The perpetual American put option for jump-diffusions: Implications for equity premiums.

Knut K. Aase *
Norwegian School of Economics and Business Administration
5045 Bergen, Norway
and
Centre of Mathematics for Applications (CMP),
University of Oslo, Norway.
Knut.Aase@NHH.NO

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Abstract

In this paper we solve an optimal stopping problem with an infinite time horizon, when the state variable follows a jump-diffusion. Under certain conditions our solution can be interpreted as the price of an American perpetual put option, when the underlying asset follows this type of process.

The probability distribution under the risk adjusted measure turns out to depend on the equity premium, which is not the case for the standard, continuous version. This difference is utilized to find intertemporal, equilibrium equity premiums.

We apply this technique to the US equity data of the last century, and find an indication that the risk premium on equity was about two and a half per cent if the risk free short rate was around one per cent. On the other hand, if the latter rate was about four per cent, we similarly find that this corresponds to an equity premium of around four and a half per cent.

The advantage with our approach is that we need only equity data and option pricing theory, no consumption data was necessary to arrive at these conclusions.

Various market models are studied at an increasing level of complexity, ending with the incomplete model in the last part of the paper.

KEYWORDS: Optimal exercise policy, American put option, perpetual option, optimal stopping, incomplete markets, equity premiums, CCAPM.

1 Introduction.

In the first part of the paper we consider the perpetual American put option when the underlying asset pays no dividends. This is known to be the same mathematical

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problem as pricing an infinite-lived American call option, when the underlying asset pays a continuous, proportional dividend rate, as shown by Samuelson (1965).

The market value of the corresponding European perpetual put option is known to be zero, but as shown by Merton (1973a), the American counterpart converges to a strictly positive value. This demonstrates at least one situation where there is a difference between these two products in the situation with no dividend payments from the underlying asset.

We analyze this contingent claim when the underlying asset has jumps in its paths. We start out by solving the relevant optimal stopping problem for a general jump-diffusion, and illustrate the obtained result by several examples.

It turns out that in the pure jump model the probability distribution under the risk adjusted measure depends on the equity premium, which is not the case for the standard, continuous version. This difference is utilized to find equity premiums when the two different models are calibrated to yield the same perpetual option values, and have the same volatilities.

We utilize this methodology to the problem of estimating the equity premiums in the twentieth century. This has been a challenge in both finance and in macro economics for some time. The problem dates back to the paper by Mehra and Prescott (1985), introducing the celebrated "equity premium puzzle". Closely related there also exists a so called "risk-free rate puzzle", see e.g., Weil (1989), and both puzzles have been troublesome for the consumption-based asset pricing theory.

The problem has its root in the small estimate of the covariance between equities and aggregate consumption, and the small estimate of the variance of aggregate consumption, combined with a large estimate of the equity premium. Using a representative agent equilibrium model of the Lucas (1978) type, the challenge has been to reconcile these values with a reasonable value for the relative risk aversion of the representative investor (the equity premium puzzle), and also with a reasonable value for his subjective interest rate (the risk free rate puzzle). Mehra and Prescott (1985) estimated the short term interest rate to one one cent, and the equity premium was estimated to around six per cent.

McGrattan and Prescott (2003) re-examine the equity premium puzzle, taking into account some factors ignored by the Mehra and Prescott: Taxes, regulatory constraints, and diversification costs - and focus on long-term rather than short-term savings instruments. Accounting for these factors, the authors find that the difference between average equity and debt returns during peacetime in the last century is less than one per cent, with the average real equity return somewhat under five per cent, and the average real debt return almost four per cent. If these values are correct, both puzzles are solved at one stroke (see e.g., Aase (2004)).

From these studies it follows that there is some confusion about the appropriate value of the equity premium of the last century, at least what numerical value to apply in models. It also seems troublesome to agree on the value of the short term interest rate for this period.

Our results for the US equity data of the last century indicate an equity premium of around 2.5 per cent if the risk free short rate has been about one per cent. If the latter rate has been around four per cent, on the other hand, we find that this corresponds to an equity premium of around 4.4 per cent. Both these values are somewhat in disagreement with the two above studies. Our value of around 2.5 per cent equity premium yields a more reasonable coefficient of relative risk aversion than the one obtained by Mehra and Prescott (1985). If, on the other hand, the average real debt return was around 4 per cent during this time period, our 4.4 per cent risk premium differs somewhat from the 1 per cent estimate in McGrattan and Prescott (2003).

The estimates we present in Section 7 are based on a rather simple model, and should be considered with some care. However, the methodology to produce these estimates is the innovative part. An advantage with our approach is that we do not need consumption data to obtain equilibrium intertemporal equity premiums, as the quality of these data has been questioned.

Another candidate to produce intertemporal risk premiums without consumption data is the ICAPM of Merton (1973b). This model, on the other hand, requires a large number of state variables to be identifiable, which means that empirical testing of the ICAPM quickly becomes difficult.

Further attempts to overcome the inaccuracies in consumption data include Campbell (1993) and (1996). Briefly explained, a log-linear approximation to the representative agent's budget constraint is made and this is used to express unanticipated consumption as a function of current and future returns on wealth. This expression is then combined with the Euler equation resulting from the investor's utility maximization to substitute out consumption of the model. As is apparent, our approach is rather different from this line of research.

The paper is organized as follows: Section 2 presents the model, Section 3 the American perpetual option pricing problem, Section 4 the solution to this problem in general, Section 5 treats adjustments to risk, Section 6 compares the solutions to this problem for the standard, continuous model and a pure jump model, and also presents comparative statics for this latter model, Section 7 applies the theory to infer about historical equity premiums, Section 8 presents solutions for a combined jump diffusion, Section 9 discusses a model where there are different possible jump sizes, Section 10 combines the latter case with a continuous component, and Section 11 treats the incomplete model, where jump sizes are continuously distributed. Section 12 concludes.

2 The Model

First we establish the dynamics of the assets in the model: There is an underlying probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ satisfying the usual conditions, where Ω is the set of states, $\mathcal{F}$ is the set of events, $\mathcal{F}_t$ is the set of events observable by time t, for any $t \geq 0$, and P is the given probability measure, governing the probabilities of events related to the stochastic price processes in the market. On this space is defined one locally riskless asset, thought as the evolution of a bank account with dynamics

$$d\beta_t = r\beta_t dt, \qquad \beta_0 = 1,$$

and one risky asset satisfying the following stochastic differential equation

$$dS_t = S_{t-}[\mu dt + \sigma dB_t + \alpha \int_R \eta(z)\tilde{N}(dt, dz)], \qquad S_0 = x > 0. \qquad (1)$$

Here B is a standard Brownian motion, $\tilde{N}(dt, dz) = N(dt, dz) - \nu(dz)dt$ is the compensated Poisson random measure, $\nu(dz)$ is the Lévy measure, and $N(t, U)$

is the number of jumps which occur before or at time t with sizes in the set U of real numbers. The process $N(t,U)$ is called the Poisson random measure of the underlying Lévy process. The function $\alpha\eta(z) \geq -1$ for all values of z. We will usually choose $\eta(z) = z$ for all z, which implies that the integral is over the set $(-1/\alpha, \infty)$. The Lévy measure $\nu(U) = E[N(1,U)]$ is in general a set function, where E is the expectation operator corresponding to the probability measure P. In our examples we will by and large assume that this measure can be decomposed into $\nu(dz) = \lambda F(dz)$ where λ is the frequency of the jumps and $F(dz)$ is the probability distribution function of the jump sizes. This gives us a finite Lévy measure, and the jump part becomes a compound Poisson process.

This latter simplification is not required to deal with the optimal stopping problem, which can be solved for any Lévy measure ν for which the relevant equations are well defined. The processes B and N are assumed independent. Later we will introduce more risky assets in the model as need arises.

The stochastic differential equation (1) can be solved using Itô's lemma, and the solution is

$$\begin{aligned} S(t) = S(0) \exp\big\{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t \\ - \alpha \int_0^t \int_R \eta(z)\nu(dz)ds + \int_0^t \int_R \ln(1 + \alpha\eta(z))N(ds,dz)\big\}, \end{aligned} \tag{2}$$

which we choose to call a geometric Lévy process. From this expression we immediately see why we have required the inequality $\alpha\eta(z) \geq -1$ for all z; otherwise the natural logarithm is not well defined. This solution is sometimes labeled a "stochastic" exponential, in contrast to only an exponential process which would result if the price Y was instead given by $Y(t) = Y(0)\exp(Z_t)$, where $Z_t = (X_t - \frac{1}{2}\sigma^2 t)$, and the accumulated return process X_t is given by the arithmetic process

$$X_t := \mu t + \sigma B_t + \alpha \int_0^t \int_R \eta(z)\tilde{N}(ds,dz). \tag{3}$$

Clearly the process Y can never reach zero in a finite amount of time if the jump term is reasonably well behaved [1], so there would be no particular lower bound for the term $\alpha\eta(z)$ in this case. We have chosen to work with stochastic exponential processes in this paper. There are several reasons why this is a more natural model in finance. On the practical side, bankruptcy can be modeled using S, so credit risk issues are more readily captured by this model. Also the instantaneous return $\frac{dS(t)}{S(t-)} = dX_t$, which equals ($\mu dt$ + "noise"), where μ is the rate of return, whereas for the price model Y we have that

$$\frac{dY(t)}{Y(t-)} = \big(\mu + \int_R (e^{\alpha\eta(z)} - 1 - \alpha\eta(z))\nu(dz)\big)\,dt + \sigma dB_t + \int_R (e^{\alpha\eta(z)} - 1)\tilde{N}(dt,dz),$$

which is in general different from dX_t, and as a consequence we do not have a simple interpretation of the rate of return in this model. [2]

[1] i.e., if it does not explode. The Brownian motion is known not to explode.

[2] If the exponential function inside the two different integrals can be approximated by the two first terms in its Taylor series expansion, which could be reasonable if the Lévy measure ν has short and light tails, then we have $\frac{dY(t)}{Y(t-)} \approx dX_t$.

3 The optimal stopping problem

We want to solve the following problem:

$$\phi(s,x) = \sup_{\tau \geq 0} E^{s,x}\Big\{e^{-r(s+\tau)}(K - S_\tau)^+\Big\}, \tag{4}$$

where $K > 0$ is a fixed constant, the exercise price of the put option, when the dynamics of the stock follows the jump-diffusion process explained above. By $E^{s,x}$ we mean the conditional expectation operator given that $S(s) = x$, under the given probability measure P.

For this kind of dynamics the financial model is in general not complete, so in our framework the option pricing problem may not have a unique solution, or any solution at all. There can be several risk adjusted measures Q, and it is not even clear that the pricing rule must be linear, so none of these may be appropriate for pricing the option at hand. If one is, however, the pricing problem may in some cases be a variation of the solution to the above problem, since under any appropriate Q the price S follows a dynamic equation of the type (1), with r replacing the drift parameter μ, and possibly with a different Lévy measure $\nu(dz)$. Thus we first focus our attention on the problem (4).

There are special cases where the financial problem has a unique solution; in particular there are situations including jumps where the model either is, or can be made complete, in the latter case by simply adding a finite number of risky assets. We return to the the different situations in the examples.

4 The solution of the optimal stopping problem

In this section we present the solution to the optimal stopping problem (4) for jump-diffusions. As with continuous processes, there is an associated optimal stopping theory also for discontinuous processes. For an exposition, see e.g., Øksendal and Sulem (2004). In order to employ this, we need the characteristic operator $\bar{\mathcal{A}}$ of the process S when $r > 0$. It is

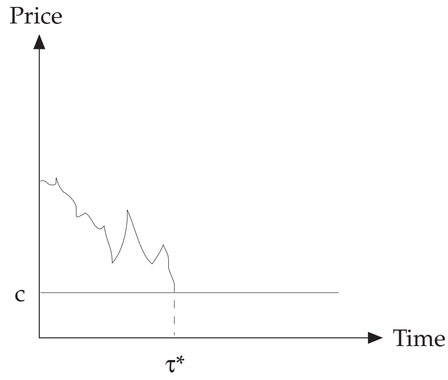
$$\bar{\mathcal{A}}\varphi(s,x) = \frac{\partial \varphi(s,x)}{\partial s} + x\mu \frac{\partial \varphi(s,x)}{\partial x} + \frac{1}{2}x^2\sigma^2 \frac{\partial^2 \varphi(s,x)}{\partial x^2} - r\varphi(s,x)$$
$$+ \int_R \{\varphi(s, x + \alpha x \eta(z)) - \varphi(s,x) - \alpha \frac{\partial \varphi(s,x)}{\partial x} x\eta(z)\}\nu(dz).$$

With a view towards the verification theorem - a version for jump-diffusion processes exists along the lines of the one for continuous processes - we now conjecture that the continuation region $\mathcal{C}$ has the following form

$$\mathcal{C} = \Big\{(x,t) : x > c\Big\},$$

where the trigger price c is some constant. The motivation for this is that for any time t the problem appears just the same, from a prospective perspective, implying that the trigger price $c(t)$ should not depend upon time. See Figure 1. In order to apply the methodology of optimal stopping, consider the vector process $Z(t) = (s + t, S(t))$, where the first component is just time, the process Z starting

Figure 1: Continuation Region



Continuation region of the perpetual American put option.

in the point (s,x). We only need to consider the characteristic operator $\mathcal{A}$ of the process Z, which is

$$\mathcal{A} = \bar{A} + r\varphi.$$

In the continuation region $\mathcal{C}$, the relevant variational inequalities reduce to the partial integro-differential-difference equation $\mathcal{A}\varphi = 0$, or

$$\frac{\partial \varphi}{\partial s} + \mu x \frac{\partial \varphi}{\partial x} + \frac{1}{2}x^2\sigma^2\frac{\partial^2 \varphi}{\partial x^2}$$
$$+ \int_R \{\varphi(s, x + \alpha x\eta(z)) - \varphi(s,x) - \alpha\frac{\partial \varphi}{\partial x}x\eta(z)\}\nu(dz) = 0.$$

Furthermore we conjecture that the function $\varphi(s,x) = e^{-rs}\psi(x)$. Substituting this form into the above equation allows us to cancel the common term e^{-rs}, and we are left with the equation

$$\begin{aligned} & - r\psi(x) + \mu x\frac{\partial \psi(x)}{\partial x} + \frac{1}{2}x^2\sigma^2\frac{\partial^2\psi(x)}{\partial x^2} + \\ & \int_R \{\psi(x + \alpha x\eta(z)) - \psi(x) - \alpha\frac{\partial \psi(x)}{\partial x}x\eta(z)\}\nu(dz) = 0 \end{aligned} \tag{5}$$

for the unknown function ψ.

Thus we were successful in removing time from the PDE, and reducing the equation to an ordinary integro-differential-difference equation.

The equation is valid for $c \le x < \infty$. Given the trigger price c, let us denote the market value $\psi(x) := \psi(x;c)$. The relevant boundary conditions are then

$$\psi(\infty;c) = 0 \quad \forall c > 0 \tag{6}$$
$$\psi(c;c) = K - c \quad \text{(exercise)} \tag{7}$$

We finally conjecture a solution of the form $\psi(x) = a_1 x + a_2 x^{-\gamma}$ for some constants a_1, a_2 and γ. The boundary condition (6) implies that $a_1 = 0$, and the boundary condition (7) implies that $a_2 = (K-c)c^{\gamma}$. Thus the conjectured form of the market value of the American put option is the following

$$\psi(x;c) = \begin{cases} (K-c)\big(\frac{c}{x}\big)^{\gamma}, & \text{if } x \geq c; \\ (K-x), & \text{if } x < c. \end{cases} \tag{8}$$

In order to determine the unknown constant γ, we insert the function (8) in the equation (5). This allows us to cancel the common term $x^{-\gamma}$, and we are left with the following nonlinear, algebraic equation for the determination of the constant γ:

$$-r - \mu\gamma + \frac{1}{2}\sigma^2\gamma(\gamma+1) + \int_R \{(1+\alpha\eta(z))^{-\gamma} - 1 + \alpha\gamma\eta(z)\}\nu(dz) = 0 \tag{9}$$

This is a well defined equation in γ, and the fact that we have successfully been able to cancel out the variables x and s, is a strong indication that we actually have found the solution to our problem.

If this is correct, it only remains to find the trigger price c, and this we do by employing the "high contact" or "smooth pasting" condition (e.g., McKean (1965))

$$\frac{\partial\psi(c;c)}{\partial x}\Big|_{x=c} = -1.$$

This leads to the equation

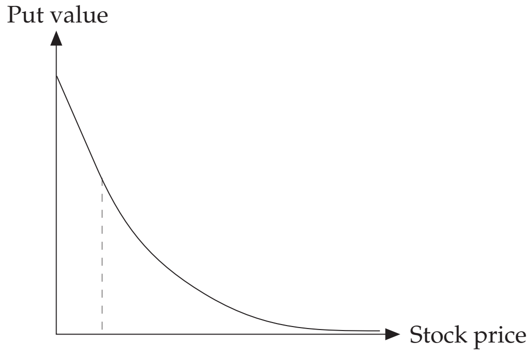
$$(k-c)c^{\gamma}(-\gamma c^{-\gamma-1}) = -1,$$

which determines the trigger price c as

$$c = \frac{\gamma K}{\gamma+1}, \tag{10}$$

where γ solves the equation (9). See Figure 2.

Figure 2: Perpetual American put option value



Market value of a perpetual American put option as a function of stock price.

We can now finally use the verification theorem of optimal stopping for jump-diffusions (see e.g. Øksendal and Sulem (2004)) to prove that this is *the* solution to our problem.

If, instead, we had used the exponential pricing model Y defined in Section 2, where $Y(t) = Y(0)\exp(Z_t)$, $Z_t = (X_t - \frac{1}{2}\sigma^2 t)$ and the the accumulated return process X_t is given by the arithmetic process in equation (3), this problem also has a solution, the above method works, and the corresponding equation for γ is given by

$$\begin{aligned}
&- r - \gamma\big(\mu + \int_R (e^{\alpha\eta(z)} - 1 - \alpha\eta(z))\nu(dz)\big) \\
&+ \frac{1}{2}\sigma^2\gamma(\gamma+1) + \int_R \big(e^{-\gamma\alpha\eta(z)} - 1 + \gamma(e^{\alpha\eta(z)} - 1)\big)\nu(dz) = 0.
\end{aligned} \tag{11}$$

5 Risk adjustments

While the concept of an equivalent martingale measure is well known in the case of diffusion price processes with a finite time horizon $T < \infty$, the corresponding concept for jump price processes is less known. In addition we have an infinite time horizon, in which case it is not true that the "risk neutral" probability measure Q is equivalent to the given probability measure P.

Suppose P and Q are two probability measures, and let $P_t := P|_{\mathcal{F}_t}$ and $Q_t := Q|_{\mathcal{F}_t}$ denote their restrictions to the information set $\mathcal{F}_t$. Then P_t and Q_t are equivalent for all t if and only if $\sigma^P = \sigma^Q$ and the Lévy measures ν^P and ν^Q are equivalent.

We now restrict attention to the pure jump case, where the diffusion matrix $\sigma = 0$. Let $\theta(s,z) \leq 1$ be a process such that

$$\xi(t) := \exp\big\{\int_0^t \int_R \ln(1-\theta(s,z))N(ds,dz) + \int_0^t \int_R \theta(s,z)\nu(dz)ds\big\} \tag{12}$$

exists for all t. Define Q_t by

$$dQ_t(\omega) = \xi(t)dP_t(\omega)$$

and assume that $E(\xi(t)) = 1$ for all t. Then there is a probability measure Q on $(\Omega, \mathcal{F})$ and if we define the random measure $\tilde{N}^Q$ by

$$\tilde{N}^Q(dt,dz) := N(dt,dz) - (1-\theta(t,z))\nu(dz)dt,$$

then

$$\int_0^t \int_R \tilde{N}^Q(ds,dz) = \int_0^t \int_R N(ds,dz) - \int_0^t \int_R (1-\theta(s,z))\nu(dz)ds$$

is a Q local martingale.

This result can be used to prove the following version of Girsanov's theorm for jump processes:

Theorem 1 *Let S_t be a 1-dimensional price process of the form*

$$dS_t = S_{t-}[\mu dt + \alpha \int_R \eta(z)\tilde{N}(dt,dz)].$$

Assume there exists a process $\theta(z) \leq 1$ such that

$$\alpha \int_R \eta(z)\theta(z)\nu(dz) = \mu \qquad a.s. \tag{13}$$

and such that the corresponding process ξ_t given in (12) (with $\theta(s,z) \equiv \theta(z)$ for all s) exists, and having $E(\xi_t) = 1$ for all t. Consider a measure Q such that $dQ_t = \xi(t)dP_t$ for all t. Then Q is a local martingale measure for S.

Proof. By the above cited result and the equality (13) we have that

$$\begin{aligned} dS_t &= S_{t-}[\mu dt + \alpha \int_R \eta(z)N(dt,dz) - \alpha \int_R \eta(z)\nu(dz)dt] \\ &= S_{t-}[\mu dt + \alpha \int_R \eta(z)\{\tilde{N}^Q(dt,dz) + (1-\theta(z)\nu(dz)dt\} - \alpha \int_R \eta(z)\nu(dz)dt] \\ &= S_{t-}[\alpha \int_R \eta(z)\tilde{N}^Q(dt,dz) + \{\mu - \alpha \int_R \theta(z)\eta(z)\nu(dz)\}dt] \\ &= S_{t-}\big[\alpha \int_R \eta(z)\tilde{N}^Q(dt,dz)\big], \end{aligned}$$

which is a local Q-martingale. □

We will call Q a risk adjusted probability measure, and θ the market price of risk (when we use the bank account as a numeraire). The above results can be extended to a system of n-dimensional price processes, see e.g., Øksendal and Sulem (2004) for results on a finite time horizon, Sato (1999), Chan (1999) and Jacod and Shiryaev (2002) for general results, and Huang and Pagès (1992) or Revuz and Yor (1991)) for results on the infinite time horizon.

Recall that the computation of the price of an American option must take place under a risk adjusted, local martingale measure Q in order to avoid arbitrage possibilities. Under any such measure Q all the assets in the model must have the same rate of return, equal to the short term interest rate r. Thus we should replace the term μ by r in equation (5). However, this may not be the only adjustment required when jumps are present. Typically another, but equivalent, Lévy measure $\nu^Q(dz)$ will appear instead of $\nu(dz)$ in equation (5). We return to the details in the following sections.

6 Two different models of the same underlying price process.

In this section we illustrate the above solution for two particular models of a financial market. We start out by recalling the solution in the standard lognormal continuous model, used by Black and Scholes and Merton.

6.1 The standard continuous model: $\alpha = 0$.

Since the equation (9) has to be solved under a risk adjujsted, local martingale measure Q in order for the solution to be the price of an American put option, we know that this is achieved in this model by replacing the drift rate μ by the interest rate r, and this is the only adjustment for Q required in the standard model. The equation for γ then reduces to

$$-r - r\gamma + \frac{1}{2}\sigma^2\gamma(\gamma + 1), \tag{14}$$

which is a quadratic equation. It has the two solutions $\gamma_1 = 2r/\sigma^2$ and $\gamma_2 = -1$. The solution γ_2 is not possible, since the boundary condition $\psi(\infty; c) = 0$ for all c, simply can not hold true in this case. Thus the solution is $\gamma = \frac{2r}{\sigma^2}$, as first obtained by Merton (1973a).

Comparative statics can be derived from the expression for the market value in (8). The results are directly comparable to the results for the finite-lived European put option: The put price ψ increases with K, ceteris paribus, and the put price decreases as the stock price x increases. Changes in the volatility parameter have the following effects: Let $v = \sigma^2$, then

$$\frac{\partial\psi}{\partial v} = \begin{cases} \frac{c}{v}\left(\frac{c}{x}\right)^{\gamma}\ln\left(\frac{x}{c}\right), & \text{if } x \geq c; \\ 0, & \text{if } x < c. \end{cases} \tag{15}$$

Clearly this partial derivative is positive as we would expect.

Similarly, but with opposite sign, for the interest rate r:

$$\frac{\partial\psi}{\partial r} = \begin{cases} -\frac{2c}{\gamma v}\left(\frac{c}{x}\right)^{\gamma}\ln\left(\frac{x}{c}\right), & \text{if } x \geq c; \\ 0, & \text{if } x < c. \end{cases} \tag{16}$$

The effect of the interest rate on the perpetual put is the one we would expect, i.e., a marginal increase in the interest rate has, ceteris paribus, a negative effect on the perpetual put value.

Notice that we have used above that

$$\frac{\partial\psi}{\partial \gamma} = \begin{cases} -\frac{c}{\gamma}\left(\frac{c}{x}\right)^{\gamma}\ln\left(\frac{x}{c}\right), & \text{if } x \geq c; \\ 0, & \text{if } x < c, \end{cases} \tag{17}$$

in other words the price is a decreasing function of γ when $x := S_t \geq c$, a result we will make use of below.

From the derivation in Section 4 we notice that the relationship (17) is true also in the jump-diffusion model, and because of this property, one can loosely think of the parameter γ as being inversely related to the "volatility" of the pricing process, properly interpreted.

6.2 A discontinuous model: The jump component is proportional to a Poisson process.

In this section we assume that $\nu(dz)$ is the frequency λ times the Dirac delta function at z_0, i.e., $\nu(dz) = \lambda\delta_{\{z_0\}}(z)dz$, $z_0 \in R\backslash\{0\}$ so that all the jump sizes are identical and equal to z_0 (which means that N is a Poisson process, of frequency λ, times z_0). First we consider the pure jump case ($\sigma^2 = 0$). We choose the function $\eta(z) \equiv z$ in this part, and the range of integration accordingly changes from $R := (-\infty, \infty)$ to $(-1/\alpha, \infty)$.

Using the results of Section 5, we find by Theorem 1 the market price of risk $\theta(z)$ from equation (13) after we have used the risk free asset as a numeraire. Thus θ must satisfy the equation

$$\alpha \int_{-1/\alpha}^{\infty} z\theta(z)\nu(dz) = \mu - r, \tag{18}$$

Due to the form of the Lévy measure $\nu(dz)$, this equation reduces to

$$\theta(z) = \begin{cases} \frac{\mu-r}{\alpha z_0 \lambda}, & \text{if } z = z_0; \\ 0, & \text{otherwise.} \end{cases} \tag{19}$$

This could be compared to the familiar Sharpe ratio $\frac{\mu-r}{\sigma}$ in the standard lognormal case. Here the term $\alpha^2 z_0^2 \lambda$ is the variance rate corresponding to the term σ^2 in the geometric Brownian motion model. This model is complete, and there is only one solution to the above equation (18).

Consider the risk adjusted probability measure Q. If we derive the dynamics of the discounted process $\bar{S}_t := e^{-rt}S_t$, this process has drift zero under the measure Q, corresponding to the market price of risk in (18), or equivalently, S has drift r under Q. Thus we must replace μ by r in the equation (9). It turns out that this is *not* the only adjustment to Q we have to perform here: Consider two Poisson processes with intensities (frequencies) λ and λ^Q and jump sizes z_0 and z_0^Q corresponding to two measures P and Q respectively. Only if $z_0 = z_0^Q$ can the corresponding P_t and Q_t be equivalent. This means that changing the frequency of jumps amounts to "reweighting" the probabilities on paths, and no new paths are generated by simply shifting the intensity. However, changing the jump sizes generates a different kind of paths. The frequency of a Poisson process can be modified without changing the "support" of the process, but changing the sizes of jumps generates a new measure which assigns nonzero probability to some events which were impossible under the old one. Thus $z_0 = z_0^Q$ is the only possibility here.

Turning to the frequency under λ^Q under Q, recall that $\nu(dz) = \lambda\delta_{\{z_0\}}(z)dz$ so we have here

$$\tilde{N}^Q(dt, dz) := N(dt, dz) - (1 - \theta(z))\lambda\delta_{\{z_0\}}(z)dzdt.$$

Since the term $(1 - \theta(z)) = (1 - \theta_0)$ where $\theta_0 := (\mu - r)/\alpha\lambda z_0$ is a constant, we must interpret the term $\lambda(1-\theta_0)$ as the frequency of jumps under the risk adjusted measure Q, or

$$\lambda^Q := \lambda(1 - \theta_0) = \lambda + \frac{r - \mu}{\alpha z_0}. \tag{20}$$

Here we notice an important difference between the standard continuous model and the model containing jumps. While it is a celebrated fact that the probability distribution under Q in the standard model does not depend on the drift parameter μ, in the jump model it does. This will have as a consequence that values of options must also depend on μ in the latter type of models. For the American perpetual put option we see this as follows: The equation for γ is

$$\lambda^Q(1 + \alpha z_0)^{-\gamma} = (r - \lambda^Q \alpha z_0)\gamma + \lambda^Q + r. \tag{21}$$

This equation is seen to depend on the drift parameter μ through the term λ^Q given in equation (20). Thus the parameter γ depends on μ, and finally so does the option value given in equation (8).

Let us briefly recall the argument why the drift parameter can not enter into the pricing formula for any contingent claim in the standard model: If two underlying asset existed with different drift terms μ_1 and μ_2 but with the same volatility parameter σ, there would simply be arbitrage. In the jump model different drift terms lead to different frequencies λ_1^Q and λ_2^Q through the equation (20), but this also leads to different volatilities of the two risky assets, since the volatility (under Q) depends upon the jump frequency (under Q). Thus no inconsistency arises when the drift term enters the probability distribution under Q in the jump model.

Let us denote the equity premium by $e_p := (r - \mu)$. We may solve the equations (20) and (21) in terms of e_p. This results in a linear equation for e_p, and the solution is

$$e_p = \alpha z_0 \left\{ \frac{r(\gamma + 1)}{(1 + \alpha z_0)^{-\gamma} - (1 - \alpha z_0 \gamma)} - \lambda \right\}. \tag{22}$$

Although this formula indicates a very simple connection between the equity premium and the parameters of the model, it is in some sense circular, since the parameter γ on the right hand side is not exogenous, but depends on all the parameters of the model. We will demonstrate later how this formula may be used to infer about historical risk premiums.

Let us focus on the equation (21) for γ. This equation is seen to have a positive root γ_d where the power function to the left in equation (21) crosses the straight line to the right in (21). If $z_0 > 0$, there exists exactly one solution if $r < \lambda^Q \alpha z_0$ for positive interest rate $r > 0$. If $r \geq \lambda^Q \alpha z_0 > 0$ there is no solution. If $-1 < z_0 < 0$, the equation has exactly one solution for $r > 0$, provided $\alpha z_0 > -1$.

Example 1. Here we illustrate different solutions to the equations for γ, first without risk adjustments, but where we calibrate the variance rates of the two noise terms. Here we recall that the variance of a compound Poisson process X_t is $\text{var}(X_t) = \lambda t E(Z^2)$, where Z is the random variable representing the jump sizes. We can accomplish this by choosing $\alpha = \lambda = \sigma = 1$, when the jump size parameter $z_0 = 1$, noticing that z_0^2 here corresponds to $E(Z^2)$. Fixing the short term interest rate $r = .06$, we get the solution $\gamma_d = .20$ of equation (21), while the corresponding solution to the equation (14) is $\gamma_c = .12$. Suppose the exercise price $K = 1$. Then we can compute the trigger price $c_c = .11$ in the continuous model, while $c_d = .17$ in the discontinuous model. This means that without any risk adjustments of the discontinuous model, it is optimal to exercise earlier using this model than using the continuous model, at least for this particular set of parameter values. $\square$

Using the respective formulas for the prices of the American put option in the two cases of Example 1, by the formula for the price $\psi(x, c)$ in equation (8) of Section 4 it is seen that the price ψ^c based on the continuous model is larger then the price ψ^{du} based on the discontinuous model with no risk adjustments, or $\psi^c(x; c_c) > \psi^{du}(x; c_d)$ for all values $x > c_c$ of the underlying risky asset, $\psi^c(x; c_c) = \psi^{du}(x; c_d)$ for $x \leq c_c$. According to option pricing theory, this ought to mean that there is "less volatility" in the jump model without risk adjustment than in the continuous counterpart. Thus risk adjustments of the frequency λ^Q must mean that $\lambda^Q > \lambda$, when $z_0 > 0$.

Example 2. Consider on the other hand the case where $z_0 < 0$, and let us pick $z_0 = -.5$. Now $\gamma_d = .29$ for the same set of parameter values as above. In order to properly calibrate the variance rates of the two models, we compare to the continuous model having $\sigma^2 = \lambda z_0^2 = .25$ or $\sigma = .50$. This yields $\gamma_c = .48$, which means that the situation is reversed from the above. The price commanded by the continuous model has decreased more than the corresponding price derived using discontinuous dynamics, without risk adjustments. Thus risk adjustments of the frequency λ^Q must now ($z_0 < 0$) mean that $\lambda^Q < \lambda$. □

From these numerical examples it seems like we have the following picture: When the jumps are all positive (and identical) and we do not adjust for risk, the jump model produces put option values reflecting less risk than the continuous one. When the jump sizes are all negative (and identical), and we continue to consider the risk neutral case, the situation is reversed. These conclusions seem natural for a put option, since price increases in the underlying tend to lower the value of this insurance product. In Example 1 only upward, sudden price changes are possible for the underlying asset, whereas the downward movement stemming from the compensated term in the price path is slower and predictable. Thus a put option that is not adjusted for risk ought to have less value under such dynamics, than in a situation where only negative, sudden price changes can take place.

In Section 7 we consider numerical results after risk adjustments of the jump model. This leads to some interesting results. Next we consider two examples.

6.3 A calibration exercise: Two initial examples.

We would like to use the above two different models for the same phenomenon to infer about equity premiums in equilibrium. In order to do this, we calibrate the two models, which we propose to do in two steps. First we ensure that the martingale terms have the same variances in both models, just as in examples 1 and 2 above. Second, both models ought to yield the same option values.

Example 3. Let us choose $\sigma = .165$ and $r = .01$. The significance of these particular values will be explained below.

Our calibration consists in the following two steps: (i) First, the matching of volatilities gives the equation $z_0^2 \alpha^2 = \sigma^2 = .027225$. We set the parameter $\alpha = 1$, and start with $z_0 = .01$, i.e., each jump size is positive and of size one per cent. The compensated part of the noise term consists of a negative drift, precisely "compensating" for the situation that all the jumps are positive. Then means that $\lambda = 272.27$ which is a roughly one jump each trading day on the average, where the time unit is one year.

(ii) Second, we calibrate the prices. Because of the equation (8), this is equivalent to equating the values of γ. Thus we find the value of λ^Q that yields γ_d as a solution of equation (21) equal to the value γ_c resulting from solving the equation (14) for the standard, continuous model. For the volatility $\sigma = .165$, the latter value is $\gamma_c = .73462$. By trying different values of λ^Q in equation (21), we find that the equality in prices is obtained when $\lambda^Q = 274.73$.

Finally, from the equation (20) for the risk adjusted frequency λ^Q we can solve for the equity premium $e_p = (r - \mu)$, which is found to be .0248, or about 2.5 per cent..

Equivalent to the above is to use the formula for the equity premium e_p in equation (22) directly, using $\gamma = \gamma_c$ in this equation and the above parameters values of α, z_0, r and λ.

Is this value dependent of our choice for the jump size z_0? Let us instead choose $z_0 = .1$. This choice gives the value of the frequency $\lambda = 2.7225$, the risk adjusted frequency $\lambda^Q = 2.9700$ and the value for the equity premium $(r - \mu) = .0248$, or about 2.5 per cent again. Choosing the more extreme value $z_0 = 1.0$, i.e., the upward jump sizes are all 100 per cent of the current price, gives the values $\lambda = .0272$, the risk adjusted frequency $\lambda^Q = .0515$ and the equity premium $(r - \mu) = .0248$, again exactly the same value for this quantity. $\square$.

In the above example the value of $\sigma = .165$ originates from an estimate of the volatility for the Standard and Poor's composite stock price index during parts of the last century. Thus the value of the equity premium around 2.5 per cent has independent interest in financial economics and in macro economics.

Notice that $\mu < r$ in equilibrium. This is a consequence of the fact that we are analyzing a perpetual put option, which can be thought of as an insurance product. The equilibrium price of a put is larger than the expected pay-out, because of risk aversion in the market. For a call option we have just the opposite, i.e., $\mu > r$, but the perpetual call option is of no use for us here, since its market value is zero.

Notice that in the above example we have two free parameters to choose, namely z_0 and α. The question remains how robust this procedure is regarding the choice of these parameters. The example indicates that our method is rather insensitive to the choice of the jump size parameter z_0. In the next section, after we have studied the comparative statics for the new parameters λ^Q, z_0 and α, we address this problem in a more systematic way, but let us just round off with the following example.

Example 4. Set $\sigma = .165$ and $r = .01$, and consider the case when $z_0 < 0$. First choose $z_0 = -.01$ (and $\alpha = 1$). i.e., each jump size is negative and of size one per cent all the time. The compensated part of the noise term will now consist of a positive drift, again "compensating" for the situation that all the jumps are negative. Then we get $\lambda = 272.27$ and the risk adjusted frequency is now $\lambda^Q = 269.78$. This gives for the equity premium $(r - \mu) = .0247$, or again close to 2.5%.

The value $z_0 = -.1$ gives $\lambda = .0273$, the risk adjusted frequency $\lambda^Q = 2.473$ and an estimate for the equity premium is $(r - \mu) = .0251$. The more extreme value of $z_0 = -.5$, i.e., each jump results in cutting the price in half, provides us with the values $\lambda = .1089$, $\lambda^Q = .0585$ and $(r - \mu) = .0252$. This indicates a form of robustness regarding the choice of the jump size parameter. $\square$

6.4 Comparative statics for the pure jump model

In this section we indicate some comparative statics for the pure jump model considered above. We have here three new parameters λ^Q, α and z_0 to concentrate on. In addition we have the drift parameter μ. Since we do not have a closed form solution, we have to rely on numerical methods. Here the result (17) is useful, since we only have to find the effects on the parameter γ in order to obtain the conclusions regarding the option values.

First let us consider the jump size parameter z_0, and we start with negative jumps. The results are reported in tables 1 and 2.

We notice from these that when the jump sizes are large and negative, the parameter γ is small and close to zero, meaning that the corresponding option value

z_0	-.999	-.99	-.90	-.80	.-70	- .60	-.50	-.40	-.30
γ	$9.9 \cdot 10^{-3}$	.016	.0412	.071	.113	.179	.290	.495	.941

Table 1: The parameter γ for different negative values of the jump sizes z_0: $\lambda^Q = 1$, $\alpha = 1$, r = .06

z_0	-.20	-.10	-.05	.-.03	-.02	-.01	-.001	-.0001
γ	2.18	8.19	27.36	62.27	115.40	311.35	5882.94	85435.51

Table 2: The parameter γ for different negative values of the jump sizes z_0: $\lambda^Q = 1$, $\alpha = 1$, r = .06

is close to its upper value of K, regardless of the value of the underlying stock S_t. As the jump sizes become less negative, the parameter γ increases, ceteris paribus, meaning that the corresponding option values decrease. As the jump sizes become small in absolute value, γ grows large, reflecting that the option value decreases towards its lowest possible value, which is $\psi_l(x, K) = (K - x)^+$ when $c = K$. Notice that there exists a solution γ to equation (21) across the whole range of z_0-values in $(-1, 0)$, which follows from our earlier observations.

For positive values of the jump size z_0 parameter, we have the following:

z_0	.0600003	.060005	.061	.062	.065	.07	.10	.30	.50
γ	$.353 \cdot 10^7$	$.212 \cdot 10^6$	1060	530	212	106	24	1.86	.693

Table 3: The parameter γ for different positive values of the jump sizes z_0: $\lambda^Q = 1$, $\alpha = 1$, r = .06

From tables 3 and 4 we see that small positive jump sizes have the same effect on γ as small negative jump sizes, giving low option values. As the jump size parameter z_0 increases, the value of the option increases towards its upper value of K.

These tables show that increasing the absolute value of the jump sizes, has the effect of decreasing the values of γ, which means that the values of the option increase. Here the jump size can not be decreased lower than -1, which is a singularity of the equation (21) when $\alpha = 1$. Notice from Table 3 that equation (21) only has a solution when $.06 < z_0$, which is consistent with the requirement $r < \lambda^Q \alpha z_0$ in this situation.

Turning to the risk adjusted frequency parameter λ^Q, under Q, we have the following. Tables 5 and 6 show that as the risk adjusted frequency λ^Q increases, the parameter γ decreases. Increasing the frequency means increasing the "volatility" of the underlying stock and this should imply increasing option prices, which is also the conclusion here following from the result (17). Note from Table 5 how the requirement $r < \lambda^Q \alpha z_0$ comes into play: There is no solution γ of the equation (21) for $z_0 \leq .06$ for these parameter values, in agreement with our earlier remarks.

From these latter two tables we are also in position to analyze how the option price depends on the drift parameter μ. Suppose μ decreases. Then, ceteris paribus, λ^Q increases, and the tables indicate that the parameter γ decreases and accordingly the put option value increases. Thus a decrease of the (objective) drift rate μ makes the put option more valuable, which seems reasonable, since this makes it more likely

z_0	.80	1	2	6	10	20	100	10 000
γ	.299	.204	.068	.015	.0079	.0035	$.63 \cdot 10^{-3}$	$.60 \cdot 10^{-5}$

Table 4: The parameter γ for different positve values of the jump sizes z_0: $\lambda^Q = 1$, $\alpha = 1$, r $= .06$

λ^Q	.0600003	.060005	.061	.062	.070	.10	.80	.90
γ	$4.0 \cdot 10^5$	$.2.4 \cdot 10^3$	121.00	61.00	13.00	3.82	.26	.23

Table 5: The parameter γ for different values of the risk adjusted jump frequency λ^Q: $z_0 = 1$, $\alpha = 1$, r $= .06$

that the option gets in the money. Notice that this kind of logic does not apply to the standard model, which may be considered a weakness.

Finally we turn to the parameter α. From the dynamic equation of the stock (1) one may be led to believe that this parameter plays a role similar to the parameter σ. Also from the equation (21) for γ it appears only in the product αz_0. In the present model the variance rate of the jump model is $\lambda\alpha^2 z_0^2$, so in this situation this product, after multiplication by λ, corresponds to the parameter σ^2 in the continuous model. However, when we consider jumps of different sizes the parameters α and z_0 can be disentangled.

It is not common to allow the parameter α to have negative values, unlike for the jump size z_0-parameter. Here we think of α as playing a role similar to σ, and this latter parameter is by convention positive. If we were to make a table of γ for varying values of α, we would confine ourselves to positive values, in which case the table would be identical to the corresponding table for z_0 (if $z_0 = 1$) by the above remarks, so we omit it here. In general α affects the support of the jump sizes, a matter we return to in the last section of the paper.

The conclusion for the parameter α is that as α increases, ceteris paribus, the option value increases, and approaches in the end, uniformly in x, its upper value of K.

6.5 Related risk adjustments of frequency

Risk adjustments of the frequency has been discussed earlier in the academic literature, in particular in insurance, see e.g., Aase (1999). This type of adjustment is, however, often referred to as something else in most of the actuarial literature; typically it is called a "loading" on the frequency. The reason for this is that in part of this literature there is no underlying financial model, and prices of insurance products are exogenous. In life insurance, for instance, the mortaliity function used for pricing purposes is usually not the statistically correct one, i.e., an estimate $\hat{\lambda}$ of λ, but a different one depending on the nature of the contract. For a whole life insurance product, where the insurer takes on mortality risk, the employed frequency is typically larger than $\hat{\lambda}$, while for an endowment insurance, such as a pension or annuity, it is typically smaller.

One way to interpret this is as risk adjustments of the mortality function, increasing the likelihood of an early death for a whole life product, and increasing the likelihood of longevity for a pensioner. Both these adjustments are in favor of

λ^Q	1	1.5	2	3	10	100	1000	10000
γ	.2042	.1341	.1000	.0661	.0196	.0019	.0002	$2 \cdot 10^{-5}$

Table 6: The parameter γ for different values of the risk adjusted jump frequency λ^Q: $z_0 = 1$, $\alpha = 1$, r = .06

the insurer, making the contract premiums higher, but must at the same time also be accepted by the insured in order for these life insurance contracts to be traded. Thus one may loosely interpret these adjustments as market based risk adjustments, although this is not the interpretation allowed by most traditional actuarial models, for reasons explained above.

A different matter is hedging in the present model. This is not so transparent as in the standard model, and has been solved using Malliavin calculus, see Aase, Øksendal, Ubøe and Privault (2000) and Aase, Øksendal and Ubøe (2001), for details.

7 Implications for equity premiums.

7.1 Introduction

In this section we turn to the problem of estimating the premium on equity of the twentieth century mentioned in the introduction. As indicated in examples 3 and 4, we suggest to use the results of the present paper to infer about the equity risk premium. The situation is that we have two complete financial models of about the same level of simplicity. We adapt these two models to the Standard and Poor's composite stock price index for the time period mentioned above, and compute the value of an American perpetual put option written on any risky asset having the same volatility as this index. Since the two models are both complete, and at about the same level of sophistication, we make the assumption that the theoretical option prices so obtained are approximately equal. Now we use the fact that the standard continuous model provides option prices that do not depend on the actual risk premium of the risky asset, but the jump model does. Exactly this difference between these two models enables us to find an estimate, based on calibrations, of the relevant equity premium. This corresponds to a no-arbitrage value, and since both the financial models are complete, these values are also consistent with a financial equilibrium, and can alternatively be thought of as equilibrium risk premiums.

7.2 The calibration

We now perform the calibration indicated in examples 3 and 4. Starting with the two no-arbitrage models of the previous section, we recall that this is carried out in two stages. First we match the volatilities in the two models under the given probability measure P: This gives the equation $\sigma_S^2 = z_0^2\alpha^2\lambda$. (We set α equal to one without loss of generality.) This step is built on a presumption that there is a linear relationship between equity premiums and volatility in equilibrium.

The consumption based capital asset pricing model (CCAPM) is a general equilibrium model, different from the option pricing model that we consider, where aggregate consumption is the single state variable. As a consequence, the instantaneous correlation between consumption and the stock index is equal to one for the continuous model, and this leads to a linear relationship between equity premiums and volatility. As noted by several authors, there is consistency between the option pricing model and the general equilibrium framework (e.g., Bick (1987), Aase (2002)). For the discontinuous model this linear relationship is not true in general (e.g., Aase (2004)), but holds with good approximation for the model of Section 6, a point we return to at the end of this section.

Second, we calibrate the prices. Because of the equation (8), this is equivalent to equating the values of γ. This will make our comparisons independent of the strike price K, as well as of the maturity of the option, since we consider a perpetual. Thus we find the value of λ^Q that yields γ_d as a solution to equation (21) equal to the value γ_c resulting from solving the equation (14) for the standard, continuous model.

Finally we infer e_p from equation (20) for the risk adjusted frequency λ^Q. Since $\lambda^Q = \lambda + (r - \mu)/z_0\alpha$, this procedure only depends on the value of the jump size z_0 that we have chosen. As noticed before, the required computations may be facilitated by using the formula (22) for the equity premium e_p, where we substitute the value $\gamma = \gamma_c$.

The following tables indicate that the procedure is rather insensitive to the choice of z_0. We start with the short rate equal to one per cent:

z_0	- 0.9	- 0.7	- 0.5	- 0.3	- 0.1	- 0.01	- 0.001
λ	.03361	.05556	.10890	.30250	2.7225	272.25	27225
λ^Q	.00461	.01912	.05847	.21911	2.4737	269.77	27200
e_p	0.0261	0.0255	0.0252	0.0250	0.0249	0.0248	0.0250

Table 7: The equity premium e_p, the jump frequency λ and the risk adjusted jump frequency λ^Q for various values of the jump size parameter z_0. The short term interest rate $r = .01$, and $\gamma_c = \gamma_d = .73462$.

z_0	0.001	0.01	0.1	0.5	1.0	10	100
λ	27225	272.25	2.7225	.10890	.02723	.00027	$.27225{\cdot}10^{-5}$
λ^Q	27249	274.73	2.9700	.15810	.05157	.00266	$.23927{\cdot}10^{-3}$
e_p	0.0240	0.0248	0.0248	0.0246	0.0245	0.0239	0.0237

Table 8: The equity premium e_p, the jump frequency λ and the risk adjusted jump frequency λ^Q for various values of the jump size parameter z_0. The short term interest rate $r = .01$, and $\gamma_c = \gamma_d = .73462$.

From tables 7 and 8 we notice that the value of the equity premium is rather stable, and fluctuates very little around .025. Even for the extreme values $z_0 = -.9$, $-.7$ and $-.5$ the values of the equity premium is rather close to 2.5 per cent. These latter values of z_0 correspond to a crash economy, where a dramatic downward adjustment occurs very rarely. (For a related, but different, discrete time model of a crash economy, see e.g., Rietz (1988)).

Also the extreme values of z_0 at the other end, 1.0, 10 and 100 corresponding to a bonanza economy with sudden upswings of 100, 1000 and 10,000 per cent respectively, provide values of the equity premium of around 2.4 per cent. We conclude that the values of the equity premium found by this method is robust with respect to the jump size parameter z_0 at the level of accuracy needed here. This holds for the short interest rate $r = .01$.

Tables 9 and 10 give a similar picture for the interest rate $r = .04$, but now the equity premium has changed to about 4.4 per cent.

z_0	- 0.9	- 0.7	- 0.5	- 0.3	- 0.1	- 0.01	- 0.001
λ	.03361	.05556	.1089	.30250	2.7225	272.25	27225
λ^Q	.00018	.00503	.030321	.1623	2.2821	267.78	27181
e_p	0.0301	0.0354	0.0393	0.0421	0.0440	0. 0447	0.0436

Table 9: The equity premium e_p, the jump frequency λ and the risk adjusted jump frequency λ^Q for various values of the jump size parameter z_0. The short term interest rate $r = .04$, and $\gamma_c = \gamma_d = 2.93848$.

z_0	0.001	0.01	0.1	0.5	1.0	10	100
λ	27225	272.25	2.7225	.10890	.02723	.00027	$.27225 \cdot 10^{-5}$
λ^Q	27270	276.75	3.1774	.20379	.07615	.00555	$.53796 \cdot 10^{-3}$
e_p	0.0452	0.0449	0.0455	0.0474	0.0489	0.0528	0.0535

Table 10: The equity premium e_p, the jump frequency λ and the risk adjusted jump frequency λ^Q for various values of the jump size parameter z_0. The short term interest rate $r = .04$, and $\gamma_c = \gamma_d = 2.93848$.

Table 11 gives the connection between the short interest rate and the equity premium in our approach. Although we have only performed the calculations for the jump size value $z_0 = -.01$, we get an indication of this relationship.

r	0.00001	0.001	0.01	0.02	0.03	0.04	0.05	0.06
e_p	1.8%	1.9%	2.5%	3.1%	3.8%	4.5%	5.2%	5.8%

Table 11: The equity premium e_p, as a function of the short term interest rate. $z_0 = -.01, \alpha = 1$.

The tables are consistent with the CCAPM for this particular jump process, a fact we now demonstrate. To this end let us recall an expression for the CCAPM for jump-diffusions (eq. (29) in Aase (2004)):

$$\begin{aligned} e_p =& (RRA)\big(\sigma_c \cdot \sigma_R + \lambda \int_{-1}^{\infty} \int_{-1}^{\infty} z_R z_c F(dz_r, dz_c)\big) \\ &- \lambda (RRA)(RRA+1) \int_{-1}^{\infty} \int_{-1}^{\infty} z_R z_c^2 F(dz_r, dz_c) + \cdots \end{aligned} \tag{23}$$

Here (RRA) stands for the coefficient of relative risk aversion, assumed to be a constant, σ_R is the volatility of the stock index, σ_c the standard deviation of aggregate consumption, and F is the joint probability distribution function of the jumps in the stock index and aggregate consumption. If there are no jump terms, we notice that e_p is proportional to the volatility parameter σ_R. The next term inside the parenthesis is the jump analogue of the first term, and then higher order terms follow. Neglecting the latter for the moment, we notice that for the pure jump model of this section, equation (23) can be written

$$e_p = (RRA)\left(\sqrt{\lambda z_{0,R}^2} \cdot \sqrt{\lambda z_{0,c}^2}\right)$$

(where we have set $\alpha = 1$). Also here we see that e_p is proportional to $(\lambda z_{0,R}^2)^{1/2}$, the volatility of the stock index, neglecting higher order terms. Thus our assumption that this is the case, holds approximately in the model at hand, and the results of this section are accordingly seen to be consistent with the CCAPM.

7.3 The relation to the classical puzzles.

From our results we can say something about the two puzzles mentioned in the introduction. We may reexamine the two puzzles using the above model, and find values for the parameters of the representative agent's utility function for different values of the equity premiums and short term interest rates, calibrated to the first two moments of the US consumption-equity data for the period 1889-1978. Below we present the results without going into details.

Consider first the case where $r = .01$ and the equity premium is 2.5 per cent. (This is, as noted above, not consistent with the Mehra and Prescott (1985) study, where $r = .01$ and the equity premium was 6 per cent.) The jump model can explain a relative risk aversion coefficient of 2.6, which must be considered as a plausible numerical value for this quantity. Turning to the Mehra and Prescott (1985)-case, this value is estimated to 10.2 using the continuous model, which is simply a version of the equity premium puzzle.

For the reexamined values presented in the McGrattan and Prescott (2003) study, the short term interest rate was estimated to be four per cent, with an equity premium of only one per cent. This is not consistent with our approach, which gives the equity premium of about 4.4 per cent in this situation. Our case corresponds to a relative risk aversion of around 3.3, estimated using a jump model.

In both situations above we still get a (slightly) negative value for the subjective interest rate.

8 A combination of the standard model and a Poisson process

We now introduce diffusion uncertainty in the model of the previous section. We choose the standard Black and Scholes model as before for the diffusion part. Taking a look at the equation (9) for γ, at first sight this seems like an easy extension of the last section, including one more term in this equation. But is is more to it than that. First we should determine the market price of risk. We have now two sources of uncertainty, and by "Girsanov type" theorems this would lead to an equation of the form

$$\sigma\theta_1 + \alpha z_0 \lambda \theta_2(z_0) = \mu - r,$$

where θ_1 is the market price of diffusion risk and $\theta_2(z)$ is the market price of jump size risk for any z. This constitutes only one equation in two variables, and has consequently infinitely many solutions, so this model is not complete. The problem is that there is too much uncertainty compared to the number of assets. In the present situation we can overcome this difficulty by introducing one more risky asset in the model. Hence we assume that the market consists of one riskless asset as before, and two risky assets with price processes S_1 and S_2 given by

$$dS_1(t) = S_1(t-)[\mu_1 dt + \sigma_1 dB(t) + \alpha_1 \int_A z_1 \tilde{N}(dt, dz)], \tag{24}$$

where $S_1(0) = x_1 > 0$ and

$$dS_2(t) = S_2(t-)[\mu_2 dt + \sigma_2 dB(t) + \alpha_2 \int_A z_2 \tilde{N}(dt, dz)], \tag{25}$$

where $S_2(0) = x_2 > 0$. Here the set of integration $A = (-1/\alpha_1, \infty) \times (-1/\alpha_2, \infty)$, and $z = (z_1, z_2)$ is two-dimensional. We now choose the following Lévy measure:

$$\nu(dz) = \lambda \delta_{z_{1,0}}(z_1) \delta_{z_{2,0}}(z_2) dz_1 dz_2$$

meaning that at each time τ of jump, the relative size jump in S_1 is $z_{1,0}$ units multiplied by α_1, and similarly the percentage jump in S_2 is $z_{2,0}$ units times α_2. (One could perhaps say that the jump sizes are independent, but since there is just one alternative jump size for each "probability distribution", we get the above interpretation.)

These joint jumps take place with frequency λ. These returns have a covariance rate equal to $\sigma_1\sigma_2$ from the diffusion part and $\lambda\alpha_1\alpha_2 z_{1,0} z_{2,0}$ from the jump part, so the risky assets display a natural correlation structure stemming from both sources of uncertainty. This gives an appropriate generalization of the model of the previous section.

In order to determine the market price of risk for this model, we are led to solving the following two equations:

$$\sigma_1\theta_1 + \alpha_1 \int_A z_1 \theta_2(z) \nu(dz) = \mu_1 - r,$$

and

$$\sigma_2\theta_1 + \alpha_2 \int_A z_2 \theta_2(dz) \nu(dz) = \mu_2 - r.$$

Using the form of the Lévy mesure indicated above, the market price of jump size risk $\theta_2(z) = \theta_2$ a constant when $z = z_0 := (z_{1,0}, z_{2,0})$ and zero for all other values of z. The above two functional equations then reduce to the following set of linear equations

$$\sigma_1\theta_1 + \lambda\alpha_1 z_{1,0}\theta_2 = \mu_1 - r,$$

and

$$\sigma_2\theta_1 + \lambda\alpha_2 z_{2,0}\theta_2 = \mu_2 - r,$$

which leads to the solution

$$\theta_1 = \frac{(\mu_1 - r)\alpha_2 z_{2,0} - (\mu_2 - r)\alpha_1 z_{1,0}}{\sigma_1 \alpha_2 z_{2,0} - \sigma_2 \alpha_1 z_{1,0}}$$

for the market price of diffusion risk, and

$$\theta_2 = \frac{(\mu_1 - r)\sigma_2 - (\mu_2 - r)\sigma_1}{\lambda(\sigma_2 \alpha_1 z_{1,0} - \sigma_1 \alpha_2 z_{2,0})} \tag{26}$$

for the market price of jump size risk, where $\theta_2(z) = \theta_2 I_{\{z_0\}}(z)$, the function $I_B(z)$ being the indicator function of the set B. Here $(\sigma_2\alpha_1 z_{1,0} - \sigma_1\alpha_2 z_{2,0}\sigma_2) \neq 0$, and the constant $\theta_2 \leq 1$. This solution is unique, so the model is complete provided the parameters satisfy the required constraints.

Consider the risk adjusted probability measure Q determined by the pair $(\theta_1, \theta_2(z))$ via the localized, standard density process for the infinite horizon situation of Section 5. If we define $\tilde{N}^Q(dt, dz) := N(dt, dz) - (1 - \theta_2(z))\nu(dz)dt$, and $B^Q(t) := \theta_1 t + B(t)$, then $\int_0^t \int_A \tilde{N}^Q(dt, dz)$ is a local Q-martingale and B^Q is a Q-Brownian motion. The first risky asset can be written under Q,

$$dS_1(t) = S_1(t-)[rdt + \sigma_1 dB^Q(t) + \alpha_1 \int_A z_1 \tilde{N}^Q(dt, dz)], \tag{27}$$

and thus $\bar{S}_1(t) := S_1(t)e^{-rt}$ is a local Q-martingale. A similar result holds for the second risky asset.

We are now in the position to find the solution to the American put problem. Consider the option written on the first risky asset. It follows from the above that the equation for γ can be written

$$\lambda^Q(1 + \alpha_1 z_{1,0})^{-\gamma} = (r - \lambda^Q \alpha_1 z_{1,0})\gamma - \frac{1}{2}\sigma_1^2\gamma(\gamma + 1) + \lambda^Q + r, \tag{28}$$

where $\lambda^Q := \lambda(1 - \theta_2)$, and θ_2 is given by the expression in (26). Again we have dependence from the drift term(s) μ on the risk adjusted probability distribution. Here both of the parameters of the second risky asset enter into the expression for the risk adjusted frequency λ^Q, which means that the market price of jump risk must be determined in this model from equation (26) in order to price the American perpetual put option.

In the case when $z_{1,0} > 0$ (and $\alpha_1 > 0$), this equation can be seen to have one positive solution for $r > 0$. (When $r \leq 0$ there is a range of parameter values where the equation has two positive solutions, then one solution, and finally no solutions.) When $-1 < z_{1,0} < 0$, there is exactly one solution when $r > 0$ (and no positive solutions when $r \leq 0$).

Example 5. In order to compare this situation to the two pure models considered in examples 1 and 2, let us again choose the parameter values such that the variance rates of all three models are equal, but we do not risk adjust the pure jump model, neither do we risk adjust the jump part of the model of this section. This means that we have set $\theta_0 = 0$ and $\theta_2 = 0$. This is accomplished, for example, by choosing $\alpha = 1$ and $\lambda = .7$, $\sigma = .55$ and $z_0 = 1$. For $r = .06$, we get the solution $\gamma_{d,c} = .17$ to the equation (28), while the solution to the equation (21) is $\gamma_d = .20$, and the corresponding solution to the equation (14) is $\gamma_c = .12$. Thus the present solution lies between the two first numerical cases considered in Example 1.

Considering the situation when $z_0 < 0$, we now calibrate to the situation of Example 2. Then we can choose $\alpha = \lambda = 1$ and $\sigma^2 + \lambda\alpha^2 z_0^2 = .25$, which is accomplished, for example, by choosing $\sigma^2 = .125$ and $z_0 = -.35355$. This gives the solution $\gamma_{d,c}$ to the equation (28) equal to $\gamma_{d,c} = .3989$, while the solution to the equation (21) is $\gamma_d = .29$, and the corresponding solution to the equation (14) is $\gamma_c = .48$, still using the same value for the short interest rate. Thus the present solution also lies between the two pure cases in Example 2. □

As a preliminary conclusion to this example we may be led to consider a combined jump-diffusion model as a compromise between the two pure counterparts.

Turning to calibration, with two risky assets we quickly get many parameters, and it is not obvious that we can proceed as before. We choose to equate both the drift rates and the variance rates, but use different characteristics for the latter. This way the market price of risk parameters will be well defined. According to the CCAPM for jump-diffusions we may then get different equity premiums, but the discrepancies will be small if the jump sizes are small, so we shall ignore them here. This may lead to a small discrepancy from the results obtained in Section 6.

Example 6. We choose $\alpha_1 = \alpha_2 = 1$, and consider the two equations $\sigma_c^2 = \sigma_1^2 + \lambda z_{1,0}^2 = .027225$ and $\sigma_c^2 = \sigma_2^2 + \lambda z_{2,0}^2 = .027225$, where we choose $\sigma_1 = .01$ ($\sigma_1 \leq .165$ is the obvious constraint here), and $z_{1,0} = .1$, $z_{2,0} = .01$. This leads to $\lambda = 2.7125$ and $\sigma_2 = .1642$. Then we calibrate the solution $\gamma_{c,d}$ to the equation (28) to the value for the standard continuous model $\gamma_c = .73462$ for the US data, and find that this corresponds to the risk adjusted frequency $\lambda^Q = 2.9593$. Assuming that $(\mu_1 - r) \approx (\mu_2 - r) := (\mu - r)$, the relationship $\lambda^Q = \lambda(1 - \theta_2)$ can now be written, using the expression (26) for the market price of jump risk θ_2:

$$\lambda^Q \approx \lambda + (r - \mu)\left(\frac{\sigma_2 - \sigma_1}{\sigma_2\alpha_1 z_{1,0} - \sigma_1\alpha_2 z_{2,0}}\right).$$

The only unknown quantity in this equation is the equity premium, which leads to the estimate $(r - \mu) = .0261$ when $r = .01$. The market price of diffusion risk $\theta_1 = -.14$, and the market price of jump risk is $\theta_2 = -.09$.

The same procedure when $r = .04$ leads to $\lambda^Q = 3.1658$, and the estimate $(r - \mu) = .048$. This confirms the results of the previous section, within the expected margin of error. Now the market prices of risk are $\theta_1 = -.26$ and $\theta_2 = -.17$. □

9 Different jump sizes.

We now turn to the situation where several different jump sizes can occur in the price evolution of the underlying asset. Suppose the Lévy measure ν is supported on n different points $a_1, a_2, \cdots, a_n$, where $-1 < a_1 < a_2 < \cdots < a_n < \infty$, $a_i \neq 0$ for all i. In our interpretation we may think of the jump size distribution function $F(dz)$ as having n simple discontinuities at each of the numbers $a_1, a_2, \cdots, a_n$ with sizes of the discontinuities equal to $p_1, p_2, \cdots, p_n$, p_i being of course the probability of the jump size a_i, $i = 1, 2, \cdots, n$.

A purely mechanical extenison of the model in section 6.2 leads to an incomplete model, since by proceeding this way we end up with one equation of the type

$$\alpha\lambda \sum_{i=1}^{n} a_i \theta(a_i) p_i = \mu - r,$$

containing n unknown market price of risk parameters $\theta(a_1), \theta(a_2), \cdots, \theta(a_n)$. Instead we consider the following market. A riskless asset exists as before, and n risky assets exist having price processes $S(t) = (S_1(t), S_2(t), \cdots, S_n(t))$ given by

$$\frac{dS_i(t)}{S_i(t-)} = \mu_i dt + \sum_{j=1}^{n} \alpha_{i,j} \int_{-1/\alpha_{i,j}}^{\infty} z\tilde{N}_j(dt, dz), \qquad i = 1, 2, \cdots, n. \tag{29}$$

Here N_j is a Poisson process, of frequency λ_j, times a_j, independent of N_i for $i \neq j$, and $\tilde{N}_j$ is the corresponding compensated process. If we define the return rate process R_i of asset i by $dR_i(t) = \frac{dS_i(t)}{S_i(t-)}$, this means that the jump distribution of R_i is $\alpha_{i,1}a_1$ with probability $p_1 := \frac{\lambda_1}{\sum_{j=1}^n \lambda_j}$, $\alpha_{i,2}a_2$ with probability $p_2 := \frac{\lambda_2}{\sum_{j=1}^n \lambda_j}, \cdots, \alpha_{i,n}a_n$ with probability $p_n := \frac{\lambda_n}{\sum_{j=1}^n \lambda_j}$. The covariance rate between returns R_i and R_j is given by $(\sum_{k=1}^n \alpha_{i,k}\alpha_{j,k}\lambda_k a_k^2)$, which can vary freely because of the relatively large freedom of choice of the parameters $\alpha_{i,j}$. Also note that jumps occur in any of the price processes with frequency $\lambda := \sum_{i=1}^n \lambda_i$.

This model gives us the following n equations to determine the market price of risk processes $\theta(z)$:

$$\sum_{k=1}^{n} \alpha_{i,k}\lambda_k \int_{-1/\alpha_{i,k}}^{\infty} z\theta_k(z)\delta_{\{a_k\}}(z)dz = \mu_i - r, \qquad i = 1, 2, \cdots, n. \tag{30}$$

Since $\delta_{\{a_k\}}(z)$ are the Dirac delta distributions at the points $\{a_k\}$, this system of equations reduce to the following system of n linear equations in n unknowns

$$\sum_{k=1}^{n} \alpha_{i,k} a_k \lambda_k \theta_k = \mu_i - r, \qquad i = 1, 2, \cdots, n, \tag{31}$$

where

$$\theta_i(z) = \begin{cases} \theta_i(a_i) := \theta_i, & \text{if } z = a_i; \\ 0, & \text{otherwise.} \end{cases} \tag{32}$$

This system of equations has a unique solution if the associated coefficient determinant is non-vanishing. The solution to the system (31) is

$$\theta = A^{-1}(\mu - r) \tag{33}$$

where θ is the vector of θ_i's, A^{-1} is the inverse of the matrix A with element $a_{i,j} = \alpha_{i,j}\lambda_j a_j$, $i, j = 1, 2, \cdots, n$, and $(\mu - r)$ is the vector with i-th element equal to $(\mu_i - r)$, $i = 1, 2, \cdots, n$. A unique solution exists when $\det(A) \neq 0$, which is equivalent to $\det(\tilde{\alpha}) \neq 0$, where $\tilde{\alpha}$ is the matrix with (i, j)-th element $\alpha_{i,j}$. As an illustration, if $n = 2$ this means that the requirement is $(\alpha_{1,1}\alpha_{2,2} - \alpha_{2,1}\alpha_{1,2}) \neq 0$.

We now turn to the density process associated with the change of probability measure from P to Q. It is given by

$$\xi(t) := \exp\Big\{ \sum_{i=1}^{n}\sum_{j=1}^{n} \int_0^t \int_{-1/\alpha_{i,j}}^{\infty} [\ln(1-\theta_j)N_j(ds,dz) + \theta_j\lambda_j\delta_{\{a_j\}}(z)dzdt]\Big\}. \quad (34)$$

This means that the restriction Q_t of Q to $\mathcal{F}_t$ is given by $dQ_t(\omega) = \xi(t)dP_t(\omega)$ for any given time horizon t, where P_t is the restriction of P to $\mathcal{F}_t$, and $E\xi(t) = 1$ for all t, then P_t and Q_t assign zero probability to the same events in $\mathcal{F}_t$. As before we call Q the risk adjusted measure for the price system S in the infinite time horizon case. This means that if we define the processes $N_i^Q(dt,dz) := N_i(dt,dz) - (1-\theta_i)\nu_i(dz)dt$, $i = 1,2,\cdots,n$, the processes $\int_0^t \int_{-1/\alpha_{i,j}}^{\infty} \tilde{N}_i^Q(ds,dz)$ are local Q-martingales for $j = 1,2.\cdots,n$, where the risky assets have the following dynamics under Q:

$$\frac{dS_i(t)}{S_i(t-)} = rdt + \sum_{j=1}^{n}\alpha_{i,j}\int_{-1/\alpha_{i,j}}^{\infty} z\tilde{N}_j^Q(dt,dz), \qquad i = 1,2,\cdots,n, \quad (35)$$

implying that $\bar{S}_i(t) := S_i(t)e^{-rt}$ is a zero drift local Q-martingale for all i.

This model is accordingly complete provided $\theta_i \leq 1$ for all $i = 1,2,\cdots,n$, and pricing e.g., the perpetual American put option written on, say, the first asset, is a well defined problem with a unique solution. The equation for γ for this option can be written

$$-r - r\gamma + \sum_{j=1}^{n}\lambda_j^Q[(1+\alpha_{1,j}a_j)^{-\gamma} - 1 + \alpha_{1,j}a_j\gamma] = 0, \quad (36)$$

where $\lambda_j^Q = \lambda_j(1-\theta_j)$, $j = 1,2,\cdots,n$. Note that this equation follows from equation (9) after the appropriate risk adjustments of the various frequecies, if we set $\alpha_{1,j} = \alpha$ for all j, and Lévy measure $\nu(dz) = \lambda^Q F(dz)$, where the probability distribution function F has discontinuities at the points $a_1, a_2, \cdots, a_n$ with probabilities $p_1^q, p_2^q, \cdots, p_n^q$, where $p_i^q = \frac{\lambda_i^Q}{\lambda^Q}$ and λ^Q is the frequency of the jumps under the probability Q ($\lambda^Q = \sum_i \lambda_i^Q$). Thus our model captures the general situation, under P, with a frequency of jumps equal to λ and a pdf of jump sizes F with support on n different points. Here is an example:

Example 7. We consider the case of $n = 2$, where we do not adjust for risk. The parameters $\alpha_{1,1} = \alpha_{1,2} = 1$, and $a_1 = -.5$, $a_2 = 1$ so that each price jump either cuts the price in half, or doubles the current price of the underlying asset. We let $\lambda_1 = \lambda_2 = 1$ so that the two different jump sizes are equally probable under P, and the total frequency λ of jumps equals two per time unit.

Fixing $r = .06$ as before, we get the solution $\gamma_{2d} = .12$ to the equation (36). In order to compare to the standard model and the model with only upward jumps of the same size ($= 1$) of Example 1, we find the corresponding γ-values adjusted so that the various variance rates are equal. They are $\gamma_c = .10$ for $\sigma^2 = 5/4$ and $\gamma_d = .16$ for $\lambda = 5/4, \alpha = 1$, and $z_0 = 1$ in the jump model. Since $\gamma_c < \gamma_{2d} < \gamma_d$, the corresponding American perpetual put option prices are ranked $\psi_c > \psi_{2d} > \psi_d$.

In this situation, when both upward and downward sudden jumps are possible in the price paths of the underlying asset, the corresponding put price is between the polar cases of only continuous movements or only upward jumps.

Comparing to the situation with only downward jumps of size $-.5$ of Example 2, this is calibrated to have the same variance by choosing $z_0 = -.5, \lambda = 5, \alpha = 1$ which gives $\gamma_d = .06$. Thus we get $\psi_d > \psi_c > \psi_{2d}$, so here the situation with two jumps reflects the "least risky" situation. $\square$

We notice that also in the situation with several jumps prices of contingent claims depend on the drift rates μ_i of the basic risky assets. In addition to requiring a risk adjustment of the frequencies λ_i, the *probabilities* p_i of the different jump sizes must also be risk adjusted under Q. Thus the system (31) of n linear equations in n unknowns for the market prices of jump risk θ_i must be solved in order to correctly price options and other contingent claims in this model.

9.1 Calibration when $n = 2$

Suppose we want to calibrate this model to the data from the Standard and Poor's composite stock index during the time period 1889-1979, as we did in Section 7. Since we only have estimates of the short time interest rate and the stock index volatility, we are left with too many parameters to estimate from too few data points, and can not expect to get the type of results as we did before. Nevertheless, below we present a numerical example when $n = 2$. For the model to be complete, we need one risky asset in addition to the index, so we assume the model consists of the two following risky assets:

$$\frac{dS_1(t)}{S_1(t-)} = \mu_1 dt + \alpha_{1,1} \int_{-1/\alpha_{1,1}}^{\infty} z\tilde{N}_1(dt, dz) + \alpha_{1,2} \int_{-1/\alpha_{1,2}}^{\infty} z\tilde{N}_2(dt, dz), \qquad (37)$$

and

$$\frac{dS_2(t)}{S_2(t-)} = \mu_2 dt + \alpha_{2,1} \int_{-1/\alpha_{2,1}}^{\infty} z\tilde{N}_1(dt, dz) + \alpha_{2,2} \int_{-1/\alpha_{2,2}}^{\infty} z\tilde{N}_2(dt, dz). \qquad (38)$$

From the solution (33) of the system of equations (31) when $n = 2$, we get for the market prices of risk parameters θ_1 and θ_2 the following two expressions:

$$\theta_1 = \frac{\alpha_{2,2}(\mu_1 - r) - \alpha_{1,2}(\mu_2 - r)}{\lambda_1 a_1(\alpha_{1,1}\alpha_{2,2} - \alpha_{1,2}\alpha_{2,1})}, \qquad (39)$$

and

$$\theta_2 = \frac{\alpha_{2,1}(\mu_1 - r) - \alpha_{1,1}(\mu_2 - r)}{\lambda_2 a_2(\alpha_{1,2}\alpha_{2,1} - \alpha_{1,1}\alpha_{2,2})}. \qquad (40)$$

From the equations $\lambda_i^Q = \lambda_i(1 - \theta_i), i = 1, 2$, we find the risk adjusted frequencies,

$$\lambda_1^Q = \lambda_1 + \frac{\alpha_{2,2}(r - \mu_1) - \alpha_{1,2}(r - \mu_2)}{a_1(\alpha_{1,1}\alpha_{2,2} - \alpha_{1,2}\alpha_{2,1})}, \qquad (41)$$

and

$$\lambda_2^Q = \lambda_2 + \frac{\alpha_{2,1}(r - \mu_1) - \alpha_{1,1}(r - \mu_2)}{a_2(\alpha_{1,2}\alpha_{2,1} - \alpha_{1,1}\alpha_{2,2})}. \qquad (42)$$

We must choose the constants in the matrix $\tilde{\alpha}$ such that the determinant $(\alpha_{1,1}\alpha_{2,2}-\alpha_{1,2}\alpha_{2,1}) \neq 0$. Choosing the first risky asset similar to the composite stock index, its variance rate must satisfy

$$\alpha_{1,1}^2\lambda_1 a_1^2 + \alpha_{1,2}^2\lambda_2 a_2^2 = \sigma^2, \tag{43}$$

where $\sigma^2 = 0.027225$ as for the index. The variance rate of the second risky asset is given by

$$\alpha_{2,1}^2\lambda_1 a_1^2 + \alpha_{2,2}^2\lambda_2 a_2^2. \tag{44}$$

In equilibrium there is sometimes a connection between the equity premiums and the standard deviation rate, which we now wish to utilize. By the CCAPM for jump-diffusions, while a linear relationship is almost exact for the model of Section 7, for the present model this is no longer the case. By Schwartz's inequality this linear relationship is at the best approximately true when the jump sizes are small and different in absolute value, as can be deduced from the result (23). Assuming we can use this approximation here, we get the following:

$$(\mu_2 - r) \approx (\mu_1 - r)\sqrt{\frac{\alpha_{2,1}^2\lambda_1 a_1^2 + \alpha_{2,2}^2\lambda_2 a_2^2}{\alpha_{1,1}^2\lambda_1 a_1^2 + \alpha_{1,2}^2\lambda_2 a_2^2}}. \tag{45}$$

We are now in position to derive an approximate expression for the equity premium $e_p = (r - \mu_1)$. Using (45) in the expressions (41) and (42), we get $\lambda_1^Q = \lambda_1 + k_1 e$ and $\lambda_2^Q = \lambda_2 + k_2 e$, where

$$k_1 = \frac{\alpha_{2,2} - \alpha_{1,2}\sqrt{\frac{\alpha_{2,1}^2\lambda_1 a_1^2 + \alpha_{2,2}^2\lambda_2 a_2^2}{\alpha_{1,1}^2\lambda_1 a_1^2 + \alpha_{1,2}^2\lambda_2 a_2^2}}}{a_1(\alpha_{1,1}\alpha_{2,2} - \alpha_{1,2}\alpha_{2,1})},$$

and

$$k_2 = \frac{\alpha_{2,1} - \alpha_{1,1}\sqrt{\frac{\alpha_{2,1}^2\lambda_1 a_1^2 + \alpha_{2,2}^2\lambda_2 a_2^2}{\alpha_{1,1}^2\lambda_1 a_1^2 + \alpha_{1,2}^2\lambda_2 a_2^2}}}{a_2(\alpha_{1,2}\alpha_{2,1} - \alpha_{1,1}\alpha_{2,2})}.$$

Inserting these expressions in the equation (36) for γ when $n = 2$, we get a linear equation for e_p, which solution is

$$\begin{aligned} e_p = &\big[r(\gamma+1) + \lambda_1\big((1 - a_1\alpha_{1,1}\gamma) - (1 + a_1\alpha_{1,1})^{-\gamma}\big) \\ &+ \lambda_2\big((1 - a_2\alpha_{1,2}\gamma) - (1 + a_2\alpha_{1,2})^{-\gamma}\big)\big] / \big[k_1\big((1 + a_1\alpha_{1,1})^{-\gamma} - (1 - a_1\alpha_{1,1}\gamma)\big) \\ &+ k_2\big((1 + a_2\alpha_{1,2})^{-\gamma} - (1 - a_2\alpha_{1,2}\gamma)\big)\big]. \end{aligned} \tag{46}$$

A numerical example is the following.

Example 8. Choosing the parameters $\alpha_{1,1} = \alpha_{2,2} = \alpha_{1,2} = 1$ and $\alpha_{2,1} = 2$, the absolute value of the determinant $|\tilde{\alpha}|$ equals one, so the risk premiums are well defined. We choose $a_1 = 0.02$ and $a_2 = -0.01$, and $p_1 = 0.5$, and consider first the case where the short term interest rate $r = 0.01$. Since $p_1 = \lambda_1/(\lambda_1 + \lambda_2)$, we obtain that $\lambda_1 = \lambda_2 = 54.45$ from equation (43). From the relation (45) we find that $(r-\mu_2) = 1.84(r-\mu_1)$, and this enables us to compute the market price of risk parameters θ_1 and θ_2, and hence the risk adjusted frequencies, which are

$$\lambda_1^Q = 54.45 + 42.16(r-\mu_1), \qquad \lambda_2^Q = 54.45 - 15.69(r-\mu_1)$$

in terms of the equity premium $(r-\mu_1)$ of the index. By inserting these values in the equation (36) for γ_{2d}, we can find the value of the risk premium that satisfies $\gamma_{2d} = \gamma_c$, where γ_c is the corresponding solution for the standard model. For $r = 0.01$ this value is $\gamma_c = 0.73462$. This calibration gives the value $(r-\mu_1) = 0.0226$, or 2.26 per cent equity premium for the composite stock index. The forgoing can alternatively (and computationally less requiring) be accomplished by using $\gamma = \gamma_c$ in the expression for e given in (46), together with the other parameter values indicated.

A similar procedure for the spot rate $r = 0.04$ calibrates γ_{2d} to $\gamma_c = 2.93848$, and this gives $(r-\mu_1) = 0.041$, or an equity premium of 4.1 per cent for the stock index. Both these values are reasonably close to the values obtained in Section 7. □

In the above example the expected return per incident is $\lambda(\alpha_{1,1}a_1p_1+\alpha_{1,2}a_2p_2) = .5445$. While there is nothing pathological about this since the compensator secures that the noise term has zero expected value per unit time, we would nevertheless like to control the input to the equation (46). We calibrate the frequencies λ_1 and λ_2 using equations (I) and(II) as follows

$$(I) \qquad \lambda(\alpha_{1,1}a_1p_1 + \alpha_{1,2}a_2p_2) = R_e,$$
$$(II) \qquad \lambda(\alpha_{1,1}^2a_1^2p_1 + \alpha_{1,2}^2a_2^2p_2) = \sigma_c^2 = 0.027225.$$

for various values of R_e. Using the parameter values for $\alpha_{i,j}$ as in Example 8, some results are the following.

a_1	.015	.02	.025	.03	.006	.002	.002	.002
a_2	- .007	- .006	- .004	-.002	- .02	- .03	- .04	- .05
e_p	0.0211	0.0227	0.0239	0.0244	0.0285	0.0256	0.0254	0.0253

Table 12: The equity premium e_p when $r = 0.01$ and $R_e = 0.001$ for various values of the parameters.

The left half of Table 12 has $a_1 > |a_2|$ in which case e_p is smaller than the value .025 of Section 7, in the second half $a_1 < |a_2|$ in which case the equity premiums are larger than .025. By varying the parameters, we find considerably more variation in the values of e_p than for the simple model of Section 7 (not shown in the table). This seems natural since the present model is more complex, and it is not reasonable that a single quantity like the variance rate is sufficient to determine the equity premium. This can also be confirmed by consulting the CCAPM for this model.

Turning to the case when the interest rate $r = 0.04$ we have the following. When the absolute values of a_1 and a_2 differ the most, we get a situation similar to the one in Section 7 in which case our assumption (45) becomes reasonable, and the equity premiums are close to those obtained earlier. In general we can obtain a wide range of different risk premiums here, by varying the parameters, simply confirming that our assumptions are not valid for this model in general.

a_1	.012	.012	.012	.04	.007	.005	.003	.002
a_2	- .008	- .008	- .008	-.002	- .015	- .02	- .03	- .04
R_e	- 0.1	- 0.001	0.001	0.025	0.025	0.025	0.025	0.025
e_p	0.0315	0.0326	0.0336	0.0445	0.0570	0.0500	0.0468	0.0456

Table 13: The equity premium e_p when $r = 0.04$, for various values of the parameters.

The above example mainly serves to illustrate how to proceed in the general case. For a particular choice of jump size parameters a_i one would need estimates of the corresponding jump probabilities p_i, or frequencies λ_i. The next step is to estimate volatilities, and connect the various equity premiums using the CCAPM. This is subsequently used to compute the market prices of risk, using the solution in (33), which then yield risk adjusted frequencies $\lambda_j^Q = \lambda_j(1-\theta_j)$ necessary as inputs for the equation (36). By leaving the risk premium of the first risky asset as a free parameter, one can finally calibrate the resulting model to the standard one, for example, and thus obtain option based estimates of the equity premiums, without the use of consumption data. This would be the procedure to follow if the assumed linear relationship between the equity premium and the standard deviation rate is reasonably accurate.

However, since our results for the present model indicate that this linear relationship does not hold, the calibration to the continuous, standard model becomes less interesting. Instead one could proceed as follows: (a) Observe option prices in the market. (b) Estimate the parameters of the index from historical observations. From this one could find a market estimate of γ. Then the correct version of the CCAPM should be used to improve the approximation (45), and finally use the corresponding expression to (46) to compute e_p. This procedure would presumably need some consumption data when using the CCAPM.

10 A combination of the standard model and the jump model with different jump sizes.

By introducing also diffusion uncertainty in the model of the previous section, there will be "too much uncertainty" compared to the number of assets, but again we may use the method of Section 8 to enlarge the space of jumps and add one risky asset. This will lead to a complete model, and our valuation problem again becomes well defined. The equation for γ is now

$$-r - r\gamma + \frac{1}{2}\sigma_1^2\gamma(\gamma+1) + \sum_{j=1}^{n} \lambda_j^Q[(1+\alpha_{1,j}a_j)^{-\gamma} - 1 + \alpha_{1,j}a_j\gamma] = 0, \tag{47}$$

where σ_1 is the volatility parameter of the continuous part of the first risky asset.

Example 9. We compare the combined model of this section, not adjusted for jump risk, with the purely discontinuous model of the previous section, also not risk adjusted, and the standard, continuous model. Using the same parameter values as in the first part of Example 7, we again obtain $\gamma_{2d} = .12$, and $\gamma_c = .10$ from the standard model having the same variance rate. Choosing $\alpha_{1,1} = \alpha_{1,2} = .9$ and $\sigma^2 = .2375$, the combined model of this section has the same variance as the other two models. This gives the solution to equation (47) equal to $\gamma_{2d,c} = .11$. Thus $\gamma_c < \gamma_{2d,c} < \gamma_{2d}$, or $\psi_{2d} > \psi_{2d,c} > \psi_c$ so that the combined model fits in between the two pure models, as we also saw in Example 5. $\square$

11 The model with a continuous jump size distribution.

Finally we consider the situation with a continuous distribution for the jump sizes, and for simplicity we only consider the jump part of the model. In this case the model is incomplete as long as there is a finite number of assets, since there is "too much uncertainty" compared to the number of assets.

The case with countably many jump sizes in the underlying asset could be approached as in Section 9, by introducing more and more risky assets. In order for the market prices of risk $\theta_1, \theta_2, \cdots$ to be well defined, presumably only mild additional technical conditions need to be imposed. One line of attack is to weakly approximate any such distribution by a sequence of discrete distributions with finite supports. This would require more and more assets, and in the limit, an infinite number of primitive securities in order for the model to possibly be complete.

Here we will not elaborate further on this, but only make the assumption that the pricing rule is linear, which would be the case in a frictionless economy where it is possible to take any short or long position. This will ensure that there is *some* probability distribution and frequency for the jumps giving the appropriate value for γ, corresponding to a value for the American perpetual put option.

Below we limit ourselves to a discussion of the prices obtained this way for two particular choices of the jump distribution, where the risk adjustment is carried out mainly through the frequency of jumps.

The model is the same as in Section 2 with one risky security S and one locally riskless asset β. The risky asset has price process S satisfying

$$dS_t = S_{t-}[\mu dt + \alpha \int_{-1/\alpha}^{\infty} z\tilde{N}(dt, dz)], \tag{48}$$

where the density process of S is given by

$$\xi(t) = \exp\{\int_0^t \int_{-1/\alpha}^{\infty} \ln(1-\theta(z))N(ds, dz) + \int_0^t \int_{-1/\alpha}^{\infty} \theta(z)\lambda F(dz)ds\}, \tag{49}$$

Here $\theta(z)$ is the market price of risk process and $F(dz)$ is the distribution function of the jump sizes, assumed absolutely continuous with a probability density $f(z)$. According to our results in Section 5, if the market price of risk satisfies the following equation

$$\int_{-1/\alpha}^{\infty} z\theta(z)f(z)dz = \frac{\mu - r}{\lambda\alpha}, \tag{50}$$

then the risk adjusted compensated jump process can be written

$$\tilde{N}^Q(dt, dz) = N(dt, dz) - (1-\theta(z))\lambda f(z)dzdt. \tag{51}$$

This means that the term

$$\lambda^Q f^Q(z) := \lambda(1-\theta(z))f(z) \tag{52}$$

determines the product of the risk adjusted frequency λ^Q and the risk adjusted density $f^Q(z)$, when θ satisfies equation (50). If the market price of risk θ is a constant, there is no risk adjustment of the density $f(z)$. The densities $f(z)$ and $f^Q(z)$ are mutually absolutely continuous with respect to each other, which means in particular that the domains where they are both positive must coincide.

Clearly the equation (50) has many solutions θ, so the model is incomplete.

In solving the American perpetual put problem for this model, it follows from our previous results that the equation for γ is given by

$$-r - r\gamma + \int_{-1/\alpha}^{\infty} \{(1+\alpha z)^{-\gamma} - 1 + \alpha\gamma z\}\lambda^Q f^Q(dz) = 0, \tag{53}$$

where we have carried out the relevant risk adjustments. We now illustrate by considering two special cases for the jump density $f(z)$.

11.1 The truncated normal case

Here we analyze normally distributed returns. In our model formulation, where we have chosen the stochastic exponential, we observe from the expression (2) for S that we can not allow jump sizes less than $-1/\alpha$, so the domain of F is the interval $[-1/\alpha, \infty)$. In this case we choose to consider a truncated normal distribution at $-1/\alpha$. By and large we restrict our attention to risk adjustments associated with a constant θ only. In the present case the most straightforward risk adjustments of the normal density $f(z)$ having mean m and standard deviation s would be another normal density having mean m^Q and standard deviation s^Q, with a similar adjustment for the truncated normal distribution. Here we only notice that a joint risk adjustment of the jump distribution f to another truncated normal with parameters m^Q and s^Q, and of the frequency λ to λ^Q, means that the equity premium can be written

$$e_p = \alpha\big(\lambda^Q E^Q\{Z|m^Q, S^Q\} - \lambda E\{Z|m, s\}\big), \tag{54}$$

where the expectations are taken of the truncated normal random variable Z with respect to the parameters indicated. The above formula then follows from (50) and (52). Notice that α does not change under the measure Q, since the supports of f and f^Q must coincide.

The exponential pricing model with normal jump sizes was considered by Merton (1976). In that case the probability density of the pricing model S_t is known explicitly. In contrast to Merton, who assumed that the jump size risk was not priced, or, he did not adjust for this type of risk, we will risk adjust precisely the jump risk, and our model is the stochastic exponential, not the exponential.

Below we have calibrated this model to the continuous one using the same technique as outlined above. Since the equity premium is not proportional to the volatility of S in this model, we can not expect to confirm the simple results of Section 7. For Z a random variable with a truncated normal distribution at $-1/\alpha$, we first solve the equation $\lambda\alpha^2 E(Z^2) = \sigma_c^2 = .027225$, or

$$\lambda\alpha^2 \frac{\int_{-1/\alpha}^{\infty} z^2 \frac{1}{\sqrt{2\pi}s} e^{-\frac{1}{2}\left(\frac{z-m}{s}\right)^2} dz}{\int_{-1/\alpha}^{\infty} \frac{1}{\sqrt{2\pi}s} e^{-\frac{1}{2}\left(\frac{z-m}{s}\right)^2} dz} = \sigma_c^2$$

for various values of m and s, and find the frequency λ. Then we solve equation (53), using the relevant values for r and $\gamma = \gamma_c(r)$, to find the risk adjusted frequency λ^Q, and finally we use equation (50) to find the equity premium $e_p = (r - \mu)$, assuming θ is a constant, so that $\lambda^Q = \lambda(1-\theta)$ and $f = f^Q$. Some results are summarized in tables 14 and 15.

α	1	1	1	.01	.8	3
(m, s)	(.1, .1)	(.4, .7)	(.4, 2.0)	(10, 10)	(.01, .01)	(.01, .01)
λ	1.36	.042	0.0065	1.36	212.70	15.13
λ^Q	1.60	.079	.019	1.60	215.78	15.94
e_p	0.024	0.026	0.025	0.024	0.025	0.024

Table 14: The equity premium e_p when $r = 0.01$ and $\gamma = 0.73462$, for various values of the parameters. The jumps are truncated normally distributed.

By decreasing the parameter α we notice from the above equation that this has the effect of increasing the frequency of jumps λ. Alternatively this can be achieved by decreasing the values of m and s, as can be observed in Table 15, where the spot rate is equal to 4 per cent. A decrease in the standard deviation s, within certain limits, moves the present model closer to the one of Section 7.

α	1	1	1	.9	2	10
(m, s)	(.1, .1)	(-.01, .01)	(1.0, 0.1)	(.01, .01)	(.011, .01)	(.01, .01)
λ	1.36	136.13	0.027	136.06	30.80	1.36
λ^Q	1.79	131.62	.076	173.01	32.91	1.79
e_p	0.043	0.045	0.049	0.045	0.046	0.043

Table 15: The equity premium e_p when $r = 0.04$ and $\gamma = 2.93848$, for various values of the parameters. The jumps are truncated normally distributed.

11.2 Exponential tails

In this model the distribution of the jump sizes is an asymmetric exponential with density of the form

$$f(z) = pae^{-a|z|} I_{[-\infty,0]}(z)/(1 - e^{-a/\alpha}) + (1-p)be^{-bz} I_{[0,\infty]}(z)$$

with $a > 0$ and $b > 0$ governing the decay of the tails for the distribution of negative and positive jump sizes and $p \in [0,1]$ representing the probability of a negative jump. Here $I_A(z)$ is the indicator function of the set A. The probability distribution of returns in this model has semi-heavy (exponential) tails. Notice that we have truncated the left tail at $-1/\alpha$. The exponential pricing version of this model, without truncation, has been considered by Kou (2002).

Below we calibrate this model along the lines of the previous section. Also here we restrict attention to risk adjusting the frequency only. We then have the following expression for the equity premium:

$$e_p = \alpha(\lambda^Q - \lambda)\Big(p\big(\frac{e^{-a/\alpha}}{\alpha(1-e^{-a/\alpha})} - \frac{1}{a}\big) + (1-p)\frac{1}{b}\Big), \tag{55}$$

where the frequency is risk adjusted, but not f. A formula similar to (54) can be obtained if also the density f is to be adjusted for risk. The simplest way to accomplish this here is to consider another probability density f^Q of the same type as the above f with strictly positive parameters p^Q, a^Q and b^Q. This would constitute an absolutely continuous change of probability density, but there are of course very many other possible changes that are allowed. In finding the expression (55) we have first solved the equation (50) with a constant θ, and then substituted for the market price of risk using the equation $\lambda^Q = \lambda(1-\theta)$.

Proceeding as in the truncated normal case, we first solve the equation $\lambda\alpha^2 E(Z^2) = \sigma_c^2 = 0.027225$, which can be written

$$\lambda\alpha^2\Big(p\big(1-e^{-a/\alpha}\big)^{-1}\big(\frac{2}{a^2} - e^{-a/\alpha}\big(\frac{1}{\alpha^2} + \frac{2}{a\alpha} + \frac{2}{a^2}\big)\big) + (1-p)\big(\frac{2}{b^2}\big)\Big) = \sigma_c^2. \tag{56}$$

Then we determine reasonable parameters through the equation $\alpha E(Z) = R_e$ for various values of R_e. This equation can be written:

$$R_e := \alpha\Big(p\Big(\frac{e^{-a/\alpha}}{\alpha(1-e^{-a/\alpha})} - \frac{1}{a}\Big) + (1-p)\frac{1}{b}\Big). \tag{57}$$

In order to arrive at reasonable values for the various parameters, we solve the two equations (56) and (57) in a and b for various values of the parameters α, p and R, where we have fixed the value of $\lambda = 250$. Then for the spot rates $r = 0.01$ and $r = 0.04$ with corresponding values of $\gamma = \gamma_c(r)$ respectively, we solve the equation (53) to find the value of λ^Q. Finally we compute the value of the equity premium from the formula (55). Some results are the following:

(α, p)	(1, .45)	(1, .55)	(1, .60)	(.01, .40)	(.01, .45)	(.01, .60)
R_e	.004	-.004	-.004	.0045	.004	.0035
a	350.23	104.07	110.34	3.76	3.50	5.54
b	140.07	350.23	278.21	1.08	1.04	.87
λ^Q	255.92	243.92	244.55	255.85	252.71	257.41
e_p	0.024	0.024	0.022	0.026	0.024	0.026

Table 16: The equity premium e_p when $r = 0.01$ and $\gamma = .73462$, for various values of the parameters, where $\lambda = 250$. The jumps are truncated, asymmetric exponentials.

(α, p)	(1, .40)	(1, .45)	(1, .60)	(.01, .40)	(.01, .45)	(.01, .60)
R_e	- .0035	-.0035	-.0035	.0045	.004	.0035
a	87.29	93.62	113.58	3.76	3.50	5.54
b	554.30	420.88	224.41	1.08	1.04	.87
λ^Q	236.24	237.53	241.29	260.54	260.67	263.38
e_p	0.048	0.044	0.048	0.047	0.043	0.047

Table 17: The equity premium e_p when $r = 0.04$ and $\gamma = 2.93848$, for various values of the parameters, where $\lambda = 250$. The jumps are truncated, asymmetric exponentials.

Since the equity premium is not proportional to the volatility of S in this model, we can not expect to obtain the simple and unique results of Section 7. As in the case of several jumps in Section 9 and the truncated normal case of the previous section, we typically get a wide variety of equity premiums for a given standard deviation of the price process, as the parameters vary. There is simply too much freedom in these models to obtain the unique results of Section 7. The volatility of the stock is not a sufficient statistic for its risk premium in these models.

The tables 14-17, as well as tables 12 and 13, all identify parameters that are consistent of the simple results obtained in Section 7, and are not meant to be representative of the variation one may obtain for e_p. Obviously there is a large amount of parameter values that satisfy this. These tables primarily illustrate numerical solutions of the basic equation (9) for γ.

12 Conclusions

In this paper we have solve an optimal stopping problem with an infinite time horizon, when the state variable follows a jump-diffusion. Under certain conditions, explained in the paper, our solution can be interpreted as the price of an American perpetual put option, when the underlying asset follows this type of process.

The probability distribution under the risk adjusted measure turns out to depend on the equity premium for this type of model, which is not the case for the standard, continuous version. This difference is utilized to find intertemporal, equilibrium equity premiums in a simple model, where the equity premium is proportional to the volatility of the asset.

We applied this technique to the US equity data of the last century, and found an indication that the risk premium on equity was about two and a half per cent if the risk free short rate was around one per cent. On the other hand, if the latter rate was about four per cent, we similarly find that this corresponds to an equity premium of around four and a half per cent.

The advantage with our approach is that we needed only equity data and option pricing theory, no consumption data was necessary to arrive at these conclusions.

Various market models were studied at an increasing level of complexity, ending with the incomplete model in the last part of the paper. In these models the equity premiums are no longer proportional to the volatility of the assets, and further econometric analyses would be needed to test our simple results obtained is Section

7 of the paper. The relevant computations needed in such an analysis are explained and illustrated.

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