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An anticipative linear filtering equation

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Abstract

In the classical Kalman-Bucy filter and in the subsequent literature so far, it has been assumed that the initial value of the signal process is independent of both the noise of the signal and of the noise of the observations.The purpose of this paper is to prove a filtering equation for a linear system where the (normally distributed) initial value X_0 of the signal process X_t has a given correlation function with the noise (Brownian motion B_t) of the observation process Z_t. This situation is of interest in applications to insider trading in finance. We prove a Riccati type equation for the mean square error

$$S(t) := E[(X_t - \hat{X}_t)^2]; \qquad 0 \leq t \leq T,$$

where $\hat{X}_t$ is the filtered estimate for X_t. Moreover, we establish a stochastic differential equation for $\hat{X}_t$ based on $S(t)$. Our method is based on an enlargement of filtration technique, which allows us to put the anticipative linear filter problem into the context of a non-anticipative two-dimensional linear filter problem with a correlation between the signal noise and the observation noise.

MSC 2010: 60G35, 62M20, 93E10, 94Axx

Key words: Anticipative linear filter equation, enlargement of filtration, insider trading

1 Introduction

In the classical Kalman-Bucy filter [KB] it is assumed that the initial value of the signal process is independent of both the noise of the signal and of the noise of the observations. As far as we know, this assumption is still made in all the subsequent presentations of linear filtering. See e.g. [LS], [K], [D] and [Ø].

It is natural to ask what happens if we allow nonzero correlations between the initial signal and the noises. This question is of interest not just for mathematical curiosity, but also for applications to insider trading in finance. For example, in the recent paper [AaBØ] an anticipative version of the Kyle-Back insider trading problem is studied, where there is a possible correlation between the terminal stock price $\tilde{v}$ and the trading process z_t of the socalled "noise traders".

The purpose of this paper is to prove a filtering equation for a linear system where the (normally distributed) initial value X_0 of the signal process X_t has a given correlation function with the noise (Brownian motion B_t) of the observation process Z_t. We prove a Riccati type equation for the mean square error

$$S(t) := E[(X_t - \hat{X}_t)^2]; \qquad 0 \leq t \leq T,$$

where $\hat{X}_t$ is the filtered estimate for X_t. Moreover, we establish a stochastic differential equation for $\hat{X}_t$ based on $S(t)$. See Theorem 2.3. Our method is based on an enlargement of filtration technique, which allows us to place the anticipative linear filter problem into the context of a non-anticipative two-dimensional linear filter problem with a correlation between the signal noise and the observation noise.

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2 An anticipative linear filtering equation

Consider the following linear filtering system:

(2.1)
(Signal process) $dX_t = 0; \quad t \in [0, T]$,

(2.2)
(Observation process) $dZ_t = G(t)X_T dt + D(t)dB_t;\ t \in [0, T],\ Z_0 = 0.$

We assume that $G(t)$ and $D(t)$ are locally bounded deterministic processes and $|D(t)|$ is bounded away from 0. The process B_t is a Brownian motion, and we let $\mathcal{F}_t = \mathcal{F}_t^B$ be the filtration generated by $B_s; s \leq t$. We assume that $X_0 = X_T$ is a normally distributed $\mathcal{F}_T$-measurable random variable. For simplicity and without loss of generality we assume that

$$E[X_0] = 0.$$

However, we do *not* assume that X_0 is independent of $\{B_t\}_{t\in[0,T]}$ (which would be the classical case). Since X_0 is Gaussian, it can be given the (unique) representation

$$X_0 = \int_0^T h(t)dB_t; \tag{2.3}$$

where $h \in L^2([0, T])$ is deterministic.

This implies that the corresponding correlation function is given by

$$E[X_0 B_t] = \int_0^t h(s)ds. \tag{2.4}$$

Let $\mathcal{Z}_t$ be the σ-algebra generated by the observations Z_s; $0 \leq s \leq t$. We want to find a stochastic differential equation for the estimate (filter)

$$\hat{X}_t := E[X_t|\mathcal{Z}_t]; \qquad t \in [0, T], \tag{2.5}$$

and we want to find the mean square error process S(t), defined by

$$S(t) = E[(X_T - \hat{X}_t)^2]; \qquad t \in [0, T]. \tag{2.6}$$

To this end we follow the presentation given in [Ø, Chapter 6] for the classical case, with the necessary modifications needed in this anticipative situation.

In the following we define

$$\mathcal{H}_t = \sigma(X_T) \vee \mathcal{F}_t; \qquad t \in [0, T] \tag{2.7}$$

to be the filtration generated by $X_T = X_0$ and $B_s; s \leq t$. Then we have (see [P, p. 366]):

Lemma 2.1 (Enlargement of filtration (I))**.** *There exists an $\mathcal{H}_t$-adapted process $A(t)$ of finite variation such that $A(0) = 0$ and*

$$B_t = \tilde{B}_t + A_t, \tag{2.8}$$

where $\tilde{B}_t$ is a Brownian motion with respect to $\mathbb{H} := \{\mathcal{H}_t\}_{t\in[0,T]}$ (and with respect to the same (the original) probability measure P for B_t).

In fact, in our setting the socalled "information drift" $A(t)$ can be found explicitly, as follows (see [H, Theorem 3.1]).

Lemma 2.2 (Enlargement of filtration (II))**.** *Put*

$$m = E[X_T^2].$$

Then

$$dA_t = \alpha_t \, dt,$$

where

$$\alpha_t = \frac{h(t)(X_T - \int_0^t h(s) dB_s)}{m - \int_0^t h^2(s) ds} \tag{2.9}$$

We now consider our original filter problem (2.1) - (2.2) as a part of (the first component of) the following 2-dimensional linear filtering problem:

(signal process)

$$\begin{cases} dX_1(t) = 0; & X_1(0) = \tilde{v} \quad \text{(Gaussian, } \mathcal{F}_T\text{-measurable)} \\ dX_2(t) = h(t) dB_t; & X_2(0) = 0; \quad 0 \leq t \leq T \end{cases} \tag{2.10}$$

(observation process)

$$dZ_t = G(t) X_1(t) dt + D(t) dB_t; \qquad Z_0 = 0; \quad t \in [0, T]. \tag{2.11}$$

We want to find the two filters

$$\hat{X}_i(t) = E[X_i(t) \mid \mathcal{Z}_t]; \qquad i = 1, 2, \tag{2.12}$$

where

$$\mathcal{Z}_t = \sigma(Z_s; s \leq t) \qquad \text{is the observation filtration.} \tag{2.13}$$

As stated this problem is not a classical linear filter problem, because $X_1(t) = \tilde{v}$ is not $\mathcal{F}_t$-adapted.

However, if we introduce the $\mathcal{H}_t$-Brownian motion $\tilde{B}$ as in (2.8), so that

$$dB_t = \alpha_t dt + d\tilde{B}_t, \tag{2.14}$$

with

$$\alpha_t = \frac{h(t)[X_1(t) - X_2(t)]}{m - \int_0^t h^2(y)dy}; \qquad 0 \leq t \leq T, \tag{2.15}$$

then the observation process can be written

$$\begin{aligned} dZ_t &= G(t)X_1(t)dt + D(t)[\alpha_t dt + d\tilde{B}_t] \\ &= \Big\{ G(t)X_1(t) + \frac{D(t)h(t)[X_1(t) - X_2(t)]}{m - \int_0^t h^2(y)dy} \Big\} dt + D(t)d\tilde{B}_t. \end{aligned} \tag{2.16}$$

The signal process $X(t) = (X_1(t), X_2(t))^*$, where * denotes matrix transposed, gets the form

$$\begin{cases} dX_1(t) = 0; \qquad X_1(0) = \tilde{v} \\ dX_2(t) = h(t)[\alpha_t dt + d\tilde{B}_t] = \frac{h^2(t)[X_1(t) - X_2(t)]}{m - \int_0^t h^2(y)dy} dt + h(t)d\tilde{B}_t; \\ X_2(0) = 0 \end{cases} \tag{2.17}$$

Note that the system (2.16)–(2.17) constitutes a linear Gaussian filter problem. It is not anticipative any more, because the drift term in the observation process is adapted to the filtration $\mathcal{H}_t$ and $\tilde{B}_t$ is an $\mathcal{H}_t$-Brownian motion.

The only unusual about the system (2.16)–(2.17) is that there is a *correlation* between the signal noise and the observation noise. This case has been studied in [Kallianpur, Section 10.5]. For completeness we recall his result in the following:

Consider the following linear, multi-dimensional filter problem with noise correlation

(signal) $\quad dX(t) = [A_0(t) + A_1(t)X(t) + A_2(t)Z_t]dt + C(t)dW(t)t$

(observation) $\quad dZ_t = [C_0(t) + C_1(t)X(t) + C_2(t)Z_t]dt + D_0(t)dW(t),$

where $X(t) \in \mathbb{R}^n = \mathbb{R}^{n\times 1}, A_0(t) \in \mathbb{R}^n, A_1(t) \in \mathbb{R}^{n\times n}, A_2(t) \in \mathbb{R}^{n\times m}, C(t) \in \mathbb{R}^{n\times q}, Z_t \in \mathbb{R}^m, C_0(t) \in \mathbb{R}^m, C_1(t) \in \mathbb{R}^{m\times n}, C_2(t) \in \mathbb{R}^{m\times m}, D(t) \in \mathbb{R}^{m\times q}$ and $W(t) = (W_1(t), ..., W_q(t))^*$ is a q-dimensional Brownian motion with $q = n + m$; m and n are natural numbers.

Define the *innovation process* ν_t by

$$d\nu_t = C_1(t)(X(t) - \hat{X}(t))dt + D_0(t)dB_t; \qquad t \in [0, T]. \tag{2.18}$$

Then $\frac{1}{D_0(t)}d\nu(t)$ is a Brownian motion with respect to $\mathcal{Z}_t$ and P, and the equation for the filter $\hat{X}(t) := E[X(t) \mid \mathcal{Z}_t]$ is

$$\begin{aligned} d\hat{X}(t) &= [A_0(t) + A_1(t)\hat{X}(t) + A_2(t)Z_t]dt \\ &+ [S(t)C_1^*(t) + C(t)D_0^*(t)][D_0(t)D_0^*(t)]^{-1}d\nu_t; \qquad \hat{X}_0 = E[X_0], \end{aligned} \tag{2.19}$$

where $S(t)$ is the mean square error matrix defined by

$$S(t) = [S_{ij}(t)]_{t\le i,j\le n} \in \mathbb{R}^{n\times n}, \quad \text{with}$$
$$S_{ij}(t) = E[(X_i(t) - \hat{X}_i(t))(X_j(t) - \hat{X}_j(t))].$$

The matrix valued function $S(t)$ satisfies the Riccati equation

(2.20)
$$\begin{cases} \frac{dS(t)}{dt} = A_1(t)S(t) + S(t)A_1^*(t) + C(t)C^*(t) \\ -[S(t)C_1^*(t) + C(t)D_0^*(t)][D_0(t)D_0^*(t)]^{-1}[C_1(t)S(t) + D_0(t)C^*(t)]; \\ S(0) = E[(X(0) - E[X(0)])(X(0) - E[X(0)])^*] \end{cases}$$

We now apply this to our setting (2.16)–(2.17). Here $n = 2$, $m = 1$,

$\tilde{B}_t = W_1(t)$ and

$$A_0 = 0, \qquad A_1(t) = \begin{bmatrix} 0 & 0 \\ \frac{h^2(t)}{m-\int_0^t h^2(y)dy} & -\frac{h^2(t)}{m-\int_0^t h^2(y)dy} \end{bmatrix}, \qquad A_2 = 0,$$

$$A_3 = 0, \qquad C(t) = \begin{bmatrix} 0 \\ h(t) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix},$$

$$C_0(t) = 0, \qquad C_2(t) = 0, \qquad D(t) = D_0(t)\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \qquad \text{and}$$

$$C_1(t) = \left[G(t) + \frac{D(t)h(t)}{m - \int_0^t h^2(y)dy}, -\frac{D(t)h(t)}{m - \int_0^t h^2(y)dy} \right]$$

Therefore the filter equation is

(2.21)

$$\begin{cases} d\hat{X}(t) = \begin{bmatrix} 0 \\ \frac{h^2(t)(\hat{X}_1(t)-\hat{X}_2(t))}{m-\int_0^t h^2(y)dy} \end{bmatrix} dt + S(t) \begin{bmatrix} G(t) + \frac{D(t)h(t)}{m-\int_0^t h^2(y)dy} \\ -\frac{D(t)h(t)}{m-\int_0^t h^2(y)dy} \end{bmatrix} D^{-2}(t) d\nu_t; \\ \hat{X}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{cases}$$

i.e.

(2.22)

$$\begin{cases} d\hat{X}_1(t) = \frac{1}{D^2(t)} \Big[S_{11}(t) \Big(G(t) + \frac{D(t)h(t)}{m-\int_0^t h^2(y)dy} \Big) - S_{12}(t) \frac{D(t)h(t)}{m-\int_0^t h^2(y)dy} \Big] d\nu_t \\ \hat{X}_1(0) = 0 \end{cases}$$

and

$$(2.23) \qquad \begin{cases} d\hat{X}_2(t) = \frac{h^2(t)(\hat{X}_1(t)-\hat{X}_2(t))}{m-\int_0^t h^2(y)dy} dt \\ \quad + \frac{1}{D^2(t)} \Big[S_{21}(t) \Big(G(t) + \frac{D(t)h(t)}{m-\int_0^t h^2(y)dy} \Big) - S_{22}(t) \frac{D(t)h(t)}{m-\int_0^t h^2(y)dy} \Big] d\nu_t; \\ \hat{X}_2(0) = 0. \end{cases}$$

The Riccati equation (2.20) becomes

$$
\begin{aligned}
\frac{dS(t)}{dt} &= \begin{bmatrix} 0 & 0 \\ \frac{h^2(t)}{m-\int_0^t h^2(y)dy} & -\frac{h^2(t)}{m-\int_0^t h^2(y)dy} \end{bmatrix} S(t) \\
&\quad + S(t) \begin{bmatrix} 0 & h^2(t)(m - \int_0^t h^2(y)dy)^{-1} \\ 0 & -h^2(t)(m - \int_0^t h^2(y)dy)^{-1} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & h^2(t) \end{bmatrix} \\
&\quad - \frac{1}{D^2(t)} \Big(S(t) \begin{bmatrix} G(t) + \frac{D(t)h(t)}{m-\int_0^t h^2(y)dy} \\ -\frac{D(t)h(t)}{m-\int_0^t h^2(y)dy} \end{bmatrix} + D(t) \begin{bmatrix} 0 \\ h(t) \end{bmatrix} \Big) \times \\
&\quad \times \Big(\Big[G(t) + \frac{D(t)h(t)}{m - \int_0^t h^2(y)dy}, -\frac{D(t)h(t)}{m - \int_0^t h^2(y)dy} \Big] S(t) + D(t)[0, h(t)] \Big)
\end{aligned}
\tag{2.24}
$$

The matrix equation (2.24) is a compact formulation of the following system (2.26) - (2.28) of coupled Riccati equations in the four components $S_{ij}(t)$; $i, j = 1, 2$, where

(2.25)

$$
\begin{cases}
S_{11}(t) = E[(\tilde{v} - E[\tilde{v} \mid \mathcal{Z}_t])^2] \\
S_{12}(t) = S_{21}(t) = E[(\tilde{v} - E[\tilde{v} \mid \mathcal{Z}_t])(\int_0^t h(s)dB_s - E[\int_0^t h(s)dB_s \mid \mathcal{Z}_t])] \\
S_{22}(t) = E[(\int_0^t h(s)dB_s - E[\int_0^t h(s)dB_s \mid \mathcal{Z}_t])^2]
\end{cases}
$$

$$
\begin{aligned}
(2.26)\quad \frac{d}{dt}S_{11}(t) = &-\frac{1}{D^2(t)}\Big[S_{11}(t)\Big(G(t)+\frac{D(t)h(t)}{m-\int_0^t h^2(y)dy}\Big)\\
&- S_{12}(t)\frac{D(t)h(t)}{m-\int_0^t h^2(y)dy}\Big]^2
\end{aligned}
$$

$$
\begin{aligned}
(2.27)\quad \frac{d}{dt}S_{12}(t) = &(S_{11}(t)-S_{12}(t))\frac{h^2(t)}{m-\int_0^t h^2(y)dy}\\
&- \frac{1}{D^2(t)}\Big[S_{11}(t)\Big(G(t)+\frac{D(t)h(t)}{m-\int_0^t h^2(y)dy}\Big)\\
&- S_{12}(t)\frac{D(t)h(t)}{m-\int_0^t h^2(y)dy}\Big]\times\Big[S_{12}(t)\Big(G(t)+\frac{D(t)h(t)}{m-\int_0^t h^2(y)dy}\Big)\\
&- S_{22}(t)\frac{D(t)h(t)}{m-\int_0^t h^2(y)dy}+D(t)h(t)\Big]
\end{aligned}
$$

$$
\begin{aligned}
(2.28)\quad \frac{d}{dt}S_{22}(t) = &(S_{12}(t)-S_{22}(t))\frac{h^2(t)}{m-\int_0^t h^2(y)dy}+h^2(t)\\
&- \frac{1}{D^2(t)}\Big[S_{12}(t)\Big(G(t)+\frac{D(t)h(t)}{m-\int_0^t h^2(y)dy}\Big)\\
&- S_{22}(t)\frac{D(t)h(t)}{m-\int_0^t h^2(y)dy}+D(t)h(t)\Big]^2.
\end{aligned}
$$

We summarize our result as follows:

Theorem 2.3. *The solution*

$$
(2.29)\qquad \hat{X}(t) = (\hat{X}_1(t),\hat{X}_2(t))^* = (E[\tilde{v}\mid\mathcal{Z}_t], E[\int_0^t h(s)dB_s\mid\mathcal{Z}_t])^*
$$

of the filter problem (2.10) -(2.11) is given by (2.21) (or, equivalently, by(2.22), (2.23)), where the mean square error matrix $S(t)$ is given by (2.24)-(2.25) (or, equivalently, by (2.25) - (2.28)).

References

[AaBØ] Aase, K., Bjuland, T. and Øksendal, B. (2007): Strategic insider trading equilibrium with partially informed noise traders. Eprint, University of Oslo 18/2007.

[D] Davis, M.H.A. (1984): Lectures on Stochastic Control and Nonlinear Filtering. Tata Institute of Fundamental Research, Bombay.

[K] Kallianpur, G. (1980): Stochastic Filtering Theory. Springer.

[H] Hu, Y. (2010). An enlargement of filtration for Brownian motion. Preprint.

[KB] Kalman, R.E. and Bucy, R.S.: New results in linear filtering and prediction theory. *Trans. ASME Ser. DJ. Basic Eng.* **83** (1961), 95–108.

[LS] Liptser, R.S. and Shiryaev, A.N. (1978): Statistics of Random Processes II. Springer.

[MY] Mansuy, R. and Yor, M. (2006): Random Times and Enlargements of Filtrations in a Brownian Motion Setting. Springer Lecture Notes in Mathematics.

[Ø] Øksendal, B. (2003). *Stochastic Differential Equations.* 6th Edition. Springer.

[P] Protter, P. (2004). Stochastic Integration and Differential Equations. Second Edition. Springer.