Optimal Portfolio Selection with both Fixed and Proportional Transaction Costs for a CRRA Investor with Finite Horizon

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Abstract

In this paper we study the optimal portfolio selection problem for a constant relative risk averse investor who faces fixed and proportional transaction costs and maximizes expected utility of end-of-period wealth. We use a continuous time model and apply the method of the Markov chain approximation to solve numerically for the optimal trading policy. The numerical solution indicates that the portfolio space is divided into three disjoint regions (Buy, Sell, and No-Transaction), and four boundaries describe the optimal policy. If a portfolio lies in the Buy region, the optimal strategy is to buy the risky asset until the portfolio reaches the lower (Buy) target boundary. Similarly, if a portfolio lies in the Sell region, the optimal strategy is to sell the risky asset until the portfolio reaches the upper (Sell) target boundary. All these boundaries are functions of the investor's horizon and the composition of the investor's wealth. Some important properties of the optimal policy are as follows: As the terminal date approaches, the NT region widens. And the NT region widens as wealth declines. As the investor's wealth increases the target boundaries converge quickly to the NT boundaries in the corresponding model with proportional transaction costs only. As wealth becomes small, the target boundaries move closer to the Merton line. The closer the terminal date, the earlier this movement begins. The effects on the optimal policy from varying volatility, drift, CRRA, and the level of transaction costs are also examined.

1 Introduction

In this paper we study the optimal portfolio selection problem for a constant relative risk averse investor. The investor faces fixed and proportional transaction costs and maximizes expected utility of end-of-period wealth.

This asset allocation problem is a variant of the classical consumption-investment problem in modern finance. In the absence of transaction costs, the closed-form solution was obtained by Merton (see, for example, Merton (1971)). The two-asset problem is of particular interest. When the stock price follows a geometric Brownian motion, the solution indicates that it is optimal for the investor to keep a constant fraction in the risky asset. As time passes, the portfolio is assumed to be adjusted continuously so that this fraction is maintained. Moreover, this fraction is independent of the investor's horizon.

The introduction of transaction costs adds considerable complexity to the optimal portfolio selection problem. The problem is simplified if one assumes that the transaction costs are proportional to the amount of the risky asset traded, and there are no transaction costs on trades in the riskless asset. In this case the problem amounts to a *stochastic singular control* problem that was solved by Davis and Norman (1990). Shreve and Soner (1994) studied this problem applying the theory of viscosity solutions to Hamilton-Jacobi-Bellmann (HJB) equations (see, for example, Flemming and Soner (1993) for that theory).

In the presence of proportional transaction costs the solution indicates that the portfolio space is divided into three disjoint positive cones, which can be specified as the buying region, the selling region, and the no-transaction region. If a portfolio lies in the buying region, the optimal strategy is to buy the risky asset until the portfolio reaches the boundary of the buying region, while if a portfolio lies in the selling region, the optimal strategy is to sell the risky asset until the portfolio reaches the boundary of the selling region. If a portfolio lies in the no-transaction region, it is not adjusted at that time. The boundaries of the no-transaction region are functions of time.

The problem is often further simplified if the investor's horizon is infinite, which gives a stationary portfolio policy. Dumas and Luciano (1991) provided an exact solution to a portfolio choice problem for a CRRA investor. Akian, Menaldi, and Sulem (1996) have considered the case with a finite number of stocks, but assuming that the noise terms are uncorrelated. They characterized the value function as a unique viscosity solution of a system of variational inequalities. The variational inequalities were then discretized by finite-difference schemes and solved numerically.

Genotte and Jung (1994) solved numerically the discrete-time model for a CRRA investor with finite horizon and proportional transaction costs. They examined the optimal trading strategies for a large set of realistic parameters. Boyle and Lin (1997) extended the work of Genotte and Jung (1994) and developed analytical expressions for the investor's indirect utility function and also for the boundaries of the no-transaction region.

The solution of the optimal portfolio selection problem having a fixed cost component is based on the theory of *stochastic impulse controls* (see, for example, Bensoussan and Lions (1984) for that theory). The first application of this theory to a consumption-investment problem was done by Eastham and Hastings (1988). They developed a general theory and showed that solving this general problem requires the solution of a system of so-called quasi-variational inequalities (QVI). This initial work was extended by Hastings (1992) and Korn (1998).

In the presence of proportional and fixed transaction costs, the portfolio space can again be divided into three disjoint regions (Buy, Sell, and NT), and four boundaries describe the optimal policy. The Buy and the NT regions are divided by the lower no-transaction boundary, and the Sell and the NT regions are divided by the upper no-transaction boundary. If a portfolio lies in the Buy region, the optimal strategy is to buy the risky asset until the portfolio reaches the lower (Buy) target boundary. Similarly, if a portfolio lies in the Sell region, the optimal strategy is to sell the risky asset until the portfolio reaches the upper (Sell) target boundary. All these boundaries are functions of the investor's horizon and the composition of the investor's wealth.

The optimal portfolio selection problem having both fixed an proportional transaction costs was further developed by Øksendal and Sulem (1999) and Chancelier, Øksendal, and Sulem (2000). Øksendal and Sulem (1999) considered the optimal consumption and portfolio selection problem for a CRRA investor with infinite horizon. They formulated this problem as a combined stochastic control and impulse control problem and showed that the value function is the unique viscosity solution of the quasi-variational inequalities associated to this combined control problem. Chancelier et al. (2000) studied the same problem and showed that the problem could be reduced to an alternative sequence of combined stochastic control and optimal stopping problems. In both papers the numerical solution was obtained by discretizing the quasi-variational inequalities by finite-difference schemes and solved by using an algorithm based on policy iteration.

Schroder (1995) solved numerically the optimal consumption-investment problem of a CRRA investor with finite horizon[1], but he considered the presence of fixed transaction costs only.

In this paper we solve numerically the asset allocation problem for the investor with finite horizon applying the method of the *Markov chain approximation* (see, for example, Kushner and Dupuis (1992)). Using this

[1] Eastham and Hastings (1988) have also studied a finite horizon model and obtained an approximate solution for a fairly simple problem

method, the solution of the variational inequalities is obtained by turning the stochastic differential equations into Markov chains in order to apply the discrete-time dynamic programming algorithm.

Throughout the paper we assume that the model parameters are always chosen so as to exclude the cases with borrowing of risk-free asset and short selling of risky asset.

The rest of the paper is organized as follows. Section 2 presents the continuous-time model. Section 3 is concerned with the construction of a discrete time approximation to the continuous time price processes used in Section 2, and the solution method. For the sake of comparison and completeness, Sections 4, 5, and 6 present the models and numerical solutions for the problems in the absence of transaction costs, in the presence of proportional transaction costs only, and in the presence of fixed transaction costs only. Our main contributions are presented in Section 7, where we discuss some properties of the value function and provide the numerical analysis of the optimal policy. The effects on the optimal policy from varying volatility, drift, CRRA, and the level of transaction costs are also examined. Section 8 concludes the paper and discusses some possible extensions.

2 The Continuous Time Model

Originally, we consider a continuous-time economy with one risky and one risk-free asset. Let $(\Omega, \mathcal{F}, P)$ be a probability space with a given filtration $\{\mathcal{F}_t\}_{0\leq t\leq T}$. The risk-free asset, which we will refer to as the bank account, pays a constant interest rate of $r \geq 0$, and, consequently, the evolution of the amount invested in the bank, x_t, is given by the ordinary differential equation

$$dx_t = rx_t dt \tag{1}$$

We will refer to the risky asset as the stock, and assume that the amount invested in the stock, y_t, evolves according to a geometric Brownian motion defined by

$$dy_t = \mu y_t dt + \sigma y_t dB_t \tag{2}$$

where μ and σ are constants, and B_t is a one-dimensional $\mathcal{F}_t$-Brownian motion.

We assume that a purchase or sale of stocks of the amount ξ incurs a transaction costs consisting of a sum of a fixed cost $k \geq 0$ (independent of the size of transaction) plus a cost $\lambda|\xi|$ proportional to the transaction $(\lambda \geq 0)$. These costs are drawn from the bank account.

We suppose that at any time the investor can decide to transfer money from the bank account to the stock and conversely. The control of the investor is a pure *impulse control* $v = (\tau_1, \tau_2, \dots; \xi_1, \xi_2, \dots)$. Here $0 \leq \tau_1 < \tau_2 < \dots$ are $\mathcal{F}_t$-stopping times giving the times when the investor decides to change his portfolio, and ξ_j are $\mathcal{F}_{\tau_j}$-measurable random variables giving the sizes of the transactions at these times. If such a control is applied to the system (x_t, y_t), it gets the form

$$\begin{aligned} dx_t &= rx_t dt & \tau_i \leq t < \tau_{i+1} \\ dy_t &= \mu y_t dt + \sigma y_t dB_t & \tau_i \leq t < \tau_{i+1} \\ x_{\tau_{i+1}} &= x_{\tau_{i+1}^-} - k - \xi_{i+1} - \lambda|\xi_{i+1}| & \\ y_{\tau_{i+1}} &= y_{\tau_{i+1}^-} + \xi_{i+1} & \end{aligned} \tag{3}$$

If the investor has the amount x on the bank account, and the amount y in the stock, his *net wealth* is defined as the holdings on the bank account after selling of all shares of the stock (if the proceeds are positive after transaction costs) or closing of the short position in the stock and is given by

$$N(x,y) = \begin{cases} \max\{x + y(1-\lambda) - k, x\} & \text{if } y \geq 0, \\ x + y(1+\lambda) - k & \text{if } y < 0. \end{cases} \tag{4}$$

The *solvency region* is defined as the region where the investor's net wealth is non-negative.

$$\mathcal{S} = \{(x,y) \in \mathbf{R}^2; N(x,y) \geq 0\} \tag{5}$$

We consider an investor with a finite horizon who has utility only of terminal wealth. It is assumed that the investor has a constant relative risk aversion. In this case his utility function is of the form

$$\begin{aligned} U(W) &= \tfrac{W^\gamma}{\gamma} & \gamma < 1, \gamma \neq 0 \\ U(W) &= \ln(W) & \gamma = 0 \end{aligned} \tag{6}$$

when $(1-\gamma)$ is a measure of the investor's relative risk aversion (RRA).

The investor's problem is to choose an admissible trading strategy to maximize $E_t[U(N_T)]$ subject to (3). We define the value function at time t as

$$V(t,x,y) = \sup_{v \in \mathcal{A}(x,y)} E_t^{x,y}[U(N_T)], \tag{7}$$

where $\mathcal{A}(x,y)$ denotes the set of admissible controls which do not cause (3) to exit from $\mathcal{S}$. We define the *intervention operator* (or the maximum utility operator) $\mathcal{M}$ by

$$\mathcal{M}V(t,x,y) = \sup_{(x',y') \in \mathcal{S}} V(t,x',y') \tag{8}$$

where x' and y' are the new values of x and y. $\mathcal{M}V(t,x,y)$ represents the value of the strategy that consists in choosing the best transaction. We define the *continuation region* D by

$$D = \Big\{(x,y); V(t,x,y) > \mathcal{M}V(t,x,y)\Big\} \tag{9}$$

The continuation region is the region where it is not optimal to rebalance the investor's portfolio.

Now we intend to characterize the value function and the associated optimal strategy, assuming there exists an optimal strategy for each initial point (t,x,y). Then, if the optimal strategy is to not transact, the utility associated with this strategy is $V(t,x,y)$. On the other hand, selecting the best transaction and then following the optimal strategy gives the utility $\mathcal{M}V(t,x,y)$. Since the first strategy is optimal, its utility is greater or equal to the utility associated with the second strategy. Hence, $V(t,x,y) \geq \mathcal{M}V(t,x,y)$ with equality when it is optimal to make a transaction. Moreover, in the continuation region the application of the dynamic programming principle gives us $\mathcal{L}V(t,x,y) = 0$, where the operator $\mathcal{L}$ is defined by

$$\mathcal{L}V(t,x,y) = \frac{\partial V}{\partial t} + rx\frac{\partial V}{\partial x} + \mu y \frac{\partial V}{\partial y} + \frac{1}{2}\sigma^2 y^2 \frac{\partial^2 V}{\partial y^2}. \tag{10}$$

It is proved (see, for example, Øksendal and Sulem (1999)) that the value function is the unique viscosity solution of the quasi-variational Hamilton-Jacobi-Bellman inequalities (QVHJBI, or just QVI):

$$\max\Big\{\mathcal{L}V(t,x,y), \mathcal{M}V - V\Big\} = 0. \tag{11}$$

3 A Markov Chain Approximation of the Continuous Time Problem

It is tempting to try to solve the partial differential equation (11) by using the classical finite-difference method, but the PDE has only a formal meaning and is to be interpreted in a symbolic sense. Indeed, we do not know whether the partial derivatives of the value function are well defined, i.e., the value function has a twice continuously differentiable solution. The method of solution of such problems was suggested by Kushner (including Kushner (1977), Kushner (1990), and Kushner and Dupuis (1992)). The basic idea involves a consistent approximation of the problem by a Markov chain, and then solving an appropriate optimization problem for the Markov chain model. Unlike the classical finite-difference method, the smoothness of the solution to the HJB or QVI equations is not needed. The methods of proof of convergence are relatively simple and require the use of only some basic ideas in the theory of weak convergence of a sequence of probability measures of random processes. Some examples of proofs of convergence of the value function of the discrete time models to their continuous time counterparts are: Fitzpatrick and Flemming (1991), Davis, Panas, and Zariphopoulou (1993), and Collings and Haussmann (1998).

In practical applications there are two basic approaches to the realization of the Markov chain approximation method. Using the first approach, one constructs a discrete time approximation to the continuous time price processes used in the continuous time model. Then the discrete time program is solved by using the discrete time dynamic programming algorithm. The examples of use of this approach are Hodges and Neuberger (1989) and Davis et al. (1993). Using the second approach, one discretizes a HJB/QVI equation by applying the finite-difference approximation scheme which serves here only as a guide to the construction of a Markov chain. The coefficients of the resulting discrete equation is then used as the transition probabilities. This approach is often denoted as "finite difference" method, but the use of finite differences is just a "device" to get a Markov chain, in itself the approach is not a finite-difference method. The examples of use of this approach are Akian et al. (1996), Øksendal and Sulem (1999) and Chancelier et al. (2000).

The main objective of this section is to present a numerical procedure for computing the optimal trading policy. We will follow the first approach and are concerned with the construction of a discrete time approximation to the continuous time price processes used in the continuous time model presented in the previous section. The reason is to be able to solve the problem numerically, i.e., our discrete time utility maximization problem is a Markov chain approximation to the associated continuous time problem. The discrete time program is then solved by using the backward recursion algorithm.

Consider the partition $0 = t_0 < t_1 < \ldots < t_n = T$ of the time interval $[0, T]$ and assume that $t_i = i\Delta t$ for $i = 0, 1, \ldots, n$ where $\Delta t = \frac{T}{n}$. Let ε be a stochastic variable:

$$\varepsilon = \begin{cases} u & \text{with probability } p, \\ d & \text{with probability } 1-p. \end{cases}$$

We define the discrete time stochastic process of the stock as:

$$y_{t_{i+1}} = y_{t_i}\varepsilon \tag{12}$$

and the discrete time process of the risk-free asset as:

$$x_{t_{i+1}} = x_{t_i}\rho \tag{13}$$

If we choose $u = e^{\sigma\sqrt{\Delta t}}$, $d = e^{-\sigma\sqrt{\Delta t}}$, $\rho = e^{r\Delta t}$, and $p = \frac{1}{2}\left[1 + \frac{\mu}{\sigma}\sqrt{\Delta t}\right]$, we obtain the binomial model proposed by Cox, Ross, and Rubinstein

(1979). An alternative choice is $u = e^{\mu\Delta t + \sigma\sqrt{\Delta t}}$, $d = e^{\mu\Delta t - \sigma\sqrt{\Delta t}}$, $\rho = e^{r\Delta t}$, and $p = \frac{1}{2}$, which was proposed by He (1990). As n goes to infinity, the discrete time processes (12) and (13) converge in distribution to their continuous counterparts (2) and (1).

The following discretization scheme is proposed for the QVI (11):

$$V^{\Delta t} = \mathcal{L}(\Delta t)V^{\Delta t}, \tag{14}$$

where $\mathcal{L}(\Delta t)$ is an operator given by

$$\begin{aligned} \mathcal{L}(\Delta t)V^{\Delta t} = \max_m \Big\{ & V^{\Delta t}(t_i, x - k - (1+\lambda)m\Delta y, y + m\Delta y), \\ & V^{\Delta t}(t_i, x - k + (1-\lambda)m\Delta y, y - m\Delta y), \\ & E\{V^{\Delta t}(t_{i+1}, x\rho, y\varepsilon)\}\Big\}, \end{aligned} \tag{15}$$

where m is an integer number. We have discretized the y-space in a lattice with grid size Δy. This scheme is based on the principle that the investor's policy is the choice of the optimal transaction, that is, to buy, sell, or do nothing for a particular state given the value function for all states in the next time instant. As mentioned above, we use the binomial tree for the stock price. In addition, we need to discretize the x-space in a lattice with grid size Δx.

Theorem 1. *The solution $V^{\Delta t}$ of (14) converges locally uniformly to the unique continuous constrained viscosity solution of (11) as* $\Delta t \to 0$

The proof is based on the notion of viscosity solutions and can be made by following along the lines of the proof of Theorem 4 in Davis et al. (1993).

In all our numerical calculations we have used the following discretization parameters: $n = 20$, $\Delta x = 1$, and $\Delta y = 1$. The calculations were implemented on a standard IBM PC. Due to the lack of huge memory capacity we detect and store only the line coordinates of the four boundaries which characterize the optimal trading strategy at each period. As a result, the algorithm grows quadratically in complexity as the number of periods increases, meaning that the calculation of the optimal policy for period $n+1$ takes approximately the same time as the calculation of the optimal policies for all n previous periods.

4 Optimal Policy Without Transaction Costs

We consider the case without any transaction costs ($\lambda = 0$, $k = 0$) for the sake of comparison. The investor's problem can be rewritten as

$$V(t, x, y) = \sup_{(x,y)} E_t^{x,y}[U(x_T + y_T)] \tag{16}$$

subject to the self-financing condition

$$d(x_t + y_t) = (rx_t + \mu y_t)dt + \sigma y_t dB_t \tag{17}$$

Merton (including Merton (1969), Merton (1971), and Merton (1973)) re-parametrized the problem by introducing new variables $w_t = x_t + y_t$ (the total wealth) and $\pi_t = \frac{y_t}{w_t}$ (the fraction of the total wealth held in stock). Since transactions are costless and instantaneous we can regard π_t as a sole decision variable. The reformulated stochastic control problem becomes

$$V(w,t) = \sup_{\pi} E_t^w[U(w_T)] \tag{18}$$

subject to

$$dw_t = [(\mu - r)\pi_t + r]w_t dt + \sigma w_t \pi_t dB_t \tag{19}$$

Merton obtained a close-form solution for π_t^*

$$\pi_t^* = \frac{\mu - r}{\sigma^2(1-\gamma)} \tag{20}$$

This solution indicates that it is optimal for investor to keep a constant fraction of total wealth in the risky asset. This means, in particular, that the investor's portfolio holdings are always on the line $y_t = qx_t$, $q = \frac{\pi^*}{1-\pi^*}$, in the (x,y)-plane. This line is commonly referred to as the Merton line. One should note that the optimal policy depends neither on the investor's horizon, nor on the investor's wealth (or its composition).

5 Proportional Transaction Costs

Here we consider the case with proportional transaction costs only ($k = 0$). In this case the problem can be formulated as a *singular stochastic control* problem (see Davis and Norman (1990) and Shreve and Soner (1994)). In contrast to the no transaction cost case, at any time t the portfolio space is divided into three disjoint positive cones, which can be specified as the Buy region, the Sell region, and the No-Transaction (NT) region. If a portfolio lies either in the Buy region or in the Sell region, the optimal strategy is to buy/sell the risky asset until the portfolio reaches the closest boundary of the NT region.

The HJB-equation for this singular stochastic control problem is given by

$$\max\left\{\mathcal{L}V(t,x,y), -(1+\lambda)\frac{\partial V}{\partial x} + \frac{\partial V}{\partial y}, (1-\lambda)\frac{\partial V}{\partial x} - \frac{\partial V}{\partial y}\right\} = 0 \tag{21}$$

where $\mathcal{L}V(t,x,y)$ is defined by (10).

Inside NT the investor does not trade. Therefore in NT the value function must satisfy the HJB equation: $\max\{\mathcal{L}V(t,x,y)\} = 0$. The last two equations in (21) define the Buy and Sell region respectively. The heuristic argument for this is as following. Because in the Buy region the optimal policy is to transact to the closest NT boundary, the investor increases his indirect utility by buying some amount of stock, Δy, at the expense of lowering holdings on the bank account by $(1+\lambda)\Delta y$. Therefore $V_y > (1+\lambda)V_x$. A necessary condition for optimality is $V_y \leq (1+\lambda)V_x$. The set of (x, y) points for which the inequality holds with equality defines the boundary $\partial\mathcal{B}$ between the Buy region and the NT region. Similarly, the equation $(1-\lambda)V_x = V_y$ defines the boundary $\partial\mathcal{S}$ between the Sell region and the NT region.

The boundaries $\partial\mathcal{B}$ and $\partial\mathcal{S}$ are straight lines through the origin and can be conveniently described by their slope coefficients $q_l(t)$ (lower NT boundary) and $q_u(t)$ (upper NT boundary). The fact that these boundaries are strait lines through the origin is the result of the homothetic property[2] of the value function $V(t,x,y)$ (see, for example, Davis and Norman (1990), Theorem 3.1).

The NT boundaries are functions of the investor's horizon only and do not depend on the investor's wealth. Moreover, the NT region tends to widen as the investing horizon approaches. To help understand the optimal policy we provide numerical illustrations (see Figures (1) and (2)) with the following data: $\mu = 10\%$, $r = 5\%$, $\sigma = 25\%$ (all in annualized terms), RRA=2, and $\lambda = 0.01$.

The interested reader may consult Genotte and Jung (1994) for the detailed examination of the optimal policy for the investor with a finite horizon and a large set of realistic parameters.

6 Fixed Transaction Costs

Here we consider the case with fixed transaction costs only ($\lambda = 0$). The problem can be formulated in the same manner as in Section 2 with the correction for zero proportional transaction costs. As in the previous case, the portfolio space again can be divided into three disjoint regions, which can be specified as the Buy region, the Sell region, and the NT region. If a portfolio lies either in the Buy region or in the Sell region, the optimal strategy is to buy/sell the risky asset until the portfolio reaches the so-called "target" boundary. All these boundaries are functions of the investor's horizon and the composition of the investor's wealth so that a possible description of the

[2]that is, $V(t,\theta x,\theta y) = \theta^{\gamma}V(t,x,y)$ for $\gamma \neq 0$

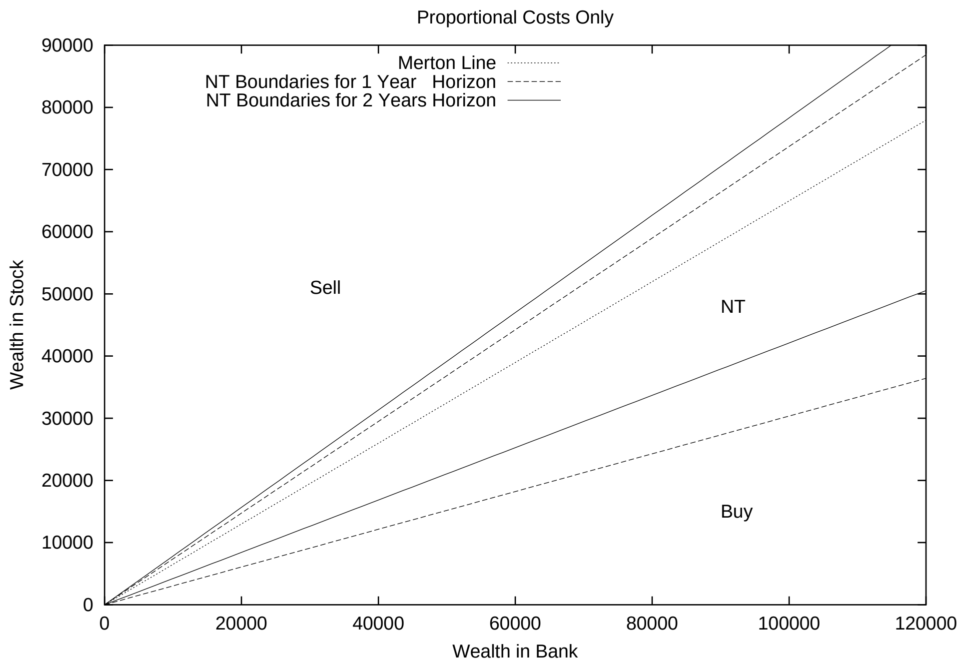

Figure 1: Optimal transaction policies for 2 different horizons

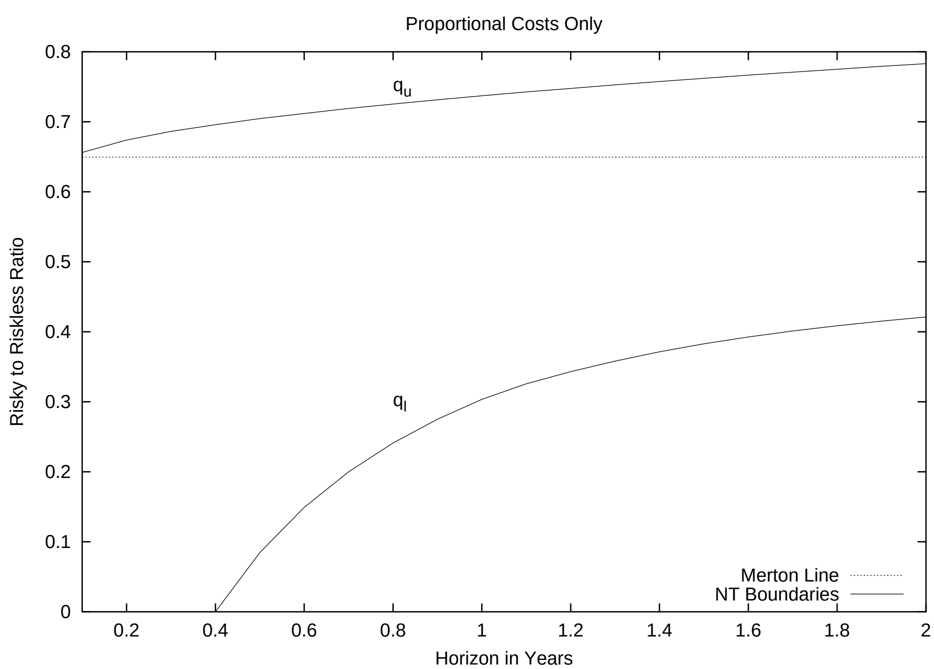

Figure 2: NT boundaries as functions of the investor's horizon

optimal policy may be given by

$$\begin{aligned} y &= q_l(x,t)x \\ y &= q^*(x,t)x \\ y &= q_u(x,t)x \end{aligned} \tag{22}$$

The first and the third equations describe the lower and the upper no-transaction boundaries respectively. The second equation describes the target boundary.

It is easy to prove that in the case of a CRRA investor the value function has the homothetic property with respect to (x, y, k)[3] (see Schroder (1995), Proposition 1). This is a very convenient property which allows to calculate the optimal policy for a single fixed fee k and then to obtain the optimal policy for another k' by simple scaling.

To illustrate the optimal policy we provide numerical calculations with the following data: $\mu = 10\%$, $r = 5\%$, $\sigma = 25\%$ (all in annualized terms), RRA=2, and $k = 1$. Figure (3) plots the optimal strategy for 2 years horizon. Figure (4) plots NT and Target boundaries as functions of the investor's horizon for $W = 400000$. Figure (5) plots NT and Target boundaries as functions of the investor's wealth for 2 years horizon.

Our numerical results agree with the findings of Schroder (1995). Some important properties of the optimal policy are as follows. The Merton line lies within the NT region. For even a small fixed fee, the NT boundaries are found to be rather wide. The optimal policy appears to converge quickly to a stationary policy as the horizon increases (see Figure (4)). The target boundary converges quickly to the Merton line as the investor's wealth increases (see Figure (5)).

The effects on the optimal policy from varying volatility, drift, and RRA are thoroughly examined in Schroder (1995).

7 Proportional and Fixed Transaction Costs

Here we consider the case with proportional and fixed transaction costs ($\lambda > 0$ and $k > 0$). Before proceeding to the numerical results, the following property of the value function can be easily established directly from the definition.

Proposition 1. *For the CRRA utility function, the value function has the homothetic property: for $\theta > 0$*

$$V(t, \theta x, \theta y, \theta k) = \theta^\gamma V(t, x, y, k) \qquad \text{if} \qquad U(W) = \frac{W^\gamma}{\gamma}, \tag{23}$$

$$V(t, \theta x, \theta y, \theta k) = \ln(\theta) + V(t, x, y, k) \qquad \text{if} \qquad U(W) = \ln(W). \tag{24}$$

[3]that is, $V(t, \theta x, \theta y, \theta k) = \theta^\gamma V(t, x, y, k)$ for $\gamma \neq 0$

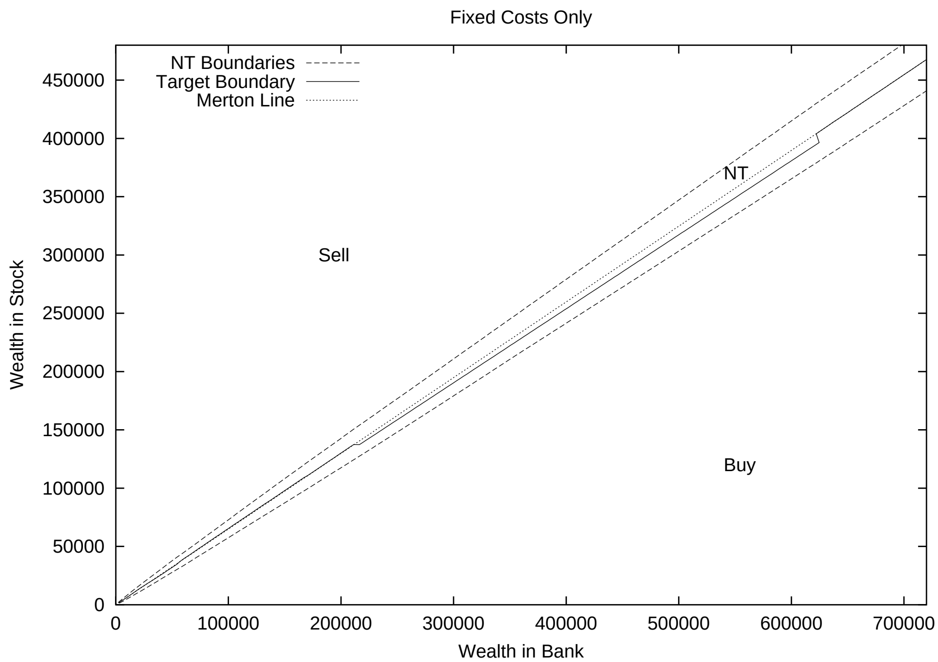

Figure 3: Optimal transaction policy for 2 years horizon

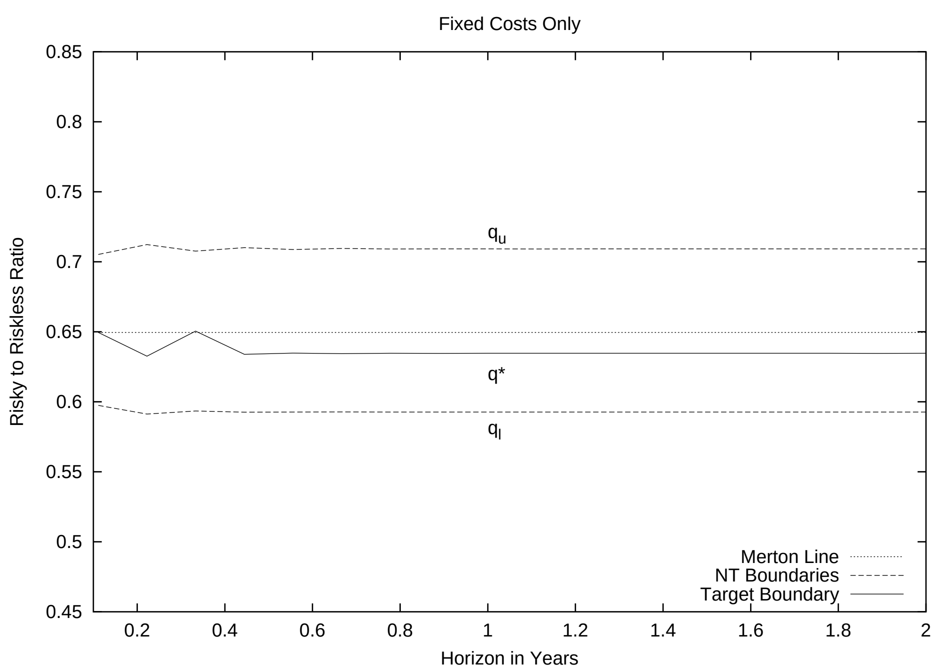

Figure 4: NT and Target boundaries as functions of the investor's horizon

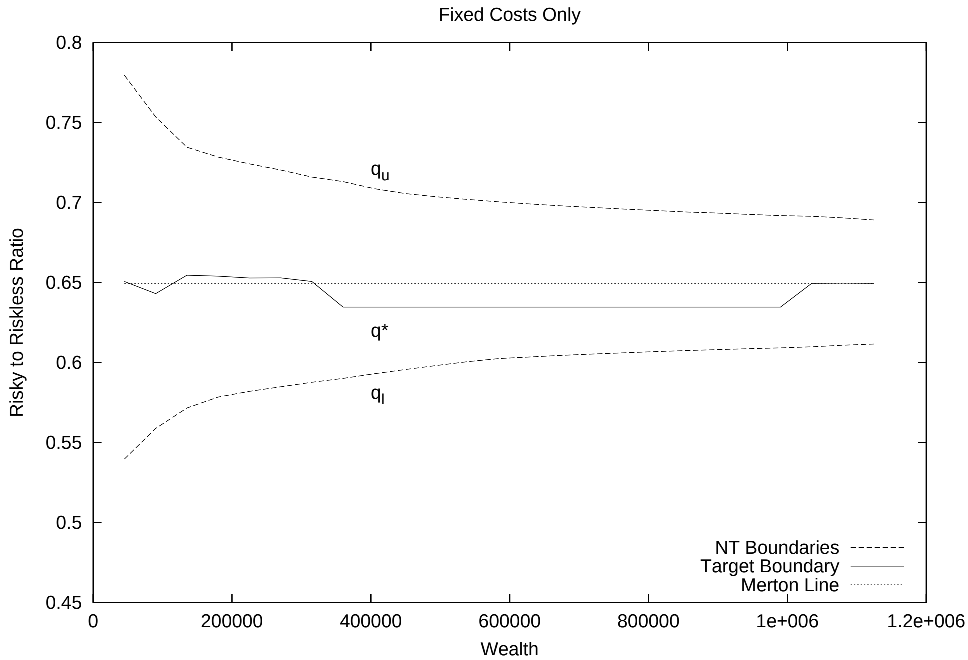

Figure 5: NT and Target boundaries as functions of the investor's wealth

Proof. The proof follows along the lines of the proof of Theorem 3.1 in Davis and Norman (1990). Denote by $\mathcal{A}(x, y, k)$ the class of admissible policies starting at $(x_t, y_t) \in \mathcal{S}$. Then it is easily checked from the equations (3) that for any $\theta > 0$

$$\mathcal{A}(\theta x, \theta y, \theta k) = \{(\theta x, \theta y, \theta k) : (x, y, k) \in \mathcal{A}(x, y)\}$$

that is, the portfolio process $(\theta x, \theta y)$ and transaction costs θk is admissible if and only if the portfolio process (x, y) and transaction costs k is admissible. Thus

$$\begin{aligned} V(t, \theta x, \theta y, \theta k) &= \sup_{\mathcal{A}(\theta x, \theta y, \theta k)} E_t^{\theta x, \theta y}[U(x_T, y_T, k)] \\ &= \sup_{\mathcal{A}(x,y,k)} E_t^{x,y}[U(\theta x_T, \theta y_T, \theta k)] \end{aligned}$$

When $U(W) = \frac{W^\gamma}{\gamma}$ we have $U(\theta x_T, \theta y_T, \theta k) = \theta^\gamma U(x_T, y_T, k)$ so that $V(t, \theta x, \theta y, \theta k) = \theta^\gamma V(t, x, y, k)$, whereas when $U(W) = \ln(W)$ then $U(\theta x_T, \theta y_T, \theta k) = \ln(\theta) + U(x_T, y_T, k)$ and $V(t, \theta x, \theta y, \theta k) = \ln(\theta) + V(t, x, y, k)$.

$\square$

In the presence of proportional and fixed transaction costs, the portfolio space can again be divided into three disjoint regions (Buy, Sell, and NT), but four (instead of three in the previous section) boundaries describe the optimal policy. As before, the Buy and NT regions are divided by the lower no-transaction boundary, and the Sell and NT regions are divided by the upper no-transaction boundary. If a portfolio lies in the Buy region, the optimal strategy is to buy the risky asset until the portfolio reaches the lower (Buy) target boundary. Similarly, if a portfolio lies in the Sell region, the optimal strategy is to sell the risky asset until the portfolio reaches the upper (Sell) target boundary.

All these boundaries are functions of the investor's horizon and the composition of the investor's wealth so that a possible description of the optimal policy may be given by

$$\begin{array}{l} y = q_l(x,t)x \\ y = q_l^*(x,t)x \\ y = q_u^*(x,t)x \\ y = q_u(x,t)x \end{array} \tag{25}$$

The first and the fourth equations describe the lower and the upper no-transaction boundaries respectively. The second and the third equations describe the lower and upper target boundaries respectively. Figure (6) illustrates the optimal strategy for $\mu = 10\%$, $r = 5\%$, $\sigma = 25\%$ (all in annualized terms), RRA=2, $k = 1$, $\lambda = 0.01$, and 2 years horizon. Figure (7) plots NT and Target boundaries as functions of the investor's horizon for $W = 200000$. Figure (8) plots NT and Target boundaries as functions of the investor's wealth for 2 years horizon.

Some important properties of the optimal policy are as follows. The NT boundaries are found to be wider than those in the model with proportional transaction costs only (as one quite logically expects). As the terminal date approaches, the NT region widens. And the NT region widens as wealth declines. The behavior of the target boundaries is rather complicated. One can observe that as the investor's wealth increases, they converge quickly to the NT boundaries in the corresponding model with proportional transaction costs only. We should note that this rate of convergence depends on the discretisation parameter n - the number of trading periods in the time interval $[0, T]$. As n grows, the rate of convergence declines. As wealth becomes small, the target boundaries move closer to the Merton line. The closer the terminal date, the earlier this movement begins.

The careful comparative statics analysis of the behavior of NT and target boundaries is beyond the scope of this paper. Indeed, every boundary is a function of many parameters and may be written as

$$y = q(t, x, \mu, r, \sigma, \gamma, \lambda, k)x$$

To do the comparative statics for every single parameter would take a huge amount of space. Besides, the presence of four boundaries makes this task

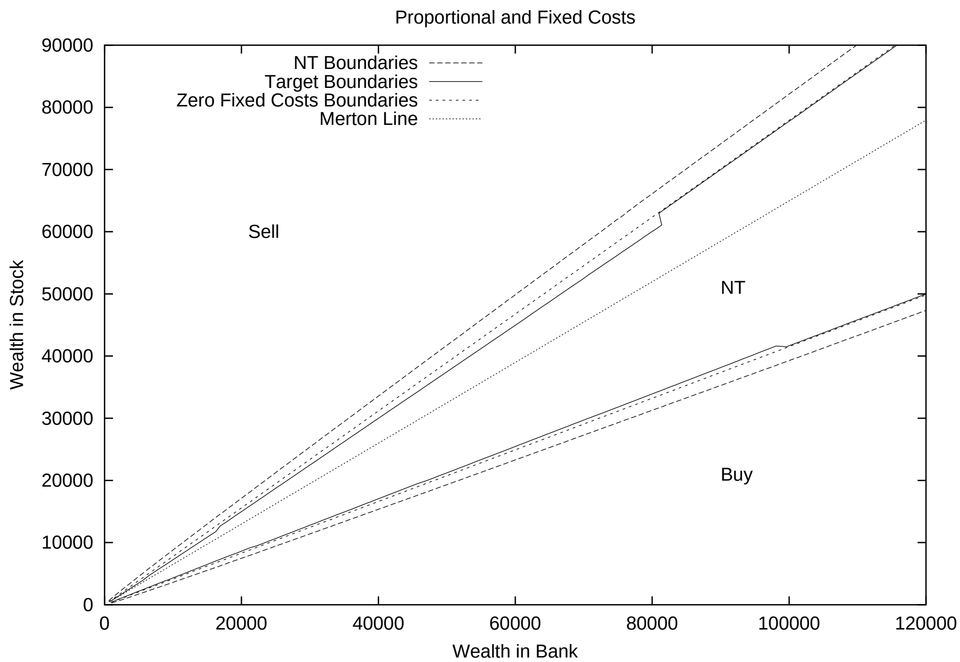

Figure 6: Optimal transaction policy for 4 years horizon

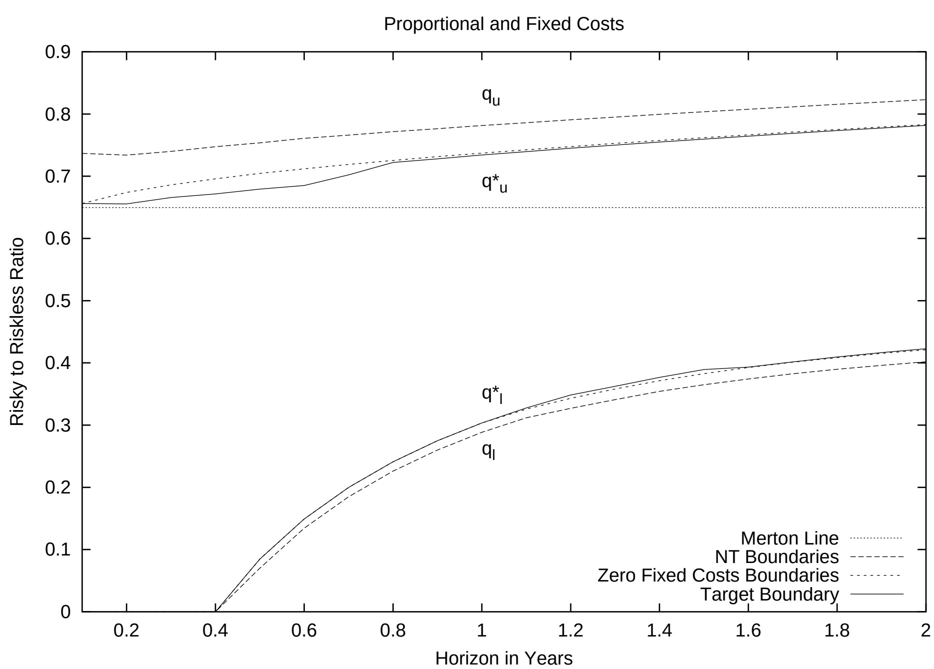

Figure 7: NT and Target boundaries as functions of the investor's horizon

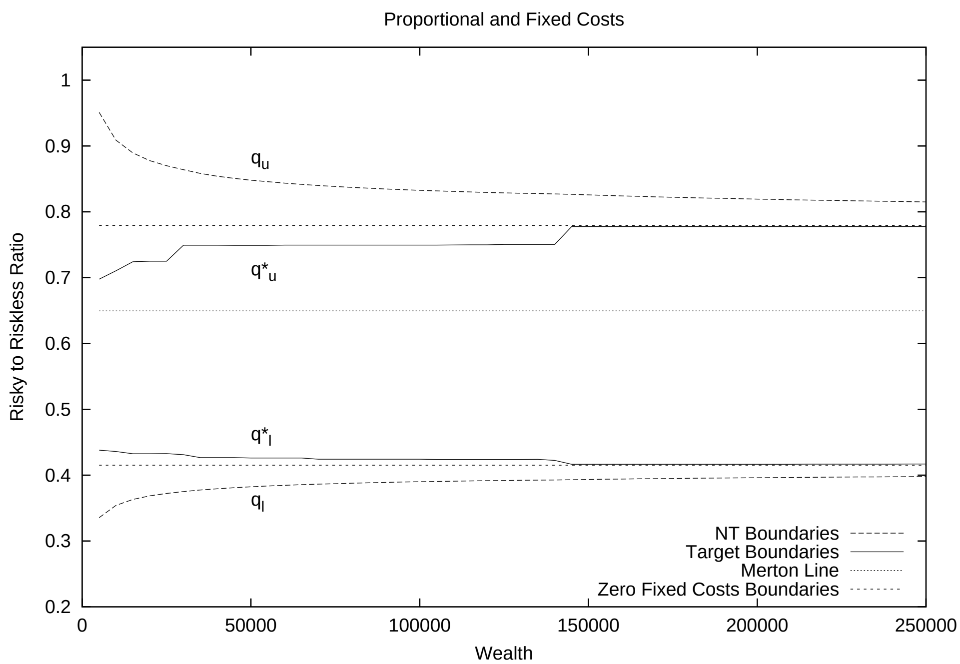

Figure 8: NT and Target boundaries as functions of the investor's wealth

rather cumbersome. Therefore we only combine comparative statics analysis for some important parameters such as the volatility σ, the drift μ, and the relative risk aversion coefficient RRA. Then we examine effects on the optimal policy from varying the level of proportional transaction costs λ. As a consequence of the homothetic property of the value function in (x, y, k), we can calculate the optimal strategy for a single fixed fee k and then obtain the optimal policy for another k' by a simple rescaling of the (x, y) axis.

In addition, we have to choose some benchmarks to make the comparisons. For this purpose we use the deviation of a boundary from the Merton line and the deviation of a boundary from the corresponding NT boundary in a model with proportional transaction costs only.

Our combined comparative statics analysis is based on the following idea. In the absence of transaction costs, the fraction of the total wealth invested in the risky asset is defined by equation (20). One can note that either doubling the volatility σ^2, the relative risk aversion coefficient RRA, or halving the risk premium $\mu - r$ has a similar effect on the optimal policy. Namely, the investor halves the fraction of his wealth in the risky asset. But what happens with the optimal policy in the presence of both fixed and proportional transaction costs?

Figures (9), (10), (11), and (12) present some results of comparison of NT and target boundaries for different volatilities, drifts, and relative risk aversion coefficients. The benchmark parameters are $\mu = 10\%$, $RRA = 1.5$, and $\sigma = 23\%$. The rest of the parameters are $r = 5\%$, $k = 1$, $\lambda = 0.01$. Figures (9) and (11) plot NT and target boundaries as functions of the investor's wealth for a 3 years horizon. Figures (10) and (12) plot NT and target boundaries as functions of the investor's horizon for $W = 60000$.

The analysis shows that either doubling volatility, RRA, or halving the risk premium has similar general consequences. The NT region narrows both in time and wealth that causes more frequent transactions. At the same time the NT region shifts downwards in the (x, y) plane causing the investor to move out of the risky stock and into the riskless bond. The target boundaries move closer both to the Merton line and to the zero fixed costs NT target boundaries.

In particular, doubling either volatility or RRA has almost the same effect on NT boundaries. As compared to these, halving the risk premium produces a wider NT region. The effect on the rate of convergence of target boundaries to the zero fixed costs NT boundaries is more distinct. The highest rate of convergence one gets by doubling the volatility. Doubling the relative risk aversion gives also a higher rate of convergence. In contrast, halving the risk premium gives a slower rate of convergence than that in the model with the benchmark parameters.

We now turn to the analysis of the effect of proportional transaction costs on the optimal trading policy. Here we limit ourselves to the presentation of the effect of transaction costs on the NT boundaries only, as the effect on the target boundaries is difficult to interpret. Figures (13) and (15) plot NT boundaries as functions of the investor's wealth for 4 years horizon. Figures (14) and (16) plot NT boundaries as functions of the investor's horizon for $W = 50000$.

As the level of proportional transaction costs becomes smaller, the deviation of NT boundaries from zero fixed costs NT boundaries increases (see Figures (15) and (16)). At the same time the NT region in the corresponding model with zero fixed transaction costs narrows. The resulting net effect is seen from Figures (13) and (14). The NT region narrows when proportional transaction costs becomes smaller.

8 Conclusions and Extensions

In this paper we study the optimal portfolio selection problem for a constant relative risk averse investor who faces fixed and proportional transaction costs and maximizes expected utility of end-of-period wealth. We use a continuous time model and apply the method of the Markov chain approximation to solve numerically for the optimal trading policy. The numerical solution indicates that the portfolio space is divided into three disjoint regions (Buy, Sell, and No-Transaction (NT)), and four boundaries describe

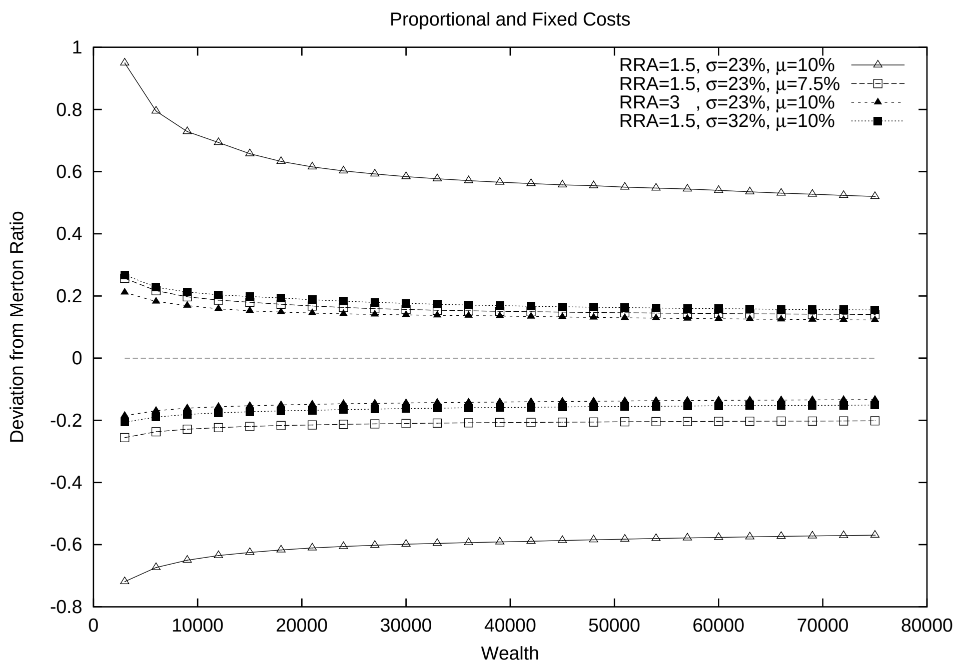

Figure 9: NT boundaries as functions of the investor's wealth

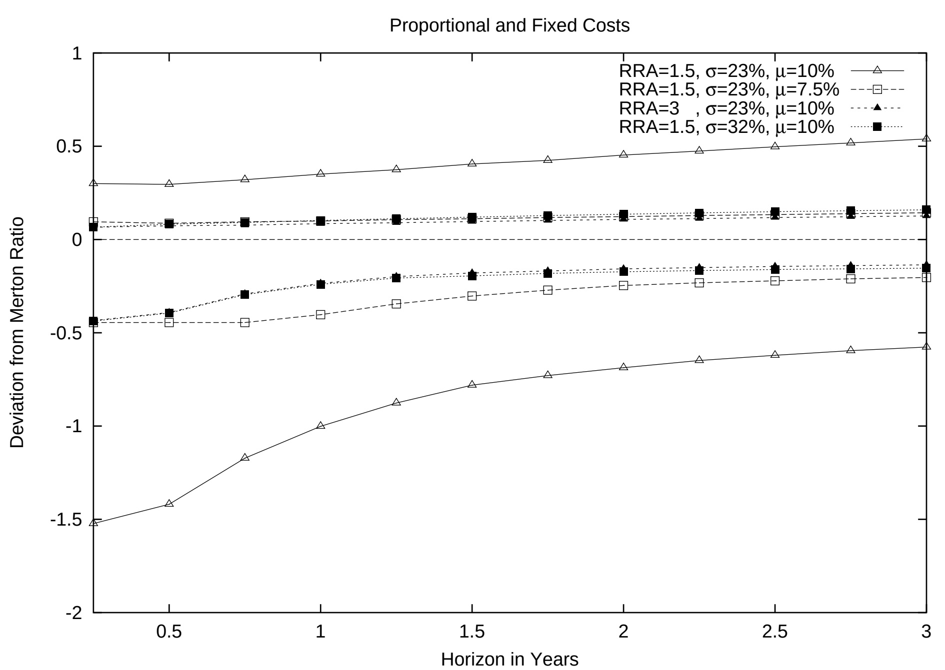

Figure 10: NT boundaries as functions of the investor's horizon

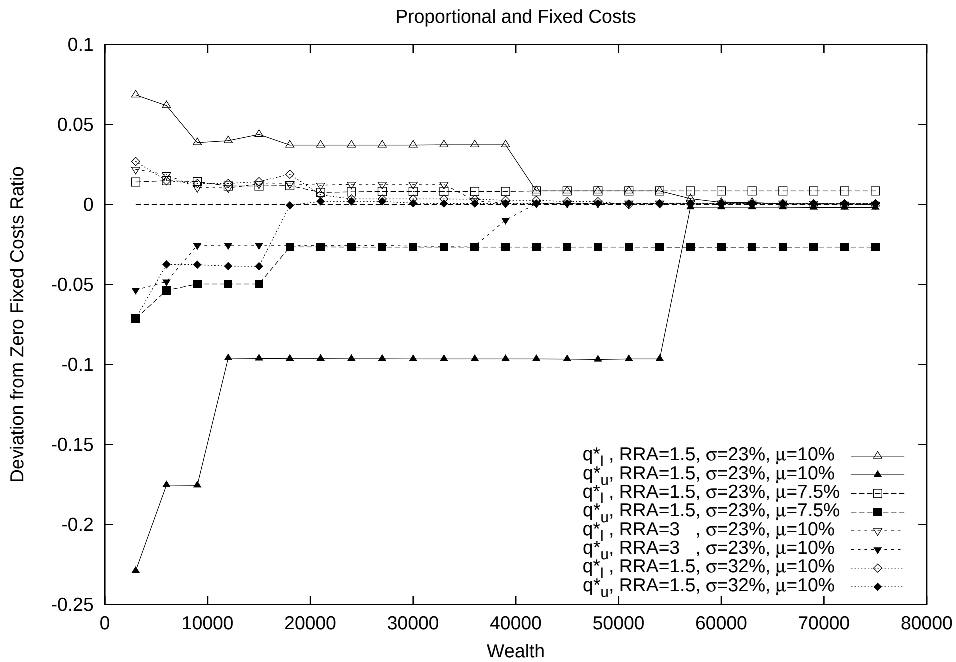

Figure 11: Target boundaries as functions of the investor's wealth

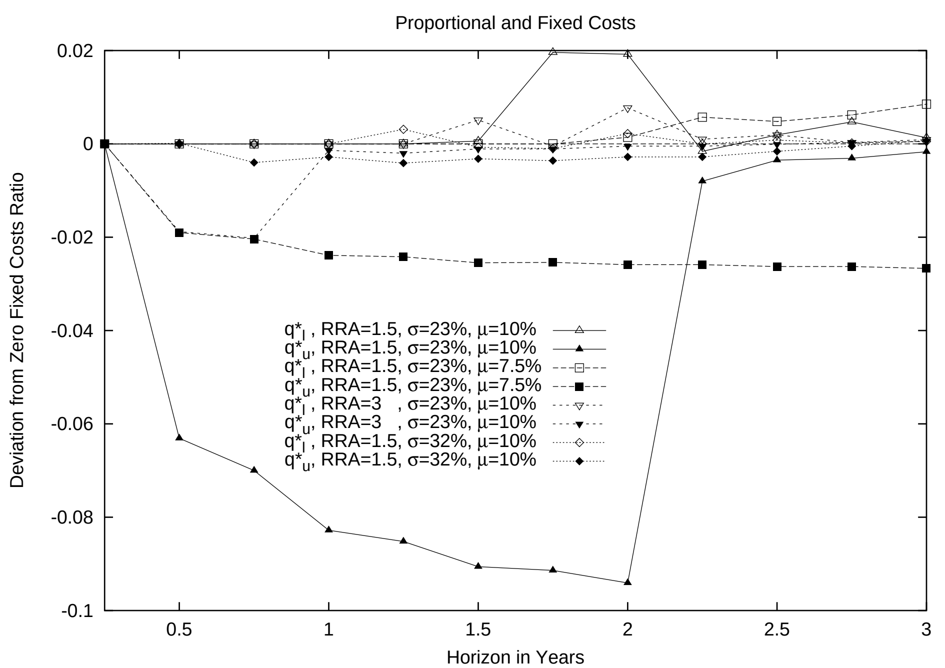

Figure 12: Target boundaries as functions of the investor's horizon

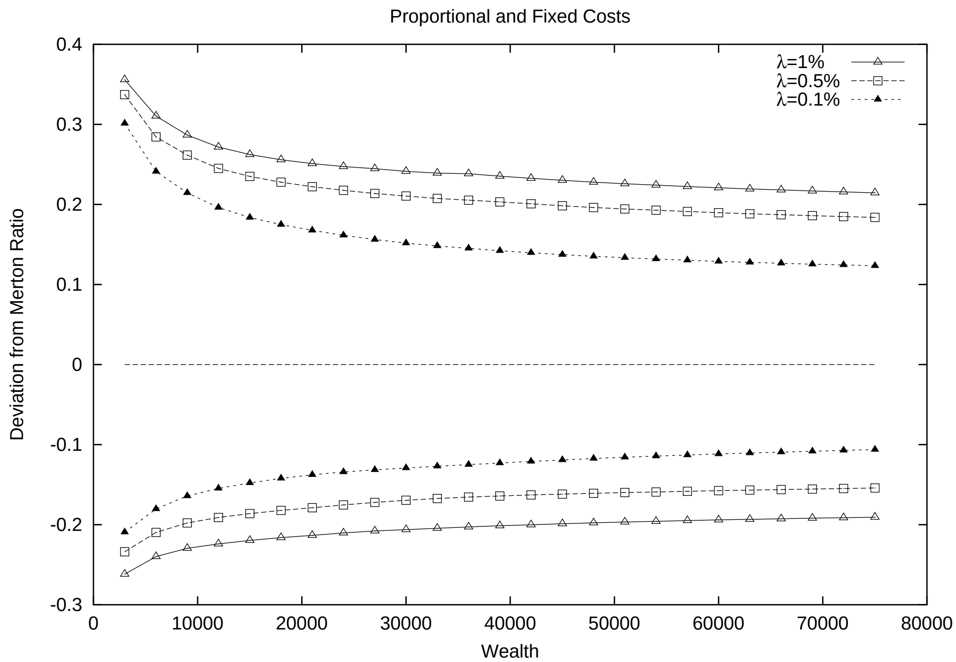

Figure 13: NT boundaries as functions of the investor's wealth

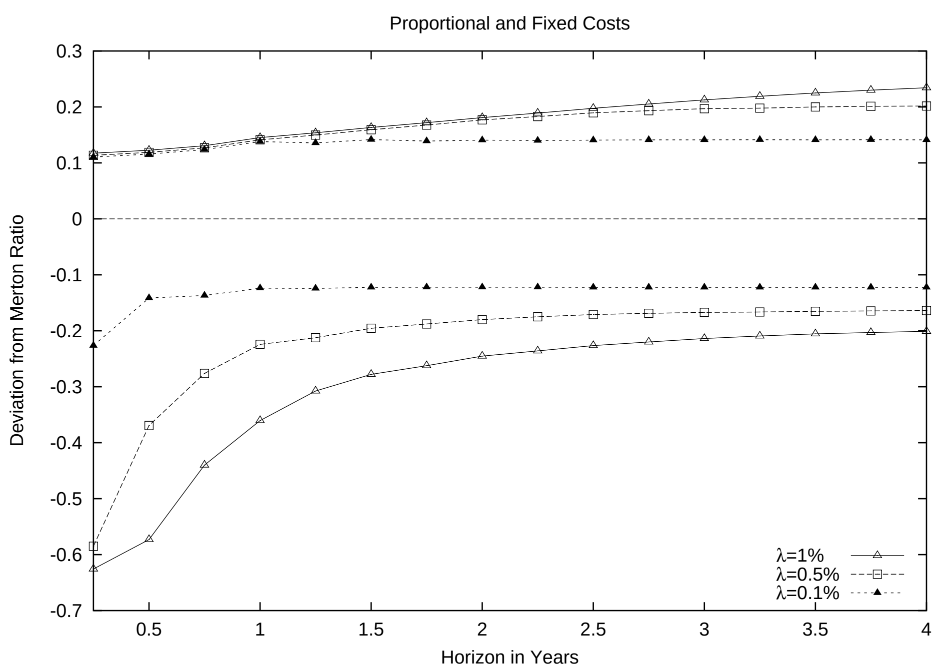

Figure 14: NT boundaries as functions of the investor's horizon

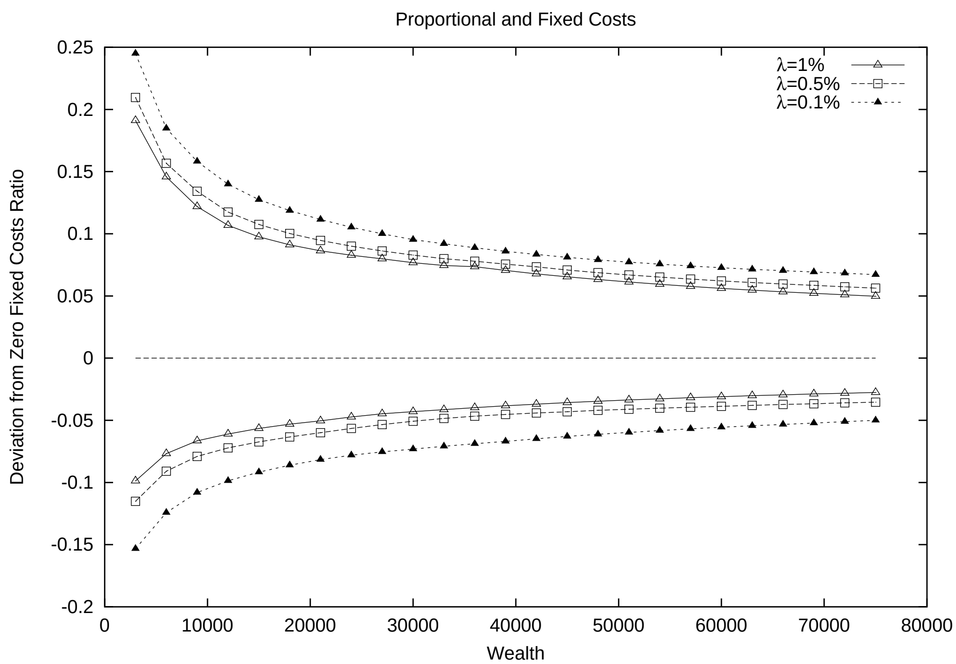

Figure 15: NT boundaries as functions of the investor's wealth

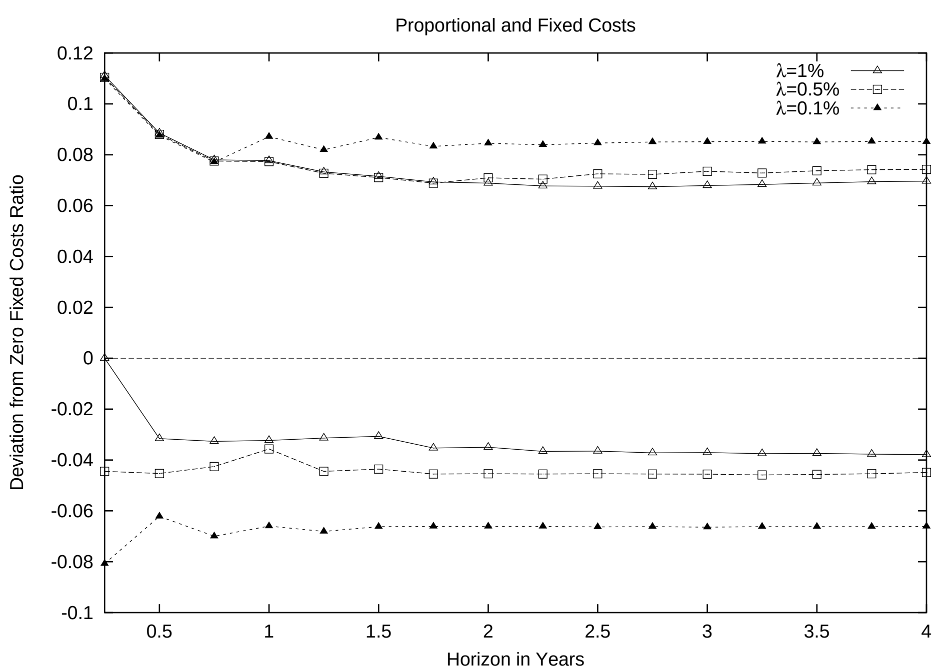

Figure 16: NT boundaries as functions of the investor's horizon

the optimal policy. If a portfolio lies in the Buy region, the optimal strategy is to buy the risky asset until the portfolio reaches the lower (Buy) target boundary. Similarly, if a portfolio lies in the Sell region, the optimal strategy is to sell the risky asset until the portfolio reaches the upper (Sell) target boundary. All these boundaries are functions of the investor's horizon and the composition of the investor's wealth. Some important properties of the optimal policy are as follows: As the terminal date approaches, the NT region widens. And the NT region widens as wealth declines. As the investor's wealth increases the target boundaries converge quickly to the NT boundaries in the corresponding model with proportional transaction costs only. As wealth becomes small, the target boundaries move closer to the Merton line. The closer the terminal date, the earlier this movement begins. The effects on the optimal policy from varying volatility, drift, CRRA, and the level of transaction costs are also examined.

Throughout the paper we assume that the model parameters are always chosen so as to exclude the cases with borrowing of the risk-free asset and short selling of stock. These two cases may be also investigated. The approach of this paper may be generalized in a straightforward manner to incorporate intermediate consumption, more general utility functions, and a more general structure of transaction costs. Another interesting extension would be the case of two or more risky assets.

However, the computational method applied for the problem we study is very time-consuming. With an acceptable amount of computational time these calculations can only be done for either rather low dimension of n - the number of periods or for "coarse" grid size for Δx and Δy. Therefore up to now the practical implementation of the numerical method could only be done for quite short investment horizons.

Finally, the utility maximization approach and numerical technique used in this paper may be successfully applied to price options in markets with both fixed and proportional transaction costs[4].

[4] this approach was pioneered by Hodges and Neuberger (1989)

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