NHH

INSTITUTT FOR FORETAKSØKONOMI

DEPARTMENT OF FINANCE AND MANAGEMENT SCIENCE

FOR 12 2008

ISSN: 1500-4066

JUNE 2008

Discussion paper

Gibbard-Satterthwaite and an Arrovian Connection

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Gibbard-Satterthwaite and an Arrovian Connection

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Abstract A very close link of G-S, the Gibbard-Satterthwaite theorem to Arrow's "impossibility" theorem is shown. G-S is derived as a corollary**:** from a strategy-proof single-seat election method F is constructed an election method G that contradicts Arrow's theorem.

Assumptions F is a preferential election method for v voters and n candidates, n>2: R = F($\mathcal{R}$) where R is the social preference relation determined by the profile $\mathcal{R} = (R_1, ..., R_v)$ and R_i is the ballot preference relation of voter i. Let P and P_i, I and I_i be the relations of strict preference and indifference associated with R and R_i. Assume that

(i) each R_i is freely chosen as one of the n! linear orderings of the candidates;

(ii) there are two I-classes, a singleton class with the unique F-winner $W_{\mathcal{R}}$ and the rest;

(iii) for every candidate X there are profiles $\mathcal{R}$ so that X = $W_{\mathcal{R}}$;

(iv) F is nondictatorial in the sense that no fixed d has $W_{\mathcal{R}}$ top-ranked in R_d for all $\mathcal{R}$.

Theorem (Gibbard 1973, Satterthwaite 1975) F is not strategy-proof.

This means that i and $\mathcal{R} = (R_1, ..., R_i, ..., R_v)$ exist so that i's preference as expressed by R_i is better served by another relation R'_i and profile $\mathcal{R}' = (R_1, ..., R'_i, ..., R_v)$, thus $W_{\mathcal{R}'} P_i W_{\mathcal{R}}$. The switch from R_i to R'_i is a strategic vote for i. The following proof by contradiction constructs another voting method G so that Q=G($\mathcal{R}$) would be linear with the same winner as R=F($\mathcal{R}$)[1].

Proof: Assume F is strategy-proof. Choose by (iii) profiles $\mathcal{S}$ and $\mathcal{T}$ so that $W_{\mathcal{S}} \neq W_{\mathcal{T}}$. Change the profile stepwise from $\mathcal{S}$ to $\mathcal{T}$, one voter switching at a time, and pick a step from $\mathcal{U}$ to $\mathcal{U}'$ where voter i by switching from R_i to R'_i causes a change: $W_{\mathcal{U}} \neq W_{\mathcal{U}'}$. Consider 3 possibilities:

1. The proof has 2 steps similar to that of Schmeidler and Sonnenschein (1978), with a more powerful conclusion (*) to step 1 and a simpler G in step 2.

(a) $W_{\mathcal{U}'} P_i W_{\mathcal{U}}$ and $W_{\mathcal{U}'} P_i' W_{\mathcal{U}}$; (b) $W_{\mathcal{U}} P_i W_{\mathcal{U}'}$ and $W_{\mathcal{U}} P_i' W_{\mathcal{U}'}$; (c) $W_{\mathcal{U}'} P_i W_{\mathcal{U}}$ and $W_{\mathcal{U}} P_i' W_{\mathcal{U}'}$.

The switch from (a) R_i to R'_i ; (b) R'_i back to R_i ; (c) R_i to R'_i is a strategic vote for i. Hence $W_{\mathcal{U}} P_i W_{\mathcal{U}'}$ and $W_{\mathcal{U}'} P_i' W_{\mathcal{U}}$. Thus, to get rid of the F-winner $W_{\mathcal{U}}$,

(*) at least one i must switch from $W_{\mathcal{U}} P_i X$ to $X P'_i W_{\mathcal{U}}$ for some X, i.e. let X overtake $W_{\mathcal{U}}$.

For given $\mathcal{R}$ and any candidate pair {A, B}, raise A and B to the top two places in each ballot so that none of them passes the other. If A becomes F-winner, write AQB. Define YQY for all Y and set $G(\mathcal{R})=Q$[1]. To complete the proof, observe the consequences C1-C8 [reason in brackets].

C1: If A is on top of every ballot of $\mathcal{R}$, then A is the F-winner $W_{\mathcal{R}}$.

[By (iii), choose $\mathcal{S}$ so that $A = W_{\mathcal{S}}$, raise A to the top of every ballot and rearrange the other candidates to obtain $\mathcal{R}$. Nobody overtakes A in any ballot. By (*) A remains F-winner.]

C2: If all top r ballot places are occupied by $A_1, ..., A_r$, one of them is the F-winner.

[If $X \notin \{A_1, ...,A_r\}$ is F-winner, raise A_1 to the top in all ballots. By C1, A_1 becomes F-winner, but X is not overtaken in any ballot and (*) is contradicted.]

C3: If A is the F-winner, then A is also G-winner: AQX for every other candidate X.

[Raise A and any X to the top two places in every ballot so that none of the two passes the other. Nobody overtakes A in any ballot, thus A remains F-winner and AQX.]

C4: Q is linear, i.e. reflexive, complete and antisymmetric.

[Apply the definition of Q and C2 with r=2.]

C5: G is IIA, "independent of irrelevant alternatives" (Arrow 1963).

[Apply the definition of Q and C2 with r=2. Rearranging ballot positions 3, ..., n will not change the F-winner. The partition $\{i: AP_iB\} \cup \{i: BP_iA\}$ of the voter set determines if AQB or BQA.]

C6: $Q=G(\mathcal{R})$ is transitive.

1. For intuitive understanding, say that "F and $\mathcal{R}$ give A an advantage over B" when AQB.

[If $G(\mathcal{R})$ has a cycle $X_1QX_2QX_3QX_1$, raise X_1, X_2, X_3 to the top 3 places in each ballot, so that no X_i overtakes an X_j. By C5, the cycle persists, which contradicts C2 and C3.]

C7: G satisfies the Pareto condition.

[If AP_iX for all i, then AQX by the definition of Q and C1.]

C8: G is nondictatorial.

[A dictator d in G is by (iv) not dictator in F. If d prefers $Y \neq A=W_{\mathcal{R}}$, C3 contradicts the dictatorship of d in G.]

Thus the assumption of a strategy-proof F implies the existence of G with properties (C4, C5, C6, C7, C8) which are mutually incompatible by Arrow’s impossibility result.

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