CLOSED FORM VALUATION OF AMERICAN OPTIONS

PETTER BJERKSUND AND GUNNAR STENSLAND

ABSTRACT. This paper presents a simple and intuitive approximation of the American call and put value. The approximation generalizes the Bjerksund-Stensland model by dividing time to maturity into two periods, each with a flat early exercise boundary. By imposing a feasible but non-optimal exercise strategy, a lower bound to the true option value is obtained. Numerical investigations indicate that the method represents an accurate and extremely computer efficient approximation to the American option value.

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1. Introduction

It is well known that the value of an American option can be represented as the solution to an optimal stopping problem. Unfortunately, there is no known closed form solution to either the American option value or the optimal exercise strategy, except for the trivial case where early exercise is non-optimal. Bjerksund and Stensland (1993a) obtain an accurate and computer efficient approximation to the American option value by imposing a feasible but non-optimal exercise strategy. In particular, they assume a flat early exercise boundary.

This paper generalises their model by assuming that time to maturity is divided into two subperiods, with one flat exercise boundary being valid for each subperiod. We derive a closed form lower bound for the American option value. Numerical investigation indicates that this lower bound represents an accurate and very computer efficient approximation of the American call and put values.

2. Assumptions

We assume a complete continuous-time Black-Scholes economy with a (positive) riskless interest rate r, where the price of the underlying asset is a geometric Brownian motion with respect to the equivalent martingale measure (EMM). In particular, let the price of the underlying asset S_t at a future date t be

$$S_t = S \exp\left\{(b - \tfrac{1}{2}\sigma^2)t + \sigma W_t\right\}, \tag{1}$$

where S is the current price, $b < r$ is the drift rate with respect to the EEM, σ is volatility, and W_t is a Brownian motion (Wiener process). Eq. (1) corresponds to the current forward price on a contract with delivery at date t being $F_{0,t} = \exp\{bt\} S$, hence b may be interpreted as cost of carry.

It is well known in the literature[1] that the value of a contingent claim can be represented as an expected discounted pay-off, where the expectation is taken with respect to the EEM, and the riskless interest rate is used for discounting. Now, consider an American call with maturity T and strike K. For a given feasible exercise strategy, represented by a stopping date $\tau \in [0, T]$, the option value from following this strategy can be written as

$$c = E_0\left[\exp\{-r\tau\}(S_\tau - K)^+\right]. \tag{2}$$

Consequently, the American call value is

$$C(S, K, T, r, b, \sigma) = \sup_{\tau \in [0,T]} E_0\left[\exp\{-r\tau\}(S_\tau - K)^+\right], \tag{3}$$

i.e., the value of following the optimal exercise strategy. Unfortunately, there is no known closed form solution to neither the early exercise strategy nor the American call value.[2] However, several approximation methods are available for the non-trivial case, ranging from simple approximation formulas to complex numerical techniques. For a survey, see Broadie and Detemple (1996).

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[1] See Harrison and Kreps (1979) and Harrison and Pliska (1981).

[2] In the case of $b \geq r$, early exercise of an American call is not optimal, hence the American call value is simply given by the European counterpart.

3. A Flat Exercise Boundary

Bjerksund and Stensland (1993a) obtain the American call value conditional on early exercise when the price of the underlying asset hits a flat boundary $X > K$ from below.[3] Given this feasible but non-optimal strategy, the American call boils down to: (i) a European up-and-out call with knock-out barrier X, strike K, and maturity date T; and (ii) a rebate $X - K$ that is received at the knock-out date if the option is knocked out prior to the maturity date.[4]

Their American call approximation is

$$\begin{aligned} & \overline{c}(S, K, T, r, b, \sigma; X) \quad (4) \\ = \; & \alpha(X)S^{\beta} - \alpha(X)\varphi(S; T \mid \beta, X, X) \\ + \; & \varphi(S, T \mid 1, X, X) - \varphi(S, T \mid 1, K, X) \\ - \; & K\varphi(S, T \mid 0, X, X) + K\varphi(S, T \mid 0, K, X), \end{aligned}$$

where[5]

$$\alpha(X) \equiv (X - K)X^{-\beta}, \quad (5)$$

$$\beta \equiv \left(\frac{1}{2} - \frac{b}{\sigma^2}\right) + \sqrt{\left(\frac{b}{\sigma^2} - \frac{1}{2}\right)^2 + 2\frac{r}{\sigma^2}}. \quad (6)$$

The function φ is given by

$$\begin{aligned} & \varphi(S, T \mid \gamma, H, X) \quad (7) \\ \equiv \; & E_0\left[e^{-rT} S_T^{\gamma} I(S_T \leq H) I(\sup_{\tau \in [0,T]} S_\tau < X)\right] \\ = \; & e^{\lambda T} S^{\gamma} \left\{ N\left(-\frac{\ln(S/H) + (b + (\gamma - \frac{1}{2})\sigma^2)T}{\sigma\sqrt{T}}\right) \right. \\ & \left. - \left(\frac{X}{S}\right)^{\kappa} N\left(-\frac{\ln(X^2/(SH)) + (b + (\gamma - \frac{1}{2})\sigma^2)T}{\sigma\sqrt{T}}\right) \right\}, \end{aligned}$$

where $H \leq X$, $I(\cdot)$ is the indicator function assuming unity if the argument is true and zero otherwise, and[6]

$$\lambda = -r + \gamma b + \tfrac{1}{2}\gamma(\gamma - 1)\sigma^2, \quad (8)$$

$$\kappa \equiv \frac{2b}{\sigma^2} + (2\gamma - 1). \quad (9)$$

The two first terms on the righthand side of Eq. (4) represent the value of the rebate component, whereas the remaining four terms represent the value of the up-and-out European call.

A strategy following from a flat early exercise boundary is clearly feasible, but not optimal. Consequently, the value from following this strategy represents a lower bound to the true option value. Numerical investigations indicate that this

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[3] The case of $S \geq X$ corresponds to immediate exercise and the option value being $S - K$.

[4] For a discussion of barrier options, see, e.g., Reiner and Rubinstein (1991a,1991b).

[5] $r > 0$ and $b < r$ ensures that $\beta > 1$.

[6] Observe that our definition on λ (c.f. our Eq. (8)) differs slightly from Bjerksund-Stensland (1993) (c.f. their Eq. (14)).

lower bound may serve as an accurate approximation. In the following, we use a slightly modified version of the closed form early exercise boundary proposed in Bjerksund-Stensland (1993a)[7]

$$X_T = B_0 + (B_\infty - B_0)\left(1 - \exp\{h(T)\}\right), \tag{10}$$

where

$$h(T) = -\left(bT + 2\sigma\sqrt{T}\right)\left(\frac{K^2}{(B_\infty - B_0)\,B_0}\right), \tag{11}$$

$$B_\infty \equiv \frac{\beta}{\beta - 1}K, \tag{12}$$

$$B_0 \equiv \max\left\{K, \left(\frac{r}{r-b}\right)K\right\}. \tag{13}$$

and β is defined by Eq. (6) above. B_∞ represents the optimal exercise boundary in the case of a perpetual American call, see Samuelson (1965).

4. A Two-step Exercise Boundary

This paper extends the flat boundary approximation above by allowing for one flat boundary X that is valid from date 0 to date t, and another flat boundary x that is valid from date t to date T, where $0 < t < T$. It is well known that the optimal boundary is a decreasing (and concave) function of calendar time, hence we take $X > x > K$. The exercise boundary for the call is composed by the solid lines in Figure 1. Observe that the exercise boundary may be viewed as a stairway with two (typically different) steps. The vertical dotted line in Figure 1 representst the no-exercise boundary, corresponding to no early exercise and the option being out-of-the-money at maturity.

(Please insert Figure 1 about here.)

To formalise, define the stopping date

$$\overline{\overline{\tau}} \equiv \inf\left\{\left\{\inf_{\tau\in[0,\infty)} : S_\tau \geq X\right\}, \left\{\inf_{\tau\in[t,\infty)} : S_\tau \geq x\right\}, T\right\}, \tag{14}$$

and define the value from following this strategy by

$$\overline{\overline{c}}(S, K, T, r, b, \sigma; X, x, t) = E_0\left[\exp\{-r\overline{\overline{\tau}}\}(S_{\overline{\overline{\tau}}} - K)^+\right]. \tag{15}$$

Observe from Eq. (14) that with $S < X$, we have the four mutually exclusive events for the stopping (pay-off) date $\overline{\overline{\tau}}$: (i) $0 < \overline{\overline{\tau}} < t$; (ii) $\overline{\overline{\tau}} = t$; (iii) $t < \overline{\overline{\tau}} < T$; and (iv) $\overline{\overline{\tau}} = T$. It can be seen from Eqs. (14)-(15) that the call can be interpreted as a portfolio of four contingent claims with the following mutually exclusive pay-offs: (i) a rebate $X - K$ received at date $\overline{\overline{\tau}}$ if $0 < \overline{\overline{\tau}} < t$; (ii) a pay-off $S_t - K$ received at date t conditional on $S_t \geq x$ and no prior pay-off; (iii) a rebate $x - K$ received at date $\overline{\overline{\tau}}$ if $t < \overline{\overline{\tau}} < T$; and (iv) a call with strike K and exercise date T conditional on no prior pay-off.

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[7]The Bjerksund-Stensland (1993a) closed form boundary is obtained by replacing Eq. (11) with
$h(T) = -\left(bT + 2\sigma\sqrt{T}\right)\left(\frac{B_0}{B_\infty - B_0}\right)$.

Proposition 1.

$$\begin{aligned}
\overline{\overline{c}} &= \alpha(X)S^{\beta} - \alpha(X)\varphi(S,t \mid \beta, X, X) \\
&+ \varphi(S,t \mid 1, X, X) - \varphi(S,t \mid 1, x, X) \\
&- K\varphi(S,t \mid 0, X, X) + K\varphi(S,t \mid 0, x, X) \\
&+ \alpha(x)\varphi(S;t \mid \beta, x, X) - \alpha(x)\Psi(S,T \mid \beta, x, X, x, t) \\
&+ \Psi(S,T \mid 1, x, X, x, t) - \Psi(S,T \mid 1, K, X, x, t) \\
&- K\Psi(S,T \mid 0, x, X, x, t) + K\Psi(S,T \mid 0, K, X, x, t),
\end{aligned}$$

where α, β, and φ are defined by Eqs. (5), (6), and (7)-(9) above. The function Ψ is given by[8]

$$\begin{aligned}
\Psi &= \Psi(S,T \mid \gamma, H, X, x, t) \\
&\equiv E_0\left[e^{-rT}S_T^{\gamma}I(S_T \le H)I(\sup_{\tau\in[0,t]} S_\tau < X)I(\sup_{\tau\in[t,T]} S_\tau < x)\right] \\
&= \exp\{\lambda T\}S^{\gamma}\left\{M\left(d_1, D_1; \sqrt{\tfrac{t}{T}}\right) - (X/S)^{\kappa}M\left(d_2, D_2; \sqrt{\tfrac{t}{T}}\right)\right. \\
&\quad \left. -(x/S)^{\kappa}M\left(d_3, D_3; -\sqrt{\tfrac{t}{T}}\right) + (x/X)^{\kappa}M\left(d_4, D_4; -\sqrt{\tfrac{t}{T}}\right)\right\},
\end{aligned}$$

with λ and κ being defined by Eqs. (8) and (9), where $M(\cdot,\cdot;\cdot)$ is the standard bivariate normal distribution function,[9] *and*

$$\begin{aligned}
d_1 &= -\frac{\ln(S/x) + (b + (\gamma - \frac{1}{2})\sigma^2)t}{\sigma\sqrt{t}}, \\
d_2 &= -\frac{\ln\left(X^2/(Sx)\right) + (b + (\gamma - \frac{1}{2})\sigma^2)t}{\sigma\sqrt{t}}, \\
d_3 &= -\frac{\ln(S/x) - (b + (\gamma - \frac{1}{2})\sigma^2)t}{\sigma\sqrt{t}}, \\
d_4 &= -\frac{\ln\left(X^2/(Sx)\right) - (b + (\gamma - \frac{1}{2})\sigma^2)t}{\sigma\sqrt{t}}, \\
D_1 &= -\frac{\ln(S/H) + (b + (\gamma - \frac{1}{2})\sigma^2)T}{\sigma\sqrt{T}}, \\
D_2 &= -\frac{\ln\left(X^2/(SH)\right) + (b + (\gamma - \frac{1}{2})\sigma^2)T}{\sigma\sqrt{T}}, \\
D_3 &= -\frac{\ln\left(x^2/(SH)\right) + (b + (\gamma - \frac{1}{2})\sigma^2)T}{\sigma\sqrt{T}}, \\
D_4 &= -\frac{\ln\left((Sx^2)/(HX^2)\right) + (b + (\gamma - \frac{1}{2})\sigma^2)T}{\sigma\sqrt{T}}.
\end{aligned}$$

Proof. See Appendices A and B. □

We suggest the following closed form exercise strategy: First, divide time to maturity into subperiods $(0,t)$ and (t,T) by

[8]Observe that $I(\sup_{\tau\in[0,t]} S_\tau < X)I(\sup_{\tau\in[t,T]} S_\tau < x)$ and $I(\overline{\overline{\tau}} = T)$ are equivalent.

[9]For a numerical approximation of the standard bivariate normal distribution function, see, e.g. Haug (1997) pp. 191-193.

$$t = \tfrac{1}{2}\left(\sqrt{5}-1\right)T, \tag{16}$$

which is motivated by concavity of the "true" exercise boundary and the "golden rule".[10] Next, use the flat boundary approximation in Eqs. (10)-(13) to determine the exercise boundary in the two subperiods $(0, t)$ and (t, T)

$$X = X_T, \tag{17}$$

$$x = X_{T-t}, \tag{18}$$

respectively.

5. Put approximation

It is well known in the literature (see, e.g., Bjerksund and Stensland (1993b) and McDonald and Schroder (1998)) that the stopping problem of an American call can be transformed into a symmetric put problem (and vice versa). In particular, the value (as well as the stopping date) of an American put on the underlying asset of Eq. (1) with current value S, strike K, and maturity T equals the value (and the stopping date) of an American call with strike S and maturity T, written on an asset with current value K, cost of carry $-b$, and volatility σ, evaluated within an economy with riskless interest rate $r - b$. Formally, the put-call transformation states

$$P(S, K, T, r, b, \sigma) = C(K, S, T, r - b, -b, \sigma). \tag{19}$$

Consequently, we calculate the American put approximations, $\overline{p}$ and $\overline{\overline{p}}$, by translating the put problem into the associated call problem, and then using the appropriate call approximation procedure as described above.

6. Numerical results

The purpose of this section is to examine the numerical properties of the two closed form approximations for representative parameter values. Rather than constructing new input data, we stick with the representative parameter values used in Barone-Adesi and Whaley (1987). The tables are organized as follows: Columns C and P contain the call and put values following from a binomial model with 3201 points on the lattice, which we use as an estimate of the "true" option value. The columns $\overline{c}$ and $\overline{p}$ are the call and put value approximations following from the flat early exercise boundary method, and the columns $\overline{\overline{c}}$ and $\overline{\overline{p}}$ are the call and put value approximations following from the two-step boundary method.

Now, a closer examination of the numerical results reveals that in case the three values of an option differ: (i) the two-step boundary method provides a stricter lower bound to the "true" option value than the flat boundary method; and (ii) the option value from the two-step boundary method is close to the average of the two other values. Consequently, a reasonable proxy of the true option value seems to be twice the option value calculated by the two-step boundary method minus the option value calculated by the flat boundary method. This proxy for the call and put are reported in columns $2\overline{\overline{c}} - \overline{c}$ and $2\overline{\overline{p}} - \overline{p}$, respectively.

[10] Recall that X and x are the exercise boundaries in subperiods $(0, t)$ and (t, T), respectively, where $X > x$. Concavity of the exercise boundary translates into $(t - 0) > (T - t)$, which in combination with the "golden rule", $(T - t)/(t - 0) = (t - 0)/(T - 0)$, leads to Eq. (16).

(Please insert Table 1 about here.)

In all tables, we fix the strike at $K = 100$. In Table 1, the cost of carry $b = -0.04$. The maximum errors of the flat and two-step boundary approximations are 0.06 and 0.03, respectively, which occurs for the call with the parameter values $r = 0.08$, $\sigma = 0.4$, $T = 0.25$, and $S = 120$. Using the flat and the two-step boundary results as control variates, the maximum error of the proxy corresponds to an overvaluation of the call by 0.02.

(Please insert Table 2 about here.)

In Table 2, we fix the cost of carry $b = 0.04$. The maximum error for the flat boundary method is 0.07, which occurs for the put with parameter values $r = 0.08$, $\sigma = 0.20$, $T = 0.5$, and $S = 100$ (i.e., at-the-money). The maximum error for the two-step boundary method is 0.04, which occurs for the put with parameter values $r = 0.08$, $\sigma = 0.40$, $T = 0.25$, and $S = 80$. This latter parameter case also leads to the maximum error of the proxy corresponding to an undervaluation of the put by 0.03.

(Please insert Table 3 about here.)

In Table 3, we let the the cost of carry b be equal to the interest rate r, corresponding to the case where the underlying is a non-dividend paying stock. In this case, it is well known that early exercise of the American call is non-optimal, hence only the put is considered. The maximum error for the flat boundary method is 0.09, which occurs for the parameter values $b = r = 0.08$, $\sigma = 0.40$, $T = 0.25$, and $S = 90$. The maximum error for the two-step boundary method is 0.04, which occurs for parameter values $b = r = 0.08$, $\sigma = 0.20$, $T = 0.5$, and $S = 100$ (i.e., at-the-money). The former parameter case also leads to the maximum error of the proxy corresponding to an overvaluation of the put by 0.03.

(Please insert Table 4 about here.)

In Table 4, we fix the parameter values $r = 0.08$, $\sigma = 0.20$, and $T = 3$, and examine the option value approximations for different levels of cost of carry b as well as different asset prices S. The case of $b = 0$ corresponds to a situation where the option is written on a futures price of a contract on future delivery. Note from Table 4 that that the maximum errors occur in this case for the in-the-money put ($S = 80$), with 0.09 for the flat boundary and 0.07 for the two-step boundary approximation methods. This parameter case also produces the maximum error of the proxy, corrsponding to an undervaluation of the put by 0.06.

From the tables, we see that the two-step boundary method represents a more accurate approximation than the flat boundary method. When comparing the method to other candidates, however, one must keep in mind that the two-step boundary (as well as the flat boundary) approximation represents a lower bound to the true option value, hence we know the sign of the approximation error. In addition, the method is closed form, and consequently very computer efficient. Indeed, more accurate methods exists, but at the expence of computer efficiency. Another advantage of working with a closed form value approximation is that approximations of the "greeks" can be readily obtained. These results (following from straight forward, but tedious algebra) are omitted here to save space.

7. Conclusions

This paper presents a closed form lower bound to the value of the American option, based on imposing a feasible but non-optimal exerise strategy. Numerical investigations indicate that this lower bound represents an accurate and very computer efficient approximation to the true American option value.

8. References

Barone-Adesi, G. and R. Whaley (1987): Efficient analytical approximation of American option values *Journal of Finance*, Vol. 42, No. 2, pp. 301-320.

Bjerksund, P. and G. Stensland (1993a): Closed-form approximation of american options, *Scandinavian Journal of Management*, Vol. 9, Suppl., pp. S88-S99.

Bjerksund, P. and G. Stensland (1993b): American Echange Options and a Put-Call Transformation: A Note, *Journal of Business Finance & Accounting*, Vol. 20, No. 9, (September), pp. 761-764.

Broadie, M. and J. Detemple (1996): American Option Valuation: New Bounds, Approximations, and a Comparison of Existing Methods *Review of Financial Studies*, Vol. 9, No. 4. (Winter, 1996), pp. 1211-1250.

Ingersoll, Jr., J.E. (1987): *Theory of Financial Decision Making*, Totowa, NJ: Rowman & Littlefield.

Harrison, J.M. and D.M. Kreps (1979): Martingale and arbitrage in multiperiod markets, *Journal of Economic Theory*, pp. 381-408.

Harrison, J.M. and S.R. Pliska (1981): Martingales and stochastic integrals in the theory of continuous trading, *Stochastic Processes and Their Applications*, pp. 313-316.

Haug, E.G. (1997): *The Complete Guide to Option Pricing Formulas*, New York: McGraw-Hill

Ju, N. (1998): Pricing an American Option by Approximating Its Early Exercise Boundary as a Multipiece Exponential Function, *Review of Financial Studies*, Vol. 11, No. 3, Fall 1998, pp. 627-646.

McDonald, R. and M. Schroder (1998): A parity result for American options. *Journal of Computational Finance*, Vol. 1, No. 3, Spring 1998.

Reiner, E. and M. Rubinstein (1991a): Breaking down the barriers, *Risk Magazine*, Vol. 4, No. 8.

Reiner, E. and M. Rubinstein (1991b): Unscrambling the Binary Code, *Risk Magazine*, Vol. 4, No. 9.

Samuelson, P.A.: Rational theory of warrant pricing, *Industrial Management Review*, (1965), pp. 13-39.

Table 1: Approx. option values. Strike $K = 100$, cost of carry $b = -0.04$									
Parameters:		American call				American put			
	$S =$	C	$\overline{c}$	$\overline{\overline{c}}$	$2\overline{\overline{c}} - \overline{c}$	P	$\overline{p}$	$\overline{\overline{p}}$	$2\overline{\overline{p}} - \overline{p}$
$r = 0.08$,	80	0.03	0.03	0.03	0.03	20.41	20.41	20.41	20.41
$\sigma = 0.20$,	90	0.58	0.57	0.58	0.58	11.25	11.25	11.25	11.25
$T = 0.25$	100	3.52	3.49	3.51	3.54	4.39	4.40	4.40	4.40
	110	10.36	10.32	10.34	10.37	1.12	1.12	1.12	1.12
	120	20.00	20.00	20.00	20.00	0.18	0.18	0.18	0.18
$r = 0.12$,	80	0.03	0.03	0.03	0.03	20.23	20.22	20.23	20.23
$\sigma = 0.20$,	90	0.57	0.57	0.57	0.58	11.14	11.14	11.14	11.14
$T = 0.25$	100	3.50	3.46	3.49	3.51	4.35	4.35	4.35	4.35
	110	10.32	10.29	10.31	10.34	1.11	1.11	1.11	1.11
	120	20.00	20.00	20.00	20.00	0.18	0.18	0.18	0.18
$r = 0.08$,	80	1.05	1.05	1.05	1.06	21.44	21.44	21.44	21.45
$\sigma = 0.40$,	90	3.26	3.25	3.26	3.27	13.92	13.91	13.91	13.92
$T = 0.25$	100	7.41	7.37	7.39	7.42	8.26	8.27	8.27	8.27
	110	13.52	13.47	13.51	13.54	4.52	4.52	4.52	4.52
	120	21.29	21.23*	21.26*	21.28	2.29	2.29	2.29	2.29
$r = 0.08$,	80	0.22	0.21	0.21	0.21	20.96	20.95	20.96	20.96
$\sigma = 0.20$,	90	1.36	1.34	1.35	1.36	12.63	12.63	12.63	12.63
$T = 0.5$	100	4.71	4.65*	4.69	4.73	6.37	6.37	6.37	6.37
	110	11.00	10.94*	10.98	11.01	2.65	2.65	2.65	2.65
	120	20.00	20.00	20.00	20.00	0.92	0.92	0.92	0.92

Notation: S: asset value; r: interest rate; σ: volatility; T: time to exercise.

C and P: Binomial call and put approximation with 3201 points on the lattice.

$\overline{c}$ and $\overline{p}$: Flat boundary call and put approximation.

$\overline{\overline{c}}$ and $\overline{\overline{p}}$: Two-step boundary call and put approximation.

$2\overline{\overline{c}} - \overline{c}$ and $2\overline{\overline{p}} - \overline{p}$: Call and put approximation using flat and two-step boundary results.

* Max. error from flat bdy. approx. 0.06; max. error from two-step bdy. approx. 0.03.

Table 2: Approx. option values. Strike $K = 100$, cost of carry $b = 0.04$									
Parameters:		American call				American put			
	$S =$	C	$\overline{c}$	$\overline{\overline{c}}$	$2\overline{\overline{c}} - \overline{c}$	P	$\overline{p}$	$\overline{\overline{p}}$	$2\overline{\overline{p}} - \overline{p}$
$r = 0.08$,	80	0.05	0.05	0.05	0.05	20.00	20.00	20.00	20.00
$\sigma = 0.20$,	90	0.85	0.85	0.85	0.85	10.22	10.19	10.21	10.23
$T = 0.25$	100	4.44	4.44	4.44	4.44	3.55	3.51	3.53	3.56
	110	11.66	11.66	11.66	11.66	0.79	0.78	0.79	0.79
	120	20.90	20.90	20.90	20.90	0.11	0.11	0.11	0.11
$r = 0.12$,	80	0.05	0.05	0.05	0.05	20.00	20.00	20.00	20.00
$\sigma = 0.20$,	90	0.84	0.84	0.84	0.84	10.20	10.17	10.19	10.20
$T = 0.25$	100	4.40	4.40	4.40	4.40	3.52	3.49	3.51	3.50
	110	11.55	11.55	11.55	11.55	0.78	0.77	0.78	0.78
	120	20.69	20.69	20.69	20.69	0.11	0.11	0.11	0.11
$r = 0.08$,	80	1.29	1.29	1.29	1.29	20.59	20.53	20.55*	20.56
$\sigma = 0.40$,	90	3.82	3.82	3.82	3.82	12.95	12.91	12.94	12.97
$T = 0.25$	100	8.35	8.35	8.35	8.35	7.47	7.42	7.45	7.47
	110	14.80	14.80	14.80	14.80	3.95	3.93	3.94	3.95
	120	22.71	22.71	22.71	22.71	1.94	1.93	1.94	1.94
$r = 0.08$,	80	0.41	0.41	0.41	0.41	20.00	20.00	20.00	20.00
$\sigma = 0.20$,	90	2.18	2.18	2.18	2.18	10.76	10.70	10.73	10.77
$T = 0.5$	100	6.50	6.50	6.50	6.50	4.77	4.70*	4.74	4.79
	110	13.42	13.42	13.42	13.42	1.74	1.71	1.72	1.74
	120	22.06	22.06	22.06	22.06	0.53	0.52	0.52	0.53

Notation: S: asset value; r: interest rate; σ: volatility; T: time to exercise.

C and P: Binomial call and put approximation with 3201 points on the lattice.

$\overline{c}$ and $\overline{p}$: Flat boundary call and put approximation.

$\overline{\overline{c}}$ and $\overline{\overline{p}}$: Two-step boundary call and put approximation.

$2\overline{\overline{c}} - \overline{c}$ and $2\overline{\overline{p}} - \overline{p}$: Call and put approximation using flat and two-step boundary results.

* Max. error from flat bdy. approx. 0.07; max. error from two-step bdy. approx. 0.04.

Table 3 : Approximated option values.					
Parameters:		American put			
$K = 100$,	$S =$	P	$\overline{p}$	$\overline{\overline{p}}$	$2\overline{\overline{p}} - \overline{p}$
$b = r = 0.08$,	80	20.00	20.00	20.00	20.00
$\sigma = 0.20$,	90	10.04	10.01	10.02	10.04
$T = 0.25$	100	3.22	3.16	3.20	3.23
	110	0.66	0.65	0.66	0.67
	120	0.09	0.09	0.09	0.09
$b = r = 0.12$,	80	20.00	20.00	20.00	20.00
$\sigma = 0.20$,	90	10.00	10.00	10.00	10.00
$T = 0.25$	100	2.93	2.86	2.90	2.93
	110	0.56	0.54	0.55	0.56
	120	0.07	0.07	0.07	0.07
$b = r = 0.08$,	80	20.32	20.28	20.30	20.33
$\sigma = 0.40$,	90	12.57	12.48*	12.54	12.60
$T = 0.25$	100	7.11	7.04	7.09	7.13
	110	3.70	3.66	3.69	3.71
	120	1.79	1.77	1.78	1.79
$b = r = 0.08$,	80	20.00	20.00	20.00	20.00
$\sigma = 0.20$,	90	10.29	10.24	10.27	10.29
$T = 0.5$	100	4.19	4.11	4.15*	4.20
	110	1.41	1.37	1.39	1.41
	120	0.40	0.39	0.39	0.40

Notation: S: asset value; K: strike; b: cost of carry;
r: interest rate; σ: volatility; T: time to exercise.
P: Binomial approx. with 3201 points on the lattice.
$\overline{p}$: Flat boundary approximation.
$\overline{\overline{p}}$: Two-step boundary approximation.
$2\overline{\overline{p}} - \overline{p}$: Approx. using flat and two-step bdy. results.
* Max error of $\overline{p}$ is 0.09; max error of $\overline{\overline{p}}$ is 0.04

Table 4: Approximated option values. Strike $K = 100$

Parameters:	$S =$	American call C	$\overline{c}$	$\overline{\overline{c}}$	$2\overline{\overline{c}} - \overline{c}$	American put P	$\overline{p}$	$\overline{\overline{p}}$	$2\overline{\overline{p}} - \overline{p}$
$r = 0.08$,	80	2.34	2.30	2.32	2.34	25.66	25.61	25.64	25.66
$\sigma = 0.20$,	90	4.75	4.71	4.74	4.76	20.08	20.04	20.07	20.09
$T = 3$,	100	8.49	8.44	8.47	8.50	15.50	15.47	15.49	15.50
$b = -0.04$	110	13.79	13.74	13.77	13.80	11.80	11.78	11.80	11.81
	120	20.88	20.85	20.86	20.88	8.89	8.87	8.88	8.89
$r = 0.08$,	80	3.98	3.95	3.97	3.99	22.21	22.12*	22.14*	22.15
$\sigma = 0.20$,	90	7.25	7.20	7.23	7.26	16.21	16.14	16.17	16.20
$T = 3$,	100	11.70	11.64	11.68	11.71	11.70	11.64	11.68	11.71
$b = 0.00$	110	17.31	17.24	17.28	17.31	8.37	8.31	8.35	8.38
	120	24.01	23.93	23.95	23.98	5.93	5.89	5.91	5.94
$r = 0.08$,	80	6.88	6.88	6.88	6.88	20.35	20.32	20.33	20.34
$\sigma = 0.40$,	90	11.49	11.49	11.49	11.49	13.50	13.43	13.47	13.50
$T = 3$,	100	17.21	17.21	17.21	17.22	8.94	8.86	8.91	8.96
$b = 0.04$	110	23.84	23.84	23.84	23.84	5.91	5.83	5.88	5.92
	120	31.17	31.16	31.16	31.17	3.90	3.83	3.87	3.90
$r = 0.08$,	80					20.00	20.00	20.00	20.00
$\sigma = 0.20$,	90					11.70	11.67	11.68	11.69
$T = 3$,	100	No early exercise				6.93	6.90	6.91	6.93
$b = 0.08$	110					4.16	4.12	4.13	4.15
	120					2.51	2.48	2.49	2.51

Notation: S: value of underlying asset; b: cost of carry ;r: interest rate;
σ: volatility; T: time to exercise.
C and P: Binomial approximation with 3201 points on the lattice.
$\overline{c}$ and $\overline{p}$: Flat boundary approximation.
$\overline{\overline{c}}$ and $\overline{\overline{p}}$: Two-step boundary approximation.
$2\overline{\overline{c}} - \overline{c}$ and $2\overline{\overline{p}} - \overline{p}$: Approximation using flat and two-step boundary results.
* Max. error from flat bdy. approx. 0.09; max. error from two-step bdy. approx. 0.07.

Appendix A. The American call approximation

This appendix derives the representation of $\bar{\bar{c}}$ as stated in the first part of Proposition 1. Define the stopping date

$$\bar{\tau}_t(x) \equiv \left\{ \inf_{\tau \in [t,\infty)} : S_\tau \geq x \right\}, \tag{A.1}$$

where $S_t < x$. It follows from Samuelson (1965) that the value of receiving the pay-off $x - K$ at the first exit date is

$$E_t\left[\exp\{-r(\bar{\tau}_t(x) - t)\}(x - K)\right] = \alpha(x) S_t^\beta, \tag{A.2}$$

where $\alpha(x)$ and β are stated in Eqs. (5) and (6).

Now, obtain the American call value approximation from following the exercise strategy given by the stopping date $\bar{\bar{\tau}}$ as follows

$$\begin{aligned}
\bar{\bar{c}} &= E_0\left[\exp\{-r\bar{\bar{\tau}}\}(S_{\bar{\bar{\tau}}} - K)^+\right] \\
&= E_0\left[\exp\{-r\bar{\bar{\tau}}\}(X - K) I(0 < \bar{\bar{\tau}} < t)\right] \\
&+ E_0\left[\exp\{-rt\}(S_t - K) I(\bar{\bar{\tau}} = t)\right] \\
&+ E_0\left[\exp\{-r\bar{\bar{\tau}}\}(x - K) I(t < \bar{\bar{\tau}} < T)\right] \\
&+ E_0\left[\exp\{-rT\}(S_T - K)^+ I(\bar{\bar{\tau}} = T)\right] \\
&= E_0\left[\exp\{-r\bar{\tau}_0(X)\}(X - K)\left(1 - I\left(\sup_{\tau\in[0,t)} S_\tau < X\right)\right)\right] \\
&+ E_0\left[\exp\{-rt\}(S_t - K) I\left(x \leq S_t \leq X\right) I\left(\sup_{\tau\in[0,t)} S_\tau < X\right)\right] \\
&+ E_0\left[\exp\{-r\bar{\tau}_t(x)\}(x - K)\left(1 - I(\sup_{\tau\in[t,T)} S_\tau < x)\right) I\left(S_t < x\right) I\left(\sup_{\tau\in[0,t)} S_\tau < X\right)\right] \\
&+ E_0\left[\exp\{-rT\}(S_T - K) I(K \leq S_T \leq x) I\left(\sup_{\tau\in[0,t)} S_\tau < X\right) I\left(\sup_{\tau\in[t,T)} S_\tau < x\right)\right] \\
&= E_0\left[\exp\{-r\bar{\tau}_0(X)\}(X - K)\right] \\
&- E_0\left[e^{-rt} E_t\left[\exp\{-r(\bar{\tau}_t(X) - t)\}(X - K)\right] I\left(\sup_{\tau\in[0,t)} S_\tau < X\right)\right] \\
&+ E_0\left[\exp\{-rt\}(S_t - K) I\left(x \leq S_t \leq X\right) I\left(\sup_{\tau\in[0,t)} S_\tau < X\right)\right] \\
&+ E_0\left[e^{-rt} E_t\left[\exp\{-r(\bar{\tau}_t(x) - t)\}(x - K)\right] I\left(S_t < x\right) I\left(\sup_{\tau\in[0,t)} S_\tau < X\right)\right] \\
&- E_0\left[e^{-rT} E_T\left[\exp\{-r(\bar{\tau}_T(x) - T)\}(x - K)\right] I\left(\sup_{\tau\in[t,T)} S_\tau < x\right) I\left(\sup_{\tau\in[0,t)} S_\tau < X\right)\right] \\
&+ E_0\left[\exp\{-rT\}(S_T - K) I(K \leq S_T \leq x) I\left(\sup_{\tau\in[0,t)} S_\tau < X\right) I\left(\sup_{\tau\in[t,T)} S_\tau < x\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \alpha(X)S^{\beta} \\
&- \alpha(X)E_0\left[e^{-rt}S_t^{\beta}I\left(S_t \le X\right)I\left(\sup_{\tau\in[0,t)} S_\tau < X\right)\right] \\
&+ E_0\left[e^{-rt}S_tI\left(S_t \le X\right)I\left(\sup_{\tau\in[0,t)} S_\tau < X\right)\right] \\
&- E_0\left[e^{-rt}S_tI(S_t \le x)I\left(\sup_{\tau\in[0,t)} S_\tau < X\right)\right] \\
&- KE_0\left[e^{-rt}I\left(S_t \le X\right)I\left(\sup_{\tau\in[0,t)} S_\tau < X\right)\right] \\
&+ KE_0\left[e^{-rt}I\left(S_t \le x\right)I\left(\sup_{\tau\in[0,t)} S_\tau < X\right)\right] \\
&+ \alpha(x)E_0\left[e^{-rt}S_t^{\beta}I\left(S_t \le x\right)I\left(\sup_{\tau\in[0,t)} S_\tau < X\right)\right] \\
&- \alpha(x)E_0\left[e^{-rT}S_T^{\beta}I\left(S_T \le x\right)I\left(\sup_{\tau\in[t,T)} S_\tau < x\right)I\left(\sup_{\tau\in[0,t)} S_\tau < X\right)\right] \\
&+ E_0\left[e^{-rT}S_TI\left(S_T \le x\right)I\left(\sup_{\tau\in[0,t)} S_\tau < X\right)I\left(\sup_{\tau\in[t,T)} S_\tau < x\right)\right] \\
&- E_0\left[e^{-rT}S_TI\left(S_T \le K\right)I\left(\sup_{\tau\in[0,t)} S_\tau < X\right)I\left(\sup_{\tau\in[t,T)} S_\tau < x\right)\right] \\
&- KE_0\left[e^{-rT}S_TI\left(S_T \le x\right)I\left(\sup_{\tau\in[0,t)} S_\tau < X\right)I\left(\sup_{\tau\in[t,T)} S_\tau < x\right)\right] \\
&+ KE_0\left[e^{-rT}I\left(S_T \le K\right)I\left(\sup_{\tau\in[0,t)} S_\tau < X\right)I\left(\sup_{\tau\in[t,T)} S_\tau < x\right)\right],
\end{aligned}
$$

which can be expressed by the functions φ and Ψ as stated in the first part of Proposition 1 above. The function φ is obtained in Bjerksund and Stensland (1993a), whereas the function Ψ is derived in the following appendix.

Appendix B. Derivation of the evaluation function Ψ

B.1. **The function.** Let $H \leq x < X$ and $S < X$. Rewrite the function Ψ as follows

$$\begin{aligned}\Psi &\equiv E_0\left[e^{-rT}S_T^{\beta}I(S_T\leq H)I\left(\sup_{\tau\in[0,t]}S_\tau<X\right)I\left(\sup_{\tau\in[t,T]}S_\tau<x\right)\right]\\ &= e^{-rT}X^{\beta}E_0\left[\exp\left\{\beta\ln\frac{S_T}{X}\right\}I\left(\ln\frac{S_T}{X}\leq\ln\frac{H}{X}\right)\right.\\ &\qquad\left.I\left(\sup_{\tau\in[0,t]}\ln\frac{S_\tau}{X}<0\right)I\left(\sup_{\tau\in[t,T]}\ln\frac{S_\tau}{X}<\ln\frac{x}{X}\right)\right]\\ &= e^{-rT}X^{\beta}E_0\left[\exp\left\{(-\beta)\left(-\ln\frac{S_T}{X}\right)\right\}I\left(-\ln\frac{S_T}{X}\geq-\ln\frac{H}{X}\right)\right.\\ &\qquad\left.I\left(\inf_{\tau\in[0,t]}-\ln\frac{S_\tau}{X}>0\right)I\left(\inf_{\tau\in[t,T]}-\ln\frac{S_\tau}{X}>-\ln\frac{x}{X}\right)\right],\end{aligned}$$

where $0 \leq -\ln(x/X) \leq -\ln(H/X)$. Now, define

$$z_\tau \equiv -\ln\frac{S_\tau}{X}$$

which is normal with expectation

$$\begin{aligned}E_0\left[z_\tau\right] &= -\ln\frac{S}{X}-(b-\tfrac{1}{2}\sigma^2)\tau,\\ \mathrm{var}_0\left[z_\tau\right] &= \sigma^2\tau.\end{aligned}$$

Consequently, we can write Ψ as

$$\Psi = e^{-rT}X^{\beta}E_0\left[\exp\{\gamma z_T\}I(z_T\geq\hat{z}_T)I(\inf_{\tau\in[0,t]}z_\tau>0)I(\inf_{\tau\in[t,T]}z_\tau>B)\right], \quad \text{(B.1)}$$

with respect to the stochastic process

$$z_\tau = z_0+\mu\tau+\sigma W_\tau,$$

with the following reinterpretations

$$\begin{aligned}z_0 &= -\ln\frac{S_0}{X}>0 &\text{(B.2)}\\ \mu &= -(b-\tfrac{1}{2}\sigma^2) &\text{(B.3)}\\ \gamma &= -\beta &\text{(B.4)}\\ B &= -\ln\frac{x}{X} &\text{(B.5)}\\ \hat{z}_T &= -\ln\frac{H}{X} &\text{(B.6)}\end{aligned}$$

B.2. **The probability density.** We need the following probability density

$$g(z_t,z_T\mid z_0)\equiv g\left(z_t\cap z_T\cap\left\{\inf_{\tau\in[0,t]}z_\tau>0\right\}\cap\left\{\inf_{\tau\in[t,T]}z_\tau>B\right\}\mid z_0\right), \quad \text{(B.7)}$$

where $z_0 > 0$ and $B > 0$.

First, use the following probability density known from the literature (e.g. Ingersoll (1987) p. 352)

$$
\begin{aligned}
f(z_t \mid z_0) &\equiv f\left(z_t \cap \left\{\inf_{\tau\in[0,t]} z_\tau > 0\right\} \mid z_0\right) \\
&= n\left(\frac{z_t - z_0 - \mu t}{\sigma\sqrt{t}}\right) - \exp\left\{\frac{-2\mu z_0}{\sigma^2}\right\} n\left(\frac{z_t + z_0 - \mu t}{\sigma\sqrt{t}}\right),
\end{aligned}
$$

where $z_0 > 0$ and $z_t > 0$, to write

$$
\begin{aligned}
g(z_t, z_T \mid z_0) &= f(z_t \mid z_0) f\left((z_T - B) \mid (z_t - B)\right) \\
&= \left\{n\left(\frac{z_t - z_0 - \mu t}{\sigma\sqrt{t}}\right) - \exp\left\{\frac{-2\mu z_0}{\sigma^2}\right\} n\left(\frac{z_t + z_0 - \mu t}{\sigma\sqrt{t}}\right)\right\} \\
&\quad \left\{n\left(\frac{z_T - z_t - \mu(T-t)}{\sigma\sqrt{T-t}}\right)\right. \\
&\quad \left. - \exp\left\{\frac{-2\mu(z_t - B)}{\sigma^2}\right\} n\left(\frac{(z_T - B) + (z_t - B) - \mu(T-t)}{\sigma\sqrt{T-t}}\right)\right\} \\
&= n\left(\frac{z_t - z_0 - \mu t}{\sigma\sqrt{t}}\right) n\left(\frac{z_T - z_t - \mu(T-t)}{\sigma\sqrt{T-t}}\right) \\
&- \exp\left\{\frac{-2\mu z_0}{\sigma^2}\right\} n\left(\frac{z_t + z_0 - \mu t}{\sigma\sqrt{t}}\right) n\left(\frac{z_T - z_t - \mu(T-t)}{\sigma\sqrt{T-t}}\right) \\
&- n\left(\frac{z_t - z_0 - \mu t}{\sigma\sqrt{t}}\right) \\
&\quad \exp\left\{\frac{-2\mu(z_t - B)}{\sigma^2}\right\} n\left(\frac{(z_T - B) + (z_t - B) - \mu(T-t)}{\sigma\sqrt{T-t}}\right) \\
&+ \exp\left\{\frac{-2\mu z_0}{\sigma^2}\right\} n\left(\frac{z_t + z_0 - \mu t}{\sigma\sqrt{t}}\right) \\
&\quad \exp\left\{\frac{-2\mu(z_t - B)}{\sigma^2}\right\} n\left(\frac{(z_T - B) + (z_t - B) - \mu(T-t)}{\sigma\sqrt{T-t}}\right).
\end{aligned}
$$

Second, use the identity

$$\exp\left\{ax - \tfrac{1}{2}a^2\right\} n(x) = n(x - a)$$

to obtain

$$
\begin{aligned}
g &= n\left(\frac{z_t - z_0 - \mu t}{\sigma\sqrt{t}}\right) n\left(\frac{z_T - z_t - \mu(T-t)}{\sigma\sqrt{T-t}}\right) \\
&- \exp\left\{\frac{-2\mu z_0}{\sigma^2}\right\} n\left(\frac{z_t + z_0 - \mu t}{\sigma\sqrt{t}}\right) n\left(\frac{z_T - z_t - \mu(T-t)}{\sigma\sqrt{T-t}}\right) \\
&- \exp\left\{\frac{-2\mu(z_0 - B)}{\sigma^2}\right\} n\left(\frac{z_t - z_0 + \mu t}{\sigma\sqrt{t}}\right) n\left(\frac{(z_T - B) + (z_t - B) - \mu(T-t)}{\sigma\sqrt{T-t}}\right) \\
&+ \exp\left\{\frac{2\mu B}{\sigma^2}\right\} n\left(\frac{z_t + z_0 + \mu t}{\sigma\sqrt{t}}\right) n\left(\frac{(z_T - B) + (z_t - B) - \mu(T-t)}{\sigma\sqrt{T-t}}\right).
\end{aligned}
$$

And third, use the identity

$$n(x) n\left(\frac{x - \rho y}{\sqrt{1 - \rho^2}}\right) = m(x, y; \rho)$$

to obtain the probability density

$$\begin{aligned} g &= m\left(\frac{z_t - z_0 - \mu t}{\sigma\sqrt{t}}, \frac{z_T - z_0 - \mu T}{\sigma\sqrt{T}}; \sqrt{\frac{t}{T}}\right) \\ &- \exp\left\{\frac{-2\mu z_0}{\sigma^2}\right\} m\left(\frac{z_t + z_0 - \mu t}{\sigma\sqrt{t}}, \frac{z_T + z_0 - \mu T}{\sigma\sqrt{T}}; \sqrt{\frac{t}{T}}\right) \\ &- \exp\left\{\frac{-2\mu(z_0 - B)}{\sigma^2}\right\} m\left(\frac{z_t - z_0 - \mu t}{\sigma\sqrt{t}}, \frac{(z_T - B) + (z_0 - B) - \mu T}{\sigma\sqrt{T}}; -\sqrt{\frac{t}{T}}\right) \\ &+ \exp\left\{\frac{2\mu B}{\sigma^2}\right\} m\left(\frac{z_t + z_0 + \mu t}{\sigma\sqrt{t}}, \frac{(z_T - B) - (z_0 + B) - \mu T}{\sigma\sqrt{T}}; -\sqrt{\frac{t}{T}}\right), \end{aligned}$$

where $B \geq 0$, $z_0 > 0$, $z_t > B$ and $z_T > B$.

Observe from Eq. (B.1) that the pay-off $\exp\{\gamma z_T\}$ is evaluated with respect to the probability density obtained just above. To simplify, apply the identity

$$\exp\{by - \tfrac{1}{2}b^2\} m(x, y; \rho) = m(x - \rho b, y - b; \rho)$$

to write the product as

$$\begin{aligned} & \exp\{\gamma z_T\} g(z_t, z_T \mid z_0) \\ = & \exp\left\{\gamma z_0 + \gamma\mu T + \tfrac{1}{2}\gamma^2\sigma^2 T\right\} \\ & \left\{ m\left(\frac{z_t - z_0 - \mu t - \gamma\sigma^2 t}{\sigma\sqrt{t}}, \frac{z_T - z_0 - \mu T - \gamma\sigma^2 T}{\sigma\sqrt{T}}; \sqrt{\frac{t}{T}}\right)\right. \\ & - \exp\left\{-2\left(\frac{\mu}{\sigma^2} + \gamma\right) z_0\right\} \\ & m\left(\frac{z_t + z_0 - \mu t - \gamma\sigma^2 t}{\sigma\sqrt{t}}, \frac{z_T + z_0 - \mu T - \gamma\sigma^2 T}{\sigma\sqrt{T}}; \sqrt{\frac{t}{T}}\right) \\ & - \exp\left\{-2\left(\frac{\mu}{\sigma^2} + \gamma\right)(z_0 - B)\right\} \\ & m\left(\frac{z_t - z_0 + \mu t + \gamma\sigma^2 t}{\sigma\sqrt{t}}, \frac{(z_T - B) + (z_0 - B) - \mu T - \gamma\sigma^2 T}{\sigma\sqrt{T}}; -\sqrt{\frac{t}{T}}\right) \\ & + \exp\left\{2\left(\frac{\mu}{\sigma^2} + \gamma\right) B\right\} \\ & \left. m\left(\frac{z_t + z_0 + \mu t + \gamma\sigma^2 t}{\sigma\sqrt{t}}, \frac{(z_T - B) - (z_0 + B) - \mu T - \gamma\sigma^2 T}{\sigma\sqrt{T}}; -\sqrt{\frac{t}{T}}\right)\right\} \end{aligned}$$

B.3. **The expectation.** Use the result just above and the symmetry of the bivariate normal distribution function

$$\int_a^\infty \int_b^\infty m(x, y; \rho)\, dy\, dx = M(-a, -b; \rho), \tag{B.8}$$

to obtain

$$\begin{aligned} \Psi &= e^{-rT} X^\beta E_0\left[\exp\{\gamma z_t\} I(z_T \geq \hat{z}_T) I\left(\inf_{\tau\in[0,t]} z_\tau > 0\right) I\left(\inf_{\tau\in[t,T]} z_\tau > B\right) \mid z_0 > 0\right] \\ &= e^{-rT} X^\beta \int\int I(z_t \geq B) I(z_T \geq \hat{z}_T) \exp\{\gamma z_T\} g(z_t, z_T \mid z_0)\, dz_t\, dz_T \end{aligned}$$

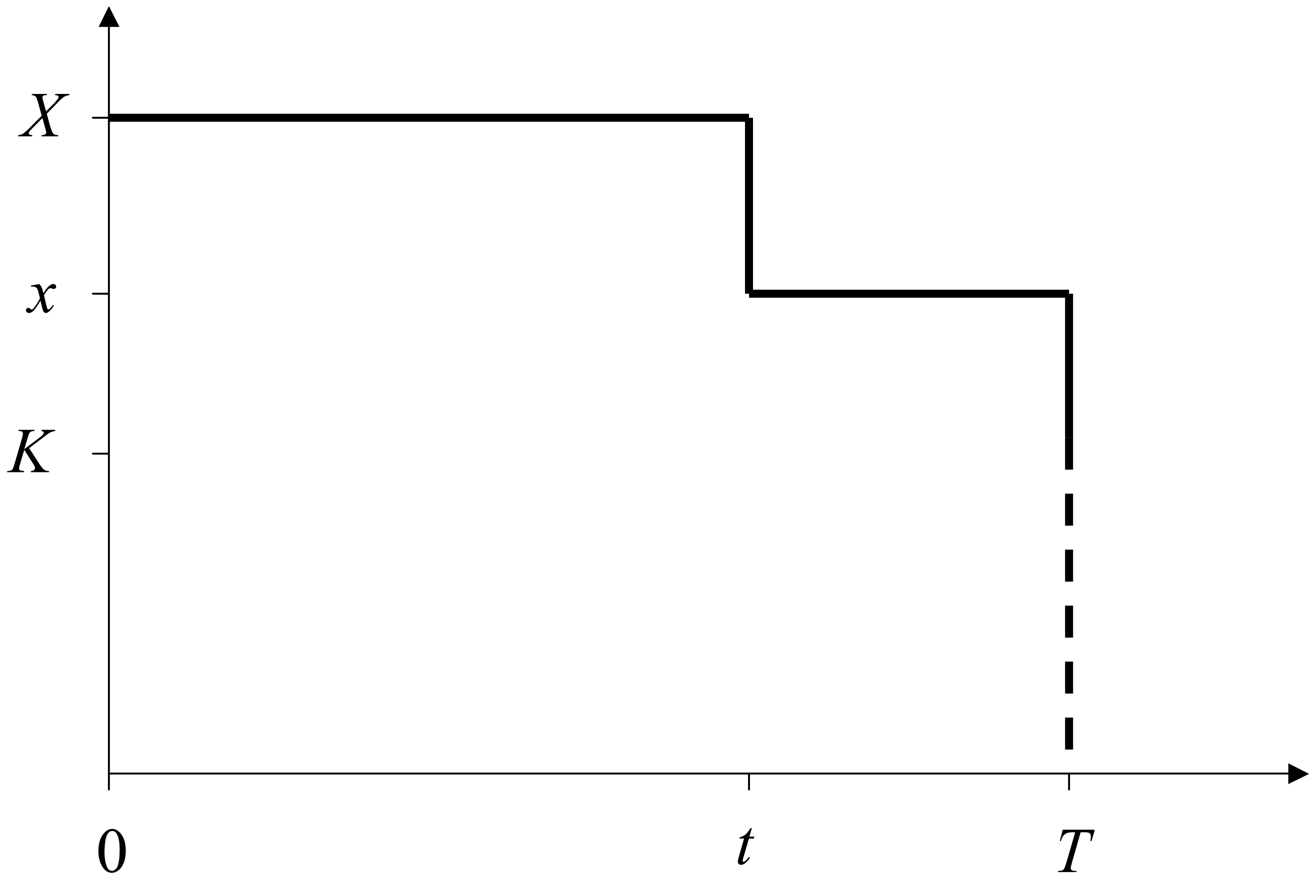
$$
\begin{aligned}
= \quad & e^{-rT} X^{\beta} \exp\left\{\gamma z_0 + \gamma\mu T + \tfrac{1}{2}\gamma^2\sigma^2 T\right\} \\
& \Bigg\{ M\left(-\frac{B - z_0 - \mu t - \gamma\sigma^2 t}{\sigma\sqrt{t}}, -\frac{\hat{z}_T - z_0 - \mu T - \gamma\sigma^2 T}{\sigma\sqrt{T}}; \sqrt{\frac{t}{T}}\right) \\
& \quad - \exp\left\{-2\left(\frac{\mu}{\sigma^2} + \gamma\right) z_0\right\} \\
& \qquad M\left(-\frac{B + z_0 - \mu t - \gamma\sigma^2 t}{\sigma\sqrt{t}}, -\frac{\hat{z}_T + z_0 - \mu T - \gamma\sigma^2 T}{\sigma\sqrt{T}}; \sqrt{\frac{t}{T}}\right) \\
& \quad - \exp\left\{-2\left(\frac{\mu}{\sigma^2} + \gamma\right)(z_0 - B)\right\} \\
& \qquad M\left(-\frac{B - z_0 + \mu t + \gamma\sigma^2 t}{\sigma\sqrt{t}}, -\frac{(\hat{z}_T - B) + (z_0 - B) - \mu T - \gamma\sigma^2 T}{\sigma\sqrt{T}}; -\sqrt{\frac{t}{T}}\right) \\
& \quad + \exp\left\{2\left(\frac{\mu}{\sigma^2} + \gamma\right) B\right\} \\
& \qquad M\left(-\frac{B + z_0 + \mu t + \gamma\sigma^2 t}{\sigma\sqrt{t}}, -\frac{(\hat{z}_T - B) - (z_0 + B) - \mu T - \gamma\sigma^2 T}{\sigma\sqrt{T}}; -\sqrt{\frac{t}{T}}\right) \Bigg\}.
\end{aligned}
$$

Finally, substitute Eqs. (B.2)-(B.6) into the above expression, and rearrange, to obtain Ψ as stated in the latter part of Proposition 1.

Department of Finance and Management Science, NHH, Helleveien 30, N-5045 Bergen, Norway.

E-mail address: petter.bjerksund@nhh.no, gunnar.stensland@nhh.no

Figure 1 : Exercise Boundary



X : Boundary from time 0 to t

x : Boundary from time t to T

K : Strike