IRREVERSIBLE INVESTMENTS REVISITED

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Abstract

A non-linear dynamic model in two state variables, two controls and three cost terms is presented for the purpose of finding the optimal combination of exploitation and capital investment in optimal renewable resource management. Non-malleability of capital is, in other words, incorporated in the model through an asymmetric convex cost-function of investment, and investments can be both positive and negative. Exploitation is controlled through the utilisation rate of available capital. A novel feature in this model is that there are fixed costs associated with the available capital whether it is utilised or not. In contrast to most of the previous literature both state variables enter the objective function.

Keywords: irreversible investments, non-malleable capital, renewable resources.

Introduction

Many renewable resources are characterized by overexploitation combined with excessive exploitation capacity. As the existing capacity has a tendency to put additional pressure on the resource, it is more and more agreed upon that models for optimal management should include capital dynamics as well as resource dynamics. By capital is here meant physical capital that can be used for the purpose of exploitation of the resource.

Smith (1968) was the first to consider capital accumulation in resource economics within a model with two capital stocks, biological and physical. The analysis of irreversible investments in physical capital, however, was initiated by the pioneering work by Clark, Clarke and Munro (1979). They developed a model to analyse the effects of irreversibility of capital investments upon optimal exploitation policies for renewable resource stocks. This is a deterministic linear model with bang-bang policies, and the main conclusion is that whereas the long-run optimal steady state is unaffected by the assumption about irreversibility the short-term optimal policies may depend significantly upon this assumption. McKelvey (1985) studied the same problem within an open access regime and found the results of Clark, Clarke and Munro to hold there as well. Charles (1983) and Charles and Munro (1985) perform stochastic analyses of the same problem and find that the effects of uncertainty can go either way with respect to investment. Boyce (1995) was the first to consider non-linearities in the objective function. He presents a model with a general non-linear utility of harvest and cost of investment functions. Neither of these functions, however, includes the resource stock, implying that all cases where the resource stock significantly affects the operating costs are ignored.

In this article we extend these ideas by presenting a deterministic non-linear dynamic model in two state variables, two controls and three cost terms. There is a cost of investment function that mimics the second-hand market for capital. That is, instead of imposing non-negativity constraints on investment to account for irreversibility, investments can take any real value but the unit price is higher when capital is bought (investment) than when it is sold (disinvestment). Non-malleability of capital is, in other words, incorporated in the model through an asymmetric convex cost-function of investment. Exploitation is controlled through the utilisation rate of available capital. In addition, a novel feature in this model is that there are fixed costs associated with the available capital whether it is utilised or not. Typically such costs are insurance, interest on capital, etc. Further, in contrast to most of the previous literature both state variables enter the objective function.

The combination of non-linearities in capital investments and fixed costs associated with capital makes this model unique and fairly realistic, which makes this a valuable extension of previous work on investigation of irreversible investments in renewable resource economics. In the following, the model is presented and investigated analytically.

The Model

The model is a dynamic optimization model with two state variables and two control variables that are strongly connected to each other. It is assumed that the objective is to maximise net present revenue. The fleet is characterized by total physical capital *k*, while the renewable resource is characterized by total biomass, *x*. The instrument used to control the capital is investment, *I,* and the instrument used to control the exploitation of the natural resource is the capital utilisation rate, $\varphi \in [0,1]$. The situation with no exploitation is represented by $\phi = 0$, and $\phi = 1$ represents the situation with exploitation at full capacity utilisation. When $0 < \phi < 1$ there is exploitation at reduced capacity utilisation. The harvest function is defined as

$$h = qx\phi k \qquad (1)$$

where q is an exogenous coefficient. The net revenue function has the form:

$$\Pi(x,k,I,\varphi) = \pi(x,h) - C(I) - K(k), \qquad (2)$$

where $\pi(x,h)$ is the net revenue associated directly with the exploitation activity, x is the stock biomass and h is the harvest rate. The term $C(I)$ comprises costs (or revenues) associated with investment (or disinvestment) I. In other words:

$$C(I) = \begin{cases} > 0, & I > 0 \\ 0, & I = 0 \\ < 0, & I < 0 \end{cases}$$

The fixed costs associated with the total level of available capital, k, whether it is utilised or not, are denoted $K(k)$. The net revenue function is assumed to be a twice continuously differentiable function, and it is further assumed that[1]

$$\begin{aligned} &\pi_h \geq 0,\ \pi_x \geq 0,\ C' > 0,\ K' > 0, \\ &\pi_{hh} < 0,\ \pi_{xx} \leq 0,\ \pi_{xh} \geq 0, \\ &C'' > 0,\ K'' \geq 0 \end{aligned} \qquad (3)$$

[1] Functional dependence is depressed for readability when it does not cause any confusion.

where subscripts denote partial derivatives. The convexity of the investment cost function, C, accounts for the non-malleability of investment. There is an asymmetric relationship between the buying price and the selling price of capital due to the assumption $C''>0$ on $I \in (-\infty, \infty)$. When $I>0$ we are buying capital, and the marginal price of capital, C', is higher then the marginal price we receive when $I<0$ and we are selling and equivalent amount of capital. In other words, the marginal price of capital is continuously increasing in investment whether it is positive or negative. The degree of malleability in this model can be controlled through the convexity of the investment cost function. By adjusting C we can have anything from almost completely malleable capital to completely non-malleable capital. As investment/disinvestment can take any value on the real axis, optimality in this control variable is an inner optimality.

The variables x and k are state variables, while I and ϕ are controls. The state equations for stock and capital are assumed to have the simple forms:

$$\dot{x} = f(x) - h \tag{4}$$

$$\dot{k} = I - bk \tag{5}$$

where $f(x)$ is the biological growth function and b is the depreciation factor for capital.

The optimization problem for the managing authority is given as follows:

$$\max_{\phi, I} \int_0^\infty e^{-\delta t} \Pi\left(k(t), x(t), I(t), \phi(t)\right) dt, \qquad \phi(t) \in [0,1], \; I(t) \in R \tag{6}$$

subject to the dynamic constraints (4) and (5) and subject to

$$\lim_{t\to\infty}(k,x) = (k^*, x^*) \neq (0,0), \quad k, x \geq 0.$$

The latter states that the management regime is obliged to establish a long term sustainable equilibrium. The current value Hamiltonian for this problem becomes:

$$H(k,x,\phi,I,\lambda,\mu) = \Pi(k,x,I,\phi) + \lambda\left[I - bk\right] + \mu\left[f(x) - qx\phi k\right] \qquad (7)$$

For simplicity we assume that the Mangasarian sufficiency theorem for infinite horizon is satisfied[2]. Hence our formulated problem has a solution. The basic properties of the functions involved are:

Basic Assumptions.

The net profit function Π *is twice continuously differential in its arguments. In addition to the properties given in (3) it is assumed that current Hamiltonian is concave in* (x,k,φ,I) *for non-negative values of* λ *and* μ.

The first-order derivatives of the Hamiltonian are:

[2] See e.g. Seierstad and SydSæther, 1987.

$$
\begin{aligned}
H_{\phi} &= [\pi_h - \mu] qxk, \\
H_I &= -C' + \lambda, \\
H_k &= [\pi_h - \mu] qx\phi - K' - b\lambda, \\
H_x &= \Pi_x + \mu(f' - qk\phi).
\end{aligned}
\tag{8}
$$

The dynamic equations for the shadow prices λ and μ become:

$$
\begin{aligned}
\dot{\lambda} &= \delta\lambda - \frac{\partial H}{\partial k} = (\delta + b)\lambda + K' - [\pi_h - \mu] qx\phi, \\
\dot{\mu} &= \delta\mu - \frac{\partial H}{\partial x} = (\delta + qk\phi - f')\mu - \Pi_x.
\end{aligned}
\tag{9}
$$

As investments can take any real value, the rate *I* that maximizes *H* must be a critical point and hence

$$\lambda = C' > 0. \tag{10}$$

As the utilisation rate is constrained by $0 \le \phi \le 1$, it gives rise to three natural regions for $k \cdot x > 0$:

Region A:	$H_{\phi} < 0$	$\mu > \pi_h(x,0)$	$\phi = 0$
Region B:	$H_{\phi} = 0$	$\mu = \pi_h(x,h)$	$0 < \phi < 1$
Region C:	$H_{\phi} > 0$	$\mu < \pi_h(x,h)$	$\phi = 1$

In the following *I* is defined as gross investment whereas the actual change in the capital level, $\dot{k}$, is defined as net investment. Further, the terms over-/undershooting will be used to describe situations where the variables have local maxima/minima. We then state some general results based on the outline above:

Proposition 1.

In regions where the capital is not fully utilised (A and B) gross investment will be increasing and there will never be overshooting with respect to capital.

Proof: In A and B the dynamic equation for the shadow price of capital is given by $\dot{\lambda} = (\delta + b)\lambda + K'$ from (9). From (10) we get $\dot{\lambda} = C''\dot{I} = (\delta + b)C' + K' > 0$, implying $\dot{I} > 0$ given the assumptions in (3). Further, inserting $\dot{k} = 0$ in the expression for $\ddot{k}$, we see that $\ddot{k} = \dot{I} = \dfrac{\dot{\lambda}}{C''} > 0$. Hence any local extreme points with respect to $k(t)$ are necessarily local minima.

The intuition behind this is that in the case of over-capitalization, disinvestment will take place at a decreasing rate and positive investment at an increasing rate. In other words, it is best to accelerate sale of capital and postpone investment. Due to the non-linearity of *C* this will not be a bang-bang operation.

Proposition 2.

In steady state the capital is fully utilised.

Proof: In steady state $\dot{k} = \dot{I} = 0$ by definition. Hence, from Proposition 1 the steady state can not be in A or B; it must be in C.

The intuition behind this is that it will always be wasteful to have unutilised capital in steady state even if the fixed costs of capital $K(k)=0$, as there will be a cost associated with the depreciation of the capital. If $K(k)>0$, there will be an additional cost associated with the idle capital.

Proposition 3.

The shadow price μ *is positive everywhere if the condition* $\delta+\frac{f}{x}>f'$ *holds in steady state. Further, in steady state* $\delta-f'$ *and* $\delta+\frac{f}{x}-f'$ *will have the same sign.*

Proof: In steady state we have from (9): $(\delta+qk\phi-f')\mu=\Pi_x$. Further, as $\dot{x}=0$ and $\phi=1$ we have $h=f=qxk$. This yields $(\delta+\frac{f}{x}-f')\mu=\Pi_x$ in steady state. As $\Pi_x>0$ from (3), we have that $\mu>0$ in steady state when $\delta+\frac{f}{x}>f'$. From (9) we have $\dot{\mu}<0$ when $\mu=0$, hence μ can not go from negative to positive. As μ is positive in steady state, it must always be positive. The last part of Proposition 3 follows from $(\delta+\frac{f}{x}-f')\mu=\Pi_x>0 \Leftrightarrow (\delta-f')\mu=\pi_x+(\pi_h-\mu)\frac{f}{x}>0$ as $\pi_h>\mu$ and $\frac{f}{x}>0$ in steady state.

As $\lambda>0$ from (10) we know that both shadow prices are positive. Further, note that the condition $\delta+\frac{f}{x}>f'$ is always fulfilled for concave growth functions like the logistic. The results in Proposition 3 turn out to be useful later.

In the following it is assumed that the shadow prices are positive everywhere, that is equivalent to $\delta - f' > 0$ in steady state which is a reasonable assumption.

Proposition 4.

The shadow price on the stock, μ, decreases when $\delta < f'$.

Proof: From (6) we have $\dot{\mu} = (\delta - f')\mu - \pi_x - (\pi_h - \mu)\cdot qk\phi < (\delta - f')\mu < 0$ as $\mu > 0$ and $(\pi_h - \mu)\cdot qk\phi \geq 0$.

Letting $V(k,x)$ be the value function, i.e. the shadow prices are given by $\lambda = V_k$ and $\mu = V_x$.

It is reasonable to assume that V is concave and $V_{kx} > 0$. This will be used in the next propositions:

Proposition 5.

In regions where the capital is not fully utilised and the stock is decreasing, capital too must be decreasing.

Proof: From the properties of the value function we get $\dot{V}_k = \dot{\lambda} = C''\cdot \dot{I} = V_{kk}\dot{k} + V_{kx}\dot{x}$ and $\dot{k} = \left(C''\cdot \dot{I} - V_{kx}\dot{x}\right)/V_{kk} < 0$ given the assumptions above.

It follows from Proposition 5 that if capital is not decreasing, we must either have full utilisation of the capital or the stock is not decreasing (or both). Therefore capital will typically be increasing when the stock is increasing and/or the capital is fully utilised.

We now introduce some useful definitions. Let *S* be defined as:

$$S(k,x) \equiv \Pi(k,x,bk,1).$$

This is equivalent to the net revenue when the physical capital is fully utilised and fixed. With this definition we can state the following proposition:

Proposition 6.

When the capital is fixed and fully utilised we have $\frac{\partial S}{\partial k} \cdot k = \delta\lambda k + \mu h$.

Proof: When $\phi = 1$ and $I = bk$ we have $S(k,x) = \pi(x,qkx) - C(bk) - K(k)$. As $\dot{\lambda} = 0$ according to (10) when capital is fixed, we have $\delta\lambda + \mu qx = \pi(k,qkx)qx - b \cdot C'(bk) - K'(k)$. The proposition then follows from $qx = \frac{h}{k}$.

The interpretation of Proposition 6 is that when the capital is fixed and fully utilised, the marginal return on capital shall equal the alternative return on capital plus the marginal return on the biological stock. The term $\frac{\partial S}{\partial k}$ is the rate of return and this is multiplied by the capital level on the left-hand side. The alternative rate of return is δ, and this is multiplied by the capital evaluated at its shadow price λ plus the harvest (which is the return on the stock) evaluated at its shadow price μ. Note also that Proposition 6 can be used to characterize the steady state where $h = f(x)$.

The marginal revenue from exploitation at a fixed stock level is given by $\pi_h(x, f(x))$ which is a function in *x* only. Integrating with respect to *x* we get $\int \pi_h(x, f(x))dx$, and this can be interpreted as the value of a stock evaluated by its marginal revenue (relative to an arbitrary reference value/point). When this is multiplied by δ, we get the alternative rate of return on the stock. On the other hand, $\pi(x, f(x))$ is the actual rate of return on the stock when it is fixed. This leads to the definition of a new term, *B*, which is the difference between the alternative rate of return and the actual rate of return on the stock:

$$B(x) \equiv \delta \int \pi_h(x, f(x))dx - \pi(x, f(x)).$$

This function turns out to be extremely useful. First, note, for example, that the classical golden rule (See e.g. Clark, 1990) used to determine a steady state is simply given by $B'(x) = 0$. This can be generalized to include capital and investment by defining

$$MC(x) \equiv K'(\tfrac{f}{qx}) + (\delta + b)C'(\tfrac{bf}{qx})$$

and

$$\eta(x) \equiv (\delta + \frac{f}{x} - f')/(qx).$$

The next proposition is useful for determining the biological stock in steady state:

Proposition 7 (Generalized Golden Rule)

The biological stock in steady state is found by solving the ordinary (algebraic) equation: $B'(x) = \eta(x) \cdot MC(x)$.

Proof: Proposition 7 follows from (10) and $\dot{x} = \dot{\mu} = \dot{\lambda} = 0$ inserted into Proposition 6.

If there is more than one solution, the preferred one is the one with highest rate of return that is the one that maximizes $S(k, x)$. Proposition 7 is a generalization of the classical Golden Rule for renewable resources (Clark, 1990; Sandal and Steinshamn, 1997). The terms derived from π is the classical Golden Rule, $(\delta - f')\pi_h = \pi_x$. It is readily seen that capital costs and investment costs enter the equation in the same manner as positive terms. Thus larger investment costs may cause the same kind of changes on the equilibrium stock. In addition, capital costs may possibly have large influence on the optimal paths, especially in region A and B.

It is well known that the resource stock is typically increasing with higher operational costs in models without capital dynamics. It is therefore relevant to ask if this also applies to capital and investment costs. This leads to the next proposition:

Proposition 8.

If and only if $\psi(x) \equiv B' - \eta \cdot MC$ *is increasing as a function of x at* $\psi = 0$*, the steady state standing stock will increase with higher capital and investment costs.*

Proof: Let MC depend on a cost parameter α, $MC = MC(x;\alpha)$, such that $\frac{\partial}{\partial\alpha}MC > 0$. From $B' = \eta \cdot MC$ in steady state it can be deducted that $\frac{\partial}{\partial x}\left[B' - \eta \cdot MC\right] \cdot \frac{\partial x}{\partial\alpha} = \eta \cdot \frac{\partial MC}{\partial\alpha}$ and hence $\frac{\partial x}{\partial\alpha} > 0$ if and only if $\frac{\partial\psi}{\partial x} > 0$ in equilibrium.

The condition that ψ is increasing when it is zero is by far the most common case. For example, in case of the widely applied logistic growth function it can be shown that this is always fulfilled. The result that the standing stock increases with higher capital and investment costs is quite intuitive. It shows, however, that increased convexity in the cost of investment function calls for a more conservative utilisation pattern.

The dynamics when the capacity is fully utilised.

In this paragraph we look at the dynamics when the capital is fully utilised, that is in region C. As it is known that the steady state must be in region C, this is in other words an analysis of the dynamics in the vicinity of steady state. In order to do so, we use a reduced state-space analysis; reduced in the sense that it is not valid outside region C.

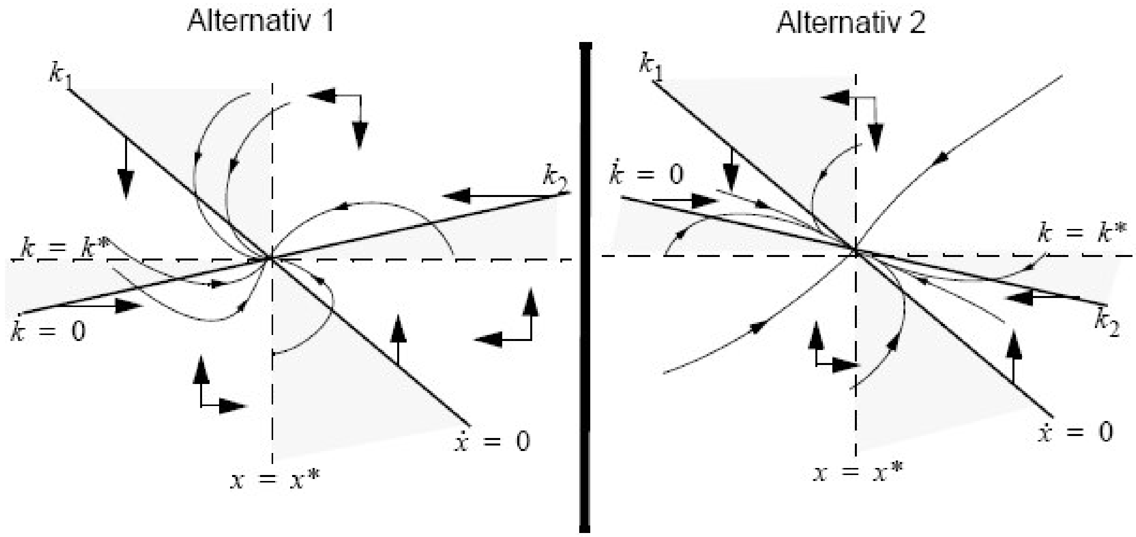

Figure 1.

Phase-diagram when the capital is fully utilised. In the left-hand panel the $\dot{k}$-isocline is increasing whereas in the right-hand panel it is decreasing. The shaded areas depict areas from where it is impossible to approach the steady state.

The graph defining $\dot{x} = 0$ is easily found as $k_1(x) = \frac{f(x)}{qx}$. This is typically a downward sloping curve, for example, if *f* is concave it will be downward sloping everywhere. We assume that f is concave in the vicinity of steady state, and therefore concentrate on downward sloping k_1. The graph defining $\dot{k} = 0$, on the other hand, is a bit more difficult to find explicitly, and we therefore look at some alternatives. Let us call this curve $k_2(x)$. In principle, we have two possibilities: k_2 can be increasing or decreasing. If k_2 is increasing in *x*, the phase-space can be divided in eight sub regions as illustrated in the left panel of Figure 1 as alternative 1. As $\dot{x}$ is negative to the right of k_1 and positive to the left of k_1, and as $\dot{k}$ is negative above k_2 and positive below k_2, it is only possible to reach the steady state from four of these sub regions as seen from the arrows in the figure. It is not possible to reach the steady state from the shaded areas such as the area between k_1 and x^* below k^*. This is also quite intuitive. If we, for example are on x^* but below k^*, the stock is already at its steady state level, but the capital is too small to keep the stock at this level even if fully utilised. Hence the stock will grow for a while, and the capital has to increase too in order to make the exploitation sufficiently large to drive the stock back to its steady state level. This is an example of overshooting with respect to the biological stock. In the first quadrant there may be overshooting with respect to capital.

If, on the other hand, if k_2 is decreasing like k_1, there are in principle two possibilities: k_2 can intercept k_1 from below or from above. However, interception from above is impossible as shown in the proof of the next proposition. We are therefore left with only one possibility, and this is illustrated as alternative 2 in the right panel of Figure 1. Again the total area is divided in eight sub regions, but, unlike alternative 1 where the four quadrants were divided in two areas each, now quadrants II and IV are divided in three sub regions each whereas quadrants I and III are not divided. In quadrants II and IV it is only possible to reach the steady state from the areas between k_1 and k_2. The steady state can in this case be reached from the whole of quadrants I and III unlike in alternative 1. This leads to the following proposition:

Proposition 9:

Let f(x) be concave at steady state. Then there will exist four sectors close to steady state from which steady state cannot be reached and there will be sectors in all quadrants from which the steady state can be reached.

The validity of this proposition is seen directly from the arrow-directions in Figure 1. It is therefore sufficient to prove that k_2 cannot intercept k_1 from above. The proof for this is given in appendix.

Summary and conclusions

This article introduces a general convex cost/revenue function for investment/disinvestment of capital in the exploitation of renewable resources. This function mimics the second-hand market for capital and is therefore a more realistic representation of irreversible investments than simple non-negativity constraints. Further, the exploitation of the resource is a function of the utilisation rate of the available capital and there is one cost associated with the capital that is actually used and one cost associated with the total available capital whether it is utilised or not. As a result of this both state variables enter the objective function.

The result is that both the steady state and the paths leading to steady state are affected by these novel features. Typically both the convexity of the cost of investment and the cost of capital will call for more conservative utilisation of the resource. It is also shown that depending on the initial conditions it is possible to approach the steady state in a variety of ways, and it is also possible to define regions from which the steady state can not be approached directly and therefore there will be so-called over- or undershooting along the paths. Actually, it is possible to have over- and undershooting in all of the four quadrants from which the steady state can be approached. This contrasts some earlier findings, e.g. Boyce (1995).

Appendix

In this appendix it is shown that k_2 cannot intercept k_1 from above in Figure 1. $\dot{k} = 0 \Rightarrow I = bk \Rightarrow \dot{I} = 0$ and further $\lambda = C' \Rightarrow \dot{\lambda} = 0$. Equation (9) for the shadow price λ now yields: $\pi_h - \mu = \frac{(\delta + b)C'(bk) + K'(k)}{qx} = \frac{MC(k)}{qx} \Rightarrow \mu = \pi_h - \frac{MC}{qx}$. Inserted into the equation for the shadow price μ, this yields:

$$\dot{\mu} = \left[\pi_{x,h} + qk\pi_{hh} + \frac{MC}{qx^2}\right]\dot{x} = (\delta - f')\left[\pi_h - \frac{MC}{qx}\right] - \pi_x - MC \cdot \frac{k}{x}$$
$$= (\delta - f')\pi_h - \pi_x - \frac{(\delta + qk - f')}{qx}MC = (\delta - f')\pi_h - \pi_x - \eta MC + \frac{f - h}{qx^2}MC$$

or

$$\left[\pi_{x,h} + qk\pi_{h,h}\right]\dot{x} = (\delta - f')\pi_h - \pi_x - \eta MC .$$

Expanding this in h at f, we get to the second order in $h - f = -\dot{x}$:

$$\dot{x}\cdot\left[\pi_{x,h} + \frac{f}{x}\pi_{h,h}\right]_{h=f} = \psi(x) + (-\dot{x})\left[(\delta - f')\pi_{h,h} - \pi_{x,h} - \frac{\eta}{qx}MC'\right]_{h=f} + O(\dot{x}^2)$$

where $\psi(x) \equiv B'(x) - \eta(x)MC(x)$ as defined in Proposition 8. To the second order in $\dot{x}$ we have

$$-A(x)\,\dot{x} = \psi(x), \qquad A(x) = -\left(\delta + \frac{f}{x} - f'\right)\pi_h(x, f) + \frac{\eta}{qx}MC'(x) > 0 .$$

This can alternatively be written

$$h = f(x) + \frac{\psi(x)}{A(x)} = qkx \Leftrightarrow k = k_2(x) = \frac{f(x)}{qx} + \frac{\psi(x)}{qxA(x)} = k_1(x) + \frac{\psi(x)}{qxA(x)} .$$

As the requirement on $\psi(x)$ is that it changes sign from negative to positive when passing through $x = x^*$, it has been established that $k_2(x)$ intercepts $k_1(x)$ locally from below.

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