Optimal Risk Sharing

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Abstract

Optimal risk sharing is considered from the perspective of the risk sharing model introduced by Karl Borch in the late 50ies.

First we introduce, in a modern setting, the main concepts from this theory. These we apply on the risk sharing problem between an insurer and an insurance customer. We motivate the development through a simple example, illustrating some fine points of this theory.

In order to explain deductibles, we separately introduce (i) costs, and (ii) moral hazard in the neoclassical model, the latter case also illustrated by an example.

KEYWORDS: Reinsurance Exchange, Equilibrium, Pareto Optimality, Representative Agent, Core Solution, Individual Rationality, Deductibles, Costs, Moral Hazard.

Introduction

We first study the following model: Let $\mathcal{I} = \{1, 2, \dots, I\}$ be a group of I reinsurers, simply termed agents for the time being, having preferences $\succeq_i$ over a suitable set of random variables. These preferences are represented by expected utility, meaning that there is a set of Bernoulli utility functions $u_i : R \to R$, such that $X \succeq_i Y$ if and only if $Eu_i(X) \geq Eu_i(Y)$. We assume smooth utility functions; here $u_i'(w) > 0, u_i''(w) \leq 0$ for all w in the relevant domains, for all $i \in \mathcal{I}$.

Each agent is endowed with a random payoff X_i called his initial portfolio. Uncertainty is objective and external, and there is no informational asymmetry. All parties agree upon the space $(\Omega, \mathcal{F}, P)$ as the probabilistic description of the stochastic environment, the latter being unaffected by their actions. Here Ω is the set of states of the world, $\mathcal{F} = \mathcal{F}^X := \sigma(X_1, X_2, \dots, X_I)$ is the set of events, and P is the common belief probability measure. It will be convenient to posit that both expected values and variances exist for all the initial portfolios, which means that all $X_i \in L^2(\Omega, \mathcal{F}, P)$, or just $X_i \in L^2$ for short.

We suppose the agents can negotiate any affordable contracts among themselves, resulting in a new set of random variables $Y_i, i \in \mathcal{I}$, representing the possible final payout to the different members of the group, or final portfolios. The transactions are carried out right away at "market prices", where $\pi(Y)$ represents the market price for any $Y \in L^2$, i.e., it signifies the group's valuation of the random variable Y relative to the other random variables in L^2. The essential objective is then to determine:

(a) The market price $\pi(Y)$ of any "risk" $Y \in L^2$ from the set of preferences of the agents and the joint probability distribution $F(x_1, x_2, \dots, x_I)$ of the random vector $X = (X_1, X_2, \dots, X_I)$.

(b) For each i, the final portfolio Y_i most preferred by him among those satisfying his budget constraint $\pi(Y_i) \leq \pi(X_i)$.

Equilibrium

Unless the functional π on L^2 is *linear*, arbitrage would be possible. Since we allow all kinds of contract formation (complete market), and the agents prefer more to less, we require that there should not be any arbitrage (i.e., contracts with positive payouts that cost nothing). Hence linearity of π follows. Also, the pricing functional π should be *positive*, meaning simply that $\pi(Z) \geq 0$ for any $Z \geq 0$ P-a.s. We know that a linear and positive functional on an L^p-space is continuous, $1 \leq p < \infty$, so by the Riesz representation theorem there exists a unique random variable $\xi \in L^2$ such that

$$\pi(Z) = E(Z\xi) \qquad \text{for all } Z \in L^2.$$

An allocation $Z = (Z_1, Z_2, \dots, Z_I)$ is called *feasible* if

$$\sum_{i=1}^{I} Z_i \leq \sum_{i=1}^{I} X_i := X_M.$$

The problem each agent is supposed to solve is the following:

$$\sup_{Z_i \in L^2} Eu_i(Z_i) \qquad \text{subject to} \qquad \pi(Z_i) \leq \pi(X_i). \tag{1}$$

Definition 1 *A competitive equilibrium is a collection* $(\pi; Y_1, Y_2, \dots, Y_I)$ *consisting of a price functional* π *and a feasible allocation* $Y = (Y_1, Y_2, \dots, Y_I)$ *such that for each* i, Y_i *solves the problem (1) and markets clear;* $\sum_{i=1}^I Y_i = \sum_{i=1}^I X_i$.

We close the system by assuming *rational expectations*. This means that the market clearing price π implied by agent behavior is assumed to be the same as the price functional π on which agent decisions are based. The main analytic issue is then the determination of equilibrium price behavior. We have the following

Theorem 1 *Suppose the preferences of the agents are strictly monotonic and convex, i.e.,* $u_i' > 0$ *and* $u_i'' \leq 0$ *for all* $i \in \mathcal{I}$, *and assume that a competitive equilibrium exists, where* $\pi(X_i) > 0$ *for each* i. *The equilibrium is then characterized by the existence of positive constants* α_i, $i \in \mathcal{I}$, *such that for the equilibrium allocation* $(Y_1, Y_2, \dots, Y_I)$

$$u_i'(Y_i) = \alpha_i \xi, \qquad a.s. \qquad for\ all \qquad i \in \mathcal{I}, \tag{2}$$

where ξ *is the Riesz representation of the pricing functional* π.

The random variable $\xi \in L^2$, here the dual space, is called the *state price deflator*.

A modern proof of this theorem, as well as many results to follow, can be found in Aase (2002), where also an extensive bibliography is given. A proof can be built on the Kuhn-Tucker Theorem and directional derivatives in function space.

Existence of equilibrium in infinite dimensional economies was dealt with by Bewley (1972). In many spaces, like L^2, the positive cone has an empty interior, creating problems with standard separating methods. A key concept here is properness, introduced by by Mas-Colell (1986). This was used in Aase (1993a) to give a set of sufficient conditions for equilibrium in the reinsurance model. Typically, the relevant conditions must restrict both preferences and the joint probability distribution $F(x)$ of X.

Pareto Optimality

Next we introduce the concept of (strong) *Pareto optimality* of an allocation.

Definition 2 *A feasible allocation* $Y = (Y_1, Y_2, \dots, Y_I)$ *is called Pareto optimal if there is no feasible allocation* $Z = (Z_1, Z_2, \dots, Z_I)$ *with* $Eu_i(Z_i) \geq Eu_i(Y_i)$ *for all* i *and with* $Eu_j(Z_j) > Eu_j(Y_j)$ *for some* j.

An important neoclassical result is that any competitive equilibrium is Pareto optimal.

In order to properly formulate our next fundamental result, consider for each nonzero vector $\lambda \in R^I_+$ of agent weights the function $u_\lambda(\cdot) : R \to R$ defined by

$$u_\lambda(v) =: \sup_{(z_1,\dots,z_I)} \sum_{i=1}^{I} \lambda_i u_i(z_i) \qquad \text{subject to} \qquad \sum_{i=1}^{I} z_i \leq v. \tag{3}$$

As the notation indicates, this function depends only on the variable v, meaning that if the supremum is attained at the point $(y_1, \dots, y_I)$, all these $y_i = y_i(v)$ and $u_\lambda(v) = \sum_{i=1}^{I} \lambda_i u_i(y_i(v))$. It is a consequence of the *Implicit Function Theorem* that under our assumptions, the function $u_\lambda(\cdot)$ is two times differentiable in v. The function $u_\lambda(v)$ is often called the supconvolution function, and is typically more "well behaved" than the individual functions $u_i(\cdot)$ that make it up.

Now we are in the position to present the announced fundamental characterization, which can be proved using the Separating Hyperplane Theorem:

Theorem 2 *Suppose u_i are concave and increasing for all i. Then Y is a Pareto optimal allocation if and only if there exists a nonzero vector of agent weights $\lambda \in R^I_+$ such that $Y = (Y_1, Y_2, \dots, Y_I)$ solves the problem*

$$\sup_{(Z_1,\dots,Z_I)} \sum_{i=1}^{I} \lambda_i E u_i(Z_i) \qquad \textit{subject to} \qquad \sum_{i=1}^{I} Z_i \leq X_M. \tag{4}$$

Next we characterize Pareto optimal allocations under the above conditions. This result is known as Borch's Theorem:

Theorem 3 *A Pareto optimum Y is characterized by the existence of nonnegative agent weights $\lambda_1, \lambda_2, \dots, \lambda_I$ and a real function $\lambda : R \to R$, such that*

$$\lambda_1 u_1'(Y_1) = \lambda_2 u_2'(Y_2) = \ldots = \lambda_I u_I'(Y_I) := \lambda(X_M) \qquad a.s. \tag{5}$$

We may identify the reciprocals of the Lagrangian multipliers α_i^{-1} in Theorem 1 with the above agent weights λ_i, i.e., $\alpha_i^{-1} = \lambda_i$, in which case the state price deflator $\xi(X) = \lambda(X_M)$ a.s.

In proving the above theorem we have used the Saddle Point Theorem and directional derivatives in function space.

Existence of Pareto optimal contracts has been treated by DuMouchel (1968). The requirements are, as we may expect, very mild.

Representative Agent

We are now in a position to introduce our last building block, the representative agent. To this end, let us recall the sup-convolution function:

$$u_\lambda(v) := \sup_{(z_1,\dots,z_I)} \sum_{i=1}^{I} \lambda_i u_i(z_i) \qquad \text{subject to} \qquad \sum_{i=1}^{I} z_i \le v. \tag{6}$$

Having this function spelled out, we consider the following problem:

$$Eu_\lambda(V) := \sup_{(Z_1,\dots,Z_I)} \sum_{i=1}^{I} \lambda_i Eu_i(Z_i) \qquad \text{subject to} \qquad \sum_{i=1}^{I} Z_i \le V. \tag{7}$$

where V and $Z_i \in L^2$ for all i.

Theorem 4 *Assume $u_i' > 0, u_i'' \le 0$ for all i, and suppose $(\pi; Y_1, Y_2, \dots, Y_I)$ is a competitive equilibrium. Then there exists a nonzero vector of agent weights $\lambda = (\lambda_1, \dots, \lambda_I)$, $\lambda_i \ge 0$ for all i such that*

(i) the equilibrium allocation $(Y_1, Y_2, \dots, Y_I)$ solves the allocation problem (7) at $V = X_M = \sum_{i=1}^{I} X_i$ in which case $Eu_\lambda(X_M) = \sum_{i=1}^{I} \lambda_i Eu_i(Y_i)$.

(ii) the collection $(\pi; X_M)$ is an equilibrium in the single-agent economy $(u_\lambda; X_M)$.

The linear pricing functional π is then given by

$$\pi(Z) = E(u_\lambda'(X_M) \cdot Z) \qquad \forall Z \in L^2,$$

that is, $u_\lambda'(X_M) = \xi$ a.s.

Here we see clearer why $\lambda(X_M) = \xi(X)$, as noted after Theorem 3, since $\xi(X_1, \dots, X_I) = \xi(X_M)$ a.s. follows from Theorem 4.

The function $E(u_\lambda(X_M))$ may be considered as a welfare function when the aggregate risk is is endogenous. Notice that this welfare function is *endogenous* by construction.

Risk tolerance and aggregation

The risk tolerance function of an agent $\rho(x) : R \to R_+$, is defined by the reciprocal of the absolute risk aversion function $R(x) = -\frac{u''(x)}{u'(x)}$, or $\rho(x) = 1/R(x)$. Let us consider the following nonlinear differential equation:

$$Y_i'(x) = \frac{R_\lambda(x)}{R_i(Y_i(x))}, \qquad x \in B, \tag{8}$$

where $R_\lambda(x) = -\frac{u_\lambda''(x)}{u_\lambda'(x)}$ is the absolute risk aversion function of the representative agent, and $R_i(Y_i(x)) = -\frac{u_i''(Y_i(x))}{u_i'(Y_i(x))}$ is the absolute risk aversion of agent i at the Pareto optimal allocation function $Y_i(x)$, $i \in \mathcal{I}$. There is a neat result connecting the risk tolerances of all the agents in the market to the risk tolerance of the representative agent in a Pareto optimal allocation. It goes as follows:

Theorem 5 *(a) The risk tolerance of the market $\rho_\lambda(X_M)$ equals the sum of the risk tolerances of the individual agents in a Pareto optimum, or*

$$\rho_\lambda(X_M) = \sum_{i \in \mathcal{I}} \rho_i(Y_i(X_M)) \qquad a.s. \tag{9}$$

(b) The real, Pareto optimal allocation functions $Y_i(x) : R \to R$, $i \in \mathcal{I}$ satisfy the nonlinear differential equations (8).

The result in (a) was found by Borch (1985); see also Bühlmann (1980) for the special case of exponential utility functions.

We round off this section with the following characterization:

Theorem 6 *The Pareto optimal sharing rules are affine if and only if the risk tolerances are affine with identical cautiousness, i.e., $Y_i(x) = A_i + B_i x$ for some constants A_i, B_i, $i \in \mathcal{I}$, $\sum_j A_j = 0$, $\sum_j B_j = 1$, $\Leftrightarrow$ $\rho_i(x_i) = \alpha_i + \beta x_i$, for some constants β and α_i, $i \in \mathcal{I}$.*

This result can be found, among other places, in Wilson (1968).

The risk exchange between an insurer and a policy holder

Consider a policy holder having initial capital w_1, a positive real number, and facing a risk X, a non-negative random variable. The insured has utility function u_1, where $u_1' > 0, u_1'' < 0$. The insurer has utility function u_2, $u_2' > 0, u_2'' \leq 0$, and initial fortune w_2, also a positive real number. These parties can negotiate an insurance contract, stating that the indemnity $I(x)$ is to be paid by the insurer to the insured if claims amount to $x \geq 0$. It seems reasonable to require that $0 \leq I(x) \leq x$ for any $x \geq 0$. Notice that this implies that no payments should be made if there are no claims, i.e., $I(0) = 0$. The premium p for this contract is payable when the contract is initialized.

We recognize that we may employ our established theory for generating Pareto optimal contracts. Doing this, Moffet (1979) was the first to show the following:

Theorem 7 *The Pareto optimal, real indemnity function I: $R_+ \to R_+$, satisfies the following nonlinear, differential equation*

$$\frac{\partial I(x)}{\partial x} = \frac{R_1(w_1 - p - x + I(x))}{R_1(w_1 - p - x + I(x)) + R_2(w_2 + p - I(x))}, \tag{10}$$

where the functions $R_1 = -\frac{u_1''}{u_1'}$, and $R_2 = -\frac{u_2''}{u_2'}$ are the absolute risk aversion functions of the insured and the insurer, respectively.

In our setting this result is an immediate consequence of Theorem 5 (b).

Example 1: Let us consider the following approach to risk sharing. Consider a potential insurance buyer who can pay a premium αp and thus receive insurance compensation $I(x) := \alpha x$ if the loss is equal to x. He then obtains the expected utility $(u' > 0, u'' < 0)$

$$U(\alpha) = E\{u(w - X - \alpha p + I(X))\}$$

where $0 \leq \alpha \leq 1$ is a constant of proportionality. It is easy to show that if $p > EX$, it will *not* be optimal to buy full insurance (i.e., $\alpha^* < 1$) (see Mossin (1968)).

The situation is the same as in the above, but now we change the premium functional to $p = (\alpha EX + c)$ instead, where c is a non-negative constant. It is now easy to demonstrate, for example using Jensen's inequality, that $\alpha^* = 1$, i.e., full insurance is indeed optimal, given that insurance at all is rational (see e.g., Borch (1990)). □

The seeming inconsistency between the solutions to these two problems has caused some confusion in the insurance literature, and we would now like to resolve this puzzle. In the two situations of Example 1 we only considered the insured's problem. Let us instead take the full step and consider both the insurer and the insurance customer at the same time, and then "optimality" simply means Pareto optimality. In doing this we want to use the above presented general risk exchange theory.

From Theorem 7 we realize the following: If $u_2'' < 0$, we notice that $0 < I'(x) < 1$ for all x, and together with the boundary condition $I(0) = 0$, by the mean value theorem we get that

$$0 < I(x) < x, \qquad \text{for all} \qquad x > 0,$$

stating that *full insurance is not Pareto optimal when both parties are strictly risk averse.* We notice that the natural restriction $0 \leq I(x) \leq x$ is not binding at the optimum for any $x > 0$, once the initial condition $I(0) = 0$ is employed.

We also notice that *contracts with a deductible d can not be Pareto optimal when both parties are strictly risk averse*, since such a contract means that

$I_d(x) = x - d$ for $x \geq d$, and $I_d(x) = 0$ for $x \leq d$ for $d > 0$ a positive real number. Thus either $I'_d = 1$ or $I'_d = 0$, contradicting $0 < I'(x) < 1$ for all x.

However, when $u''_2 = 0$ we notice that $I(x) = x$ for all $x \geq 0$: *When the insurer is risk neutral, full insurance is optimal and the risk neutral part, the insurer, assumes all the risk.* Clearly, when R_2 is uniformly much smaller than R_1, this will approximately be true even if $R_2 > 0$.

This gives a neat resolution of the above mentioned puzzle. We see that the premium p does not really enter the discussion in any crucial manner when it comes to the actual form of the risk sharing rule $I(x)$, although this function naturally depends on the parameter p.

We could now use the theory of competitive equilibrium to find p. It is given as

$$p_{ce} = \frac{E\{I(X)u'_\lambda(X_M)\}}{E\{u'_\lambda(X_M)\}}. \tag{11}$$

(see e.g. Aase (2002), (1993a)). We could also use elements of cooperative game theory and determine the core in the present situation (see e.g., Lemaire (2003), this volume). The largest premium p_a that the insured will accept is given by

$$Eu_1(w_1 - p_a - X + I_{p_a}(X)) = Eu_1(w_1 - X),$$

while the smallest premium p_b that the insurer will accept in this situation is given by

$$Eu_2(w_2 + p_b - I_{p_b}(X)) = u_2(w_2).$$

Both these follow from applying *individual rationality.* Between these two prices the price p_{ce} must lie, i.e., if p_{ce} exists, then $p_{ce} \in [p_b, p_a]$. Thus we here have a situation where the core can be parameterized by the premium p. This is illustrated by examples in Aase (2002).

For a treatment of cooperative game theory in the standard risk sharing model, see Baton and Lemaire (1981). For a characterization of the Nash bargaining solution in the standard risk sharing model, see Borch (1960a) and (1960b).

Deductibles I: Administrative Costs

In the rather neat theory demonstrated above we were not able to obtain deductibles in the Pareto optimal contracts. Since such features are observed in real insurance contracts, it would be interesting to find conditions when such contracts result. The best explanation of deductibles in insurance is, perhaps, provided by introducing costs in the model. Intuitively, when there are costs incurred from settling claim payments, costs that depend on the compensation and are to be shared between the two parties, the claim size ought to be beyond a certain minimum in order for it to be Pareto optimal to compensate such a claim. Technically speaking, the natural requirement $0 \leq I(x) \leq x$ will be binding at the optimum, and this causes a strictly positive deductible to occur.

Following Raviv (1979), let the costs $c(I(x))$ associated with the contract I and claim size x satisfy $c(0) = a \geq 0$, $c'(I) \geq 0$ and $c''(I) \geq 0$ for all $I \geq 0$; in other words, the costs are assumed increasing and convex in the indemnity payment I. We then have the following:

Theorem 8 *In the presence of costs the Pareto optimal, real indemnity function I: $R_+ \to R_+$, satisfies*

$$I(x) = 0 \qquad \textit{for} \qquad x \leq d$$

$$0 < I(x) < x \qquad \textit{for} \qquad x > d.$$

In the range where $0 < I(x) < x$ it satisfies the differential equation

$$\frac{\partial I(x)}{\partial x} = \frac{R_1(w_1 - p - x + I(x))}{R_1(w_1 - p - x + I(x)) + R_2(A)(1 + c'(I(x))) + \frac{c''(I(x))}{(1+c'(I(x)))}}, \qquad (12)$$

where $A = w_2 + p - I(x) - c(I(x))$.

Moreover, a necessary and sufficient condition for the Pareto optimal deductible d to be equal to zero is $c'(\cdot) \equiv 0$ (i.e., $c(I) = a$ for all I).

If the cost of insurance depends on the coverage, then a nontrivial deductible is obtained. Thus Arrow's (1970, Theorem 1) deductible result was not a consequence of the risk-neutrality assumption (see also Arrow (1974)). Rather it was obtained because of the assumption that insurance cost is proportional to coverage. His result is then a direct consequence of Theorem 8 above:

Corollary 1 *If $c(I) = kI$ for some positive constant k, and the insurer is risk neutral, the Pareto optimal policy is given by*

$$I(x) = \begin{cases} 0, & \textit{if } x \leq d; \\ x - d, & \textit{if } x > d. \end{cases}$$

where $d > 0$ if and only if $k > 0$.

Here we obtained full insurance above the deductible. If the insurer is strictly risk averse, a nontrivial deductible would still obtain if $c'(I) > 0$ for some I, but now there would also be coinsurance (further risk sharing) for losses above the deductible.

Risk aversion, however, is not the only explanation for coinsurance. Even if the insurer is risk neutral, coinsurance might be observed, provided the cost function is a strictly convex function of the coverage I. The intuitive reason for this result is that cost function nonlinearity substitutes for utility function nonlinearity.

To conclude this section, a strictly positive deductible occurred if and only if the insurance cost depended on the insurance payment. Coinsurance above the deductible $d \geq 0$ results from either insurer risk aversion or cost function nonlinearity. For further results on costs and optimal insurance, see e.g., Spaeter and Roger (1995).

Deductibles II: Moral Hazard

A situation involving moral hazard is characterized by a decision taken by one of the parties involved (the insured), that only he can observe. The other party (the insurer) understands what decision the insured will take, but can not force the insured to take any particular decision by a contract design, since he can not monitor the insured.

The concept of moral hazard has its origin in marine insurance. The old standard marine insurance policy of Lloyd's - known as S.G. (ship and goods) policy - covered "physical hazard", more picturesquely described as "the perils of the see". The "moral hazard" was supposed to be excluded, but it seemed difficult to give a precise definition of this concept. Several early writers on marine insurance (e.g., Dover (1957), Dinsdale (1949), Winter (1952)) indicate that situations of moral hazard were met with underwriters imposing an extra premium.

The following idea was initiated by Holmström (1979). In order to explain this, let, as in the above, $u_2(x)$ and w_2 denote the insurer's utility function and initial wealth, $u_1(x)$, $v(a)$, w_1 are similarly the corresponding utility function and initial wealth of the insured. In the latter case $v(a)$ denotes *disutility* of effort a, effort designated to minimize or avoid the loss. Only the insured can observe a. The loss facing the insured is denoted by X, having probability density function $f(x, a)$. Notice that here we deviate from the neoclassical assumption that uncertainty is exogenous, since the stochastic environment is now effected by the insured's actions.

The problem may be formulated as follows:

$$\max_{I(x),a,p} Eu_2(w_2 - I(X) + p)$$

subject to $I(x) \in [0, x]$, $p \geq 0$, and subject to

$$Eu_1(w_1 - X + I(X) - p) - v(a) \geq \bar{h}$$

and

$$a \in \text{argmax}_{a'}\{Eu_1(w_1 - X + I(X) - p) - v(a')\}$$

The first constraint is called the participation constraint (individual rationality), and the last one is called the incentive compatibility constraint. We illustrate by an example presented in Aase (2002):

Example 2: Consider the case with $u_2(x) = x$, $u_1(x) = \sqrt{x}, v(a) = a^2$ and the probability density of claims $f(x, a) = ae^{-ax}$ is exponential with parameter a (effort). Notice that $P(X > x) = e^{-ax}$ decreases as effort a increases: An increase in effort decreases the likelihood of a loss X larger than any given level x.

We consider a numerical example where the initial certain wealth of the insurer $w_1 = 100$, and his alternative expected utility $\bar{h} = 19.589$. This number equals his expected utility without any insurance. In this case the optimal effort level is $a^* = 0.3719$.

In the case of no moral hazard, we solve the problem without the incentive compatibility constraint, and obtain what is called *the first best solution*. As expected, since the insurer is risk neutral, full insurance is optimal: $I(x) = x$, and the first best level of effort $a^{FB} = .3701$, smaller than without insurance.

The expected utility of the representative agent we may denote the welfare function. Here it is:

$$Eu_2(w_2 + p - I(X)) + \lambda(Eu_1(w_1 - p - X + I(X)) - v(a))$$

$$= w_2 + 195.83,$$

since $\lambda = \lambda^{FB} = 9.8631$. Here $p^{FB} = 2.719$. Moving to the situation with moral hazard, full insurance is no longer optimal. We get a contract with a deductible d, and less than full insurance above the deductible:

$$I(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq d; \\ x + p - w_1 + (\lambda + \frac{\mu}{a} - \mu x)^2, & \text{if } d < x \leq \frac{\lambda}{\mu} + \frac{1}{a}; \\ x + p - w_1, & \text{if } x > \frac{\lambda}{\mu} + \frac{1}{a}. \end{cases}$$

Here the deductible $d = 11.24$, the second best level of effort $a = a^{SB} = .3681$. Notice that this is lower that a^{FB}. The Lagrangian multipliers of the two constraints are: $\lambda^{SB} = 9.7214$ and $\mu = 0.0353$. The second best premium in this case is $p^{SB} = 0.0147$.

Due to the presence of moral hazard there is now a welfare loss: The expected utility of the representative agent has decreased:

$$Eu_2(w_2 + p - I(X)) + \lambda^{SB}(Eu_1(w_1 - p - X + I(X)) - v(a^{SB}))$$

$$= w_2 + 187.73,$$

implying a welfare loss of 8.10 compared to the first best solution. Notice that the premium p^{SB} is here lower than for the full insurance case of the first best solution, due to a smaller liability, by the endogenous contract design, for the insurer in the situation with moral hazard. □

In the above example we have used *the first order approach*, justified in Jewitt (1988).

Again we see that deductibles may result, and also coinsurance above the deductible, when the classical model predicts full insurance and no deductible.

A slightly different point of view is the following: If the insured can gain by breaking the insurance contract, moral hazard is present. In such cases the insurance company will often check that the insurance contract is observed. Note, in the above model the insurance company *could not* observe the action a of the insured. This was precisely the cause of the problem in Example 2. Here, checking is possible, but will *cost money*, so it follows that the mere existence of moral hazard will lead to costs. This is a different type of costs from those of Example 2, but has the same origin - moral hazard. The present situation clearly invites analysis as a two-person game - played between the insurance company and its customer.

Borch 1980 takes up this challenge, and finds a Nash equilibrium in mixed strategies. In this model the insured pays the full cost of moral hazard through the premium.

It seems to have been Karl Borch's position that moral hazard ought to be met by imposing an extra premium. [1]

From the above example we note that increased premiums may not really be the point in dealing with this problem: Here we see that the premium p^{SB} is actually smaller that the premium p^{FB}. The important issue is that the

[1]Stiglitz (1983) presented a model where the premium per units of insurance offered increases under moral hazard. He deals with loss distributions that can only take two values, so this concept does not have any direct counterpart in our model.

contract creates incentives for the insured to protect his belongings. This is brought out very clearly in the above example if the first best solution is implemented when moral hazard is present. Then the insured will set his level of effort $a = 0$, i.e., to its smallest possible level, since he has no incentive to avoid the loss, resulting in a very large loss with high probability (a singular situation with a Dirac distribution at infinity). Since the second best solution *is* the best when moral hazard is present, naturally this leads to a low welfare, in particular for the insurer.

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