Equilibrium, Evolutionary Stability and Gradient Dynamics

Sjur Didrik Flåm*

Department of Economics, University of Bergen
and Norwegian School of Economics and Business Administration

Abstract. Considered here are equilibria, notably those that solve noncooperative games. Focus is on connections between evolutionary stability, concavity and monotonicity. It is shown that evolutionary stable points are local attractors under gradient dynamics. Such dynamics, while reflecting search for individual improvement, can incorporate myopia, imperfect knowledge and bounded rationality/competence.

Key words: equilibrium problems, noncooperative games, evolutionary and asymptotic stability, gradient dynamics, concavity, monotonicity.
JEL classification: C62, C72.

1. Introduction

This paper has *two* main objects: *first*, to relate a crucial condition called *evolutionary stability* to general equilibrium problems; *second*, to illustrate how such stability, when coupled with *gradient methods*, may help in locating equilibria. A subsidiary purpose is to emphasize some important links between theories of optimization and strategic behavior.

Motivation for this endeavor stems from recurrent problems in noncooperative games. Unlike convex programs such games often yield disconnected solution sets, typically composed of isolated singletons. When this happens, one might like to justify, predict or select a single *Nash equilibrium* among several. That problem has attracted much interest and spurred many studies. A main line of inquiry looks at persistence or emergence of equilibrium under perturbations. The leading idea there is that a solution becomes more plausible, comprehensible or refined if it filters through stability tests; see Harsanyi and Selten (1988) or van Damme (1991). Another, maybe more fruitful direction of investigation seeks equilibrium as the asymptotic limit, if any, of repeated play among agents who iteratively adapt to new experience; consult e.g. Ermoliev and Uryasiev (1982), Flåm (1999), Robinson (1951), or Rosen (1965).

Evolutionary game theory - presented in Samuelson (1997), Vega-Redondo (1996), Weibull (1996) and pioneered by Maynard Smith and Price (1973), (1982) - stands out in accommodating features from both these strands.[1] On one side, the prime concept of an *evolutionary stable* (ES) strategy puts focus on particularly robust Nash

*E-mail: sjur.flaam@econ.uib.no.

[1] In addition, it inspires and informs recent philosophy about social contracts and conventions; see Binmore (1994,1998).

equilibria. Loosely speaking, an ES strategy withstands moderate pressure from mutation and selection. On the other side, in symmetric two-person cases such a strategy becomes asymptotically stable under so-called *replicator dynamic;* see Hofbauer and Sigmund (1998). So, the notion ES speaks for itself on two complementary grounds: Not only will such outcomes be robust; they will also enjoy good stability properties under an important sort of dynamic.

However, that dynamic, when Darwinian in nature, fits biology better than economics. While biological agents are "hard-wired" to behave in certain ways, economic agents deliberate their choices and act intentionally. While nature selects strategists the latter choose strategies. Economic or social dynamic should therefore reflect not only growth or survival, but rationality or learning as well. Candidate dynamics include mean-value iterates (Flåm 1998), fictitious vor-Spiel (Robinson 1951), and adaptive play (Young 1998). Common to them, however, is the dominant role assigned to best responses, an assignment that hardly can achieve universal acceptance. Some natural reserves emerge because global optimization typically requires much competence and energy. Moreover, such behavior often induces nonsmoothness or instability in how play unfolds. To mitigate these features - and to accommodate more human-like, pragmatic players - we shall assume instead that *everybody, at every stage, merely seeks to better himself - and does so in fairly myopic, local manner.* But then: *Will repeated play lead to Nash equilibrium? And if so, to what sort of equilibrium?*

To come to grips with these issues Section 2 recalls and generalizes the key notion ES. Section 3 lends that notion some legitimacy by showing that *any ES point will be asymptotically stable under gradient dynamic.* This novel finding appears useful because such dynamic not only fits deliberate self-improvement; it can also accommodate bounded rationality and limited knowledge. Section 4 offers some examples.

2. Evolutionary Stability

The subsequent study will be organized around a generalized notion of evolutionary stability:

Definition 1. (Evolutionary stability) *Let X be a nonempty closed convex subset of some real Euclidean space $\mathbb{E}$, and let $\pi : X \times X \to \mathbb{R}$ be continuous. We declare $\bar{x} \in X$* ***evolutionary stable****, ES for short, (with respect to this π) if for all $x \neq \bar{x}$ in some vicinity $V \subseteq X$ of $\bar{x}$, the function $\pi(\cdot, x)$ is* ***locally***

$$\left.\begin{array}{lcl} \textbf{\textit{concave}} \text{ at } x\text{: } g \in \partial\pi(\cdot,x)(x) \Rightarrow \pi(x,x) + \langle \bar{x} - x, g\rangle & \geq & \pi(\bar{x},x) \text{ and} \\ \textbf{\textit{superior}} \text{ (strictly maximal) at } \bar{x}\text{: } \pi(\bar{x},x) & > & \pi(x,x). \end{array}\right\} \quad (1)$$

All together (1) says that

$$\pi(x,x) + \langle \bar{x} - x, g\rangle \geq \pi(\bar{x},x) > \pi(x,x)$$

whenever $x \in X$ is distinct from but sufficiently near $\bar{x}$; see Fig. 1.

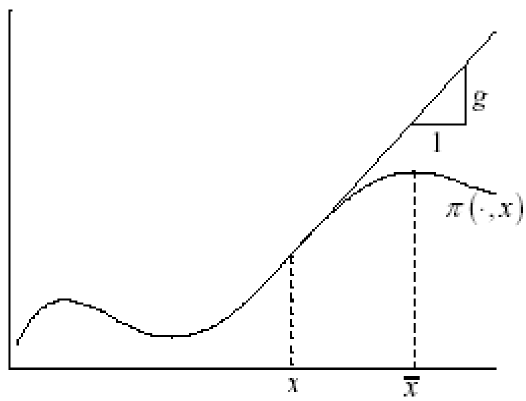

Fig. 1: Evolutionary stability.

Throughout $\langle \cdot, \cdot \rangle$ denotes an inner product on the space $\mathbb{E}$ with associated norm $\|x\| := \langle x, x \rangle^{\frac{1}{2}}$. The vector $g \in \partial\pi(\cdot, x)(x)$ is a partial, generalized gradient[2] of $\pi(\cdot, \cdot)$, taken with respect to the first variable and evaluated at the diagonal pair (x, x). Note that π satisfies (1) iff for any numbers $\alpha > 0, \beta$ the affine transformation $\alpha\pi + \beta$ does the same. Moreover, if π and $\hat{\pi}$ both satisfy (1) for the same $\bar{x}$, then so does $\pi + \hat{\pi}$.

To avoid repeated statements and qualifications, any symbol X or X_i refers henceforth to a nonempty closed, possibly unbounded convex subset of some Euclidean space, and V always denotes an X-neighborhood of an ES $\bar{x} \in X$; that is, $\{x : \|x - \bar{x}\| \leq r\} \cap X \subseteq V \subseteq X$ for some positive r. (X is thus endowed with the relative topology induced by the ambient space.)

In essence (1) spells out properties that $\pi(\cdot, x)$ should satisfy in its first variable, the point $\bar{x}$ there being distinguished. The strict inequality in (1) may be interpreted as follows: Construe the "second variable" x as a *system state* that parametrizes a "univariate" payoff function $\pi(\cdot, x)$. Then, if viewed from any prevailing state $x \neq \bar{x}$, close enough to $\bar{x}$, the "choice" $\bar{x}$ is strictly preferred to the status quo. Moreover, the somewhat myopic, linear prediction $\langle \bar{x} - x, g \rangle$ of the resulting gain $\pi(\bar{x}, x) - \pi(x, x)$ is always a positive over-estimate.

Bomze and Pötscher (1989) study instances where π is linear in the first variable. (See Examples 8 and 9 below.) The first property in (1) then comes for free and assumes a global character. Moreover, when $\pi(\cdot, x) = \langle \cdot, n(x) \rangle$, the second property in (1) says that $\langle x - \bar{x}, n(x) \rangle < 0$ for all $x \in V \setminus \bar{x}$. This inequality points towards an interior/strict version of the *normal cone* $N(\bar{x}) := \{n : \langle x - \bar{x}, n \rangle \leq 0, \forall x \in X\}$ that "stretches out" of X at $\bar{x}$. Bomze and Pötscher (op. cit.) explore this relation. They deal with cases where X is a collection of probability measures on some (sigma-field of an) underlying space of strategies.

Our main motivation for considering (1) stems from noncooperative, normal-form games, involving a finite set I of players. Individual $i \in I$ then chooses, with-

[2] See Clarke et al. (1998) or Rockafellar (1970).

out collaboration, a strategy $x_i \in X_i$ to maximize his supposedly continuous payoff $\pi_i(x_i, x_{-i})$, the profile $x_{-i} := (x_j)_{j \neq i}$ being short notation for the actions of i's rivals. Let then $X := \Pi_{i \in I} X_i$ and define $\pi : X \times X \to \mathbb{R}$ by

$$\pi(x, x') := \sum_{i \in I} \left\{ \pi_i(x_i, x'_{-i}) - \pi_i(x'_i, x'_{-i}) \right\}, \tag{2}$$

this implying $\pi(x, x) = 0$. Note that $\bar{x} \in X$ will be a *Nash equilibrium* iff it is an extremal *fixed point,* namely: $\bar{x} \in \arg\max \{\pi(x, \bar{x}) : x \in X\}$ - or equivalently, iff it satisfies the *variational inequality*

$$\pi(\bar{x}, \bar{x}) \geq \pi(x, \bar{x}) \text{ for all } x \in X. \tag{3}$$

Also note that for instance (2) the first inequality in (1) holds provided every payoff function $\pi_i(x_i, x_{-i})$ be concave and generalized differentiable in own variable x_i. The strict inequality in (1) is admittedly somewhat demanding. Example 4 below elaborates on its relevance for games.

Instead of (1) the reader might be familiar with the following alternative notion: A point $\bar{x} \in X$ is declared an *evolutionary stable strategy* (ESS) - in the sense of Maynard Smith and Price (1973), (1982) - iff

$$\forall x \in X \diagdown \bar{x} \; \exists \; \varepsilon(x) \in (0, 1] \text{ such that } \pi(\bar{x}, x^{\varepsilon}) > \pi(x, x^{\varepsilon}) \text{ whenever } \varepsilon \in (0, \varepsilon(x)]. \tag{4}$$

Here $x^{\varepsilon} := (1 - \varepsilon)\bar{x} + \varepsilon x$ is a mixed state, and (4) says that the "incumbent" $\bar{x}$ fares better against the modified ("contaminated") situation x^{ε} than does the "mutant" x.

Proposition 1. *Suppose π is locally superior at $\bar{x}$ and that*

$$\pi(x^{\varepsilon}, x^{\varepsilon}) \geq (1 - \varepsilon)\pi(\bar{x}, x^{\varepsilon}) + \varepsilon\pi(x, x^{\varepsilon}) \tag{5}$$

for any positive $\varepsilon \leq$ some positive threshold $\varepsilon(x)$. Then the ESS condition (4) *holds.*

Proof. Pick any $x \in X \backslash \bar{x}$ and a positive $\varepsilon \leq \varepsilon(x)$ so small that $x^{\varepsilon} = (1-\varepsilon)\bar{x} + \varepsilon x \in V$ and (5) holds. Then

$$\pi(\bar{x}, x^{\varepsilon}) > \pi(x^{\varepsilon}, x^{\varepsilon}) \geq (1 - \varepsilon)\pi(\bar{x}, x^{\varepsilon}) + \varepsilon\pi(x, x^{\varepsilon}),$$

whence (4) follows. □

The concavity assumptions in Definition 1 and Proposition 1 are, of course, satisfied when π is globally concave in the first variable. For a converse of Proposition 1 suppose X is bounded. Then, given any $\bar{x} \in X$ define its *opposite boundary* $obd(\bar{x})$ as follows. For every $x \in X \setminus \bar{x}$ there exists a unique number $\lambda \geq 1$ at which the half-ray $\lambda x + (1 - \lambda)\bar{x}$ leaves X. The set $obd(\bar{x})$ consists of exactly these boundary points $\lambda x + (1 - \lambda)\bar{x}$. Clearly, when $\bar{x}$ is relatively interior to X, its opposite boundary coincides with all of bdX. If X is a polyhedron and $\bar{x} \in bdX$, then $obd(\bar{x})$ comprises all of bdX less the relatively open faces to which $\bar{x}$ belongs. A convenient feature of polyhedral instances is that $\bar{x} \notin cl(obd(\bar{x}))$. We say that $\bar{x} \in X$ admits a *uniform entry barrier* $\bar{\varepsilon} \in (0, 1]$ iff (4) holds for all $x \in obd(\bar{x})$ and all positive $\varepsilon \leq \bar{\varepsilon}$. Following Hofbauer and Sigmund (1998), when π is affine in the second variable, one may prove that any ESS $\bar{x}$ admits a uniform barrier. We declare a function $f : X \to \mathbb{R}$ *strictly quasi-convex at* $\bar{x} \in X$ iff there exists a positive $\tilde{\varepsilon} \leq 1$ such that

$$\tilde{\varepsilon} \geq \varepsilon > 0 \;\; \& \;\; f(\bar{x}) > f(x) \Rightarrow f(\bar{x}) > f((1-\varepsilon)\bar{x} + \varepsilon x).$$

Proposition 2. *Let X be bounded. Suppose that $\bar{x} \in X$ admits a uniform barrier and does not belong to the closure of its opposite boundary. Then if π is strictly quasi-convex with respect to the first variable at $\bar{x}$, that point is locally superior.*

Proof. Let $\bar{\varepsilon}$ equal the minimum of the uniform barrier and the treshold $\tilde{\varepsilon}$ of strict quasi-convexity. Since $\bar{x} \notin cl(obd(\bar{x}))$, the set

$$V := \{(1-\varepsilon)\bar{x} + \varepsilon x : x \in obd(\bar{x}), 0 \leq \varepsilon \leq \min(\bar{\varepsilon}, \tilde{\varepsilon})\}$$

becomes an X-neighborhood of $\bar{x}$. That is, there exists a positive radius r such that $\{x \in X : \|x - \bar{x}\| \leq r\} \subseteq V \subseteq X$. Pick any $x^{\varepsilon} = (1-\varepsilon)\bar{x} + \varepsilon x \in V$ with $x \in obd(\bar{x})$ and $0 < \varepsilon \leq \min(\bar{\varepsilon}, \tilde{\varepsilon})$. Then (4) holds and by the strict quasi-convexity of $\pi(\cdot, x^{\varepsilon})$ at $\bar{x}$ we get the desired inequality $\pi(\bar{x}, x^{\varepsilon}) > \pi(x^{\varepsilon}, x^{\varepsilon})$. □

Bomze and Pötscher (1989), while assuming π linear in the first variable, explore existence of a uniform barrier. In that case $\bar{x}$ is declared *uninvadable*.

Propositions 1 and 2 point to a possible equivalence between (1) and (4) provided π is suitably concave in the first variable. Clearly, the implication (4) $\Rightarrow$ (3) obtains by letting $\varepsilon \to 0^+$. But more may hold here. In fact, adding appropriate linearity, we obtain straightforwardly the classical definition that any other best response x to the incumbent $\bar{x}$ fares worse than $\bar{x}$ against itself:

Proposition 3. *Suppose π is affine in the second variable. Then (4) is equivalent to*

$$\pi(\bar{x}, \bar{x}) \geq \pi(x, \bar{x}) \text{ for all } x \in X, \text{ and } x \neq \bar{x}, \pi(x, \bar{x}) = \pi(\bar{x}, \bar{x}) \Rightarrow \pi(\bar{x}, x) > \pi(x, x). \; \square \tag{6}$$

We conclude this section by considering a special but important class of problems for which equilibria are ES:

Definition 2. (Concave-convex functions) $\pi : X \times X \to \mathbb{R}$ *is called concave in the first coordinate at $\bar{x}$ if for any x in some vicinity $V \subseteq X$ of $\bar{x}$ and every $\varepsilon \in [0, 1]$, it holds with $x^{\varepsilon} := (1-\varepsilon)\bar{x} + \varepsilon x$ that*

$$\pi(x^{\varepsilon}, x) \geq (1-\varepsilon)\pi(\bar{x}, x) + \varepsilon\pi(x, x). \tag{7}$$

π is declared convex in the second coordinate at $\bar{x}$ if under the same hypothesis

$$\pi(x^{\varepsilon}, x^{\varepsilon}) \leq (1-\varepsilon)\pi(x^{\varepsilon}, \bar{x}) + \varepsilon\pi(x^{\varepsilon}, x). \tag{8}$$

Proposition 4. (Concave-convex problems have ES equilibria) *Suppose π is constant along the diagonal and concave-convex at $\bar{x}$ in the sense of* (7), (8)*. Also suppose that $\pi(\bar{x}, \cdot)$ is strictly convex at $\bar{x}$, this meaning that*

$$\pi(\bar{x}, x) > \pi(\bar{x}, \bar{x}) + \langle x - \bar{x}, \bar{\gamma} \rangle \tag{9}$$

for all x sufficiently near but $\neq \bar{x}$ and any $\bar{\gamma} \in \partial\pi(\bar{x}, \cdot)(\bar{x})$. Then, under (3) $\bar{x}$ is ES.

Proof. With no loss of generality on may replace π by the translated version $\pi(\cdot, \cdot) - \pi(\bar{x}, \bar{x})$. So, suppose that $\pi(x, x) = 0$ for all $x \in X$. As preparation for the main argument we claim that

$$-\partial\pi(\cdot, \bar{x})(\bar{x}) \subseteq \partial\pi(\bar{x}, \cdot)(\bar{x}). \tag{10}$$

To demonstrate (10) pick a vicinity $V \subseteq X$ of $\bar{x}$ such that both (7) and (8) hold. Select any $x \in V$ and let as before $x^{\varepsilon} = (1 - \varepsilon)\bar{x} + \varepsilon x$ for $\varepsilon \in (0, 1)$. Divide (8) by ε and let $\varepsilon \searrow 0$ to obtain

$$0 \leq \pi(\bar{x}, x) + \lim_{\varepsilon \searrow 0} \left[\pi(x^{\varepsilon}, \bar{x}) - \pi(\bar{x}, \bar{x})\right] / \varepsilon.$$

The concavity property (7) tells that for every $\bar{g} \in \partial\pi(\cdot, \bar{x})(\bar{x})$ it holds that

$$\lim_{\varepsilon \searrow 0} \frac{\pi(x^{\varepsilon}, \bar{x}) - \pi(\bar{x}, \bar{x})}{\varepsilon} \leq \langle x - \bar{x}, \bar{g} \rangle .$$

Combining these inequalities we get $\pi(\bar{x}, x) \geq \langle x - \bar{x}, -\bar{g} \rangle \geq \pi(\bar{x}, \bar{x}) + \langle x - \bar{x}, -\bar{g} \rangle$, whence $-\bar{g} \in \partial\pi(\bar{x}, \cdot)(\bar{x})$. This proves claim (10). We come now to the main argument. By assumption (3) there exists a gradient $\bar{g} \in \partial\pi(\cdot, \bar{x})(\bar{x})$ such that $\langle x - \bar{x}, \bar{g} \rangle \leq 0$ for all $x \in X$. Via (10) we get $-\bar{g} \in \partial\pi(\bar{x}, \cdot)(\bar{x})$. Hence by the strict local convexity (9) of $\pi(\bar{x}, \cdot)$ we have

$$\pi(\bar{x}, x) > \pi(\bar{x}, \bar{x}) + \langle x - \bar{x}, -\bar{g} \rangle \geq \pi(\bar{x}, \bar{x}) = \pi(x, x).$$

This takes care of local superiority. The other (gradient) inequality in (1) follows from the local concavity (7) of $\pi(\cdot, x)$ over V. □

3. Gradient or Adaptive Dynamics

As said, we prefer to model dynamics that reflect rationality but requires neither global perception nor much system knowledge of any concerned party. Fitting such a philosophy is the *gradient flow*

$$\frac{dx}{dt} := \dot{x} \in P_{Tx} \left[\partial\pi(\cdot, x)(x)\right], \tag{11}$$

its initial point $x(0) \in X$ being arbitrary. For simplicity we often write x instead of $x(t)$. The set $Tx := cl\mathbb{R}_{+}(X - x)$ denotes the *tangent cone* of X at $x \in X$. The *orthogonal projection* (operator) onto a nonempty closed convex set C is written P_C. It is also called the *closest approximation* in C.

(11) subscribes to Alfred Marshall's opinion that adjustments are driven by margins, emerging through linearization in the vicinity of the current state.

By a solution to the *differential inclusion* (11) we mean an absolutely continuous function $0 \leq t \mapsto x(t) \in X$ which satisfies (11) almost everywhere (a.e.); see Aubin and Cellina (1984). *We tacitly assume that from any $x(0) \in X$ there emanates a unique solution.* In that case $x(t) \in X$ for all $t > 0$. To appreciate (11) reconsider the normal-form game that gives rise to instance (2). Then (11) splits into a system

$$\frac{dx_i}{dt} := \dot{x}_i \in P_{T_i x_i}\left[\partial\pi_i(\cdot, x_{-i})(x_i)\right], \forall i, \tag{12}$$

having initial points $x_i(0) \in X_i$ which could be arbitrary or determined by historical "accident." Equation (12) portrays each player as steadily moving along a projected direction of payoff ascent. In particular, when x_i is interior to X_i, and $\pi_i(\cdot, x_{-i})$ differentiable at x_i, player i proceeds in the direction of *steepest ascent.*

System (12) requires very little input to be kept going. Basically, it suffices that each player persistently observes his marginal payoff. Nobody must know the game, his rivals, their preferences or actions. Admittedly, players behaving as described by (12) stand to be criticized: Each seemingly believes that all his rivals will stay put. No one looks for large size improvements; nobody makes a great leap towards a best response. But criticisms of such attitudes often appears unfair though. How could a player form specific beliefs about his rivals if he never gets to observe their actions? And why should he move much or swiftly - or respond optimally - if he merely knows his own payoff function up to local, linear approximation? In spite of players knowing little or moving slowly, system (12) enjoys remarkable stability. Indeed, as showed next, equilibrium may be reached - and perpetuated, not by perfect rationality, but by reasonable responses to local incentives:

Theorem 1. (Asymptotic stability of ES under gradient flows) *Any solution trajectory to* (11) *starting in a vicinity V of an ES $\bar{x}$, for which* (1) *holds, will converge to that point $\bar{x}$.*

Proof. Recall the so-called Moreau decomposition (see Aubin and Cellina 1984, Proposition 0.6.3), saying that any vector $g \in \mathbb{E}$ can be uniquely (and orthogonally) decomposed as a sum $g = \mathbf{t} + \mathbf{n}$ where $\mathbf{t} \in Tx$ is tangent and $\mathbf{n}$ is normal (orthogonal) to X at the current point x. Orthogonality means that $\langle x' - x, \mathbf{n}\rangle \leq 0, \forall x' \in X$. As a result, $P_{Tx}[g] = \mathbf{t} = g - \mathbf{n}$, and any solution $t \mapsto x(t)$ to (11) makes the map $t \mapsto \Lambda(t) := \frac{1}{2}\|x(t) - \bar{x}\|^2$ Ljapunov. Indeed, omitting reference to time t, that map satisfies

$$\dot{\Lambda} = \langle x - \bar{x}, \dot{x}\rangle = \langle x - \bar{x}, g - \mathbf{n}\rangle \leq \langle x - \bar{x}, g\rangle \leq \pi(x, x) - \pi(\bar{x}, x) < 0 \text{ a.e.}$$

Using the continuity of π this shows that $\Lambda(t) \searrow 0$, whence $x(t) \to \bar{x}$. $\square$

This result is not entirely novel. The instance when π is linear in its first variable, and X equals the standard simplex, appears in Hofbauer and Sigmund (1998) as Theorem 9.6.1. Those authors also use more general gradients, derived from a local x-dependent inner product on the tangent space to X at x. Doing so offers a double bonus: first, one may accommodate nonconvex manifolds X; second, the class of dynamics is much enlarged. In particular, applying the Shahshahani inner product on the interior of the simplex, the replicator equation comes out as gradient dynamics; see Hofbauer and Sigmund (1998, Theorem 24.3). Interesting in this regard are the results of Hopkins (1999) that replicator-like dynamics may emerge when fictitious play is aggregated over a heterogenous population.

We end this section by briefly considering use of discrete time in (11). For that purpose we must first inquire about the nature of steady states:

Proposition 5. (Characterization of steady states) *$\bar{x}$ is a rest point for (11), i.e., $0 \in P_{T\bar{x}}\left[\partial\pi(\cdot,\bar{x})(\bar{x})\right]$ if*

$$\bar{x} \in P_X\left[\bar{x} + s\bar{g}\right] \text{ for some } \bar{g} \in \partial\pi(\cdot,\bar{x})(\bar{x}) \text{ and all } s > 0. \tag{13}$$

This happens iff for some $\bar{g} \in \partial\pi(\cdot,\bar{x})(\bar{x})$ the following variational inequality holds:

$$\langle x - \bar{x}, \bar{g}\rangle \leq 0 \text{ for all } x \in X. \tag{14}$$

Finally, when $\pi(\cdot,\cdot)$ is concave in the first variable, $\bar{x}$ is a steady state iff it satisfies (3).

Proof. Let $\bar{g} \in \partial\pi(\cdot,\bar{x})(\bar{x})$ satisfy (13). Then $0 = \lim_{s\searrow 0}\left\{ P_X\left[\bar{x} + s\bar{g}\right] - \bar{x}\right\}/s = P_{T\bar{x}}\left[\bar{g}\right]$, this being a general result of Zarantonello (1971). (13) says that

$$\bar{x} = \arg\min\left\{\frac{1}{2}\left\|\bar{x} + s\bar{g} - x\right\|^2 : x \in X\right\}$$

and (14) is the associated (necessary and sufficient) optimality condition. (3) follows from standard convex analysis (Rockafellar 1970). $\square$

We now cast (11) into the following, more realistic, discrete-time process: Iteratively, at stages $k = 0, 1, \ldots$ let

$$x^{k+1} = P_X\left[x^k + s_k g^k\right], \tag{15}$$

the initial point $x^0 \in X$ being arbitrary or accidental. Here $g^k \in \partial\pi(\cdot, x^k)(x^k)$ is a partial gradient of $\pi(\cdot,\cdot)$ with respect to the first variable at (x^k, x^k). The positive sequence $(s_k)_{k=0}^{\infty}$ of step sizes is chosen a priory subject to $\lim_{k\to\infty} s_k = 0$ and $\sum_{k=0}^{\infty} s_k = +\infty$. Evidently, (15) is a dynamic version of the fixed point condition (13). In the particular setting (2) system (12) assumes the form

$$x_i^{k+1} = P_{X_i}\left[x_i^k + s_k g_i^k\right] \text{ for all } i, \tag{16}$$

where $g_i^t \in \partial\pi_i(\cdot, x_{-i}^t)(x_i^t)$ is agent i's marginal payoff. Game theoretic studies of (12) or (16) include Ermoliev and Uryasiev (1982), Flåm (1998), Rosen (1965).

Theorem 2. (Convergence to ES) *Suppose the partial gradient* $g \in \partial\pi(\cdot, x)(x)$ *is unique and depends continuously on* $x \in X$. Also *suppose that* (1) *holds for some ES* $\bar{x}$ *and vicinity* V. *Then, if the sequence* (x^k) *generated by* (15) *is bounded and enters infinitely often into* V, *we have* $x^k \to \bar{x}$.

Proof. It is known from Zarantonello (1971) that for any vector $x \in X$, and direction g we have $\lim_{s\searrow 0} \{P_X[x+sg] - x\}/s = P_{Tx}[g]$. Consequently,

$$P_X[x+sg] = x + sP_{Tx}[g] + o(s)$$

where $o(s)$ denotes a vector satisfying $\frac{\|o(s)\|}{s} \to 0$ as $s \searrow 0$. This means that (15) can be rewritten on the format

$$x^{k+1} - x^k = s_k \left\{h(x^k) + b^k\right\}$$

with $h(x) := P_{Tx}[\partial\pi(\cdot, x)(x)]$ and $\lim_{k\to\infty} b^k = 0$. Recall that by assumption the continuous vector field $\dot{x} = h(x)$ has unique solutions. The conclusion now follows from Proposition 3.1 in Benaim (1996). □

In the interest of a more global statement than Theorem 2 we introduce some additional notions, these presuming that solutions $t \mapsto x(t)$ to (11) be extended to all $t \in \mathbb{R}$. The *alfa* and *omega limit* of an initial point $\xi \in X$, denoted $\alpha(\xi)$ and $\omega(\xi)$, equals the set of all accumulation points of the solution trajectory $x(t)$, starting in ξ, when t tends to $-\infty$ and $+\infty$, respectively. We say that (11) yields *simple dynamics* if both $\alpha(\xi)$ and $\omega(\xi)$ are rest points for every $\xi \in X$.

A rest point e^1 of (11) is said to go to another rest point e^2 via a connecting orbit $\mathcal{O}$ if for some $\xi \in \mathcal{O}$ we have $e^1 = \alpha(\xi)$ and $e^2 = \omega(\xi)$. A *cycle* is a finite set of such points $e^1, \ldots, e^n, (e^{n+1} = e^1)$ together with orbits $\mathcal{O}^1, \ldots, \mathcal{O}^n$ such that $\mathcal{O}^k$ connects e^k to e^{k+1}. Now invoking Corollary 3.7 in Benaim (1996) we get:

Theorem 3. (Convergence to a discrete limit set) *Assume that* (11) *has isolated rest points, simple dynamics and no cycle. Then a bounded sequence* (x^k) *generated by* (15) *converges toward a steady state.* □

A finite collection of Nash equilibria is generic in games. For such instances it remains a challenge to explore whether the dynamic (16) is simple with no cycles.

4. EXAMPLES

To illustrate and justify the concept ES, introduced above, this section offers several examples. The first four are concerned with noncooperative games, Example 4 being of main importance.

Example 1. (*Two-person, zero-sum games)* Let there be merely two players $i \in$

$I := \{+1, -1\}$. When individual i selects his strategy $x_i \in X_i$, he derives continuous payoff $\pi_i(x_i, x_{-i})$ which is concave in own choice x_i. Suppose payoffs sum to zero and define $\pi : X \times X \to \mathbb{R}$ by

$$\pi(x, x') := \pi_1(x_1, x'_{-1}) - \pi_1(x'_1, x_{-1}). \tag{17}$$

Then $\bar{x} = (\bar{x}_1, \bar{x}_{-1})$ is Nash equilibrium iff (3) holds, or equivalently, iff $\bar{x}$ is a max-min saddle point of π_1. When π_1 is affine in its second variable, a point $\bar{x}$ will be ES in the sense of (4) if it is a *strict Nash equilibrium*, i.e., if

$$\bar{x} = \arg\max \{\pi(x, \bar{x}) : x \in X\}. \quad \square \tag{18}$$

Example 2. (*Two-person, finite-strategy, symmetric games*) Consider a two-person, symmetric game with a finite set S of pure strategies. Denote then by $X := \Delta(S)$ the simplex consisting of all probability measures $x := (x(s))_{s \in S}$ on S. The payoff $\pi(x, x')$ to a player who uses the mixed strategy $x \in X$ in front of his rival's choice $x' \in X$, is given in terms of an already specified function $\pi : S \times S \to \mathbb{R}$ by its bilinear extension $\pi(x, x') := \sum_{s, s' \in S} x(s)\pi(s, s')x'(s')$. The interpretation of (4) goes in terms of random pair-wise matching of players from a single population. Equilibrium $\bar{x}$ is then seen as standard, common behavior prior to the appearance of a "mutation " $x \neq \bar{x}$. Specifically, one envisages that a population percentage ε starts to employ a new strategy x. Biological intuition behind (4) says that incumbent individuals, maintaining their pre-entry choice $\bar{x}$, fare better against the post-entry population mix $x^\varepsilon = (1 - \varepsilon)\bar{x} + \varepsilon x$, $0 < \varepsilon \leq \varepsilon(x)$, than does any entering mutant $x \neq \bar{x}$. Naturally, in this example *symmetric* equilibria are of particular interest. We emphasize, however, that the more general definition (1) makes no hypothesis about symmetry.

Besides (1) and (4) there is a third equivalent characterization, due to Hofbauer, Schuster and Sigmund (1979):

Proposition 6. (Local superiority of ESS) *$\bar{x}$ is ES of a two-person, finite-strategy, symmetric game iff it is locally superior.* $\square$

Example 3. (*Two-person, two-strategy, symmetric games*) This example continues and specializes the preceding one. Consider a two-person, two-strategy, symmetric game where player $i = \pm 1$ assigns probability x_i to his first (either upper row or left column) action. The 2×2 payoff matrix may, without loss of generality, be assumed diagonal with entries a_1, a_2. Thus (16) reads

$$x_i^{k+1} = P_{[0,1]}\left[x_i^k + s_k\left\{(a_1 + a_2)x_{-i}^k - a_2\right\}\right], \; i = \pm 1. \tag{19}$$

A well studied setting accommodates *one* homogeneous population of players. The initial point then naturally lies on the diagonal, and instead of (19) we could just as well have used the one-dimensional system $x^{k+1} = P_{[0,1]}\left[x^k + s_k\left\{(a_1 + a_2)x^k - a_2\right\}\right]$, evolving in $[0, 1]$. A more general scenario admits *two* populations, starting maybe at different levels and moving in the state space $[0,1]^2$. In any case, assuming $a_1 a_2 \neq 0$, there are *three* cases to consider: *First*, when $a_1 a_2 < 0$, a *Prisoner's Dilemma* emerges with *one* strict equilibrium whence ES. *Second*, when both a_1 and a_2 are positive, a *Coordination Game* comes on stage, having three Nash equilibria, all symmetric. The two pure ones are ES whereas the mixed one $x_i = a_2/(a_1 + a_2)$ is not. *Third,* if both a_1 and a_2 are negative, we get an antagonistic *Hawk-Dove Game*, having two strict, pure, asymmetric Nash equilibria and a symmetric, mixed one $x_i = a_2/(a_1 + a_2)$. Given merely one population, the mixed equilibrium is the only ES. Present two populations, that equilibrium is no longer ES.

Example 4. (*Noncooperative games in strategic form*) Extending the preceding examples we now come to the setting which provided our principal motivation. Let the set I of noncooperative players be finite and have at least two members. Inequality (4) applied to (2) yields the following condition, introduced by Taylor (1979):

$$x \in X \setminus \bar{x}, 0 < \varepsilon \leq \varepsilon(x) \Rightarrow \sum_{i \in I} \pi_i(\bar{x}_i, x_{-i}^{\varepsilon}) > \sum_{i \in I} \pi_i(x_i, x_{-i}^{\varepsilon}).$$

To insist on local superiority certainly shrinks the class of fitting games. (There are games for which no ES strategy exists; see Weibull (1996, page 39). Nonetheless, these conditions hold more often than might first be believed. Included are the instances for which (2) is concave-convex - as brought out by Proposition 4. As one example we advocate games with bilinear interaction:

Proposition 7. (Games with bilinear interaction have ES equilibria) *Suppose each individual $i \in I$ has a utility function of the form*

$$\pi_i(x_i, x_{-i}) = U_i(x_i) + \left\langle x_i, \sum_{j \in I \setminus i} A_{ij} x_j \right\rangle + U_{-i}(x_{-i}),$$

U_i being concave, twice differentiable, and A_{ij} a $dimx_i \times dimx_j$ matrix. If the grand matrix, featuring A_{ij} in block entry ij and $\nabla^2 U_i(x_i)$ in diagonal block entry ii, is negative semi-definite, then the game is concave-convex in the sense of (7) *and* (8). *When moreover, the said block matrix is negative definite, the strict convexity condition* (9) *holds, and every Nash equilibrium is ES.*

Proof. (2) assumes the form

$$\pi(x, x') = \sum_i \left\{ U_i(x_i) - U_i(x_i') + \left\langle x_i - x_i', \sum_{j \in I \setminus i} A_{ij} x_j' \right\rangle \right\}.$$

Evidently, this π is concave in its first variable x, and (strict) convexity in its second variable x' obtains by the fact that $\nabla^2 \pi(x, \cdot)$ is negative semi-definite (definite). $\square$

The setting of Proposition 7 does not imply uniqueness of Nash equilibrium: Consider a concave single-agent optimization problem with linear constraints. The set of saddle points need then not reduce to a singleton.

The last result was stated in a global fashion, appropriate, for example, in Cournot type oligopolies with affine inverse demand and strictly convex production costs. More generally, we believe that bilinear, local approximation around equilibrium often satisfies the curvature assumptions mentioned in Proposition 7.

Example 5. (*Single-agent optimization, concave programming*) In the simple situation of a single decision maker, say Robinson Crusoe, isolated from the intricacies of strategic interaction, let $\Pi : X \to \mathbb{R}$ be his payoff function and define $\pi(x, x') := \Pi(x)$. If Π is strictly concave and super-differentiable on X, then evidently, $\bar{x} \in X$ is ES iff $\bar{x} = \arg\max \Pi$. More generally, suppose Π is twice continuously differentiable with negative definite Hessian $\frac{\partial^2}{\partial x^2}\Pi$ at every local maximum. Then $\bar{x} \in X$ will be ES iff it is a strict local maximum.

Example 6. (*Variational inequalities*) Let here $\pi(x, x') := \langle x - x', m(x') \rangle$ for some continuous mapping $m : X \to \mathbb{E}$. Then π is affine in its first variable, and (3) amounts to the variational inequality $\langle x - \bar{x}, m(\bar{x}) \rangle \leq 0, \forall x \in X$. Local superiority holds if m is strictly monotone decreasing near $\bar{x}$, that is, if $\langle x - \bar{x}, m(x) - m(\bar{x}) \rangle < 0$ for all $x \in V \setminus \bar{x}$.

Example 7. (*Equilibrium problems,* Blum and Oettli (1994), Flåm and Antipin (1997), Konnov and Schaible (2000)) Many instances make π nil along the diagonal. So, in view of the important instance (2) assume here that $\pi(x, x) = 0, \forall x \in X$. Then, (3) is called a (primal) *equilibrium problem.* Associated to the latter is a *dual equilibrium problem:* namely, to find $\bar{x} \in X$ such that $\pi(\bar{x}, x) \geq 0, \forall x \in X$. Denote by X^P, X^D the two corresponding solution sets. One declares π *monotone* if $\pi(x, x') + \pi(x', x) \geq 0$ for all pairs $x, x' \in X$ - and *pseudo-monotone* if $\pi(x, x') \leq 0 \Rightarrow \pi(x', x) \geq 0$. Pseudo-monotonicity straightforwardly implies $X^P \subseteq X^D$. If $\pi(x, \cdot)$ is lower semicontinuous from the right along any ray, and $\pi(\cdot, x)$ is strictly quasi-concave, then $X^D \subseteq X^P$; see Bianchi and Schaible (1996). Note that a solution $\bar{x}$ to (3) becomes locally superior under strict local pseudo-monotonicity, that is, when $\pi(x, \bar{x}) \leq 0 \Rightarrow \pi(\bar{x}, x) > 0, \forall x \in V \diagdown \bar{x}$.

Example 8. (*Competitive equilibrium,* Flåm and Sandvik (2000)) Let the vector x record prices of various goods, produced and/or consumed by price-taking (competitive) economic agents. Those agents presumably generate (single-valued, continuous, 0-homogeneous) excess demand $E(x)$. By Walras law $x \cdot E(x) = 0$. Define here $\pi(x, x') := x \cdot E(x')$ and choose X as a suitable part of the non-negative orthant. For example, let X equal the standard unit simplex. Then, if (3) holds, or equivalently, if $E(\bar{x}) \leq 0$, we call $\bar{x}$ a (free disposal) equilibrium. The first inequality in (1) holds trivially, and $\bar{x}$ becomes locally superior if excess demand generated by $x \in V \diagdown \bar{x}$ is too costly under $\bar{x}$, i.e., if $\bar{x} \cdot E(x) > 0$. Process (11) amounts to the classical Walrasian tâtonnement.

Example 9. *(Malthusian dynamics)* When $\pi(x, x') := x \cdot F(x')$ for some continuous mapping $F : X \to \mathbb{E}$, the concavity requirement in (1) has no bite. Friedman (1991) discusses such instances when X is the finite product of standard simplices $X_i \subset \mathbb{E}_i, i \in I$. Let then $D_i(x_i)$ be the diagonal matrix formed by $x_i \in X_i$ and set $F_i(x) := D_i(x_i)\,[f_i(x) - x_i \cdot f_i(x)\mathbf{1}_i]$ where $\mathbf{1}_i := (1, 1, ..)$ has appropriate dimension. Since the force $F_i(x)$ so defined belongs to the tangent cone of X_i at x_i, the dynamics (11) simplify to a coupled set of replicators: $\dot{x}_i = F_i(x), i \in I$, called *Malthusian* by Friedman (op. cit.). Local superiority means here that

$$\bar{x} \cdot F(x) > x \cdot F(x) \tag{20}$$

for all $x \in X \diagdown \bar{x}$ sufficiently near $\bar{x}$. For frequency-dependent stability of two interacting species condition (20) has been studied in more general form by Cressman (1992, 1996).

Acknowledgement Generous economic support from Ruhrgas, Røwdes fond and Meltzers høyskolefond is gratefully acknowledged. I thank an anonymous referee for very helpful remarks and suggestions.

REFERENCES

[1] Aubin, J. P. and A. Cellina (1984). *Differential Inclusions*, Springer-Verlag, Berlin .

[2] Benaim, M. (1996).A dynamical system approach to stochastic approximations, *SIAM J. Control and Optimization* 34, 437-472.

[3] Bianchi, M. and S. Schaible (1996). Generalized monotone bifunctions and equilibrium problems, *J. Opt. Theory and Applic.* 90, 31-43.

[4] Binmore, K. (1994, 1998). *Game Theory and the Social Contract*, The MIT Press, vol. I: *Playing Fair*, Vol. II: *Just Playing.*

[5] Blum, E. and W. Oettli (1994). From optimization and variational inequalities to equilibrium problems, *Mathematics Student* 63, 123-145.

[6] Bomze, I. M. and B. M. Pötscher (1989). *Game Theoretical Foundations of Evolutionary Stability*, Lecture Notes in Economics and Mathematical Systems 324, Springer-Verlag, Berlin.

[7] Clarke, F. H., Yu. S. Ledyaev, R.J. Stern and P. R. Woleski (1998). *Nonsmooth Analysis and Control Theory*, Springer-Verlag, Berlin.

[8] Cressman, R. (1992). *The Stability Concept of Evolutionary Game Theory*, Lecture Notes in Biomathematics 94, Springer-Verlag, Berlin.

[9] Cressman, R. (1996). Frequency-dependent stability for two-species interactions, *Theoretical Population Biology* 49, 189-210.

[10] Ermoliev, Yu. M. and S. P. Uryasiev (1982). Nash equilibrium in n-person games, (in Russian) *Kibernetika* 3, 85-88.

[11] Flåm, S. D. and A. S. Antipin (1997). Equilibrium programming using proximal-like algorithms, *Mathematical Programming* 78, 29-41.

[12] Flåm, S. D. (1998). Restricted attention, myopic play, and the learning of equilibrium, *Annals of Operations Research* 82, 473-482.

[13] Flåm, S. D. (1998). Averaged predictions and the learning of equilibrium play, *Journal of Economic Dynamics and Control* 22, 833-848.

[14] Flåm, S. D. (1999). Learning equilibrium play: a myopic approach, *Computational Optimization and Appl.*14, 87-102.

[15] Flåm, S. D. and B. Sandvik (2000). Competitive equilibrium: Walras meets Darwin, *Optimization* 47, 137-153.

[16] Friedman, D. (1991). Evolutionary games in economics, *Econometrica* 59, 3, 637-666.

[17] Harsanyi, J. C. and R. Selten (1988). *A General Theory of Equilibrium Selection*, The MIT Press, Cambridge, MA.

[18] Hofbauer, J., P. Schuster and K. Sigmund (1979). A note on evolutionary stable strategies and game dynamics, *Journal of Theoretical Biology* 81, 609-612.

[19] Hofbauer, J. and K. Sigmund (1990). Adaptive dynamics and evolutionary stability, *Applied Math. Letters* 3, 75-79.

[20] Hofbauer, J. and K. Sigmund (1988). *The Theory of Evolution and Dynamical System*, Cambridge University Press.

[21] Hofbauer, J. and K. Sigmund (1998). *Evolutionary Games and Population Dynamics*, Cambridge University Press.

[22] Hopkins, E. (1999). Learning, matching and aggregation, *Games and Economic Behavior* 26, 1, 79-110.

[23] Konnov, I. V. and S. Schaible (2000). Duality for equilibrium problems under generalized monotonicity, *J. Opt. Theory and Applic.* 104, 2, 395-408.

[24] Maynard Smith, J. (1982). *Evolution and the Theory of Games*, Cambridge University Press, Cambridge.

[25] Maynard Smith, J. and G. R. Price (1973). The logic of animal conflict, *Nature* 246, 15-18

[26] Robinson, J. (1951). An iterative method for solving a game, *Ann. Math.* 54, 296-301.

[27] Rockafellar, R. T. (1970). *Convex Analysis*, Princeton University Press.

[28] Rosen, J. B. (1965). Existence and uniqueness of equilibrium points for concave n-person games, *Econometrica* 33, 520-534.

[29] Samuelson, L.(1997). *Evolutionary Games and Equilibrium Selection*, The MIT Press.

[30] Taylor, P. (1979). Evolutionary stable strategies and game dynamics, *Journal of Applied Probability* 16, 76-83.

[31] van Damme, E. (1991). *Stability and Perfection of Nash Equilibria*, Springer-Verlag, Berlin.

[32] Vega-Redondo, F. (1996). *Evolution, Games and Economic Behavior*, Oxford University Press.

[33] Weibull, J. W. (1996). *Evolutionary Game Theory*, The MIT Press, Cambridge, Massachusetts.

[34] Zarantonello, E. H. (1971). Projections on convex sets in Hilbert spaces and spectral theory, in E. H. Zarantonello (ed.) *Contributions to Nonlinear Functional Analysis*, Academic Press, New York (1971).

[35] Young, P. (1998). *Individual Strategy and Social Choice,* Princeton University Press.