# Prospect Theory and the Size and Value Premium Puzzles

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Abstract:

Using canonical data for the US stock and bond markets, we show that the kinked piecewise-exponential value function can rationalize the cross-section of stock returns in addition to the level of the equity premium, while the kinked piecewise-power value function of Tversky and Kahneman can explain only the latter.

THE CUMULATIVE PROSPECT THEORY, CPT, (Tversky and Kahneman (1992)) summarizes several violations of the Expected Utility Theory by means of a kinked, piecewise power value function and a reverse S-shaped function transforming the probabilities. In Financial Economics, CPT has been successful in explaining the *equity premium puzzle* or the historically favourable risk-return trade-off of stocks relative to bonds.[1] As shown by Benartzi and Thaler (1995), for a yearly holding period, the CPT statistic of the stock index is not significantly different from the CPT statistic of the bond index. However, as shown in this paper, the same explanation does not rationalize the *size premium puzzle* (Mehra and Prescott (1985)) or the historically favourable risk-return trade-off of small cap stocks relative to large cap stocks (first documented by Banz (1981)) and the *value premium puzzle* or the favourable returns of value stocks relative to growth stocks (first documented by Basu (1977)). Fama and French (1992), (1993)) provide a rigorous empirical analysis of these phenomena.

Nevertheless, the three puzzles can be explained simultaneously if we replace the piecewise-power value function of Tversky and Kahneman with a piecewise-exponential value function. The new value function has a kinked and convex-concave shape (reflecting loss aversion and risk seeking for losses), just as the original value function. However, for large outcomes, the piecewise exponential value function exhibits more curvature and hence the function discourages extreme risk taking[2].

The two versions of value functions, the piecewise-power and the piecewise-exponential compare as follows:

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[1] This is a narrow, portfolio-oriented interpretation of the equity premium puzzle that does not account for the intertemporal investment-consumption problem.

[2] This feature of the piecewise-exponential value function turned out to be important also for proving existence of competitive equilibria in the CAPM based on prospect theory proposed in DeGiorgi, Hens and Levy (2003).

$$v(x) = \begin{cases} b^+ x^a & \text{for } x \geq 0 \\ -b^-(-x)^a & \text{for x} < 0 \end{cases} \text{ and } v(x) = \begin{cases} -l^+ \exp(-a\,x) + l^+ & \text{for } x \geq 0 \\ l^- \exp(a\,x) - l^- & \text{for x} < 0 \end{cases}.$$

where $0 \leq a \leq 1$ and the $b^+$, $l^+$and $b^-$, $l^-$ are positive numbers.

Kahneman and Tversky (1979) report median values for $a$ and $\frac{b^-}{b^+}$ of about 0.88 and 2.25 respectively. Figure 1 shows that our proposal, choosing the parameters $a \approx 0.2$ and , $l^+ = 6.52$ and $l^-$=14.7 (so that $\frac{l^-}{l^+} \approx 2.25$) approximates the Tversky and Kahneman (1992) utility index very well for values around zero. We presume that the experimental evidence given for the value function specification of Kahneman and Tversky (1979) foremost concerns the shape of the utility function around zero. Note also that the utility function we propose is different to that of Kahneman and Tversky (1979) for very high stakes because it is less linear than theirs. Indeed our function is bounded above by $l^+$and it is bounded below by $-l^-$. The piecewise-exponential value function is supported by the laboratory results obtained by Bosch-Domenech and Silvestre (2003), who experimentally find that decision makers usually show risk aversion for larger amounts, for both gains and losses.

**[Insert Figure 1 about here]**

## I. Data

In order to stay as close as possible to the original equity premium studies of Mehra and Prescott (1985) and Benartzi and Thaler (1995) we consider real returns on equity and bonds. However, there are two differences. First, we consider an extended sample from 1927 to 2002, including the bull market of the 1990s and the equity bear market that followed in the early 2000s. Second, we expand the investment universe and include portfolios sorted on market capitalization (ME) and book-to-market-equity ratio (B/M) in the analysis.

The stock market portfolio is proxied by the CRSP all-share index, a value-weighted average of common stocks listed on NYSE, AMEX, and NADAQ. The bond index is defined as the intermediate government bond index maintained by Ibbotson. This index closely matches the 5-year Government bond index employed by Benartzi and Thaler (1995). We use the canonical decile portfolios formed on ME and the decile portfolios formed on B/M. For detailed data description and selection procedures we refer to Fama and French (1992) (1993). We use monthly and annual real returns for the period from January 1927 to December 2002 (912 months). Bond and inflation data are obtained from Ibbotson Associates and the stock portfolio data from Kenneth French's online data library.

Table I presents some basic descriptive statistics of the stock portfolios and bond and equity indices. Clearly, stocks outperform bonds during our 76-year sample period by about 6 percent on an annual basis. However, stocks are riskier which is reflected in a low minimum (-40% in the worst year) and a high standard deviation. Contrary, bonds offer downside protection (-17% in the worst year), but the upside potential is limited. Small and value firms offer higher average returns and higher variance, combined with positive skewness. Puzzling is the BM8 and BM9 portfolios, which combine high average returns with a minimum return above -50% and a maximum return in excess of 100%. Clearly, these portfolios seem far more attractive than the all-equity index.

**[Insert Table I about here]**

## II. Methodology

We test if the market portfolio of risky assets is the optimal portfolio for a representative investor who obeys to the rules of (1) the mean-variance framework, (2) the CPT or (3) our extension of CPT. The standard approach to test if the market portfolio is optimal is to analyze the first-order condition or the Euler equation. This approach is valid for the mean-variance framework, because the first-order condition is necessary and sufficient for establishing the maximum in this framework. By contrast, the first-order condition gives only a necessary optimality condition for the CPT and our extension of CPT. Both models allow for allows for local risk seeking and hence there may be minima and local maxima, which will also satisfy the first-order condition.

There exist various multivariate global optimization methods for locating the global optimum if the objective function is not concave (see, for example, Horst and Pardalos (1995)). Unfortunately, these methods generally are computationally too demanding for high dimension problems such as ours (we use 22 assets).

To circumvent this problem, we analyze the various objective functions (Sharpe ratio, CPT statistic, adjusted CPT statistic) at all the individual benchmark portfolios. This approach can be seen as a very rough grid search; the individual assets are excluded from the analysis and only the 22 benchmark portfolios are seen as a discrete approximation to the investment possibilities set.

Thus, for each benchmark portfolio, we compute the Sharpe ratio, the CPT statistic and the adjusted CPT statistic. To account for sampling variation, we use the bootstrap methodology to compute the p-value for the null that the benchmark portfolio is equally attractive as the market portfolio.

## III. Test results

Contrary to Benartzi and Thaler (1995), the CPT statistic of the bond index is significantly higher than the CPT statistic of the stock index. This is due to the inclusion of the equity bear market in the early 2000s. Further, CPT cannot rationalize the size and value effects. Specifically, while the CPT statistic of the stock market index is –1.590, the CPT statistic of size portfolio 1 is 2.290 (0.03) and that of B/M portfolio 8 is 2.083 (0.00). In large part, these high values are explained by the favourable upside potential of small caps and value stocks. For example, the ME 1 portfolio of small caps has a maximum return of 155.29% and the BM1 portfolio of value stocks has a maximum of 113.53%. Interestingly, there is no corresponding downside risk for the small caps and value stocks. Apparently, the return distribution is positively skewed and highly correlated in downside markets, which limits the downside risk and the potential for downside risk reduction by means of portfolio diversification. These properties make the small cap and value stock portfolios very attractive for the CPT investor, who overweighs small probabilities and whose marginal value function diminishes very slowly.

Using the piecewise-exponential value function, all three puzzles disappear. The bond index does not achieve a significantly higher CPT+ statistic than the stock index. Also, the size and value effects disappear; no benchmark portfolio achieves a significantly higher CPT+ statistic than the market portfolio. Because the marginal function of the piecewise-exponential value function decreases much faster than the piecewise-power value function, CPT+ assigns a much lower weight to the upside potential of the small caps and value stocks. In brief, the piecewise-exponential value function succeeds in explaining away the equity premium, size premium and value premium puzzle at the same time.

**[Insert Table II about here]**

To further illustrate our point, Figure 2 shows the effect of underweighting or overweighting the size portfolio 1 and/or the B/M portfolio 8 on the value of the three objective functions (Sharpe ratio, CPT statistic and CPT+statistic). Panel A shows the Sharpe ratio as a function of the weights assigned to the two portfolios. Adding a position in the value portfolio yields a substantial increase in the Sharpe ratio. While the market portfolio yields a Sharpe ratio of 0.38, investing 100\% in the value portfolio yields a Sharpe ratio of 0.45. By contrast, adding a position in the small cap portfolio has a small negative effect. Panel B shows the results if we replace the Sharpe ratio with the CPT statistic. The value premium persists. The market portfolio yields a CPT statistic of -1.59, while investing 100\% in the value portfolio yields a CPT statistic of 2.08. Consistent with the results in Table II, we also see the profitability of a small cap strategy. Specifically, investing 100\% in the small cap portfolio yields a CPT statistic of 2.29 and investing 100\% in the value portfolio and 100\% in the small cap portfolio (and shorting the market portfolio) even yields a CPT statistic as high as 5.36. Panel C displays the results for the extended CPT. The improvement possibilities from value strategies become less pronounced.

**[Insert Figure 2 about here]**

## IV. Conclusion

While CPT can rationalize the equity premium puzzle (at least in the Benartzi and Thaler sample), it fails to explain the size and value premium puzzles. By contrast, the LGH extension of CPT, based on a kinked, convex-concave, piecewise-exponential value function rationalizes all three puzzles simultaneously. Hence, the key elements of CPT (loss aversion, convexity for losses and probability transformation) seem to explain the key asset pricing puzzles provided we mitigate the extreme risk seeking implied by CPT.

# References


Banz, Rolf W., 1981, The relationship between return and market value of common stocks, *Journal of Financial Economics* 9, 3-18.

Basu, Sanjoy, 1977, Investment performance of common stocks in relationship to their price-earnings ratios: A test of the efficient market hypothesis, *Journal of Finance* 32, 663-682.

Bosch-Domenech, A. and J. Silvestre (2003) ``Reflections on Gains and Losses: A 2x2x2x7 Experiment", Working Paper, Universitat Pompeu Fabra and University of California at Davis.

Benartzi, S., and R. H. Thaler, 1995, Myopic Loss Aversion and the Equity Premium Puzzle, *Quarterly Journal of Economics* 110, 73-92.

Fama, Eugene F., and Kenneth R. French, 1992, The cross-section of expected stock returns, *Journal of Finance* 47, 427-465.

Fama, Eugene F., and Kenneth R. French, 1993, Common risk-factors in the returns on stocks and bonds, *Journal of Financial Economics* 33, 3-56.

Horst, Reiner and Panos M. Pardalos (1995), *Handbook of Global Optimization*, Kluwer, Dordrecht.

De Giorgi, Enrico, Thorsten Hens and Haim Levy ,, 2003, Existence of CAPM equilibria with Prospect Theory Preferences, Zurich NCCR-Working Paper No. 85. http://ssrn.com/abstract=420184.

Mehra, Rajnish and EDward C. Prescott, 1985, THe equity premium: a puzzle, Journal of Monetary Economics 15, 145-161.

Tversky, A., and D. Kahneman, 1992, Advances in Prospect-Theory - Cumulative Representation of Uncertainty, *Journal of Risk and Uncertainty* 5, 297-323.

**Figure 1**

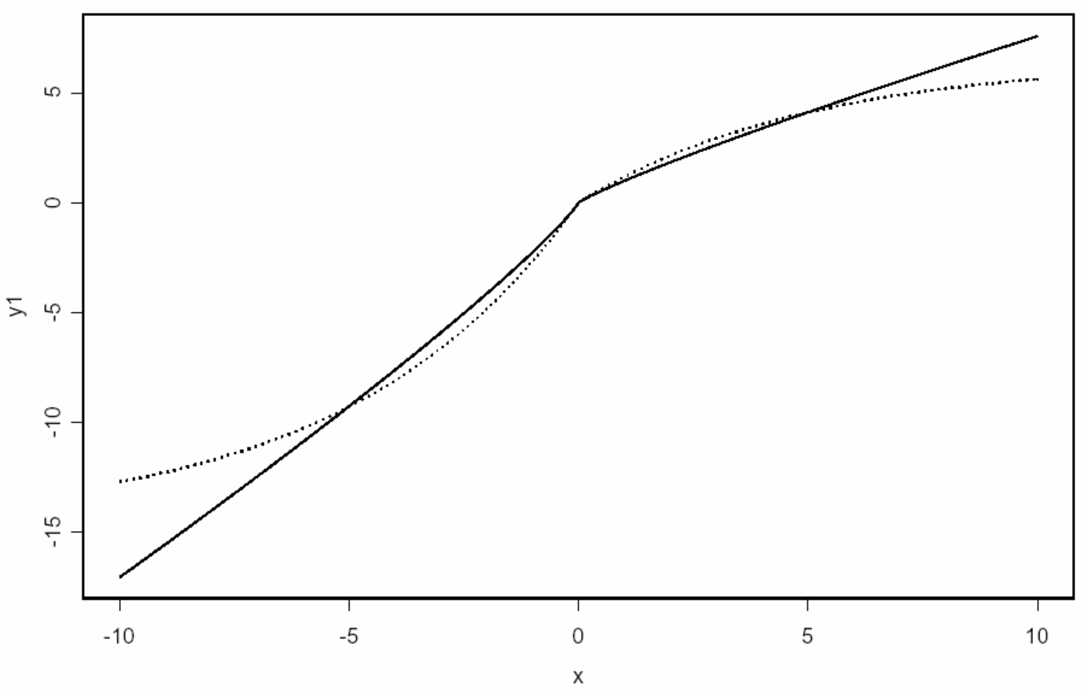


Figure 1: Tversky and Kahneman (1992) utility index (full line) and $u(x) = -\lambda^+ e^{-\alpha x} + \lambda^+$ for $x \geq 0$ and $u(x) = \lambda^- e^{\alpha x} - \lambda^-$ for $x < 0$ (dotted line), where $\lambda^+ = 6.52$, $\lambda^- = 14.7$ and $\alpha \approx 0.2$.

## Figure 2

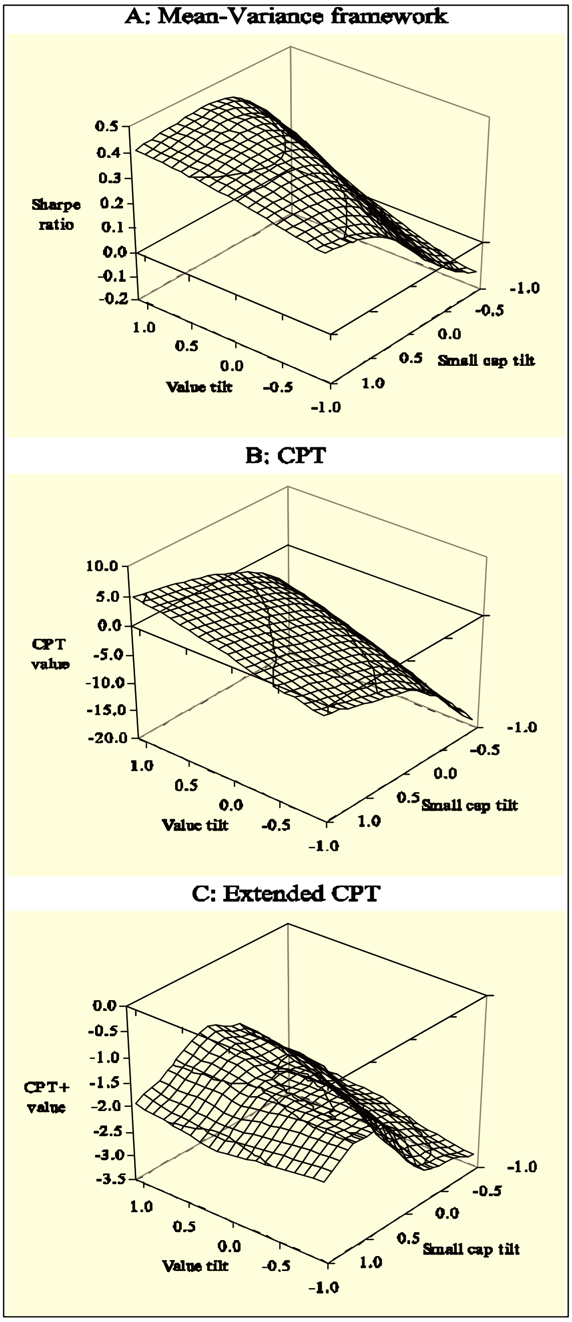


Figure 2: The figure shows the effect of underweighting or overweighting size portfolio 1 (small cap tilt) and/or B/M portfolio 8 (value tilt) on the Sharpe ratio (Panel A), the CPT statistic (Panel B) and the CPT+ statistic (Panel C). Every combination represents a portfolio consisting of a position in the small caps and the value stocks and the remainder in the market portfolio. For example, (0.4,-0.3) represents investing 40\% in small caps, -30\% (a short position) in value stocks and 90\% in the market portfolio. The market portfolio yields a CPT+ statistic of -1.50, while investing 100\% in the value portfolio yields a CPT+ statistic of -1.16, a relatively small improvement. Also, the size effect completely disappears; in fact, small cap strategies yield a substantially lower CPT+ statistic than the market portfolio. For example, investing 100\% in the small cap portfolio yields a CPT+ statistic of -2.17 and investing 100\% in the value portfolio and 100\% in the small cap portfolio (and shorting the market portfolio) yields a CPT statistic of -1.88. Clearly, in Panel C, it is more plausible than in Panel A and B that the market portfolio is the global optimum and that the small remaining improvement possibilities are due to sampling error.

**Table I**

## Descriptive Statistics

The table shows descriptive statistics (average, standard deviation, skewness, **excess** kurtosis and max and min) for the annual real returns of the value-weighted CRSP all-share market portfolio, the intermediate government bond index of Ibbotson and the size and value decile portfolios from Kenneth French' data library. The sample period is from January 1927 to December 2002 (76 yearly observations).

| | Avg. | Stdev. | Skew. | Kurt. | Min | Max |
|---|---|---|---|---|---|---|
| Equity | 8.59 | 21.05 | -0.19 | -0.36 | -40.13 | 57.22 |
| Bond | 2.20 | 6.91 | 0.20 | 0.59 | -17.16 | 22.19 |
| Small | 16.90 | 41.91 | 0.92 | 1.34 | -58.63 | 155.29 |
| 2 | 13.99 | 37.12 | 0.98 | 3.10 | -56.49 | 169.71 |
| 3 | 13.12 | 32.31 | 0.69 | 2.13 | -57.13 | 139.54 |
| 4 | 12.53 | 30.56 | 0.46 | 0.83 | -51.48 | 115.32 |
| 5 | 11.91 | 28.49 | 0.44 | 1.60 | -49.57 | 119.40 |
| 6 | 11.65 | 27.46 | 0.31 | 0.61 | -49.69 | 102.17 |
| 7 | 11.09 | 25.99 | 0.30 | 1.14 | -47.19 | 102.06 |
| 8 | 10.15 | 23.76 | 0.29 | 1.19 | -42.68 | 94.12 |
| 9 | 9.63 | 22.33 | 0.02 | 0.46 | -41.68 | 78.15 |
| Large | 8.06 | 20.04 | -0.22 | -0.52 | -40.13 | 48.74 |
| Growth | 7.84 | 23.60 | 0.02 | -0.64 | -44.92 | 60.35 |
| 2 | 8.77 | 20.41 | -0.27 | -0.27 | -39.85 | 55.89 |
| 3 | 8.52 | 20.56 | -0.10 | -0.47 | -38.00 | 51.90 |
| 4 | 8.25 | 22.49 | 0.49 | 2.39 | -45.02 | 96.33 |
| 5 | 10.29 | 22.82 | 0.36 | 1.92 | -51.55 | 93.77 |
| 6 | 10.05 | 23.04 | 0.19 | 0.63 | -54.39 | 73.57 |
| 7 | 11.00 | 24.73 | 0.18 | 1.22 | -51.13 | 97.91 |
| 8 | 12.82 | 27.01 | 0.67 | 1.95 | -46.56 | 113.53 |
| 9 | 13.71 | 29.08 | 0.56 | 1.85 | -47.42 | 123.72 |
| Value | 13.32 | 33.05 | 0.43 | 1.40 | -59.78 | 134.46 |

**Table II**

## Test Results

The table shows for each benchmark portfolio the Sharpe ratio, the CPT statistic and the adjusted CPT statistic with the piecewise-exponential value function. Also, the table reports the bootstrap p-value. Cells with numbers in bold face refer to portfolios that yield a significantly higher value than the market portfolio at a 5% significance level.

| | MV | | CPT | | CPT+ | |
|---|---|---|---|---|---|---|
| | Statistic | p-value | statistic | p-value | statistic | p-value |
| Equity | 0.380 | | -1.590 | | -1.496 | |
| Bond | **0.329** | **0.007** | **-0.788** | **0.008** | -1.105 | 0.240 |
| Small | 0.384 | 0.140 | **2.290** | **0.030** | -2.172 | 0.933 |
| 2 | 0.357 | 0.317 | 1.053 | 0.085 | -1.981 | 0.888 |
| 3 | 0.384 | 0.215 | 0.654 | 0.085 | -1.749 | 0.749 |
| 4 | 0.387 | 0.212 | 0.278 | 0.066 | -1.509 | 0.514 |
| 5 | 0.394 | 0.180 | 0.197 | 0.070 | -1.411 | 0.377 |
| 6 | 0.400 | 0.153 | **0.101** | **0.043** | -1.441 | 0.413 |
| 7 | 0.402 | 0.142 | **0.076** | **0.033** | -1.416 | 0.347 |
| 8 | 0.403 | 0.140 | **-0.006** | **0.020** | -1.342 | 0.233 |
| 9 | 0.404 | 0.116 | **-0.552** | **0.035** | -1.322 | 0.224 |
| Large | 0.376 | 0.457 | -1.767 | 0.741 | -1.427 | 0.279 |
| Growth | 0.308 | 0.821 | -2.673 | 0.863 | -2.012 | 0.920 |
| 2 | 0.410 | 0.104 | -1.352 | 0.410 | -1.286 | 0.129 |
| 3 | 0.392 | 0.219 | -1.299 | 0.251 | -1.503 | 0.516 |
| 4 | 0.336 | 0.591 | -0.695 | 0.158 | -1.484 | 0.465 |
| 5 | 0.420 | 0.075 | **0.502** | **0.039** | -0.985 | 0.059 |
| 6 | 0.403 | 0.137 | -0.176 | 0.147 | -1.380 | 0.336 |
| 7 | 0.419 | 0.076 | -0.018 | 0.101 | -1.234 | 0.273 |
| 8 | **0.447** | **0.027** | **2.083** | **0.003** | **-1.163** | 0.233 |
| 9 | **0.449** | **0.026** | **1.905** | **0.008** | **-1.098** | 0.203 |
| Value | 0.383 | 0.174 | -0.050 | 0.202 | -1.422 | 0.436 |