership is concentrated/dispersed, i.e., the structure of B.

A number of data sets with $n = 5$ were generated in a random manner. The nonzero elements of A were drawn from a uniform discrete distribution in the interval 1, ... ,10. The density of A, i.e., the probability that a particular element A_{ij} is nonzero, was set equal to the values 0.1, 0.4, 0.7, or 1.0. After A had been determined, we set $c_j := \sum_{i \in R} A_{ij}$ for all $j \in P$. The *total amount* of resource i was initially set to $b_{iN} := \sum_{j \in P} A_{ij}$, which was then distributed among the players according to the ownership profiles shown in Figure 5.1, where the x's indicate[9] which players are allocated positive amounts of each resource. For resource i denote these players by N_i. Let $\beta_{ik} \sim U(0,1)$ be a random number corresponding to resource i and player k. Then the amount of resource i given to player k is given by

$$\left\lceil b_{iN} \frac{\beta_{ik}}{\sum_{\ell \in N_i} \beta_{i\ell}} \right\rceil$$

Profile 1 implies a relatively even distribution of the resources among the players, and may be seen as an extreme case. At the other extreme we find profile 4, where the entire amount of each resource is given to a single player. In the former case, the increased profits resulting from cooperation are modest, while in the latter cooperation is essential. Profiles 2 and 3 are located somewhere in between the two extremes. Note that according to these profiles, the resource bundles of player 1 and 2 are complements, and

[9] Row numbers from 1-10 refer to data sets with $r = 10$, and row numbers 1-100 to datasets with $r = 100$.

		Player				
Rows		1	2	3	4	5
1	1-10	x	x	x	x	x
2	11-20	x	x	x	x	x
3	21-30	x	x	x	x	x
4	31-40	x	x	x	x	x
5	41-50	x	x	x	x	x
6	51-60	x	x	x	x	x
7	61-70	x	x	x	x	x
8	71-80	x	x	x	x	x
9	81-90	x	x	x	x	x
10	91-100	x	x	x	x	x

(a) Profile 1

		Player				
Rows		1	2	3	4	5
1	1-10	x		x		
2	11-20		x	x		
3	21-30	x		x		
4	31-40		x	x		
5	41-50	x			x	
6	51-60		x		x	
7	61-70	x			x	
8	71-80		x			x
9	81-90	x				x
10	91-100		x			x

(b) Profile 2

		Player				
Rows		1	2	3	4	5
1	1-10	x		x		
2	11-20		x	x		
3	21-30	x		x		
4	31-40		x	x		
5	41-50	x			x	
6	51-60		x		x	
7	61-70	x				
8	71-80		x			x
9	81-90	x				x
10	91-100		x			

(c) Profile 3

		Player				
Rows		1	2	3	4	5
1	1-10	x				
2	11-20	x				
3	21-30		x			
4	31-40		x			
5	41-50			x		
6	51-60			x		
7	61-70				x	
8	71-80				x	
9	81-90					x
10	91-100					x

(d) Profile 4

Figure 5.1: Ownership distribution profiles

this is also the case for 3-5. Profile 3 differs from profile 2 in that player 4 does not own anything of resource 7 (61-70), and that player 4 does not own anything of resource 10 (91-100). Hence, profile 3 should, a priori, give greater benefits from cooperation than does profile 2.

Some properties/special cases regarding the data sets should be mentioned. First, note that if the ownership of resources is highly concentrated, and the density of A is high, we will have zero profits for many coalitions. In the extreme case of profile 4, where each resource has a single owner, we will have

$$v(S) = 0 \qquad\qquad \forall S \neq N, \tag{5.1}$$

if all entries of A are nonzero. Hence, positive profits can only be made if all the players pool their resources. In Figures 5.2 and 5.4, a "$\star$" after the problem name indicates that (5.1) is satisfied.

On the other hand, in the case where A is sparse, the game v will in many cases be *additive*, i.e.,

$$v(S) + v(T) = v(S \cup T) \qquad \forall S, T \subset N \text{ s.t. } S \cap T = \emptyset. \tag{5.2}$$

To see why this is the case, consider the special case where every column of A has at most one nonzero entry. Then a unit of resource i should be used to produce the product that gives the highest profit contribution per unit that it consumes of resource i, i.e., the product, among those for which $A_{ij} > 0$, such that $\frac{c_j}{A_{ij}}$ is greatest. Assuming that there is at least on product such that $A_{ij} > 0$, the value of one unit of resource i is the constant

$$w_i := \max_{\substack{j \in P \\ A_{ij} > 0}} \frac{c_j}{A_{ij}},$$

and this constant is independent of who the owner of resource i is. Hence, the total profit that can be made by a coalition S can be found by simply summing the value of its resources, i.e.,

$$v(S) = \sum_{i \in R} w_i b_i(S) = \sum_{i \in R} \sum_{k \in S} w_i b_{ik} = \sum_{k \in S} \sum_{i \in R} w_i b_{ik},$$

which clearly satisfies (5.2). For additive games, the core consists of a single point. If u^* is an optimal solution to the dual of $LP(N)$, we know from

Profile	Problem		d	p^*	ε_1	ε_5	ε_{10}	ε_{20}	ε_{30}
1	P1D10A	∘	0.1	15	0.361	0.180	0.122	0.000	0.000
	P1D10B	∘	0.1	14	0.391	0.224	0.133	0.000	0.000
	P1D10C	∘	0.1	17	0.418	0.151	0.069	0.000	0.000
	P1D40A	∘	0.4	13	0.394	0.306	0.137	0.000	0.000
	P1D40B	∘	0.4	18	0.348	0.212	0.130	0.000	0.000
	P1D40C	∘	0.4	11	0.339	0.172	0.069	0.000	0.000
	P1D70A		0.7	16	0.237	0.154	0.122	0.000	0.000
	P1D70B		0.7	18	0.333	0.159	0.086	0.000	0.000
	P1D70C		0.7	21	0.446	0.197	0.134	0.009	0.000
	P1D100A		1.0	31	0.456	0.180	0.131	0.068	0.003
	P1D100B		1.0	30	0.367	0.199	0.104	0.042	0.000
	P1D100C		1.0	26	0.281	0.151	0.088	0.018	0.000
2	P2D10A	∘	0.1	23	0.780	0.593	0.463	0.207	0.000
	P2D10B	∘	0.1	25	0.747	0.620	0.481	0.219	0.000
	P2D10C	∘	0.1	23	0.630	0.602	0.439	0.213	0.000
	P2D40A		0.4	26	0.715	0.575	0.456	0.147	0.000
	P2D40B		0.4	26	0.627	0.574	0.435	0.127	0.000
	P2D40C		0.4	27	0.704	0.596	0.459	0.136	0.000
	P2D70A		0.7	25	0.752	0.573	0.294	0.039	0.000
	P2D70B		0.7	26	0.774	0.639	0.418	0.080	0.000
	P2D70C		0.7	24	0.679	0.512	0.275	0.041	0.000
	P2D100A		1.0	11	0.502	0.304	0.019	0.000	0.000
	P2D100B		1.0	7	0.606	0.107	0.000	0.000	0.000
	P2D100C		1.0	11	0.385	0.200	0.004	0.000	0.000
3	P3D10A	∘	0.1	25	0.713	0.639	0.500	0.282	0.000
	P3D10B	∘	0.1	25	0.868	0.607	0.494	0.202	0.000
	P3D10C	∘	0.1	25	0.735	0.627	0.475	0.265	0.000
	P3D40A		0.4	29	0.838	0.652	0.547	0.299	0.000
	P3D40B		0.4	29	0.850	0.636	0.497	0.258	0.000
	P3D40C		0.4	30	0.800	0.671	0.525	0.232	0.000
	P3D70A		0.7	21	0.796	0.540	0.347	0.101	0.000
	P3D70B		0.7	22	0.837	0.518	0.410	0.114	0.000
	P3D70C		0.7	21	0.707	0.483	0.271	0.039	0.000
	P3D100A		1.0	8	0.631	0.165	0.000	0.000	0.000
	P3D100B		1.0	5	0.586	0.000	0.000	0.000	0.000
	P3D100C		1.0	8	0.509	0.213	0.000	0.000	0.000
4	P4D10A		0.1	31	0.880	0.721	0.619	0.405	0.042
	P4D10B	∘	0.1	31	0.849	0.718	0.620	0.416	0.151
	P4D10C	∘	0.1	31	0.876	0.733	0.609	0.412	0.124
	P4D40A		0.4	30	0.812	0.792	0.596	0.360	0.000
	P4D40B		0.4	27	0.832	0.772	0.589	0.253	0.000
	P4D40C		0.4	27	0.815	0.787	0.590	0.327	0.000
	P4D70A		0.7	11	0.789	0.660	0.203	0.000	0.000
	P4D70B		0.7	12	0.813	0.689	0.276	0.000	0.000
	P4D70C		0.7	8	0.819	0.649	0.000	0.000	0.000
	P4D100A	⋆	1.0	1	0.000	0.000	0.000	0.000	0.000
	P4D100B	⋆	1.0	1	0.000	0.000	0.000	0.000	0.000
	P4D100C	⋆	1.0	1	0.000	0.000	0.000	0.000	0.000

Figure 5.2: Results for column aggregation, with $n = 5$, $p = 100$, and $r = 10$

Owen (1975) that the entire core is given by the point $(u^*)^T B$. In Figures 5.2 and 5.4, a "$\circ$" after the problem name indicates that (5.1) is satisfied.

For the data sets shown in Figure 5.2, where $p = 100$ and $r = 10$, column aggregation was performed. Initially, the weight matrix G consisted of a single column corresponding to an optimal primal solution of $LP(N)$, and new columns were added by repeatedly solving MIP^G, where p^* is the number of columns needed in order to have $\epsilon^G = \max_{S \subset N,\, S \neq \emptyset}\{v(S) - v^G(S)\} = 0$, i.e., in order for the games v and v^G to coincide. Note that the number of coalitions is $2^n - 1 = 31$, which is an upper bound on the number of columns needed. In the table of Figure 5.2 is also reported ε_t, the value of $\epsilon^G/v(N)$ when t columns have been added to G.

The results in Figure 5.2 indicate that the effect on p^* of varying the density of A is ambiguous. If ownership is concentrated, as in profile 4, increasing density has a negative effect on p^*, whereas when ownership is dispersed, the effect is positive. Figure 5.3, based on four of the datasets, can help explain this phenomenon. Each row in the four respective diagrams corresponds to a coalition, and the coalitions have been sorted according to their size, as indicated by the numbers to the left of the diagrams. The $|$'s and $\bullet$'s represent nonzero elements of optimal primal solutions to $LP(S)$ for every nonempty coalition $S \subseteq N$. Let $\Omega(G)$ denote the set of coalitions that correspond to columns included in G. The $\bullet$'s correspond to coalitions that are members of $\Omega(G)$, while the $|$'s correspond to coalitions outside of $\Omega(G)$[10].

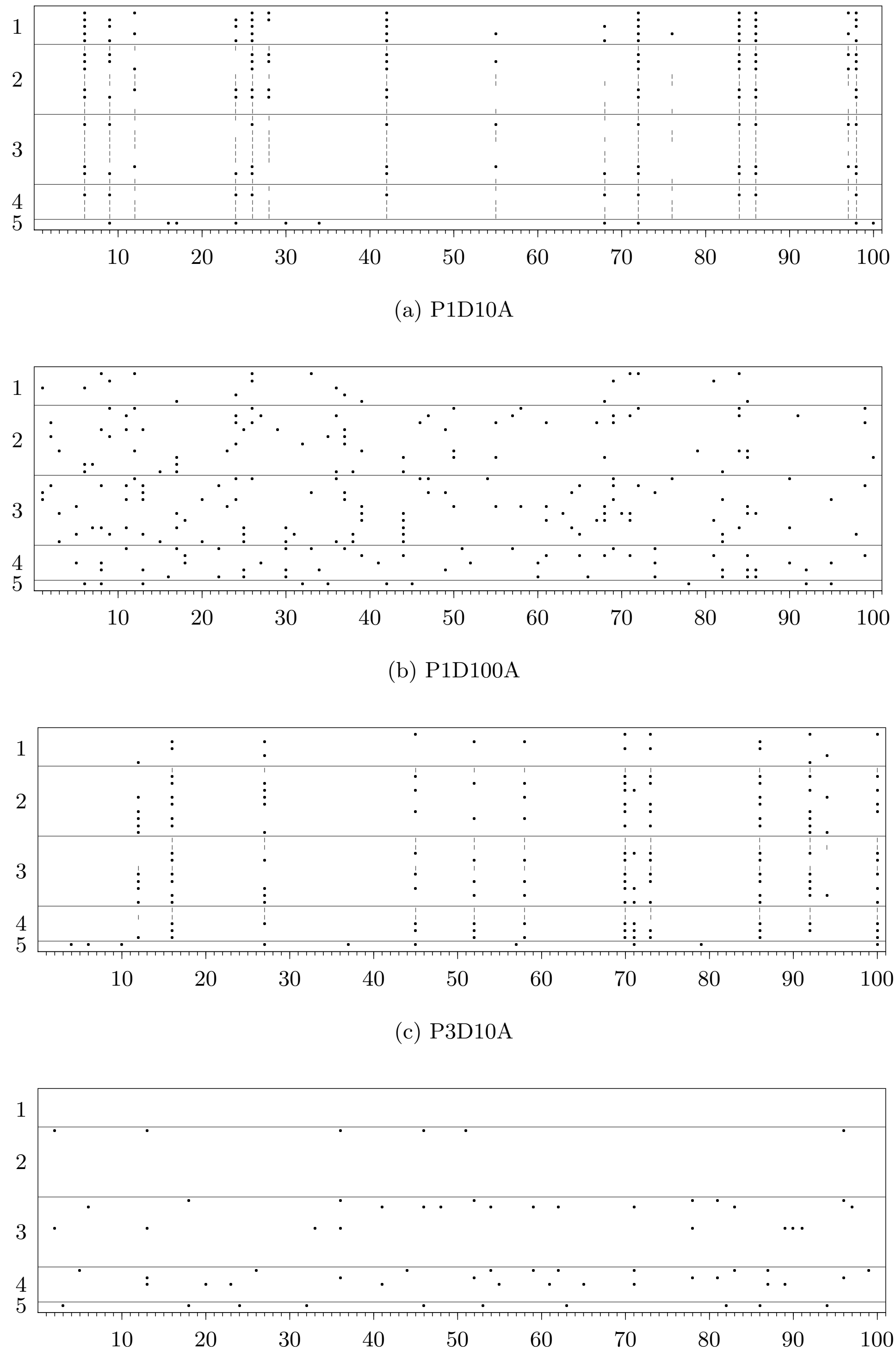
We see that increased density

(i) leads to more variation among the production plans of the various coalitions, and

(ii) makes it more difficult for small coalitions to produce anything at all, i.e., there are fewer nonzero entries for small coalitions.

When ownership is dispersed, such as for profile 1, effect (i) is dominant.

[10] In this latter case, because of Proposition 3.2(iii), we solved $LP^G(S)$ and used GX to obtain an optimal solution of $LP(S)$. Hence, if there are multiple optimal solutions to $LP(S)$, we ensure that the chosen solution can indeed be expressed as a linear combination of the columns of G.



(a) P1D10A

(b) P1D100A

(c) P3D10A

(d) P3D100A

Figure 5.3: Nonzero elements of primal solutions

Proposition 3.2(iii) indicates that greater variation among the primal solutions of various coalitions makes the column aggregation approach less successful. When ownership is relatively concentrated, as for profile 3, effect (ii) dominates. If a coalition S cannot produce anything, we will have $v(S) = 0$, hence $0 \leq v^G(S) \leq v(S) \Rightarrow v^G(S) = v(S) = 0$ for any choice of the weight matrix $G \in \mathbb{R}_+^{p \times \bar{p}}$.

For the data sets of Figure 5.4, where $n = 5$, $p = 10$, and $r = 100$, row aggregation was performed. Initially, H consisted of one row corresponding to an optimal dual solution of $LP(N)$, and new rows were added by repeatedly solving MIP^H. The number r^* indicates the number of rows that had to be included in H in order to make $v = v^H$. We also report ε_t, the value of $\epsilon^H / v(N)$ when t rows have been added to H.

The results in Figure 5.4 indicate that increased density of A makes the row aggregation approach more successful, i.e., r^* decreases, except for profile 1, where r^* is close to or at the upper bound $2^n - 1$. In order to explain this, consider Figure 5.5, which is similar to Figure 5.3, except that the nonzero elements of optimal *dual* solutions are indicated for every coalition. The $\bullet$'s correspond to $S \in \Omega(H)$, where $\Omega(H)$ is the set of coalitions corresponding to rows of H, and the $|$'s correspond to $S \notin \Omega(H)$[11].

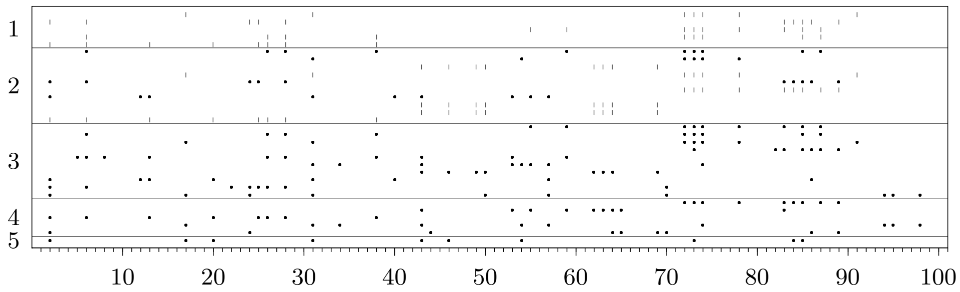
We see that increased density of A, for the examples shown in Figure 5.5, has the effect of decreasing the number of nonzero entries. To see why this is the case, note that a relatively dense A-matrix makes decisions on different products/resources more interdependent. The number of bottlenecks, and hence the number of positive dual prices, will be fewer, as seen for dataset P2D100D and P4D100D. This makes it easier to express the dual solutions of *all* coalitions as combinations of the dual solutions of a relatively small subset of the coalitions.

More concentrated ownership seems to work in the same direction as increased density of A, but we have no good explanation for this phenomenon at present.

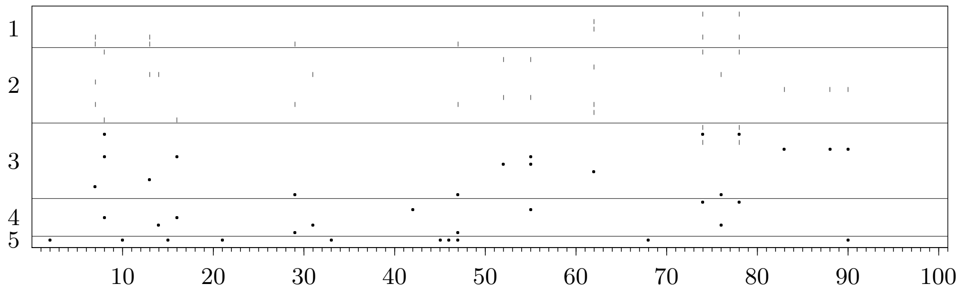
[11] In this latter case, we solved $LP^G(S)$ and, considering Proposition 3.4(iii), used $U^T H$ in order to obtain an optimal solution to $LP(S)$. Hence, if there are multiple optimal dual solutions, we choose one that can be expressed as a linear combination of the rows of H.

Profile	Problem		Density	r^*	ε_1	ε_5	ε_{10}	ε_{20}	ε_{30}
	P1D10D		0.1	31	0.218	0.097	0.040	0.026	0.006
	P1D10E		0.1	31	0.153	0.088	0.050	0.020	0.001
	P1D10F		0.1	31	0.155	0.085	0.043	0.020	0.003
	P1D40D		0.4	31	0.282	0.167	0.081	0.036	0.011
	P1D40E		0.4	31	0.319	0.177	0.076	0.040	0.005
1	P1D40F		0.4	31	0.337	0.197	0.094	0.039	0.003
	P1D70D		0.7	31	0.303	0.186	0.083	0.047	0.017
	P1D70E		0.7	31	0.342	0.210	0.140	0.050	0.007
	P1D70F		0.7	31	0.382	0.202	0.141	0.046	0.009
	P1D100D		1.0	30	0.328	0.239	0.141	0.065	0.000
	P1D100E		1.0	31	0.317	0.208	0.118	0.046	0.002
	P1D100F		1.0	31	0.509	0.201	0.133	0.043	0.002
	P2D10D		0.1	20	0.646	0.404	0.156	0.000	0.000
	P2D10E		0.1	21	0.646	0.332	0.119	0.014	0.000
	P2D10F		0.1	17	0.707	0.484	0.205	0.000	0.000
	P2D40D		0.4	17	0.816	0.502	0.104	0.000	0.000
	P2D40E		0.4	17	0.865	0.443	0.065	0.000	0.000
2	P2D40F		0.4	17	0.734	0.457	0.311	0.000	0.000
	P2D70D		0.7	15	0.852	0.572	0.053	0.000	0.000
	P2D70E		0.7	15	0.867	0.581	0.063	0.000	0.000
	P2D70F		0.7	16	0.844	0.437	0.055	0.000	0.000
	P2D100D		1.0	14	0.927	0.492	0.010	0.000	0.000
	P2D100E		1.0	14	0.791	0.713	0.015	0.000	0.000
	P2D100F		1.0	12	0.942	0.397	0.002	0.000	0.000
	P3D10D		0.1	17	0.715	0.246	0.111	0.000	0.000
	P3D10E		0.1	18	0.639	0.403	0.135	0.000	0.000
	P3D10F		0.1	18	0.725	0.275	0.178	0.000	0.000
	P3D40D		0.4	11	0.743	0.540	0.017	0.000	0.000
	P3D40E		0.4	10	0.869	0.481	0.000	0.000	0.000
3	P3D40F		0.4	16	0.811	0.390	0.257	0.000	0.000
	P3D70D		0.7	9	0.880	0.464	0.000	0.000	0.000
	P3D70E		0.7	9	0.889	0.510	0.000	0.000	0.000
	P3D70F		0.7	11	0.889	0.492	0.045	0.000	0.000
	P3D100D		1.0	8	0.918	0.398	0.000	0.000	0.000
	P3D100E		1.0	9	0.959	0.589	0.000	0.000	0.000
	P3D100F		1.0	10	0.857	0.563	0.000	0.000	0.000
	P4D10D		0.1	12	1.000	0.337	0.060	0.000	0.000
	P4D10E		0.1	9	0.928	0.460	0.000	0.000	0.000
	P4D10F		0.1	10	0.935	0.388	0.000	0.000	0.000
	P4D40D	$\star$	0.4	6	1.000	0.507	0.000	0.000	0.000
	P4D40E	$\star$	0.4	6	0.836	0.732	0.000	0.000	0.000
4	P4D40F	$\star$	0.4	6	0.859	0.730	0.000	0.000	0.000
	P4D70D	$\star$	0.7	6	1.000	0.521	0.000	0.000	0.000
	P4D70E	$\star$	0.7	6	0.950	0.527	0.000	0.000	0.000
	P4D70F	$\star$	0.7	6	1.000	0.383	0.000	0.000	0.000
	P4D100D	$\star$	1.0	6	0.996	0.211	0.000	0.000	0.000
	P4D100E	$\star$	1.0	6	1.000	0.229	0.000	0.000	0.000
	P4D100F	$\star$	1.0	6	0.989	0.620	0.000	0.000	0.000

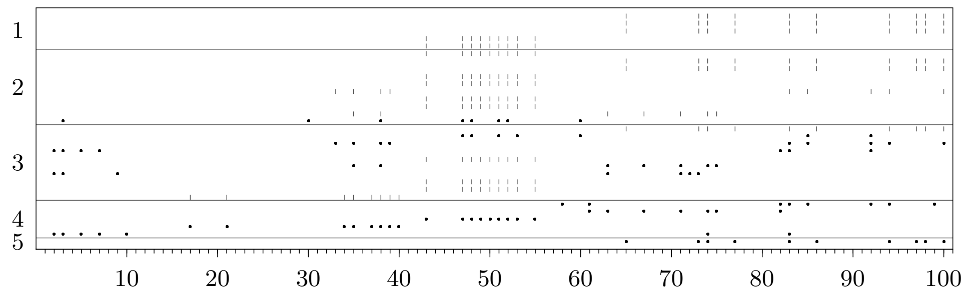
Figure 5.4: Results for row aggregation, with $n = 5$, $p = 10$, and $r = 100$



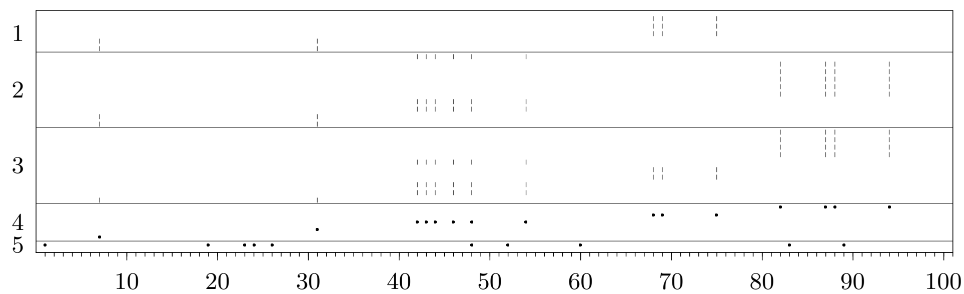
(a) P2D10D



(b) P2D100D



(c) P4D10D



(d) P4D100D

Figure 5.5: Nonzero elements of dual solutions

6 Conclusion

We have shown how the dimensions of linear production games may be reduced by aggregating over columns or rows. In Section 3 we showed that by choosing weights corresponding to optimal solutions of the primal (dual) corresponding to particular coalitions, we can make the aggregated games coincide with the original games for those coalitions. This can be used to create a new game, easier to handle computationally, whose core form a superset (subset) of the original core. This introduces a possible error, and in Section 4 we provide a method, by solving a mixed integer programming problem, for quantifying this error in the special case where the players pool their resources by simply adding them. The solution of this problem can also be used to improve the bound on the original game, by adding a new column (row) to the weight matrix, and suggests a procedure by which the bound can successively be improved. In section 5 we tested this procedure on a number of examples. The examples differ with respect to how concentrated the ownership of the resources are, and the density of the technology matrix A. The results indicate that for column aggregation, the aggregation approach is suitable for problems where ownership is relatively concentrated (dispersed) and A is dense (sparse). Row aggregation seems to be suitable for cases where ownership is relatively concentrated *and* A is dense.

We know from Owen (1975) that some core points can be obtained from the dual solution corresponding to the grand coalition. An interesting question is, if we have constructed H such that $v^H = v$, whether the dual solutions included in H can be given an interpretation in relation to the core of the original game.

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