Discussion paper

The investment horizon problem: A resolution

BY
KNUT K. AASE

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Knut K. Aase [*]

Norwegian School of Economics and Business Administration
5045 Bergen, Norway
and
Centre of Mathematics for Applications (CMA),
University of Oslo, Norway.
Knut.Aase@NHH.NO

June 16, 2009

Abstract

In the canonical model of investments, the optimal fractions in the risky assets do not depend on the time horizon. This is against empirical evidence, and against the typical recommendations of portfolio managers. We demonstrate that if the intertemporal coefficient of relative risk aversion is allowed to depend on time, or the age of the investor, the investment horizon problem can be resolved. Accordingly, the only standard assumption in applied economics/finance that we relax in order to obtain our conclusion, is the state and time separability of the intertemporal felicity index in the investor's utility function. We include life and pension insurance, and we also demonstrate that preferences aggregate.

KEYWORDS: The investment horizon problem, complete markets, life and pension insurance, dynamic programming, Kuhn-Tucker, directional derivatives, time consistency, aggregation

I Introduction

One of the central issues in asset pricing is the allocation of capital between different asset classes and in particular the choice between equity and

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bond investments. This asset allocation problem has received a great deal of attention in the financial economics literature but no consensus regarding its solution seems to have been reached. The modern formulation of the problem stems from Mossin (1968), Samuelson (1969), and Merton (1969). They found necessary conditions for the optimal portfolio choice of an investor to be constant over the life cycle, i.e., independent of both age and wealth. Mossin called this myopia in portfolio choice. Both Mossin (1968), in a discrete-time model, and Merton (1971), in the continuous-time version, have shown that, under certain standard assumptions, the portfolio choice decision can be made independently of the consumption versus savings decision. The assumptions are: 1) asset returns are i.i.d., 2) agents have additively separable constant relative risk aversion (CRRA) utility, 3) agents have no non-tradeable assets, and 4) markets are frictionless and complete. If portfolio choice is going to depend on age and/or on wealth, then one or more of these standard assumptions must be relaxed.

In this paper we investigate the effects of relaxing the assumption about the state and time separability of the felicity index, by letting the coefficient of relative risk aversion $\gamma := \gamma(t)$ be a continuous function of time t. Still the felicity index is of the standard form $u(x, t)$, but can no longer be written $u(x, t) = u(x)h(t)$, say. We demonstrate that if investors maximize the expected utility of consumption over their lifetimes, then, with this modification, the length of an investor's remaining horizon has a predictable effect on the optimal proportion to invest in stocks. Mathematically we can still solve the optimal consumption/investment problem with this assumption relaxed.

In order to briefly explain our results, recall that in the canonical model with one risky and one risk-free asset, the optimal fraction φ of wealth in the risky asset should be maintained constant according to $\varphi = \frac{1}{\gamma} \frac{\mu - r}{\sigma^2}$ at each time t, where μ is the return rate on the risky asset, σ its volatility and r is the risk-free rate of interest.

Under our assumption we demonstrate that $\varphi(t) = \frac{1}{\gamma(\tilde{t}_t)} \frac{\mu - r}{\sigma^2}$, where $\tilde{t}_t$ is a random quantity that can be determined from the investor's information set at each time t. Moreover $\tilde{t}_t > t$ for each t, making the optimal portfolio ratio $\varphi(t)$ both time and state dependent. The consequences of this result are several, and we shall return to the details later. Here we only point out that if the risk aversion function $\gamma(t)$ is increasing with time, then our result implies that individuals should invest more in the risky asset when they have a longer horizon, i.e., when they are young, and gradually move into bonds as they grow older. This is in agreement with advice from investment professionals, and also with empirical studies of actual behavior, but contradicts the results of the canonical model with a constant γ. It seems natural,

with this assumption, that the investor should pick some average time in the remaining horizon when deciding on today's portfolio choice.

One of the reasons for the advice that younger people should hold a higher fraction in equities is the tendency for stocks to outperform bonds or bills over the long run, despite the higher stock market volatility. This should not be mistaken as a "time diversification" advice, which is a different but related issue, typically arising after each down-turn in the stock market (e.g., Delong (2008), Bodie (2009)). For example, following the 2008/09 market crash it is evident that many people around the world have lost their pensions, partly or entirely. For many old people it seems obvious that they have too short remaining life times to regain what has been lost.

Paul A. Samuelson has explained, in many articles over the years, what is wrong with time diversification, and our results are in agreement with his main conclusions. In Samuelson (1989a) for example, he demonstrates that under the standard assumptions 1) - 4) cited above, the optimal portfolio strategy based on maximizing expected utility of consumption over the investor's lifetime, beats various buy-and-hold strategies by clear margins.

Under the assumption that the risk aversion increases as the consumer grows older, our results imply that the ratios in the risky assets will decrease with age, but not necessarily in a monotonic fashion. Furthermore, these ratios will depend on the state of the economy, the wealth level of the agent at any time during the investment period, his subjective impatience rate, and mortality. Thus the portfolio choice decision can no longer be separated from the consumption versus saving decision.

The terminology varies, however, and as an example, Eeckhoudt, Gollier and Schlesinger (2005) calls the time horizon problem the "time diversification" argument.

Several papers have questioned the validity of the reasons investment professionals give for their advice about cautiousness at old age. Jagannathan and Kocherlakota (1996) claim that the advice is correct only for people who have labor income that is relatively uncorrelated with stock returns. This idea is explored thoroughly in Bodie, Merton, and Samuelson (1992). Labor income can often be thought of as an implicit risk-free investment if present value of human capital is not too risky. The present value of human capital decreases over time and so if the investor wants a constant fraction of total wealth in the risky asset, he must move out of equities as he ages. However, the young also faces a longer consumption horizon, which must be taken into account, since it is the difference that counts.

Empirically Ameriks and Zeldes (2004) find a hump-shaped age effect in the fraction of all household financial assets invested in equity. The predicted equity share starts below 10% in the mid 20s, peaks at 20% in the late 40s and

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50s and declines again to 10% in the late 70s. Conditional on participation, this is 20% in the mid 20s, peaks at 50% around age 50 and declines to 30% in late 70s. Storesletten, Telmer and Yaron (2007) construct a model in which the share invested in the risky asset is hump-shaped over the investment horizon. They also incorporate a production economy and calibrate their model quite well to the U.S data in a modified version of the Constantinides and Duffie (1996) model. A similar hump effect is obtained through different means in Constantinides, Donaldson, and Mehra (2002), who construct an OLG model where they distinguish between young, middle-aged and old agents. In their model the young cannot borrow with human capital as collateral because of moral hazard and adverse selection. Notice that a hump-shaped age effect can be obtained in our model by simply assuming a hump-shape for the function $\gamma(t)$.

Samuelson (1989b) has explained the horizon effect by assuming that the agent maximizes expected logarithmic utility of terminal wealth, and is anxious not to fall below a "subsistence" level. Mossin (1968) considers a multiperiod model with no intermediate consumption, where the objective is to maximize expected utility of wealth at the end of the horizon. For harmonic absolute risk aversion (HARA) utility functions, where the absolute risk tolerance is linear in wealth, he characterizes the horizon problem as follows: The horizon effect is positive, i.e., investors reduce their holdings of the risky assets over time, or negative according to as the relative risk aversion is increasing or decreasing in wealth. While there seems to be no definite argument for or against decreasing relative risk aversion in wealth, it is a common agreement that absolute risk aversion is decreasing in wealth. In Macroeconomics however, according to Campbell and Viceira (2002) is power utility's property of wealth independent relative risk aversion attractive, and is required to explain the stability of financial variables in the face of secular economic growth. As a consequence of Mossin's results, the quadratic utility function has a positive horizon effect, however this utility function exhibits increasing absolute risk aversion. In a two-period model Gollier (1995) extends Mossin's result to convex absolute risk tolerance functions.

Our results can be seen to be consistent with the literature on counter-cyclical risk aversion introduced in Campbell and Cohcrane (1999), related to habit formation. When $\gamma(t)$ is an increasing function larger than one, we show that the investment ratios in the risky assets decrease when wealth increases, and vice versa. This effect seems to be empirically documented.

Another strand of the literature has pursued the impact on predictability of returns on portfolio choice. Examples include Poterba and Summers (1988), Hakansson (1971), Kim and Omberg (1996) and Wachter (2002), where they consider HARA utility functions. In short, a number of attempts

have been made to explain asset allocation decisions over the life cycle.

An interesting question which is pursued in a two-period model in Gollier and Zeckhauseer (2002) is whether we can say anything about age and portfolio choice in the absence of predictability and background risks. That is, can we say that people with certain utility functions unambiguously will shift their investments from stocks to bonds or vice versa as they age? Qvigstad Sørensen and Aase (2008) find that portfolio choice depends on the integral of expectations of the absolute risk tolerance of the direct utility function over the horizon. They also recover both Mossin's results and those of Gollier (1995) in a continuous-time model with intermediate consumption and mortality included, allowing for pension insurance as well as access to a securities market.

Related to these last two references there is the question weather one can separate the relative risk tolerance of the individual's indirect utility function from the corresponding direct one. In the canonical model they are the same. In our model, since the utility index is not time and state separable, the above separation turns out to be true. Since the indirect utility function, the optimal future utility, is the one that affects investments, this separation is important in explaining the horizon effect.

In Section 4 we show the aggregation property with several agents, which is important for equilibrium. This is another obvious weakness of the canonical model.

The paper is organized as follows: In Section 2 the model is presented, mortality is introduced and the consumption/investment problem is formulated. Section 3 presents the solution to this problem. In Section 4 the main theorem is formulated and discussed, and Section 5 concludes.

II   The Model

II-A   The financial primitives

We consider a consumer who has access to a securities market, and pension insurance. The securities market can be described by a price vector $X = (X^{(0)}, X^{(1)}, \cdots, X^{(N)})$ signifying the spot prices at each time $t \geq 0$ of the securities, here modeled as an Itô process with values in R^{N+1}. For each $n = 1, 2, \cdots, N$ we assume that

$$dX_t^{(n)} = \mu_n X_t^{(n)} \, dt + X_t^{(n)} \sigma^{(n)} \, dB_t, \qquad X_0^{(n)} > 0, \qquad t \in [0, T], \quad (1)$$

where $\sigma^{(n)}$ is the n-th row of a matrix σ in $R^{N \times d}$ consisting of constants, with linearly independent rows, and where μ_n is a constant. Here d is the

dimension of the Brownian motion B. For simplicity we assume that $d = N$. Underlying there is a probability space $(\Omega, \mathcal{F}, P)$ and an increasing information filtration $\mathcal{F}_t$ generated by the d-dimensional Brownian motion in the usual way. This implies, in particular, that each price process $X_t^{(n)}$ is a geometric Brownian motion of the sort used in the Black and Scholes model of option pricing. We suppose that $\sigma^{(0)} = 0$, so that $r = \mu_0$ is the risk free interest rate. T is the finite horizon of the economy.

The state price deflator is denoted by π and is given by

$$\pi_t = \xi_t e^{-rt}, \tag{2}$$

where the density process ξ has the representation

$$\xi_t = \exp(-\eta' \cdot B_t - \frac{t}{2}\eta' \cdot \eta), \tag{3}$$

and η' means the transpose of the vector η. Here η is the market-price-of-risk for the discounted price process $X_t e^{-rt}$, defined by

$$\eta = \sigma^{-1}\nu, \tag{4}$$

and ν is the vector with n-th component $(\mu_n - r)$, the excess rate of return on security n, $n = 1, 2, \cdots, N$.

The consumer/investor is represented by an endowment process e and a utility function $U : L_+ \to R$, where the set $L = \{c : c_t \text{ is } \mathcal{F}_t\text{-adapted, and } E(\int_0^T c_t^2 \, dt) < \infty\}$, and L_+, the positive cone of L, is the set of consumption rate processes. The specific form of the function U is the following time additive one given by

$$U(c) = E\{ \int_0^{T_x} u(c_t, t) \, dt\}, \tag{5}$$

where T_x is the remaining lifetime of an x-year old consumer. We assume that the probability distribution $F^x(t) = P(T_x \leq t)$ does not depend upon the probability distribution of the risky securities. In order to avoid unnecessary technicalities, we assume the support of T_x is finite and given by the set $(0, \tau)$ where the constant $\tau < T$.[1] Notice that the consumer has no bequest motive, an issue we consider later.

The intertemporal utility index $u(x, t)$ in (5) is usually assumed to be a separable function in state and time, i.e., $u(x, t) = g(x)h(t)$ where g and h are two real functions. This assumption is made primarily for computational

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[1]Formally the probability space is enlarged to accommodate this life distribution.

convenience, in particular when dynamic programming is employed. For example will this often allow one to use the separation method when solving the partial differential equation associated to the Hamilton-Jacobi-Bellman equation of dynamic programming. In applied economics and finance the most common assumption is that of a constant coefficient of intertemporal relative risk aversion γ, i.e., that $u(x,t) = \frac{1}{1-\gamma}x^{(1-\gamma)}e^{-\rho t}$ where ρ is the subjective interest rate.

In this paper we make the assumption that the relative risk aversion $\gamma := \gamma(t)$ is a continuous function of time. This will allow us to choose other properties for this function, i.e., inverted U-shape, or just an increasing function. It should be emphasized that this does not imply that we are assuming what we are going to show, namely that the optimal portfolio weights in the risky securities, associated to the consumer's optimal life-time consumption problem, decrease with age. At this stage it is far from clear that this will be the result, let alone that it is possible to solve the problem with this assumption. The following assumption is made about the function $u(x,t)$:

Assumption 1

$$u(x,t) = \begin{cases} \frac{1}{1-\gamma(t)}x^{(1-\gamma(t))}e^{-\rho t}, & \text{if } \gamma(t) \neq 1; \\ \ln(x)e^{-\rho t}, & \text{if } \gamma(t) = 1. \end{cases} \tag{6}$$

where $\gamma : [0,\tau) \to R_+$ is a continuous and strictly positive function of time.

Suppose, for example, that $\gamma(t)$ is increasing in time t. The implication of Assumption 1 is that the agent understands that the time to recover in the future from adverse effects in the risky securities is limited, and plans ahead for this by deciding to act gradually more risk averse as time increases.

Assumption 1 could alternatively be descriptive, or purely normative. Notice that (6) satisfies time consistency (Johnsen and Donaldson (1985)).

The elasticity of intertemporal substitution in consumption can be shown to be approximately equal to $\frac{1}{\gamma(t)}$ (without uncertainty and in discrete time), but this is no longer an exact relationship between these two quantities. This indicates on one hand that our assumption does not deviate too much from the canonical model, but on the other our assumption loosens up this strict, inverse relationship between these two key quantities, which is often sought in modern representation of preferences.

Before we continue, we shall say a few words about mortality and the random variable T_x.

II-B   Mortality

Yaari (1965), Hakansson (1969) and Fisher (1973) were of the first to introduce an uncertain lifetime into the theory of the consumer. The remaining lifetime of an x year old consumer/investor at time zero, T_x, has cumulative probability distribution function $F^x(t) = P(T_x \leq t)$, $t \geq 0$, and the survival function we denote by $\bar{F}^x(t) = P(T_x > t)$. By conditioning on what happens at an intermediate time t we have that

$$P(T_x > t + s) = P(T_x > t + s | T_x > t)P(T_x > t)$$
$$+ P(T_x > t + s | T_x \leq t)P(T_x \leq t)$$

which is, since the second term is zero

$$\bar{F}^x(t + s) = P(T_x > t + s | T_x > t)P(T_x > t). \tag{7}$$

By ignoring adverse selection effects in the population buying pension insurance[2], it is reasonable to assume that

$$P(T_x > t + s | T_x > t) = P(T_{x+t} > s), \tag{8}$$

in which case we obtain the function equation

$$\bar{F}^x(t + s) = \bar{F}^x(t)\bar{F}^{x+t}(s). \tag{9}$$

This equation is known to have a solution on the form

$$\bar{F}^x(t) = \frac{l(x + t)}{l(x)} \tag{10}$$

for some function $l(\cdot)$ of one variable only. The decrement function $l(x)$ can be interpreted as the expected number alive in age x from a population of $l(0)$ newborne.

The force of mortality or death intensity is defined as

$$\mu_x(t) = \frac{f_x(t)}{1 - F^x(t)} = -\frac{d}{dt} \ln \bar{F}^x(t), \qquad F^x(t) < 1 \tag{11}$$

where $f_x(t)$ is the probability density function of T_x. Integrating yields the survival function in terms of the force of mortality

$$\bar{F}^x(t) = \frac{l(x + t)}{l(x)} = \exp\Big\{ - \int_0^t \mu_x(u)\, du \Big\}. \tag{12}$$

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[2]In earlier times selection effects were sometimes modeled by actuaries at this stage, but this is rarely done today.

Suppose $y \geq 0$ a.s. is a process in L. Then the formula

$$E\left(\int_0^{T_x} y_t dt\right) = \int_0^\tau E(y_t) \frac{l(x+t)}{l(x)} dt = \int_0^\tau E(y_t) e^{-\int_0^t \mu_x(u)du} dt \qquad (13)$$

follows essentially from integration by parts, our independence assumption regarding mortality and the Fubini Theorem. We also have the formulas

$$\mu_x(t) = -\frac{l'(x+t)}{l(x+t)}, \qquad \text{and} \qquad f_x(t) = -\frac{l'(x+t)}{l(x)} = \frac{l(x+t)}{l(x)} \mu_{x+t} \qquad (14)$$

where $l'(x+t)$ is the derivative of $l(x+t)$ with respect to t. Notice from this that that we may write $\mu_x(t) = \mu(x+t) = \mu_{x+t}$, where the latter equality is just notational.

We emphasize that it is assumption (6) that is the crucial one for our results, not the assumption that the remaining lifetime of the consumer is stochastic. This latter choice is made in order to give a natural formulation of the consumption/investment problem.

II-C   The Consumption/Investment Problem

In order to formulate this problem, first note that a trading strategy $\theta = (\theta^{(0)}, \theta^{(1)}, \cdots, \theta^{(N)})$ is an adapted stochastic process for which the stochastic integral $\int \theta dX$ exists. For the moment consider the fixed, non-random time horizon τ. Here we follow Duffie (2001), Ch 9. Given an initial wealth $w > 0$, we then say that (c, θ) is budget-feasible, denoted $(c, \theta) \in \Lambda(w)$, if c is a consumption choice in L_+ and θ is a trading strategy satisfying

$$\theta_t \cdot X_t = w + \int_0^t \theta_s \cdot dX_s - \int_0^t \pi_s c_s ds \geq 0, \qquad \text{a.s.} \qquad t \in [0, \tau], \qquad (15)$$

and

$$\theta_\tau \cdot X_\tau \geq 0 \qquad \text{a.s.} \qquad (16)$$

The first equation (15) says that the current market value $\theta_t \cdot X_t$ of the trading strategy is nonnegative and equal to its initial value w, plus gains/losses from security trade, net of consumption purchases to date. The nonnegative wealth restriction can be viewed as a credit constraint, also extending to the terminal date τ in (16).

From our point of view, the main invention a pension insurance market brings into this model is to remove the last wealth restriction. The terminal restriction $\theta_\tau \cdot X_\tau \geq 0$ (almost surely) is replaced by an expectation, namely $\frac{1}{\pi_x} E(\theta_{T_x} \cdot X_{T_x} \pi_{T_x}) = 0$, which is, of course, less demanding. This new restriction assumes "fair pricing" of pension insurance at market values. As

a consequence, the individual's lifetime consumption can be increased when the individual is allowed to gamble on his/her own life length, via a market for pension insurance or life annuities.

Conceptually this is equivalent to the following: Imagine that a population of individuals in age x exchange their endowment processes for the optimal consumption rate processes, where the insurer was informed about their objectives and attitudes towards risk. From pooling over the individuals, the insurance company can then promise a consumption stream as long as each individual is alive, and only then.

The consumer's problem is, for each initial wealt w, to solve

$$\sup_{(c,\theta)} U(c) \tag{17}$$

subject to (15) and the expectation version of (16).

This problem can be reformulated in terms of the fractions $\varphi' = (\varphi^{(1)}, \varphi^{(2)}, \cdots, \varphi^{(N)})$ of total wealth held in the risky securities:

$$\varphi_t^{(n)} = \begin{cases} \frac{\theta_t^{(n)} X_t^{(n)}}{\theta_t \cdot X_t}, & \text{if } \theta_t \cdot X_t \neq 0; \\ 0, & \text{if } \theta_t \cdot X_t = 0, \end{cases} \tag{18}$$

for $n = 1, 2, \cdots, N$. The individual's wealth process at time t, $W_t = \theta_t \cdot X_t$, satisfies the stochastic differential equation

$$dW_t = \big(W_t(\varphi_t' \cdot \nu + r) - \pi_t c_t\big)dt + W_t \varphi_t' \cdot \sigma dB_t, \qquad W_0 = w. \tag{19}$$

The first order condition for the problem (17) subject to (19) is given by the Bellman equation, which in the present situation takes on the form (see e.g. Aase and Qvigstad (2008))

$$\sup_{(c,\varphi)} \big\{ \mathcal{D}^{(c,\varphi)} J(w,t) - \mu_x(t) J(w,t) + u(c,t) \big\} = 0, \tag{20}$$

with boundary condition

$$EJ(w, T_x) = 0, \qquad w > 0, \tag{21}$$

where $J(w,t)$ is the indirect utility function of the consumer at time t when the wealth $W_t = w$, and the differential operator $\mathcal{D}^{(c,\varphi)}$ is given by

$$\mathcal{D}^{(c,\varphi)} J(w,t) = J_w(w,t)(w\varphi \cdot \nu + rw - \pi c) + J_t(w,t) \tag{22}$$

$$+ \frac{w^2}{2} \varphi' \cdot (\sigma \cdot \sigma') \cdot \varphi \, J_{ww}(w,t).$$

This is a non-standard dynamic programing problem, a so called non autonomous problem. If the function $u(\cdot, t)$ is strictly concave and twice continuously differentiable on $(0, \infty)$, we know that the optimal ratios $\varphi(w, t)$ in the risky assets are given at any time t by

$$\varphi(w, t) = -\frac{J_w(w, t)}{w J_{ww}(w, t)} (\sigma\sigma')^{-1}\nu \qquad \text{for all } t, \tag{23}$$

where the first factor is the relative risk tolerance of the investor's indirect utility function. The problem is to determine the function $J(w, t)$ and its first two partial derivatives with respect to wealth.

Instead of solving this problem directly, we solve an equivalent one. As is well known (e.g., Cox and Huang (1989) or Pliska (1986)), in a complete market the dynamic program (17) - (22) has the same solution as the following simpler, yet more general problem

$$\sup_c U(c), \tag{24}$$

subject to

$$E\left\{ \int_0^{T_x} \pi_t c_t \, dt \right\} \le E\left\{ \int_0^{T_x} \pi_t e_t \, dt \right\} := w \tag{25}$$

where e is the endowment process of the individual. The pension insurance element secures the consumer a consumption stream as long as needed, but only if it is needed. This makes it possible to compound risk-free payments at a higher rate of interest than r, namely at the rate $(r + \mu_x(t))$ at time t.

The optimal wealth process W_t associated with a solution c^* to the problem (24)-(25) can be implemented by some adapted and allowed trading strategy θ^* or φ^*, since the marketed subspace M is equal to L. Without mortality this is well-known, and by introducing the new random variable T_x it still holds. In principal mortality corresponds to a new state of the economy, which should be priced, but the insurer can diversify this risk away by pooling over the agents, all in age x, so its corresponding Arrow-Debreu state price is equal to $\exp\{-\int_0^t \mu_x(u)du\}$, which contains no extra random component in π. Accordingly, adding the pension insurance contract in an otherwise complete model has no implications for the state price π other than this, and thus the model is still complete.

III   The Solution to the Problem

The constrained optimization problem (24)-(25) can be solved by Kuhn-Tucker and a variational argument. To this end, we notice that the La-

grangian of the problem is

$$\mathcal{L}(c,\lambda) = E\left\{ \int_0^{T_x} \left(u(c_t, t) - \lambda(\pi_t(c_t - e_t)) \right) dt \right\}, \tag{26}$$

where the felicity index $u(x,t)$ is given by (6). By our assumptions the optimal solution c^* to the problem (24)-(25) satisfies $c_t^* > 0$ a.s. for a.a $t \in [0, T_x)$, in which case the first order conditions involve the existence of a Lagrange multiplyer, a specific value λ, such that c^* maximizes $\mathcal{L}(c,\lambda)$ and complementary slackness holds.

Denoting the directional derivative of $\mathcal{L}(c,\lambda)$ in the "direction" $h \in L$ by $\bigtriangledown\mathcal{L}(c,\lambda; h)$, the first order condition of this unconstrained problem becomes

$$\bigtriangledown\mathcal{L}(c^*,\lambda; h) = 0 \qquad \text{for all } h \in L, \tag{27}$$

which is equivalent to

$$E\left\{ \int_0^\tau \left(\left(c_t^{-\gamma(t)} e^{-\rho t} - \lambda \pi_t \right) h(t) \right) \frac{l(x+t)}{l(x)} dt \right\} = 0, \qquad \text{for all } h \in L, \tag{28}$$

where the survival probability $P(T_x > t) = \frac{l(x+t)}{l(x)}$. In the derivation of (28) we have used the formula (13).

In order for (28) to hold true for all processes $h \in L$, it must be the case that the optimal consumption process is given by

$$c_t^* = \left(\lambda e^{\rho t} \pi_t \right)^{-\frac{1}{\gamma(t)}} \qquad \text{a.s.}, \qquad t \geq 0. \tag{29}$$

In this expression everything is known except the specific value of the Lagrange multiplyer λ, and, as usual, this quantity is determined by the budget constraint. Since $\frac{\partial}{\partial x} u(x,t) > 0$, complementary slackness implies that the budget constraint (25) holds with equality, so

$$E\left\{ \int_0^{T_x} c_t^* \pi_t dt \right\} = w$$

or, by formula (13)

$$\int_0^\tau \left(\lambda e^{\rho t} \right)^{-\frac{1}{\gamma(t)}} E\left(\pi_t^{(1-\frac{1}{\gamma(t)})} \right) \frac{l(x+t)}{l(x)} dt = w, \tag{30}$$

where we have used (29) for the optimal consumption process c^*. First we want to establish that equation (30) determines the Lagrange multiplyer λ uniquely for each value of initial wealth $w > 0$. In order to see this, first

notice that from (2) and (3) and the moment generating function of the multivariate normal distribution

$$E\left(\pi_t^{\left(1-\frac{1}{\gamma(t)}\right)}\right) =$$

$$E\left\{\exp\left[-r\left(1-\frac{1}{\gamma(t)}\right)t - \frac{t}{2}\eta'\cdot\eta\left(1-\frac{1}{\gamma(t)}\right) - \left(1-\frac{1}{\gamma(t)}\right)\eta'\cdot B_t\right]\right\} =$$

$$\exp\left\{-r\left(1-\frac{1}{\gamma(t)}\right)t - \frac{t}{2}\eta'\cdot\eta\left(1-\frac{1}{\gamma(t)}\right)\frac{1}{\gamma(t)}\right\}.$$

Inserting this expression in equation (30) we obtain

$$\int_0^\tau \lambda^{-\frac{1}{\gamma(t)}} \exp\left\{-\left[\rho\frac{1}{\gamma(t)} + r(1-\frac{1}{\gamma(t)})\right.\right.$$
$$\left.\left. + \frac{1}{2}\eta'\cdot\eta(1-\frac{1}{\gamma(t)})(\frac{1}{\gamma(t)})\right]t\right\}\frac{l(x+t)}{l(x)}dt = w \quad (31)$$

Next we make use of the First Mean Value Theorem for Integrals, which says:

Proposition 1 *Let $f(x)$ and $g(x)$ be two integrable functions, where $f(x)$ is continuous and $g(x)$ does not change sign in the integration interval (a,b). Then there exists a number $d \in (a,b)$ such that*

$$\int_a^b f(x)g(x)dx = f(d)\int_a^b g(x)dx. \quad (32)$$

Notice that the equality in (32) is exact and not merely an approximation.

Using this theorem, equation (31) can be written

$$\lambda^{-\frac{1}{\gamma(t')}} \int_0^\tau \exp\left\{-\left[\rho\frac{1}{\gamma(t)} + r(1-\frac{1}{\gamma(t)})\right.\right.$$
$$\left.\left. + \frac{1}{2}\eta'\cdot\eta(1-\frac{1}{\gamma(t)})(\frac{1}{\gamma(t)})\right]t\right\}\frac{l(x+t)}{l(x)}dt = w \quad (33)$$

for some $t' \in (0,\tau)$. Since $\gamma(t') > 0$, we see that the left-hand side of equation (33) defines a continuous function of λ, say $f(\lambda)$, that satisfies $f(\lambda) \to +\infty$ when $\lambda \to 0^+$, and $f(\lambda) \to 0$ when $\lambda \to +\infty$. Moreover the function $f : R_+ \to R_+$ is invertible and on-to. Thus, for any $w > 0$ there exists one (and only one) $\lambda > 0$ such that equation (33) holds true, which was to be shown. We work with this value of λ from now on.

Having determined the optimal consumption rate c_t^*, the last step in our approach is to find the optimal wealth process W_t associated with c^*, and its

corresponding Itô process representation. This we subsequently compare to the representation given in (19), where the portfolio fractions φ correspond to the solution of the problem (17)-(22). By the equivalence between the two problem formulations in Section 2c, since an Itô process representation is unique, this will give us the final equation from which to determine the optimal portfolio fractions φ.

To this end, the optimal wealth process W_t associated to c^* of (29) with λ satisfying (30), or equivalently (33), is given by

$$
W_t = \frac{1}{\pi_t} E_t\Big\{ \int_t^{T_x} \pi_s c_s^* ds \Big\} =
$$

$$
\frac{1}{\pi_t} E_t\Big\{ \int_t^{T_x} \big(\lambda e^{\rho s}\big)^{-\frac{1}{\gamma(s)}} \pi_s^{(1-\frac{1}{\gamma(s)})} ds \Big\} =
$$

$$
\frac{1}{\pi_t} \int_t^{\tau} \big(\lambda e^{\rho s}\big)^{-\frac{1}{\gamma(s)}} E_t\Big\{ \exp\Big[-r\big(1-\frac{1}{\gamma(s)}\big)s
$$

$$
-\frac{s}{2}\eta' \cdot \eta\big(1-\frac{1}{\gamma(s)}\big) - \big(1-\frac{1}{\gamma(s)}\big)\eta' \cdot B_s \Big]\Big\} \frac{l(x+s)}{l(x+t)} ds. \quad (34)
$$

Here E_t means conditional expectation given the information filtration $\mathcal{F}_t \vee (T_x > t)$, i.e., given the financial information available at time t and the fact that the individual is alive then. Also note that $P(T_{x+t} > s - t) = \frac{l(x+s)}{l(x+t)}$ for $s > t$. The conditional expectation in (34) is found as follows, for any $s > t$:

$$
E_t\Big\{ \exp\Big[-r\big(1-\frac{1}{\gamma(s)}\big)s - \frac{s}{2}\eta' \cdot \eta\big(1-\frac{1}{\gamma(s)}\big) - \big(1-\frac{1}{\gamma(s)}\big)\eta' \cdot B_s \Big]\Big\} =
$$

$$
\exp\Big\{ -r\big(1-\frac{1}{\gamma(s)}\big)s - \frac{s}{2}\eta' \cdot \eta\big(1-\frac{1}{\gamma(s)}\big) - \big(1-\frac{1}{\gamma(s)}\big)\eta' \cdot B_t \Big\} \cdot
$$

$$
E_t\Big\{ \exp\Big(-\big(1-\frac{1}{\gamma(s)}\big)\eta' \cdot (B_s - B_t)\Big)\Big\} =
$$

$$
\exp\Big\{ -r\big(1-\frac{1}{\gamma(s)}\big)s - \frac{s}{2}\eta' \cdot \eta\big(1-\frac{1}{\gamma(s)}\big) - \big(1-\frac{1}{\gamma(s)}\big)\eta' \cdot B_t \Big\} \cdot
$$

$$
\exp\Big(\frac{1}{2}\big(1-\frac{1}{\gamma(s)}\big)^2 \eta' \cdot \eta(s-t)\Big) =
$$

$$
\exp\Big\{ -r\big(1-\frac{1}{\gamma(s)}\big)t - \frac{t}{2}\eta' \cdot \eta\big(1-\frac{1}{\gamma(s)}\big) - \big(1-\frac{1}{\gamma(s)}\big)\eta' \cdot B_t \Big\} \cdot
$$

$$
\exp\Big\{ -\big(r + \frac{1}{2}\eta' \cdot \eta - \frac{1}{2}\eta' \cdot \eta\big(1-\frac{1}{\gamma(s)}\big)\big)\big(1-\frac{1}{\gamma(s)}\big)(s-t)\Big\} =
$$

$$
\pi_t^{\big(1-\frac{1}{\gamma(s)}\big)} \cdot \exp\Big\{ -\big(r + \frac{1}{2}\eta' \cdot \eta - \frac{1}{2}\eta' \cdot \eta\big(1-\frac{1}{\gamma(s)}\big)\big)\big(1-\frac{1}{\gamma(s)}\big)(s-t)\Big\}.
$$

In the above we have used the independence of the Brownian increment $(B_s - B_t)$ from the information in $\mathcal{F}_t$, the multinormal distribution of this increment as well as the expression for the state price deflator π_t given in (2) and (3).

Going back to the expression for the optimal wealth in equation (34), we now obtain

$$W_t = \int_t^\tau \left(\lambda e^{\rho s}\right)^{-\frac{1}{\gamma(s)}} \pi_t^{-\frac{1}{\gamma(s)}} \exp\left\{ -\left(r + \frac{1}{2}\eta' \cdot \eta \right.\right.$$
$$\left.\left. -\frac{1}{2}\eta' \cdot \eta\left(1 - \frac{1}{\gamma(s)}\right)\right)\left(1 - \frac{1}{\gamma(s)}\right)(s - t)\right\} \frac{l(x+s)}{l(x+t)} ds. \quad (35)$$

First notice by (2), (3) and Itô's lemma that

$$d\pi_t = -\pi_t\left(rdt + \eta'dB_t\right), \qquad \pi_0 = 1,$$

and hence, again using Itô's formula, for any fixed $s > t$ it follows that

$$d\pi_t^{-\frac{1}{\gamma(s)}} = \left(\frac{r}{\gamma(s)} + \frac{1}{2}\frac{1}{\gamma(s)}\left(\frac{1}{\gamma(s)} + 1\right)\eta' \cdot \eta\right)\pi_t^{-\frac{1}{\gamma(s)}} dt + \pi_t^{-\frac{1}{\gamma(s)}}\frac{1}{\gamma(s)}\eta'dB_t. \quad (36)$$

Let the function $g(s,t)$ be defined by

$$g(s,t) := \left(\lambda e^{\rho s}\right)^{-\frac{1}{\gamma(s)}} \pi_t^{-\frac{1}{\gamma(s)}} \exp\left\{ -\left(r + \frac{1}{2}\eta' \cdot \eta \right.\right.$$
$$\left.\left. -\frac{1}{2}\eta' \cdot \eta\left(1 - \frac{1}{\gamma(s)}\right)\right)\left(1 - \frac{1}{\gamma(s)}\right)(s - t)\right\} \frac{l(x+s)}{l(x+t)}. \quad (37)$$

Then we may write

$$W_t = \int_t^\tau g(s,t)\, ds,$$

which means that

$$dW_t = -g(t,t)dt + \int_t^\tau d_t g(s,t)\, ds.$$

Using the definition of $g(s,t)$ in (37) and the result in (36), we obtain the following stochastic differential equation for the wealth W_t

$$dW_t = \mu_W(t)dt + \left(\int_t^\tau \left(\lambda e^{\rho s}\right)^{-\frac{1}{\gamma(s)}} \frac{1}{\gamma(s)}\pi_t^{-\frac{1}{\gamma(s)}} \cdot\right.$$
$$\exp\left\{ -\left(r + \frac{1}{2}\eta' \cdot \eta - \frac{1}{2}\eta' \cdot \eta\left(1 - \frac{1}{\gamma(s)}\right)\right) \cdot\right.$$
$$\left.\left. \left(1 - \frac{1}{\gamma(s)}\right)(s - t)\right\} \frac{l(x+s)}{l(x+t)} ds\right)\eta'dB_t, \quad (38)$$

where the drift term is given by

$$
\mu_W(t) = -\left(\lambda e^{\rho t}\right)^{-\frac{1}{\gamma(t)}} \pi_t^{-\frac{1}{\gamma(t)}} + \int_t^\tau \left(\lambda e^{\rho s}\right)^{-\frac{1}{\gamma(s)}} \pi_t^{-\frac{1}{\gamma(s)}} \cdot
$$
$$
\exp\left\{-\left(r + \frac{1}{2}\eta'\cdot\eta - \frac{1}{2}\eta'\cdot\eta\left(1 - \frac{1}{\gamma(s)}\right)\right)\left(1 - \frac{1}{\gamma(s)}\right)(s-t)\right\}\cdot
$$
$$
\left(\frac{1}{\gamma(s)}(r + \eta'\cdot\eta) + \mu_x(t)\right)\frac{l(x+s)}{l(x+t)}ds. \quad (39)
$$

In the above we have used (14) in the evaluation of the derivative of $P(T_{x+t} > s - t) = \frac{l(x+s)}{l(x+t)}$ with respect to current time t, i.e.,

$$
\frac{\partial}{\partial t}\left(\frac{l(x+s)}{l(x+t)}\right) = -\frac{l(x+s)}{l(x+t)^2}l'(x+t) = \mu_x(t)\frac{l(x+s)}{l(x+t)},
$$

which shows that this time effect on the survival probability positively influences the drift term of wealth. The reason is simply that the decrement function $l(x)$ is decreasing with age, which implies, among other things, that the one-year death probability is strictly positive. As an example, the probability of surviving 10 years for a 45 year old consumer is smaller than the corresponding probability of the same person, now one year older, to survive another 9 years. This is because the 45 year old has a positive probability of dying before reaching age 46.

Using (29) we see that the first term in (39) is equal to $-c_t^*$, and reflects the negative effect from current consumption on the remaining wealth of the agent.

From the expression (35) for the wealth $W(t)$ and from the diffusion term for $W(t)$ given in (38), it follows from the First Mean Value Theorem for Integrals in Proposition 1 that

$$
dW(t) = \mu_W(t)dt + \sigma_W(t)dB_t \quad (40)
$$

where

$$
\sigma_W(t) = W(t)\frac{1}{\gamma(\tilde{t}_t)}\eta'. \quad (41)
$$

The quantity $\tilde{t}_t > t$ for all $t \in (0, \tau)$, $\tilde{t}_t \in (t, \tau)$ and $\tilde{t}_t$ is $\mathcal{F}_t \vee (T_x > t)$-measurable. By Proposition 1, for each $t > 0$ this random quantity is deter-

mined by the following equation

$$\frac{1}{\gamma(\tilde{t}_t)} \int_t^\tau \left(\lambda e^{\rho s}\right)^{-\frac{1}{\gamma(s)}} \pi_t^{-\frac{1}{\gamma(s)}} \exp\left\{-\left(r + \frac{1}{2}\eta' \cdot \eta\right.\right.$$
$$\left.\left. -\frac{1}{2}\eta' \cdot \eta\left(1 - \frac{1}{\gamma(s)}\right)\right)\left(1 - \frac{1}{\gamma(s)}\right)(s-t)\right\}\frac{l(x+s)}{l(x+t)}ds =$$
$$\int_t^\tau \left(\lambda e^{\rho s}\right)^{-\frac{1}{\gamma(s)}}\frac{1}{\gamma(s)}\pi_t^{-\frac{1}{\gamma(s)}} \exp\left\{-\left(r + \frac{1}{2}\eta' \cdot \eta - \frac{1}{2}\eta' \cdot \eta\left(1 - \frac{1}{\gamma(s)}\right)\right)\cdot\right.$$
$$\left.\left(1 - \frac{1}{\gamma(s)}\right)(s-t)\right\}\frac{l(x+s)}{l(x+t)}ds. \quad (42)$$

From this relationship we notice that in addition to depending on wealth W_t and current time t, the random time $\tilde{t}_t$ depends upon the subjective rate ρ, the state of the economy through the state price deflator π_t, the market price of risk η, as well as the other parameters of the problem including mortality through the probability distribution of the remaining lifetime T_{x+t} of the agent, whose age is $(x+t)$ at time t.

Finally, by comparing the stochastic differential equation for the optimal wealth $W(t)$ in (38) with the analogous equation given in (19), where the portfolio fractions φ correspond to the solution of the problem (17)-(22), by complete markets it follows that

$$\varphi_t'\sigma = \frac{1}{\gamma(\tilde{t}_t)}\eta',$$

or by the use of the relation (4) for the market-price-of-risk η, we have that the optimal portfolio ratios are given by

$$\varphi(t) = \frac{1}{\gamma(\tilde{t}_t)}(\sigma\sigma')^{-1}\nu. \quad (43)$$

This is the appropriate generalization of the standard result stating that

$$\varphi = \frac{1}{\gamma}(\sigma\sigma')^{-1}\nu$$

does not depend on current time t, nor of the wealth W_t of the agent, when the relative risk aversion γ is a constant.

The solution in (43) can accommodate many of the observed empirical facts and recommendations by portfolio managers. Assuming for example that the function $\gamma(t)$ is increasing in time t, then the investor will gradually invest a smaller fraction of his/her wealth in the risky assets as time runs, since the random time $\tilde{t}_t$ is always larger that current time t. However, this

relationship may not be strictly monotonic, since $\tilde{t}_t$ is also state dependent, as noted above. Moreover this stochastic time also depends on mortality.

It seems fairly intuitive that the investor, under Assumption 1, will choose a time between now and τ in the investment strategy, given his/her choice to act in a more risk averse manner in the future. This represents a clear refinement of the canonical model, in a more realistic direction in this regard. In the next section we state the main theorem of the paper.

We notice from the representation of the wealth in (35) - (41) that it is a indeed a Markov process, so in principal one can solve the problem by the use of dynamic programming. Technically this would amount to guessing the solution, and then using The Verification Theorem.

IV   The Main Result

We start by formulating the conclusions of the previous section. Define the process $Y(t)$ by the following expression

$$
Y(t) = \int_t^\tau \left(\lambda e^{\rho s}\right)^{-\frac{1}{\gamma(s)}} \frac{1}{\gamma(s)} \pi_t^{-\frac{1}{\gamma(s)}} \exp\Big\{ -\Big(r + \frac{1}{2}\eta' \cdot \eta
$$
$$
- \frac{1}{2}\eta' \cdot \eta\big(1 - \frac{1}{\gamma(s)}\big)\Big)\Big(1 - \frac{1}{\gamma(s)}\Big)(s-t)\Big\} \frac{l(x+s)}{l(x+t)} ds. \quad (44)
$$

Then we have the following:

Theorem 1 *Suppose the relative risk aversion $\gamma(t)$ of the agent is a continuous function of time t. Then the optimal portfolio fractions $\varphi(t)$ in the risky securities are time and wealth dependent, and given by*

$$
\varphi(t) = \frac{1}{\gamma(\tilde{t}_t)}(\sigma\sigma')^{-1}\nu,
$$

where the random time $\tilde{t}_t$, always larger than current time t, and in the information set $\mathcal{F}_t \vee (T_x > t)$, is determined at each time t by the equation

$$
\gamma(\tilde{t}_t) = \frac{W(t)}{Y(t)}. \quad (45)
$$

Here the agent's optimal wealth $W(t)$ is given by equation (35), where the Lagrange multiplyer λ is found from (33), and $Y(t)$ is defined in (44).

Equation (45) is just a rewriting of (42) using (44).

IV-A Discussion of the Theorem

IV-A.1 General Remarks

Equation (45) of Theorem 1 gives us the opportunity to study some properties of the optimal investment strategy. First notice that the portfolio choice decision can not be separated from the consumption versus savings decision. Here the portfolio weights φ_t depend on wealth, or on consumption, which is not the case for the canonical model. So relaxing the time and state separability in the utility function, gives us a resolution of the horizon problem, at the cost of this "Fisher" separation.

Second, observe that when the function $\gamma(t) = \gamma$ for all t, then $\gamma(\tilde{t}_t) = \gamma$ a.s. for all $t \in (0, \tau)$ and the cannonical solution results.

Third, notice that when the investment horizon, call it τ, is deterministic as often is the case in standard financial models, then we still have a solution to the horizon problem provided our assumption (6) is maintained, namely that $\gamma(t)$ is time varying. Still $\tilde{t}_t$ can be found from equation (42), merely by setting the force of mortality $\mu_x(u) = 0$ for all $u \in [0, \tau]$. This is accomplished in this equation by setting the survival probability equal to one, or $l(x+s) = l(x+t)$ for all $s \in (t, \tau)$. Merely allowing the time horizon to be random is not contributing to the solution of the horizon problem.

IV-A.2 Comparative Statics

We can also derive some comparative statics results, starting with equation (42), or (45). First we make the assumption that $\gamma(t) > 1$ for all $t \in (0, \tau)$ and $\gamma(t)$ is an increasing function.

Except from t and W_t, perhaps the most interesting effect on the optimal investment strategy results from a shock in the state price π_t. Also the effect from changing the survival probability is of interest. Both these quantities are dynamic, so we will get a more correct picture when the dynamics of γ_t is taken into account. This we do below.

First notice that an increase in wealth $W(t)$ implies, ceteris paribus, that $\gamma(\tilde{t}_t)$ increases in order for (45) to hold. Thus $\tilde{t}_t$ increases, implying that $\varphi(t)$ decrease. A positive shock to π_t on the other hand, leads to an increase in φ_t, and a decrease in the optimal consumption rate c_t^*, all else equal.

This leads to a counter-cyclical investment behavior, consistent with the effects from including habit formation in the preferences.

A similar wealth effect also materializes for general HARA utility functions when the relative risk aversion is an increasing function of consumption (see e.g., Aase and Quigstad Sørensen (2008)).

Turning to the subjective rate ρ, an increase in this quantity leads to, ceteris paribus, an increase in the fractions held in the risky securities. The more impatient the investor is, the higher ratios are held in the risky assets.

When time t increases, the decrement function $l(x+t)$ decreases. Also the mortality is not state dependent. Therefore we consider the effect on $\varphi(t)$ from a decrease in $l(x+t)$, all else equal: It is negative, telling us that the survival element causes the agent to move into less risky investments, relatively speaking, as time goes.

When the function $\gamma(t)$ is smaller than one, the above conclusions are all reversed. In this regard the value of one for the relative risk aversion is a border case, and explains why the agent is sometimes called *risk tolerant* when $\gamma < 1$.

When the force of mortality depends on both age and wealth, Aase and Qvigstad Sørensen (2008) obtain a condition under which the young should invest a higher fraction in the risky asset ($N = d = 1$) than the old. The condition is satisfied when the force of mortality is separable in time and wealth.

Bodie and Crane (1997) find among individuals in a surevey, that the portion of total assets held in equities declines with age and rises with wealth. This would be consistent with our model for risk tolerant individuals and increasing $\gamma(\cdot)$. Summers et al. (2006) find that individuals' portfolios become more risk-seeking with age, taking account of asset accumulation. This could be consistent with our model with $\gamma(\cdot)$ decreasing with time, and larger than one, illustrating the flexibility obtained by allowing risk aversion to depend on time.

IV-A.3   The Relative Risk Tolerance

It follows from (23) and Theorem 1 that the relative risk tolerance of the indirect utility function at time t is given by

$$-\frac{J_w(W(t),t)}{W(t)J_{ww}(W(t),t)} = \frac{1}{\gamma(\tilde{t}_t)},$$

while the relative risk tolerance of the direct utility function at time t is $\frac{1}{\gamma(t)}$. These are different, and under the assumption that the function $\gamma(t)$ is increasing, the former is smaller that the latter, and non-increasing with time according to Theorem 1, explaining the positive time effect.

IV-A.4   The Pure Time Discount Rate

It can be seen from the expression for the wealth given in (35) that pure time discounting at time t of future consumption at time s occurs according to

the rate

$$-\frac{1}{\gamma(s)}\Big(\rho(s-t)-r(s-t)\Big)-\Big(r(s-t)+\int_0^{s-t}\mu_{x+t}(u)du\Big).$$

When the consumer is impatient and $\rho > r$, the future time discounting is higher than for a more patient agent. This effect decreases if $\gamma(\cdot)$ is an increasing function. Notice how mortality acts as an addition to the risk-fee rate.

IV-A.5 The Dynamics of the Optimal Consumption

It is instructive to take a look at the dynamics of the optimal consumption rate c_t^*. Assuming that $\gamma(t)$ is differentiable, by Itô's formula

$$dc_t^* = c_t^*\Big(\frac{1}{\gamma(t)}(r-\rho)+\frac{\gamma'(t)}{\gamma(t)^2}\big(\rho t+\ln(\pi_t)\big)+\frac{1}{2}\frac{1}{\gamma(t)}\Big(\frac{1}{\gamma(t)}+1\Big)\eta'\cdot\eta\Big)dt$$
$$+ c_t^*\Big(\frac{1}{\gamma(t)}\eta'\Big)dB(t), \quad (46)$$

where $\gamma'(t)$ is the derivative of $\gamma(t)$ with respect to time.

The first term in the drift is the familiar, ordinary differential equation for c_t^* when there is only a credit market available. Solved together with the budget constraint, one can analyze various insurance contracts depending on the nature of this constraint. It tells us that the time evolution of c_t^* is strictly increasing, or decreasing, depending on the sign of $(r-\rho)$. Adding the third term in the drift and the diffusion term, we get the analogous results when a stock market is included, where c_t^* becomes a geometric Brownian motion.

Finally, adding the second term in the drift gives us the current model with time varying risk aversion. Because $\ln(\pi_t)$ appears in the drift, the optimal consumption rate is no longer a log-normal process.

Using (2) and (3) this second term can be written

$$\frac{\gamma'(t)}{\gamma(t)^2}\big(\rho t+\ln(\pi_t)\big) = \frac{\gamma'(t)}{\gamma(t)^2}\Big((\rho-r)-\frac{t}{2}\eta'\cdot\eta-\eta'B(t)\Big).$$

Because of the presence of the Brownian motion, the sign of this term is ambiguous. However, when $\gamma'(t) > 0$, the presence of this term will dampen the effect from the first term of the drift for low values of t.

IV-A.6 An Alternative Optimization Criterion

If instead of maximizing the expected utility of life time consumption, as we do, the criterion is to maximize the expected utility of end of period

consumption, with our assumption of a time varying relative risk aversion and, for simplicity, a fixed time horizon τ, the optimal investment rule would be

$$\varphi(t) = \frac{1}{\gamma(\tau)}(\sigma\sigma')^{-1}\nu \qquad \text{for all } t \leq \tau.$$

Thus the end of period preference would dominate the investment rule at each time t in the investment period. Also here there is a difference between the indirect and the direct utility functions, still there is no "running time" effect. This shows that this distinction in the objective criterion matters under our assumption; the criterion we have chosen is clearly the most relevant one in our setting.

IV-A.7   Extensions to Life Insurance

In the model there is no bequest. Introducing demand for life insurance via a bequest v for positive wealth at the time of death T_x, will not change matters very much. Here we present a very brief sketch. Suppose the objective criterion is

$$U(c, Z) = E\Big(\int_0^{T_x} u(c_t, t)dt + v(Z_{T_x}, T_x)\Big)$$

where Z is a nonnegative random variable describing terminal lump-sum consumption, which we interpret as the amount of life insurance payable upon death of the individual. Standard life insurance contracts postulate Z to be a known quantity, say 1, but we can actually solve the problem where Z is a decision variable, so that the the amount paid to the heirs is both time and state dependent. For simplicity of exposition, assume that $v = u$.

Proceeding as in Section 3, the first order condition for optimality in Z turns out to be

$$v'(Z_t, t) = \lambda\pi_t \qquad \text{for } t = T_x,$$

which means that the optimal amount of life insurance is given by

$$Z_t = \big(\lambda e^{\rho t}\pi_t\big)^{-\frac{1}{\gamma(t)}}, \qquad \text{for } t = T_x.$$

Optimality in c is as before. The Lagrange multiplier λ is found from an equation like (33), with the only change that the conditional survival probability $P(T_x > t)$ is replaced by $(P(T_x > t) + f_x(t))$, i.e., the term $\frac{l_{x+t}}{l_x}$ is replaced by the sum $\frac{l_{x+t}}{l_x}(1 + \mu_{x+t})$.

Theorem 1 is still valid, except that in the corresponding expressions for $W(t)$ and $Y(t)$, the conditional survival probability $P(T_x > s | T_x > t) = \frac{l_{x+s}}{l_{x+t}}$, for $s > t$, is replaced by the sum

$$(P(T_x > s | T_x > t) + P(T_x \in (s, s+ds) | T_x > t),$$

where the conditional probability density

$$P(T_x \in (s, s+ds)|T_x > t) = \frac{P(T_x \in (s, s+ds))}{P(T_x > t)} = \frac{l_{x+s}}{l_{x+t}}\mu_{x+s}.$$

In other words, the term $\frac{l_{x+s}}{l_{x+t}}$ is replaced by $\frac{l_{x+s}}{l_{x+t}}(1 + \mu_{x+s})$. There will naturally be a change to the drift $\mu_W(t)$, but still Theorem 1 is valid with the above changes. The economic effect is that the consumer's lifetime consumption will be reduced with an amount corresponding to the actuarial value of this life insurance contract. The optimal investment policy is still of the type given in Theorem 1. This life insurance contract with a state and time dependent insured amount would constitute an innovation in the market for life insurance contracts.

A standard life insurance contract with $Z = 1$ a.s. is of course simpler to analyze. In this case there is no optimization in the Z-variable, and Theorem 1 will take on the following changes: In the equation for λ the initial wealth w is replaced by $(w - \bar{A}_x)$, where $\bar{A}_x = \int_0^\tau e^{-rt}\frac{l_{x+t}}{l_x}\mu_{x+t}dt$ is the actuarial value of the life insurance contract at time zero, when the consumer is in age x. The equation for the consumer's wealth $W(t)$ at time t will get an addition by the amount $\bar{A}_{x+t}$, the remaining net value of the life insurance contract when the consumer has reached age $x+t$, and by Thiele's differential equation of actuarial science, to the drift term $\mu_W(t)$ we will have to add the term

$$\bar{A}'_{x+t} = \bar{A}_{x+t}r - \mu_{x+t}(1 - \bar{A}_{x+t})$$

where prime means differentiation with respect to time. Otherwise Theorem 1 is unchanged. We notice that the effect from this life insurance contract on the consumer's life time consumption is to lower the consumption with exactly the actuarial value $\bar{A}_x$ of this contract.

IV-A.8 Aggregation

For the canonical model aggregation does not work unless all agents have the same preferences. Below we demonstrate that the preferences of Assumption 1 aggregate.

Imagine there are I agents in the model indexed by $i \in \{1, 2, \cdots, I\}$, endowed with outstanding shares, say $\bar{\theta}_i$, with different risk aversion functions $\gamma_i(\cdot)$ and subjective rates ρ_i. The present model is easier to reconcile with an equilibrium than the canonical one. First, a positive shock to the state price π_t leads to a decrease in the optimal consumption for risk averse agents, and an increase in consumption for risk tolerant ones. In contrast, for the canonical model an increase in π_t leads to a decrease in each individual's

wealth $W_i(t)$, and accordingly a decrease in consumption for all the agents, by the wealth effect. This monolithic behavior will also tend to carry over to the stock market, meaning that all the investors should buy and sell at about the same time. This can not be consistent with a general equilibrium model that makes realistic assumptions about asset supplies.

As the comparative statics show, in the present model a positive shock in the state price leads to an increased demand for the risky assets for risk averse persons, and a decreased demand for risk tolerant ones.

Second, the investment behavior of the representative agent is related to the relative risk tolerance of this agent's indirect utility function. Recall that in a Pareto optimum, the sum of the individual absolute risk tolerances is equal to the absolute risk tolerance of the representative agent. This would require an equality of the form

$$\frac{1}{W(t)} \sum_{i=1}^{I} \frac{W_i(t)}{\gamma_i(\tilde{t}_t^i)} = \frac{1}{\gamma_{RA}(\tilde{t}_t^{RA})} \tag{47}$$

for some $\tilde{t}_t \in (t, \tau)$ and some function $\gamma(\cdot)$. Here $W(t) = \sum_{i=1}^{I} W_i(t)$ is the aggregate wealth.

In (47) the function $\gamma_{RA}(\tilde{t}_t^{RA})^{-1}$ is a convex combination of the individual $\gamma_i(\tilde{t}_t^i)^{-1}$ with time dependent, continuous and $\mathcal{F}_t$-measurable weights. All the processes $\tilde{t}_t^i$ are $\mathcal{F}_t$-measurable, and hence so is $\tilde{t}_t^{RA}$. Furthermore the function $\gamma_{RA}(\cdot)$ is continuous. Hence, the relative risk aversion of the representative agent's indirect utility function is of the same type as the individual in Theorem 1. Notice that if one of the agents is close to risk neutral, this agent may only dominate in the representation (47) if his or her relative wealth is not too low, in which case the representative agent may be close to risk neutral as well.

The corresponding representation for the canonical model is

$$\frac{1}{W(t)} \sum_{i=1}^{I} \frac{W_i(t)}{\gamma_i} = \frac{1}{\gamma}, \tag{48}$$

for some positive constant γ. We note that this equality can not hold for (almost) all $t \in (0, \tau)$ almost surely unless $\gamma_i = \gamma$ for all i, since otherwise the left-hand side is both state and time varying, while the right-hand side is a constant.

While asset prices and wealth change with time in a random manner, so does the process $\tilde{t}_t^{RA}$ in (47). The question is if there exist a set of clearing prices of the class prescribed by our model such that the representative agent retains his portfolio $\bar{\theta} = \sum_i \bar{\theta}_i$ unchanged as time goes.

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This is a problem with both models, but most severe for the canonical one.

IV-B   The dynamics of the relative risk aversion

When the function $\gamma(\cdot)$ is smooth and invertible (e.g., monotonic), and has a smooth inverse, the random process $\tilde{t}_t$ is an Itô-process, since it can be written as a smooth function of a ratio of two positive Itô-processes, in which case its representation can be found from (42). From a study of this representation one should be able to separate the effects of the random, exogenous shocks from the pure time increment. More directly, we are interested in the variations of the risk aversion $\gamma(\tilde{t}_t)$ itself, also an Itô-processes. In order to see what is involved, we next present the dynamic representation of $\gamma(\tilde{t}_t)$.

We seek the Itô-processes representation

$$d\gamma(\tilde{t}_t) = \mu_\gamma dt + \sigma_\gamma dB(t)$$

of $\gamma(\tilde{t}_t)$ under the assumptions of the last section. Starting with (42)

$$\gamma(\tilde{t}_t) = \frac{W(t)}{Y(t)} := \frac{\int_t^\tau h(s,t)\pi_t^{-\frac{1}{\gamma(s)}}ds}{\int_t^\tau h(s,t)\frac{1}{\gamma(s)}\pi_t^{-\frac{1}{\gamma(s)}}ds},$$

where $h(s,t)$ is the strictly positive, deterministic function depending on all the parameters of the problem given by $h(s,t)\pi_t^{-\frac{1}{\gamma(s)}} = g(s,t)$, where $g(s,t)$ is defined in (37). We already know the Itô-representation of the wealth $W(t)$ given in (38)-(41). The corresponding representation for the process Y is

$$dY(t) = \mu_Y(t)dt + \sigma_Y(t)dB(t),$$

or

$$dY(t) = \Big(-h(t,t)\frac{1}{\gamma(t)}\pi_t^{-\frac{1}{\gamma(t)}} + \int_t^\tau \frac{1}{\gamma(s)}\pi_t^{-\frac{1}{\gamma(s)}}\Big(h(s,t)\cdot$$
$$\Big(\frac{r}{\gamma(s)} + \frac{1}{2}\frac{1}{\gamma(s)}\big(\frac{1}{\gamma(s)} + 1\big)\Big) + h_t(s,t)\Big)ds\Big)dt$$
$$+ \Big(\int_t^\tau h(s,t)\big(\frac{1}{\gamma(s)}\big)^2 \pi_t^{-\frac{1}{\gamma(s)}}ds\Big)\eta' dB_t, \quad (49)$$

where $h_t(s,t)$ is the derivative of $h(s,t)$ with respect to t. By Itô's formula we may write

$$d\gamma(\tilde{t}_t) = \frac{1}{Y(t)}dW(t) - \frac{W(t)}{Y(t)^2}dY(t) + \frac{W(t)}{Y(t)^3}(dY(t))^2 - \frac{dW(t)dY(t)}{Y(t)^2},$$

or, in standard form

$$d\gamma(\tilde{t}_t) = \Big(\frac{1}{Y(t)}\mu_W(t) - \frac{W(t)}{Y(t)^2}\mu_Y(t) + \frac{W(t)}{Y(t)^3}\sigma_Y(t) \cdot \sigma_Y(t)$$
$$- \frac{\sigma_W(t) \cdot \sigma_Y(t)}{Y(t)^2} \Big) dt + \Big(\frac{1}{Y(t)}\sigma_W(t) - \frac{W(t)}{Y(t)^2}\sigma_Y(t) \Big) dB(t). \quad (50)$$

From this representation we see that the dynamics of the risk aversion is rather involved, and will of course depend on the shape of the function $\gamma(\cdot)$. For example can the diffusion term be written

$$\sigma_\gamma(t) = \Big(\frac{1}{Y(t)}\sigma_W(t) - \frac{W(t)}{Y(t)^2}\sigma_Y(t) \Big) = \Big(1 - \frac{\gamma(\tilde{t}_t)}{\gamma(t_t^*)} \Big)\eta', \quad (51)$$

where t_t^* is determined by $\sigma_Y(t) = \frac{1}{\gamma(t_t^*)}\sigma_W(t)$, using Proposition 1.

Assuming as above that $\gamma(t)$ is increasing and larger than one, it can be seen that $\gamma(t_t^*) \geq \gamma(\tilde{t}_t)$ a.s. Since the vector of market-prices-of-risk η has positive elements in this model, the diffusion term of $\gamma(\tilde{t}_t)$ is a.s. non-negative. Recall that B is a vector of independent Brownian motions, so the sign of a "shock" in $B(t)$ is rather ambiguous. Therefore, let us for the moment assume that $N = d = 1$, so that there is only one risky asset, and one source of exogenous shocks. Assuming the drift $\mu_\gamma(t)$ is positive, then a positive increment $dB_t > 0$ implies that $d\gamma_t > 0$, and consequently, from (43) it is seen that the investor will reduce the fraction in the risky security. This is consistent with the comparative statics result of the last section, since a positive shock in B leads to a negative change in the state price π.

The drift term can be written

$$\mu_\gamma(t) = \frac{\mu_W(t)}{Y(t)} \Big(1 - \frac{\gamma(\tilde{t}_t)}{\gamma(t_t^{**})} \Big) + \Big(\frac{\gamma(\tilde{t}_t)}{\gamma(t_t^*)^2} - \frac{1}{\gamma(t_t^*)} \Big)\eta' \cdot \eta,$$

where $\mu_Y(t) = \frac{1}{\gamma(t_t^{**})}\mu_W(t)$ follows by a slight extension of Proposition 1.

When $dB(t) < 0$, the increment $d\gamma_t$ may be positive or negative depending on the relative sizes of the terms involved. However, negative shocks in γ has a limited global effect on the evolution of the random process $\tilde{t}_t$, since the restriction $\tilde{t}_t > t$ must be met according to Theorem 1, which means that the drift term $\mu_\gamma(t)$ must, on the average, be positive. Since the last term in the expression for his drift is negative, and since $\mu_W(t)$ becomes negative as the agent grows older as can be seen from (39), eventually the term $(1 - \frac{\gamma(\tilde{t}_t)}{\gamma(t_t^{**})})$ must become negative. The result is that when the consumer/investor grows old enough, the current, optimal consumption c_t^* dominates all the other effects and risky investments decrease.

Notice that when γ is a constant, then both the drift and the diffusion terms are readily seen to be identically equal to zero from the above representation, as the case should be, and the canonical model results.

These questions may be further analyzed by e.g., numerical techniques based on simulations, or analytic methods. The above dynamic representation for γ is well suited for this.

V   Conclusions

The paper investigates the effect of horizon length on portfolio choice. We have considered the problem of maximizing the utility of a consumption stream over the life cycle of an individual, who can invest in a complete market. In our model the income is generated by an endowment process.

In this setting we have demonstrated that if the intertemporal coefficient of relative risk aversion is allowed to be time dependent, the investment horizon problem can be resolved. When this coefficient is increasing with time, for example, the individual will invest smaller fractions of his wealth in the risky securities as he grows older.

At each time t the optimal portfolio ratios depend on the realization of an $\mathcal{F}_t \vee (T_x > t)$-measurable random variable $\tilde{t}_t(\omega)$, with probability distribution depending on the joint distribution of the state price and the remaining lifetime of the individual. This means that, in addition to current time and wealth, the optimal portfolio ratios depend on the market-price-of-risk, the risk-free rate, the subjective discount rate, and mortality. Thus the portfolio choice decision can no longer be separated from the consumption versus savings decision.

Comparative statics show the following: When the risk aversion is an increasing function larger than one, an increase in wealth implies a decrease in the ratios held in the risky assets, ceteris paribus, and a decrease in the state price leads to an increase in the optimal consumption, and a decrease in the risky exposure. An increase in the impatience rate leads the agent to invest higher ratios in the risky securities. The time effect from mortality can be obtained from a decrease in the decrement function. It leads the agent to choose a less risky exposure as as he/she grows older, all else equal. When the individuals are risk tolerant, i.e., when $\gamma < 1$, these comparative statics conclusions are all reversed, which is promising for an extension to equilibrium. We have demonstrated that the aggregation property holds for this kind of utility.

The analysis also enables us to study the investment behavior in the presence of the exogenous shocks in the stock market. The results are only sug-

gestive at this point, but indicate that a positive shock leads the investor to reduce the fraction invested in a risky asset, and increase consumption, while a negative shock will reduce these effects, possibly resulting in an increased exposure in the risky asset in some states of the world. On the average, however, the risky exposure will decrease as the consumer gets older, provided $\gamma(t)$ is an increasing, continuous function.

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