# A general stochastic calculus approach to insider trading

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**Proposed running head** A general stochastic calculus approach to insider trading

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**Abstract**

The purpose of this paper is to present a general stochastic calculus approach to insider trading. In a market driven by a standard Brownian motion $B(t)$ on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, P)$, by an *insider* we mean a person who has access to a filtration (information) $\mathbb{G} = \{\mathcal{G}_t\}_{0\leq t\leq T}$ which is strictly bigger than the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{0\leq t\leq T}$ of $B(t)$. In this context an insider strategy is represented by a $\mathcal{G}_t$-adapted process $\phi(t)$ and we interpret the portfolio of an insider as the forward integral $\int_0^\infty \phi(t,\omega)dB^-(t)$ defined in [18].
We consider an optimal portfolio problem with logarithmic utility for an insider with access to a general information $\mathcal{G}_t \supset \mathcal{F}_t$ and show that if the value of this problem is finite and an optimal insider portfolio $\pi^*(t)$ exists, then $B_t$ is a $\mathcal{G}_t$-semimartingale, i.e. the enlargement of filtration property holds. This is a partial converse of previously known results in this field.

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# 1 Introduction

How do we model the hedging by an insider in finance? Let $\{B(t)\}_{t\geq 0} = \{B(t,\omega)\}_{t\geq 0, \omega\in\Omega}$ be a standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, P)$ . By an *insider* we mean a person who has access to a filtration $\mathbb{G} = \{\mathcal{G}_t\}_{0\leq t\leq T}$ which is strictly bigger than the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{0\leq t\leq T}$ of $B(t)$. Therefore the question is how to interpret integrals of the form

$$\int_0^T \phi(t,\omega)dB(t) \tag{1.1}$$

where $\phi$ is assumed to be adapted to $\mathcal{G}_t \supset \mathcal{F}_t$.
A natural, and the most common, approach to this question is to assume that $\mathcal{G}_t$ is such that $B(t)$ is a semimartingale with respect to $\mathcal{G}_t$. In this case we can write

$$B(t) = \widehat{B}(t) + A(t), \quad 0 \leq t \leq T \tag{1.2}$$

where $\widehat{B}(t)$ is a $\mathcal{G}_t$-Brownian motion and $A_t$ is a continuous $\mathcal{G}_t$-adapted finite variation process.
If $A_t$ has the form

$$A(t) = \int_0^t \alpha(u)du \tag{1.3}$$

then the process $\alpha(\cdot)$ is called the *information drift* ([9]). In general, if a relation of the form (1.2) holds, then it is natural to define

$$\int_0^T \phi(t,\omega)dB(t) = \int_0^T \phi(t,\omega)d\widehat{B}(t) + \int_0^T \phi(t,\omega)dA(t) \tag{1.4}$$

because both terms of the right-hand side are well-defined.

**Example 1.1** *Let $T_0 \geq T$ and*

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma(B(T_0)); \quad 0 \leq t \leq T \tag{1.5}$$

*i.e. $\mathcal{G}_t$ is the $\sigma$-algebra generated by $\mathcal{F}_t$ and the terminal value $B(T_0)$. Then it can be shown that (see e.g. [11])*

$$\widehat{B}(t) := B(t) - \int_0^t \frac{B(T_0) - B(s)}{T_0 - s} ds; \quad 0 \leq t \leq T \tag{1.6}$$

*is a $\mathcal{G}_t$-Brownian motion. So in this case (1.2) holds with*

$$A(t) := \int_0^t \frac{B(T_0) - B(s)}{T_0 - s} ds; \quad 0 \leq t \leq T \tag{1.7}$$

In general, there are several difficulties with this approach:

(i) How do we know if (1.2) is possible?
(ii) If (1.2) is possible, how do we find $A_t$?
(iii) What do we do if (1.2) is not possible?

Partial answers to (i) and (ii) can be found in the contributions to the book of Jeulin and Yor ([11]). See also [9].
The purpose of this paper is to present a more general approach to insider trading which does not assume that (1.2) holds. One of our main results is in fact a kind of converse: we consider an optimal portfolio problem with logarithmic utility for an insider with access to the information $\mathcal{G}_t \supset \mathcal{F}_t$ and show that if the value of this problem is finite and an optimal insider portfolio $\pi^*(t)$ exists, then in fact (1.2) and (1.3) hold, with $\alpha(t)$ closely related to $\pi^*(t)$. See Theorem 3.5.

# 2 Some preliminaries

A general reference for this section is [15]. See also [17].

## 2.1 The Wiener-Itô chaos expansion theorem

We first recall the classical Wiener-Itô chaos expansion theorem.

A function $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ is called *symmetric* if

$$g(x_{\sigma_1}, \ldots, x_{\sigma_n}) = g(x_1, \ldots, x_n) \tag{2.1}$$

for all permutations $\sigma$ of $(1, \ldots, n)$. If in addition

$$||g||^2_{L^2(\mathbb{R}^n_+)} = \int_{\mathbb{R}^n_+} g^2(x_1, \ldots, x_n)dx_1 \ldots dx_n < \infty \tag{2.2}$$

we say that $g \in \hat{L}^2(\mathbb{R}^n_+)$, the space of the symmetric square integrable functions on $\mathbb{R}^n_+ = \mathbb{R}_+ \times \cdots \times \mathbb{R}_+$, where $\mathbb{R}_+ = [0, \infty)$.

If $g \in \hat{L}^2(\mathbb{R}^n_+)$, we define the $n$-fold iterated Itô integral $I_n(g)$ of $g$ by

$$I_n(g) := n! \int_0^\infty (\int_0^{x_n} (\ldots (\int_0^{x_2} g(x_1, \ldots, x_n)dB(x_1))dB(x_2) \ldots) \ldots dB(x_n) \tag{2.3}$$

Then we have

$$E[I_n(g_n)I_m(g_m)] = \begin{cases} 0 & \text{if} \quad n \neq m \\ n!||g_n||^2_{L^2(\mathbb{R}^n_+)} & \text{if} \quad n = m \end{cases} \tag{2.4}$$

If $n = 0$ we set $I_0(g_0) = g_0$ if $g_0$ is constant.

**Theorem 2.1 (The Wiener-Itô chaos expansion)** *Let $F(\omega)$ be an $\mathcal{F}_\infty$-measurable random variable such that $E[F^2] < \infty$ where $E = E_P$ denotes the expectation with respect to $P$. Then there exists a unique sequence $(f_n)_{n\in\mathbb{N}} \in \hat{L}^2(\mathbb{R}_+^n)$ such that*

$$F(\omega) = \sum_{n=0}^{\infty} I_n(f_n) \tag{2.5}$$

*Moreover, we have the isometry*

$$E[F^2] = \sum_{n=0}^{\infty} n! \|f_n\|^2_{\hat{L}^2(\mathbb{R}_+^n)} \tag{2.6}$$

## 2.2 The Skorohod Integral

Suppose that $\phi(t,\omega)$ is a stochastic process such that

$$\phi(t,\omega) \text{ is } \mathcal{F}_\infty\text{-measurable for all } t \geq 0 \tag{2.7}$$

and

$$E\left[\phi^2(t,\omega)\right] < \infty \text{ for all } t \geq 0 \tag{2.8}$$

Then for each $t$ we can apply the Wiener-Itô chaos expansion to $F(\omega) := \phi(t,\omega)$ and we get that there exist functions $f_n(\cdot,t) \in \hat{L}^2(\mathbb{R}_+^n)$ such that

$$\phi(t,\omega) = \sum_{n=0}^{\infty} I_n(f_n(\cdot,t))$$

Here $f_n(t_1,\ldots,t_n,t)$ is symmetric with respect to the first $n$ variables $t_1,\ldots,t_n$. Therefore the *symmetrization* $\tilde{f}_n$ of $f_n$ as a function of all the $(n+1)$ variables $t_1,\ldots,t_n,t$ is given by, with $t_{n+1} = t$,

$$\tilde{f}_n(t_1,\ldots,t_{n+1}) = \frac{1}{n+1}[f_n(t_1,\ldots,t_{n+1}) + \cdots + f_n(t_2,\ldots,t_{n+1},t_1)] \tag{2.9}$$

where we only sum over those permutations $\sigma$ of the indices $(1,\ldots,n+1)$ which interchange the *last* component with one of the others and leave the rest in place.

**Definition 2.2 (Skorohod Integral)** *Suppose $\phi(t,\omega)$ satisfies (2.7), (2.8) and has a chaos expansion*

$$\phi(t,\omega) = \sum_{n=0}^{\infty} I_n(f_n(\cdot,t)) \tag{2.10}$$

*Assume that*

$$\sum_{n=0}^{\infty}(n+1)!\|\tilde{f}_n\|^2_{L^2(\mathbb{R}_+^{n+1})} < \infty \tag{2.11}$$

*Then we define the Skorohod integral of $\phi$ by*

$$\int_0^{\infty}\phi(t,\omega)\delta B(t) := \sum_{n=0}^{\infty} I_{n+1}(\tilde{f}_n) \tag{2.12}$$

*If (2.11) holds, we say that $\phi$ is Skorohod integrable.*

Note that if $\phi$ is Skorohod integrable then

$$E\left[\int_0^{\infty}\phi(t,\omega)\delta B(t)\right] = 0 \tag{2.13}$$

and

$$E\left[(\int_0^{\infty}\phi(t,\omega)\delta B(t))^2\right] = \sum_{n=0}^{\infty}(n+1)!\|\tilde{f}_n\|^2_{L^2(\mathbb{R}_+^{n+1})} \tag{2.14}$$

**Example 2.3** *We have $\int_0^T B(T,\omega)\delta B(t) = B^2(T) - T$, where in general*

$$\int_0^T \phi(t,\omega)\delta B(t) := \int_0^{\infty}\chi_{[0,T]}(t)\phi(t,\omega)\delta B(t)$$

## 2.3 The Malliavin Derivative

**Definition 2.4** *Let $F(\omega) \in L^2(P)$ have the expansion*

$$F(\omega) = \sum_{n=0}^{\infty} I_n(f_n)$$

*We say that $F$ is Malliavin differentiable and write $F \in \mathbb{D}_{1,2}$ if*

$$\sum_{n=1}^{\infty} nn!\|f_n\|^2_{L^2(\mathbb{R}_+^n)} < \infty \tag{2.15}$$

*In this case, we define the Malliavin derivative of $F$ at $t$, $D_tF(\omega)$, by*

$$\sum_{n=1}^{\infty} nI_{n-1}(f_n(\cdot,t)) \tag{2.16}$$

*Note that*

$$E\left[\int_0^\infty (D_t F(\omega))^2 dt\right] = \sum_{n=1}^\infty n n! \|f_n\|^2_{L^2(\mathbb{R}_+^n)} \tag{2.17}$$

**Example 2.5** *We have*

$$D_t(\int_0^\infty f(s)dB(s)) = f(t) \qquad \textit{for a.a. } t$$

*if $f$ is a deterministic function in $L^2(\mathbb{R}_+)$.*

## 2.4 The Wick product

**Definition 2.6** *Let $F \in L^2(P)$, $G \in L^2(P)$ have the expansion*

$$F(\omega) = \sum_{n=0}^\infty I_n(f_n), \quad G(\omega) = \sum_{m=0}^\infty I_m(g_m)$$

*Then we define the Wick product $(F \diamond G)(\omega)$ by the expansion*

$$(F \diamond G)(\omega) = \sum_{m,n=0}^\infty I_{n+m}(f_n \widehat{\otimes} g_m) = \sum_{k=0}^\infty I_k(\sum_{n+m=k} f_n \widehat{\otimes} g_m) \tag{2.18}$$

*when convergent in $L^2(P)$. Here $\widehat{\otimes}$ denotes symmetrized tensor product, i.e. $f_n \widehat{\otimes} g_m$ is the symmetrization with respect to the $n+m$ variables $x_1, \dots, x_{n+m}$ of the tensor product*

$$(f_n \otimes g_m)(x_1, \cdots, x_n, x_{n+1}, \cdots, x_{n+m}) := f_n(x_1, \cdots, x_n) g_m(x_{n+1}, \cdots, x_{n+m}) \tag{2.19}$$

**Example 2.7** *If $f(t), g(t) \in L^2(\mathbb{R}_+)$ are deterministic, we have*

1. $\int_0^\infty f(t)dB(t) \diamond \int_0^\infty g(t)dB(t) = \int_0^\infty f(t)dB(t) \cdot \int_0^\infty g(t)dB(t) - \int_0^\infty f(t)g(t)dt.$
   *In particular,*

$$B(T) \diamond B(T) = B^2(T) - T$$

2. $$\exp^\diamond\left(\int_0^\infty f(t)dB(t)\right) := \sum_{n=0}^\infty \frac{1}{n!}(\int_0^\infty f(t)dB(t))^{\diamond n} = \exp\left(\int_0^\infty f(t)dB(t) - \frac{1}{2}\int_0^\infty f^2(t)dt\right)$$

3. *$E[F \diamond G] = E[F] \cdot E[G]$, when defined. We remark that here independence is not required.*

If one of the factors is Gaussian, then there is a simple and useful relation between the Wick product and the ordinary product. Let $F \in \mathbb{D}_{1,2}$ and let $h(t) \in L^2(\mathbb{R}_+)$ be deterministic. Then

$$F \cdot \int_0^\infty h(t)dB(t) = F \diamond \int_0^\infty h(t)dB(t) + \int_0^\infty h(t)D_t F dt \tag{2.20}$$

Hence, by (2.13) and 3. above

$$E\left[F \cdot \int_0^\infty h(t)dB(t)\right] = E\left[\int_0^\infty h(t)D_t F dt\right] \tag{2.21}$$

One reason for the importance of the Wick product is that it is closely related to Itô and Skorohod integrals. Let $\phi(t,\omega)$ be càdlàg (i.e. right-continuous with left limits) and Skorohod integrable. Then

$$\int_0^T \phi(t)\delta B(t) = \lim_{\Delta t_j \to 0} \sum_j \phi(t_j) \diamond \Delta B(t_j) \quad \text{in } L^2 \tag{2.22}$$

where $\Delta B(t_j) = B(t_{j+1}) - B(t_j)$, $\Delta t_j = t_{j+1} - t_j$, $\{t_j\}_j$ being a partition of $[0,T]$.

## 2.5 The forward integral

For more information about the forward integral, we refer to [18].

**Definition 2.8** *Let $\phi(t,\omega)$ be a measurable process. The forward integral of $\phi$ is defined by*

$$\int_0^\infty \phi(t,\omega)dB^-(t) = \lim_{\epsilon \to 0} \int_0^\infty \phi(t,\omega)\frac{B(t+\epsilon) - B(t)}{\epsilon}dt \tag{2.23}$$

*if convergent in probability. If the limit exists in $L^2(P)$ we write $\phi \in Dom_2\delta^-$.*

Note that if $\phi$ is càdlàg then

$$\int_0^T \phi(t,\omega)dB^-(t) = \lim_{\Delta t_j \to 0} \sum_j \phi(t_j) \cdot \Delta B(t_j) \tag{2.24}$$

To see this, we argue as follows. We may assume that $\phi(t,\omega) = \sum_{j=1}^n \phi(t_j,\omega)\chi_{[t_j,t_{j+1})}(t)$. Then

$$\int_0^\infty \phi(t,\omega)dB^-(t) = \lim_{\epsilon\to 0}\int_0^\infty \phi(t,\omega)\frac{B(t+\epsilon)-B(t)}{\epsilon}dt =$$
$$\sum_{j=1}^n \phi(t_j)\lim_{\epsilon\to 0}\int_{t_j}^{t_{j+1}} \frac{B(t+\epsilon)-B(t)}{\epsilon}dt =$$
$$\sum_{j=1}^n \phi(t_j)\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{t_j}^{t_{j+1}} (\int_t^{t+\epsilon} dB_u)dt =$$
$$\sum_{j=1}^n \phi(t_j)\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{t_j}^{t_{j+1}} (\int_{u-\epsilon}^{u} dt)dB_u =$$
$$\sum_{j=1}^n \phi(t_j)\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{t_j}^{t_{j+1}} \epsilon\; dB_u =$$
$$\sum_{j=1}^n \phi(t_j)(B(t_{j+1}-B(t_j))$$

If we combine (2.24) with (2.22) and (2.21) we get the following relation between the forward integral, the Skorohod integral and the Malliavin derivative:

**Lemma 2.9** *Let $\phi \in Dom_2\delta^-$ be Skorohod integrable and càdlàg and assume that $\phi(s) \in \mathbb{D}_{1,2}$ for all $s \in [0,T]$. Then*

$$\int_0^T \phi(t,\omega)dB^-(t) = \int_0^T \phi(t,\omega)\delta B(t) + \int_0^T D_{t^+}\phi(t)dt \tag{2.25}$$

*where*

$$D_{t^+}\phi(t) = \lim_{s\to t^+} D_s\phi(t)$$

*In particular,*

$$E\left[\int_0^T \phi(t,\omega)dB^-(t)\right] = E\left[\int_0^T D_{t^+}\phi(t)dt\right] \tag{2.26}$$

Proof. Combining (2.24), (2.20), (2.22) we get

$$\int_0^T \phi(t,\omega)dB^-(t) = \lim_{\Delta t_j\to 0}\sum_j \phi(t_j)\cdot\Delta B(t_j) =$$
$$\lim_{\Delta t_j\to 0}\left\{\sum_j \phi(t_j)\diamond\Delta B(t_j) + \sum_j\int_{t_j}^{t_{j+1}} D_s\phi(t_j)ds\right\} =$$
$$\int_0^T \phi(t,\omega)\delta B(t) + \int_0^T D_{t^+}\phi(t)dt$$

□

**Remark 2.10** *Note that if $\phi(t,\omega)$ is $\mathcal{F}_t$-adapted, then*

$$D_{t^+}\phi(t) = 0$$

*because $D_s\phi(t) = 0$ for all $s > t$.*

We now explain how the forward integral appears naturally in insider modeling. Let $\mathcal{G}_t \supset \mathcal{F}_t$ as in Section 1 and assume that $B(t)$ is a semimartingale with respect to $\mathcal{G}_t$, so that (1.2) holds, i.e.

$$B(t) = \widehat{B}(t) + A(t); \qquad 0 \le t \le T$$

where $\widehat{B}(t)$ is a $\mathcal{G}_t$-adapted Brownian motion, $A_t$ is a $\mathcal{G}_t$-adapted finite variation continuous process. Then we have

**Lemma 2.11** *Let $\phi(s,\omega)$ be as in Lemma 2.9. Then*

$$\int_0^T \phi(t)d\widehat{B}(t) + \int_0^T \phi(t)dA_t = \int_0^T \phi(t)\delta B(t) + \int_0^T D_{t^+}\phi(t)dt \tag{2.27}$$

PROOF. By equation (1.2) and Lemma 2.9, we get

$$\begin{aligned}\int_0^T \phi(t)d\widehat{B}(t) + \int_0^T \phi(t)dA_t = \lim_{\Delta t_j\to 0}\sum_j \phi(t_j)\cdot(\Delta\widehat{B}(t_j) + \Delta A(t_j)) = \\ \lim_{\Delta t_j\to 0}\sum_j \phi(t_j)\cdot\Delta B(t_j) = \\ \lim_{\Delta t_j\to 0}\left\{\sum_j \phi(t_j)\diamond\Delta B(t_j) + \sum_j\int_{t_j}^{t_{j+1}} D_s\phi(t_j)ds\right\} = \\ \int_0^T \phi(t,\omega)\delta B(t) + \int_0^T D_{t^+}\phi(t)dt\end{aligned}$$

□

**Corollary 2.12** *Let $\phi$ be as in Lemma 2.11. Then*

$$\int_0^T \phi(t)d\widehat{B}(t) + \int_0^T \phi(t)dA_t = \int_0^T \phi(t)dB^-(t) \tag{2.28}$$

In view of Corollary 2.12 we see that if (1.2) holds, then it is natural to interpret "$\int_0^T \phi(t,\omega)dB(t)$" as $\int_0^T \phi(t,\omega)dB^-(t)$ in insider trading model, when $\phi(t)$ is $\mathcal{G}_t$-adapted. From now on we adopt this as our mathematical model in insider trading in general, without assuming that (1.2) holds. Thus in (1.1) we put

$$\int_0^T \phi(t,\omega)dB(t) := \int_0^T \phi(t)dB^-(t) = \int_0^T \phi(t,\omega)\delta B(t) + \int_0^T D_{t^+}\phi(t)dt \tag{2.29}$$

for all processes $\phi(t,\omega)$ which are Skorohod-integrable and such that $D_{t^+}\phi(t)$ exists for a.a. $t$ and

$$E\left[(\int_0^T |D_{t^+}\phi(t)|dt)^2\right] < \infty \tag{2.30}$$

**Definition 2.13** *A stochastic process $\phi(t,\omega)$ is called an admissible insider portfolio if*

1. *$\phi(t)$ is $\mathcal{G}_t$-adapted*
2. *$\phi(t)$ is Skorohod-integrable over $[0,T]$*
3. *$D_{t^+}\phi(t)$ exists for almost all $t \in [0,T]$ and*

$$E\left[(\int_0^T |D_{t^+}\phi(t)|dt)^2\right] < \infty$$

*The set of all admissible portfolios is denoted by $\mathcal{A}$.*

# 3 Optimal portfolio of an insider

Suppose that our financial market has the form

$$\textbf{Bond price} \quad dS_0(t) = r(t)S_0(t)dt; \quad S_0(0) = 1 \tag{3.1}$$

$$\textbf{Stock price} \quad dS_1(t) = S_1(t)[\mu(t)dt + \sigma(t)dB(t)]; \quad S_1(0) = x \tag{3.2}$$

Here $r(t), \mu(t), \sigma(t)$ are $\mathcal{F}_t$-adapted processes, where as before $\mathcal{F}_t$ is the filtration of the Brownian motion $B(t)$.
Now fix an *insider filtration* $\mathbb{G} := \{\mathcal{G}_t\}_{0\le t\le T} \supset \{\mathcal{F}_t\}_{0\le t\le T} := \mathbb{F}$. Let $\pi(t)$ be a portfolio denoting the fraction of the wealth invested in the stock at time $t$ by an insider. Thus $\pi(t)$ is a *$\mathcal{G}_t$-adapted* stochastic process. The corresponding wealth $X(t) = X^{(\pi)}(t)$ of the insider at time $t$ will then satisfy the equation

$$\begin{aligned} dX(t) = r(t)(1-\pi(t))X(t)dt + \pi(t)X(t)[\mu(t)dt + \sigma(t)dB^-(t)] = \\ X(t)\left[\{r(t) + (\mu(t) - r(t))\pi(t)\}dt + \sigma(t)\pi(t)dB^-(t)\right]; \quad X(0) = x_0 \end{aligned} \tag{3.3}$$

Fix a terminal time $T > 0$ and a utility function

$$U : \mathbb{R} \longrightarrow [-\infty, \infty)$$

assumed to be concave, nondecreasing and upper semicontinuous. Consider the following *insider optimal portfolio problem*:

**PROBLEM 3.1**: Find $V_T^{\mathbb{G}} \in \mathbb{R}$ and $\pi^* \in \mathcal{A}$ such that

$$V_T^{\mathbb{G}} := \sup_{\pi \in \mathcal{A}} E\left[U(X^{(\pi)}(T))\right] = E\left[U(X^{(\pi^*)}(T))\right] \tag{3.4}$$

We recall that here $\mathcal{A}$ denotes the set of *admissible portfolios* (Definition 2.13). We call $V_T^{\mathbb{G}} \leq \infty$ the *value* of the optimal portfolio problem and $\pi^* \in \mathcal{A}$ the *optimal portfolio* (if it exists).

This problem was first studied by Pikovski and Karatzas ([12]). They assume that

$$U(x) = \log x \tag{3.5}$$

and that $\mathcal{G}_t$ has the form

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma(L); \quad 0 \leq t \leq T \tag{3.6}$$

for some fixed random variable $L$. They also assume that there exists a $\mathcal{G}_t$-adapted process $\alpha(t)$ such that

$$\widehat{B}(t) = B(t) - \int_0^t \alpha(s)ds \tag{3.7}$$

is a $\mathcal{G}_t$-Brownian motion.
Subsequently, this problem has been studied by many authors, but to the best of our knowledge they all assume that (3.6) and (3.7) hold. See for example Leon, Navarro and Nualart [14] and Imkeller [9] and the references therein. The recent paper Corcuera et al. [3] has a different, but related assumption. The purpose of our paper is to study Problem 3.1 for a general filtration $\mathcal{G}_t \supset \mathcal{F}_t$, without assuming (3.6) or (3.7).

We first prove the following result of independent interest:

**Theorem 3.1** *Let $\xi(t)$ and $\eta(t)$ be $\mathcal{G}_t$-adapted processes such that $\int_0^t (|\xi(s)| + \eta^2(s))\,ds < \infty$ and $\eta \in \mathcal{A}$ for all $t > 0$. Then the equation*

$$dX(t) = X(t)[\xi(t)dt + \eta(t)dB^-(t)]; \quad X(0) = x_0 \tag{3.8}$$

*has the unique solution*

$$X(t) = x_0 \exp\left(\int_0^t \left\{\xi(s) - \frac{1}{2}\eta^2(s)\right\} ds + \int_0^t \eta(s)dB^-(s)\right); \quad t \geq 0 \tag{3.9}$$

**Remark 3.2** *Theorem 3.1 is an extension of Theorem 2.13 in [14], where the same solution formula is obtained (by a different method) for the special case when (3.6) holds, i.e.*

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma(L); \quad 0 \leq t \leq T$$

*$L$ being a fixed $\mathcal{F}_T$-measurable random variable.*

**Remark 3.3** *For notational simplicity we use the notation*

$$D_t f(t) \quad \textit{for} \quad D_{t^+} f(t)$$

*from now on.*

PROOF. 1.*Existence*

Put

$$Y(t) = x_0 \exp\left(\int_0^t \left\{\xi(s) - \frac{1}{2}\eta^2(s)\right\} ds + \int_0^t \eta(s) dB^-(s)\right) =$$
$$x_0 \exp\left(\int_0^t \left\{\xi(s) - \frac{1}{2}\eta^2(s) + D_s\eta(s)\right\} ds + \int_0^t \eta(s)\delta B(s)\right) \tag{3.10}$$

The by the Itô formula for Skorohod integrals (Theorem 6.1, [16]) we get

$$dY(t) = Y(t)\left[(\xi(t) - \frac{1}{2}\eta^2(t) + D_t\eta(t))dt + \eta(t)\delta B(t)\right] +$$
$$+\frac{1}{2}Y(t)\eta^2(t)dt + Y(t)\eta(t)\left[D_t f(t) + \int_0^t D_t\eta(u)\delta B(u)\right] dt =$$
$$Y(t)\xi(t)dt + Y(t)D_t\eta(t))dt + Y(t)\eta(t)\delta B(t) + Y(t)\eta(t)\left[D_t f(t) + \int_0^t D_t\eta(u)\delta B(u)\right] dt \tag{3.11}$$

where

$$f(t) = \int_0^t \left\{\xi(s) - \frac{1}{2}\eta^2(s) + D_s\eta(s)\right\} ds \tag{3.12}$$

Now (see (2.25))

$$Y(t)\eta(t)\delta B(t) = Y(t)\eta(t)dB^-(t) - D_t(Y(t)\eta(t))dt =$$
$$Y(t)\eta(t)dB^-(t) - Y(t)D_t\eta(t)dt - D_tY(t)\eta(t)dt \tag{3.13}$$

Therefore (3.11) can be written

$$dY(t) = Y(t)\xi(t)dt + Y(t)\eta(t)dB^-(t) +$$
$$+ \left(-D_tY(t)\eta(t) + Y(t)\eta(t)\left[D_tf(t) + \int_0^t D_t\eta(u)\delta B(u)\right]\right) dt$$

So $X_t := Y_t$ satisfies equation (3.8) if and only if

$$D_tY(t) = Y(t)\left[D_tf(t) + \int_0^t D_t\eta(u)\delta B(u)\right]$$

i.e.

$$D_t(\log Y(t)) = D_tf(t) + \int_0^t D_t\eta(u)\delta B(u) \tag{3.14}$$

By (3.10) we see that

$$D_t(\log Y(t)) = D_tf(t) + D_t\left(\int_0^t \eta(u)\delta B(u)\right) =$$
$$D_tf(t) + D_t\left(\int_0^T \chi_{[0,t]}(u)\eta(u)\delta B(u)\right) =$$
$$D_tf(t) + \int_0^t D_t\eta(u)\delta B(u) + \lim_{s\to t^+}\eta(s)\chi_{[0,t]}(s) =$$
$$D_tf(t) + \int_0^t D_t\eta(u)\delta B(u)$$

The last line is (3.14). This proves that the process $X_t = Y_t$ given by (3.9) solves equation (3.8).

2. *Uniqueness*
Let $X_1(t)$ be some solution of (3.8). Then

$$dX_1(t) = \{X_1(t)\xi(t) + D_t(X_1(t)\eta(t))\}\, dt + \eta(t)\delta B(t)$$

Define

$$Z(t) = \exp\left(-\int_0^t \left\{\xi(s) - \frac{1}{2}\eta^2(s)\right\} ds - \int_0^t \eta(s)dB^-(s)\right) =$$
$$\exp\left(\int_0^t \left\{-\xi(s) - \frac{1}{2}\eta^2(s)\right\} ds + \int_0^t (-\eta(s))dB^-(s)\right)\exp\left(\int_0^t \eta^2(s)ds\right)$$

Then by the multi-dimensional Itô formula for Skorohod integrals ([16], Theorem 6.4) and by Part 1

$$
\begin{aligned}
dZ(t) &= Z(t)\left[(-\xi(t)+\eta^2(t))dt - \eta(t)dB^-(t)\right] \\
&= \left\{Z(t)(-\xi(t)+\eta^2(t)) - D_t(Z(t)\eta(t))\right\}dt - \eta(t)\delta B(t)
\end{aligned}
$$

Hence by the multi-dimensional Itô formula again

$$
\begin{aligned}
d\left(X_1(t)Z(t)\right) &= X_1(t)dZ(t) + Z(t)dX_1(t) - Z(t)X_1(t)\eta(t)^2 dt \\
&+ \left[X_1(t)\eta(t)D_tZ(t) - \eta(t)Z(t)D_tX_1(t)\right]dt \\
&= X_1(t)\left\{\left[Z(t)(-\xi(t)+\eta^2(t)) - D_t(Z(t)\eta(t))\right]dt - \eta(t)\delta B(t)\right\} \\
&+ Z(t)\left\{\left[\xi(t)X_1(t) + D_t(X_1(t)\eta(t))\right]dt + \eta(t)\delta B(t)\right\} \\
&- X_1(t)Z(t)\eta(t)^2dt + \eta(t)\left[X_1(t)D_tZ(t) - Z(t)D_tX_1(t)\right]dt \\
&= 0
\end{aligned}
$$

Hence $X_1(t)Z(t)$ is constant and therefore

$$
X_1(t) = X_1(0)Z^{-1}(t) = X(t) \quad \text{(defined in (3.9))}
$$

□

We now return to Problem 3.1. We consider the case when (3.5) holds, i.e. $U(x) = \log x$. By Theorem 3.1 the solution $X(t) = X^{(\pi)}(t)$ of the wealth equation (3.3) is

$$
X(t) = x_0 \exp\left(\int_0^t \left\{r(s) + (\mu(s)-r(s))\pi(s) - \frac{1}{2}\sigma^2(s)\pi^2(s)\right\}ds + \int_0^t \sigma(s)\pi(s)dB^-(s)\right) \tag{3.15}
$$

where $t \geq 0$. Hence

$$
\begin{aligned}
&E\left[\log\frac{X(T)}{x_0}\right] = \\
&E\left[\int_0^T \left\{r(s) + (\mu(s)-r(s))\pi(s) - \frac{1}{2}\sigma^2(s)\pi^2(s)\right\}ds + \int_0^T \sigma(s)\pi(s)dB^-(s)\right] = \\
&E\left[\int_0^T \left\{r(s) + (\mu(s)-r(s))\pi(s) - \frac{1}{2}\sigma^2(s)\pi^2(s) + D_s(\sigma(s)\pi(s))\right\}ds\right]
\end{aligned} \tag{3.16}
$$

Now we write

$$
\pi(s) = \frac{\mu(s)-r(s)}{\sigma^2(s)} + \frac{\alpha(s)}{\sigma(s)} \tag{3.17}
$$

for some $\mathcal{G}_t$-adapted process $\alpha(t)$. Substituting ( 3.17) in (3.16), we obtain

$$E\left[\log\frac{X(T)}{x_0}\right]=E\left[\int_0^T\left\{r(s)+\frac{1}{2}\frac{(\mu(s)-r(s))^2}{\sigma^2(s)}+D_s\alpha(s)-\frac{1}{2}\alpha^2(s)\right\}ds\right]$$

Therefore, to maximize $E\left[\log\frac{X^{(\pi)}(T)}{x_0}\right]$ over all $\mathcal{G}_t$-adapted processes $\pi(t)\in\mathcal{A}$ it suffices to maximize

$$H(\alpha):=E\left[\int_0^T\left\{D_s\alpha(s)-\frac{1}{2}\alpha^2(s)\right\}ds\right] \tag{3.18}$$

for all $\mathcal{G}_t$-adapted processes $\alpha(t)\in\mathcal{A}$.

**Remark 3.4** *Note that if $\mathcal{G}_t=\mathcal{F}_t$ then $D_s\alpha(s)(=D_{s+}\alpha(s))=0$ for all $\alpha\in\mathcal{A}$ and therefore it is optimal to choose $\alpha=0$ in this case. Hence, in this case we have the well-known result*

$$V_T^{\mathbb{F}}=E\left[\int_0^T\left\{r(s)+\frac{1}{2}\frac{(\mu(s)-r(s))^2}{\sigma^2(s)}\right\}ds\right],\quad \pi^*(s)=\frac{\mu(s)-r(s)}{\sigma^2(s)} \tag{3.19}$$

To maximize (3.18), we apply a variational argument.
Suppose $\alpha(t)$ maximizes $H(\alpha)$. Then if $y\in\mathbb{R}$ and $\theta(t)$ is another $\mathcal{G}_t$-adapted process in $\mathcal{A}$ we have that the function

$$y\longrightarrow H(\alpha+y\theta)$$

is maximal for $y=0$. Therefore

$$\begin{aligned}&0=\frac{d}{dy}H(\alpha+y\theta)_{y=0}=\\&\frac{d}{dy}E\left[\int_0^T\left\{D_s(\alpha(s)+y\theta(s))-\frac{1}{2}(\alpha(s)+y\theta(s))^2\right\}\right]_{y=0}=\\&E\left[\int_0^T\{D_s\theta(s)-\alpha(s)\theta(s)\}\,ds\right]\end{aligned} \tag{3.20}$$

Now fix $t\in[0,T)$ and apply (3.20) to the process

$$\theta(s)=\chi_{[t,t+h)}(s)\theta(t);\quad 0\le s\le T$$

where $h>0$ is a constant such that $t+h\le T$ and $\theta(t)$ is $\mathcal{G}_t$-measurable and Malliavin differentiable. Then (3.20) and (2.26) give

$$\begin{aligned}0=E\left[\int_t^{t+h}\{D_u\theta(t)-\alpha(u)\theta(t)\}\,du\right]=\\E\left[\theta(t)\int_t^{t+h}1\cdot\delta B(u)-\theta(t)\int_t^{t+h}\alpha(u)du\right]=\\E\left[\theta(t)\left\{B(t+h)-B(t)-\int_t^{t+h}\alpha(u)du\right\}\right]\end{aligned}$$

Since this holds for all $\mathcal{G}_t$-measurable Malliavin differentiable $\theta(t)$ we conclude that

$$E\left[\left(B(t+h)-B(t)-\int_t^{t+h}\alpha(u)du\right)\Big|\mathcal{G}_t\right]=0$$

This is equivalent to saying that the process

$$\widehat{B}(t):=B(t)-\int_0^t\alpha(u)du;\quad 0\le t\le T$$

is a $\mathcal{G}_t$-martingale and hence a $\mathcal{G}_t$-Brownian motion. We have proved

**Theorem 3.5** *Suppose that there exists an optimal portfolio $\pi^*(t)$ for Problem 3.1 when $U(x)=\log x$ with $V_T^{\mathbb{G}}<\infty$. Then $\pi^*(s)$ has the form*

$$\pi^*(s)=\frac{\mu(s)-r(s)}{\sigma^2(s)}+\frac{\alpha(s)}{\sigma(s)} \tag{3.21}$$

*where $\alpha(t)$ is a $\mathcal{G}_t$-adapted process such that*

$$\widehat{B}(t):=B(t)-\int_0^t\alpha(u)du \tag{3.22}$$

*is a $\mathcal{G}_t$-Brownian motion. The corresponding value is*

$$V_T^{\mathbb{G}}=\log x_0+E\left[\int_0^T\left\{r(s)+\frac{1}{2}\frac{(\mu(s)-r(s))^2}{\sigma^2(s)}+D_s(\alpha(s))-\frac{1}{2}\alpha^2(s)\right\}ds\right] \tag{3.23}$$

**Remark 3.6** *This result provides a kind of converse of the result of Pikovski and Karatzas ([12],[13]) and others, in the sense that if $V_T^{\mathbb{G}}<\infty$ then (3.22) is in fact necessary for the existence of the optimal portfolio.*

Now that (3.22) and (3.23) are established, we get from (3.15) and (2.28) that

$$\begin{aligned}
&E\left[\log\frac{X(T)}{x_0}\right]=\\
&E\left[\int_0^T\left\{r(s)+(\mu(s)-r(s))\pi(s)-\frac{1}{2}\sigma^2(s)\pi^2(s)\right\}ds+\int_0^T\sigma(s)\pi(s)d\widehat{B}(s)+\int_0^T\sigma(s)\pi(s)\alpha(s)ds\right]=\\
&E\left[\int_0^T\left\{r(s)+(\mu(s)-r(s))\pi(s)-\frac{1}{2}\sigma^2(s)\pi^2(s)+\sigma(s)\pi(s)\alpha(s)\right\}ds\right]
\end{aligned} \tag{3.24}$$

since

$$E\left[\int_0^T\sigma(s)\pi(s)d\widehat{B}(s)\right]=0 \tag{3.25}$$

Using the substitution (3.17) we transform this to

$$E\left[\log\frac{X(T)}{x_0}\right]=E\left[\int_0^T\left\{r(s)+\frac{1}{2}\frac{(\mu(s)-r(s))^2}{\sigma^2(s)}+\frac{1}{2}\alpha^2(s)+\alpha(s)\frac{(\mu(s)-r(s))}{\sigma(s)}\right\}ds\right] \tag{3.26}$$

by the same calculation as the one following (3.17). Note that since $\beta(t):=\dfrac{(\mu(t)-r(t))}{\sigma(t)}$ is $\mathcal{F}_t$-adapted, we have by (3.22)

$$E\left[\int_0^T\beta(t)\alpha(t)dt\right]=E\left[\int_0^T\beta(t)dB(t)-\int_0^T\beta(t)d\widehat{B}(t)\right]=0 \tag{3.27}$$

Therefore (3.26) gives

$$E\left[\log\frac{X(T)}{x_0}\right]=E\left[\int_0^T\left\{r(s)+\frac{1}{2}\frac{(\mu(s)-r(s))^2}{\sigma^2(s)}+\frac{1}{2}\alpha^2(s)\right\}ds\right] \tag{3.28}$$

Combining this with Theorem 3.5 and (3.19) we get

**Theorem 3.7** *Suppose that there exists an optimal portfolio $\pi^*(t)$ for Problem 3.1 when $U(x)=\log x$ with $V_T^{\mathbb{G}}<\infty$. Then, as in Theorem 3.5,*

$$\pi^*(s)=\frac{\mu(s)-r(s)}{\sigma^2(s)}+\frac{\alpha(s)}{\sigma(s)} \tag{3.29}$$

*where $\alpha(t)$ is such that*

$$\widehat{B}(t):=B(t)-\int_0^t\alpha(u)du \tag{3.30}$$

*is a $\mathcal{G}_t$-Brownian motion. The corresponding value is*

$$V_T^{\mathbb{G}}=V_T^{\mathbb{F}}+\frac{1}{2}E\left[\int_0^T\alpha^2(s)ds\right] \tag{3.31}$$

Here $V_T^{\mathbb{F}}$ represents the value of the honest trader and $\frac{1}{2}E\left[\int_0^T\alpha^2(s)ds\right]$ the *additional value (utility)* obtained by the insider. Theorem 3.7 represents a partial converse of Theorem 2.1 in [9].

As an illustration, we give the following well-known example.

**Example 3.8** *Suppose $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(B(T_0))$ for some constant $T_0 \geq T$. Then we have seen in Example 1.1 that*

$$\alpha(s) := \frac{B(T_0) - B(s)}{T_0 - s} \tag{3.32}$$

*satisfies (3.30). It was proved in [12] that in this case the additional utility for the insider is*

$$\frac{1}{2}E\left[\int_0^T \alpha^2(s)ds\right] = \frac{1}{2}\int_0^T \frac{1}{T_0 - s}ds = \frac{1}{2}\log\left(\frac{T_0}{T_0 - T}\right) \tag{3.33}$$

*If $T_0 = T$ then $V_T^{\mathbb{G}} = \infty$!*

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