Optimal Consumption and Portfolio in a Jump Diffusion Market

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Revised June 13, 2001

Abstract

We consider the problem of optimal consumption and portfolio in a jump diffusion market consisting of a bank account and a stock, whose price is modelled by a geometric Lévy process.

We show that in the absence of transaction costs, the solution in the jump diffusion case has the same form as in the pure diffusion case solved by Merton [M]. In particular, the optimal portfolio is to keep a constant fraction of wealth invested in the stock. This constant is smaller than the corresponding optimal fraction in the pure diffusion case.

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1 Introduction

In this paper we study the problem of optimal consumption and investment policy in a jump diffusion market consisting of a bank account and a stock, whose price is modelled by a geometric Lévy process.

Suppose the bank account gives a fixed interest rate $r > 0$. Then the price $P_1(t)$ of this asset is given by

$$dP_1(t) = rP_1(t)dt \text{ for } t \geq 0; \quad P_1(0) = p_1 > 0. \tag{1}$$

Let $P_2(t)$ denote the price of the stock at time t. Assume that $P_2(t)$ is a càdlàg process satisfying the following stochastic differential equation

$$dP_2(t) = \alpha P_2(t)dt + \sigma P_2(t)dW(t) + P_2(t^-) \int_{-1}^{\infty} \eta \tilde{N}(dt, d\eta); P_2(0^-) = p_2 > 0. \tag{2}$$

Here α and σ are positive constants, $W(t)$ is a Wiener process (Brownian motion) on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ and

$$\tilde{N}(t, U) = N(t, U) - tq(U); \quad t \geq 0, U \in \mathcal{B}(-1, \infty)$$

is the compensator of a homogeneous Poisson random measure $N(t, U)$ on $\mathbf{R}^+ \times \mathcal{B}(-1, \infty)$ with intensity measure $E[N(t, U)] = tq(U)$, where $dq(\eta)$ is the Lévy measure associated to N. We denote by $\mathcal{B}(-1, \infty)$ the Borel σ-algebra on $(-1, \infty)$. We assume that

$$\int_{-1}^{\infty} |\eta| q(d\eta) < \infty. \tag{3}$$

Note that since we only allow jump sizes η which are bigger than -1, the process $P_2(t)$ will remain positive for all $t \geq 0$, a.s. See e.g. Bensoussan and Lions [BL], Jacod and Shiryaev [JS] and Protter [P] for more information about such stochastic differential equations.

We assume that at any time t the investor can choose a rate $c(t) \geq 0$ of consumption taken from the bank account. We also assume that we can transfer money at any time from one asset to the other without transaction costs. Let $X(t), Y(t)$ denote the amount of money invested in bank and stock respectively. Let

$$u(t) = \frac{Y(t)}{X(t) + Y(t)} \tag{4}$$

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be the fraction of the total wealth invested in stock at time t.

Define the performance criterion by

$$J^{c,u}(x,y) = E^{x,y}\left[\int_0^\infty e^{-\delta t}\frac{c^\gamma(t)}{\gamma}dt\right] \tag{5}$$

where $\delta > 0, \gamma \in (0,1)$ are constants and $E^{x,y}$ is the expectation with respect to the probability law $P^{x,y}$ of $(X(t), Y(t))$ when $(X(0^-), Y(0^-)) = (x,y) \in \mathbf{R}^2$. The problem is to find V and $(c^*, u^*) \in \mathcal{A}$ such that

$$V(x,y) = \sup_{(c,u)\in\mathcal{A}} J^{c,u}(x,y) = J^{c^*,u^*}(x,y) \tag{6}$$

where $\mathcal{A}$ is the family of *admissible* controls (see Section 2). In the special case when the stock price is a geometric Brownian motion (i.e., $\tilde{N} = 0$) this problem was first studied by Merton [M]. He proved that if

$$\delta > \gamma\left[r + \frac{(\alpha - r)^2}{2\sigma^2(1-\gamma)}\right] \tag{7}$$

then the value function $V_0(x,y)$ is given by

$$V_0(x,y) = K_0(x+y)^\gamma \tag{8}$$

where

$$K_0 = \frac{1}{\gamma}\left[\frac{1}{1-\gamma}\left(\delta - \gamma r - \frac{\gamma(\alpha-r)^2}{2\sigma^2(1-\gamma)}\right)\right]^{\gamma-1}. \tag{9}$$

Moreover, the corresponding optimal consumption c_0^* is given (in feedback form) by

$$c_0^*(x,y) = (K_0\gamma)^{\frac{1}{\gamma-1}}(x+y) \tag{10}$$

and the corresponding optimal portfolio is to keep the fraction $Y(t)/(X(t)+Y(t))$ of wealth invested in the stocks constantly equal to the value

$$u_0^* = \frac{\alpha - r}{(1-\gamma)\sigma^2} \quad \text{at all times.} \tag{11}$$

In other words, it is optimal to perform transactions in such a way that the state $(X(t), Y(t))$ is always situated on the line $y = \dfrac{u_0^*}{1 - u_0^*}x$ in the (x,y)-plane (the Merton line).

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In Section 2 we extend the results of Merton to the case when the stock price is a *geometric Lévy process* (i.e., given by (2)) We prove that the value function $V(x, y)$ still has the same form, namely

$$V(x, y) = K(x + y)^{\gamma} \tag{12}$$

but with a different constant K (under an assumption similar to (7). The corresponding optimal consumption c^* is given by

$$c^*(x, y) = (K\gamma)^{\frac{1}{\gamma - 1}} \cdot (x + y) \tag{13}$$

and the optimal portfolio is to keep the fraction $Y(t)/(X(t) + Y(t))$ constantly equal to a value u^* (see Theorem 2.3).

We also prove that if $N \neq 0$ then $V(x, y) < V_0(x, y)$, $c^*(x, y) > c_0^*(x, y)$ and $u^* < u_0^*$ (see Figure 1).

Actually, the introduction of the jump term involving the integral with respect to $\tilde{N}$ has the same effect on the solution as increasing the volatility σ.

Our work is also inspired by the paper [Z], where an optimal stopping problem for a jump diffusion process is solved.

After this paper was written, we became aware of a paper by K.Aase [Aa]. He actually solves the same optimal consumption and portfolio problem, albeit in a slightly different jump diffusion market.

2 Optimal portfolio and consumption

As in Merton [M] we reduce the dimension of the problem by introducing the total wealth process

$$Z(t) = X(t) + Y(t) \tag{14}$$

as a new state variable and by representing the portfolio by the fraction

$$u(t) = \frac{Y(t)}{Z(t)} \tag{15}$$

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of the wealth invested in stock at time t. Using (1-2) we see that the dynamics of $Z(t)$ is given by

$$dZ(t) = ([r(1 - u(t)) + \alpha u(t)]Z(t) - c(t))dt + \sigma u(t)Z(t)dW(t)$$

$$+ uZ(t^-) \int_{-1}^{\infty} \eta \tilde{N}(dt, d\eta), \quad Z(0^-) = z = x + y \geq 0. \tag{16}$$

We now consider $(c(t), u(t))$ as our control and we call it *admissible* and write $(c(t), u(t)) \in \mathcal{A}$ if:

(i) the processes $c(t), u(t)$ are predictable,

(ii) $c(t, \omega) \geq 0$ for a.e. (t, ω),

(iii) $u(t, \omega) \in [0, 1]$ for a.e. (t, ω),

(iv) If the initial endowment $z \in \mathcal{S}$, then $Z(t) \in \mathcal{S}$ for all $t \geq 0$, where $\mathcal{S}$ is the solvency region:

$$\mathcal{S} = \{z \in \mathbf{R}; z \geq 0\}.$$

Remark 2.1. The constraint $u(t) \in [0, 1]$ is necessary to ensure that $Z(t)$ remains nonnegative.

The problem can now be formulated as follows: Find the function $V(z)$ and the corresponding optimal strategy $(c^*(t), u^*(t)) \in \mathcal{A}$ such that

$$V(z) = \sup_{(c(t), u(t)) \in \mathcal{A}} J^{c,u}(z) = J^{c^*, u^*}(z) \tag{17}$$

where

$$J^{c,u}(z) = E^z \left[\int_0^{\infty} e^{-\delta t} \frac{c^{\gamma}(t)}{\gamma} dt \right]. \tag{18}$$

Note that the generator $\hat{A}$ of the time-space process $d\hat{Z}(t) = (dt, dZ(t))$ when applying the (Markov) control (c, u) is given by

$$\hat{A}\varphi(s, z) = \frac{\partial \varphi}{\partial s} + ([r + (\alpha - r)u]z - c)\frac{\partial \varphi}{\partial z} + \frac{1}{2}\sigma^2 u^2 z^2 \frac{\partial^2 \varphi}{\partial z^2}$$

$$+ \int_{-1}^{\infty} \left[\varphi(s, z + uz\eta) - \varphi(s, z) - \frac{\partial \varphi}{\partial z}(s, z)uz\eta \right] dq(\eta).$$

Therefore we can formulate a Hamilton-Jacobi-Bellman (HJB) verification theorem as follows (see e.g. [BL] or [F,Th.III.4] for a proof).

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Theorem 2.2. Verification theorem.

a) Let $v(z) : [0, \infty) \to [0, \infty)$ be a twice continuously differentiable function such that

$$H(c, u, z) \quad := \quad \frac{1}{\gamma}c^\gamma - \delta v(z) + ([r + (\alpha - r)u]z - c)v'(z)$$

$$+ \frac{1}{2}\sigma^2 u^2 z^2 v''(z) + \int_{-1}^{\infty} [v(z + uz\eta) - v(z) - v'(z)uz\eta]dq(\eta) \leq 0 \tag{19}$$

for all $c \geq 0$, $u \in [0, 1]$.

Then

$$v(z) \geq V(z). \tag{20}$$

b) Suppose, in addition to (19), that for all $z \geq 0$ there exist $c^*(z) \geq 0, u^*(z) \in [0, 1]$ such that

$$H(c^*(z), u^*(z), z) = 0. \tag{21}$$

Suppose $(c^*, u^*) \in \mathcal{A}$ and that

$$\lim_{R \to \infty} E^z[e^{-\delta T_R} v(Z^*(T_R))] = 0 \text{ for all; } z \geq 0 \tag{22}$$

where $T_R = \min(R, \inf\{t > 0; |Z^*(t)| \geq R\})$ and $Z^*(t)$ is the wealth process obtained by using the control (c^*, u^*). Then

$$v(z) = V(z) \text{ for all } z \geq 0$$

and the control (c^*, u^*) is optimal.

Our main result is the following:

Theorem 2.3. *Let u^* be the maximum point over the interval $[0, 1]$ of the function*

$$h(u) := (\alpha - r)u\gamma - \frac{1}{2}\sigma^2 u^2 \gamma(1 - \gamma) + \int_{-1}^{\infty} [(1 + u\eta)^\gamma - 1 - u\gamma\eta]dq(\eta); u \in [0, 1] \tag{23}$$

and assume that

$$\lambda := \delta - r\gamma - h(u^*) > 0, \tag{24}$$

$$\alpha > r \text{ and } \|q\| = \int_{-1}^{\infty} dq(\eta) < \infty, \tag{25}$$

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and

$$\delta - u^* \gamma [2\alpha - r - \|q\| - \int_{-1}^{\infty} \eta dq(\eta)] - h(u^*) > 0. \tag{26}$$

Set

$$K = \frac{1}{\gamma} \left(\frac{1 - \gamma}{\lambda} \right)^{1 - \gamma}. \tag{27}$$

Then the value function $V(z)$ for problem (17) is given by

$$V(z) = K z^{\gamma}. \tag{28}$$

The corresponding optimal consumption rate is

$$c^*(z) = (K\gamma)^{\frac{1}{\gamma - 1}} z \tag{29}$$

and the optimal portfolio is to keep the fraction u constantly equal to the maximum point u^* of the function $h(u)$ defined in (23).

Proof. The first order conditions for maximality of $H(c, u, z)$ as a function of c is

$$c^{\gamma - 1} - v'(z) = 0 \tag{30}$$

i.e.,

$$c = c^*(z) = (v'(z))^{\frac{1}{\gamma - 1}}, \text{if } v'(z) > 0. \tag{31}$$

Let us try to put

$$v(z) = C z^{\gamma} \tag{32}$$

for some constant C. By (31) this gives

$$c^*(z) = (C\gamma)^{\frac{1}{\gamma - 1}} z \tag{33}$$

and

$$\frac{1}{\gamma}(c^*(z))^{\gamma} - c^*(z)v'(z) = \frac{1 - \gamma}{\gamma}(v'(z))^{\frac{\gamma}{\gamma - 1}} = \frac{1 - \gamma}{\gamma}(C\gamma)^{\frac{\gamma}{\gamma - 1}} z^{\gamma}.$$

Substituted into (19) and (21) this gives

$$\sup_{u \in [0,1]} \left\{ \frac{1 - \gamma}{\gamma}(C\gamma)^{\frac{\gamma}{\gamma - 1}} - \delta C + \quad [r + (\alpha - r)u]\gamma C - \frac{1}{2}\sigma^2 u^2 \gamma(1 - \gamma)C \right. \\ \left. + C \int_{-1}^{\infty} [(1 + u\eta)^{\gamma} - 1 - u\gamma\eta]dq(\eta) \right\} = 0. \tag{34}$$

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The first order condition for maximality of the function h defined in (23) is

$$g(u) := \alpha - r - u\sigma^2(1 - \gamma) - \int_{-1}^{\infty}[1 - (1 + u\eta)^{\gamma-1}]\eta dq(\eta) = 0. \tag{35}$$

Since $g(0) = \alpha - r > 0$ and $g(1) = \alpha - r - \sigma^2(1 - \gamma) - \int_{-1}^{\infty}[1 - (1 + \eta)^{\gamma-1}]\eta dq(\eta)$, we see that if $g(1) < 0$, i.e. if

$$0 < \alpha - r < \sigma^2(1 - \gamma) + \int_{-1}^{\infty}[1 - (1 + \eta)^{r-1}]\eta dq(\eta), \tag{36}$$

then there exists $u^* \in (0, 1)$ such that $g(u^*) = 0$. Otherwise the maximum point u^* of $h(u)$ over the interval $[0, 1]$ is either $u^* = 0$ or $u^* = 1$. Note that u^* is constant, both with respect to t and x.

With this choice $u = u^*$ substituted into (34) we get

$$\frac{1 - \gamma}{\gamma}(C\gamma)^{\frac{\gamma}{\gamma-1}} - C\lambda = 0, \tag{37}$$

where

$$\lambda := \delta - [r + (\alpha - r)u^*]\gamma + \frac{1}{2}\sigma^2(u^*)^2\gamma(1 - \gamma) - \int_{-1}^{\infty}[(1 + u^*\eta)^{\gamma} - 1 - u^*\gamma\eta]dq(\eta). \tag{38}$$

When $\lambda > 0$, i.e. when

$$\delta > [r + (\alpha - r)u^*]\gamma - \frac{1}{2}\sigma^2(u^*)^2\gamma(1 - \gamma) + \int_{-1}^{\infty}[(1 + u^*\eta)^{\gamma} - 1 - \gamma u^*\eta]dq(\eta), \tag{39}$$

(which is (24)), then (37) has the solution

$$C = K := \frac{1}{\gamma}\left(\frac{1 - \gamma}{\lambda}\right)^{1-\gamma}. \tag{40}$$

With this choice of C we see that the function

$$v(z) := Kz^{\gamma} \tag{41}$$

satisfies (19). We conclude that

$$V(z) \leq Kz^{\gamma} \text{ for all } z \geq 0. \tag{42}$$

Moreover, by our choice of c^*, u^* we also have that (21) holds. So the proof is complete if we can verify that (22) holds, i.e., that

$$\lim_{R \to \infty} E^z[e^{-\delta T_R}(Z^*(T_R))^{\gamma}] = 0 \text{ for all } z \geq 0. \tag{43}$$

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To this end, note that by (16) we have

$$dZ^*(t) \le u^*\alpha Z^*(t)dt + u^*\sigma Z^*(t)dW(t) + Z^*(t^-)u^* \int_{-1}^{\infty} \eta d\tilde{N}(dt, d\eta).$$

Therefore, by well-known comparison principles we get

$$Z^*(t) \le L(t) \text{ for all } t \ge 0$$

where $L(t)$ is the geometric Lévy process given by

$$dL(t) = u^*\alpha L(t)dt + u^*\sigma L(t)dW(t) + L(t^-)u^* \int_{-1}^{\infty} \eta d\tilde{N}(dt, d\eta), \tag{44}$$

with initial value $L(0^-) = Z^*(0^-) = z$.

The solution of (44) is

$$L(t) = z \exp\{(u^*\alpha - \frac{1}{2}(u^*\sigma)^2 - u^*\|q\|)t + u^*\sigma W(t) + \int_0^t \int_{-1}^{\infty} \ln(1 + u^*\eta)N(ds, d\eta)\} \tag{45}$$

where $\|q\| = q(-1, \infty)$.

In general we have that

$$E[\exp\{\int_0^{T_R} \int_{-1}^{\infty} \ln(1 + h(s, \eta))N(ds, d\eta)\}] = E[\exp\{\int_0^{T_R} \int_{-1}^{\infty} h(s, \eta)q(d\eta)ds\}]$$

(assuming the integrals exist). If $\gamma > 0$ is a constant we can write

$$\gamma \ln(1 + h(s, \eta)) = \ln(1 + (1 + h(s, \eta))^{\gamma} - 1),$$

and hence

$$E[\exp\{\int_0^{T_R} \int_{-1}^{\infty} \gamma \ln(1 + h(s, \eta))N(ds, d\eta)\}] =$$
$$E[\exp\{\int_0^{T_R} \int_{-1}^{\infty} ((1 + h(s, \eta))^{\gamma} - 1)q(d\eta)ds\}].$$

Using this in (45) we get

$$\begin{aligned} E[L(T_R)^{\gamma}] &= z^{\gamma} E[\exp\{\gamma(u^*\alpha - \frac{1}{2}(u^*\sigma)^2 - u^*\|q\|)T_R \\ &+ \frac{1}{2}\gamma^2(u^*\sigma)^2 T_R + \int_{-1}^{\infty}((1 + u^*\eta)^{\gamma} - 1)q(d\eta) \cdot T_R\}]. \end{aligned}$$

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Hence

$$E[e^{-\delta T_R}(Z^*(T_R))^\gamma] \leq z^\gamma E[\exp\{(-\delta + \gamma u^*\alpha - \frac{1}{2}(u^*\sigma)^2\gamma(1-\gamma) - \gamma u^*\|q\|$$
$$+ \int_{-1}^{\infty} ((1+u^*\eta)^\gamma - 1)q(d\eta))T_R\}]$$

which tends to 0 as R goes to ∞, because by (26) the coefficient of T_R in the exponent is negative. $\square$

We compare our solution $V(z), c^*, u^*$ in the jump diffusion case to the Merton solution $V_0(z), c_0^*, u_0^*$ in the pure diffusion case ($N = 0$):

Corollary 2.4. *Assume that the conditions of Theorem 2.3 hold and let $V(z), c^*, u^*$ be as given there. Let $V_0(z), c_0^*, u_0^*$ be the corresponding solution when there are no jumps, i.e., when $N = 0$. Then we have*

$$V(z) \leq V_0(z) \quad \forall z \geq 0$$

$$c^*(z) \geq c_0^*(z) \quad \forall z \geq 0$$

and

$$u^* \leq u_0^*.$$

Proof. Let $h(u)$ be as in (23) and let $h_0(u)$ be the corresponding function with $q = 0$, i.e.,

$$h_0(u) = (\alpha - r)u\gamma - \frac{1}{2}\sigma^2 u^2\gamma(1-\gamma).$$

Then

$$h(0) = h_0(0) = 0 \text{ and}$$

$$h'(u) = \gamma[\alpha - r - \sigma^2 u(1-\gamma) - \int_{-1}^{\infty}[1 - (1+u\eta)^{\gamma-1}]\eta dq(\eta)$$

and therefore we see that

$$h'(u) \leq h_0'(u) \quad \text{for all } u \in [0, 1].$$
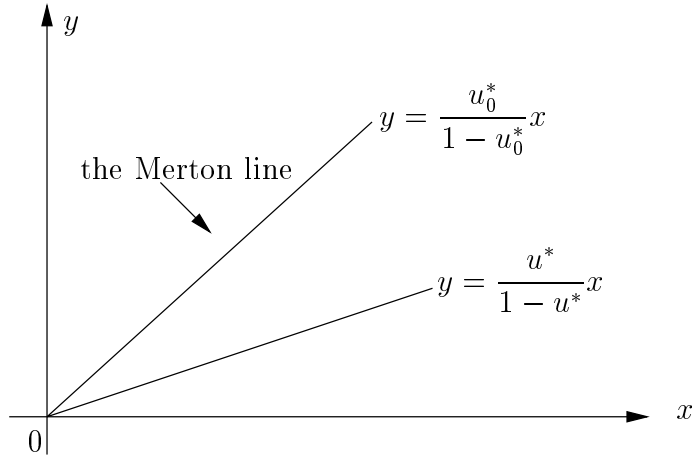
We conclude that

$$u^* \leq u_0^*$$

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y

$y = \dfrac{u_0^*}{1 - u_0^*} x$

the Merton line

$y = \dfrac{u^*}{1 - u^*} x$

x

0

Figure 1: Optimal policy in the jump diffusion and the pure diffusion market

and

$$h(u^*) \le h_0(u_0^*).$$

Therefore, if λ is as in (24) and λ_0 is the corresponding constant for $q = 0$, i.e.,

$$\lambda_0 = \delta - r\gamma - h_0(u_0^*)$$

then

$$\lambda_0 \le \lambda.$$

Therefore, by (27),

$$K = \frac{1}{\gamma}\left(\frac{1 - \gamma}{\lambda}\right)^{1-\gamma} \le K_0 := \frac{1}{\gamma}\left(\frac{1 - \gamma}{\lambda_0}\right)^{1-\gamma}.$$

Hence by (28)

$$V(z) = Kz^\gamma \le K_0 z^\gamma = V_0(z)$$

and by (29)

$$c^*(z) = (K\gamma)^{\frac{1}{\gamma-1}} z \ge (K_0\gamma)^{\frac{1}{\gamma-1}} z = c_0^*(z).$$

$\square$

Remark 2.5. The results of Corollary 2.4 show that the effect of introducing jumps in the model is the same as the effect of increasing the volatility: the value function *decreases*, the optimal

consumption rate (as a function of the current wealth) *increases* and the optimal fraction of the wealth to be kept in the risky investment *decreases*.

The first two statements may appear contradictory, because the value funtion is by definition the expected value of the discounted utility of the optimal consumption rate. However, since the consumption rates $c^*(z)$ and $c_0^*(z)$ are given in *feedback form* (Markovian controls) the actual consumption rates are given by

$$c^*(Z^*(t)) \text{ and } c_0^*(Z_0^*(t))$$

respectively, where $Z_0^*(t)$ is the optimal wealth process when $q = 0$. And since we may have $Z^*(t) \leq Z_0^*(t)$ we may well have

$$c^*(Z^*(t)) \leq c_0^*(Z_0^*(t))$$

even though

$$c^*(z) \geq c_0^*(z) \quad \text{for all } z \geq 0.$$

The problem with transaction costs is treated in a forthcoming paper [FØS].

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