Evolutionary Stable Investment in Stock Markets*

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Abstract

This paper studies the performance of portfolio rules in incomplete markets for long-lived assets with endogenous prices. The dynamics of wealth shares in the process of repeated reinvestment of wealth is modelled as a random dynamical systems. The performance of a portfolio rule is determined by the wealth share eventually conquered in competition with other rules. We derive necessary and sufficient conditions for the evolutionary stability of portfolio rules when dividends are Markov or, in particular, i.i.d. These local stability conditions leads to a unique evolutionary stable strategy for which an explicit representation is given. It is further demonstrated that mean-variance optimization is not evolutionary stable while the CAPM-rule always imitates the best portfolio rule and survives.

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1 Introduction

We study an incomplete asset market where a finite number of portfolio rules manage capital by iteratively reinvesting in a fixed set of long-lived assets. In every period assets pay dividends according to the realization of a stationary Markov process in discrete time. In addition to the exogenous wealth increase due to dividends, portfolio rules face endogenously determined capital gains or losses. Portfolio rules are encoded as non-negative vectors of expenditure shares for assets. The set of portfolio rules considered is not restricted to those generated by expected utility maximization. It may as well include investment rules favored by behavioral finance models. Indeed any portfolio rule that is adapted to the information filtration is allowed in our framework. Portfolio rules compete for market capital that is given by the total value of all assets in every period in time. The endogenous price process provides a *market selection mechanism* along which some strategies gain market capital while others lose.

The power of evolutionary ideas in finance has been recognized by Friedman (1953) and Fama (1965) a long time ago. They argued that the market naturally selects for rational strategies, which, in effect, would lead to market efficiency. Rigorous applications of evolutionary reasoning to financial markets, however, are quite recent. Many time series properties of asset prices, for example, have found an explanation by evolutionary reasoning based on computer simulations (see for example Arthur, Holland, LeBaron, Palmer, and Taylor (1997), LeBaron, Arthur, and Palmer (1999), Brock and Hommes (1997), and Lux (1994), among others). For alternative approaches based on replicator dynamics and evolutionary game theory see Farmer and Lo (1999) and Friedman (2001).

The aim of our paper is to contribute to a Darwinian theory of portfolio selection, or evolutionary portfolio theory. This theory views asset markets as being stratified according to the portfolio rules that investors use to manage wealth. With every such rule (mean-variance rule, growth-optimal rule, CAPM-rule, naive diversification, prospect theory based rules, relative-dividends rule, for example) a certain amount of wealth is being managed. In our model the impact of any such rule on market prices is proportional to the amount of wealth managed by the rule. In a Darwinian model two forces are at work: one reducing the variety of species and one increasing it. In our model the first such force is the endogenous return process acting as a market selection mechanism that determines the evolution of wealth managed by the portfolio rules. Secondly, any system of portfolio rules that is selected by the market selection process is checked for its evolutionary stability, i.e. it is checked whether the innovation of a new portfolio rule with very little initial wealth can grow against the incumbent rule.

The Darwinian theory of asset markets seems to describe very well a modern asset market in which most of the available capital is invested by delegated management. Indeed investors typically choose funds by the portfolio rules, also called "styles," according to which the money is invested. Style consistency appears nowadays to be one of the most important features in monitoring fund managers.

In this paper we derive a description of the market selection process from a random dynamical systems perspective. In each period in time the evolution of the distribution of market capital, i.e. wealth shares, is determined by a map that depends on the exogenous process determining the asset payoffs. An equilibrium in this model is provided by a distribution of wealth shares across portfolio rules that is invariant under the market selection process. It turns out that (provided there are no redundant assets) every invariant distribution of market shares is generated by a *monomorphic* population, i.e. all traders with strictly positive wealth use the same portfolio rule at such equilibrium. A criterion for evolutionary stability as well as evolutionary instability is derived for such monomorphic populations. Roughly speaking a portfolio rule is evolutionary stable if it has the highest exponential growth rate in any population where itself determines market prices. This implies that an evolutionary stable investment strategy is robustness against the entry of new portfolio rules. In a sense an evolutionary stable population plays the "best response against itself."

The stability criterium for the robustness of invariant distributions with respect to the entry of new portfolio rules singles out one portfolio rule, denoted λ^*, that is the unique *evolutionary stable strategy*, i.e. it drives out any mutation. Moreover, any other investment strategy can successfully be invaded by a slightly changed strategy. According to this rule one should divide wealth proportionally to the expected relative dividends of the assets. An explicit formula for this rule is given—applicable in actual markets.

The effect of this rule on asset prices is equalization of assets' expected relative returns—in particular asset pricing is *log-optimal* (Long Jr. 1990), i.e. the same prices would be obtained in a standard asset pricing model with a representative consumer having a logarithmic von Neumann–Morgenstern utility function. Hence the portfolio rule λ^* could also be obtained as the outcome of a completely rational market. Indeed λ^* is a simple value strategy that practitioners favor for long run investments (for a similar strategy see e.g. Spare and Ciotti (1999)).

One implication of our main results is that a rational market is evolutionary stable while an irrational market is evolutionary unstable. In particular we show that any irrational market can already be destabilized by small changes in the existing strategies. A further implication of our evolutionary stability results is that among all proportional investment strategies only λ^* can be a candidate for a rule that starting from any initial distribution of wealth gathers total market wealth in the long-run in competition with any set of other portfolio rules. Indeed, global stability of the rule λ^* has recently been demonstrated for the case of short-lived assets (Evstigneev, Hens, and Schenk-Hoppé 2002). Simulations with simple strategies show that also with long-lived assets λ^* is the unique portfolio rule which among all simple strategies is able to gather total market wealth (Hens, Schenk-Hoppé, and Stalder 2002). An analytical proof of this finding is still warranted.

We also apply the stability criterium obtained here to demonstrate that mean-variance optimization can be invaded by any completely diversified portfolio rule while the CAPM-rule, which prescribes buying the market portfolio, is able to always imitate the best portfolio rule and thus survives.

Our approach complements the recent work by Blume and Easley (2001) and Sandroni (2000) who consider an infinite horizon stochastic exchange economy with short-lived assets and complete markets. Agents maximize expected discounted utility from consumption over the infinite time horizon. The solution concept used in this literature is a competitive equilibrium with rational expectations. It turns out that those consumers who predict correctly the probability of the occurrence of the states of the world will drive out all other consumers. However, as Blume and Easley (2001) made perfectly clear, this result is ultimately linked to Pareto-efficiency. Complete markets are therefore essential in their approach.

Besides considering the more general case of incomplete markets, the approach presented here is also quite different as it pursues a dynamical systems perspective which is not compatible with correct anticipation of future prices as in a competitive equilibrium with rational expectations. This is simply because in a rational expectations equilibrium the outcomes anticipated for the future determine the current outcome (time is running backwards) while in a dynamical system the outcome of the current period determines the future outcomes (time is running forwards). Moreover, we consider consumption and portfolio decisions as two separate aspects of investments that should be kept conceptually distinct. Since this paper focuses on the portfolio selection problem, we assume that all rules considered have some identical and exogenously given consumption rate. As Epstein and Zin (1989) have argued this is well compatible with expected utility maximization.

While with rational expectations equilibrium allocations in the case of short-lived and of long-lived assets are equivalent, allowing for rational and irrational strategies the case of long-lived assets is very different from the case of short-lived assets. With all types of behavior prices can depart from their fundamental values (bubbles and crashes) which is a potential threat to rational strategies. Indeed recent results in behavioral finance, for example Shleifer (2000, Chap. 2.2), show that under specific circumstances noise traders can earn a higher average rate of return than rational arbitrageurs. This phenomenon, called "noise trader risk", is one of the core questions in evolutionary finance. It can only be answered in a general model of the market selection process allowing for all types of strategies.

Our approach relates closely to the classical finance approach to maximize the expected growth rate of wealth for some exogenously given return process. In a sense we show which portfolio rule turns out to maximize the expected growth rate of wealth in a model with endogenously determined returns. Hakansson (1970), Thorp (1971), Algoet and Cover (1988), and Karatzas and Shreve (1998), among others, have explored this maximum growth perspective. Computing the maximum growth portfolio is a non-trivial problem. Even if one restricts attention to i.i.d. returns, when markets are incomplete, there is no explicit solution to this investment problem in general. Numerical algorithms to compute the maximum growth portfolio have been provided by Algoet and Cover (1988) and Cover (1984, 1991). Our result is interesting also in this respect because the simple portfolio rule that we obtain shows that considering the equilibrium consequences of expected growth rate maximization does not make matters more complicated but rather much easier. Indeed, as mentioned above, the portfolio rule λ^* can be characterized as the unique portfolio rule that maximizes its growth rate of wealth in a population in which the rule itself determines the returns.

The next section presents the economic model which has the mathematical structure of a random dynamical system. The model is based on Lucas (1978)'s infinite horizon asset market model with long-lived assets and a single perishable consumption good. In this model we introduce heterogenous portfolio rules that are adapted to the information filtration, and we study the resulting sequence of short run equilibria. In section 3 we define the long run equilibrium concepts and different stability notions. In particular we define invariant distributions of relative wealth and show that those are characterized by monomorphic populations, i.e. an invariant distribution of relative wealth arise if and only if all investors use the same portfolio rule. Then we define evolutionary stability of invariant distributions of relative wealth as those being robust to the innovation of new strategies. Section 4 contains the main results. For various degrees of complexity on the dividend process and the portfolio rules we show that the relative dividends rule λ^* is the unique evolutionary stable strategy. Section 5 analyzes the evolutionary stability of portfolio rules based on mean-variance optimization. We study the issue of under-diversified portfolios, and discuss the implication of the

CAPM investment strategy. Section 6 concludes.

2 An Evolutionary Stock Market Model

This section introduces an infinite horizon asset market model with long-lived assets and a single perishable consumption good, as in the seminal paper Lucas (1978).

There are $K \geq 1$ long-lived assets and cash. Time is discrete and denoted by $t = 0, 1, \ldots$. Each asset $k = 1, ..., K$ pays off a dividend per share at the beginning of every period and before trade takes place in this period. $D_t^k \geq 0$ denotes the total dividend paid to all shareholders of asset k at the beginning of period t. We assume that $\sum_k D_t^k > 0$.[1] D_t^k depends on the history of states of the world $\omega^t = (..., \omega_0, ..., \omega_t)$ where $\omega_t \in S$ is the state revealed at the beginning of period t. It is for technical convenience (and without loss of generality) to assume infinite histories. S is assumed to be finite, and every state is drawn with some strictly positive probability.

Dividend payoffs are in terms of cash. Cash is only used to buy consumption goods—in particular it cannot be used to store value. Assets are issued at time 0. The initial supply of every asset k, s_0^k, is normalized to 1. At any period in time the supply remains constant: $s_t^k = s_0^k$. The supply of cash s_t^0 is given by the total dividends of all assets.

There are finitely many portfolio rules (also referred to as investment strategies) indexed by $i = 1, ..., I$, $I \geq 2$, each is pursued by an investor. The portfolio rule of investor i is a time- and history-dependent proportional strategy, denoted by $\lambda_t^i(\omega^t) = (\lambda_{t,k}^i(\omega^t))_{k=0,...,K}$ with $0 \leq \lambda_{t,k}^i(\omega^t) \leq 1$ for all k and $\sum_{k=0}^K \lambda_{t,k}^i(\omega^t) = 1$. For each $k \geq 1$, $\lambda_{t,k}^i(\omega^t)$ is the fraction of the wealth investor i assigns to the purchase of the risky asset k in period t, while $\lambda_{t,0}^i(\omega^t)$ is the fraction of wealth held in cash. Investment strategies are distinct across investors[2].

In the following discussion we assume that everything is well-defined. In particular prices are assumed to be strictly positive. A general result along with sufficient conditions are provided the full derivation of the model.

For a given portfolio rule $\lambda_t^i(\omega^t)$ and wealth w_t^i, the portfolio purchased by investor i at the beginning of period t is given by

$$\theta_{t,k}^i = \frac{\lambda_{t,k}^i(\omega^t)\, w_t^i}{p_t^k} \qquad k = 0, 1, ..., K. \tag{1}$$

[1]This assumption avoids "dead" periods in which no dividends are paid.

[2]The case of investors pursuing the same portfolio rule can be handled as follows: Investors with the same strategy set up a fund with claims equal to their initial share.

$\theta^i_{t,0}$ is the units of cash and $\theta^i_{t,k}$ is the units of assets held by investor i. Since we have normalized the supply of the long-lived assets to 1, $\theta^i_{t,k}$ is the percentage of all shares issued of asset k that investor i purchases. p^k_t denotes the market clearing price of asset k in period t. We normalize the price for cash $p^0_t = 1$ in every period t. The price of the consumption good is also the numeraire.

For any portfolio holdings of agents $(\theta^i_t)_{i=1,...,I}$ the market equilibrium conditions for cash and long-lived assets are given by

$$\sum_{i=1}^{I} \theta^i_{t,k} = s^k_t, \qquad k = 0, ..., K, \tag{2}$$

where the supply of the risky assets is $s^k_t = 1$, while the supply of cash is given by

$$s^0_t = \sum_{k=1}^{K} D^k_t(\omega^t) > 0 \tag{3}$$

with strict positivity by the assumption that at least one asset pays a dividend.

The budget constraint of investor i in every period $t = 0, 1, ...$

$$\sum_{k=0}^{K} p^k_t \, \theta^i_{t,k} = w^i_t \tag{4}$$

is fulfilled since the fractions $\lambda^i_{t,k}(\omega^t)$, $k = 0, ..., K$, add up to one, see (1).

Since the consumption good is perishable, the wealth of investor i (in terms of the price of the consumption good) at the beginning of period $t+1$ and after dividends are payed turns out to be given by

$$w^i_{t+1} = \sum_{k=1}^{K} (D^k_{t+1}(\omega^{t+1}) + p^k_{t+1}) \, \theta^i_{t,k} \tag{5}$$

Wealth can change over time because of dividend payments and capital gains. Since the cash $\theta^i_{t,0}$ held by every investor is consumed, the amount of cash available in any one period stems only from the current's period dividend payments.

The market-clearing price p^k_t for the risky assets ($k \geq 1$) can be derived from (2) by inserting (1). One finds

$$p^k_t = \sum_{i=1}^{I} \lambda^i_{t,k}(\omega^t) \, w^i_t = \lambda_{t,k}(\omega^t) \, w_t \tag{6}$$

where $\lambda_{t,k} = (\lambda^1_{t,k}, ..., \lambda^I_{t,k})$ and $w_t^T = (w_t^1, ..., w_t^I)$.

Inserting (1) and (6) in (5) one obtains

$$w^i_{t+1} = \sum_{k=1}^{K} \left(D^k_{t+1}(\omega^{t+1}) + \lambda_{t+1,k}(\omega^{t+1})\, w_{t+1}\right) \frac{\lambda^i_{t,k}(\omega^t)\, w^i_t}{\lambda_{t,k}(\omega^t)\, w_t} \tag{7}$$

This is an implicit equation for the wealth of each investor i, w^i_{t+1}, for a given distribution of wealth w_t across investors. It is convenient for the further analysis to define

$$A^i_t = \sum_{k=1}^{K} D^k_{t+1}(\omega^{t+1}) \frac{\lambda^i_{t,k}(\omega^t)\, w^i_t}{\lambda_{t,k}(\omega^t)\, w_t}, \qquad \text{and} \qquad B^{i,k}_t = \frac{\lambda^i_{t,k}(\omega^t)\, w^i_t}{\lambda_{t,k}(\omega^t)\, w_t} \tag{8}$$

The time index refers to the dependence on wealth: A^i_t and $B^{i,k}_t$ both depend on the wealth in period t. (7) can now be written as

$$w^i_{t+1} = A^i_t + \sum_{k=1}^{K} B^{i,k}_t\, \lambda_{t+1,k}(\omega^{t+1})\, w_{t+1} \tag{9}$$

and thus

$$w_{t+1} = A_t + B_t\, \Lambda_{t+1}(\omega^{t+1})\, w_{t+1} \tag{10}$$

where $\Lambda_{t+1}(\omega^{t+1})^T = (\lambda_{t+1,1}(\omega^{t+1})^T, ..., \lambda_{t+1,K}(\omega^{t+1})^T) \in \mathbb{R}^{I\times K}$ is the matrix of portfolio rules, and $B_t \in \mathbb{R}^{I\times K}$ is the matrix of portfolios in period t. $A^T_t = (A^1_t, ..., A^I_t) \in \mathbb{R}^I$ are the dividends payments, and $B_t\, \Lambda_{t+1}(\omega^{t+1})\, w_{t+1}$ are the capital gains.

Solving (9) gives an explicit law of motion governing the distribution of wealth across strategies. One has

$$w_{t+1} = \left[\mathrm{Id} - B_t\, \Lambda_{t+1}(\omega^{t+1})\right]^{-1} A_t \tag{11}$$

(assuming existence of the inverse matrix) with Id being the identity matrix in $\mathbb{R}^{I\times I}$. The following result ensures that the evolution of wealth (11) is well-defined.

The following assumptions are imposed.

(A.1) Every investor consumes but less than entire wealth: $0 < \lambda^i_{t,0}(\omega^t) < 1$ *for all* i, t *and* ω^t.

(A.2) There is at least one investor with a complete-diversification strategy, i.e. there is a j *such that* $\lambda^j_{t,k}(\omega^t) > 0$ *for all* $k = 1, ..., K$, t *and* ω^t.

Proposition 1 *Suppose $w_0 > 0$, (A.1) holds, and (A.2) is satisfied for some investors with $w_0^j > 0$. Then (11) is well-defined in all periods in time and, for every $i = 1, ..., I$, $w_t^i > 0$ if and only if $w_0^i > 0$.*

Proof of Proposition 1. It suffices to prove the following: Suppose $w_t > 0$, (A.1) holds, and (A.2) is satisfied for some investor with $w_t^j > 0$. Then (11) is well-defined, $w_{t+1} > 0$, and, moreover, $w_{t+1}^i > 0$ if and only if $w_t^i > 0$ for every $i = 1, ..., I$.

We show first that the matrix $C := \mathrm{Id} - B_t\, \Lambda_{t+1}(\omega^{t+1})$ is invertible by proving that it has a column dominant diagonal Murata (1977, Corollary p. 22). C has entries

$$C_{jj} = 1 - \sum_{k=1}^{K} \bar{\lambda}_k^j \frac{\lambda_k^j w^j}{\lambda_k w} \quad \text{and} \quad C_{ij} = - \sum_{k=1}^{K} \bar{\lambda}_k^j \frac{\lambda_k^i w^i}{\lambda_k w} \quad (i \neq j)$$

on the diagonal and off-diagonal, respectively, where $\bar{\lambda}_k^i = \lambda_{t+1,k}^i(\omega^{t+1})$, $\lambda_k^i = \lambda_{t,k}^i(\omega^t)$, and $w = w_t$ for notational ease. All entries are well-defined because prices $\lambda_k w \geq \lambda_k^j w^j > 0$ (for some j) by our assumption.

The condition for a column dominant diagonal is in particular satisfied, if for every $j = 1, ..., I$,

$$|C_{jj}| > \sum_{i \neq j} |C_{ij}| \tag{12}$$

Off-diagonal entries are obviously non-positive, i.e. $C_{ij} \leq 0$ for $i \neq j$. The diagonal elements are strictly positive, i.e. $C_{jj} > 0$, since $0 \leq \lambda_k^j w^j / (\lambda_k w) \leq 1$ and therefore

$$C_{jj} \geq 1 - \sum_{k=1}^{K} \bar{\lambda}_k^j = 1 - (1 - \bar{\lambda}_0^j) = \bar{\lambda}_0^j > 0$$

according to assumption (A.1).

Thus (12) is equivalent to,

$$1 > \sum_{i=1}^{I} \sum_{k=1}^{K} \bar{\lambda}_k^j \frac{\lambda_k^i w^i}{\lambda_k w} \tag{13}$$

Since the right-hand side of the last equation is given by

$$\sum_{k=1}^{K} \bar{\lambda}_k^j \sum_{i=1}^{I} \frac{\lambda_k^i w^i}{\lambda_k w} = \sum_{k=1}^{K} \bar{\lambda}_k^j = 1 - \bar{\lambda}_0^j$$

and $\bar{\lambda}_0^j > 0$ by assumption, (13) holds true. Thus C is invertible.

The matrix C has strictly positive diagonal entries and non-positive off-diagonal entries. Thus, Murata (1977, Theorem 23, p. 24) ensures that $w_{t+1} \geq 0$ if $A_t \geq 0$ (see (8) for the definition of A_t). Clearly, $A_t \geq 0$ if $w_t \geq 0$. This observation implies $\lambda_{t+1,k}(\omega^{t+1})\, w_{t+1} \geq 0$ for all k.

Let (A.2) hold for investor j and let $w_t^j > 0$. Investor js portfolio is completely diversified, i.e. $\theta_{t,k}^j > 0$ for all k. Thus, $A_t^j > 0$ because at least one asset pays a strictly positive dividend. Equation (7) implies, together with the above result that prices in period $t+1$ are non-negative, $w_{t+1}^j > 0$.

By assumption (A.2) this finding implies $\lambda_{t+1,k}^j(\omega^{t+1}) > 0$ for all k. Since for each investor with $w_t^i > 0$, $B_{t,k}^i > 0$ for some k, (7) further implies that $w_{t+1}^i > 0$ for every investor with $w_t^i > 0$. Obviously, $w_{t+1}^i = 0$ if $w_t^i = 0$. This completes the proof. □

Proposition 1 ensures that the evolution of the wealth distribution on $\mathbb{R}_+^I$ is well-defined: for given w_t, (11) yields the distribution of wealth w_{t+1} in the subsequent period in time.[3] We can state the law of motion in the convenient form

$$w_{t+1} = f_t(\omega^{t+1}, w_t) \tag{14}$$

where

$$f_t(\omega^{t+1}, w_t) = \left[\mathrm{Id} - \left[\frac{\lambda_{t,k}^i(\omega^t) w_t^i}{\lambda_{t,k}(\omega^t) w_t}\right]_i^k \Lambda_{t+1}(\omega^{t+1})\right]^{-1} \left[\sum_{k=1}^K D_{t+1}^k(\omega^{t+1}) \frac{\lambda_{t,k}^i(\omega^t) w_t^i}{\lambda_{t,k}(\omega^t) w_t}\right]_i$$

The final step is to derive the law of motion for the investors' market shares. This will complete the derivation of the evolutionary stock market model.

The following assumption is imposed throughout the remainder of this paper.

(B.1) All investors have the same rate of consumption: $\lambda_{t,0}^i(\omega^t) = \lambda_{t,0}(\omega^t)$.

It is clear that, other things being equal, a smaller rate of consumption leads to a higher growth rate of wealth. Without assumption (B.1) the evolution of wealth would be biased in favor of investors with a high saving rate. Since we want to analyze the relative performance of different asset allocation rules no rule should have an disadvantage in terms of the rate at which wealth is withdrawn from it.

[3] Of course assumptions (A.1) and (A.2) are needed. Further, it is convenient to define $w_{t+1} = (0, ..., 0)$, if $w_t = (0, ..., 0)$.

Aggregating (7) over investors, one finds

$$\begin{aligned} W_{t+1} &= \sum_{k=1}^{K} D_{t+1}^{k}(\omega^{t+1}) + \sum_{k=1}^{K} \lambda_{t+1,k}(\omega^{t+1})\, w_{t+1} \\ &= D_{t+1}(\omega^{t+1}) + (1-\lambda_{t+1,0}(\omega^{t+1}))\, W_{t+1} \end{aligned} \tag{15}$$

where $D_{t+1}(\omega^{t+1}) = \sum_{k=1}^{K} D_{t+1}^{k}(\omega^{t+1})$ is the aggregate dividend payment. The last equality holds because $\sum_{k=1}^{K} \lambda_{t+1,k}\, w_{t+1} = \sum_{i=1}^{I} \sum_{k=1}^{K} \lambda_{t+1,k}^{i}\, w_{t+1}^{i} = (1-\lambda_{t+1,0}) \sum_{i=1}^{I} w_{t+1}^{i}$.

Equation (15) implies

$$W_{t+1} = \frac{D_{t+1}(\omega^{t+1})}{\lambda_{t+1,0}(\omega^{t+1})} \tag{16}$$

The economy grows (or declines) with rate $D_{t+1}(\omega^{t+1})/(\lambda_{t+1,0}(\omega^{t+1})\, W_t)$. The growth rate is thus the ratio of the rate at which additional wealth is injected by dividends, $D_{t+1}(\omega^{t+1})/W_t$, to the rate at which wealth is withdrawn from the process for consumption, $\lambda_{t+1,0}(\omega^{t+1})$.

The market share of investor i is $r_t^i = w_t^i/W_t$. Using (16) and exploiting the particular structure of the variables (8) that define the law of motion (14), we obtain

$$r_{t+1} = \frac{\lambda_{t+1,0}(\omega^{t+1})}{D_{t+1}(\omega^{t+1})}\, f_t(\omega^{t+1}, r_t) \tag{17}$$

or, equivalently,

$$r_{t+1} = \lambda_{t+1,0}(\omega^{t+1}) \left(\mathrm{Id} - \left[\frac{\lambda_{t,k}^{i}(\omega^t) r_t^i}{\lambda_{t,k}(\omega^t) r_t} \right]_i^k \Lambda_{t+1}(\omega^{t+1}) \right)^{-1} \left[\sum_{k=1}^{K} d_{t+1}^{k}(\omega^{t+1}) \frac{\lambda_{t,k}^{i}(\omega^t) r_t^i}{\lambda_{t,k}(\omega^t) r_t} \right]_i$$

where

$$d_{t+1}^{k}(\omega^{t+1}) = \frac{D_{t+1}^{k}(\omega^{t+1})}{D_{t+1}(\omega^{t+1})}$$

is the relative dividend payment of asset k. Equation (17) is referred to as the *market selection process*.

The wealth of an investor i in any period in time can be derived from her market share and the aggregate wealth, defined by (16), as

$$w_{t+1}^{i} = \frac{D_{t+1}(\omega^{t+1})}{\lambda_{t+1,0}(\omega^{t+1})}\, r_{t+1}^{i} \tag{18}$$

The further analysis is restricted to the stationary case. We make the following assumptions.

(B.2) Stationary (ergodic) strategies, i.e. $\lambda^i_{t,k}(\omega^t) = \lambda^i_k(\omega^t)$, *for all* $i = 1, ..., I$ *and* $k = 0, 1, ..., K$.

(B.3) Stationary (ergodic) relative dividend payments $d^k_t(\omega^t) = d^k(\omega_t)$, *for all* $k = 1, ..., K$.

Assumption (B.3) is fulfilled, for instance, if $D^k_{t+1}(\omega^{t+1}) = d^k(\omega_{t+1})\, W_t$ with $W_t = \sum_i w^i_t$, i.e. the dividend payment of every asset has an idiosyncratic component $d^k(\omega_{t+1})$ (depending only on the state of nature in the respective period) and an aggregate component W_t. Dividends grow or decline with the same rate as aggregate wealth.

The last two assumptions ensure that the calender date does not enter in strategies and dividends, i.e. the model becomes stationary; only the observed history matters.

Under these assumptions, the market selection process (17) generates a *random dynamical system* (Arnold 1998) on the simplex $\Delta^I = \{r \in \mathbb{R}^I \mid r^i \geq 0, \sum_i r^i = 1\}$. For any initial distribution of wealth $w_0 \in \mathbb{R}^I_+$, (17) defines the path of market shares on the event tree with branches ω^t. The initial distribution of market shares is given by $(r^i_0)_i = (w^i_0/W_0)_i$. Formally, this can be stated as follows.

Let Ω denote the set of all realizations $\omega \in S^{\mathbb{Z}}$. Denote the right-hand side of (17) by $h(\omega^{t+1}, r_t) : \Delta^I \to \Delta^I$ (it is stationary by assumptions (B)). Define $\varphi(t, \omega, r) = h(\omega^{t+1}, \cdot) \circ \ldots \circ h(\omega^1, r)$ for all $t \geq 1$, and $\varphi(0, \omega, r) = r$. In words, $\varphi(t, \omega, r)$ is the vector of wealth shares of all investors at time t when the initial distribution of market shares is r and the sequence of realizations of states is $\omega \in S^{\mathbb{Z}}$.

3 Evolutionary Stability

This section introduces the stability concepts needed to analyze the long run behavior of the wealth shares under the market selection process.

Given a random dynamical system for a set of stationary and adapted trading strategies (λ^i), one is particularly interested in those wealth shares that evolve in a stationary fashion over time. Here we restrict ourselves to deterministic distributions of market shares that are fixed under the market selection process (17).[4] To specify this notion, we recall the definition of a deterministic fixed point in the framework of random dynamical systems.

[4]See e.g. Schenk-Hoppé (2001) for an application of stochastic invariant distributions (random fixed points).

Let a set of strategies (λ^i) be given, and denote by φ the associated random dynamical system.

Definition 1 *$\bar{r} \in \Delta^I$ is called a (deterministic)* fixed point *of φ if, for all $\omega \in \Omega$ and all t,*

$$\bar{r} = \varphi(t, \omega, \bar{r}). \tag{19}$$

The distribution of market shares $\bar{r}$ is said to be invariant *under the market selection process (17).*

By the definition of $\varphi(t, \omega, r)$ the condition (19) is equivalent to $\bar{r} = \varphi(1, \omega, \bar{r})$ for all ω, i.e. a deterministic state is fixed under the one-step map if and only if it is fixed under all t-step maps.

It is straightforward to see that the state in which one investor possesses the entire market does not change over time. In any set of trading strategies each unit vector in Δ^I (i.e. each vertex) is a fixed point. This follows from Proposition 1 which shows that $r^i = 0$ implies $\varphi^i(t, \omega, r) = 0$.

Proposition 2 *Suppose the dividend and capital gains matrix has full rank at a deterministic fixed point. Then all investors use the same portfolio rule.*

Proof. The result does not require conditions (B.2) and (B.3). (7) and (16) give

$$r^i_{t+1} = \sum_{k=1}^{K} (\lambda_0 \, d^k_{t+1}(\omega_{t+1}) + q^k_{t+1}(\omega^{t+1})) \, \frac{\lambda^i_{t,k}(\omega^t) r^i_t}{q^k_t(\omega^t)} \tag{20}$$

with

$$q^k_t(\omega^t) = \sum_{i=1}^{I} \lambda^i_{t,k}(\omega^t) \, r^i_t \tag{21}$$

Suppose $r^i_{t+1} = r^i_t = r^i > 0$ for all i. Then equation (20) can be written as

$$\left(\sum_{k=1}^{K} \left[\lambda_0 \, d^k_{t+1}(\omega_{t+1}) + q^k_{t+1}(\omega^{t+1}) \right] \, \frac{\lambda^i_{t,k}(\omega^t)}{q^k_t(\omega^t)} - 1 \right) r^i = 0 \tag{22}$$

If the dividend and capital gain matrix

$$\lambda_0 \, d^k_{t+1}(\omega_{t+1}) + q^k_{t+1}(\omega^{t+1})$$

has full rank (as a function of k and ω_{t+1} for each given history ω^t), then (22) implies $\lambda^i_{t,k}(\omega^t) = q^k_t(\omega^t)$. In light of (21), this means that for all for all $i = 1, ..., I$

$$\lambda^i_{t,k} = \sum_{j=1}^{I} \lambda^j_{t,k} \, r^j$$

Hence $\lambda^i = \lambda^j$ for all i, j. $\square$

We are particularly interested in *stable* fixed points of the market selection process. Loosely speaking, stability means that small perturbations of the market shares' initial distribution do not have a long-run effect. If an invariant distribution of market shares is stable, every sample path starting in a neighborhood of this fixed point at time zero is asymptotically identical to the sample path of the invariant distribution of the wealth shares.

Since fixed points are associated to unique trading strategies (the total wealth being concentrated on this trading strategy), the natural definition of a trading strategy's stability is that of the fixed point's stability. We will need different notions of stability, defined as follows.

In the following definition we assume that for any given incumbent strategy λ^i, the mutant strategy λ^j is distinct in the sense that with strictly positive probability $\lambda^j \neq \lambda^i$. Moreover, as a matter of notation the first entry in the tuple of relative wealth shares $r = (r^i, r^j)$ refers to the incumbent's strategy, while the second refers to the entrant's wealth share.

Definition 2 *A trading strategy λ^i is called* evolutionary stable *if, for all λ^j, there is a random variable $\varepsilon > 0$ such that $\lim_{t\to\infty} \varphi^i(t, \omega, r) = 1$ (almost surely) for all $r^i \geq 1 - \varepsilon$ ($r^j = 1 - r^i \leq \varepsilon$).*

For each evolutionary stable distribution of market shares there exits an entry barrier (a random variable here) below which the new portfolio rule does not drive out the incumbent player. Any perturbation, if sufficiently small, does not change the long-run behavior of the distribution of market shares. The market selection process asymptotically leaves the mutant with no market share. Finally, a corresponding local stability criterion is introduced.

Definition 3 *A trading strategy λ^i is called* locally evolutionary stable *if for all λ^j there exists a random variable $\delta(\omega) > 0$ such that λ^i is evolutionary stable for all portfolio rules λ^j with $\|\lambda^i(\omega) - \lambda^j(\omega)\| < \delta(\omega)$ for all ω.*

A locally evolutionary stable distribution of market shares is evolutionary stable with respect to local mutations. That is, the strategies that can be pursued by the mutants are limited to small deviations from existing strategies.

4 The Main Results

4.1 Simple Strategies

A particular case of the trading strategies considered in this paper are those being constant over time. This section studies their evolutionary stability. To analyze evolutionary stability of a trading strategy one has to consider the random dynamical system (17) with an incumbent (with market share r_t^1) and a mutant (with market share $r_t^2 = 1 - r_t^1$). The resulting one-dimensional system is given by

$$r_{t+1}^1 = \frac{\lambda_0}{\delta}\left[\left(1 - \sum_{k=1}^K \lambda_k^2 \theta_k^2\right)\sum_{k=1}^K d^k(\omega_{t+1})\theta_k^1 + \left(\sum_{k=1}^K \lambda_k^2 \theta_k^1\right)\sum_{k=1}^K d^k(\omega_{t+1})\theta_k^2\right] \quad (23)$$

with

$$\begin{aligned} \theta_k^1 &= \frac{\lambda_k^1\, r_t^1}{\lambda_k^1\, r_t^1 + \lambda_k^2\,(1 - r_t^1)}, \qquad \theta_k^2 = \frac{\lambda_k^2\,(1 - r_t^1)}{\lambda_k^1\, r_t^1 + \lambda_k^2\,(1 - r_t^1)} \\ \delta &= \left(1 - \sum_{k=1}^K \lambda_k^1\, \theta_k^1\right)\left(1 - \sum_{k=1}^K \lambda_k^2\, \theta_k^2\right) - \left(\sum_{k=1}^K \lambda_k^2\, \theta_k^1\right)\left(\sum_{k=1}^K \lambda_k^1\, \theta_k^2\right) \end{aligned}$$

λ^1 and λ^2 are fixed vectors of percentages with $\lambda_0^1 = \lambda_0^2 = \lambda_0$.

The derivative of the right-hand side of (23) (denoted by $h(\omega_{t+1}, r_t^1)$ which now only depends on ω_{t+1} but does not depend on the history ω^t) with respect to r_t^1 evaluated at $r_t^1 = 1$ can be derived employing some elementary algebra. One finds

$$\frac{\partial h(\omega_{t+1}, r_t^1)}{\partial r_t^1}\left(r_t^1 = 1\right) = 1 - \lambda_0 + \lambda_0 \sum_{k=1}^K d^k(\omega_{t+1})\,\frac{\lambda_k^2}{\lambda_k^1} \quad (24)$$

¿From (24) one can read off the exponential growth rate of the wealth of investor 2 in a small neighborhood of $r^1 = 1$, i.e. the state in which investor 1 owns the total market wealth. This growth rate determines the local stability of this steady state. If the growth rate is negative, investor 2 looses her wealth and the market share of investor 1 tends to one. In this case the portfolio rule λ^1 is stable in the pool (λ^1, λ^2). If the growth rate is positive, investor 2 gains wealth and the market share of investor 1 falls. In this case the portfolio rule λ^1 is not stable.

The growth rate is given by

$$g_{\lambda^1}(\lambda^2) = \mathbb{E}\ln\left[1 - \lambda_0 + \lambda_0 \sum_{k=1}^K d^k(\omega_0)\,\frac{\lambda_k^2}{\lambda_k^1}\right] \quad (25)$$

$\mathbb{E}$ denotes the expected value, i.e. integration with respect to the invariant probability measure $\mathbb{P}$ of the stationary dividend process.

We have the following result.

Theorem 1 *Let the state of nature be determined by an ergodic process and let the matrix of relative dividends d have full rank. Suppose investors only employ simple strategies, i.e. $\lambda(\omega) \equiv \lambda \in \Delta^{K+1}$. Then the simple strategy $\lambda^\star$ defined by, $\lambda_0^\star = \lambda_0$, and*

$$\lambda_k^* = (1-\lambda_0)\, \mathbb{E} d^k(\omega_0) = (1-\lambda_0) \sum_{s\in S} p_s\, d^k(s) \tag{26}$$

for $k = 1, ..., K$, is evolutionary stable, and no other strategy is locally evolutionary stable.

Proof of Theorem 1. Obviously $\lambda^\star$ is a completely mixed strategy, i.e. $\lambda_k^\star > 0$ for all k, and one has $\sum_{k=1}^K \lambda_k^\star = 1 - \lambda_0$.

It is convenient to use equation (25) to define the auxiliary function,

$$g_\beta(\alpha) := \mathbb{E} \ln \left(1 - \lambda_0 + \lambda_0 \sum_{k=1}^{K} d^k(\omega_0) \frac{\alpha_k}{\beta_k} \right) \tag{27}$$

with strategies normalized by $1-\lambda_0$ to make $\sum_{k=1}^K \alpha_k = \sum_{k=1}^K \beta_k = 1$. For each fixed strategy $\beta \in \mathrm{int}\Delta^{\mathrm{K}} \subset \mathbb{R}^{\mathrm{K}}$, $g_\beta : \mathrm{int}\Delta^{\mathrm{K}} \to \mathbb{R}$. $g_\beta(\alpha)$ is the Lyapunov exponent of the distribution of wealth that assigns total wealth to the 'status quo' population that plays strategy β in a market in which α is the only the alternative strategy.

The first assertion of the theorem follows if we can show that $g_{\lambda^\star}(\alpha) < 0$ for all $\alpha \in \mathrm{int}\Delta^{\mathrm{K}}$ with $\alpha \neq \lambda^\star$. We will prove that $g_\beta(\alpha)$ is strictly concave for all $\beta \in \mathrm{int}\Delta^{\mathrm{K}}$ and that $g_{\lambda^\star}(\alpha)$ takes on its maximum value at $\alpha = \lambda^\star$.

To ensure strict concavity it suffices to show that $\alpha \mapsto g_\beta(\alpha)$ is strictly concave on the space $\mathbb{R}^K_{++}$, because restriction of the domain to the linear subspace $\mathrm{int}\Delta^{\mathrm{K}}$ preserves strict concavity. The function $\ln \sum_{k=1}^K (d^k(\omega_0)\, \alpha_k / \beta_k)$ is concave for all ω and—due to the no-redundancy assumption of full rank for d—strictly concave on a set of positive measure. Therefore $g_\beta(\alpha)$ is strictly concave for each fixed $\beta \in \mathrm{int}\Delta^{\mathrm{K}}$.

$\lambda^\star$ is the unique maximum of $g_{\lambda^\star}(\alpha)$ on $\mathrm{int}\Delta^{\mathrm{K}}$ if all directional derivatives at this point are zero. To ensure this property, one needs the partial derivatives of $g_\beta(\alpha)$. The derivative with respect to the ith component α_i is given by

$$\frac{\partial g_\beta(\alpha)}{\partial \alpha_i} = \lambda_0 \mathbb{E} \frac{d^i(\omega_0)/\beta_i}{1 - \lambda_0 + \lambda_0 \sum_{k=1}^K d^k(\omega_0) \frac{\alpha_k}{\beta_k}}$$

Observe that interchanging integration and differentiation is allowed because $\ln(\sum_{k=1}^{K} d^k(\omega_0)\,\alpha_k/\beta_k)$ is integrable for each fixed α. The last equation implies

$$\frac{\partial g_{\lambda^\star}(\lambda^\star)}{\partial \alpha_i} = \lambda_0 \mathbb{E}\frac{d^i(\omega)}{\lambda_i^\star} = \lambda_0 \mathbb{E}\frac{d^i(\omega)}{(1-\lambda_0)\mathbb{E}d^i} \equiv \frac{\lambda_0}{1-\lambda_0}$$

for all $i = 1, ..., K$, since $\sum_{k=1}^{K} d^k(\omega) = 1$ for all ω.

The directional derivative of $g_{\lambda^\star}$ in direction $(d\alpha_1, ..., d\alpha_K)$ with the restriction $\sum_{k=1}^{K} d\alpha_k = 0$ (which is a vector in the simplex) is equated as

$$\sum_{i=1}^{K} \frac{\partial g_{\lambda^\star}(\lambda^\star)}{\partial \alpha_i}\, d\alpha_i = 0.$$

Hence any portfolio rule different to $\lambda^\star$ is not evolutionary stable.

Let finally prove that any strategy $\beta \neq \lambda^\star$ with $\beta \in \mathrm{int}\Delta^{\mathrm{K}}$ cannot be locally evolutionary stable. A strategy $\beta \neq \lambda^\star$ is not locally evolutionary stable, if for any neighborhood of β there exists an α such that $g_\beta(\alpha) > 0$. It suffices to show that the directional derivative of g_β at β is strictly positive in one direction.

Since $\beta \neq \lambda^\star$ and both are points in the simplex there exists $i \neq j$ with $\beta_i > \lambda_i^\star$ and $\beta_j < \lambda_j^\star$. Note that we have assumed a minimum of two assets.

The directional derivative of g_β at β in the direction $d\alpha$ given by $d\alpha_i = -1/2$, $d\alpha_j = 1/2$, and zero otherwise, is given by,

$$\sum_{k=1}^{K} \frac{\partial g_\beta(\beta)}{\partial \alpha_k}\, d\alpha_k = \sum_{k=1}^{K} \frac{\mathbb{E}d^k}{\beta_k}\, d\alpha_k = \frac{1}{2}\left(\frac{\lambda_j^\star}{\beta_j} - \frac{\lambda_i^\star}{\beta_i}\right) > 0.$$

□

4.2 Stationary Strategies

We next allow for trading strategies that depend on past observations. The market selection process for two investors with stationary portfolio rules is given by

$$r_{t+1}^1 = \frac{\lambda_0}{\delta_{t+1}}\left(\Big[1 - \sum_{k=1}^{K}\lambda_{t+1,k}^2\theta_{t,k}^2\Big]\sum_{k=1}^{K} d_{t+1}^k\theta_{t,k}^1 + \Big[\sum_{k=1}^{K}\lambda_{t+1,k}^2\theta_{t,k}^1\Big]\sum_{k=1}^{K} d_{t+1}^k\theta_{t,k}^2\right) \quad (28)$$

where $\lambda_{t,k}^i = \lambda_k^i(\omega^t)$, $d_{t+1}^k = d^k(\omega_{t+1})$ and

$$\theta_{t,k}^1 = \frac{\lambda_{t,k}^1\, r_t^1}{\lambda_{t,k}^1\, r_t^1 + \lambda_{t,k}^2\,(1-r_t^1)} \quad , \quad \theta_{t,k}^2 = \frac{\lambda_{t,k}^2\,(1-r_t^1)}{\lambda_{t,k}^1\, r_t^1 + \lambda_{t,k}^2\,(1-r_t^1)}$$

$$\delta_{t+1} = \left[1-\sum_{k=1}^{K}\lambda^1_{t+1,k}\theta^1_{t,k}\right]\left[1-\sum_{k=1}^{K}\lambda^2_{t+1,k}\theta^2_{t,k}\right] - \left[\sum_{k=1}^{K}\lambda^2_{t+1,k}\theta^1_{t,k}\right]\left[\sum_{k=1}^{K}\lambda^1_{t+1,k}\theta^2_{t,k}\right]$$

The derivative of the right-hand side of (28) with respect to r^1_t evaluated at $r^1_t = 1$ turns out to be (after some lengthy but elementary calculations)

$$\frac{\partial h(\omega^{t+1}, r^1_t)}{\partial r^1_t}\Big|_{r^1_t=1} = \sum_{k=1}^{K}\left(\lambda^1_k(\omega^{t+1}) + \lambda_0\, d^k(\omega_{t+1})\right)\frac{\lambda^2_k(\omega^t)}{\lambda^1_k(\omega^t)} \qquad (29)$$

The exponential growth rate of investor 2's market share in a small neighborhood of the state in which investor 1 owns the total market wealth $r^1 = 1$ is given by

$$g_{\lambda^1}(\lambda^2) = \mathbb{E}\ln\left[\sum_{k=1}^{K}\left(\lambda^1_k(\omega^1) + \lambda_0\, d^k(\omega_1)\right)\frac{\lambda^2_k(\omega^0)}{\lambda^1_k(\omega^0)}\right] \qquad (30)$$

Observe that in contrast to the case of simple strategies, capital gains are possible in a fixed point due to changes in the strategy of the incumbent.

4.2.1 IID Case

The first stationary-strategy case to be analyzed in detail is the one in which state of nature is an i.i.d. process. According to the assumptions made in the general model, the number of possible states in any one period S is finite and, under the additional i.i.d. property, each state is drawn with some probability $p_s > 0$. The history up to some period in time, say t, does not have an effect on the distribution of the state ω_{t+1} (which is distributed according to $(p_1, ..., p_S)$).

Two strategies λ^1, λ^2 are distinct, i.e. $\lambda^1 \neq \lambda^2$, if $\lambda^1(\omega^0) \neq \lambda^2(\omega^0)$ on a set of strictly positive probability.

Theorem 2 *Suppose the state of nature is i.i.d. Define the portfolio rule* λ^* *by* $\lambda^*_0 = \lambda_0$*, and, for* $k = 1, ..., K$*,*

$$\lambda^*_k = (1-\lambda_0)\,\mathbb{E}d^k = (1-\lambda_0)\sum_{s\in S} p_s\, d^k(s). \qquad (31)$$

Instability results

(i) *No strategy is stable against* λ^**, i.e.* $g_\lambda(\lambda^*) \geq 0$*.*

(ii) *If* $(d^k(s))^k_s$ *has full rank, then every strategy* $\lambda \neq \lambda^*$ *is locally unstable* *with respect to λ^*, i.e. $g_\lambda(\lambda^*) > 0$.*

Stability results

(iii) *λ^* is not unstable, i.e. $g_{\lambda^*}(\lambda) \le 0$ for every strategy λ.*

(iv) *If $((1-\lambda_0)\,\mathbb{E}d^k + \lambda_0\, d^k(s))_s^k$ has full rank, then λ^* is locally stable, i.e. $g_{\lambda^*}(\lambda) < 0$ for every strategy $\lambda \neq \lambda^*$.*

Proof of Theorem 2. We first show (ii), pointing out how to derive assertion (i) in the proof. Suppose $(d^k(s))_s^k$ has full rank.

Normalizing strategies with $1-\lambda_0$ to make $\sum_{k=1}^K \lambda_k = 1$ for notational simplicity, assertion (ii) requires to show that

$$g_\lambda(\lambda^*) = \int_{S^{\mathbb{N}}} \tilde{g}(\lambda^*, \lambda, \omega^0)\, \mathbb{P}^0(d\omega^0) > 0 \tag{32}$$

with

$$\tilde{g}(\lambda^*, \lambda, \omega^0) \;\; = \;\; \sum_{s\in S} p_s \ln \left[\sum_{k=1}^K \Big((1-\lambda_0)\, \lambda_k(\omega^0, s) + \lambda_0\, d^k(s) \Big) \frac{\mathbb{E}d^k}{\lambda_k(\omega^0)} \right]$$

where $\mathbb{P}^0(d\omega^0)$ is the product measure on $S^{\mathbb{N}}$. This is the representation of the growth rate (30) in the i.i.d. case.

The Jensen inequality yields

$$\tilde{g}(\lambda^*, \lambda, \omega^0) \;\; \ge \;\; (1-\lambda_0) \sum_{s\in S} p_s \ln \left[\sum_{k=1}^K \lambda_k(\omega^0, s)\, \frac{\mathbb{E}d^k}{\lambda_k(\omega^0)} \right] \tag{33}$$

$$+ \,\lambda_0 \sum_{s\in S} p_s \ln \left[\sum_{k=1}^K d^k(s)\, \frac{\mathbb{E}d^k}{\lambda_k(\omega^0)} \right] \tag{34}$$

Consider the term (34) first. If $\lambda(\omega^0) = \lambda^*$, then (34) is equal to zero because $\sum_k d^k(s) = 1$. For any $\lambda(\omega^0) \neq \lambda^*$, one has

$$\sum_{s\in S} p_s \ln \left[\sum_{k=1}^K d^k(s)\, \frac{\mathbb{E}d^k}{\lambda_k(\omega^0)} \right] > 0$$

see Hens and Schenk-Hoppé (2003, Theorem 1). The strict inequality follows from the full rank assumption. Without this assumption the term (34) can be equal to zero even if the strategies are distinct when evaluated at ω^0.

Consider next the term on the right of (33). Application of the Jensen inequality yields

$$\ln \left(\sum_{k=1}^K \lambda_k(\omega^0, s)\, \frac{\mathbb{E}d^k}{\lambda_k(\omega^0)} \right) \ge \sum_{k=1}^K \mathbb{E}d^k \;\ln \left(\frac{\lambda_k(\omega^0, s)}{\lambda_k(\omega^0)} \right)$$

Thus

$$\int_{S^{\mathbb{N}}} \sum_{s\in S} p_s \ln\left(\sum_{k=1}^{K} \lambda_k(\omega^0, s)\,\frac{\mathbb{E}d^k}{\lambda_k(\omega^0)}\right)\,\mathbb{P}^0(d\omega^0)$$
$$\geq \sum_{k=1}^{K} \mathbb{E}d^k \left(\int_{S^{\mathbb{N}}} \ln \lambda_k(\omega^1)\,\mathbb{P}^0(d\omega^1) - \int_{S^{\mathbb{N}}} \ln \lambda_k(\omega^0)\,\mathbb{P}^0(d\omega^0)\right) = 0$$

The assumption that $\lambda(\omega^0) \neq \lambda^*$ with strictly positive probability yields validity of (32), i.e. $g_\lambda(\lambda^*) > 0$. This proves the instability results.

We next prove the stability results (iii) and (iv). Analogously to the above, we only consider (iv) in detail and point out how to derive (iii).

Using the normalization of strategies introduced in the proof of (ii), the growth rate can be written as

$$g_{\lambda^*}(\lambda) = \int_{S^{\mathbb{N}}} \bar{g}(\lambda(\omega^0), \lambda^*)\,\mathbb{P}^0(d\omega^0) \tag{35}$$

with

$$\bar{g}(\lambda, \lambda^*) = \sum_{s\in S} p_s \,\ln\left[\sum_{k=1}^{K}\left(1-\lambda_0+\lambda_0\,\frac{d^k(s)}{\mathbb{E}d^k}\right)\,\lambda_k\right] \tag{36}$$

$\bar{g}(\lambda, \lambda^*)$ is a concave function of $\lambda \in \Delta^K$. Moreover, if $[(1-\lambda_0)\,\mathbb{E}d^k + \lambda_0\, d^k(s)]$ has full rank then so has $[(1-\lambda_0)\,\mathbb{E}d^k + \lambda_0\, d^k(s)]/\mathbb{E}d^k = (1-\lambda_0) + \lambda_0\,(d^k(s)/\mathbb{E}d^k)$ and, thus, $\bar{g}(\lambda, \lambda^*)$ is even strictly concave in $\lambda \in \Delta^K$.

Assertion (iv) is immediate if we can show that the maximum of $\lambda \mapsto \bar{g}(\lambda, \lambda^*)$ over Δ^K is equal to zero, and further that this maximum is attained at $\lambda = \lambda^*$. Under the full rank assumption, strict concavity ensures that the maximizer $\lambda = \lambda^*$ is unique. Clearly, $\bar{g}(\lambda^*, \lambda^*) = 0$. This implies $\bar{g}(\lambda, \lambda^*) < 0$ if $\lambda \neq \lambda^*$. Thus, if $\lambda(\omega^0) \neq \lambda^*$ with strictly positive probability, then $g(\lambda, \lambda^*) < 0$.

If the rank is not full, then any variation of λ that leaves $\sum_k [(1-\lambda_0)\,\mathbb{E}d^k + \lambda_0\, d^k(s)]/\mathbb{E}d^k\,\lambda_k$ unchanged for all s (i.e. in the direction of linearly dependent columns) also attains the maximum. Thus we can only conclude that $g_{\lambda^*}(\lambda) \leq 0$, which is (iii).

We use the property that $\bar{g}(\lambda, \lambda^*)$ is continuously differentiable in λ. $\lambda = \lambda^*$ is a critical point of $\bar{g}(\lambda, \lambda^*)$ if

$$\sum_{n=1}^{K} \frac{\partial \bar{g}(\lambda, \lambda^*)}{\partial \lambda_n}\Big|_{\lambda=\lambda^*} d\lambda_n = 0 \tag{37}$$

for all $\sum_n d\lambda_n = 0$. Then the tangent space at λ^* is parallel to the simplex Δ^K. Concavity ensures that $\bar{g}(\lambda, \lambda^*)$ cannot take on values higher than $\bar{g}(\lambda^*, \lambda^*) = 0$.

One has

$$\frac{\partial \bar{g}(\lambda, \lambda^*)}{\partial \lambda_n} = \sum_{s\in S} p_s \frac{((1-\lambda_0)\,\mathbb{E}d^n + \lambda_0\, d^n(s))\,/\mathbb{E}d^n}{\sum_{k=1}^K ((1-\lambda_0)\,\mathbb{E}d^k + \lambda_0\, d^k(s))\ \lambda_k/\mathbb{E}d^k}$$

Thus

$$\begin{aligned}\sum_{n=1}^K \frac{\partial \bar{g}(\lambda, \lambda^*)}{\partial \lambda_n}\Big|_{\lambda=\lambda^*} d\lambda_n &= \sum_{n=1}^K \sum_{s\in S} p_s \frac{(1-\lambda_0)\,\mathbb{E}d^n + \lambda_0\, d^n(s)}{\mathbb{E}d^n}\, d\lambda_n \\ &= \sum_{n=1}^K \frac{(1-\lambda_0)\,\mathbb{E}d^n + \lambda_0\, \mathbb{E}d^n}{\mathbb{E}d^n}\, d\lambda_n = \sum_{n=1}^K d\lambda_n = 0\end{aligned}$$

This ensures $\bar{g}(\lambda(\omega^0), \lambda^*) \leq 0$ for all $\lambda(\omega^0)$, and $\bar{g}(\lambda(\omega^0), \lambda^*) < 0$ for all $\lambda(\omega^0) \neq \lambda^*$. Thus, if $\lambda(\omega^0) \neq \lambda^*$ on a set of strictly positive measure, $g_{\lambda^*}(\lambda) < 0$. This proves (iv). $\square$

Lemma 1 *Suppose $(d^k(s))_s^k$ has full rank. Then full rank condition in Theorem 2 (iv) (i.e. $((1-\lambda_0)\,\mathbb{E}d^k + \lambda_0\, d^k(s))_s^k$ has full rank) holds, if*

$$\min_s p_s > (1/2 - \lambda_0)/(1-\lambda_0) \tag{38}$$

In particular, (38) is fulfilled if $\lambda_0 > 1/2$. The condition holds for all $0 < \lambda_0 < 1$ in the two state i.i.d. case with uniform distribution ($S = \{1, 2\}$ and $p_1 = p_2 = 1/2$).

Proof of Lemma 1. The matrix $((1-\lambda_0)\,\mathbb{E}d^k + \lambda_0\, d^k(s))_s^k$ can be written as $[(1-\lambda_0)\,\pi + \lambda_0\,\mathrm{Id}]\,d$ where π is the matrix with all rows given by $(p_1, ..., p_S)$ and $d = (d^k(s))_s^k$. If d has full rank then it suffices to show that $[(1-\lambda_0)\,\pi + \lambda_0\,\mathrm{Id}]$ also has full rank.

The condition for a column dominant diagonal of the transposed of $((1-\lambda_0)\,\mathbb{E}d^k + \lambda_0\, d^k(s))_s^k$ with weighing vector $(1, ..., 1)$ reads $(1-\lambda_0)\,p_s + \lambda_0 > \sum_{u\neq s}(1-\lambda_0)\,p_u$. The last term is equal to $(1-\lambda_0)\,(1-p_s)$. This inequality is equivalent to (38). $\square$

4.2.2 Markov Case

This section extends the main result in the i.i.d. case to the model with Markov dividend payments.

Let Π denote the matrix of transition probabilities.

The growth rate of strategy λ^2 at λ^1-prices (30) is equal to

$$g_{\lambda^1}(\lambda^2) = \int_{S^N} \sum_{s\in S} \pi_{\omega_0 s} \ln \left[\sum_{k=1}^{K} \left(\lambda_k^1(\omega^0 s) + \lambda_0\, d^k(s)\right) \frac{\lambda_k^2(\omega^0)}{\lambda_k^1(\omega^0)} \right] \mathbb{P}(d\omega^0) \quad (39)$$

If also *strategies were Markov* then we would have,

$$g_{\lambda^1}(\lambda^2) = \sum_{s,\tilde{s}\in S} p_s\, \pi_{s\tilde{s}} \ln \left[\sum_{k=1}^{K} \left(\lambda_k^1(\tilde{s}) + \lambda_0\, d^k(\tilde{s})\right) \frac{\lambda_k^2(s)}{\lambda_k^1(s)} \right] \quad (40)$$

The convention for notation of conditional expected values is $\mathbb{E}(\lambda^* \mid s) = \sum_{s'} \pi_{ss'}\, \lambda^*(s')$.

Theorem 3 *Suppose the state of nature is governed by a Markov process with strictly positive transition probabilities $\pi_{s\tilde{s}} > 0$ for all $s, \tilde{s}$. Define the stationary portfolio rule λ^* by $\lambda_0^* = \lambda_0$ and, for all $s \in S$ and all $k = 1, ..., K$,*

$$\lambda^* = (1-\lambda_0)\, \lambda_0\, [\mathrm{Id} - (1-\lambda_0)\, \pi]^{-1}\ \pi\, d \quad (41)$$

Instability results

(i) *Every strategy $\lambda \neq \lambda^*$ is not stable against some arbitrarily close strategy μ, i.e. $g_\lambda(\mu) \geq 0$.*

(ii) *If, in addition to (i), $([(1-\lambda_0)\lambda_k(\omega^0, s) + \lambda_0\, d^k(s)]/\lambda_k(\omega^0))_s^k$ has full rank for almost all ω^0, then $g_\lambda(\mu) > 0$.*

Stability results

(iii) *λ^* is not unstable, i.e. $g_{\lambda^*}(\lambda) \leq 0$ for every strategy λ.*

(iv) *If $([(1-\lambda_0)\,\mathbb{E}(\lambda^* \mid s) + \lambda_0\,\mathbb{E}(d^k \mid s)]/\lambda^*(s))_s^k$ has full rank, then λ^* is locally stable, i.e. $g_{\lambda^*}(\lambda) < 0$ for every strategy $\lambda \neq \lambda^*$.*

The existence of the inverse in (41) follows immediately from the property that $\mathrm{Id} - (1-\lambda_0)\,\pi$ has a row dominant diagonal (recall that $0 < \lambda_0 < 1$).

Remark 1 *(i) λ^* is a Markov strategy.*

(ii) The strategy λ^* defined in (41) is a fundamentalist's portfolio rule in the sense that the fraction of wealth assigned to each asset is equal to the

expected value of the discounted (relative) dividends. To see this, consider the alternative representation of λ^*,

$$\lambda^* = \lambda_0 \sum_{m=1}^{\infty} (1-\lambda_0)^m \, \pi^m \, d \tag{42}$$

Equivalence of (41) and (42) is immediate by noting that the latter is a geometric series.

The strategy λ^* satisfies

$$\mathbb{E}(\lambda_k^* \mid s) + \lambda_0 \, \mathbb{E}(d^k \mid s) = 1/(1-\lambda_0) \, \lambda_k^*(s)$$

for all k and all s. It "balances" capital and dividend gains.

Proof of Theorem 3. As in the proof of Theorem 2 we normalize all strategies to achieve $\sum_{k=1}^{K} \lambda_k = 1$.

Using (39), we can write the growth rate of strategy μ at λ-prices as

$$g_\lambda(\mu) = \int_{S^N} \tilde{g}(\mu(\omega^0), \lambda(\omega^0), \omega_0) \, \mathbb{P}(d\omega^0) \tag{43}$$

with

$$\tilde{g}(\mu(\omega^0), \lambda(\omega^0), \omega_0) = \sum_{s \in S} \pi_{\omega_0 s} \, \ln \left[\sum_{k=1}^{K} \left[(1-\lambda_0) \lambda_k(\omega^0, s) + \lambda_0 \, d^k(s) \right] \, \frac{\mu_k(\omega^0)}{\lambda_k(\omega^0)} \right]$$

Analogously to the discussion in the proof of Theorem 2, one finds that, for every fixed ω^0, $\tilde{g}$ is a concave function of $\mu(\omega^0) \in \Delta^K$. Under the full rank assumption in (ii), it is even strictly concave.

One further has

$$\frac{\partial \tilde{g}}{\partial \mu_n(\omega^0)} = \sum_{s \in S} \pi_{\omega_0 s} \frac{\left[(1-\lambda_0) \, \lambda_n(\omega^0, s) + \lambda_0 \, d^n(s) \right] / \lambda_n(\omega^0)}{\sum_{k=1}^{K} \left[(1-\lambda_0) \, \lambda_k(\omega^0, s) + \lambda_0 \, d^k(s) \right] \, \mu_k(\omega^0) / \lambda_k(\omega^0)}$$

Thus

$$\begin{aligned} & \sum_{n=1}^{K} \left(\frac{\partial \tilde{g}}{\partial \mu_n(\omega^0)} \bigg|_{\mu(\omega^0) = \lambda(\omega^0)} \right) \, d\mu_n(\omega^0) \\ & \quad = \sum_{n=1}^{K} \frac{\sum_{s \in S} \pi_{\omega_0 s} \, \left[(1-\lambda_0) \, \lambda_n(\omega^0, s) + \lambda_0 \, d^n(s) \right]}{\lambda_n(\omega^0)} \, d\mu_n(\omega^0) \end{aligned} \tag{44}$$

for every $d\mu_1(\omega^0), ..., d\mu_K(\omega^0)$ with $\sum_{n=1}^{K} d\mu_n(\omega^0) = 0$.

Denote by $\tilde{\Omega}$ the set of strictly positive measure on which $\lambda \neq \lambda^*$. We show that for each $\omega^0 \in \tilde{\Omega}$, (44) is strictly positive for some $(d\mu_1(\omega^0), ..., d\mu_K(\omega^0))$ with $\sum_{n=1}^K d\mu_n(\omega^0) = 0$. This property implies that there is some μ with $\tilde{g}(\mu(\omega^0), \lambda(\omega^0), \omega_0) > 0$ on $\tilde{\Omega}$. Defining $\mu(\omega^0) = \lambda(\omega^0)$ for all ω^0 such that $\lambda(\omega^0) = \lambda^*(\omega^0)$, we obtain a strategy μ that is arbitrarily close to λ. By construction this strategy satisfies $g_\lambda(\mu) > 0$, which verifies assertion (i). Measurability of μ follows from the fact that, due to finiteness of S, the sigma algebra of the probability space under consideration is the power set, and, thus, every function is measurable.

It is clear that (44) is strictly positive for some $(d\mu_1(\omega^0), ..., d\mu_K(\omega^0))$ with $\sum_{n=1}^K d\mu_n(\omega^0) = 0$ if and only if

$$\frac{\sum_{s\in S} \pi_{\omega_0 s}\, \left[(1-\lambda_0)\, \lambda_n(\omega^0, s) + \lambda_0\, d^n(s)\right]}{\lambda_n(\omega^0)} \tag{45}$$

is not constant in n (for given ω^0).

We will show that (45) is constant, i.e.

$$\sum_{s\in S} \pi_{\omega_0 s}\, \left[(1-\lambda_0)\, \lambda_n(\omega^0, s) + \lambda_0\, d^n(s)\right] = c\lambda_n(\omega^0)$$

for all n, if and only if $\lambda = \lambda^*$.

Taking the sum over n on both sides of the last equality shows that $c = 1$. The condition that (45) is constant therefore becomes

$$\sum_{s\in S} \pi_{\omega_0 s}\, \left[(1-\lambda_0)\, \lambda_n(\omega^0, s) + \lambda_0\, d^n(s)\right] = \lambda_n(\omega^0) \tag{46}$$

It is important to point out that assertion (iii) and (iv) are immediate once this result is derived. The procedure is completely analogous to the proof of Theorem 2. The growth rate $\tilde{g}(\lambda(\omega^0), \lambda^*(\omega_0)\omega_0)$ is a concave function (strictly concave under the full rank assumption in (iv)). Thus if (46) is constant if and only if $\lambda = \lambda^*$, then $g_{\lambda^*}(\lambda)$ takes on its maximum, which is zero, at $\lambda = \lambda^*$. This is (iii) resp. (iv).

To show that (46) holds if and only if $\lambda = \lambda^*$, we need to consider three distinct cases: (a) $\lambda(\omega^0)$ does not depend on ω^0; (b) $\lambda(\omega^0)$ depends only on a finite history, i.e. $\lambda(\omega^0) = \lambda(\omega_{-T}, ..., \omega_0)$ for some $T \geq 0$; and (c) $\lambda(\omega^0)$ depends on an infinite history.

Case (a): (45) becomes

$$(1-\lambda_0)\, \lambda_n + \lambda_0\, \mathbb{E}(d^n \mid \omega_0) = \lambda_n \tag{47}$$

which is equivalent to $\mathbb{E}(d^n \mid \omega_0) = \lambda_n$. If, as we have assumed in the Theorem, the dividend process is a non-degenerate Markov process, μ has to depend on ω_0. This is a contradiction.

Case (b): Obviously, if $\lambda(\omega^0)$ is only a function of ω_0, then $\lambda = \lambda^*$. For a strategy $\lambda(\omega^0)$ that depends on a history of length $T \geq 1$ (45) becomes

$$\sum_{s \in S} \pi_{\omega_0 s} \left[(1-\lambda_0)\, \lambda_n(\omega_{-T+1}, ..., \omega_0, s) + \lambda_0\, d^n(s)\right] = \lambda_n(\omega_{-T}, ..., \omega_0) \qquad (48)$$

If λ_n would vary with ω_{-T}, (48) could not hold for all ω^0. Thus (48) implies that $\lambda(\omega^0) = \lambda(\omega_{-T+1}, ..., \omega_0)$. Repeated application shows that $\lambda(\omega^0) = \lambda(\omega_0)$. However, this implies $\lambda = \lambda^*$, as discussed above.

Case (c): (45) reads

$$\sum_{\omega_1 \in S} \pi_{\omega_0 \omega_1} \left[(1-\lambda_0)\, \lambda_n(\omega^1) + \lambda_0\, d^n(\omega_1)\right] = \lambda_n(\omega^0) \qquad (49)$$

An analogous equation holds with $\lambda_n(\omega^1)$ on the right-hand side,

$$\sum_{\omega_2 \in S} \pi_{\omega_1 \omega_2} \left[(1-\lambda_0)\, \lambda_n(\omega^2) + \lambda_0\, d^n(\omega_2)\right] = \lambda_n(\omega^1) \qquad (50)$$

Inserting (50) in (49) yields

$$\lambda_n(\omega^0) = (1-\lambda_0)^2\, \pi^2_{\omega_0\omega_2}\, \lambda_n(\omega^2) + \lambda_0 \left[(1-\lambda_0) \sum_{\omega_2} \pi^2_{\omega_0\omega_2}\, d^n(\omega_2) + \sum_{\omega_1} \pi^1_{\omega_0\omega_1}\, d^n(\omega_1)\right] \qquad (51)$$

where $\pi^m_{\omega_0\omega_m} = \sum_{\omega_1,\ldots,\omega_m} \pi_{\omega_0\omega_1} \ldots \pi_{\omega_{m-1}\omega_m}$.

Repeating this procedure and observing that

$$(1-\lambda_0)^m \sum_{\omega_m} \pi^m_{\omega_0\omega_m}\, \lambda_n(\omega^m) \to 0 \text{ as } m \to \infty$$

we find

$$\lambda_n(\omega^0) = \frac{\lambda_0}{1-\lambda_0} \sum_{m=1}^{\infty} (1-\lambda_0)^m \sum_{\omega_m} \pi^m_{\omega_0\omega_m}\, d^n(\omega_m) \qquad (52)$$

Thus $\lambda_n(\omega^0)$ is a function of ω_0 only, implying that $\lambda = \lambda^*$, as discussed in case (b). The equivalence of (52) and the definition of λ^* in the Theorem 3 has been established in Remark 1. □

5 Mean-Variance Optimization

In this section we analyze the evolutionary stability of portfolio rules based on mean-variance optimization. The mutual fund theorem states that given all investors build portfolios according to the mean-variance-criterion, then every investor will hold a combination of the riskless asset and the market portfolio in any capital market equilibrium. Even though it is very questionable whether indeed all investors use mean-variance-optimization, investing a big share of wealth in the market portfolio is a very common behavior.

We extend our previous model by incorporating a strategy that enables an investor to buy the market portfolio. This extension relates our model to the classical CAPM[5] results.

It is well known that in practice mean-variance portfolios are often under-diversified, i.e. they typically put positive weight on very few assets only. To cure this defect it is then usually suggested to modify the mean-variance portfolio by devoting some positive but small share of the budget on every asset in the portfolio, ensuring that the portfolio is completely mixed. We show in the section 5.2 that this commonly used “quick fix” of the under-diversification problem is indeed an improvement of the mean-variance portfolio.

5.1 The CAPM strategy

This subsection uses the very general framework with non-autonomous strategies (17). However, we make assumption (A.1).

Buying the market portfolio means to imitate the “population mix.” In particular if one investment strategy gains then the market-portfolio investor assigns more weight to the corresponding strategy. The investor holding the market portfolio has constant market share.

Consider an investor, say $\gamma = I + 1$, who wants to buy a fraction of the market portfolio. The market portfolio is the vector of total stocks of each asset, here the initial supply of one unit of each asset. In terms of an investment strategy buying the market portfolio requires to divide wealth proportional to the asset prices, i.e.,

$$\lambda^{\gamma}_{t,k} = (1 - \lambda_{t,0}) \frac{p^k_t}{\sum_{l=1}^{K} p^l_t}$$

where $k = 1, ..., K$. This trading strategy depends on the equilibrium prices in the current period. An investor who buys the market portfolio has therefore to give a demand function to the auctioneer. This calls for an extension of our previous analysis.

[5] See also Sciubba (1999) for an analysis of CAPM-trading rules in the original Blume and Easley (1992) setup with diagonal securities.

Suppose all other investors pursue trading strategies $\lambda_t^i(\omega^t) \in \Delta^K$, $i = 1, ..., I$. Then the market-clearing condition (6) becomes,

$$p_t^k = \lambda_{t,k}\, w_t + (1 - \lambda_0)\, \frac{p_t^k}{\sum_{l=1}^K p_t^l}\, w_t^\gamma \tag{53}$$

where $w_t = (w_t^1, ..., w_t^I)$. Since $\sum_{l=1}^K p_t^l = (1 - \lambda_{t,0})\, W_t$, we obtain for the normalized price $q_t^k = p_t^k / W_t$ the condition

$$q_t^k = \lambda_{t,k}\, r_t + q_t^k\, r_t^\gamma \tag{54}$$

Thus

$$q_t^k = \frac{1}{1 - r_t^\gamma}\, \lambda_{t,k}\, r_t \tag{55}$$

The portfolio of the CAPM investor is therefore given by

$$\theta_{t,k}^\gamma = r_t^\gamma \quad \text{for all } k = 1, ..., K \tag{56}$$

Writing (17) in the form of (7) gives for investor γ

$$r_{t+1}^\gamma = \sum_{k=1}^K \left(\lambda_{t+1,0}\, d_{t+1}^k(\omega^{t+1}) + q_{t+1}^k(\omega^{t+1})\right)\, r_t^\gamma \tag{57}$$

Since

$$\sum_{k=1}^K d_{t+1}^k(\omega^{t+1}) = 1 \quad \text{and} \quad \sum_{k=1}^K q_{t+1}^k(\omega^{t+1}) = 1 - \lambda_{t+1,0} \tag{58}$$

one finds that

$$r_{t+1}^\gamma = r_t^\gamma.$$

Summarizing our findings we can state the following result.

Proposition 3 *The market share of a CAPM investor is constant in any population in which all other players pursue simple strategies. In particular, a CAPM investor will never vanish nor dominate the market.*

The intuition behind this result is given by the representation of the normalized market-clearing price in the model with only simple strategies, $q_t^k = \sum_{i=1}^I \lambda_k^i\, r_t^i$. The normalized equilibrium price equals the relative market wealth invested in that asset. If one player dominates the market in the long-run and asymptotically own the entire market wealth, the asset price will reflect the trading strategy of this investor. The CAPM investor mimics this strategy because he distributes his wealth according to the relative value of the assets.

¿From an evolutionary point of view it can be concluded that investing in the market portfolio is a strategy with strong resistance against the market selection mechanism. Hence even though buying the market portfolio may not be in accordance with mean-variance optimization (because not everybody uses it) it is a convenient rule which automatically imitates the most successful trading strategy!

5.2 Diversification

In this subsection we show that under-diversification is fatal for evolutionary stability. This result follows readily from the expressions of the Lyapunov-exponent, equation (24), that we have derived above.

Corollary 1 *Suppose $\hat{\lambda}$ is an under-diversified strategy, i.e. for at least one k, $\hat{\lambda}_k(\omega^0) = 0$ and $d^k(\omega_1) > 0$ on a set of strictly positive probability. Denote by $\hat{\lambda}^\varepsilon_k := (1-\varepsilon)\hat{\lambda}_k + (1-\lambda_0)\varepsilon/K$, $0 < \varepsilon \leq 1$, the corresponding ε-completed strategy. Then $\hat{\lambda}^\varepsilon$ is robust against $\hat{\lambda}$-mutants for all sufficiently small $\varepsilon > 0$, i.e. the distribution of wealth shares that assigns total wealth to the $\hat{\lambda}^\varepsilon$-player is stable in the population $(\hat{\lambda}^\varepsilon, \hat{\lambda})$.*

Even though using the "quick fix" to prevent under-diversification is better than investing according to the under-diversified portfolio rule, it is clear from the main result Theorem 1, that ε-completed under-diversified simple strategies are not locally stable (if they do not coincide with $\lambda^\star$). However, we next show that the situation for ε-completed portfolio rules $\hat{\lambda}^\varepsilon$ is even worse. Any completely mixed simple strategy drives out $\hat{\lambda}^\varepsilon$ for all small enough $\varepsilon > 0$.

Corollary 2 *Given any completely diversified strategy λ^c (uniformly in ω bounded away from zero) and any under-diversified strategy $\hat{\lambda}$. Then $\hat{\lambda}^\varepsilon$, defined in Corollary 1, is not robust against λ^c-mutants for all sufficiently small $\varepsilon > 0$, i.e. the distribution of wealth shares that assigns total wealth to the $\hat{\lambda}^\varepsilon$-player is not stable in the population $(\hat{\lambda}^\varepsilon, \lambda^c)$.*

Proof of Corollary 1. We need to show that the growth rate, defined in (30), satisfies

$$\mathbb{E} \ln \left(\sum_{k:\hat{\lambda}(\omega^0)_k > 0}^{K} \left[\hat{\lambda}^\varepsilon_k(\omega^1) + \lambda_0\, d^k(\omega_1) \right] \frac{\hat{\lambda}_k(\omega^0)}{(1-\varepsilon)\hat{\lambda}_k(\omega^0) + (1-\lambda_0)\,\varepsilon/K} \right) < 0$$

for all small $\varepsilon > 0$. The left-hand side of this equation is strictly increased by omitting ε/K in the denominator. We thus obtain the sufficient condition,

$$\mathbb{E}\ln\left(\sum_{k:\hat{\lambda}(\omega^0)_k>0}\left[\hat{\lambda}_k^{\varepsilon}(\omega^1)+\lambda_0\,d^k(\omega_1)\right]\right)\leq\ln(1-\varepsilon) \tag{59}$$

One has

$$\sum_{k:\hat{\lambda}_k(\omega^0)>0}\left[\hat{\lambda}_k^{\varepsilon}(\omega^1)+\lambda_0\,d^k(\omega_1)\right]\leq(1-\lambda_0)+\lambda_0\sum_{k:\hat{\lambda}_k(\omega^0)>0}d^k(\omega_1)\leq 1$$

for all events, where last inequality is strict for all ω such that $d^k(\omega_1) > 0$ while $\hat{\lambda}_k(\omega^0) = 0$ for some k. Since the latter holds according to the assumption in the corollary, the left-hand side of (59) turns out to be strictly negative. Therefore (59) is satisfied for all small enough ε. □

Proof of Corollary 2. One has to show that

$$\mathbb{E}\ln\left(\sum_{k=1}^{K}\left[\hat{\lambda}_k^{\varepsilon}(\omega^1)+\lambda_0\,d^k(\omega_1)\right]\frac{\lambda_k^c(\omega^0)}{\hat{\lambda}_k^{\varepsilon}(\omega^0)}\right)>0 \tag{60}$$

for all small $\varepsilon > 0$.

Noting that the bracketed term in (60) is equal to

$$\sum_{k:\hat{\lambda}_k(\omega^0)>0}\frac{\left(\hat{\lambda}_k^{\varepsilon}(\omega^1)+\lambda_0\,d^k(\omega_1)\right)\lambda_k^c(\omega^0)}{(1-\varepsilon)\hat{\lambda}_k(\omega^0)+(1-\lambda_0)\varepsilon/K}+\sum_{k:\hat{\lambda}_k(\omega^0)=0}\frac{\left(\hat{\lambda}_k^{\varepsilon}(\omega^1)+\lambda_0\,d^k(\omega_1)\right)\lambda_k^c(\omega^0)}{(1-\lambda_0)\varepsilon/K}$$

The first term is bounded away from zero uniformly in ε. The second term tends to infinity as $\varepsilon \to 0$. Since for some k, $\hat{\lambda}_k(\omega^0) = 0$ with strictly positive probability according to our assumptions, one finds that the left-hand side of (60) tends to infinity as $\varepsilon \to 0$. □

6 Conclusion

We have studied the evolution of wealth shares of portfolio rules in incomplete markets with long-lived assets. Prices are determined endogenously. The performance of a portfolio rule in the process of repeated reinvestment of wealth is determined by the wealth share eventually conquered in competition with other portfolio rules. Using random dynamical systems theory, we derived necessary and sufficient conditions for the evolutionary stability of portfolio rules. In the case of Markov (in particular i.i.d.) payoffs these local stability conditions lead to a simple portfolio rule that is the unique evolutionary stable strategy. This rule possesses an explicit representation as it invests proportionally to the expected relative dividends. Moreover, it is demonstrated that mean-variance optimization is not evolutionary stable while the CAPM-rule always imitates the best portfolio rule and survives.

These results are first steps towards a general evolutionary theory of portfolio selection. Future research may focus on the following generalizations of our results:

- simultaneous mutants:
  The concept of evolutionary stability that we have used considers one mutant at a time. By restricting attention to simpler stage games like bi-matrix games, evolutionary game theory has derived very interesting results for the case of arbitrary many mutants (see Friedman (2001), for example.) Since for the case of short-lived assets Hens and Schenk-Hoppé (2003) have recently demonstrated evolutionary stability of λ^* with arbitrary many mutants there is some hope to also get this result for the case of long-lived assets.

- non-stationary dividends:
  Having shown the advantages of the relative dividends rule λ^* one may wonder why portfolio selection in real stock markets is still under debate. One reason may be that dividend are not stationary ergodic. Indeed for Dow-Jones data, for example, we have found severe non-stationarity in the dividends process. It is interesting to see how λ^* could be adapted to locally stationary processes for example.

- global stability:
  With long-lived assets λ^* is the unique evolutionary stable strategy. Also we have shown that λ^* can destabilize any incumbent portfolio rule. However, so far we have not shown that ,as in the case of short-lived assets, starting from any initial position, the market selection process converges to λ^*.

- price dependent portfolio rules:
  The most general portfolio rules we have allowed for are stationary strategies adapted to the information filtration. Simple momentum rules like "buy (sell) if prices have gone up (down)" are not in the class of rules we studied since through the price path non-stationarities can enter the portfolio rules. Note that momentum rules were for example studied in interaction with rules based on rational expectations in Brock

and Hommes (1997). It is very interesting to see whether our market selection process (17) may generate results similar to theirs.

- short sales:
  We have ruled out short sales for the technical reason to ensure that prices remain positive. Alternatively one could let 0 be a lower barrier for asset prices. This would however change the market selection process in a non-smooth way.

- liquidity shocks:
  We have studied consumption proportionally to wealth. If one of the investor is for example an insurance company then from time to time she will need to withdraw large amounts of cash independent of the current wealth.

- alternative market structures:
  In our paper the market is organized by a batched auction. In every period every portfolio rule submits its budget shares to some auctioneer who then determines prices as the wealth average of the strategies portfolio rules. Some financial markets are indeed organized this way. For example the opening and the closing auctions on the German stock exchange operate similarly. However, more common market structures are double-auctions or a market maker systems. As Bottazzi, Dosi, and Rebesco (2003) have recently shown, the dynamics of evolutionary models may well depend on the specific market organization chosen. It is interesting to see how the wealth selection process studied here is affected by the choice of the market structure.

- strategic interaction:
  The best strategy is of course the strategy that tries to exploit the other strategies in the market.[6] A natural question to ask is therefore what the Nash-equilibrium is in a game corresponding to the market selection process (17). Obviously, the Nash equilibrium depends on the preferences of the investors. Recently Hens, Reimann, and Vogt (2003) have shown that all investors playing λ^* in every period is a Nash-equilibrium if all investors maximize expected utility functions with correct beliefs about the occurrence of the states provided assets are short lived and there is no aggregate risk, i.e. $\sum_k d^k(s) = \sum_k d^k(z)$ for all s, z, or assets are short lived and all investors have logarithmic preferences. Moreover, for the case of Arrow-securities, Alós-Ferrer and

[6] "The best plan is ... to profit by the folly of others." Pliny the Elder, from John Barlett, comp. Familiar Quotations, 9th ed. 1901.

Ania (2003) have shown that in this setting λ^* is the unique evolutionary stable strategy in the sense of Schaffer (1988). These results show that our approach can also be viewed as an evolutionary justification for Nash equilibria of certain market games. It is interesting to see whether these results still hold for general payoff structures.

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