European Option Pricing and Hedging with both Fixed and Proportional Transaction Costs

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Abstract

In this paper we extend the utility based option pricing and hedging approach, pioneered by Hodges and Neuberger (1989) and further developed by Davis, Panas, and Zariphopoulou (1993), for the market where each transaction has a fixed cost component. We present a model, where investors have a $CARA$ utility and finite time horizons, and derive some properties of reservation option prices. The model is then numerically solved for the case of European call options. We examine the effects on the reservation option prices and the corresponding optimal hedging strategies of varying the investor's ARA and the drift of the risky asset. Our examination suggests distinguishing between two major types of investors behavior: the net investor and the net hedger, in relation to the pricing and hedging of options. The numerical results of option pricing and hedging for both of these types of investors are presented. We also try to reconcile our findings with such empirical pricing biases as the bid-ask spread, the volatility smile and the volatility term-structure.

1 Introduction

The break-trough in option valuation theory starts with the publication of two seminal papers by Black and Scholes (1973) and Merton (1973). In both papers authors introduced a continuous time model of a complete friction-free market where a price of a stock follows a geometric Brownian motion. They presented a self-financing, dynamic trading strategy consisting of a riskless security and a risky stock, which replicate the payoffs of an option. Then they argued that the absence of arbitrage dictates that the option price be equal to the cost of setting up the replicating portfolio.

In the presence of transaction costs in capital markets the absence of arbitrage argument is no longer valid, since perfect hedging is impossible. Due to the infinite variation of the geometric Brownian motion, the continuous replication policy incurs an infinite amount of transaction costs over any trading interval no matter how small it might be. A variety of approaches have been suggested to deal with the problem of option pricing and hedging with transaction costs. We maintain that the utility based approach, pioneered by Hodges and Neuberger (1989), produces the most "optimal" polices. The rationale under this approach is as follows. Since entering an option contract involves an unavoidable element of risk, in pricing and hedging options, one must consider the investor's attitude toward risk. The other alternative approaches are mainly preference-free and concerned with the "financial engineering" problem of either replicating or super-replicating option payoffs. These approaches are generally valid only in a discrete-time model with a relatively small number of time intervals.

The key idea behind the utility based approach is the indifference argument. The writing price of an option is defined as the amount of money that makes the investor indifferent, in terms of expected utility, between trading in the market with and without writing the option. In a similar way, the purchase price of an option is defined as the amount of money that makes the investor indifferent between trading in the market with and without buying the option. These two prices are also referred to as the investor's *reservation write price* and the investor's *reservation purchase price*. In many respects a reservation option price is determined in a similar manner to a *certainty equivalent* within the expected utility framework, which is an entirely traditional approach to pricing in economics.

The utility based option pricing approach is perhaps not entirely satisfactory due to some apparent drawbacks: First, the method does not price options within a general equilibrium framework, and, hence, instead of a unique price one gets two price bounds that depend on the investor's utility function, which is largely unknown. Second, the linear pricing rule from the complete and frictionless market does not apply to the reservation option prices. Generally, the unit reservation purchase price decreases in the number of options, and the unit reservation write price increases in the number of options. Nevertheless, the method is well-defined in contrast to ad-hoc delta hedging in the presence of transaction costs, and, moreover, it yields a narrow price band which is much more interesting than the extreme bounds of a super-replicating strategy[1]. Some attractive features of these bounds are as follows. It can be proved that in a friction-free market the two reservation prices coincide with the Black-Sholes price. The bounds are robust with respect to the choice of utility function since the level of absolute risk aversion seems to be the only important determinant. Judging against the best possible tradeoff between the risk and the costs of a hedging strategy, the utility based approach seems to achieve excellent empirical performance (see Martellini and Priaulet (2000), Clewlow and Hodges (1997), and Mohamed (1994)). Quite often one points out that the numerical calculations of reservation option prices are very time-consuming. Considering the exploding development within the computer industry this problem gradually becomes less and less important. All these suggest that the utility based approach is a very reasonable and applicable option pricing method.

The starting point for the utility based option pricing approach is to consider an investor who faces transaction costs and maximizes expected utility of end-of-period wealth. The introduction of transaction costs adds considerable complexity to the utility maximization problem[2] as opposed to the case with no transaction costs. The problem is simplified if one assumes that the transaction costs are proportional to the amount of the risky asset traded, and there are no transaction costs on trades in the riskless asset. In this case the problem amounts to a *stochastic singular control* problem that was solved by Davis and Norman (1990). Shreve and Soner (1994) studied

[1] Shreve, Soner, and Cvitanic (1995) proved, in particular, that in a continuous time model with proportional transaction costs the costs of buying one share of stock is the cheapest super-replicating policy

[2] In this paper we consider the two-asset problem only

this problem applying the theory of viscosity solutions to Hamilton-Jacobi-Bellman (HJB) equations (see, for example, Flemming and Soner (1993) for that theory).

In the presence of proportional transaction costs the solution indicates that the portfolio space is divided into three disjoint regions, which can be specified as the Buy region, the Sell region, and the no-transaction (NT) region. If a portfolio lies in the Buy region, the optimal strategy is to buy the risky asset until the portfolio reaches the boundary between the Buy region and the NT region, while if a portfolio lies in the Sell region, the optimal strategy is to sell the risky asset until the portfolio reaches the boundary between the Sell region and the NT region. If a portfolio lies in the NT region, it is not adjusted at that time.

The problem is further simplified if the investor's utility function is of the exponential type ($CARA$ investor). In this case the option price and hedging strategy are independent of the investor's holdings in the risk-free asset and the computational complexity is considerably reduced.

In all the papers with numerical solutions (except for Andersen and Damgaard (1999)) the authors used the method of the *Markov chain approximation* (see, for example, Kushner and Dupuis (1992)). Using this method, the solution to the utility maximization problem is obtained by turning the stochastic differential equations into Markov chains in order to apply the discrete-time dynamic programming algorithm.

Hodges and Neuberger (1989) introduced the approach and calculated numerically optimal hedging strategies and reservation prices of European call options using a binomial lattice, without really proving the convergence of the numerical method. For simplicity they chose the drift of the risky asset equal to the risk-free rate of return. Davis et al. (1993) rigorously analyzed the same model, showed that the value function of the problem is a unique viscosity solution of a fully nonlinear variational inequality. They proved the convergence of discretization schemes based on the binomial approximation of the stock price, and presented computational results for the reservation write price of an option. Whalley and Wilmott (1997) did an asymptotic analysis of the model of Hodges and Neuberger (1989) and Davis et al. (1993) assuming that transaction costs are small. They show that the optimal hedging strategy is to hedge to a particular bandwidth[3]. Clewlow and

[3]That is, the optimal strategy is not to rehedge until the position moves out of the line

Hodges (1997) extended the earlier work of Hodges and Neuberger (1989) by presenting a more efficient computational method, and a deeper study of the optimal hedging strategy.

Constantinides and Zariphopoulou (1999a) considered an infinite horizon economy with multiple securities having time stationary returns, a constant interest rate, and any number of derivatives. In this model upper and lower bounds on reservation write and purchase prices, respectively, are obtained for the class of investors with time additive preferences and a utility function of the power type. Constantinides and Zariphopoulou (1999b) derived analytic bounds on the reservation write price of a European-style contingent claim.

Andersen and Damgaard (1999) were the first to compute the reservation prices of European-style options in a market with two risky securities and an investor with HARA utility. They suggested using the method of convex optimization. Unfortunately, using this method the calculations are highly time-consuming and were implemented for a 9-period model only. They found that the reservation option prices based on the exponential utility function is a good approximation of the reservation prices implied by HARA utility function with the same initial level of absolute risk aversion. Damgaard (2000a) and Damgaard (2000b) computed the reservation prices of European and American-style options for an investor having a HARA utility. He examined how the reservation prices and corresponding portfolio policies depend on the risk aversion coefficient, the level of the investor's initial wealth, and the drift of the underlying risky asset.

To the best of our knowledge, no one has calculated reservation option prices and hedging strategies in the market with a fixed cost component[4]. The solution to the utility maximization/optimal portfolio selection problem where each transaction has a fixed cost component is more complicated and is based on the theory of *stochastic impulse controls* (see, for example, Bensoussan and Lions (1984) for that theory). The first application of this theory to a consumption-investment problem was done by Eastham and Hastings (1988). They developed a general theory and showed that solving this general problem requires the solution of a system of so-called quasi-

with the perfect hedge position by a certain amount

[4] Clewlow and Hodges (1997) made computations for a 3-period model in the market with both fixed and proportional transaction costs, without really presenting a continuous-time model for this case

variational inequalities (QVI). This initial work was extended by Hastings (1992) and Korn (1998), and was further developed by Øksendal and Sulem (1999) and Chancelier, Øksendal, and Sulem (2000).

In this paper we extend the works of Hodges and Neuberger (1989), Davis et al. (1993), and Clewlow and Hodges (1997), who computed reservation option prices in the model with a $CARA$ investor and the presence of proportional transaction costs only. First, we formulate the option pricing and hedging problem for the $CARA$ investor in the market with both fixed and proportional transaction costs and derive some properties of reservation option prices. Then we numerically solve the problem for the case of European call options applying the method of the Markov chain approximation. The solution indicates that in the presence of both fixed and proportional transaction costs, most of the time, the portfolio space can again be divided into three disjoint regions (Buy, Sell, and NT), and the optimal policy is described by four boundaries. The Buy and the NT regions are divided by the lower no-transaction boundary, and the Sell and the NT regions are divided by the upper no-transaction boundary. If a portfolio lies in the Buy region, the optimal strategy is to buy the risky asset until the portfolio reaches the Buy target boundary. Similarly, if a portfolio lies in the Sell region, the optimal strategy is to sell the risky asset until the portfolio reaches the Sell target boundary.

Our examination of the effects on the reservation option prices and the corresponding optimal hedging strategies of varying the investor's ARA and the drift of the risky asset suggests distinguishing between two major types of investors behavior in relation to the pricing and hedging of options: the *net investor* and the *net hedger*. The net investor, as well as the net hedger, has his own pattern of pricing and hedging options. Both the net investor's reservation option prices are above the BS-price, and they are very close to each other. The net investor overhedges both long and short option positions as compared to the BS-strategy. The net hedger's reservation purchase price is generally below the BS-price, and the net hedger's reservation write price is above the BS-price. Here the difference between the two prices depends on the level of the net hedger's absolute risk aversion and the level of transaction costs. Judging against the BS-strategy, the net hedger underhedges out-of-the-money and overhedges in-the-money long option positions. When the net hedger writes options, his strategy is quite the opposite. The net hedger overhedges out-of-the-money and underhedges in-the-money short option positions. The remarkable features of the net hedger's strategy are jumps to zero in target amounts in the stock when the stock price decreases below some certain levels. And at these levels the NT region widens.

We point out on two possible resolutions of the question: Under what circumstances will a writer and a buyer agree on a common price for an option? In the model with both fixed and proportional transaction costs under certain model parameters there occurs a situation when the reservation purchase price is higher than the reservation write price. The other possibility arises when a writer and a buyer, both of them being net investors in the underlying stocks, face different transaction costs in the market.

We also try to reconcile our findings with such empirical pricing biases as the bid-ask spread, the volatility smile and the volatility term structure. Our general conclusion here is that these empirical phenomena could not be accounted for solely by the presence of transaction costs.

The rest of the paper is organized as follows. Section 2 presents the continuous-time model and the basic definitions. In Section 3 we derive some important properties of the reservation option prices. Section 4 is concerned with the construction of a discrete time approximation of the continuous time price processes used in Section 2, and the solution method. The numerical results for European-style call options are presented in Section 5. Section 6 concludes the paper and discusses some possible extensions.

2 The Continuous Time Formulation

Originally, we consider a continuous-time economy, similar to that of Øksendal and Sulem (1999), with one risky and one risk-free asset. Let $(\Omega, \mathcal{F}, P)$ be a probability space with a given filtration $\{\mathcal{F}_t\}_{0 \le t \le T}$. The risk-free asset, which we will refer to as the bank account, pays a constant interest rate of $r \ge 0$, and, consequently, the evolution of the amount invested in the bank, x_t, is given by the ordinary differential equation

$$dx_t = r x_t dt \tag{1}$$

We will refer to the risky asset to as the stock, and assume that the price of the stock, S_t, evolves according to a geometric Brownian motion defined by

$$dS_t = \mu S_t dt + \sigma S_t dB_t \tag{2}$$

where μ and σ are constants, and B_t is a one-dimensional $\mathcal{F}_t$-Brownian motion.

The investor holds x_t in the bank account and the amount y_t in the stock at time t. We assume that a purchase or sale of stocks of the amount ξ incurs a transaction costs consisting of a sum of a fixed cost $k \geq 0$ (independent of the size of transaction) plus a cost $\lambda|\xi|$ proportional to the transaction ($\lambda \geq 0$). These costs are drawn from the bank account.

If the investor has the amount x in the bank account, and the amount y in the stock, his *net wealth* is defined as the holdings in the bank account after either selling of all shares of the stock (if the proceeds are positive after transaction costs) or closing of the short position in the stock and is given by

$$X_t(x,y) = \begin{cases} \max\{x_t + y_t(1-\lambda) - k, x_t\} & \text{if } y_t \geq 0, \\ x_t + y_t(1+\lambda) - k & \text{if } y_t < 0. \end{cases} \tag{3}$$

We suppose that at any time the investor can decide to transfer money from the bank account to the stock and conversely. The control of the investor is a pure *impulse control* $v = (\tau_1, \tau_2, \ldots; \xi_1, \xi_2, \ldots)$. Here $0 \leq \tau_1 < \tau_2 < \ldots$ are $\mathcal{F}_t$-stopping times giving the times when the investor decides to change his portfolio, and ξ_j are $\mathcal{F}_{\tau_j}$-measurable random variables giving the sizes of the transactions at these times. If such a control is applied to the system (x_t, y_t), it gets the form

$$\begin{aligned} dx_t &= r x_t dt & \tau_i \leq t < \tau_{i+1} \\ dy_t &= \mu y_t dt + \sigma y_t dB_t & \tau_i \leq t < \tau_{i+1} \\ x_{\tau_{i+1}} &= x_{\tau_{i+1}^-} - k - \xi_{i+1} - \lambda|\xi_{i+1}| & \\ y_{\tau_{i+1}} &= y_{\tau_{i+1}^-} + \xi_{i+1} & \end{aligned} \tag{4}$$

We consider an investor with a finite horizon $[0,T]$ who has utility only of terminal wealth. It is assumed that the investor has a constant absolute risk aversion. In this case his utility function is of the form

$$U(\gamma, W) = -\exp(-\gamma W), \tag{5}$$

where γ is a measure of the investor's absolute risk aversion (ARA), which is independent of the investor's wealth.

2.1 Utility Maximization Problem without Options

The investor's problem is to choose an admissible trading strategy to maximize $E_t[U(X_T)]$, i.e., the expected utility of his net terminal wealth, subject to (4). We define the value function at time t as

$$V(t, x, y) = \sup_{v \in \mathcal{A}(x,y)} E_t^{x,y}[U(\gamma, X_T)], \tag{6}$$

where $\mathcal{A}(x, y)$ denotes the set of admissible controls available to the investor who starts at time t with an amount of x in the bank and y holdings in the stock. We assume that the investor's portfolio space is divided into two disjoint regions: a *continuation region* and an *intervention region*. The intervention region is the region where it is optimal to make a transaction. We define the *intervention operator* (or the maximum utility operator) $\mathcal{M}$ by

$$\mathcal{M}V(t, x, y) = \sup_{(x', y') \in \mathcal{A}(x,y)} V(t, x', y') \tag{7}$$

where x' and y' are the new values of x and y. In other words, $\mathcal{M}V(t, x, y)$ represents the value of the strategy that consists in choosing the best transaction. The continuation region is the region where it is not optimal to rebalance the investor's portfolio. We define the continuation region D by

$$D = \big\{(x, y); V(t, x, y) > \mathcal{M}V(t, x, y)\big\} \tag{8}$$

Now, by giving heuristic arguments, we intend to characterize the value function and the associated optimal strategy. If for some initial point (t, x, y) the optimal strategy is to not transact, the utility associated with this strategy is $V(t, x, y)$. Choosing the best transaction and then following the optimal strategy gives the utility $\mathcal{M}V(t, x, y)$. The necessary condition for the optimality of the first strategy is $V(t, x, y) \geq \mathcal{M}V(t, x, y)$. This inequality holds with equality when it is optimal to rebalance the portfolio. Moreover, in the continuation region, the application of the dynamic programming principle gives $\mathcal{L}V(t,x,y)=0$, where the operator $\mathcal{L}$ is defined by

$$\mathcal{L}V(t,x,y)=\frac{\partial V}{\partial t}+rx\frac{\partial V}{\partial x}+\mu y\frac{\partial V}{\partial y}+\frac{1}{2}\sigma^2y^2\frac{\partial^2 V}{\partial y^2}. \tag{9}$$

The subsequent theorem formalizes this intuition.

Theorem 1. *The value function is the unique constrained viscosity solution of the quasi-variational Hamilton-Jacobi-Bellman inequalities (QVHJBI, or just QVI):*

$$\max\Big\{\mathcal{L}V, \quad \mathcal{M}V-V\Big\}=0 \tag{10}$$

with the boundary condition

$$V(T,x,y)=U(\gamma,X_T).$$

The proof can be made by following along the lines of the proof in Øksendal and Sulem (1999) with corrections for no consumption, and our finite horizon.

The amount of x_T is given by

$$x_T=\frac{x}{\delta(T,t)}-\sum_{i=0}^{n}\frac{(k+\xi_i+\lambda|\xi_i|)}{\delta(T,\tau_i)}, \tag{11}$$

where $\delta(T,t)$ is the discount factor defined by

$$\delta(T,t)=\exp(-r(T-t)), \tag{12}$$

and $t\le\tau_1<\tau_2<\ldots<\tau_n<T$. Therefore, taking into consideration our utility function defined by (5), we can write

$$V(t,x,y)=\exp(-\gamma\frac{x}{\delta(T,t)})Q(t,y), \tag{13}$$

where $Q(t,y)$ is defined by $Q(t,y)=V(t,0,y)$. It means that the dynamics of y through time is independent of x. This representation suggests transformation of (10) into the following QVI for the value function $Q(t,y)$:

$$\max\Big\{\mathcal{D}Q(t,y), \sup_{y'\in\mathcal{A}(y)}\exp\left(\gamma\frac{k-(y-y')+\lambda|y-y'|}{\delta(T,t)}\right)Q(t,y')-Q(t,y)\Big\}=0, \tag{14}$$

where y' is the new value of y, $\mathcal{A}(y)$ denotes the set of admissible controls available to the investor who starts at time t with y holdings in the stock, and the operator $\mathcal{D}$ is defined by

$$\mathcal{L}Q(t,y) = \frac{\partial Q}{\partial t} + \mu y \frac{\partial Q}{\partial y} + \frac{1}{2}\sigma^2 y^2 \frac{\partial^2 Q}{\partial y^2}. \tag{15}$$

This is an important simplification that reduces the dimensionality of the problem. Note that the function $Q(t,y)$ is evaluated in the two-dimensional space $[0,T] \times \mathbb{R}$.

In the absence of any transaction costs the solution for the optimal trading strategy is given by

$$y^*(t) = \frac{\delta(T,t)}{\gamma} \frac{(\mu - r)}{\sigma^2}, \tag{16}$$

by using the result in Davis et al. (1993).

The numerical calculations show that in the presence of both fixed and proportional transaction costs, most of the time, the portfolio space can be divided into three disjoint regions (Buy, Sell, and NT), and the optimal policy is described by four boundaries. The Buy and NT regions are divided by the lower no-transaction boundary, and the Sell and NT regions are divided by the upper no-transaction boundary. If a portfolio lies in the Buy region, the optimal strategy is to buy the risky asset until the portfolio reaches the Buy target boundary. Similarly, if a portfolio lies in the Sell region, the optimal strategy is to sell the risky asset until the portfolio reaches the Sell target boundary.

However, there is generally a time interval, say $[\tau_1, \tau_2)$[5], when the NT region consists of two disjoint sub-regions which, in their turn, divide either the Buy region (when $\mu > r$) or the Sell region (when $\mu < r$) into two parts. Nevertheless, as in the former case, the target boundaries are unique. The rationale for the existence of a second (minor) NT sub-region can be explained in terms of fixed transaction costs. Recall how we define the investor's net wealth (see equation (3)). If the investor's holdings in the stock are positive, he will sell all his shares of the stock on the terminal date only if the proceeds are positive after transaction costs. Putting it another way, the rational investor will not sell his shares of the stock if $y(1-\lambda) < k$.

[5] $\tau = T - t$ represents the time remaining until the terminal date

Suppose for the moment that $y_\tau \to 0^+$ for some $\tau \in [\tau_1, \tau_2)$. Consider the two alternatives: (i) No trade at τ and thereafter up to the terminal date, and (ii) buy a certain number of shares of the stock at τ in order to move closer to the optimal level of holdings in the model with no transaction costs. In the former case it is almost sure that at the terminal date the holdings in the stock will not exceed the fixed transaction fee k. That is, $y_0(1-\lambda) < k$ $a.s.$, and, thus, it is not optimal to sell shares of the stock. Hence, in the first alternative the investor does not pay any transaction costs. In the second alternative the investor pays at least round trip transaction costs equal to $2k$ (we ignore the time value of money). It turns out that the first alternative is better than the second one when the terminal date is close.

All the NT and target boundaries are functions of the investor's horizon and do not depend on the investor's holdings in the bank account, so that a possible description of the optimal policy for $\tau \in (0, \tau_1) \cup [\tau_2, \infty)$ may be given by

$$\begin{gathered} y = y_u(\tau) \\ y = y_l^*(\tau) \\ y = y_u^*(\tau) \\ y = y_l(\tau), \end{gathered} \tag{17}$$

where the first and the forth equations describe the upper and the lower NT boundaries respectively, and the second and the third equations describe the target boundaries. For $\tau \in [\tau_1, \tau_2)$ a possible description of the optimal policy may be given by

$$\begin{gathered} y = y_u(\tau) \\ y = y_l^*(\tau) \\ y = y_u^*(\tau) \\ y = y_l(\tau) \\ y = y_{2u}(\tau) \\ y = y_{2l}(\tau) = 0. \end{gathered} \tag{18}$$

The first and the forth equations describe the upper and the lower boundaries of the main NT sub-region. The second and the third equations describe the target boundaries. The last two equation characterize the minor NT sub-region which lies in between $y = y_{2u}(\tau) < k$ and $y = y_{2l}(\tau) = 0$. It is always the case that $y_l < y_l^* < y_u^* < y_u$ and $y_{2l} < y_{2u}$. The minor NT region is largely insignificant. Further we will not pay any attention to it in order to keep focus and concentration only on important issues.

The analysis of the optimal portfolio policy without options for a CARA investor with a finite horizon and a large set of realistic parameters, as well as the illustration of the case where the NT region consists of two disjoint sub-regions, is beyond the scope of this paper. The interested reader may consult Zakamouline (2002) for details.

If the function $Q(t, y)$ is known in the NT region, then

$$Q(t,y) = \begin{cases} \exp\left(\gamma \frac{k-(1-\lambda)(y-y_u^*)}{\delta(T,t)}\right) Q(t, y_u^*) & \forall y(t) \geq y_u(t), \\ \exp\left(\gamma \frac{k+(1+\lambda)(y_l^*-y)}{\delta(T,t)}\right) Q(t, y_l^*) & \forall y(t) \leq y_l(t). \end{cases} \tag{19}$$

This follows from the optimal transaction policy described above. That is, if a portfolio lies in the Buy or Sell region, then the investor performs the minimum transaction required to reach the closest target boundary.

2.2 Utility Maximization Problem with Options

Now we introduce a new asset, a cash settled European-style option contract with expiration time T and payoff $g(S_T)$ at expiration. For the sake of simplicity, we assume that these options may be bought or sold only at time zero. This means that there is no trade in options thereafter.

Consider an investor who trades in the riskless and the risky assets and, in addition, buys $\theta > 0$ options. This investor we will refer to as the buyer of options. The buyer's problem is to choose an admissible trading strategy to maximize $E_t[U(X_T + \theta g(S_T))]$ subject to (4). We define his value function at time t as

$$J^b(t, x, y, S, \theta) = \sup_{v \in \mathcal{A}_\theta^b(x,y)} E_t^{x,y}[U(\gamma, X_T + \theta g(S_T))], \tag{20}$$

where $\mathcal{A}_\theta^b(x, y)$ denotes the set of admissible controls available to the buyer who starts at time t with an amount of x in the bank and y holdings in the stock.

Definition 1. The unit reservation purchase price of θ European-style options is defined as the price P_θ^b such that

$$V(t, x, y) = J^b(t, x - \theta P_\theta^b, y, S, \theta) \tag{21}$$

In other words, the reservation purchase price, P_θ^b, is the highest price at which the investor is willing to buy options, and when the investor is indifferent between the two alternatives: (*i*) a utility maximization problem where he trades in the riskless and risky assets only, and (*ii*) a utility maximization problem where the investor, in addition, buys options at price P^b_θ.

Consider now an investor who trades in the riskless and the risky assets and, in addition, writes $\theta > 0$ options. This investor we will refer to as the writer of options. The writer's problem is to choose an admissible trading strategy to maximize $E_t[U(X_T - \theta g(S_T))]$ subject to (4). We define his value function at time t as

$$J^w(t, x, y, S, \theta) = \sup_{v \in \mathcal{A}^w_\theta(x,y)} E^{x,y}_t[U(\gamma, X_T - \theta g(S_T))], \tag{22}$$

where $\mathcal{A}^w_\theta(x, y)$ denotes the set of admissible controls available to the writer who starts at time t with an amount of x in the bank and y holdings in the stock.

Definition 2. The unit reservation write price of θ European-style options is defined as the compensation P^w_θ such that

$$V(t, x, y) = J^w(t, x + \theta P^w_\theta, y, S, \theta) \tag{23}$$

That is, the reservation write price, P^w_θ, is the lowest price at which the investor is willing to sell options, and when the investor is indifferent between the two alternatives: (*i*) a utility maximization problem where he trades in the riskless and risky assets only, and (*ii*) a utility maximization problem where the investor, in addition, writes options at price P^w_θ.

The solutions to problems (21) and (23) provide the unique reservation option prices and the optimal strategies. We interpret the difference in the two trading strategies, with and without options, as "hedging" the options.

Theorem 2. *The value functions of both problems (20) and (22) are the unique viscosity solutions of the quasi-variational Hamilton-Jacobi-Bellman inequalities:*

$$\max\left\{\overline{\mathcal{L}}J, \quad \mathcal{M}J - J\right\} = 0 \tag{24}$$

with the boundary conditions

$$J^b(T,x,y,S,\theta) = U(\gamma, X_T + \theta g(S_T))$$
$$J^w(T,x,y,S,\theta) = U(\gamma, X_T - \theta g(S_T))$$

where the operator $\overline{\mathcal{L}}$ *given by*

$$\overline{\mathcal{L}}J = \frac{\partial J}{\partial t} + rx\frac{\partial J}{\partial x} + \mu y \frac{\partial J}{\partial y} + \mu S \frac{\partial J}{\partial S} + \frac{1}{2}\sigma^2 y^2 \frac{\partial^2 J}{\partial y^2} + \sigma^2 y S \frac{\partial^2 J}{\partial y \partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 J}{\partial S^2}. \tag{25}$$

The proof can be carried out by following along the lines of the proof of Theorem (1).

As in the case of the optimal portfolio selection problem without options, we can show that the dynamics of y through time is independent of x. Therefore

$$\begin{aligned} J^b(t,x,y,S,\theta) &= \exp(-\gamma \tfrac{x}{\delta(T,t)}) H^b(t,y,S,\theta), \\ J^w(t,x,y,S,\theta) &= \exp(-\gamma \tfrac{x}{\delta(T,t)}) H^w(t,y,S,\theta), \end{aligned} \tag{26}$$

where $H^b(t,y,S,\theta)$ and $H^w(t,y,S,\theta)$ are defined by $H^b(t,y,S,\theta) = J^b(t,0,y,S,\theta)$ and $H^w(t,y,S,\theta) = J^w(t,0,y,S,\theta)$ respectively. This also suggests transformation of (24) into the following QVI for the value function $H(t,y,S,\theta)$:

$$\max\left\{\overline{\mathcal{D}}H, \sup_{y'\in\mathcal{A}(y)} \exp\left(\gamma \frac{k - (y-y') + \lambda|y-y'|}{\delta(T,t)}\right) H(t,y',S,\theta) - H(t,y,S,\theta)\right\} = 0, \tag{27}$$

where y' is the new value of y, $\mathcal{A}(y)$ denotes the set of admissible controls available to the investor who starts at time t with y holdings in the stock, and the operator $\overline{\mathcal{D}}$ is defined by

$$\overline{\mathcal{D}}H = \frac{\partial H}{\partial t} + \mu y \frac{\partial H}{\partial y} + \mu S \frac{\partial H}{\partial S} + \frac{1}{2}\sigma^2 y^2 \frac{\partial^2 H}{\partial y^2} + \sigma^2 y S \frac{\partial^2 H}{\partial y \partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 H}{\partial S^2}. \tag{28}$$

Again we have reduced the dimensionality of the problem by one. Note that the function $H(t,y,S,\theta)$ is evaluated in the three-dimensional space $[0,T] \times \mathbb{R} \times \mathbb{R}^+$. Consequently, after all the simplifications, the unit reservation purchase price is given by (follows from (21) and (26))

$$P^b_\theta(t,S) = \frac{\delta(T,t)}{\theta\gamma} \ln\left(\frac{H^b(t,y,S,\theta)}{Q(t,y)}\right), \tag{29}$$

and the unit reservation write price is given by (follows from (23) and (26))

$$P_\theta^w(t,S) = \frac{\delta(T,t)}{\theta\gamma}\ln\left(\frac{Q(t,y)}{H^w(t,y,S,\theta)}\right). \tag{30}$$

In practical applications one usually assumes that the investor has zero holdings in the stock at time zero.

In the absence of any transaction costs the solution for the optimal trading strategy for the writer of options in (y,S)-plane is given by (see, for example, the result in Davis et al. (1993))

$$y_w^*(t,S,\theta) = \theta S\frac{\partial P_{BS}(t,S)}{\partial S} + \frac{\delta(T,t)}{\gamma}\frac{(\mu-r)}{\sigma^2}, \tag{31}$$

and the solution for the optimal trading strategy for the buyer of options is given by

$$y_b^*(t,S,\theta) = -\theta S\frac{\partial P_{BS}(t,S)}{\partial S} + \frac{\delta(T,t)}{\gamma}\frac{(\mu-r)}{\sigma^2}, \tag{32}$$

where $P_{BS}(t,S)$ is the price of one option in a market with no transaction costs (i.e., the Black-Sholes price).

As in the case without options, in the presence of both fixed and proportional transaction costs the portfolio space again can be divided into three disjoint regions[6] (Buy, Sell, and NT) may be given by

$$\begin{aligned} y &= y_u(\tau,S)\\ y &= y_l^*(\tau,S)\\ y &= y_u^*(\tau,S)\\ y &= y_l(\tau,S). \end{aligned} \tag{33}$$

Section 5 of this paper provides illustrations of the optimal portfolio strategy with options.

If the function $H(t,y,S,\theta)$ (here we suppress the superscripts w and b) is known in the NT region, then

$$H(t,y,S,\theta) = \begin{cases} \exp\left(\gamma\frac{k-(1-\lambda)(y-y_u^*)}{\delta(T,t)}\right)H(t,y_u^*,S,\theta) & \forall y(t,S)\geq y_u(t,S),\\ \exp\left(\gamma\frac{k+(1+\lambda)(y_l^*-y)}{\delta(T,t)}\right)H(t,y_l^*,S,\theta) & \forall y(t,S)\leq y_l(t,S). \end{cases} \tag{34}$$

[6] To put it more precisely, some of them may have sub-regions. Recall our stipulation that in the presentation we do not pay any attention to the minor NT sub-region

That is, according to the optimal transaction policy, if a portfolio lies in the Buy or Sell region, then the investor performs the minimum transaction required to reach the closest target boundary.

3 No-Arbitrage Bounds and Properties of the Reservation Prices

3.1 No-Arbitrage Bounds in Presence of Transaction Costs

First of all we want to derive upper and lower bounds for option prices that do not depend on any particular assumptions about the investor's utility function[7]. We, namely, want to adjust the no-arbitrage pricing bounds derived in Merton (1973) for the presence of both fixed and proportional transaction costs. We will consider cash settled call and put options with exercise price K.

From both the definition of an option and the absence of arbitrage condition, we have that

$$P_\theta(t, S) \geq 0, \tag{35}$$

where $P_\theta(t, S)$ is a unit option price of a position of θ options for both the buyer and the writer.

Proposition 1. *The upper bound for the price of a call option is given by*

$$P_\theta(t, S) \leq S(t)\frac{1+\lambda}{1-\lambda} + \frac{k}{\theta}(1 + \delta(t, T)) \tag{36}$$

Here we use the condition that the option can never be worth more than the stock. If this relationship is not true, an arbitrager can make a riskless profit by buying $\frac{\theta}{1-\lambda}$ stocks and selling θ call options. The upper bound for a put option price is the same as in the case of no transaction costs, i.e., K.

For European call and put options we can derive tighter lower bounds than the relationship (35).

Proposition 2. *A lower bound for the price of a European call option is*

$$P_\theta(t, S) \geq \max\left[0, S(t)\frac{1-\lambda}{1+\lambda} - \frac{k}{\theta}(1 + \delta(t, T)) - K\delta(t, T)\right] \tag{37}$$

[7] The only requirement that the investor prefers more to less

This proposition is an extension of Theorem (1) in Merton (1973) in the presence of both fixed and proportional transaction costs. If this relationship is not true, an arbitrager can make a riskless profit by shorting $\frac{\theta}{1+\lambda}$ stocks, buying θ call options, and investing the proceeds risk-free.

Proposition 3. *A lower bound for the price of a European put option is*

$$P_\theta(t,S) \geq \max\left[0, K\delta(t,T) - S(t)\frac{1+\lambda}{1-\lambda} - \frac{k}{\theta}(1+\delta(t,T))\right] \qquad (38)$$

If this relationship is not true, an arbitrager can make a riskless profit by borrowing $\theta K\delta(t,T)$ at the risk-free rate, and buying $\frac{\theta}{1-\lambda}$ stocks and θ put options.

Note, that in all the relationships, due to the presence of a fixed transaction fee, the bounds depend on the number of options. These bounds converge to the bounds in the market with only proportional transaction costs when the number of options goes to infinity.

3.2 Properties of the Reservation Option Prices

Let's for the moment write the investor's value function of the utility maximization problem without options as $V(t,\gamma,x,y,k)$, and the corresponding value function of the utility maximization problem with options as $J(t,\gamma,x,y,k,S,\theta)$. By this we want to emphasize that both the value functions depend on the investor's coefficient of absolute risk aversion and the fixed transaction fee.

Theorem 3. *For an investor with the exponential utility function and an initial endowment* (x,y) *we have*

$$V(t,\gamma,x,y,k) = V(t,\theta\gamma,\frac{x}{\theta},\frac{y}{\theta},\frac{k}{\theta}), \qquad (39)$$

$$J(t,\gamma,x,y,k,S,\theta) = J(t,\theta\gamma,\frac{x}{\theta},\frac{y}{\theta},\frac{k}{\theta},S,1). \qquad (40)$$

Proof. Both these relationships can be easily established from the form of the exponential utility function. In particular, the portfolio process $\{\frac{x_s}{\theta},\frac{y_s}{\theta}; s > t\}$ is admissible given the initial portfolio $(\frac{x_t}{\theta},\frac{y_t}{\theta})$ and fixed transaction cost fee $\frac{k}{\theta}$ if and only if $\{x_s, y_s; s > t\}$ is admissible given the initial portfolio (x_t, y_t) and fixed transaction cost fee k. Furthermore, $U(\gamma, X_T) = U(\theta\gamma, \frac{X_T}{\theta})$ and $U(\gamma, X_T \pm \theta P_\theta) = U(\theta\gamma, \frac{X_T}{\theta} \pm P_\theta)$.

Corollary 4. *For an investor with the exponential utility function and an initial holding in the stock y we have*

$$Q(t,\gamma,y,k) = Q(t,\theta\gamma,\frac{y}{\theta},\frac{k}{\theta}), \tag{41}$$

$$J(t,\gamma,y,k,S,\theta) = J(t,\theta\gamma,\frac{y}{\theta},\frac{k}{\theta},S,1). \tag{42}$$

Proof. This follows from Theorem (3) and the definitions of the value functions Q and H.

Theorem 5. *For an investor with exponential utility function we have that*

1. *An investor with an initial holding in the stock y, ARA coefficient γ, and the fixed transaction fee k has a unit reservation purchase price of θ options equal to his reservation purchase price of one option in the case where he has an initial holding in the stock $\frac{y}{\theta}$, ARA coefficient $\theta\gamma$ and the fixed transaction fee $\frac{k}{\theta}$. That is,*

$$P^b_\theta(t,S) = \frac{\delta(T,t)}{\gamma}\ln\left(\frac{H^b(t,\theta\gamma,\frac{y}{\theta},\frac{k}{\theta},S,1)}{Q(t,\theta\gamma,\frac{y}{\theta},\frac{k}{\theta})}\right). \tag{43}$$

2. *An investor with an initial holding in the stock y, ARA coefficient γ, and the fixed transaction fee k has a unit reservation write price of θ options equal to his reservation write price of one option in the case where he has an initial holding the in stock $\frac{y}{\theta}$, ARA coefficient $\theta\gamma$ and the fixed transaction fee $\frac{k}{\theta}$. That is,*

$$P^w_\theta(t,S) = \frac{\delta(T,t)}{\gamma}\ln\left(\frac{Q(t,\theta\gamma,\frac{y}{\theta},\frac{k}{\theta})}{H^w(t,\theta\gamma,\frac{y}{\theta},\frac{k}{\theta},S,1)}\right). \tag{44}$$

Proof. This follows from Theorem (3), the definitions of the value functions Q and H, Corollary (4), and equations (21) and (23).

As mentioned above, in the practical applications of the utility based option pricing method one assumes that the investor has zero holdings in the stock at time zero, i.e., $y = 0$, hence $\frac{y}{\theta} = 0$ as well. In this case Theorem (5) says that the resulting unit reservation option price and the corresponding optimal hedging strategy[8] in the model with the triple of

[8] Here, the hedging strategy per option. For θ options the strategy must be re-scaled accordingly

parameters (γ, k, θ) will be the same as in the model with $(\theta\gamma, \frac{k}{\theta}, 1)$. That is, instead of calculating a model with θ options we can calculate a model with 1 option only. All we need is adjusting the two parameters for θ: the absolute risk aversion from γ to $\theta\gamma$, and the fixed transaction fee from k to $\frac{k}{\theta}$.

Corollary 6. *For an investor with exponential utility function, an initial holding in the stock $y = 0$, and the fixed transaction fee $k = 0$ we have that*

1. *The unit reservation purchase price, $P^b_\theta(t, S)$, is decreasing in the number of options θ.*
2. *The unit reservation write price, $P^w_\theta(t, S)$, is increasing in the number of options θ.*

The result in Corollary (6) is quite intuitive. When there are transaction costs in the market, holding options involves an unavoidable element of risk. Therefore, the greater number of options the investor holds, the more risk he takes. When, in particular, there are only proportional transaction costs, according to the pricing formulas in Theorem (5) an increase in θ corresponds only to an increase in the investor's "pseudo" $ARA = \theta\gamma$. Consequently, the more options the risk averse investor has to buy, the less he is willing to pay per option. Similarly, the seller of options will demand a unit price which is increasing in the number of options. When the fixed transaction fee $k \neq 0$, the dependence of the unit reservation price on the number of options is not obvious. The unit reservation write price can, for example, first decrease[9] and then increase when the number of options increases. Note, in particular, that the linear pricing rule from the complete and frictionless market does not apply to the reservation option prices.

Corollary 7. *The unit reservation option price in the market with both fixed and proportional transaction costs converges to the price in the market with only proportional transaction costs when the number of options goes to infinity. That is,*

$$\lim_{\theta \to \infty} P^b_\theta(t, S, \lambda, k) = P^b_\theta(t, S, \lambda, 0) \tag{45}$$

$$\lim_{\theta \to \infty} P^w_\theta(t, S, \lambda, k) = P^w_\theta(t, S, \lambda, 0) \tag{46}$$

[9]Note, that the fixed transaction fee per option is decreasing in the number of options

We conjecture that as $\theta \to \infty$ the reservation option prices converge to the corresponding stochastic dominance bounds that were derived in Constantinides and Perrakis (2000). The utility based reservation option prices are generally tighter bounds on option prices than the stochastic dominance bounds which are valid for any non-decreasing and concave utility function.

4 A Markov Chain Approximation of the Continuous Time Problem

The main objective of this section is to present numerical procedures for computing the investor's value functions and the corresponding optimal trading policies. It is tempting to try to solve the partial differential equations (10) and (24) by using the classical finite-difference method, but the PDEs have only a formal meaning and are to be interpreted in a symbolic sense. Indeed, we do not know whether the partial derivatives of the value functions are well defined, i.e., the value functions have twice continuously differentiable solutions. The method of solution of such problems was suggested by Kushner (see, for example, Kushner and Martins (1991) and Kushner and Dupuis (1992)). The basic idea involves a consistent approximation of the problem by a Markov chain, and then the solution of an appropriate optimization problem for the Markov chain model. Unlike the classical finite-difference method, the smoothness of the solution to the HJB or QVI equations is not needed.

First, according to the the Markov chain approximation method, we construct discrete time approximations of the continuous time price processes used in the continuous time model presented in Section 2. Then the discrete time program is solved by using the discrete time dynamic programming algorithm (i.e., backward recursion algorithm).

Consider the partition $0 = t_0 < t_1 < \ldots < t_n = T$ of the time interval $[0, T]$ and assume that $t_i = i\Delta t$ for $i = 0, 1, \ldots, n$ where $\Delta t = \frac{T}{n}$. Let ε be a stochastic variable:

$$\varepsilon = \begin{cases} u & \text{with probability } p, \\ d & \text{with probability } 1 - p. \end{cases}$$

We define the discrete time stochastic process of the stock as:

$$S_{t_{i+1}} = S_{t_i}\varepsilon \tag{47}$$

and the discrete time process of the risk-free asset as:

$$x_{t_{i+1}} = x_{t_i}\rho \tag{48}$$

If we choose $u = e^{\sigma\sqrt{\Delta t}}$, $d = e^{-\sigma\sqrt{\Delta t}}$, $\rho = e^{r\Delta t}$, and $p = \frac{1}{2}\left[1 + \frac{\mu}{\sigma}\sqrt{\Delta t}\right]$, we obtain the binomial model proposed by Cox, Ross, and Rubinstein (1979). An alternative choice is $u = e^{\mu\Delta t + \sigma\sqrt{\Delta t}}$, $d = e^{\mu\Delta t - \sigma\sqrt{\Delta t}}$, $\rho = e^{r\Delta t}$, and $p = \frac{1}{2}$, which was proposed by He (1990). As n goes to infinity, the discrete time processes (47) and (48) converge in distribution to their continuous counterparts (2) and (1).

The following discretization scheme is proposed for the QVI (10):

$$\mathbb{V}^{\Delta t} = \mathcal{O}(\Delta t)\mathbb{V}^{\Delta t}, \tag{49}$$

where $\mathcal{O}(\Delta t)$ is an operator given by

$$\begin{aligned}\mathcal{O}(\Delta t)\mathbb{V}^{\Delta t}(t_i, x, y) = \max\Big\{ & \max_m \mathbb{V}^{\Delta t}(t_i, x - k - (1+\lambda)m\Delta y, y + m\Delta y), \\ & \max_m \mathbb{V}^{\Delta t}(t_i, x - k + (1-\lambda)m\Delta y, y - m\Delta y), \\ & E\{\mathbb{V}^{\Delta t}(t_{i+1}, x\rho, y\varepsilon)\}\Big\},\end{aligned} \tag{50}$$

where m runs through the positive integer numbers, and

$$\begin{aligned}&\mathbb{V}^{\Delta t}(t_i, x - k - (1+\lambda)m\Delta y, y + m\Delta y) \\ &\qquad = E\left\{\mathbb{V}^{\Delta t}(t_{i+1}, (x - k - (1+\lambda)m\Delta y)\rho, (y + m\Delta y)\varepsilon)\right\}\end{aligned} \tag{51}$$

$$\begin{aligned}&\mathbb{V}^{\Delta t}(t_i, x - k + (1-\lambda)m\Delta y, y - m\Delta y) \\ &\qquad = E\left\{\mathbb{V}^{\Delta t}(t_{i+1}, (x - k + (1-\lambda)m\Delta y)\rho, (y - m\Delta y)\varepsilon)\right\},\end{aligned} \tag{52}$$

as at time t_i we do not know yet the value function. Instead, we use the known values at the next time instant, t_{i+1}. Here we have discretized the y-space in a lattice with grid size Δy, and the x-space in a lattice with grid size

Δx[10]. This scheme is a dynamic programming formulation of the discrete time problem. The solution procedure is as follows. Start at the terminal date and give the value function values by using the boundary conditions as for the continuous value function over the discrete state space. Then work backwards in time. That is, at every time instant t_i and every particular state (x, y), by knowing the value function for all the states in the next time instant, t_{i+1}, find the investor's optimal policy. This is carried out by comparing maximum attainable utilities from buying, selling, or doing nothing.

Theorem 8. *The solution $\mathbb{V}^{\Delta t}$ of (49) converges locally uniformly to the unique continuous constrained viscosity solution of (10) as $\Delta t \to 0$*

The proof is based on the notion of viscosity solutions and can be made in the same manner as the proof of Theorem (4) in Davis et al. (1993).

The following discretization scheme is proposed for the QVI (24):

$$\mathbb{J}^{\Delta t} = \overline{\mathcal{O}}(\Delta t)\mathbb{J}^{\Delta t}, \tag{53}$$

where $\overline{\mathcal{O}}(\Delta t)$ is an operator given by

$$\begin{aligned}\overline{\mathcal{O}}(\Delta t)\mathbb{J}^{\Delta t}(t_i, x, y, S) = \max\Big\{ & \max_m \mathbb{J}^{\Delta t}(t_i, x - k - (1+\lambda)m\Delta y, y + m\Delta y, S), \\ & \max_m \mathbb{J}^{\Delta t}(t_i, x - k + (1-\lambda)m\Delta y, y - m\Delta y, S), \\ & E\{\mathbb{J}^{\Delta t}(t_{i+1}, x\rho, y\varepsilon, S\varepsilon)\}\Big\},\end{aligned} \tag{54}$$

where m runs through the positive integer numbers, and

$$\begin{aligned}&\mathbb{J}^{\Delta t}(t_i, x - k - (1+\lambda)m\Delta y, y + m\Delta y, S) \\ &\qquad = E\left\{\mathbb{J}^{\Delta t}(t_{i+1}, (x - k - (1+\lambda)m\Delta y)\rho, (y + m\Delta y)\varepsilon, S\varepsilon)\right\}\end{aligned} \tag{55}$$

$$\begin{aligned}&\mathbb{J}^{\Delta t}(t_i, x - k + (1-\lambda)m\Delta y, y - m\Delta y, S) \\ &\qquad = E\left\{\mathbb{J}^{\Delta t}(t_{i+1}, (x - k + (1-\lambda)m\Delta y)\rho, (y - m\Delta y)\varepsilon, S\varepsilon)\right\}.\end{aligned} \tag{56}$$

[10]It is supposed that $\lim_{\Delta t \to 0} \Delta y \to 0$, and $\lim_{\Delta t \to 0} \Delta x \to 0$, that is, $\Delta y = c_y \Delta t$, and $\Delta x = c_x \Delta t$ for some constants c_y and c_x

The principle behind this scheme is the same as for the discretization scheme (49). As before, we have discretized the y-space in a lattice with grid size Δy, and the x-space in a lattice with grid size Δx. In addition, we use a binomial tree for the stock price process.

Theorem 9. *The solution $\mathbb{J}^{\Delta t}$ of (53) converges locally uniformly to the unique continuous constrained viscosity solution of (24) as $\Delta t \to 0$*

The proof follows along similar arguments as in Theorem (8).

Also in the discrete time framework the dynamics of y through time is independent of x. Therefore (13) and (26) can be written as follows:

$$\begin{aligned} \mathbb{V}^{\Delta t}(t,x,y) &= \exp(-\gamma \tfrac{x}{\delta(T,t)})\mathbb{Q}^{\Delta t}(t,y), \\ \mathbb{J}^{b,\Delta t}(t,x,y,S,\theta) &= \exp(-\gamma \tfrac{x}{\delta(T,t)})\mathbb{H}^{b,\Delta t}(t,y,S,\theta), \\ \mathbb{J}^{w,\Delta t}(t,x,y,S,\theta) &= \exp(-\gamma \tfrac{x}{\delta(T,t)})\mathbb{H}^{w,\Delta t}(t,y,S,\theta). \end{aligned} \tag{57}$$

The discretization scheme for the function $\mathbb{Q}^{\Delta t}(t,y)$ is derived from (49) and (57) to be

$$\begin{aligned} \mathbb{Q}^{\Delta t}(t_i,y) = \max\Big\{ &\max_m \exp\left(\gamma \frac{k+(1+\lambda)m\Delta y}{\delta(T,t_i)}\right)\mathbb{Q}^{\Delta t}(t_i, y+m\Delta y), \\ &\max_m \exp\left(\gamma \frac{k-(1-\lambda)m\Delta y}{\delta(T,t_i)}\right)\mathbb{Q}^{\Delta t}(t_i, y-m\Delta y), \\ &E\{\mathbb{Q}^{\Delta t}(t_{i+1}, y\varepsilon)\}\Big\}. \end{aligned} \tag{58}$$

As in the continuous time case, if the value function $\mathbb{Q}^{\Delta t}(t_i,y)$ is known in the NT region, then it can be calculated in the Buy and Sell region by using the discrete space version of (19):

$$\mathbb{Q}^{\Delta t}(t_i,y) = \begin{cases} \exp\left(\gamma \frac{k-(1-\lambda)(y-y_u^*)}{\delta(T,t_i)}\right)\mathbb{Q}^{\Delta t}(t_i,y_u^*) & \forall y(t_i) \geq y_u(t_i), \\ \exp\left(\gamma \frac{k+(1+\lambda)(y_l^*-y)}{\delta(T,t_i)}\right)\mathbb{Q}^{\Delta t}(t_i,y_l^*) & \forall y(t_i) \leq y_l(t_i). \end{cases} \tag{59}$$

In the same manner we can derive from (53) and (57) the discretization schemes for the value functions $\mathbb{H}^{b,\Delta t}(t,y,S,\theta)$ and $\mathbb{H}^{w,\Delta t}(t,y,S,\theta)$.

Davis et al. (1993) and Damgaard (2000b) used only one discretization scheme analogous to (53) for calculating both the value functions[11] $\mathbb{Q}$ and

[11] Hodges and Neuberger (1989) and Clewlow and Hodges (1997) avoided the evaluation of the value function V by choosing $\mu = r$

$\mathbb{H}$ ($\mathbb{V}$ and $\mathbb{J}$ in the work of Damgaard (2000b), since for the HARA utility function one cannot reduce the dimensionality of the problem). We propose to use different discretization schemes as the evaluation of the value function without options is a much easier task than the evaluation of the value function with options. Consequently, our method of calculating the value function $\mathbb{Q}$ is much more efficient. Moreover, the proposed discretization schemes describe only the basic structure of the algorithm we employ. In the practical realization, this algorithm is very time-consuming. At first, we detect the boundaries of the NT region. Afterwards we estimate the value function inside the NT region. Outside of the NT region, in the utility maximization problem without options, the value function is calculated via (59). In the utility maximization problem with options we use the discrete space version of (34).

5 Numerical Results

In this section we present the results of our numerical computations of reservation purchase and write prices and the corresponding hedging strategies for European call options. In most of our calculations we used the following model parameters: the risky asset price at time zero $S_0 = 100$, the strike price $K = 100$, the volatility $\sigma = 20\%$, the drift $\mu = 10\%$, and the risk-free rate of return $r = 5\%$ (all in annualized terms). The options expire at $T = 1$ year. The proportional transaction costs $\lambda = 1\%$ and the fixed transaction fee $k = 0.5$. The discretization parameters of the Markov chain, depending on the investor's ARA, are: $n \in [100, 150]$ periods of trading, and the grid size $\Delta y \in [0.001, 0.1]$. For high levels of the investor's ARA we cannot increase the number of periods of trading beyond some threshold as the values of the exponential utility are either overflow or underflow. However, this is not an issue for calculating the prices of put options.

The number of options is always 1 in all our calculations. Recall that, according to Theorem (5), the resulting unit reservation option price and the corresponding optimal hedging strategy in the model with the triple of parameters $(\gamma, k, 1)$ will be the same as in the model with $(\frac{\gamma}{\theta}, \theta k, \theta)$. This means if, for example, we choose $\gamma = 1$, $k = 0.5$, and $\theta = 1$, then we get the same unit reservation option price as in the model with $\gamma = 0.01$, $k = 50$, and $\theta = 100$.

5.1 The Sensitivity to γ and μ

In this subsection we are primarily interested in how reservation option prices depend on the measure of the investor's absolute risk aversion γ and the drift of the risky asset μ. Hodges and Neuberger (1989), Davis et al. (1993), and Clewlow and Hodges (1997) operated only with $\gamma = 1$ and found that the reservation purchase price is below, and the reservation write price is above the corresponding Black-Sholes price. Lo, Mamaysky, and Wang (2000) calibrated γ in their model to be between 0.0001 and 5.0. We see that $\gamma = 1$ lies in the upper end of the interval and corresponds to a very high risk aversion. Damgaard (2000a) and Damgaard (2000b) studied the sensitivity of reservation option prices to the investor's relative risk aversion (RRA) coefficient and the level of the investor's initial wealth. He found that the above mentioned pattern, when the reservation purchase price is below, and the reservation write price is above the BS-price, is valid only for either low levels of the investor's initial wealth or high levels of RRA. When either the investor's initial wealth increases or RRA decreases, both the reservation option prices approach the horizontal asymptote located above the BS-price. Either a higher wealth or a lower RRA for a $HARA$ utility corresponds to a lower ARA. This suggests that the level of ARA influences the reservation option prices in a not straightforward manner. He also found that reservation option prices are to some extent sensitive to the drift of the underlying asset.

Figures (2) and (1) show the dependence of reservation option prices on the level of the investor's absolute risk aversion for two different stock drifts. When the stock drift is equal to the risk-free interest rate (see Figure (1)), the reservation write price is always above the BS-price and the reservation purchase price is always below the BS-price. The reservation prices are located more or less symmetrically on each side of the BS-price. The parameter γ seems to influence only the magnitude of the deviation of reservation option prices from the corresponding BS-prices. The deviation is greater in the model with both fixed and proportional transaction costs than in the model with proportional transaction costs only.

As it is seen from Figure (2), when $\mu > r$ both reservation option prices are located above the BS-price for low values of γ. Here the reservation option prices are virtually independent of the choice of γ. Besides, the difference between them is very small. For high values of γ the pattern of

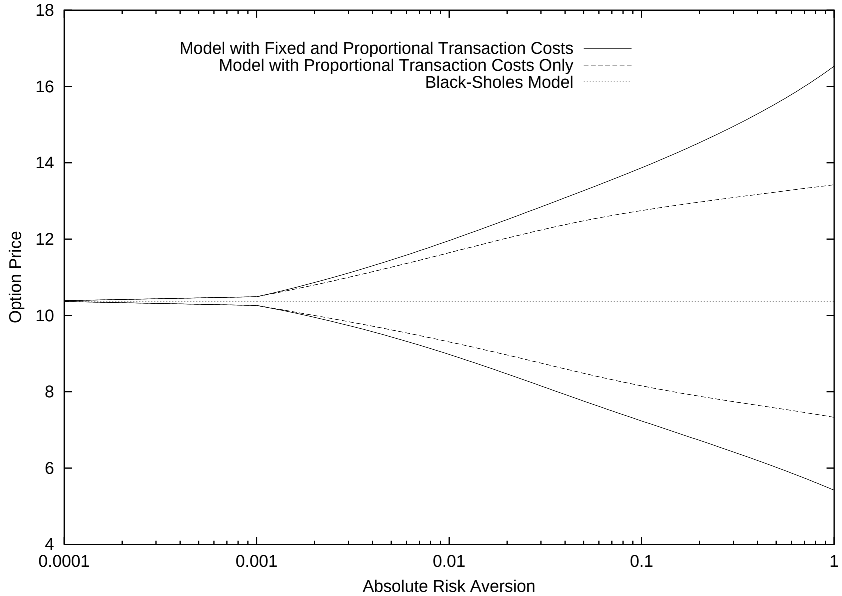

Figure 1: Reservation option prices versus γ for $\mu = r = 5\%$

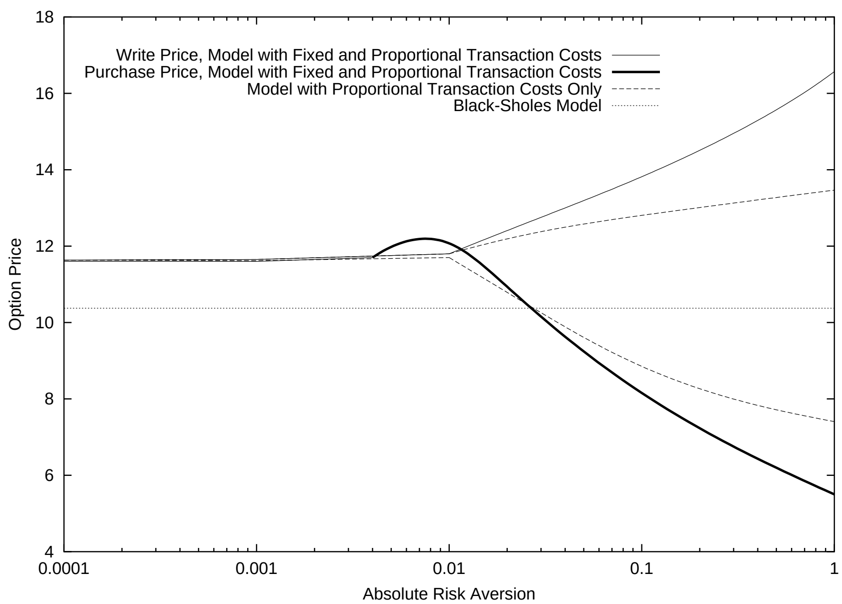

Figure 2: Reservation option prices versus γ for $\mu = 10\%$ and $r = 5\%$

the reservation option prices resembles the case when $\mu = r$. Note, that for $\gamma = 1$ the reservation option prices are practically the same for both $\mu = 10\%$ and $\mu = 5\%$. This suggests that for high values of γ the reservation option prices are almost independent of the drift of the underlying stock.

When the transaction costs have a fixed fee component there is an interval $\gamma \in (0.006, 0.014)$ for which the reservation purchase price is higher than the reservation write price, $P^b > P^w$. This situation occurs when the optimal number of shares bought at time zero in the model without options is close to the optimal number of shares sold[12] in order to hedge the risk of the long option position in the subsequent model with options (i.e., the net purchase of the stock is close to zero). In this case it is optimal for the buyer not to invest in the stock. We can interpret this situation as follows: The buyer moves his risky investment into options and goes out of the stock market. In this case, buying options on the stock instead of buying the stock saves the buyer from both fixed and proportional transaction costs. Thus, the reservation purchase price goes up, and may exceed, under certain model parameters, the reservation write price. The maximum reservation purchase price is attained when the optimal amount invested in the stock in the model without options is equal to the optimal number of shares sold in order to reduce the risk of options in the model with options. Note that in the model with proportional transaction costs only, the reservation purchase price seems to be always lower than the reservation write price, $P^b < P^w$.

Figure (3), together with the study of the transactions the investor makes, helps understand the dependence of the reservation option prices on the drift of the underlying stock for low values of γ. The figure shows the deviation of the reservation option prices from the BS-price versus the drift of the underlying stock for $\gamma = 0.001$ and two levels of proportional transaction costs.

For low μ, the investor does not invest in the stock due to the presence of transaction costs and the short investment horizon. The price of an option here is the discounted expected payoff from the option plus/minus some risk premium. When μ rises above some threshold value, the investor begins to invest in the stock. The presence of an option results in a correction for the

[12] Recall that we interpret the difference in the optimal amounts invested in the stock in the model with options and in the model without options as a hedge against the risk of options

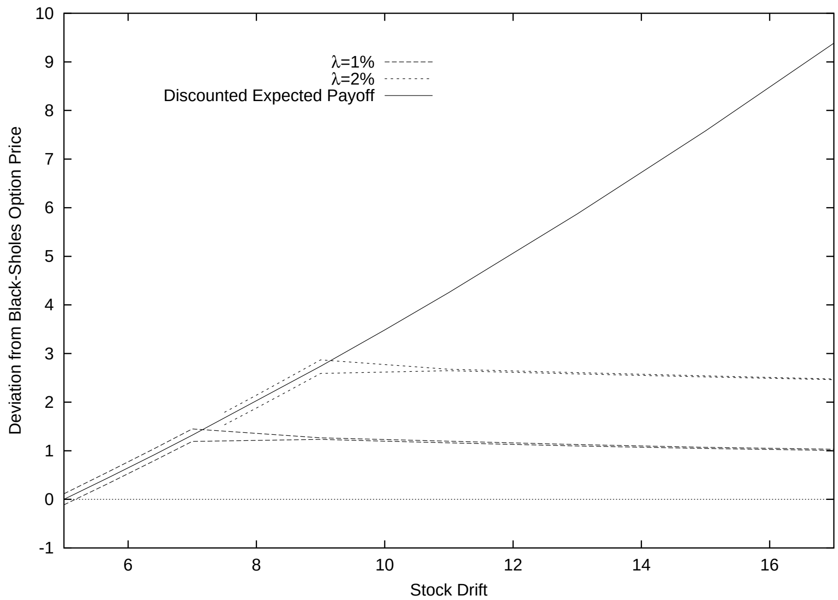

Figure 3: Deviation of the reservation option prices from the BS-price versus the drift of the underlying stock for $\gamma = 0.001$

number of shares of the stock bought to offset the additional risk from the option. Note that the level of the fixed transaction fee does not influence the price of an option, because the investor pays the same fixed costs regardless of the presence of an option. Both the reservation option prices are located above the corresponding BS-price, and the reservation write price is higher than the reservation purchase price. Figure (3) indicates that the level of proportional transaction costs explains the magnitude of the deviation of a reservation option price from the BS-price. The intuition behind this is as follows. For low values of γ and sufficiently high values of μ, an option serves as a substitute for the stock. Holding an option saves the buyer from some transaction costs. On the contrary, the writer adds extra transaction costs[13] into the option price. Both of these effects drive the reservation option price up.

A closer look at the interaction between the investor's absolute risk aversion coefficient γ[14], the drift of the risky asset μ, and the other model param-

[13] Note that the writer needs to buy some additional quantity of the stock to offset the risk from the option

[14] Recall the interplay between the number of options and the "pseudo" risk aversion

eters suggests distinguishing between two major types of investors behavior in relation to the pricing and hedging of options: the *net investor* and the *net hedger*.

Definition 3. By *net investors* we mean those investors for whom the number of shares of the stock held in the utility maximization problem without options is *greater* than the additional number of shares of the stock either bought or sold in order to hedge away the risk of options in the subsequent utility maximization problem with options.

Definition 4. By *net hedgers* we mean those investors for whom the number of shares of the stock held in the utility maximization problem without options is *less* than the additional number of shares of the stock either bought or sold in order to hedge away the risk of options in the subsequent utility maximization problem with options.

Roughly speaking, by making such a distinction between investors we suppose that the investor's overall portfolio problem can be separated into an investment problem and a hedging problem[15]. After such a separation it is possible to determine which problem is "bigger" in terms of the funds used to resolve every problem. Note that the main criterion in distinguishing between a net investor and a net hedger is the absolute difference in the numbers of shares of the stock in the utility maximization problems with and without options, in relation to the number of shares of the stock in the utility maximization problem without options. For the investor with the number of shares of the stock held in the utility maximization problem without options greater than the additional number of shares of the stock either bought or sold in the subsequent utility maximization problem with options, the "net" behavior is investing. On the contrary, for a net hedger the hedging problem is prevailing. In other words, a larger part of the net hedger's investment in the stock is devoted to hedge away the risk of options. In either case the additional transaction costs, or possible savings on transaction costs, are included in the option price.

Let's elaborate on this a bit further. Recall the optimal strategies for an investor, a writer of options, and a buyer of options. The optimal investor policy (without options) requires selling some shares of the stock when the

[15]Clearly, the possibility of such a separation is obvious in the no transaction costs case. Just look at equations (16), (31), and (32)

stock price goes up and buying additional shares of the stock when the stock price falls down (see equation 16). The hedging of the short option position requires purchasing additional shares of the stock when the stock price increases and selling some shares of the stock when the stock price decreases. On the contrary, the hedging of the long option position requires selling short additional number of shares of the stock when the stock price rises and decreasing the short position in the stock when the stock price falls.

Consider an investor who invests in the risky asset and, in addition, writes some number of options. It is easy to see that the investing and hedging decisions work in the opposite directions. We call this investor a net investor if we see that his optimal overall portfolio strategy (the net of the sum of the two strategies) requires selling some shares of the stock when the stock price goes up and buying additional shares of the stock when the stock price falls down. For a net investor the hedging strategy is "absorbed" by the investing strategy, which means that no additional transaction costs are added to the option price except, roughly, the round trip transaction costs to buy additional shares at time zero and sell them on the terminal date. For a net hedger all the excessive hedging transaction costs are added to the option price. The higher risk aversion, the more often an investor hedges an option. This means that the reservation write price increases as risk aversion increases.

Consider now an investor who invests in the risky asset and, in addition, buys some number of options. For him the investing and hedging decisions work in the same direction. We call this investor a net investor as long as his optimal overall portfolio strategy does not require short selling of the stock. The basic idea here is that it is optimal for an investor to take some certain amount of risk, depending on the investor's level of risk aversion, if the risk is properly rewarded (again, see equation 16). We can consider an option as another risky investment opportunity available to the investor. Since the payoffs from a call option and the stock are positively correlated, an option serves as a substitute for the stock. Investing in options causes the investor to invest less in the stock in order to maintain the amount of undertaken risk at the optimal level. Thus, it reduces transaction costs payed in the stock market, and these savings increase the reservation purchase price. Again, it turns out that, as in the case of a writer, the savings on options are approximately equal to the saved round trip transaction costs of not buying some number of shares. The rest of transactions, between the time zero and the terminal date, are roughly the same. This is true as long as the risk of options does not exceed the optimal level, beyond which the investor is involved in extensive hedging by shorting the stock. These hedging transaction costs are subtracted from the option price. In this case, a higher risk aversion results in a lower reservation purchase price.

To summarize: An option contract presents an additional risk to the investor. This risk needs to be hedged away by some proper trading strategy. All the transaction costs of the hedging strategy are either added to or deducted from the option price. The fact is that some of the investors (we call them net investors), either writers[16] or buyers of options, can effectively manage the risk of options in markets with transaction costs, but the others (we call them net hedgers) cannot. Moreover, to some extent, buying options is not just taking a risk, but an investment opportunity that might save some transaction costs.

As a result, net investors and net hedgers have different patterns of option pricing and hedging. Both of the net investor's reservation option prices are above the BS-price, and they are very close to each other. Approximately, the net investor's reservation option price can be calculated using the formula

$$P = P_{BS} + 2\Delta_{BS}(0)S_0\lambda, \tag{60}$$

where $\Delta_{BS}(0)$ is the BS-delta of the option at time zero. Indeed, putting into the formula the values of the terms we get $P = 10.44 + 2 * 0.64 * 100 * 0.01 = 11.72$, which is slightly above the reservation write price that is equal to 11.67. To a large extent, this relationship is insensitive to the level of absolute risk aversion and the amount of the fixed transaction fee. The net hedger's reservation purchase price is generally below the BS-price, and the net hedger's reservation write price is above the BS-price. Here the size of the difference between the two prices depends on the level of the absolute risk aversion and the level of transaction costs. In addition, the net hedger's optimal policy could easily be converted to control limits comparable to the BS-delta. This is not true for net investors, whose optimal policy can be considered only as an overall portfolio policy.

[16]Intuitively, writing a call option an a stock is not risky for those who have this stock in their portfolios

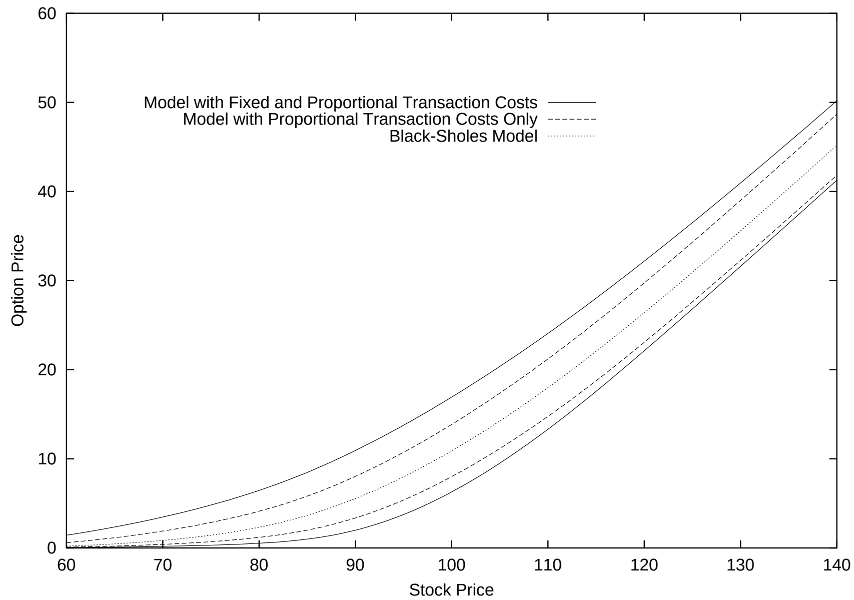

Figure 4: Bounds on reservation option prices versus the price of the underlying stock for a net hedger with $\gamma = 1$

For the chosen model parameters, $\gamma \approx 0.01$ can serve as a point of division between the net investor's and the net hedger's behavior (see Figure (2)).

5.2 Bounds on Reservation Option Prices and Optimal Trading Strategy for Net Hedgers

Figure (4) shows the bounds on reservation option prices versus the price of the underlying stock for a net hedger with $\gamma = 1$. Figures (5) and (6) show the optimal strategy control limits for the buyer and the writer of options respectively. For the sake of comparison we provide the corresponding BS-prices and the BS-delta curves for hedging options, as well as the bounds on reservation option prices and the corresponding optimal strategy control limits in the model with proportional transaction costs only.

As it was described in Section 2, in the presence of both fixed and proportional transaction costs, most of the time, the investor's portfolio space can be divided into three disjoint regions (Buy, Sell, and NT), and the optimal policy is described by four boundaries. If a portfolio lies in the Buy region,

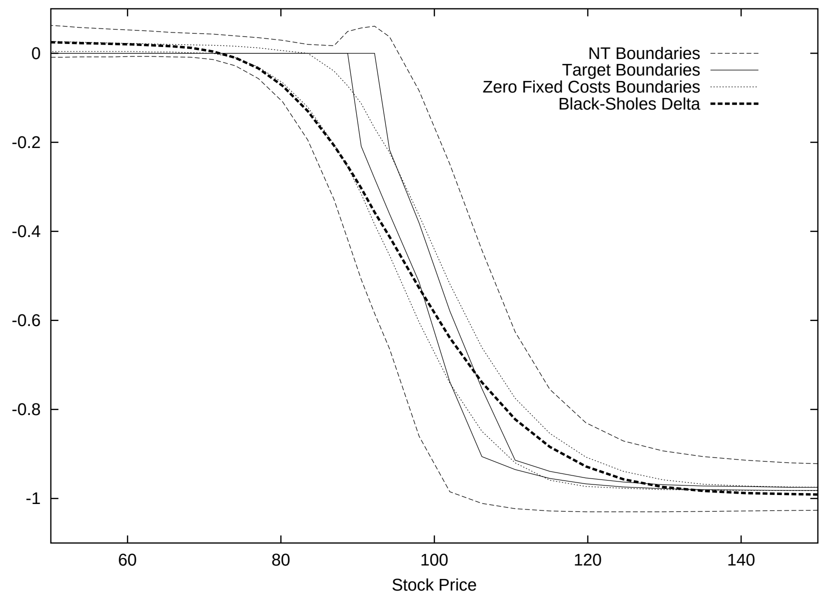

Figure 5: Optimal strategy control limits of a long call option for a net hedger with $\gamma = 1$

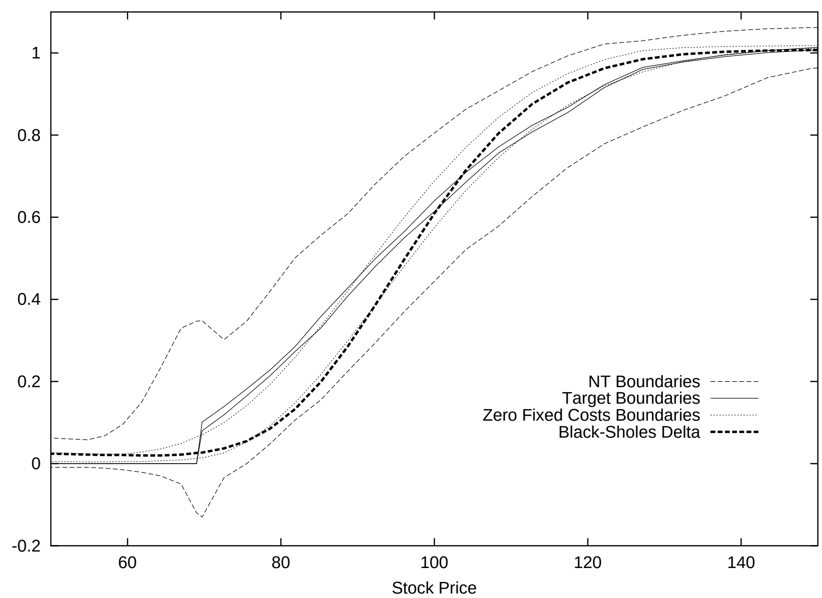

Figure 6: Optimal strategy control limits of a short call option for a net hedger with $\gamma = 1$

the optimal strategy is to buy the risky asset until the portfolio reaches the Buy target boundary. Similarly, if a portfolio lies in the Sell region, the optimal strategy is to sell the risky asset until the portfolio reaches the Sell target boundary. In Figures (5) and (6) all the four boundaries are converted to control limits comparable to the BS-delta, so the numerical results of this subsection may be considered as an extension of the results presented in Hodges and Neuberger (1989) and Clewlow and Hodges (1997). All the model parameters are approximately the same except for the introduction of a fixed cost component.

We define *moneyness* as the ratio between the strike price and the futures stock price, i.e., $M = \frac{K}{Se^{(T-t)r}}$. We will refer to an option as at-the-money if its $M = 1$, out-of-the-money if its $M > 1$, and in-the-money if $M < 1$. As it is seen from Figures (5) and (6), the target *average*[17] delta for at-the-money options is very close to the Black-Sholes delta. The buyer of options underhedges out-of-the-money options and overhedges in-the-money options. On the contrary, the writer of options overhedges out-of-the-money options and underhedges in-the-money options. These observation are also valid for the model with proportional transaction costs only, but in that model the degree, to which the hedger over/underhedges an option, is less. Moreover, the target boundaries in the model with a fixed cost component are closer to each other than the NT boundaries in the model with no fixed cost component. Note, that the NT region and the distance between the two target boundaries are larger for the buyer than for the writer of options. This reflect the fact that an option is more risky for the writer than for the buyer. Hence, the writer hedges an option more frequently, and, thereby, charges a greater risk premium[18] than the buyer.

The most remarkable features of the net hedger's strategies are jumps to zero in target deltas when the stock price decreases below some certain levels. At these levels the NT regions widen. Especially the picture is clear for the writer of options (see Figure (6)) at $S \approx 70$. This behavior is fairly easy to understand. When transaction costs have a fixed fee component, it is not optimal to transact to some levels below a certain threshold. Instead, it is better to liquidate the stock position. The decision "to hedge or not

[17]That is, the average between Sell and Buy targets

[18]We interpret the notion of risk premium as the absolute value of the difference between a reservation option price and the BS-price

to hedge" is somewhat crucial, so it is, to some extent, better not to hurry with a transaction, but wait and see what happens with the stock price.

5.3 Reservation Option Prices and Optimal Trading Strategy for Net Investors

Figure (7) shows how reservation option prices depend on the price of the underlying stock for a net investor with $\gamma = 0.001$. For the net investor the two reservation option prices almost coincide. From the figure one can note that both of the reservation option prices are always above the corresponding BS-prices. As the underlying stock price increases, the deviation of a reservation option price from the BS-price also increases.

Figures (8) and (9) show the NT and target boundaries as functions of the stock price for the buyer and the writer of options respectively. For the sake of comparison we provide the optimal strategy curve in the absence of any transaction costs, as well as the NT boundaries in the model with proportional transaction costs only.

We define the investor's *delta* of one option of the position of θ options as

$$\Delta_\theta = \frac{y(0,\theta) - y(0)}{S_0\theta}, \tag{61}$$

where $y(0)$ and $y(0,\theta)$ are the investor's wealths invested in the stock at time zero in the utility maximization problem without options and with θ options respectively. Our numerical results show that both the writer and the buyer of options always overhedge options as compared to the Black-Sholes delta.

There has been one unresolved question in the utility based option pricing framework with transaction costs: Under what circumstances will a writer and a buyer agree on a common price for an option? Generally in the model with only proportional transaction costs $P_\theta^w > P_\theta^b$ if all parameters are the same for all the calculations. In the model with both fixed and proportional transaction costs under certain model parameters there occurs a situation when the reservation purchase price is higher than the reservation write price. Thus, the agreement is possible. We indicate another possibility for such an agreement. It exploits the fact that the reservation purchase price for the net investor lies above the BS-price. Note, that this possibility exists also for the case with proportional transaction costs only.

The other possibility for the agreement might arise in the situation when

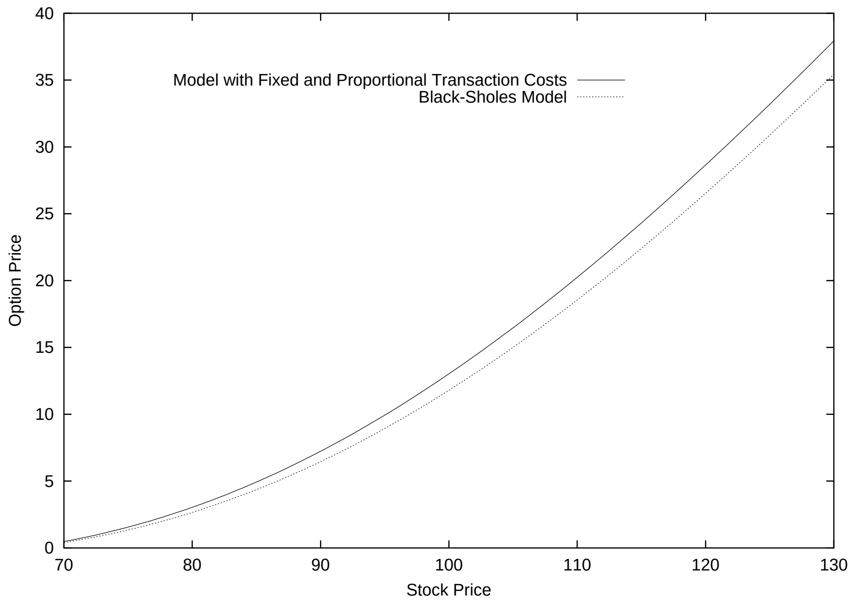

Figure 7: Reservation option price versus the price of the underlying stock for a net investor with $\gamma = 0.001$

a writer and a buyer, both of them being net investors in the underlying stocks, face different transaction costs in the market. Indeed, in real markets the commissions one pays on purchase, sale, and short borrowing are negotiated and depend on the annual volume of trading, as well as on the investor's other trading practices. In order to model realistic transaction costs one usually distinguishes between two classes of investors: large and small (see, for example, Dermody and Prisman (1993)). Large investors are defined as those who frequently make large trades in "blocks" (defined as 10,000 shares or more) via the block trading desks or brokerage houses. Large investors usually face transaction costs schedule with no minimum fee specified. In contrast, small investors are defined as those who use retail brokerage firms and often trade in 100-share round lots. For small investors there is a minimum fee on any trade. The main point is that the small investors have higher commission rates than the large ones. In other words, the large investors have lower level of proportional transaction costs than the small ones. In this case the reservation write price might be less than the reservation purchase price [19], i.e., the large investors could sell options

[19]This situation could be easily deduced from either Figure (3) or pricing formula 60

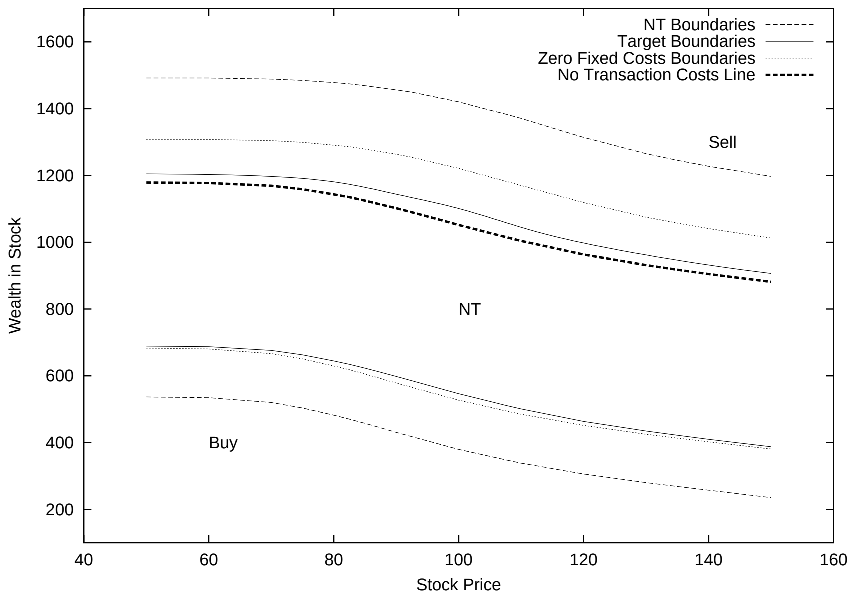

Figure 8: Optimal strategy control limits of 2 long call options for a net investor with $\gamma = 0.001$

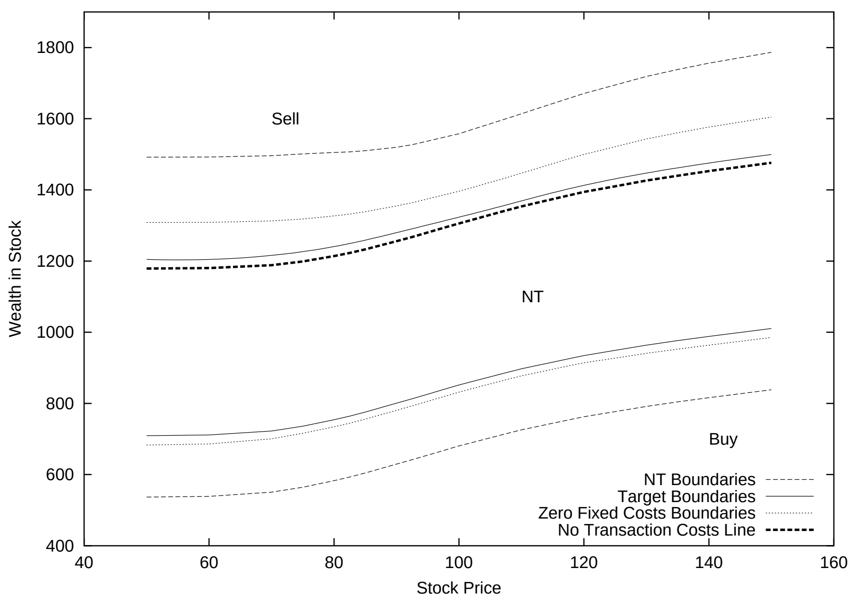

Figure 9: Optimal strategy control limits of 2 short call options for a net investor with $\gamma = 0.001$

to the small investors at an acceptable price.

5.4 Volatility Smile, Term Structure and Bid-Ask Spread

Given the assumptions in the Black-Sholes model, all option prices on the same underlying asset with the same expiration date but different exercise prices should have the same implied volatility. However, the shape of implied (from the market prices of traded contracts) volatility resembles either a smile or a skew. A skew is a general pattern for equity options. The implied volatility decreases as the strike price increases. This means, for example, that in-the-money call options are overpriced as compared to the theoretical Black-Sholes option price. In addition, the implied volatility depends also on the maturity of the option (the so-called term-structure of implied volatility). For equity options the implied volatility is usually an increasing function of option maturity. The size of a bid-ask spread shows also a consistent pattern: the bid-ask spread is lowest for at-the-money options. For either out-of-the-money or in-the-money options the bid-ask spread is higher. For both deep out-of-the-money and deep in-the-money options the bid-ask spread is approximately two times as large as for at-the-money options (see, for example, Peña, Rubio, and Serna (1999)). There were launched many possible explanations for the smile and the bid-ask spread in options prices, but in this subsection we want to reconcile the implications of the presence of transaction costs to the forms of the volatility smile, its term-structure, and the bid-ask spreads in the option pricing model we employ.

The presence of the bid-ask spread in option prices is an essential feature of the utility based option pricing model. The model gives two different option prices, one for the writer of options and the other for the buyer of options. The reservation write price and the reservation purchase price could be interpreted as the ask price and the bid price respectively. The ask price is always higher than the bid price. As to the functional form of the bid-ask spread, the utility based option pricing model implies quite opposite dependence on the moneyness of options. In particular, the bid-ask spread should be largest for at-the-money options. Both the net investors and the net hedgers need to hedge at-the-money options more often, and here a larger bid-ask spread reflects higher transaction costs. Out-of-the-money and in-the-money options require less hedging and, thus, lower transaction costs. Therefore, these options have lower bid-ask spread. The magnitude of the bid ask spread depends on the level of transaction costs and the risk aversion of the market agents. In fact, for the model parameters we use, the bid-ask spreads for the net investors with $\gamma < 0.01$ is actually less than the typical empirical bid-ask spread.

To this end let's assume that the option price in the market is the average of the reservation write and purchase prices. This resulting option price, as a function of the strike price, differs from the BS-price in a way that could be interpreted in terms of the implied volatility. Figure (10) shows some possible forms of the volatility smile. For the net hedgers the form of the volatility smile is a standard smile. The rationale for this form is the difference in the writer's and buyer's behavior. The writer, namely, hedges options more often than the buyer. The difference in frequency of hedging is more substantial for out-of-the-money and in-the-money options. Thus, the reservation write price drives up the average price for these options. For the net investors the form of the implied volatility is a classical skew. As mentioned above, for the net investors-buyers an option serves as a substitute for the stock. Holding an option saves the buyer from some transaction costs. On the contrary, the writer adds extra transaction costs to the option price. Both of the effects increase the option price. The less the strike price, the more the buyer saves and the writer adds. Note, that the form of the implied volatility is determined mainly by the type of a buyer, because the reservation write prices for both the net investor and the net hedger are quite close to each other, especially for deep in-the-money options.

Unfortunately, the steepness of the theoretical implied volatility smile is not high enough to explain the empirical facts[20]. If we take, for example, the net investors and calculate the implied volatilities for $M = 1$ and $M = 0.9$ we get 22.4% and 24.0% respectively. The difference between them is 1.6%. The differences among the empirical implied volatilities, however, are much larger (up to $5 - 10\%$) to be accounted for by transaction costs.

The form of the volatility term-structure, when it is an increasing function of option maturity, could also be accounted for by transaction costs. The longer the maturity the more transactions are carried out in order to hedge an option. As the writer of options hedges more often than the buyer, the reservation write price increases more than the reservation purchase price

[20] Note, that we use 1% proportional transaction costs rate which is higher than the realistic rate

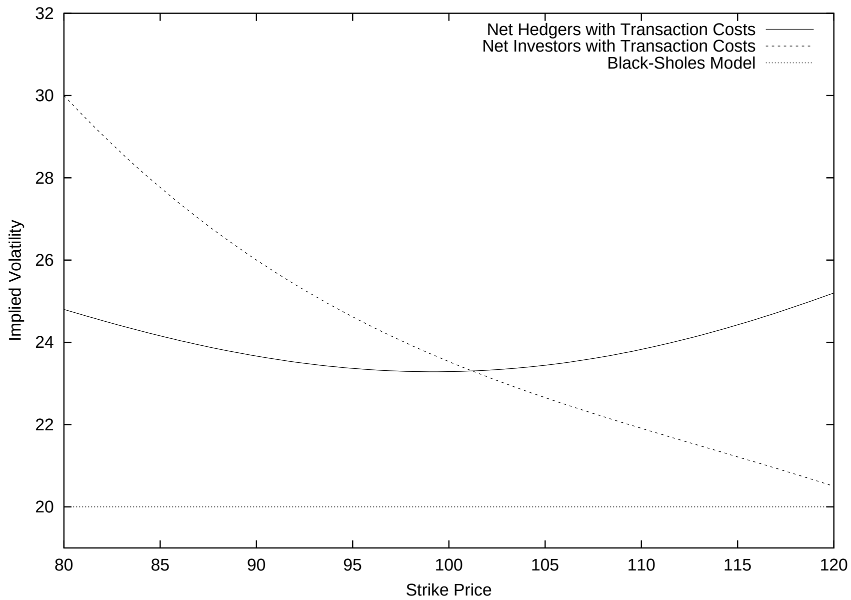

Figure 10: The possible forms of implied volatility as a function of the strike price

decreases. As a result, the average reservation option price increases when the time to maturity increases.

Our general conclusion here is that these empirical pricing biases could not be accounted for solely by the presence of transaction costs, even in the presence of a fixed cost component. It seems quite clear that something else is going on. Thus, our findings agree with those of Constantinides (1997).

6 Conclusions and Extensions

In this paper we extended the utility based option pricing and hedging approach, pioneered by Hodges and Neuberger (1989), for the market where each transaction has a fixed cost component. We formulated the continuous time option pricing and hedging problem for the $CARA$ investor in the market with both fixed and proportional transaction costs. Then we numerically solved the problem applying the method of the Markov chain approximation. The solution indicates that in the presence of both fixed and proportional transaction costs, most of the time, the portfolio space can be divided into three disjoint regions (Buy, Sell, and NT), and the optimal policy is described by four boundaries. The Buy and the NT regions are divided by the lower no-transaction boundary, and the Sell and the NT regions are divided by the upper no-transaction boundary. If a portfolio lies in the Buy region, the optimal strategy is to buy the risky asset until the portfolio reaches the Buy target boundary. Similarly, if a portfolio lies in the Sell region, the optimal strategy is to sell the risky asset until the portfolio reaches the Sell target boundary. All these boundaries are functions of time in the problem where the investor trades in the riskless and risky asset only. And all these boundaries are functions of both time and the price of the risky asset in the problem where the investor, in addition, writes/buys an option.

Our examination of the effects on the reservation option prices and the corresponding optimal hedging strategies of varying the investor's ARA and the drift of the risky asset suggested distinguishing between two major types of investors behavior in relation to the pricing and hedging of options: the *net investor* and the *net hedger*. The net investor, as well as the net hedger, has his own pattern of pricing and hedging options. Both the net investor's reservation option prices are above the BS-price, and they are very close to each other. The net investor overhedges both long and short option positions as compared to the BS-strategy. The net hedger's reservation purchase price is generally below the BS-price, and the net hedger's reservation write price is above the BS-price. Here the difference between the two prices depends on the level of the net hedger's absolute risk aversion and the level of transaction costs. Judging against the BS-strategy, the net hedger underhedges out-of-the-money and overhedges in-the-money long option positions. When the net hedger writes options, his strategy is quite the opposite. The net hedger overhedges out-of-the-money and underhedges in-the-money short option positions. The remarkable features of the net hedger's strategy are jumps to zero in target amounts in the stock when the stock price decreases below some certain levels. And at these levels the NT region widens.

We pointed out on two possible resolutions of the question: Under what circumstances will a writer and a buyer agree on a common price for an option? In the model with both fixed and proportional transaction costs under certain model parameters there occurs a situation when the reservation purchase price is higher than the reservation write price. The other possibility arises when a writer and a buyer, both of them being net investors in the underlying stocks, face different transaction costs in the market.

We also tried to reconcile our findings with such empirical pricing biases as the bid-ask spread, the volatility smile and the volatility term structure. Our general conclusion here is that these empirical phenomena could not be accounted for solely by the presence of transaction costs.

As it was conjectured by Davis et al. (1993) and showed in Andersen and Damgaard (1999), Damgaard (2000a), Damgaard (2000b), the reservation option prices are approximately invariant to the specific form of the investor's utility function, and mainly only the level of absolute risk aversion plays an important role. In particular, we calculated the reservation option prices for low levels of ARA with the same parameters as in the papers by Damgaard (2000a) and Damgaard (2000b) and obtained practically the same values. As a result, it seems to be of a little practical interest to calculate the reservation option prices and optimal hedging strategies using other utility functions besides the exponential one. These calculations will be very time-consuming, and, moreover, the optimal hedging strategy will be difficult to interpret because of its three-dimensional (x, y, S)-form.

As it was suggested by Davis et al. (1993) and presented in Davis and Zariphopoulou (1995), the utility based option pricing approach could also be applied to the pricing of American-style options. The problem of finding the reservation write price of an American-style option is somewhat tricky, because it is the buyer of option who chooses the optimal exercise policy. Therefore, the writer's problem must be treated from both the writer's and the buyer's perspective simultaneously. The problem of finding the reservation purchase price is simpler, since it suffices to consider the buyer's problem alone. Damgaard (2000a) calculated the reservation purchase prices of American-style call options for the case of the investor with $HARA$ utility and proportional transaction costs only. We believe that it is of a great practical interest to calculate both the reservation option prices and the corresponding optimal hedging/exercise policies for the markets with a general transaction costs structure, as the majority of traded option contracts are of American-style. This is an interesting area for future research.

Another interesting extension could be the calculation of reservation option prices in economies with more than one risky asset. We conjecture that for the $CARA$ utility and two risky assets the problem can be solved quite efficiently.

References

Andersen, E. D. and Damgaard, A. (1999). "Utility Based Option Pricing with Proportional Transaction Costs and Diversification Problems: an Interior-Point Optimization Approach", *Applied Numerical Mathematics*, *29*, 395–422.

Bensoussan, A. and Lions, J.-L. (1984). *Impulse Control and Quasi-Variational Inequalities.* Gauthier-Villars, Paris.

Black, F. and Scholes, M. (1973). "The Pricing of Options and Corporate Liabilities", *Journal of Political Economy*, *81*(3), 637–654.

Chancelier, J.-P., Øksendal, B., and Sulem, A. (2000). "Combined Stochastic Control and Optimal Stopping, and Application to Numerical Approximation of Combined Stochastic and Impulse Control", Preprint, Department of Mathematics, University of Oslo.

Clewlow, L. and Hodges, S. (1997). "Optimal Delta-Hedging under Transaction Costs", *Journal of Economic Dynamics and Control*, *21*, 1353–1376.

Constantinides, G. M. (1997). "Transaction Costs and the Implied Volatility Smile", Working Paper, University of Chicago.

Constantinides, G. M. and Perrakis, S. (2000). "Stochastic Dominance Bounds on Derivatives Prices in a Multiperiod Economy with Proportional Transaction Costs", Working Paper, University of Chicago, University of Ottawa and Concordia University.

Constantinides, G. M. and Zariphopoulou, T. (1999a). "Bounds on Derivative Prices in an Intertemporal Setting with Proportional Transaction Costs and Multiple Securities", Working Paper, Graduate School of Business, University of Chicago, Chicago.

Constantinides, G. M. and Zariphopoulou, T. (1999b). "Bounds on Prices of Contingent Claims in an Intertemporal Economy with Proportional Transaction Costs and General Preferences", *Finance and Stochastics*, *3*, 345–369.

Cox, J. M., Ross, S. A., and Rubinstein, M. (1979). "Option Pricing: A Simplified Approach", *Journal of Financial Economics*, *7*, 229–263.

Damgaard, A. (2000a). "Computation of Reservation Prices of Options with Proportional Transaction Costs", Working Paper, Department of Accounting, Finance and Law, University of Southern Denmark, Odense, Denmark.

Damgaard, A. (2000b). "Utility Based Option Evaluation with Proportional Transaction Costs", Working Paper, Department of Accounting, Finance and Law, University of Southern Denmark, Odense, Denmark.

Davis, M. H. A. and Norman, A. R. (1990). "Portfolio Selection with Transaction Costs", *Mathematics of Operations Research*, *15*(4), 676–713.

Davis, M. H. A., Panas, V. G., and Zariphopoulou, T. (1993). "European Option Pricing with Transaction Costs", *SIAM Journal of Control and Optimization*, *31*(2), 470–493.

Davis, M. H. A. and Zariphopoulou, T. (1995). "American Options and Transaction Fees", In: *Mathematical Finance* eds. M.H.A. Davis et al., Springer-Verlag, New York.

Dermody, J. C. and Prisman, E. Z. (1993). "No Arbitrage and Valuation in Markets with Realistic Transaction Costs", *Journal of Financial and Quantitative Analysis*, *28*(1), 65–80.

Eastham, J. and Hastings, K. (1988). "Optimal Impulse Control of Portfolios", *Mathematics of Operations Research*, *13*, 588–605.

Flemming, W. and Soner, H. M. (1993). *Controlled Markov Processes and Viscosity Solutions*. Springer-Verlag, New York.

Hastings, K. (1992). "Impulse Control of Portfolios with Jumps and Transaction Costs", *Communications in Statistics - Stochastic Models*, *8*, 222–239.

He, H. (1990). "Convergence from Discrete to Continuous Time Contingent Claim Prices", *Review of Financial Studies*, *3*, 523–546.

Hodges, S. D. and Neuberger, A. (1989). "Optimal Replication of Contingent Claims under Transaction Costs", *Review of Futures Markets*, *8*, 222–239.

Korn, R. (1998). "Portfolio Optimization with Strictly Positive Transaction Costs and Impulse Controls", *Finance and Stochastics*, *2*, 85–114.

Kushner, H. J. and Dupuis, P. G. (1992). *Numerical Methods for Stochastic Control Problems in Continuous Time*. Springer-Verlag, New York.

Kushner, H. J. and Martins, L. F. (1991). "Numerical Methods for Stochastic Singular Control Problems", *SIAM Journal of Control and Optimization*, *29*(6), 1443–1475.

Lo, A., Mamaysky, H., and Wang, J. (2000). "Asset Prices and Trading Volume under Fixed Transaction Costs", Working Paper, MIT.

Martellini, L. and Priaulet, P. (2000). “Optimal Dyanmic Hedging in the Presence of Transaction Costs: an Empirical Investigation”, Working Paper, University of Southern California.

Merton, R. C. (1973). “Theory of Rational Option Pricing”, *BELL Journal of Economics*, *4*, 141–183.

Mohamed, B. (1994). “Simulations of Transaction Costs and Optimal Rehedging”, *Applied Mathematical Finance*, *1*, 49–63.

Øksendal, B. and Sulem, A. (1999). “Optimal Consumption and Portfolio with both Fixed and Proportional Transaction Costs”, Preprint, Department of Mathematics, University of Oslo.

Peña, I., Rubio, G., and Serna, G. (1999). “Why do We Simle? On the Determinants of the Implied Volitility Function”, *Journal of Banking and Finance*, *23*, 1151–1179.

Shreve, S. and Soner, H. M. (1994). “Optimal Investment and Consumption with Transaction Costs”, *The Annals of Applied Probability*, *4*, 609–692.

Shreve, S. E., Soner, H. M., and Cvitanic, J. (1995). “There is no Nontrivial Hedging Portfolio for Option Pricing with Transaction Costs”, *The Annals of Applied Probability*, *5*(2), 327–355.

Whalley, A. E. and Wilmott, P. (1997). “An Asymptotic Analysis of an Optimal Hedging Model for Option Pricing with Transaction Costs”, *Mathematical Finance*, *7*(3), 307–324.

Zakamouline, V. I. (2002). “Optimal Portfolio Selection with Transaction Costs for a CARA Investor with Finite Horizon”, Discussion paper, The Norwegian School of Economics and Business Administration.