Making Prospect Theory Fit for Finance

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Abstract

This paper gives a survey over a common aspect of prospect theory that occurred to be of importance in a series of recent papers developed by Enrico De Giorgi, Thorsten Hens, Janos Mayer, Haim Levy, Thierry Post, Marc Oliver Rieger and Mei Wang. The common aspect of these papers is that the value function of the prospect theory of Kahneman and Tversky (1979) and similarly that of Tversky and Kahneman (1992) has to be re-modelled if one wants to apply it to portfolio selection. Instead of the piecewise power value function, a piecewise *negative exponential* function should be used. This functional form is still compatible with laboratory experiments but it has the following advantages over and above Kahneman and Tversky`s piecewise power function:

1. The Bernoulli Paradox does not arise for lotteries with finite expected value.
2. No infinite leverage/robustness problem arises.
3. CAPM-equilibria with heterogeneous investors and prospect utility do exist.
4. It is able to simultaneously resolve the following asset pricing puzzles: the equity premium, the value and the size puzzle.

1. Introduction

THE CUMULATIVE PROSPECT THEORY (CPT) of Tversky and Kahneman (1992) summarizes several violations of the expeccted utility hypothesis. First of all, while expected utility is already based on a formal representation of a decision problem, CPT has two stages. The first stage is an editing phase in which the given representation of the decision problem is transformed into a formal decision. The

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second stage is an evaluation phase in which, based on a value and a probability weighting function, the lottery with the highest value is chosen.
In this paper, as in most finance papers, we assume that the editing phase is already completed and we thus only consider the valuation phase. This makes our results comparable to expected utility. The main building blocs of prospect theory that distinguishes it from expected utility theory are then:

1. Investors evaluate assets according to gains and losses relative to a given reference point..
2. Investors dislike losses by a factor of 2.25 as compared to their liking of gains.
3. Investors have changing risk aversion because their value functions are S-shaped with turning point at the origin.
4. Investors` probability assessments are biased in the way that extremely small probabilities (extremely high probabilities) are over- (under-) valued.

While CPT describes very well the choice of agents among a restricted set of lotteries it has some shortcomings when it is transferred to describe the solutions to portfolio selection problems. This is because the set of lotteries that can be generated by portfolio selection is quite large – it is typically uncountable and unbounded. Unfortunately, while the mathematical representation of prospect theory suggested by Tversky and Kahneman did well in the laboratory, it is not appropriate for portfolio selection problems. The main point of this survey is to argue that the prospect theory of Tversky and Kahneman (1992) and similarly that of Kahneman and Tversky (1979) has to be re-modelled if one wants to apply prospect theory in finance. Instead of modelling the value function by a piecewise power function a piecewise negative exponential function should be used. The main difference between the piecewise power function and the piecewise negative exponential function concerns large outcomes. In fact, while both functions are assumed to have a kinked and convex-concave shape with turning point at the origine, the piecewise negative exponential function exhibits more curvature and thus discourage extreme risk taking. As a consequence, while the suggested functional form also satisfies the main features of prospect theory and is still compatible with laboratory experiments, it has the following advantages over and above Kahneman and Tversky`s piecewise power function:

1. The St. Petersburg Paradox does not arise for lotteries with finite expected value.
2. No infinite leverage/robustness problem arises.
3. CAPM-equilibria with heterogeneous investors and prospect utility do exist.
4. It is able to explain the disposition effect with myopic optimization.
5. It is able to simultaneously resolve the following puzzles:
   The equity premium, the value and the size puzzle.

The St. Petersburg paradox is associated with the birth of expected utility theory. It shows that for lotteries with infinite expected monetary value people are not willing to pay an infinite sum of money. This observation led Bernoulli (1738) to postulate that people value lotteries not by their expected monetary value but rather by their expected *utility* of the monetary values it delivers. Assuming a sufficiently decreasing marginal utility of wealth, which is for example the case with a logarithmic utility, Bernoulli (1736) resolved the St. Petersburg Paradox. However, for any unbounded utility a lottery can be found that would still result in an infinite valuation of its monetary payoff. Hence, Bernoulli`s (1738) suggestion only solved the particular paradox arising from the particular game played in St. Petersburg but similar paradoxes occur for any unbounded utility. One solution is to only admit lotteries with bounded expected monetary value. However, as Rieger and Wang (2004) show, see section 3 of this survey, this solution is not sufficient to rule out the paradox for CPT. Since the piecewise exponential value function that we proposes in this survey is bounded this paradox would no longer arise for prospect theory.

As we show in section 4, the piecewise power function also has the problem that for almost all asset prices the investors` optimal portfolios are unbounded. With this functional from, the marginal utility of wealth does not decrease sufficiently fast. As an effect, existence of competitive equilibria, for example in the CAPM, cannot be ensured with the piecewise power function for economies with heterogeneous investors. As we show in section 5, with heterogeneous investors, CAPM equilibria do however exist if CPT is based on the piecewise negative exponential function. Of course, for any given investor one is able to find asset prices such that his portfolio selection problem has a solution. Indeed one can still use the standard argument common in the asset pricing literature where for a single representative investor asset prices are chosen such that the investor holds the market portfolio. However, as De Giorgi, Hens and Levy (2004) show this "decision support argument" for asset prices is not robust, since already small changes of the asset prices would lead the investor to choose totally different portfolios. Hence, with heterogeneous investors having piecewise power utilities there will not be a common vector of asset price for which all investors find a solution to their portfolio selection problem.

In Finance CPT has been successful in explaining the *equity premium puzzle*, i.e. the historically favourable risk-return trade-off of stocks relative to bonds. As shown by Benartzi and Thaler (1995), for a yearly holding period, the CPT statistic of the stock index is not significantly different from the CPT statistic of the bond index. However, as shown in De Giorgi, Hens and Post (2005), see section 7 of this survey, the same explanation does not rationalize the *size premium puzzle*, i.e. the historically favourable risk-return trade-off of small cap stocks relative to large cap stocks (first documented by Banz (1981)). Neither is it able to solve the *value premium puzzle* i.e. the favourable returns of value stocks relative to growth stocks (first documented by Basu (1977)). Fama and French (1992), (1993)) provide a rigorous empirical analysis of these phenomena. Nevertheless, the three puzzles can be explained simultaneously if we replace the piecewise-power value function of Tversky and Kahneman with a piecewise negative exponential value function. In fact, as discussed above, the new value function has a kinked and convex-concave shape (reflecting loss aversion and risk seeking for losses), just as the original value function. However, for large outcomes, the piecewise negative exponential value function exhibits more curvature hence the function discourages investment opportunities which provide extreme losses, also when this are coupelet with huge gains.

Previous work on prospect theory in portfolio selection has mainly focussed on the impact of loss aversion[2]. Following the seminal analysis of Bernartzi and Thaler (1995), Barberis, Huang and Santos (2001), Gomes (2003) and Berkelaar,

[2] An exception is Barberis and Huang (2004) who consider loss aversion, changing risk aversion and probability weighting simultaneously.

Kouvenberg and Post (2003) have studied how loss aversion affects single period portfolio decisions. Berkelaar, Kouvenberg and Post (2003) consider a multi-period model where the price dynamics are described by Ito processes (a generalization of Gomes 2003) and investors possess prospect theory utility indexes. The main result of their paper is that investors with prospect theory preferences follow a partial portfolio insurance strategy. Moreover, the initial portfolio weights on stocks typically increases with the investment horizon. If the investment horizon is short, then investors with loss aversion strongly reduce their holdings in stocks (myopic loss aversion, see Bernatzi and Thaler (1995) compared to investors with smooth power utility, while when the investment horizon is long, they strongly invest on stocks, since there is time to make up their losses, i.e. investors face gain opportunities. In these papers the value function is piecewise linear and the probability transformation is not considered. Moreover, these models consider a single representative investor.

This paper is organized as follows: In the next section we lay down a model that is sufficiently general to embed the papers we want to give a survey of. Then we report the results on the St. Petersburg paradox, the infinite leverage problem, the existence of CAPM equilibria, the disposition effect and the asset pricing puzzles.

2. The Model

Since the various aspects of CPT that we want to bring together in this survey come from quite different settings it is important to first lay down a model that is sufficiently general to embody all these aspects as special cases.

The description of the model follows Duffie (1988, section I.11). There are two points in time, t = 0, called today and t = 1, called tomorrow. The uncertainty tomorrow is modelled by a probability space (M, M, h). Consider *L*, the space of real-valued measurable functions on (M, M, h). We endow *L* with the scalar product $x \cdot y = \int_M x(m)y(m)dh$ and with the norm $\|x\| = \sqrt{x \cdot x}$. The consumption set will be the subset of *L* with finite norm, $L^2(h) = \{x \in L \mid \|x\|^2 < \infty\}$. Let $E_x(x) = \int_M x(m)x(dm)$ denote the expected value of the random variable $x \in L^2(h)$ under the probability measure $x \sim h$. In the case $x = h$ we denote $\mathrm{E}_x(x)$ by $m(\mathrm{x})$. For any two random variables $x, y \in L^2(h)$, let $cov(x,y) = m(xy) - m(x)m(y)$ be the covariance and let $s(x) = \sqrt{\mathrm{cov}(x,x)}$ be the standard deviation. The price space is also $L^2(h)$. Let the marketed subspace, X, be generated as the span of $(A_j)_{j=0,1,\ldots,J}$, a collection of securities in $L^2(h)$, one of which, say j = 0, is the risk free asset. Let q_j denote the price of asset j, j=0,1,…,J. Then the gross return of asset j is defined as $R_j = \frac{A_j}{q_j}$. There are i = 1,2,…,I investors being endowed with initial wealth w^i. Using the existing assets, investors transform their initial wealth into random wealth which they totally use for consumption in t = 1. All agents have already decided to invest w^i on the financial market and they evaluate the consumption in t = 1 by utility functions

$U^i : X \to R$, i = 1,2,...,I that are monotonically increasing and continuous. Hence the agents` optimization problem can be defined as

$$\max_{q \in R^{J+1}} \mathrm{U}^i\left(\sum\nolimits_{j=0}^{J} A_j q_j\right) \text{ s.t. } \sum_{\mathrm{j}=0}^{\mathrm{J}} \mathrm{q}_{\mathrm{j}} q_j = w^i.$$

Equivalently the optimization problem can be written in terms of returns:

$$\max_{l \in R^{J+1}} \mathrm{U}^i\left(\left(\sum\nolimits_{j=0}^{J} R_j\, l_j\right) w^i\right) \text{ s.t. } \sum_{\mathrm{j}=0}^{\mathrm{J}} l_j = 1.$$

These optimization problems do only have a solution if there are no arbitrage opportunities. Since investors' preferences are monotone, any non-negative attainable consumption x with non-positive price represents an arbitrage opportunity. Thus, any no arbritrage condition should ensure that every non-negative attainable consumption *x with non-positive price* must be zero:

$$L_+^2 \cap \left\{ x \in L^2(h) \,\middle|\, x = \sum\nolimits_{j=0}^{J} A_j q_j \text{ and } \sum_{\mathrm{j}=0}^{\mathrm{J}} \mathrm{q}_{\mathrm{j}} q_j \le w^i \right\} = \{\varnothing\}.$$

This condition can be seen as a minimal restriction for the asset prices q_j, j=0,1,…,J. One immediate implication is that the risk free asset needs to have a positive price, i.e. $q_0 > 0$. By the Dalang-Morton-Willinger Theorem (see for example Delbaen (1999)), there exists a probability measure, called the equivalent martingale measure, $p \sim h$ on (M, M) such that $\frac{\mathrm{q_j}}{\mathrm{q_0}} = E_p(A_j)$, for all j=1,2,...,J. Defining $q_0 = \frac{1}{1+r}$ we obtain $\mathrm{q_j} = \frac{1}{1+r} E_p(A_j), j = 1,...,J.$ Equivalently we get $E_p(R_j - (1+r)) = 0$, for all j = 1,...,J. That is to say under the equivalent martingale measure there are no expected excess returns.

In the case of prospect theory the utility function U^i is defined by a reference point $RP^i \in R$, a value function $v^i : R \to R$ and a probability transformation $T^i : [0,1] \to [0,1]$: $U^i(x) = \int_R v^i(x - RP^i)\, d(T^i \circ \mathrm{N}(x))$, where N denotes the cumulative distribution of $\mathrm{x} \in \mathrm{L}^2(h)$. We assume the following general properties of v^i and T^i:

1. v^i is a two times differentiable function on $\mathrm{R} \backslash \{0\}$, strictly increasing on R, strictly concave on (0,8) and strictly convex on (-8 ,0).
2. T^i is a differentiable, non-decreasing function from [0,1] onto [0,1] with $T^i(p) = p$ for p = 0 and p=1 and with $T^i(p) < p$ $\left(T^i(p) > p\right)$ for p large (small).

Kahneman and Tversky`s (1992) model of CPT and also our suggestion share these general properties. The weighting function is assumed to be given by $T^i(p) = \dfrac{p^{g^i}}{\left(p^{g^i} + (1-p)^{g^i}\right)^{1/g^i}}$, where the median of g^i is about 0.65.

Kahneman and Tversky (1979) and (1992) have suggested for the value function to consider the piecewise power function. Instead we propose the piecewise exponential function:

$$v^i(x)=\begin{cases} b^{i+}x^{a^i} & \text{for } x\geq 0 \\ -b^{i-}(-x)^{a^i} & \text{for x} < 0 \end{cases} \text{ and } v^i(x)=\begin{cases} -l^{i+}\exp(-a^i x)+l^{i+} & \text{for } x\geq 0 \\ l^{i-}\exp(a^i x)-l^{i-} & \text{for x} < 0 \end{cases}.$$

Where $0\leq a^i \leq 1$ and the b^{i+}, $l^{i+}$and b^{i-}, l^{i-} are positive numbers. Kahneman and Tversky (1979) report median values for a^i and $\frac{b^{i-}}{b^{i+}}$ of about 0.88 and 2.25 respectively. Figure 1 shows that our proposal, choosing our parameters $a^i \approx 0.2$ and , $l^{i+}= 6.52$ and $l^{i-}=14.7$ (so that $\frac{l^{i-}}{l^{i+}}\approx 2.25$) approximates the Tversky and Kahneman (1992) utility index very well for values around zero. We presume that the experimental evidence given for the value function specification of Kahneman and Tversky (1979) foremost concerns the shape of the utility function around zero. Note also that the utility function we propose is different to that of Kahneman and Tversky (1979) for very high stakes because it is less linear than theirs. Indeed our function is bounded above by l^{i+} and it is bounded below by $-l^{i-}$. Our theoretical analysis is supported by the laboratory results obtained by Bosch-Domenech and Silvestre (2003), who experimentally find that decision makers usually show risk aversion for larger amounts, for both gains and losses.

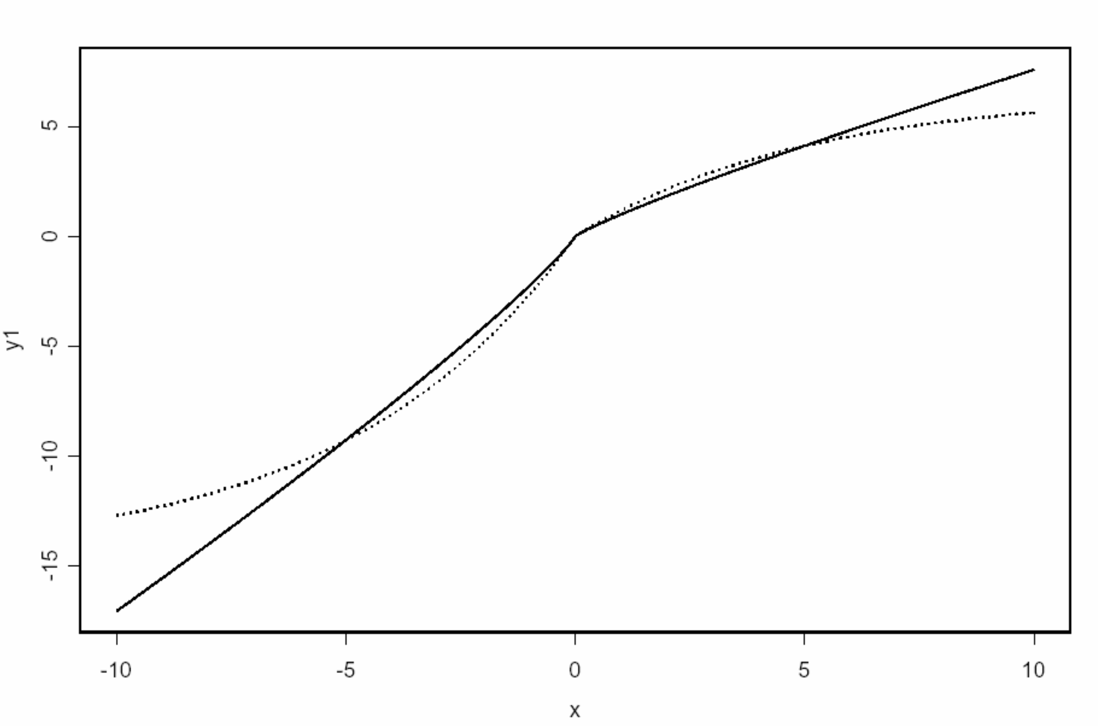

Figure 1: Tversky and Kahneman (1992) utility index (full line) and $u(x) = -\lambda^+ e^{-\alpha x} + \lambda^+$ for $x \geq 0$ and $u(x) = \lambda^- e^{\alpha x} - \lambda^-$ for $x < 0$ (dotted line), where $\lambda^+ = 6.52$, $\lambda^- = 14.7$ and $\alpha \approx 0.2$.

3. The St. Petersburg Paradox [Rieger and Wang (2004)]

The St. Petersburg paradox is usually explained by the following example: A player is reluctant to pay enormous amounts of money for a gamble in which he gets 2^n ducats when the coin lands "heads" on the ground for the first time at the n-th throw. Note that the gamble has an infinite expectation:

$$E_h(x) = \frac{1}{2}\cdot 2 + \frac{1}{4}\cdot 4 + ... + \frac{1}{2^n}\cdot 2^n + \cdots = 1 + 1 + \cdots = \infty.$$

This example already dates back to Bernoulli (1736). The solution of this problem is usually to replace the formula of expected value with the one of expected utility, in which a strictly concave utility function makes the subjective utility of the large outcomes no longer high enough compensate the very low probability associated with them.

It is, however, important to keep in mind that for gambles with infinite expected value, the strict concavity of the utility function alone cannot guarantee the expected utility to be finite. For example, if the gamble from above offers 2^{2^n} ducats when the coin lands "heads" for the first time at the n-th throw, then with a strictly concave utility function like $u(x) = x^{0.88}$, the expected utility is still infinite. The St. Petersburg paradox can be resolved by allowing only for "realistic" gambles: Indeed, under the assumption of a finite expected value, a (not necessarily strictly) concave utility function is sufficient to guarantee the expected utility is finite. Even though this statement is almost trivial in the framework of expected utility theory, it turns out to be false in the context of CPT. In fact Rieger and Wang (2004)[3] show that with CPT a gamble with *finite* expected value can have *infinite* prospect utility – independent of the concavity of the value function. This is possible, since the probability weighting function suggested by Kahneman and Tversky (1979) has infinite slope at zero and since the slope of the value function does not decrease much for high. The gamble Rieger and Wang (2004) construct is as follows: The probability measure of possible outcomes is given by $\text{p}(x) = \begin{cases} 0 & \text{x} \le 1 \\ \text{Cx}^{-k} & \text{x}>1 \end{cases}$ where C = $\int_0^\infty \text{x}^{-k} dx$. For ? close to 2, they show that for values of the risk aversion parameter a and the probability weighting parameter ? that are consistent with the experimental literature indeed CPT utility is infinite.. Note that this problem does not arise from the usual S-shape of the value function in CPT, since the example works with only considering positive outcomes (i.e. the concave part). Rieger and Wang (2004) suggest curing this problem by choosing a probability weighting function that has finite slope at 0. Alternatively, one could replaced the piecewise power value function by the piecewise negative exponential value function. This also solves the version of the St. Petersburg Paradox pointed out by Rieger and Wang (2004) because the piecewise negative exponential value function is bounded.

[3] See also Blavatskyy (2004).

4. The Infinite Leverage/Robustness Problem [De Giorgi and Hens (2005)]

So far we have shown that given the probability weighting function of Kahneman and Tversky (1979) the value function should better be bounded. Here we will argue that the boundedness is also important to solve an infinite leverage problem arising with the piecewise power function. Moreover, choosing a power function to model the agents` risk aversion leads to a robustness problem that does not occur for the piecewise exponential function. The infinite leverage problem is the observation that without imposing short sales constraints on the risk free asset (i.e. a borrowing constraint) a prospect utility maximizer always finds a portfolio of risky assets that he would like to leverage infinitely in order to obtain infinite utility: Recall that maximizing a power function in the expected utility approach is certainly possible without imposing a borrowing constraint. If markets are arbitrage free any portfolio of risky assets delivers gross returns that have a positive probability to obtain both, positive and negative excess returns. Hence scaling any such portfolio will let the utility tend to negative infinity. In the prospect theory case however, negative excess returns are not punished enough to avoid infinite leveraging. The robustness problem is the observation that small changes of the value function parameters lead to drastic changes in the asset allocation. As we show next these two problem are closely related for the piecewise power value function.

To make this point, start from the optimization problem written in terms of returns.

$\max_{l \in R^{J+1}} \mathrm{U}^i\left(\left(\sum_{j=0}^{J} R_j l_j\right) w^i\right)$ s.t. $\sum_{j=0}^{J} l_j = 1$. Assuming CPT we can write:

$U^i(x) = \int_R v^i(x - RP^i)\, d(T^i \circ \mathrm{N}_l(x))$, x $=\left((1-l_0)\sum_{j=1}^{J} R_j l_j + l_0 R_0\right) w^i, \sum_{j=1}^{J} l_j = 1$.

Now suppose, for the reference point being equal to zero, for some portfolio of risky assets, say $\tilde{l} \in \mathrm{R}^J$ with $\sum_{j=1}^{J} \tilde{l}_j = 1$, we obtain a positive prospect value:

$\int_R v^i(x)\, d(T^i \circ \mathrm{N}_{\tilde{l}}(x)) > 0$, for some x $=\left((1-l_0)\sum_{j=1}^{J} R_j \tilde{l}_j + l_0 R_0\right) w^i$, with $\sum_{j=1}^{J} \tilde{l}_j = 1$.

The existence of such a portfolio follows from weak conditions on asset prices. For example, in case of Gaussian distributed returns, the existence of a risky portfolio with positive prospect value is ensured if the expected return of the market portfolio exceeds the risk-free rate of return.
Referring to Kahneman and Tversky`s (1979) and (1992) specification of the value function we can rewrite the utility function as:

$$U^i(x) = \int_R d(x) b^i(x) \left(d(x)(x - RP^i)\right)^{a^i} d(T^i \circ \mathrm{N}_{\tilde{l}}(x)),$$

where $d(\mathrm{x}) = \begin{cases} +1 & x \geq RP^i \\ -1 & x < RP^i \end{cases}$, $b^i(\mathrm{x}) = \begin{cases} b^{i+} & x \geq RP^i \\ b^{i-} & x < RP^i \end{cases}$

Now, fix a portfolio $\tilde{l} \in \mathrm{R}^J$ as defined above and consider what happens if the leverage is increased, i.e. $l_0 \to -\infty$. For any such portfolio we obtain the payoffs:

$x - RP^i = \left[\left(\sum_{j=1}^{J} R_j \tilde{l}_j\right) w^i + \frac{l_0 R_0 w^i - RP^i}{(1-l_0)}\right](1-l_0)$. Hence the term $(1-l_0)^{a^i}$ factors out from the utility computation and by the choice of the portfolio $\tilde{l} \in \mathbb{R}^J$ in the limit for $l_0 \to -\infty$ it is multiplied with the positive term

$$\int_R d(x)b^i(x)\left(d(x)(x-R_0)w^i\right)^{a^i} d(T^i \circ N_{\tilde{l}}(x)),\text{where } x = \sum_{j=1}^J R_j \tilde{l}_j.$$

Note that this term is independent of $\tilde{l}$. Hence $l_0 \to -\infty$ gives infinite utility.[4]

With a bounded utility, infinite utility is impossible but still one may want to infinitely leverage the portfolio. For example with the piecewise exponential function there is no optimal leverage since the utility is increasing for all positive payoffs. However since the utility values are bounded above the increases become more and more negligible so that for any small epsilon there are portfolios that cannot be improved by other portfolios by more than epsilon. Hence an investor with a piecewise exponential value function who will be satisfied (up to some epsilon) with a finite leverage.

A problem closely related to the infinite leverage problem is the robustness problem. The simplest case where this problem arises with the piecewise power function occurs when the reference point is the wealth obtained from investing all initial wealth in the risk free asset. In this case we obtain: $x - RP^i = \left[\left(\sum_{j=1}^J R_j \tilde{l}_j\right) - R_0\right] w^i (1 - l_0)$. Hence when on changing the parameters the excess return term $\left(\sum_{j=1}^J R_j \tilde{l}_j\right) - R_0$ crosses zero, the asset allocation jumps from no risk-free asset to only risky free assets. This is a non-intuitive property for portfolio choice. Moreover, in asset pricing models with a representative investor who is induced to hold the market portfolio for some prices, on changing the parameters slightly the representative agent will depart from his choice drastically. That is to say, the standard "decision support argument" is not robust with the piecewise power function. As De Giorgi and Hens (2005) show also the robustness problem does not occur with the piecewise negative exponential value function, because this functional from prohibits to factor out the fraction invested in the risk free asset.

5. Existence of CAPM-equilibria [De Giorgi, Hens and Levy (2004)]

So far we have basically argued that a good value function for prospect theory should be bounded and should not allow to factor out the fraction of wealth invested in the risk free asset. Here we now argue that the exponential function is a very convenient function when prospect theory should be combined with the CAPM. Assuming that the payoffs (and thus the returns) are normally distributed it is easy to see that a prospect utility is actually also a mean-variance utility. Let $m_l = \sum_{j=0}^J m(R_j) l_j$ be the expected return of a portfolio and let accordingly

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[4] Indeed, $\lim_{l_0 \to -\infty} \frac{x - RP^i}{1 - l_0} = \left[\left(\sum_{j=1}^J R_j \tilde{l}_j\right) - R_0\right] w^i$. Hence if for the market portfolio R_M we have $m(R_M) > 1 + r,$ then already for $\tilde{l} = l_M$ the prospect utility divided by *(1-? 0)* is positive so that infinite leveraging gives infinite utility.

$s_l^2 = \sum_{j=0}^{J} \sum_{k=0}^{J} l_j \operatorname{cov}(R_j, R_k) l_k$ be the variance of a portfolio. Denoting by $\Phi_{m,s}$ the cumulative normal distribution we can write the agent`s optimization problem as:

$$U^i(x) = \int_R v^i(x - RP^i)\, d(T^i \circ \Phi_{m_l, s_l}(x)), \text{ where } x = \left(\sum_{j=0}^{J} R_j l_j\right) w^i, \text{with} \sum_{j=0}^{J} l_j = 1 .$$

Hence in this case the CPT utility function is a function of mean and variance only. Moreover, standardising the normal distribution reveals that this function is certainly increasing in mean but due to the convexity of the value function for losses it need not be decreasing in variance.

$$V^i(m_l, s_l) = \int_R v^i\left((\text{x-RP}^i)s_l + m_l\right) d(T^i \circ \hat{\Phi}(x)), \text{ where } \hat{\Phi}(x) = \Phi\left(\frac{x - m_l}{s_l}\right).$$

Since there is a risk free asset and since agents` mean-variance utility is increasing in the mean, agents will only choose portfolios on the capital market line in the mean-standard deviation diagram, i.e. on the straight line through the risk free asset and the tangential portfolio (see Figure 1b). Hence two-fund separation holds and in a capital asset market equilibrium the security market line theorem will hold. As a consequence, excess returns are determined by covariance with respect to the market portfolio, which is given by the total amount of payoffs available in the economy:

$$m(R_j) - (1+r) = \frac{\operatorname{cov}(R_j, R_M)}{s^2(R_M)}\left(m(R_M) - (1+r)\right), \text{ where } R_M = \sum_{j=1}^{J} R_j \bar{l}_j. \quad \text{With} \quad \bar{l}_j$$

being the supply i.e. the total market capitalization of asset j.

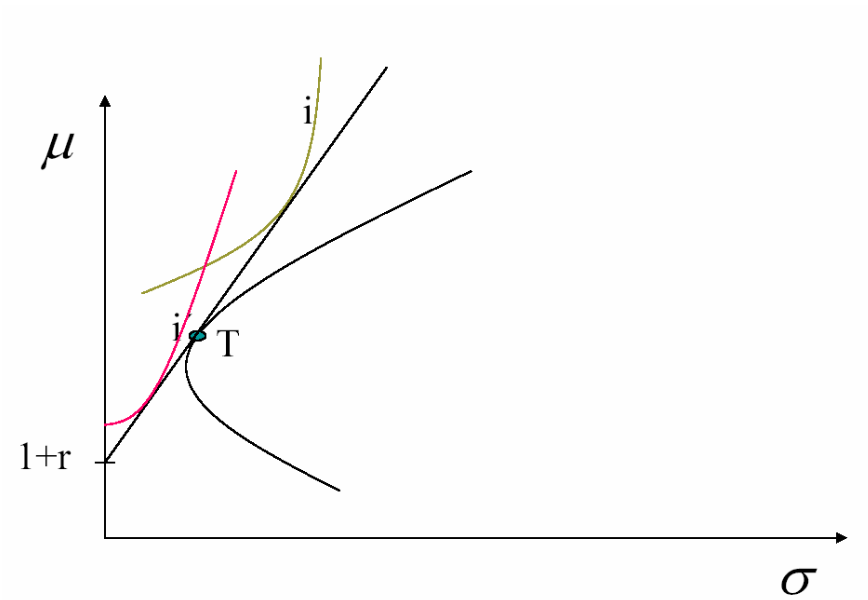

Figure 1b: Two-Fund Separation Theorem

For the piecewise power function of Kahneman and Tversky the indifference curve in the mean-standard deviation diagram looks rather strange (see Figure 3). For values of the mean below the reference point (which we have taken to be the risk free rate) the utility function is not quasi-concave and the indifference curves are not sufficiently upward sloping while for values above the reference point we get quasi-concavity but downward sloping indifference curves. It should not come as a surprise that the infinite leverage problem, described in the previous section, also holds in this more restricted setting.

With the piecewise negative exponential value function indifference curves look much nicer, as Figure 3 reveals. No infinite leverage problem occurs and the area where quasi-concavity cannot be ensured can be ruled out as possible equilibrium allocations if one is willing to assume that the expected return of the market portfolio is higher than the risk free rate. If the latter is chosen as the reference rate of return, this assumption means that one is more likely to find returns in the gain region of the value function. As a result of these assumptions (after a long sequence of careful computations) indeed De Giorgi, Hens and Levy (2004) are able to show the existence of CAPM-equilibria for any set of prospect theory investors.

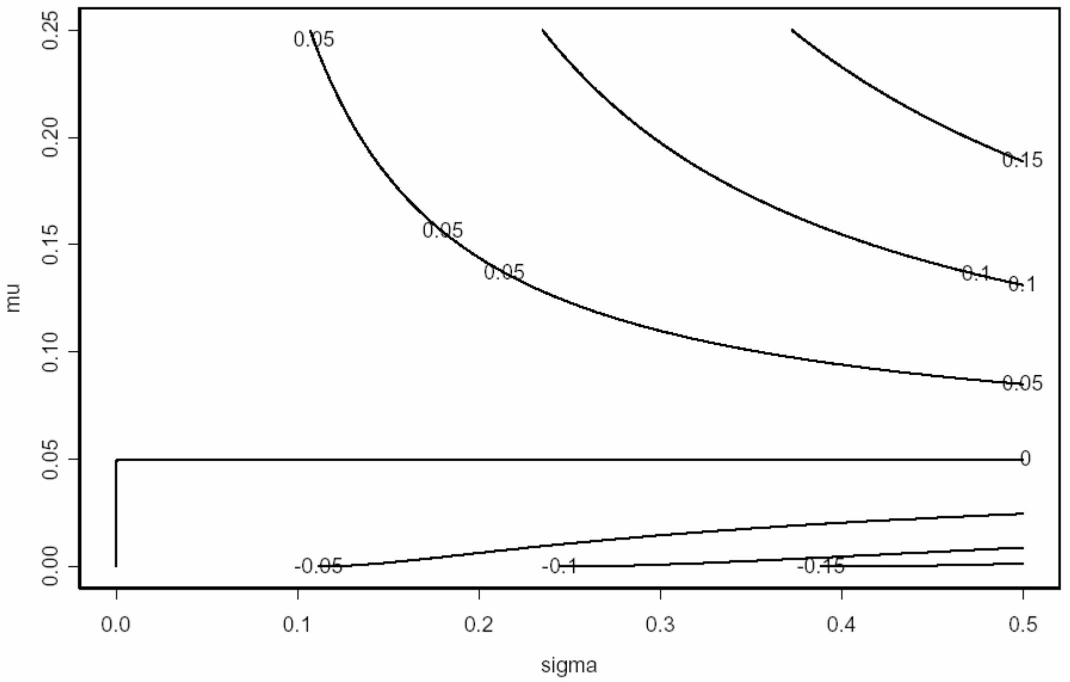

Figure 2: Indifference curves in the mean and standard deviation space for the utility function induced by Tversky and Kahnemann (1992) utility index and probability transformation.

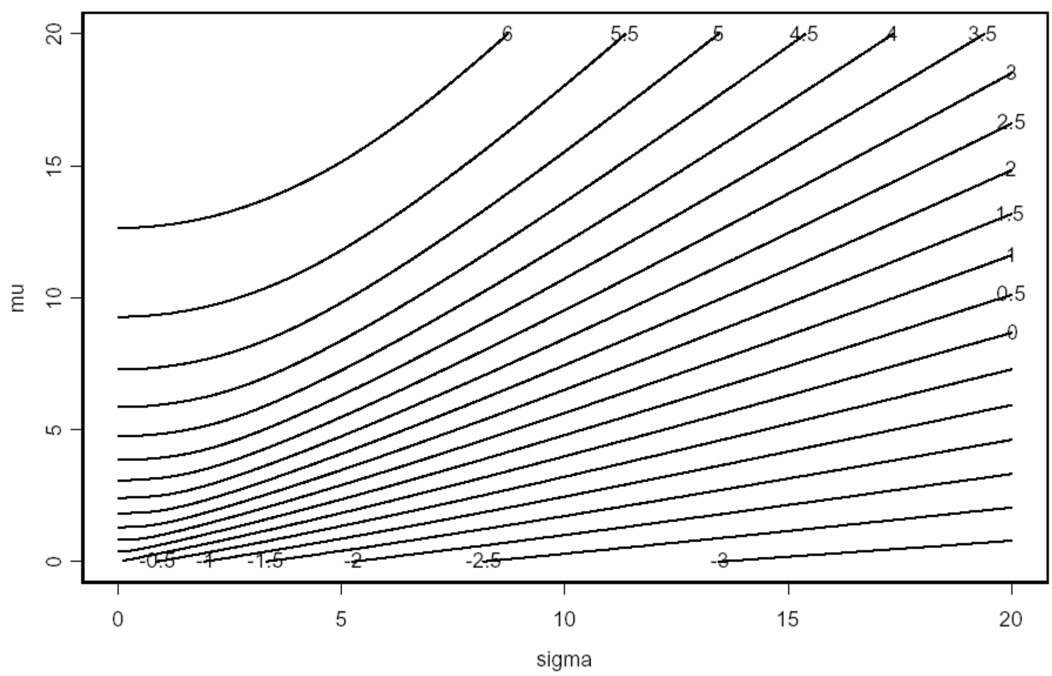

Figure 3: Indifference curves in the mean and standard deviation space for the utility function induced by $u(x) = -\lambda^+ e^{-\alpha x} + \lambda^+$ for $x \geq 0$ and $u(x) = \lambda^- e^{\alpha x} - \lambda^-$ for $x < 0$ where $\lambda^+ = 6.52$, $\lambda^- = 14.7$ and $\alpha \approx 0.2$.

6. The Equity Premium -, the Size - and the Value Puzzle

[De Giorgi, Hens and Post (2005)]

In order to stay as close as possible to the original equity premium studies of Mehra and Prescott (1985) and Benartzi and Thaler (1995) we consider real returns on equity and bonds. However, there are two differences. First, De Giorgi, Hens and Post (2005) consider an extended sample including the bull market of the 1990s and the equity bear market that followed in the early 2000s. Second, they expand the investment universe and include portfolios sorted on market capitalization (ME) and book-to-market-equity ratio (B/M) in the analysis.

The stock market portfolio is proxied by the CRSP all-share index, a value-weighted average of common stocks listed on NYSE, AMEX, and NADAQ. The bond index is defined as the intermediate government bond index maintained by Ibbotson. This index closely matches the 5-year Government bond index employed by Benartzi and Thaler (1995). De Giorgi, Hens and Post (2005) use the canonical decile portfolios formed on ME and the decile portfolios formed on B/M. For detailed data description and selection procedures we refer to Fama and French (1992) (1993). De Giorgi, Hens and Post (2005) use monthly and annual real returns for the period from January 1927 to December 2002 (912 months). Bond and inflation data are obtained from Ibbotson Associates and the stock portfolio data from Kenneth French's online data library.

Table I presents some basic descriptive statistics of the stock portfolios and bond and equity indices. Clearly, stocks outperform bonds during our 76-year sample period by about 6 percent on an annual basis. However, stocks are riskier which is reflected in a low minimum (-40% in the worst year) and a high standard deviation. Contrary, bonds offer downside protection (-17% in the worst year), but the upside potential is limited. Small and value firms offer higher average returns and higher variance, combined with positive skewness. Puzzling is the BM8 and BM9 portfolios, which combine high average returns with a minimum return above -50% and a maximum return in excess of 100%. Clearly, these portfolios seem far more attractive than the all-equity index.

[Insert Table I about here]

De Giorgi, Hens and Post (2005) basically intend to test whether the market portfolio of risky assets is the optimal portfolio for a representative investor who obeys to the rules of (1) the mean-variance framework, (2) the piecewise power CPT or (3) the piecewise exponential CPT. The standard approach to test if the market portfolio is optimal is to analyze the first-order condition or the Euler equation. This approach is valid for the mean-variance framework, because the first-order condition is necessary and sufficient for establishing the maximum in this framework. By contrast, the first-order condition gives only a necessary optimality condition for CPT. Both models allow for local risk seeking and hence there may be minima and local maxima, which will also satisfy the first-order condition.

There exist various multivariate global optimization methods for locating the global optimum if the objective function is not concave (see, for example, Horst and Pardalos (1995)). Unfortunately, these methods generally are computationally too demanding for high dimension problems such as ours (we use 22 assets).

To circumvent this problem, De Giorgi, Hens and Post (2005) simply analyzed the various objective functions (Sharpe ratio, CPT statistic, adjusted CPT statistic) at all the individual benchmark portfolios. This approach can be seen as a very rough grid search; the individual assets are excluded from the analysis and only the 22 benchmark portfolios are seen as a discrete approximation to the investment possibilities set.

Thus, for each benchmark portfolio, they compute the Sharpe ratio, the CPT statistic and the adjusted CPT statistic. To account for sampling variation, we use the bootstrap methodology to compute the p-value for the null that the benchmark portfolio is equally attractive as the market portfolio.

Contrary to Benartzi and Thaler (1995), the CPT statistic of the bond index is significantly higher than the CPT statistic of the stock index. This is due to the inclusion of the equity bear market in the early 2000s. Further, CPT cannot rationalize the size and value effects. Specifically, while the CPT statistic of the stock market index is –1.590, the CPT statistic of size portfolio 1 is 2.290 (0.03) and that of B/M portfolio 8 is 2.083 (0.00). In large part, these high values are explained by the favourable upside potential of small caps and value stocks. For example, the ME 1 portfolio of small caps has a maximum return of 155.29% and the BM1 portfolio of value stocks has a maximum of 113.53%. Interestingly, there is no corresponding downside risk for the small caps and value stocks. Apparently, the return distribution is positively skewed and highly correlated in downside markets, which limits the downside risk and the potential for downside risk reduction by means of portfolio diversification. These properties make the small cap and value stock portfolios very attractive for the CPT investor, who overweighs small probabilities and whose marginal value function diminishes very slowly.

Using the piecewise exponential value function, all three puzzles are resolved. The bond index does not achieve a significantly higher CPT+ statistic than the stock index. Also, the size and value effects disappear; no benchmark portfolio achieves a significantly higher CPT+ statistic than the market portfolio. Because the marginal function of the piecewise negative exponential value function decreases much faster than the piecewise-power value function, CPT+ assigns a much lower weight to the upside potential of the small caps and value stocks. In brief, the piecewise-exponential value function succeeds in explaining away the equity premium, size premium and value premium puzzle at the same time.

[Insert Table II about here]

7. Other modifications of Prospect Theory

The modification of the value function that we proposed here is one important aspect of making prospect theory applicable to finance. As mentioned above, one may also want to change the probability weighting function in order to avoid an infinite slope at zero. An even more fundamental point arises from the fact that prospect theory has been designed to describe choices between lotteries while in many finance applications data are given that are not represented as lotteries (probability distributions). In a typical application, for a finite number of dates t = 1, 2 ,…,T a sample of a finite number of asset returns $R_t^k, k = 1,2,...K.$ is given. A lottery on the other hand is a representation in which each observation of returns gets assigned the relative frequency or the likelihood of that observation. The resulting probability distribution depends on which returns are seen to be sufficiently similar to be seen as one observation. Unfortunately, this decision (which in the case of representing data by a histogram is the selection of the band width) is not innocuous for the prospect theory of Kahneman and Tversky (1979) since the probability weighting function distorts the relative frequencies obtained. In the extreme case, for example, in which every observation is seen as being different to any other observation, all returns would be equally likely and the probability weighting function would not change the relative weight. However, if some returns get grouped together they get a different likelihood than others and the weighting function distorts the asset allocation. Hens, Mayer, Rieger and Wang [2005] started looking into this issue and suggested to "repair" prospect theory in order to avoid unreasonable dependence on the way data is grouped to lotteries.

8. Conclusion

We have argued that for various reasons instead of modelling the value function of prospect theory by a piecewise power function a piecewise negative exponential function should be used. This functional form is still compatible with laboratory experiments but it has the following advantages over and above Kahneman and Tversky`s piecewise power function:

1. The Bernoulli Paradox does not arise for lotteries with finite expected value.

2. No infinite leverage/robustness problem arises.
3. CAPM-equilibria with heterogeneous investors and prospect utility do exist.
4. It is able to simultaneously resolve the following asset pricing puzzles:
The equity premium, the value and the size puzzle.

Modelling prospect theory with the piecewise negative exponential function makes it fit for applications to finance like portfolio selection. From this re-modelling of prospect theory we expect a series of new results, as for example a new explanation of the asset allocation puzzle (see De Giorgi, Hens and Mayer (2005)). Our contribution to these new results should however not be overemphasized since we are “standing on the shoulders of giants”: Daniel Kahneman and Amos Tversky.

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Table I

Descriptive Statistics

The table shows descriptive statistics for the annual real returns of the value-weighted CRSP all-share market portfolio, the intermediate government bond index of Ibbotson and the size and value decile portfolios from Kenneth French' data library. The sample period is from January 1927 to December 2002 (76 yearly observations).

	Avg.	Stdev.	Skew.	Kurt.	Min	Max
Equity	8.59	21.05	-0.19	-0.36	-40.13	57.22
Bond	2.20	6.91	0.20	0.59	-17.16	22.19
Small	16.90	41.91	0.92	1.34	-58.63	155.29
2	13.99	37.12	0.98	3.10	-56.49	169.71
3	13.12	32.31	0.69	2.13	-57.13	139.54
4	12.53	30.56	0.46	0.83	-51.48	115.32
5	11.91	28.49	0.44	1.60	-49.57	119.40
6	11.65	27.46	0.31	0.61	-49.69	102.17
7	11.09	25.99	0.30	1.14	-47.19	102.06
8	10.15	23.76	0.29	1.19	-42.68	94.12
9	9.63	22.33	0.02	0.46	-41.68	78.15
Large	8.06	20.04	-0.22	-0.52	-40.13	48.74
Growth	7.84	23.60	0.02	-0.64	-44.92	60.35
2	8.77	20.41	-0.27	-0.27	-39.85	55.89
3	8.52	20.56	-0.10	-0.47	-38.00	51.90
4	8.25	22.49	0.49	2.39	-45.02	96.33
5	10.29	22.82	0.36	1.92	-51.55	93.77
6	10.05	23.04	0.19	0.63	-54.39	73.57
7	11.00	24.73	0.18	1.22	-51.13	97.91
8	12.82	27.01	0.67	1.95	-46.56	113.53
9	13.71	29.08	0.56	1.85	-47.42	123.72
Value	13.32	33.05	0.43	1.40	-59.78	134.46

Table II

Test Results

The table shows for each benchmark portfolio the Sharpe ratio, the CPT statistic and the adjusted CPT statistic with the piecewise-exponential value function. Also, the table reports the bootstrap p-value. Cells that are colored gray refer to portfolios that yield a significantly higher value than the market portfolio at a 5% significance level.

	MV		CPT		CPT+	
	Statistic	p-value	statistic	p-value	statistic	p-value
Equity	0.380		-1.590		-1.496	
Bond	0.329	0.007	-0.788	0.008	-1.105	0.240
Small	0.384	0.140	2.290	0.030	-2.172	0.933
2	0.357	0.317	1.053	0.085	-1.981	0.888
3	0.384	0.215	0.654	0.085	-1.749	0.749
4	0.387	0.212	0.278	0.066	-1.509	0.514
5	0.394	0.180	0.197	0.070	-1.411	0.377
6	0.400	0.153	0.101	0.043	-1.441	0.413
7	0.402	0.142	0.076	0.033	-1.416	0.347
8	0.403	0.140	-0.006	0.020	-1.342	0.233
9	0.404	0.116	-0.552	0.035	-1.322	0.224
Large	0.376	0.457	-1.767	0.741	-1.427	0.279
Growth	0.308	0.821	-2.673	0.863	-2.012	0.920
2	0.410	0.104	-1.352	0.410	-1.286	0.129
3	0.392	0.219	-1.299	0.251	-1.503	0.516
4	0.336	0.591	-0.695	0.158	-1.484	0.465
5	0.420	0.075	0.502	0.039	-0.985	0.059
6	0.403	0.137	-0.176	0.147	-1.380	0.336
7	0.419	0.076	-0.018	0.101	-1.234	0.273
8	0.447	0.027	2.083	0.003	-1.163	0.233
9	0.449	0.026	1.905	0.008	-1.098	0.203
Value	0.383	0.174	-0.050	0.202	-1.422	0.436