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Voces Populi and the Art of Listening

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Abstract

The strategy most damaging to many preferential election methods is to give insincerely low rank to the main opponent of one’s favorite candidate. Theorem 1 determines the 3-candidate Condorcet method that minimizes the number of noncyclic profiles allowing this strategy. Theorems 2, 3, and 4 establish conditions for an anonymous and neutral 3-candidate single-seat election to be monotonic and still avoid this strategy completely. Plurality elections combine these properties; among the others "conditional IRV" gives the strongest challenge to the plurality winner. Conditional IRV is extended to any number of candidates. Theorem 5 is an impossibility of Gibbard-Satterthwaite type, describing 3 specific strategies that cannot all be avoided in meaningful anonymous and neutral elections.

1 Preference and sincerity

With preferential elections, not only the first choice of the voters may influence the result. When a candidate cannot hope to be the first choice of a voter group, it may still be worth an effort to obtain a high ranking from the group through campaigning and building alliances. The link between a candidate or party and the voters is fundamentally different in preferential elections and in elections where voters may support only one alternative, because the incentives for political behaviour are different. But the incentive to campaign for second and third ranks from politically adjacent voter groups is based on the assumption that voters will express their sincere preference in their ballots. However, depending on profile and election method, voters may have counter-incentives to vote strategically. According to the Gibbard-Satterthwaite theorem (Gibbard 1973, Satterthwaite 1975), one cannot get rid of the possibility of strategic voting in elections, but the theorem is based on a very broad definition of strategic voting. Some kinds of strategic voting should be regarded as less destructive than others. Arguably, one of them may even sometimes be regarded as useful. An election method should be assessed according to the incentives it gives to the most destructive kinds. How may such incentives be avoided?

Notation In a preferential election voter i (i = 1, 2, ..., v) expresses in the ballot an ordinal preference as a binary relation R_i defined in the set of candidates; the social preference is

$$R = R(R_1, R_2, \dots, R_v).$$[1]

Associated to R are the relations P (strict preference) and I (indifference):

$$X\ P\ Y \text{ means } (X\ R\ Y \text{ and not } Y\ R\ X),\quad X\ I\ Y \text{ means } (X\ R\ Y \text{ and } Y\ R\ X),$$

and similarly for the ballot preferences R_i. Here R_i is supposed to be a complete ordering, i.e. a ranking with equal preference allowed, but equality is handled by means of symmetrization.

1. Voter preferences R_i extend from the set of candidates C to the set of possible social preference relations R over C: voter i in a single-seat election prefers R to R^* if XR_iY, where X and Y are winners in R and R^* respectively. If $(R(R_1', R_2, ..., R_v))\ R_1\ (R(R_1, R_2, ..., R_v))$, voter 1 may vote strategically R_1' instead of R_1. The preference relation R_1 is thus better represented by R_1' than by itself.

A ballot relation ABC(DEF), which in Hill's notation (Hill 2001) means a shared 4.-6. rank ($AP_iBP_iCP_iD\ I_iE\ I_iF$) is then counted as the 3! compatible ballots with linear orderings, each ballot of weight 1/6: i.e. ABCDEF, ABCDFE, ABCFDE, ABCFED, ABCEFD, ABCEDF. A practical way to include incomplete ballots is to count ABC as ABC(DEF), etc.

1.1 Why preferential elections?

Elections give people opportunities to express a sincere ideal opinion or to vote instrumentally in order to really influence a social preference relation, but this choice between idealism or realism may be an unpleasant dilemma. Plurality elections, which are commonly used for single-seat constituencies in UK and USA, are often criticized for making voting more unpleasant than it has to be. The predicament of Nader's supporters in the US presidential election 2000 has received much attention: should they vote for Nader or for a major candidate? To some extent preferential elections make it possible for a voter to express first preference for a minor candidate, and also give real support the most acceptable major candidate.

However, hard decisions cannot always be avoided. In the didactic words of The Jenkins Commission (1998), "In many situations of life a decision has to be made in favour of a second or third best choice and there is no inherent reason why what has often to be applied to jobs, houses, even husbands and wives should be regarded as illegitimate when it comes to voting".

Pressure, as experienced by the Nader supporters, to vote for a major candidate, is generally considered as the main explanation for "Duverger's law", i.e. that plurality elections favor a two-party system. It is a matter of political science discussion (e.g. Cox 1997) how strong this effect actually is, and a matter of opinion if it is in society's best interest. When subsequent ranks are allowed to influence the outcome of an election, another incentive is created for parties and candidates: Gore's campaign organizers might have negotiated with Nader's organization for subsidiary support rather than complain that Nader voters threw the election to Bush.

To the extent that politically adjacent voter groups approach each other for subsidiary support rather than attack each other for de facto causing the opposite side to win, the driving force behind Duverger's law is reduced. However, the consequence for the seat distribution is not the only issue.[2] The political seat distribution will depend on the particular election method, but small voter groups may obtain considerable political influence because of the value their subsidiary support has for the major candidates. Preferential elections are therefore likely to influence the political climate and landscape in a very different way than both plurality elections and other elections where a voter supports just one party or candidate (Reilly 2002a). Such influence may well be regarded as the main justification for preferential elections, but it will be reduced if the election method gives incentives for a voter not to rank the candidates sincerely.

1.2 Strategic voting

In the fora where preferential election methods are discussed, much attention is devoted to three types of strategic voting. They are all built on violations of the IIA-axiom (Independence of Irrelevant Alternatives): candidate X may pass candidate Y in the social ranking without passing Y in any ballot ranking. In a 3-candidate single seat election they work as follows:

Strategy 1) Some voters switch their ranking from ABC to BAC so B can win instead of C.

Strategy 2) Some voters switch their ranking from ABC to ACB so A can win instead of B.

Strategy 3) Some voters switch their ranking from ABC to BAC so A can win instead of C.

The strategies are popularly called, respectively, "compromising", "burying", and "push-over". Other switches may also be strategic options, for certain election methods and given certain profiles. Saari (2003) discusses the most common preferential election methods and describes all the switches a voter may do away from a "sincere" ranking ABC and thereby conceivably

2. To secure representation for e.g. ethnical minority groups in an assembly without distorting its political composition, the full electorate may choose among minority candidates for a number of reserved positions, as it is done in Indian elections (Pande 2003). A similar idea may be implemented in STV-elections (Stensholt 2004); if the rules and results for the first seats imply that only some of the candidates (e.g. women) are eligible for the last seats; the remaining voting weight attached to each ballot then gives an advantage to a voter group that still is politically under-represented.

improve the outcome according to the sincere ranking.

In most election methods in the STV-family (Single Transferable Vote), there may occur profiles (i.e. preference distributions) that allow strategy 3. For real elections with the single-seat version of STV, called IRV (Instant Runoff Voting) or AV (Alternative Vote), it has been estimated that 0.9% of all 3-candidate profiles may allow strategy 3 (Stensholt 2002). However, to actually exploit this opportunity seems unrealistic as it requires reliable information about the preference profile, accurate planning, accurate execution, and the absence of counter strategy attempts. Therefore the possibility of strategy 3 does not necessarily destroy the beneficial influence of IRV on the political climate. But consider the reverse effect: Some voters sincerely change their preference to ABC, and C wins, but their original ranking BAC would have let A win anyway. These voters are likely to see the election result as a punishment for honesty, and distrust the method. This violation of IIA is very unreasonable: a voter should be able to trust that the best way to support a candidate is to give the candidate top rank on the ballot.

Monotonicity A preferential election is called *monotonic* if a candidate, X, by moving upwards in one ballot ranking, without any other changes in any ballot ranking, can only do better or equally well in relation to any other candidate. Thus, for any other candidate Y,

X I Y before means X R Y afterwards, X P Y before means X P Y afterwards.

Monotonic elections avoid strategy 3.

Profiles that allow strategy 2 are common in positional election methods like the Borda Count, and also in Condorcet methods. It is likely that some voter groups will attempt strategy 2. Unlike the violation of IIA exploited in strategy 3, the violation behind strategy 2 is not not necessarily unreasonable, because a switch from ABC to ACB might also be caused by a sincere change in some voters' assessment of the merit gap between A and B. However, the switch may also be an act of cunning, and incentives to attempt strategy 2 undermine the intended effect of preferential voting on the political climate.

Respect for ballot rankings We will say that a preferential election *respects ballot rankings* if it has the following property: A candidate, X, after a permutation in one ballot of the candidates ranked under X in that ballot, without any other changes in any ballot ranking, does equally well in relation to any other candidate, i.e. for any other candidate Y,

Y P X, X I Y, or X P Y before the permutation remains Y P X, X I Y, or X P Y afterwards[3].

Elections that respect ballot rankings avoid strategy 2.

Plurality voting is frequently and unfairly criticized for its obvious urge to apply strategy 1[4]. The underlying violation of IIA is in itself quite reasonable. When voters switch from ABC to BAC because B is considered more likely than A to defeat C, this cannot in any way be seen as an undemocratic act of cunning. A BAC-vote in recognition of B's ability to unite a large voter group is "insincere" only in a technical sense. But if an ABC-ballot gives B the same advantage over C as a BAC-ballot does, preferential voting may have the intended beneficial influence on the political climate. Political cooperation between A and B will then be encouraged with gentler means than a pressure to use strategy 1, which means to sacrifice A in order to avoid C.

For a preferential election method to be proof against strategy 1, it suffices that it has the following property, symmetric to respect for ballot rankings: A candidate, X, after a permutation in a single ballot of the candidates ranked before X in that ballot, does equally well in relation to any other candidate, i.e. for any other candidate Y,

Y P X, X I Y, or X P Y before the permutation remains Y P X, X I Y, or X P Y afterwards.

Elections with this property avoid strategy 3 as well. The property does not imply monotonicity, since it does not rule out that switching from ABC to BAC could let A win instead of B. Respect for ballot rankings and its symmetric companion together would imply IIA. Because of Arrow's

3. The phrases "later-no-harm/later-no-help" introduced by Woodall (1996) with a different but similar meaning, have also been used in the same meaning as "respect for ballot rankings".
4. As one might expect, strategy 1 is common in plurality elections with a close contest between two major candidates; a study of British parliamentary elections is found in Alvarez et al (2001)

impossibility theorem (Arrow, 1963) that combination of axioms is not realistic.

However, a good impossibility result is delicate. Arrow’s IIA may be seen as a defence against unwanted strategic voting; IIA is then an overkill, but for an overkill, it is not extreme. Two essential parts of IIA, respect for ballot rankings and no strategy 3, are satisfied by plurality voting. If the axiomatic method is properly applied, then, in the words of Sen (1999a), "It is therefore to be expected that constructive paths in social choice theory derived from axiomatic reasoning, would tend to be paved on one side by impossibility results (opposite to the side of multiple possibilities)”. Below we explore the possibility of having elections that, like plurality elections, sustain the two axioms, respect for ballot rankings and monotonicity, but also exert less pressure on voters to attempt strategy 1 than the plurality method does.

2 Strategy types and their impact on elections

Three strategies are discussed. Preferential elections may be seen as ways to reduce the need for strategy 1 which is the only available voting strategy in plurality elections. An incentive to apply strategy 2, which under some circumstances may award voters for giving bottom rank to their sincere second choice, is a particularly unfortunate property of many preferential election methods. The very much unwanted strategy 3 is in itself much more unreasonable than strategy 2, and although it is not a practical tool for voters in any election method, its theoretic possibility in a family of election methods causes criticism.

2.1 Strategy 1

A plurality election is an extreme case of preferential election, where only first places count. Plurality elections give strong incentives to apply strategy 1.

Example 1 Consider e.g. a 3-candidate profile in the set {Bush, Gore, Nader} that was estimated from opinion polls taken before the US presidential election 2000. Standardized to 1000 voters, the profile was

$$(|BGN|, |BNG|, |NBG|, |NGB|, |GNB|, |GBN|) = (424,\ 22,\ 33, 100,\ 40, 381),$$

where |XYZ| voters rank XYZ[5]. A spatial model fits this profile well: each candidate is assigned an ideal point and voters are uniformly distributed in a disc, ranking the candidate set {B, G, N} according to distance. However, the pictogram of figure 1 (Stensholt 1996) is exact and unique up to rotations and reflections. It uses three non-concurrent chords; the smaller triangle that these chords form, the better the spatial model approximates the pictogram.

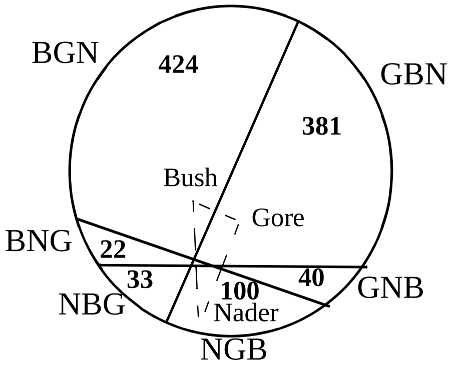

Figure 1 *1000 voter pictogram of estimated profile in the US presidential election 2000. The chords form a triangle covering 0.000433 of the disc, and are close to the mid-normals of the candidate triangle shown in dashed line.*

In most states, the plurality winner wins all delegates to the Electoral College. With the profile of figure 1, a plurality election puts the 133 N-preferrers under heavy pressure not to "waste their votes", i.e. to attempt strategy 1 and vote GNB or BNG. Certainly many N-preferrers instead abstained from voting. Abstentions and sincere preference changes aside, obviously many N-preferrers also voted strategically for a major candidate, B or G. Strategic voting helped G more than B, and so G became the national plurality winner with 51.00 mill. votes over B (50.46 mill.) and N (2.83 mill.), but B still won the majority in the Electoral College.[6]

Strategy 1 is possible in all common preferential election methods, but usually for fewer profiles

5. Author’s estimate: data from the National Election Study "feeling thermometer” (Burden 2003).
6. Hartvigsen (2006) studies an organized attempt to persuade some G-preferrers to vote N in states where G was safe and the same number of N-preferrers to vote G in states where G was in danger.

than in plurality elections. To varying degree, other methods reduce the pressure on, say, NBG-preferrers either to abstain or to support their most acceptable major candidate with BNG. Can a voting method let an NBG-preferrer always vote NBG and still influence the B-G contest exactly as by voting BNG? If this idea is pushed too far, it will, like other utopian ideas, have unfortunate consequences. In particular, very natural procedures let strategy 2 enter the scene.

2.2 Strategy 2

Notation A family of preferential election methods can be described as follows: The preference relation R_i is expressed as an nxn-matrix $M^i = (m^i_{xy})$[7], and the social preference relation $R = R(R_1, R_2, \ldots, R_v)$ is calculated from the Dodgson matrix, i.e. the sum $M = (m_{xy}) = \Sigma_i M^i$.

Example 2 Let B, G, N in figure 1 be candidates 1, 2, 3 and let $(m^i_{xy}, m^i_{yx}) = (1, 0)$ mean that voter i ranks candidate X before candidate Y. Rankings BGN, BNG, NBG, NGB, GNB, GBN are expressed, respectively, with ballot matrices M^i =

$$\begin{bmatrix}0&1&1\\0&0&1\\0&0&0\end{bmatrix}, \begin{bmatrix}0&1&1\\0&0&0\\0&1&0\end{bmatrix}, \begin{bmatrix}0&1&0\\0&0&0\\1&1&0\end{bmatrix}, \begin{bmatrix}0&0&0\\1&0&0\\1&1&0\end{bmatrix}, \begin{bmatrix}0&0&0\\1&0&1\\1&0&0\end{bmatrix}, \begin{bmatrix}0&0&1\\1&0&1\\0&0&0\end{bmatrix}.$$

The Dodgson matrix is the sum of the ballot matrices for the 1000 voters in figure 2:

$$M = \begin{bmatrix}0&479&827\\521&0&845\\173&155&0\end{bmatrix}$$

In the Condorcet relation G beats B (521-479) and N (845-155). The Borda points are obtained as row sums (1306, 1366, 328). In fact, the voters may cast a single vote or a double vote in the

7. Since the ballot matrix M^i is uniquely determined by the ballot ranking R_i, it is enough that each voter produces a preference list. Some election methods, that also lets a voter express intensity of preference, may correspond to a choice among different M^i for the same ordinal preference R_i, this is likely to complicate the issue of strategic voting by creating other strategy types.

B-G Borda contest. If r BGN-preferrers vote BNG and s GBN-preferrers vote GNB, both sides attempting strategy 2 by exercising their double vote, the matrix sum above is changed to

$$\begin{bmatrix} 0 & 479 & 827-s \\ 521 & 0 & 845-r \\ 173+s & 155+r & 0 \end{bmatrix} \text{ with row sums } \begin{bmatrix} 1306-s \\ 1366-r \\ 328+s+r \end{bmatrix}.$$

A voting war between B and G may make N Borda-winner and even Condorcet-winner. In the case of a Condorcet method, only B, being number 2 in the sincere Condorcet ranking GBN, can use strategy 2 to change the result. B must then create a Condorcet cycle (r>345). However, the result then depends on r, s, and the particular Condorcet method.

Example 3 In approval voting (Weber 1995, Brams and Fishburn 2003), each R_i is a complete ordering with exactly two indifference classes. In a 4-candidate election, a sincere ABCD-preferrer is supposed to choose between three ballots: A(BCD), (AB)(CD), (ABC)D. Letting A, B, C, D be candidates 1, 2, 3, 4, these preferences may be expressed with ballot matrices M^i =

$$\frac{1}{2}\cdot\begin{bmatrix} 0&2&2&2\\0&0&1&1\\0&1&0&1\\0&1&1&0 \end{bmatrix}, \quad \frac{1}{2}\cdot\begin{bmatrix} 0&1&2&2\\1&0&2&2\\0&0&0&1\\0&0&1&0 \end{bmatrix}, \quad \frac{1}{2}\cdot\begin{bmatrix} 0&1&1&2\\1&0&1&2\\1&1&0&2\\0&0&0&0 \end{bmatrix};$$

there are 4, 6, and 4 possible ballot matrices similar to the first, second and third of these. The election is tallied as a Borda Count, ranking by row sums of M. This is equivalent to treating indifference classes by means of symmetrization. With the idea of strategy 2, some sincere ABCD-preferrers may vote A(BCD) instead of (AB)(CD) in order to make A win instead of B.[8]

Example 4 All positional election methods belong to the matrix family. In elections to the parliament of Nauru, a voter gives P_r = 1/r points to the candidate with rank r (Reilly 2002b).

8. Making A win instead of B, however, is not a strict improvement according to an (AB)(CD)-ballot; (AB)(CD) is not better represented by A(BCD) than by itself. On the other hand, to compare approval voting to other methods, we should use the ABCD-preferrer as a common reference, and consider a successful transfer from (AB)(CD) to A(BCD) as strategy in a wider sense.

The point difference between the candidates ranked r and r+1 is then 1/r(r+1). The ordering ABCD may be expressed as a weighted sum of the matrices of example 3:

$$M^i = \frac{1}{8}\cdot\begin{bmatrix}0&2&2&2\\0&0&1&1\\0&1&0&1\\0&1&1&0\end{bmatrix} + \frac{1}{24}\cdot\begin{bmatrix}0&1&2&2\\1&0&2&2\\0&0&0&1\\0&0&1&0\end{bmatrix} + \frac{1}{48}\cdot\begin{bmatrix}0&1&1&2\\1&0&1&2\\1&1&0&2\\0&0&0&0\end{bmatrix} = \frac{1}{48}\cdot\begin{bmatrix}0&15&17&18\\3&0&11&12\\1&7&0&10\\0&6&8&0\end{bmatrix}$$

the first, second, and third component gives the appropriate advantage P_1-P_2, P_2-P_3, P_3-P_4 points, respectively, to {A} over {B, C, D}, {A, B} over {C,D}, and {A, B, C} over {D}.[9] There are 4! different ballot matrices, $M^j = Q_j^{-1}M^iQ_j$, where Q_j is a permutation matrix Q_i= I. A ballot contributes to the entry pair (m^i_{xy}, m^i_{yx}) according to how it ranks candidates X and Y; here it contributes (15/48, 3/48) to the entry pair of its two highest ranked candidates.

The incentive to carry on the mutually destructive strategy 2 described in example 2 is clear in the Borda Count and in other positional methods. In the Nauru method of example 4, the point difference 1/r(r+1) between ranks r and r+1 decreases with increasing r. Thus the incentive for strategy 2 may be weaker and the incentive for strategy 1 stronger than in the Borda Count.[10]

Remark on matrix elections Condorcet's principle of pairwise comparisons can always be applied to the Dodgson matrix, so that X R Y whenever $m_{xy} \geq m_{yx}$. For any positional system, the ballot matrices may be chosen as in example 4, i.e. a weighted sum of approval voting ballots as shown in example 3. Then there will not be cycles. It is enough to check that a 3-cycle A R B R C R A with at least one P cannot occur in approval voting. Letting |(AB)C| denote the

9. Only the differences between row sums count in a Borda type tally, but if one likes to see row sums 1, 1/2, 1/3, and 1/4 in a ballot, it suffices to add -2 to the diagonal entries of the right hand integer matrix.
10. In the annual Eurovision Song Contest, the national votes are counted in a positional method, with 12, 10, 8, 7, 6, 5, 4, 3, 2, 1 points for the ten best, but a national ballot is itself determined by the viewers' votes, and a participating viewer supports just one song. Thus strategy 2 is avoided. The Eurovision method is indirect, like the US presidential election, but a nation's Eurovision ballot is not similar to the seat distribution of a state in the Electoral College.

number of voters with preference (AB)C etc, A R B, B R C, C R A mean

$$|(CA)B|+|A(BC)| \geq |(BC)A|+|B(CA)|,$$

$$|(AB)C|+|B(CA)| \geq |(CA)B|+|C(AB)|,$$

$$|(BC)A|+|C(AB)| \geq |(AB)C|+|A(BC)|,$$

respectively. If one or more of the three Rs were a P then there would be one or more strict inequalities, and summation would yield a contradiction.

With sincere voting from a large number of independent voters, the probability that a profile with a Condorcet cycle will occur, is very small (Gehrlein 2002), but profiles that allow strategy 2 in a Condorcet method are common. Strategy 2 then involves creating a Condorcet cycle.[11]

Example 5 The three profiles in figure 2 illustrate the vulnerability of all Condorcet methods to strategy 2. The profiles are, respectively, (|ABC|, |ACB|, |CAB|, |CBA|, |BCA|, |BAC|) =

(10, 24, 22, 11, 19, 14), (20, 14, 22, 11, 10, 23), (20, 14, 11, 22, 19, 14).

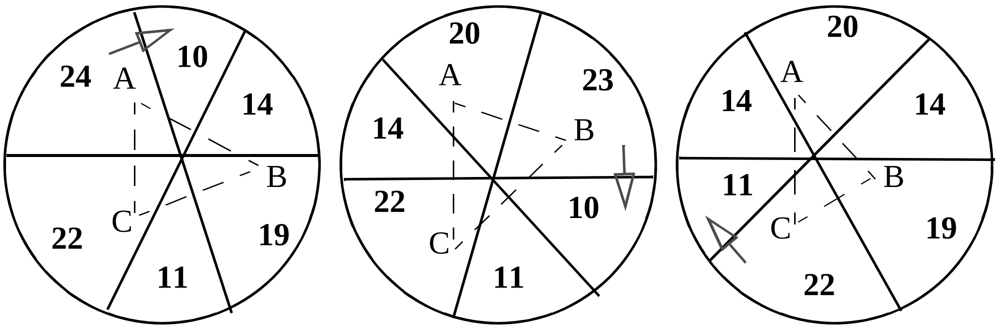

Figure 2 *Condorcet rankings are respectively* CAB, ABC, *and* BCA.

A, B, *and* C, *respectively may create the cycle of figure 3.*

The point is that number 2 in the Condorcet rankings above may create the same cyclic profile, shown in figure 3, (20, 14, 22, 11, 19, 14):

11. However, as long as ballot matrices are weighted sums of approval matrices (example 3), the voters may even have freedom to express preference intensities without creating cycles in a Condorcet tally.

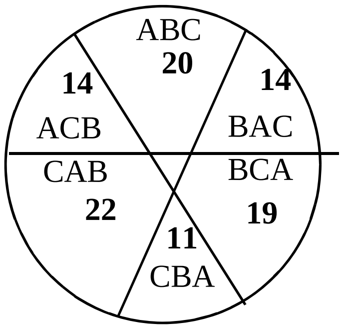

Figure 3 C *beats* A *beats* B *beats* C.

If the particular extension rule declares A, B, or C winner in the profile of figure 3, then the supporters of A, B, or C, respectively, may win by strategy 2 as shown in figure 2.[12]

As in Example 5, every Condorcet cycle may come from successful strategic voting for A, B, or C, starting from a suitable non-cyclic profile. As long as the Condorcet winner has less than 50% of the top ranks, the supporters of number 2 in the Condorcet ranking can always create a Condorcet cycle.

A 3-candidate Condorcet method is determined by the extension rule that picks the winner in a cycle. The discussion groups for Condorcet methods focus on what cycle-break rules that are most reasonable, given the deplorable fact that no matter how the rule is defined, the winner must lose some pairwise contest. However, it is known that different Condorcet extensions also differ in the probability that a profile which allows strategy 2, will occur. Consider the 3-candidate Condorcet method defined by the following cycle-break rule:

The winner is the candidate who defeats the plurality winner.

Theorem 1: This rule minimizes the number of non-cyclic profiles that allow strategy 2.

Proof: Let (|ABC|, |ACB|, |CAB|, |CBA|, |BCA|, |BAC|) = (p, q, r, s, t, u) with

12. In a cyclic profile pictogram the chord triangle T covers the center; to achieve this, the strategy usually creates a large T. In figure 3 T covers =.0065 of the disc. Pictograms with such a large T hardly occur in real elections with many independent voters. But all pictograms of figure 2 are quite realistic: strategy 2 in a Condorcet method does not require any rare profile property.

$$p + q + r + s + t + u = 1 .$$

Consider a cycle where A P B, B P C, and C P A. Then

$$p + q + r > s + t + u \,, \;\; t + u + p > q + r + s \,, \;\; r + s + t > u + p + q$$

Candidate A can reach this cycle by means of strategy 2, moving x voters from ACB to ABC, starting from the profile $(p - x, q + x, r, s, t, u)$. This starting profile is noncyclic if C beats B, i.e. $q + x + r + s > t + u + p - x$. Obviously $x \le p$. Thus, for A's strategy parameter x,

$$t + u + p - q - r - s < 2x \le 2p$$

The length of the interval for 2x is

$$p + q + r + s - t - u = 1 - 2(t + u) .$$

The same cycle can be reached from a noncyclic profile with strategy 2 by B and C, respectively

$$(p, q, r, s, t - y, u + y) \text{ and } (p, q, r - z, s + z, t, u) ,$$

with strategy parameters y for B and z for C. The intervals for 2y and 2z have lengths

$$1 - 2(r + s) \text{ and } 1 - 2(p + q) .$$

Now $t + u, r + s, p + q$ are the numbers of top-ranks for the candidate defeated by, respectively, A, B, C. When the cycle-break win is awarded to the candidate who defeats the plurality winner in a pairwise contest, the number of noncyclic profiles that allow strategy 2 is minimized. □

The rule of theorem 1 makes C win in the profile of figure 3 by defeating the plurality winner A. Similarly, awarding the cycle-break win to the candidate who directly defeats the plurality loser will maximize the number of non-cyclic profiles where successful strategy 2 is possible.

For n-candidate elections, n>3, there must be a rule for how to split up a "Smith set" of more than 3 candidates, i.e. the smallest set of candidates that beat all candidates outside the set in pairwise contests. One way is to tally the candidate triples using the cycle-break rule of theorem 1, and rank the candidates according to the number of triples they win.

In matrix elections, the social relation R is a function of the Dodgson matrix M. A Condorcet election without cycle-breaks is a matrix election. Some Condorcet methods are matrix elections and some are not; that depends on the particular extension rule. A couple of methods named after Nanson (Nanson 1882) eliminate one or more candidates according to their Borda scores, and repeat eliminating until a winner remains.[13] Since Borda scores are found from the Dodgson matrix M, Nanson’s methods are matrix methods. The method in theorem 1 requires the plurality scores, which are not found from M, and is therefore not a Dodgson matrix method.

Except in an extreme case like plurality elections, an entry m_{xy} has contributions from ballots that express very different preference relations R_i, and a marginal change in any m_{xy} may change the outcome at some profile. Thus all voters have reason to consider the effect of their contribution to every m_{xy}. Matrix elections and Condorcet methods are therefore generally vulnerable to strategy 2.

The most common elections in the STV-family avoid strategy 2, but unfortunately they open for strategy 3 in certain profiles.

2.3 Strategy 3

The basic idea behind the STV methods is to tally the ballots in several rounds; each round ends with a candidate being eliminated or elected. In order to be elected a candidate must get enough support in terms of top-ranks. If no candidate gets elected, one candidate is eliminated. For the development of STV-ideas, see (Tideman, 1995).

In the most thematic STV methods, the criterion for elimination is also exclusively based on the top-ranks (Stensholt 2004). In any round, the only information available to the tally process is then the current top-rank in each ballot; only when the currently top-ranked candidate in a ballot

13. Nanson originally eliminated all candidates with less than average Borda sum at the same time; Baldwin’s later modification eliminates only the candidate with lowest Borda sum before recalculation. In both procedures at least one candidate in the Smith set will escape from elimination, as shown e.g. in Stensholt (2004, p 419).

is eliminated or elected, is the ballot's support transferred to its second-ranked candidate.

When used in multi-seat constituencies, each ballot counts with a certain weight (voting power) which is reduced every time the ballot contributes to the election of a candidate. A suitable weight reduction gives a reasonably proportional representation of various voter groups, to the extent that the profile reflects the group interests.

By letting the tally process ignore the ranks k+1, k+2, ..., n in an original ballot as long as any of the first k candidates remain in the race, the commonly used STV versions make strategy 2 impossible. However, strategy 3 may become possible. Consider an IRV election with candidates A, B, and C, where A is clear plurality winner, C is Condorcet winner, and B is Condorcet loser but not too far behind C in terms of top-ranks. Two such profiles are shown in figure 4. If some A-supporters switch from ABC to BAC, they may obtain that C instead of B is eliminated in tally round 1, and that A defeats B instead of losing to C in tally round 2, despite the initial sacrifice of top-ranks. Such strategy may be a fine balancing act:

Example 6 Consider first the left hand profile of figure 4 as showing sincere preferences:

$$(|ABC|, |ACB|, |CAB|, |CBA|, |BCA|, |BAC|) = (11, 30, 16, 14, 21, 8).$$

Let t of the 11 ABC-preferrers be assigned to vote strategically BAC. A's party must choose t>1 in order to eliminate C and t<7 in order to win against B in round 2.

In the right hand profile,

$$(|ABC|, |ACB|, |CAB|, |CBA|, |BCA|, |BAC|) = (2, 43, 15, 16, 22, 2),$$

there are not enough ABC-voters available for strategy 3, but some of the 43 ACB-voters may also vote BAC. Let ACB $\rightarrow$ BAC mean a transfer of some votes from ACB to BAC, etc., and decompose it as ACB $\rightarrow$ ABC $\rightarrow$ BAC. Since strategy 2 is impossible, the first component has no effect. As an IRV-strategy for A, ACB $\rightarrow$ BAC is therefore also labelled as strategy 3, extending the definition of section 1.2. In the second profile 8 or 9 A-preferrers must vote BAC.

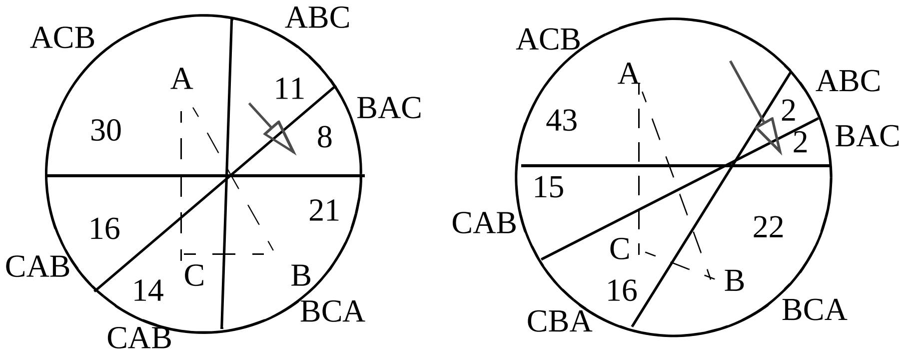

Figure 4 *IRV: Strategy 3 exploits the non-monotonicity and lets* A *(plurality winner) win instead of* C *(Condorcet winner). The right hand profile is nearly single-peaked, as only 4 voters rank* C *last.*

Strategy 3 is very risky, particularly for the ACB-preferrers; if too many of them vote BAC, they make B win instead of C. In real 3-candidate elections with a large number of independent voters, profiles which allow strategy 3 in IRV are likely to occur more often than profiles with a Condorcet cycle. However, strategy 2 in a Condorcet method does not depend on a random occurrence of a cycle; it requires a profile where no candidate has 50% of the top ranks, i.e. where a cycle can be created. A candidate who expects to be second in a Condorcet ranking, then has a clear incentive to attempt strategy 2 in many Condorcet methods.

2.4 Other voting strategies

In multi-seat STV many kinds of "free ride" may occur (Schulze, 2004). Consider e.g. two voter groups with rankings ABX... and BAY... . Assume A is elected in round 1 and B in round 2. In the standard versions of STV, the ABX voters have their weight reduced twice, since they first contribute to elect A, and then to elect B, while the BAY voters do not contribute to the election of A and "pay" only when B is elected. This creates an incentive to help X to the third seat instead of Y with a strategic transfer of votes from ABX to BAX. This weakness is eliminated in the Meek version of STV, where the BA voters, after election of B in round 2, pay the AB voters a compensation for their election of A. After election of A and B, an AB-voter and a BA-voter have had the same satisfaction, and they carry the same weight into tally round 3.

The "no-show paradox" (Fishburn and Brams 1983, Stensholt 2004) occurs when a voter group obtains a better result by not participating than by voting sincerely.

Example 7 Suppose 8 ACB-preferrers actually vote BAC in the second profile of figure 4, obtaining the elimination of C. If u of the 22 BCA-preferrers then abstain, the profile becomes

$$(|ABC|, |ACB|, |CAB|, |CBA|, |BCA|, |BAC|) = (2, 35, 15, 16, 22\text{-}u, 10),$$

and the numbers of top ranks for A, B, and C are 37, 32-u, 31. With $1<u<3$ the abstainers obtain the elimination of B, and C will still defeat A in round 2. Here the no-show effect may be seen as a counter-strategy that the BCA-preferrers perform against strategy 3 from the A-preferrers; with their counter-strategy they obtain that C wins instead of A. However, with $u>2$ they throw the win to A. The BCA-voters have a much more reliable weapon in strategy 1 since any number of them may vote CBA without risking to help A.

It has been shown that the no-show effect also occurs in all Condorcet methods with $n>3$ candidates (Moulin 1988).

Entering or withdrawing a candidate in an election may have significant effect on the result. In plurality elections it is clearly disadvantageous to enter two politically similar candidates, popularly called "clones". By splitting the votes, a clone may throw the election to the opposite side. However, in the Borda Count it is advantageous to enter a clone: If the A-party enters the clone A’, and r and r’ AB-preferrers vote AA’B and A’AB, then they give A and A’ respectively r and r’ extra points compared to B. Even if there should also be r+r’ BA-preferrers, it is unlikely that both of these advantages will be neutralized by the BA’A- and the BAA’-votes; thus either A or A’ gets an advantage over B[14].

14. Usually a "clone" of candidate A means a candidate A’ ranked immediately after A in all ballots. Then entering A’ as a candidate gives A maximal advantage, but also cloning in the sense used here will, with high probability, be a disadvantage to B.

The usual election model assumes a fixed voter set and a fixed candidate set. Both the no-show and the cloning effect violate that assumption. Most election methods one cares to study are homogeneous, in the sense that the social preference depends on the relative profile. With restriction to homogeneous methods it is meaningful to discuss the no-show effect axiomatically. However, in order to discuss the cloning effect in an axiomatic setting, one shold axiomatically link together election models for different numbers of candidates.

3 Elections that are proof to strategies 2 and 3

How can an n-candidate preferential election with $n \geq 3$ candidates give a complete and transitive social relation R and combine monotonicity with respect for ballot rankings? In the analysis below, the two symmetry conditions, neutrality and anonymity, are also assumed. Anonymity means that if two voters switch ballots, the social preference R remains the same; R is then determined by the number of ballots in each of the n! linear ranking categories. Neutrality means that if two candidates are switched in all ballot preferences R_i, they are switched in the social preference R.

3.1 *Possibilities and impossibilities in 3-candidate elections*

Consider profiles with 3-candidates, A, B, and C, and voters who have linear preference relations. Standardize so that

$$|ABC| + |ACB| + |CAB| + |CBA| + |BCA| + |BAC| = 1.$$

Lemma 1 Assume a 3-candidate election method gives a reflexive, complete and transitive social relation R and is anonymous, neutral, monotonic, and respects ballot rankings. If the plurality ranking is BCA, then B R A, i.e. A cannot strictly beat B in relation R.

Proof: Assume A P B, i.e. that A strictly beats B. The sequence of profile transformations in Table 1 change two components at a time. Here an arrow, $\rightarrow$, also means a profile change, e.g. 0 $\rightarrow$ 1 caused by the vote transfer CBA $\rightarrow$ CAB, etc. In profile 0, by assumption,

$$p+q \;\leq\; r+s \;\leq\; t+u.$$

Table 1 Transformations and conclusions about social preference:

Profile	ABC	ACB	CAB	CBA	BCA	BAC	definitions	preference
0	p	q	r	s	t	u		A P B
1	p	q	r_1	s_1	t	u	s_1 = min(p,s), r_1 = r+s-s_1	A P B
2	s_1	q_1	r_1	s_1	t	u	q_1 = p+q-s_1	A P B
3	s_1	q_2	q_2	s_1	t	u	q_2 = (q_1+r_1)/2	A P B
4	s_1	q_2	q_2	s_1	t_1	t_1	t_1 = (t+u)/2	A P B
5	p_1	q_2	q_2	s_1	t_1	1/6	p_1 = s_1+t_1-1/6	A P B
6	p_1	q_2	q_2	p_1	1/6	1/6		C I A P B
7	1/6	1/6	q_2	p_1	1/6	1/6		C I A P B
8	1/6	1/6	1/6	1/6	1/6	1/6		C I A P B

Each transformation is a transfer of votes between two neighbor categories of voters:

0 $\rightarrow$ 1 from CBA to CAB cannot harm A (monotonicity);

1 $\rightarrow$ 2 from ABC to ACB cannot harm A (respect for ballot rankings);

2 $\rightarrow$ 3 from CAB to ACB cannot harm A (monotonicity);

3 $\rightarrow$ 4 between BAC and BCA cannot help B (respect for ballot rankings);

4 $\rightarrow$ 5 from BAC to ABC cannot harm A or help B (monotonicity);

5 $\rightarrow$ 6 from BCA to CBA cannot help B (monotonicity), in 6 neutrality implies C I A ;

6 $\rightarrow$ 7 between ABC and ACB cannot harm or help A (respect for ballot rankings);

7 $\rightarrow$ 8 between CAB and CBA cannot harm or help C (respect for ballot rankings), and

C I A together with C P B imply A P B.

The conclusion C I A P B in profile 8 violates neutrality. □

The possibility of having another winner than the plurality winner appears from the next result. Let the plurality scores of A, B, C, respectively, be

$$\alpha = p+q, \ \ \beta = t+u, \ \ \gamma = r+s$$

Also define $\delta = q+\gamma$**;** thus $\delta > 1/2$ when C beats B in Condorcet's sense. Clearly

$$p + \beta + \delta = 1$$

Theorem 2 Assume that an election method for 3 candidates A, B, and C gives a reflexive, complete and transitive social relation, that it is anonymous, neutral, and monotonic, and that it respects ballot rankings. Assume the plurality ranking is BCA, i.e. $\beta \geq \gamma \geq \alpha$. Then,

(i) if $\gamma < 1/3$ or $\delta < 1/2$, then B must win (alone or jointly);

(ii) if $\gamma \geq 1/3$ and $\delta \geq 1/2$, then B or C must win (alone or jointly).

Proof: The lemma states that B R A. Hence A cannot win alone. Thus it suffices to show statement (i). If B does not win alone or jointly, then C P B R A. Consider a sequence of transfers starting from profile 0.

There are two cases, $\delta = q+r+s < 1/2$ in Table 2 and $\gamma = r+s < 1/3$ in Table 3. They overlap, but are treated independently.

Table 2 First case, $q+r+s < 1/2$, i.e. B beats C in Condorcet's relation

Profile	ABC	ACB	CAB	CBA	BCA	BAC	definitions	preference
00	p	q	r	s	t	u		C P B
09	p	q	r	s	s	u_1	u_1 = t+u-s	C P B
10	p_1	q	r	s	s	r	p_1 = p+u_1-r	C P B
11	q	p_1	r	s	s	r		C P B

Here $q+r+s < 1/2$ implies $q+r+s < t+u+p = u_1+s+p = s+r+p_1$ and so $q < p_1$. Consider 3 transfers:

00 $\rightarrow$ 09 between BCA and BAC cannot help B (respect for ballot rankings);

09 $\rightarrow$ 10 from BAC to ABC cannot help B (monotonicity);

10 $\rightarrow$ 11 from ABC to ACB cannot help B (monotonicity).

But the last transformation may also be done by transposition of B and C in all ballots, and so by neutrality B P C in profile 11. The contradiction proves the first case.

Table 3 Second case, r+s < 1/3

Profile	ABC	ACB	CAB	CBA	BCA	BAC	definitions	preference
00	p	q	r	s	t	u		C P B
09	p	q	r	s	s	u_1	u_1 = t+u-s	C P B
12	p_1	q	r	s	s	u_2	p_1 = 1/3-q, $u_2 = u_1$+p-p_1	C P B

Two transfers are made so that A passes C in terms of top-ranks, while B remains plurality winner. Since r+s < t+u = s+u_1, u_1 > r.

00 $\rightarrow$ 09 is as above.

09 $\rightarrow$ 12 from BAC to ABC cannot help B (monotonicity).

In profile 12, |BCA|+|BAC| > 1/3 = |ABC|+|ACB| > |CAB|+|CBA|. C P B contradicts lemma 1, and proves the second case. □

Lemma 2 Assume that an election method for 3 candidates gives a reflexive, complete and transitive social relation R, that it is anonymous, neutral, and monotonic, and that it respects ballot rankings. If the election also does not allow strategy 1 in any profile, then only a Condorcet winner can win the election alone.

Proof: Respect for ballot rankings and monotonicity means that also strategies 2 and 3 are ruled out. Consider an election with candidates X, Y, and Z, where strategies 1, 2, and 3 are impossible. Assume Z P Y and Z P X, without Z being Condorcet winner. Assume, for contradiction, that X be at least equal to Z in the pairwise Condorcet comparison:

$$|XZY| + |XYZ| + |YXZ| \geq |YZX| + |ZYX| + |ZXY|$$

Those who prefer X to Z may transfer votes between the categories XZY and XYZ and between the categories XYZ and YXZ to obtain

$$|XZY| = |ZXY|, \quad |XYZ| = |ZYX|, \quad |YXZ| \geq |YZX|$$

The first kind of transfer cannot upset the result Z P X (respect for ballot rankings). The second kind cannot upset it because strategies 1 and 3 are not available. A final transfer from YXZ to YZX switches |YXZ| and |YZX|, and cannot help X because of monotonicity. Thus we still have Z P X. However, the final switch of profiles may also be done by switching X and Z in all ballots, and neutrality implies X P Z in the last profile. □

Theorem 3 Assume that an election method for 3 candidates gives a reflexive, complete and transitive social relation, that it is anonymous, neutral, and monotonic, and that it respects ballot rankings. If strategy 1 is unavailable in all profiles, the method fails to give a single winner whenever the Condorcet winner is also the plurality loser.

Proof: By Lemma 1, a plurality loser can never win the election alone. By Lemma 2, only a Condorcet winner can do it. □

A single seat election method is too indecisive for practical use if it fails to produce a winner whenever the Condorcet winner also happens to be the plurality loser. In all elections considered here, the social relation R is defined by means of a finite number of inequalities; in a 3-candidate election they may be written with a finite number of linear expressions L_i:

$$L_i(|ABC|, |ACB|, |CAB|, |CBA|, |BCA|, |BAC|) > K_i \quad or \quad \geq K_i \, .$$

At most, one should tolerate indecisiveness as a consequence of equalities, i.e. at a "thin" set in the profile space. Thus, to achieve respect for ballot rankings and monotonicity, we must accept that some profiles will allow strategy 1.

Theorem 2 shows that in order to combine monotonicity with respect for ballot rankings, an election must pay attention to the plurality ranking BCA: B must win unless

$$|BCA| + |BAC| + |ABC| \leq \tfrac{1}{2} \text{ and } |CBA| + |CAB| \geq \tfrac{1}{3} \, .$$

If both of these conditions are satisfied, one may eliminate A from the race and have an immediate runoff between B and C. The subsidiary preference of the A-preferrers then become available to the tally officials without causing disrespect for ballot rankings. In the following 3-candidate election method the IRV/STV-rules are modified:

Conditional IRV Let $\alpha \le \gamma \le \beta$.

If $\gamma < 1/3$ or $\beta > 1/2$, then B wins without runoff.

If $\alpha < 1/3 \le \gamma \le \beta \le 1/2$, then A is eliminated, and there is an instant runoff between B and C for first place; the ballot of an A-preferrer counts as a full vote.

If $\alpha = \beta = \gamma = 1/3$, a 3-way tie is declared (and some tie-break may be invoked).

Conditional STV is like conditional IRV except that it includes an immediate runoff for second place between A and C if $\gamma < 1/3$ or $\beta > 1/2$; according to standard STV rules the voting power of the B-preferrers is reduced in order to obtain an approximately proportional representation on two seats (e.g. Stensholt 2004).

Thus in conditional STV there is exactly one runoff.

Conditional IRV and STV with 3 candidates are for single seat and double seat elections, respectively. Compared to standard IRV, the plurality winner (B) enjoys additional protection in conditional IRV, because there the plurality runner up (C) must qualify as a challenger by getting at least 1/3 of the top ranks.

There are other possibilities, which offer B extra protection in case of a challenge from C. One may, e.g. count the subsidiary vote of an A-preferrers as k votes, $0 \le k \le 1$. Thus there is a family of 3-candidate single seat election methods. The plurality election, $k = 0$, is one extreme and gives B maximal protection, while conditional IRV, $k = 1$, is the other extreme and gives B minimal protection. By including a runoff for second place when the plurality winner is not challenged, we obtain a similar famliy of two-seat elections, but in 3-candidate standard two-seat STV, the plurality winner will be elected to first seat without a runoff.

The vector (α, β, γ) contains the relative numbers of top-ranks. In figure 5 we interpret them as barycentric coordinates, and represent (α, β, γ) as a point in the standard simplex (Saari, 1994, p.32). The Plurality winner always wins with more than 1/2 of the top-ranks, and also with more than 1/3 of the top-ranks unless another candidate also have 1/3 or more of the top-ranks. If (α, β, γ) is in one of the rhombic boxes there is an instant runoff for first or second place.

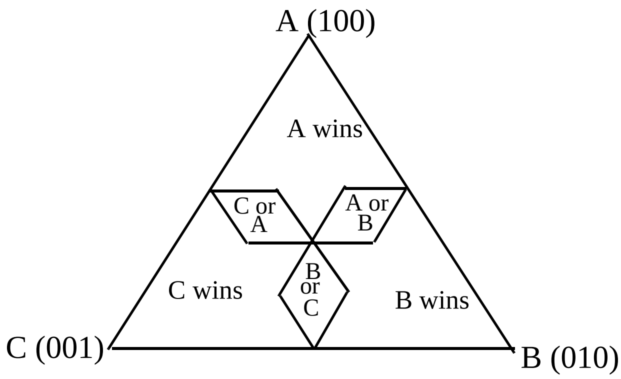

Figure 5 *There is an instant runoff for first place if* (α, β, γ) *belongs to one of the rhombic boxes, between two candidates with at least 1/3 and at most 1/2 of the topranks.*

Each election in the two families gives a reflexive, complete and transitive social relation. It is anonymous and neutral and respects ballot rankings since the subsidiary ranking in a ballot only counts when the fate of the top-ranked candidate is clear. Moreover,

Theorem 4 Each election in the two families is monotonic.

Proof: It is enough to consider an election with candidates {X, Y, Z} in the second family, i.e. with a runoff for second place if there is none for first place, because it picks the same candidate for first place as the corresponding election in the first family. An attempt to help X by exploiting nonmonotonicity moves a ballot from YXZ to YZX or from XYZ to YXZ, thereby changing the profile from (i) to (ii), say. It must be shown that this cannot help X in relation to

Y or to Z. A move from YXZ to YZX cannot help X, because no top-rank is changed, and so the runoffs are the same in (i) and (ii), and X can only be weakened by the move. However, a move of a ballot from XYZ to YXZ may cause one runoff to appear and another to disappear. Without ties, there are 6 possible social rankings according to top-ranks, as shown in figure 6:

	1	2	3	4	5	6
nr 1 in top-ranks	X	X	Z	Z	Y	Y
nr 2 in top-ranks	Y	Z	X	Y	Z	X
nr 3 in top-ranks	Z	Y	Y	X	X	Z

Figure 6 *In conditional 3-candidate STV there is an instant runoff for first place between the two best in top-ranks if number 2 has at least 1/3 of them; otherwise there is a runoff for second place between number 2 and 3. In conditional 3-candidate IRV there is no instant runoff for second place. The figure illustrates that no candidate can be helped by strategy 3: if, e.g., the gap between X and Y is shortened in column 1, then X may be challenged by Y.*

There is an instant runoff for second place if and only if number 2 in top-ranks is closer to number 3 than to number 1. The transition from profile (i) moves X slightly down and Y slightly up in the count of top-ranks. If the runoffs in (i) and (ii) are for the same place, they are between the same two candidates, but X will be weaker in (ii) than in (i), and therefore cannot be helped by the strategic attempt. The only possibility to help X would be to make a switch between runoff for first and runoff for second place. That cannot happen in columns 2 and 5. A runoff with X and Y for first place may appear in column 1 or disappear in column 6; in neither case can it help X. A runoff with X and Y for second place may appear in column 3 or disappear in column 4; in neither case can it help X. The same argument works with equal number of top-ranks for number 1 and 2 or for number 2 and 3. □

3.2 *Plurality vs Condorcet*

Example 8 Two 3-candidate profiles are shown in figure 7:

(|ABC|, |ACB|, |CAB|, |CBA|, |BCA|, |BAC|) = (10, 14, 27, 10, 12, 27);

(|XYZ|, |XZY|, |ZXY|, |ZYX|, |YZX|, |YXZ|) = (22, 18, 03, 25, 27, 05).

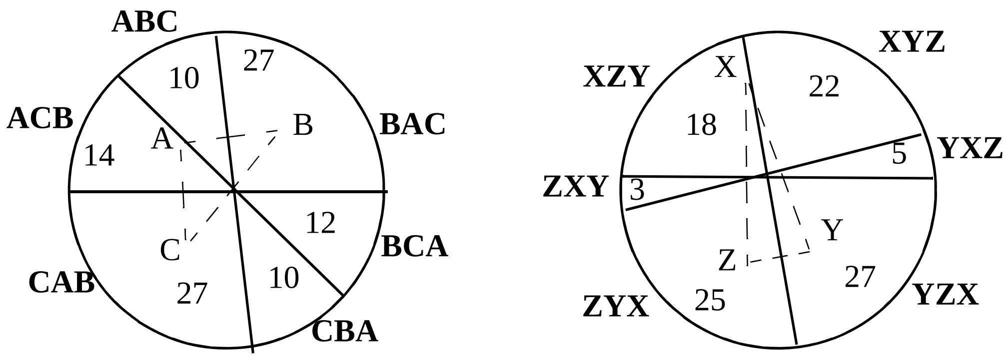

Figure 7 A *is Condorcet winner and plurality loser, so by theorem 3 no candidate can win if the election method is monotonic, respects ballot rankings and does not permit strategy 1.* X *is plurality winner and Condorcet loser;* {Y,Z} *is a "solid coalition" in the sense that >50% of the voters rank* X *last.*

For both profiles a spatial model in the shape of a candidate triangle fits well. Since the tiny triangle formed by the chords does not cover the circle center, there is no Condorcet cycle.

The first profile has plurality ranking BCA and Condorcet ranking ACB. Moreover, A is anti-plurality winner, i.e. has the smallest number of bottom-ranks, and may be seen as the natural compromise candidate between the wing candidates B and C. However, both standard and conditional IRV eliminate A and after a runoff for first place give ranking C P B I A when there is no need to separate nonelected candidates. Standard 2-seat STV gives first seat to B without runoff and final ranking B P C P A, while conditional STV eliminates A and lets the A-preferrers influence the final ranking, C P B P A. If the two seats have equal status, both 2-seat elections may be modified to give B I C P A.

The second profile has plurality ranking XYZ and Condorcet ranking YZX. A "solid coalition" of 52 voters prefers both Y and Z over X. Standard IRV eliminates Z and picks Y after runoff with X; it lets the solid majority win silently with sincere voting. Neither Y nor Z have 1/3 of the top-ranks, thus conditional IRV picks X without instant runoff. This possibility, however, gives the solid majority an incentive either to use strategy 1 or to break out of our 3-candidate model and promote a common candidate. Standard 2-seat STV gives first seat to X and second seat to Y; conditional STV eliminates Z and then gives Y the first and X the second seat.

The Condorcet relation ranks the candidates according to distance from the center. As long as the poltical reality shows profiles that are well described by our spatial model, it is likely that the circle center also corresponds to a political center in reality and in public perception. Each candidate has an incentive to appear to be located closer to the center than other candidates. This raises the question of what effect Condorcet methods in real elections will have on the political landscape and the public perception of it. Condorcet methods are currently used in some organizations but there is little experience from political elections.

Both standard and conditional IRV should be seen as variations on the plurality theme: The influence of the political center is often indirect; a compromise center candidate is perhaps not elected, but candidates from "left" and "right" compete for subsidiary support from the center voters. Thus, as in plurality elections, a candidate cannot move far out on a left-right scale and still win in a single-seat constituency. Some countries have a long tradition for STV and its single seat version IRV in their political elections. The STV approach emphasizes primary support but often gives decisive power to the central voters through their subsidiary votes. The candidates have to balance two incentives. They need good primary support, which they will not obtain by crowding into the political center, and they also need subsidiary support from the center voters.

3.3 n-*candidate elections,* n>3

Consider all n-candidate elections, n>3, that accept all linear ballot rankings, produce a reflexive, complete and transitive social relation R, are neutral and monotonic, and respect ballot rankings. These methods all have a weak Pareto property:

Lemma 3 Neutrality and monotonicity imply that if A P_i B for all i, then A R B.[15]

Proof: If A P_i B for all i and still B P A, then make a new profile by performing a sequence of neighbor transpositions in each ballot, moving B upwards and A downwards until they have changed place. A and B have switched places in the new social ranking and, by neutrality, A P B. On the other hand, by monotonicity, the social ranking B P A is preserved. □

Theorem 5 Consider an n-candidate election method, n $\geq$ 3, which allows all linear ballot rankings, produces a reflexive, complete and transitive social relation, is anonymous, neutral, and monotonic, and respects ballot rankings. Then some profiles allow strategy 1- or the method fails to produce a single winner whenever the Condorcet winner is last in top-ranks.

Proof: Consider profiles where 3 candidates A, B, and C are ranked above all other candidates in all ballot rankings. By lemma 3, A, B, and C are at least equal to all others in the social preference relation. By respect for ballot rankings, the restriction of R to {A, B, C} depends only on the restriction of each R_i to {A, B, C}. The conclusion follows from theorem 3. □

This is a result of the Gibbard-Satterthwaite type (Gibbard 1973, Satterthwaite 1975). With the additional assumptions of anonymity and neutrality, there is a more specific conclusion than in

15. Usually a stronger Pareto version is assumed: if AP_iB for all i, then APB. It implies that if AP_iBP_iC for all i, then APBPC, and there are at least three social indifference classes. The common version of Arrow's impossibility theorem assumes this strong Pareto property, and so does not deny the existence of nondictatorial elections that combine IIA with a complete social relation R with two indifference classes: elected and nonelected candidates. It just states that such elections cannot be obtained by first having an election satisfying the usual axioms (IIA, complete R, and strong Pareto), and then (if necessary) fusing some indifference classes. In particular, imposing the strong Pareto axiom may be not the best way to start an axiomatic study of single seat elections. With the weaker Pareto, the dictator in Arrow's theorem becomes a voter d with veto power, i.e. that AP_dB implies ARB.

the original theorem: In a reasonably decisive election method, strategy 1, known from plurality elections, cannot be completely avoided without the more obnoxious strategies 2 or 3 associated with disrespect for ballot rankings or nonmonotonicity becoming available at certain profiles.

The very possibility of successful strategy 1 will occasionally create an unpleasant pressure on some voters to vote against their political conscience. That pressure is often regarded as the main objection to plurality elections. One purpose of all preferential election methods is to reduce this pressure by letting the voters' subsequent rankings influence the social preference. Obviously, a plurality election is monotonic and respects ballot rankings. Can another monotonic election method that respects ballot rankings be more lenient with the voters? Conditional IRV for n=3 may be generalized:

Conditional IRV for n>3 candidates Let $N(X_i)$ be the number of top-ranks for candidate X_i and order the candidates accordingly: $N(X_1) \geq N(X_2) \geq \ldots \geq N(X_n)$.

Assume $N(X_1) > N(X_3)$ (otherwise some tie.break is needed). If $2N(X_2) \geq N(X_1)+N(X_3)$, then have an instant runoff between X_1 and X_2; otherwise let X_1 win without a challenge.

In conditional IRV only the plurality winner X_1 could have a reason to attempt strategy 3, with the idea to avoid runoff against X_2 by giving some top-ranks to a third candidate X_i. But it is easily checked that this would be too costly: if X_1 raises X_i above X_2 then X_1 drops below second place.

On the other hand, if $2N(X_2) < N(X_1)+N(X_3)$, it is easy to construct a profile where X_1 loses a runoff with X_2 but can obtain and win a runoff against X_3.

Example 9 In the two-round French presidential election 2002, the incumbent Chirac was clear plurality winner in round 1 with 20%, in front of Le Pen with 17% and Jospin with 16%. With these top-ranks, Chirac would have won without a runoff under conditional IRV. Chirac's overwhelming 82-18 win over Le Pen in the two-candidate runoff indicates that in a standard IRV election the longer process of elimination and vote transfer would have let Jospin and probably other candidates pass Le Pen. Chirac would then, at best, have reached the final tally round with a stronger opponent than Le Pen. How many Chirac-preferrers voted for Le Pen in round 1 in order to get an easier opponent in round 2? With the French rules, they would of course switch back to Chirac in round 2. The instant runoffs of standard and conditional IRV makes an attempt at strategy 3 a riskier enterprise.

4 On the choice of election method

Sen (1999b) wrote "A country does not have to be deemed fit *for* democracy; rather it has to become fit *through* democracy". Democratic ideas must be as ancient as civilization itself, and democracy develops together with a cultural framework that promotes participation through openness, access to information, fora for discussion, and tolerance for deviating opinions. Elections, and a widely accepted way to interpret them are central parts of that framework.

4.1 Vox populi, vox dei (?)

Out of its context, Alcuin's oft quoted phrase, *vox populi. vox dei*, gives the false impression of coming from a medieval democrat. In 800, Alcuin actually wrote to Charlemagne, *Nec audiendi sunt qui solent dicere vox populi, vox dei; cum tumultus vulgi semper insaniae proxima sit* (Do not listen to those who would say that the voice of the people is the voice of God, for the voice of the mob is close to madness).

The idea of elections that aggregate individual opinions across the entire society in order to reach a social choice is not likely to be found in the power circles of the medieval world. However, in Ramon Llull we find the thought that in a suitable context, a suitably restricted ecclesiastical electorate suitably prepared for its task, might find the divine truth. In *Artificum* *electiones personarum*, assumed to be written before 1283 (Hägele and Pukelsheim 2001), he describes a Condorcet method for single-seat elections: There is a sequence of pairwise votes, and Llull rules that the two candidates must leave the room while the other chapter members deliberate and vote. The candidate who wins most pair-wise comparisons is elected.

Llull in fact essentially designed Copeland's Condorcet method[16] 500 years before Condorcet[17]. It seems unlikely that this was caused by an awareness of Condorcet cycles, because the Copeland method will not pick a single winner from a Smith set of 3 candidates (i.e. a 3-cycle) or 4 candidates, unless some pairs are tied. Since Llull's concern was to elect - without fraud or simony - a prelate in a rather small electorate within a church chapter, it seems less far-fetched to ask why Llull did not keep the complete scores of each pair and add them as in a Borda Count. Was Llull aware of the strong urge to apply strategy 2 in the Borda Count?

Llull ordained strict guidelines to support the electorate in its quest for the divine truth. Each voter took an oath to always elect the person in whom three principles were best embodied: *Primum est honestas et sanctitas uites* (honesty and holiness of life); *Secundum est scientia et sapientia* (knowledge and wisdom); *Tertium est conveniens dispositio cordis* (a suitable disposition of the heart). Llull's principles clearly applies to voters as well as candidates, because the electorate coincides with the candidate set.

Vox dei may have held a more central position in the medieval world than in the rhetoric of modern mass democracies. Political leaders still claim to have a mandate from *vox populi*, but already an awareness of Condorcet cycles should raise some doubt as to whether the singular form of that phrase generally is meaningful. An electorate may be a highly polyphonic choir with many conductors. A fair listener with a perceptive ear cannot always extract any *vox populi*. Will an election method itself exert unacceptable influence on the voters' behaviour? If

16. The candidate with most pairwise victories is the Copeland winner.
17. However, in Lull's method with v voters, votes are collected for one candidate pair at a time, from the v-2 voters not in the pair.

not, may it help us to listen and make some sense out of the many *voces* it accepts as input?

4.2 Interest or judgment; decision or welfare assessment

Social choice theory is still a joint enterprise, common to political science and economics. Is it over-ambitious to cover formal elections and welfare economics with a common theory? Sen (1977. p. 53) writes: "It can be argued that some of the difficulties in the general theory of social choice arise from a desire to fit essentially different classes of group aggregation into one uniform framework and from seeking excessive generality”. Sen distinguishes between the individual preference relations expressing interest (I) or judgment (J), and between the social preference relation expressing a collective decision (D) or welfare assessment (W).

The four combinations ID, JD, IW, JW are certainly not disjoint categories. Formal elections, with their problems of strategic voting, belong on an ID-JD axis, and Llull was concerned that ecclesiastic elections should be JD rather than ID. At the W-side, welfare economists study preference relations and social welfare functions (SWFs) by means of utility functions.

Arrow’s theorem is about multivariate maps of relations and applies to both the W-side and the D-side. Strategies 1, 2, and 3 are linked to violations of the IIA-axiom, and therefore also belong to both sides. It looks like the Gibbard-Satterthwaite theorem has caused more concern on the D-side than the W-side, perhaps in part because the simpler mathematical structure of formal elections makes it easier to see the problems and to get at them.

Social choice may be made through a market mechanism or through a formal aggregation of individual preferences, as in an election or an SWF. In both cases the interface with the field of ethics is an arena of persistent debate. Examples from the same period are (Sen 1979) and the philosophical skirmish that followed (Ng 1981, Sen 1981); Llull’s exhortation to voters and candidates alike should also be listened to, as a counterpoint across categories and time.

4.3 Sincere or strategic voting

The Gibbard-Satterthwaite theorem uses a very general definition of strategic voting: It covers every case where a marginal voter i may improve the outcome according to an individual preference relation R_i by misrepresenting R_i as R_i', say. By theorem 5, strategies 1, 2, and 3 cannot all be completely avoided in an election method with satisfactory decisiveness. The usual Gibbard-Satterthwaite version is for single-seat elections that are "resolute", which mean that necessary tie-breaks are included. It extends immediately to resolute r-seat elections with a Pareto-condition; because in a profile with candidates X_1 , ..., X_{r-1} fixed in the r-1 top positions of every ballot there is just one seat contested by means of the rankings r, r+1, ... A further Gibbard-Satterthwaite generalization is found in (Duggan and Schwartz, 2000).

Strategy 3 is typical of the STV family. It is connected with eliminations in the tally process. It can hardly be exploited in real elections, but it will be upsetting when, on rare occasions, it will be found after an IRV-election that a voter group, who sincerely changed to ABC from BAC, caused C to win instead of A.

However, there are strong arguments in favor of the STV/IRV. With their stepwise tally process, where only the current top-ranked candidate in each ballot is considered, some versions of IRV and STV get rid of strategy 2, which is an important strategic possibility in most matrix elections and Condorcet methods. In many matrix elections, a profile like figure 1 is an incentive to mutually destructive use of strategy 2.

A single-seat preferential election method subdivides a profile space of high dimension into territories, one territory of victory for each candidate, with possible ties at the border. Strategy 2 can then be seen as a step crossing a border although we should expect the step to be parallell to the border. Strategy 3 means crossing a border from, say, the A-territory into the B-territory, although the direction of the step lets us expect that any crossing would be from the B-territory to the A-territory. Strategies 1, 2, and 3 may be said to cross borders in a logical, an illogical, and an antilogical direction, respectively.

Presentations of preferential election methods aimed at voters who are used to plurality elections often start by criticizing the plurality voting for producing an obviously "wrong" winner, as in the second profile in figure 7. X is clear plurality winner although a "solid coalition" of 52% of the electorate rank X last, while Y and Z are very close to each other in the political landscape. However, if the supporters of Y and Z will not cooperate towards a common goal, at least by means of strategy 1, then the technical term "solid coalition" is politically inappropriate, and it is not quite fair to blame an election method if X wins. It is not at all clear that X is the "wrong" social choice in the second profile of figure 7. [18]

For any election method, there will be profiles where some voters find that the choice between expressional and instrumental voting is far from obvious. A *conveniens dispositio cordis* may mean a willingness to compromise. That is important in many walks of life. As the Jenkins commission remarked, why should an election be an exception? It may also mean a willingness to stand up for a minor candidate, be counted, and accept the result.

4.4 Voting methods and the political landscape

Duverger’s law is the prime example of an election method (plurality election, often called "first-past-the post") being seen as shaping the political landscape, but all election methods should bee considered from that point of view. Preferential metods are used both in elections where candidates are running without party nomination and in elections with parties as political intermediaries. To the extent that small parties survive with a preferential election method in single-seat constituencies because of the negotiation power carried by their subsidiary votes, some of the political process may also be moved from hidden intra-party struggle (Caillaud and

18. Although a candidate triangle attached to a pictogram reflects an average perception of the political landscape among the voters, one should take care and not overstretch the analogy. In particular, when a spatial model is fitted, the candidate triangle is unique only up to homothetic transformations centered on the chords’ intersection point.

Tirole, 2002) to open inter-party communication.

What would be the effect of the various matrix elections and Condorcet methods? To an unknown degree, the urge to use strategy 2 will complicate the issue; the strength of this urge varies a lot with the particular election method. Perhaps the Condorcet method which resolves cycles by means of the method in theorem 1 (the candidate who defeats the plurality winner in a one-to-one contest wins in a Condorcet cycle of length 3) deserves special attention. However, if use of strategy 2 will be sufficiently limited, these methods are likely to favor the political center to a very high degree. Such a political compromise may be fine when the electorate is polarized, e.g. through ethnical division (Reilly 2002a). In other circumstances one may be concerned about a lack of diversity, e.g. if a political assembly consists of local Condorcet winners: Will a driving force, directed differently than the one behind Duverger's law, then contribute to shape a political landscape where only center candidates are noticed?

In terms of the final seat distribution in an assembly, the IRV elections are closer to plurality elections than Condorcet methods are. Conditional IRV, as described above, come closer than standard IRV, and it avoids both of the obnoxious strategies 2 and 3. It puts pressure on more voters to resort to strategy 1, particularly if the condition for an instant runoff may be not fulfilled, but it may still be much more favorable for smaller parties than a plurality election. An impression of how standard and conditional IRV might work in a real election may be obtained by superimposing figure 5 above on a barycentric plot of the vote shares (Labour, Liberal Democrat, Conservative) in 527 English constituencies 1997 (Myatt, 2007, figure 4). The conditional IRV runoffs correspond to the rhombic boxes of figure 5, and would mainly occur between Labour and Conservative, in many constituencies between Liberal Democrat and Conservative, but only in 3 constituencies between Labour and Liberal Democrat. Most of the Conservative direct wins in conditional IRV (no instant runoff) would have been with less than 50 % of the top-ranks and would have led to an instant runoff in standard IRV. However, the barycentric plot incorporates the results of strategy 1 that actually took place.

Since a compromise candidate may well be eliminated before a large potential supply of subsidiary votes has arrived, a major candidate must combine wide acceptance with enthusiastic support. Will IRV or conditional IRV for that reason tend to recruit better candidates for leadership than Condorcet methods will do?

REFERENCES

Alvarez, Ramon. M., Boehmke, Fred J., and Nagler, Jonathan. (2006): "Strategic Voting in British Elections" , *Electoral Studies* 25, 1-19

Arrow, Kenneth. J. (!963): *Social Choice and Individual values*, Cowles Foundation Monograph 12, Yale University Press

Brams, Steven J. and Fishburn, Peter C. (2003): "Going from Theory to Practice: The Mixed Success of Approval Voting", *Annual Meeting of The American Political Association*

Burden, Barry C. (2003): "Minor parties in the 2000 Presidential Election", in *Models of Voting in Presidential Elections: The 2000 U.S. Election*, ed. by Herbert F. Weisberg and Clyde Wilcox, Stanford, CA: Stanford University Press

Caillaud, Bernard and Tirole, Jean. (2002): "Parties as political intermediaries" *The Quarterly Journal of Economics* 117, 1453-1489

Cox, Gary W. (1997): *Making Votes Count. Strategic Coordination in the World's Electoral Systems*, Cambridge University Press

Duggan, John and Schwartz, Thomas. (2000) "Strategic manipulability without resoluteness or shared beliefs: Gibbard Sattertwhaite generalized", *Social Choice and Welfare* 17, 85-93

Fishburn, Peter C. and Brams, Steven J. (1983): "Paradoxes of preferential voting" *Math.*

Mag. 56, 207-214

Gehrlein, William V. (2002): "Condorcet's Paradox and the Likelihood of its Occurrence: Different Perspectives on Balanced Preferences", *Theory and Decision* 52, 171-199

Gibbard, Allan (1973): "Manipulation of Voting Schemes: A General Result", *Econometrica* **41**, 587-601

Hartvigsen, David. (2006): Vote trading in public elections, *Mathematical Social Sciences* 52, 31-48

Hägele, Günter and Pukelsheim, Friedrich. (2001): "Llull's writings on electoral Systems", *Studia Llulliana* 41, 3-38

Hill, I. D. (2001): "Difficulties with equality of preference", *Voting Matters* issue 13

Jenkins, The Rt. Hon Lord. (1998): *The Report of the Independent Commission on the Voting System*, Presented to Parliament by the Secretary of State for the Home Department by Command of Her Majesty, October 1998

Meek, Brian. (1969): "Une Nouvelle Approche du Scrutin Transférable" *Mathématique et Sciences Humaines*, 7:25 167-217: English version 1994 in *Voting Matters* issue 1

Moulin, Hervé. (1988): "Condorcet's Principle implies the No Show Paradox", *Journal of Economic Theory* 45 53-64

Myatt, David P. (2007): "On the Theory of Strategic Voting", *Review of Economic Studies* 74, 255-281.

Nanson, Edward J. (1882) "Methods of Election", *Trans Proc R Soc Victoria* 18 197-240

Ng, Yew-Kwang. (1981) "Welfarism: A Defence Against Sen's Attack"; *Economic Journal;* 91, 531-535

Pande, Rohini. (2003) "Can Mandated Political Representation Increase Policy Influence for Disadvantaged Minorities? Theory and Evidence from India" *American Economc Review*, 93, 1132-1151

Reilly, Benjamin. (2002a): "Electoral Systems for Divided Societies", *Journal of Democracy*, Vol. 13, Johns Hopkins University Press, United States, 156-170

---------- (2002b): "Social Choice in the South Seas: Electoral Innovation and the Borda Count in the Pacific Island Countries", *International Political Science Review* 23, 355-372

Saari, Donald G. (1994): *Geometry of Voting*, Studies in Economic Theory, Springer-Verlag

--------- (2003): Unsettling aspects of voting theory, *Economic theory* 22, 529-555

Satterthwaite, Mark A. (1975): "Strategy Proofness and Arrow’s Conditions: Existence and Correspondence Theorems for Voting Procedures and Social Welfare Functions", *Journal of Economic Theory* 10, 187-217

Schulze, Markus. (2004): "Free Riding", *Voting Matters* issue 18

Sen, Amartya. (1977): "Social Choice Theory: A Re-Examination", *Econometrica* 45, 53-89

------- (1979): "Personal Utilities and Public Judgements: Or What's Wrong With Welfare Economics", *Economic Journal* 89, 537-558

------- (1981) "A Reply to 'Welfarism: A Defence against Sen's Attack' "; *Economic Journal* 91, 531-535

------- (1999a): "The Possibility of Social Choice", *American Economc Review* 349-378

------- (1999b): "Democracy as a Universal Value", *Journal of Democracy* 10.3, 3-17

Stensholt, Eivind. (1996): "Circle Pictograms for Vote Vectors", *SIAM Review*, 38, 96-116

------- (2002): "Nonmonotonicity in AV", *Voting Matters* issue 15

------- (2004): "Single Transferable Votes with Tax Cuts", *SIAM Review* 46, 417-442

Tideman, Nicolaus. (1995): "The Single Transferable Vote", *The Journal of Economic Perspectives* 9 27-38

Weber, Robert J. (1995): "Approval Voting", *J. Econ Perspectives* 9(1)

Woodall, Douglas R. (1996) "Monotonicity and Single-Seat Election Rules", *Voting matters*, Issue 6