# Jump Dynamics: The Equity Premium and the Risk-Free Rate Puzzles.

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**Abstract**

The paper develops a consumption based equilibrium model, focusing on the risk premium and the risk-free interest rate. We derive testable expressions for these quantities, and confront these with sample estimates for the 20. century. Our framework is a dynamic model in continuous time, allowing for random jumps at random time points, in addition to diffusion uncertainty. Preferences are time separable and additive.

The classical equity premium puzzle and the risk-free rate puzzle are re-examined. We present values for the parameters of the representative agent's utility function for different values of risk premia and interest rates, calibrated to two first moments of the US-data of the last century. Relatively low values for agents' risk aversion are consistent with the model, but positive values of the subjective interest rate seem harder to fit.

*KEYWORDS: Consumption based CAPM, Equilibrium interest rate, The equity premium puzzle, the risk-free rate puzzle, jump/diffusions.*

## Introduction

The paper develops an expression for the difference between average equity and debt returns, and an expression for average real debt return in equilibrium, using a dynamic model in continuous time. For different values of this risk premium and the spot rate we calculate a range of values for the

relative risk aversion and the subjective interest rate of the representative agent. We take into account the small covariance between consumption and equities, and the small variance of consumption observed in the last century. The small magnitudes of these two quantities are the primary causes of the equity premium puzzle and the risk-free rate puzzle, respectively, given the choice of model. Our first question is if an alternative model could do better, and if so, by how much. In this regard we extend the preliminary examples presented in Aase (1993 a-b).

McGrattan and Prescott (2003) adjust for some factors ignored by Mehra and Prescott - taxes, regulatory constraints, and diversification costs - and focusing on long-term rather than short-term savings instruments. The new findings for the difference between average equity and debt returns of the last century, and the average real debt return, fits nicely into the permitted range. If these findings are generally accepted, both puzzles are resolved at one stroke.

As a model for the consumption rate $c$ and the equities return $R$ we employ a dynamic model in continuous time allowing for random jumps at random time points, in addition to the more familiar diffusion type. We demonstrate that this framework is well suited for economic equilibrium analysis.

One motivation for introducing jumps, is to get out of the mean-variance setting of economic modelling under uncertainty. A diffusion model is driven by the Brownian motion, a stochastic process that is Gaussian, and can thus be characterized by its first two moments. This usually results in expressions for economic quantities that only depend upon the two first moments of the relevant probability distributions, even if these are not normal. The cause of the classical mean-variance dependence, usually in a one period setting, has a different origin. Here preferences were assumed to depend only on the two first moments. Although the means and the variances have a different origin in these two settings, the results can sometimes have a striking similarity.

Recalling the related discussion between Borch, Feldstein and Tobin in 1969, Borch, for example, simply pointed out that the probability distribution of a random variable generally depends on more than only its two first moments. Similar remarks were made by Feldstein, and both authors illustrated possible shortcomings from restricting attention to only the two first moments in the representation of preferences of individual decision makers.

Here we should remember that it is a consequence of Carleman's Theorem (see e.g., Anderson (1958)) that, even in the case where a probability distribution has moments of all orders, knowledge of these moments is, in general, not enough to determine the entire probability distribution itself.

By allowing random sized jumps at random time points in the dynamic

framework, we obtain equilibrium relations that depend on other quantities of the relevant, joint probability distributions than only the two first moments. We demonstrate that this fact gives us more flexibility when trying to e.g., fit equity premia and average debt returns derived from the model, to consumption and equity data of the 20. century, than can be obtained from using Itô-processes only. We relate our findings to those of Hansen and Jagannathan (1991), Rietz (1988) and Salyer (1998).

We work with a time additive and separable set of preferences, and a key point is to try to confront the data using this type of model framework. There is, of course, a large literature discussing different preferences, such as habit formation, in the present setting, recent references being Allais (2004) or Chen and Ludvigson (2004). See also the basic papers in this direction by Haug (2001), Constantinides (1990), Detemple and Zapatero (1991) and Sundaresan (1989).

The paper is organized as follows: In section 1 we present a short version of the the premises of the economic model, and recall the expressions for the equilibrium risk premiums and equilibrium interest rate using continuous dynamics. In section 2 we introduce discontinuous dynamics in the representative agent model, and derive the relevant risk premia and interest rate in this setting. In section 3 we introduce certain simplifying assumptions, making our model suitable for calibration to the data of the 20. century. Several examples are presented throughout. Section 4 concludes.

# 1 The case of continuous dynamics

In this section we present, very briefly, the rudiments of a consumption based equilibrium model. Follow e.g., Aase (2002), the consumption space $L$ is the set of adapted processes $c$ satisfying the integrability constraint $E\left(\int_0^T c_t^2\,dt\right) < \infty$ for some fixed time horizon $T$. In this economy there are $m$ agents, each being characterized by a nonzero consumption endowment process $e^i$ in the set $L_+$ of non-negative processes in $L$, and by a strictly increasing utility function $U_i(\cdot) : L_+ \to R$.

We assume the utility functions to be time-additive with a representation $U_i(c) = E\left[\int_0^T u_i(c_t, t)\,dt\right], i = 1, 2, \ldots, m$. Consider the function $u$ defined by

$$u(y,t) = \sup_{x \in R^m} \sum_{i=1}^{m} \lambda_i u_i(x_i, t) \qquad \text{subject to} \qquad \sum_{i=1}^{m} x_i \leq y, \tag{1}$$

for non-negative constants $\lambda_i$. Conditions are well-known guaranteeing that

problem (1) has a solution, in which case the market, or *the representative agent*, has the "additive" utility function of the form

$$U(c) = E\left[\int_0^T u(c_t, t)\, dt\right].$$

## 1.1 A Model for the Short-Term Interest Rate

We now assume that the aggregate consumption rate $c_t$ follows an Itô-process with the representation

$$dc(t) = \mu_c(t)dt + \sigma_c(t)dB(t), \tag{2}$$

The uncertainty in continuous models in continuous time is usually modelled by a $d$-dimensional Brownian motion $B(t)$, which we assume for the moment.

Then we know that the spot price $p_t$ of aggregate consumption also plays the role of a *state price deflator*, where $p_t = u'(c_t, t)$. Assuming the marginal utility $u'$ smooth enough for the application of Itô's lemma, we get that

$$dp(t) = \mu_p(t)dt + \sigma_p(t)dB(t), \tag{3}$$

where $\sigma_p = u''(c_t, t)\sigma_c(t)$ and

$$\mu_p(t) = u''(c_t, t)\mu_c(t) + \frac{\partial}{\partial t}u'(c_t, t) + \frac{1}{2}u'''(c_t, t)\sigma_c(t) \cdot \sigma_c(t). \tag{4}$$

It is then known that the equilibrium spot interest rate $r_t$ is given by

$$r(t) = -\frac{\mu_p(t)}{p(t)}, \qquad \text{for } t \leq T, \tag{5}$$

where $p_t = u'(c_t, t)$, the marginal utility of the function $u$ given in equation (1), and $\mu_p$ is given in equation (4).

Let the representative agent have a Savage expected utility functional of the following additive and separable type

$$U(c) = E\left\{\int_0^T e^{-\int_0^t \rho(s)\, ds} u(c_t)\, dt\right\}, \tag{6}$$

where $\rho(t)$ is the subjective interest rate. In this case we get that

$$r(t) = \rho(t) + \left(-\frac{u''(c_t)c_t}{u'(c_t)}\right)\frac{\mu_c(t)}{c_t} - \frac{1}{2}\frac{u'''(c_t)}{u'(c_t)}\sigma_c(t) \cdot \sigma_c(t). \tag{7}$$

The second term on the right hand side is the reciprocal of the intertemporal elasticity of substitution in consumption multiplied by the expected growth rate. The term $(\frac{u'''(c_t)}{u'(c_t)})$ is the degree of *prudence* of the consumer, and $(-\frac{u''(c_t)c_t}{u'(c_t)})^{-1} := el(c_t)$ is the *intertemporal elasticity of substitution in consumption.* Since $u' > 0$, the prudence term is positive if $u''' > 0$, and the representative consumer is then called prudent. If this is the case *precautionary saving* results. From the expression (7) we notice the local mean-variance nature of this relationship: It typically depends on the mean rate $\mu_c(t)$ at any time instant $t$, as well as on the variance rate $\sigma_c(t) \cdot \sigma_c(t)$ at time $t$.

Since rational behavior dictates that the subjective interest rate $\rho(t)$ is non-negative, equation (7) gives the intertemporal restriction

$$el(c_t)^{-1} \frac{\mu_c}{c_t} \leq r_t + \frac{1}{2} \frac{u'''(c_t)}{u'(c_t)} \sigma_c(t) \cdot \sigma_c(t). \tag{8}$$

for any $t \leq T$. It is interesting to compare this inequality to the restriction on the agent's intertemporal marginal rate of substitution (IMRS) of Hansen and Jagannathan (1991). They present a restricted region for the means and standard deviations of IMRS's, consistent with historical time series data, using a non-parametric approach. With a positive subjective interest rate, only very large values of the risk aversion $\alpha$ are consistent with this region.

## 1.2 The Risk-Free Rate Puzzle

The risk-free rate puzzle is perhaps best illustrated my a simple example.

*Example 1.* Consider a (lognormal) model where the aggregate consumption rate satisfies

$$dc_t = c_t \mu_c dt + c_t(\sigma_{c,1} dB_1(t) + \sigma_{c,2} dB_2(t)), \tag{9}$$

where $\mu_c, \sigma_{c,1}$ and $\sigma_{c,2}$ are constants, and $B_1$ and $B_2$ are two independent, standard Brownian motions. Here $\sigma_c^2 := \sigma_{c,1}^2 + \sigma_{c,2}^2$.

The representative investor (consumer) has a felicity index $u(x,t) = \frac{x^{1-\alpha}}{1-\alpha} e^{-\rho t}$, $\alpha \neq 1$, $u(x) = \ln(x)$ if $\alpha = 1$, where he subjective impatience rate $\rho$ is a constant, and $\alpha$ is the coefficient of intertemporal relative risk aversion, another constant. Here $u(x) = \frac{x^{1-\alpha}}{1-\alpha}$ in the representation (6) above.

It now follows that the equilibrium short term interest rate is

$$r = \rho + \alpha \mu_c - \frac{1}{2} \alpha(1+\alpha)\sigma_c^2. \tag{10}$$

For the valid values of $\alpha$ ($\alpha > 0$) the precautionary savings hypothesis holds in this case, so this representative consumer is prudent. By its very nature,

$\rho > 0$, measuring the time preference of consumption. This quantity is often thought to be is close to one per cent. Reasonable values for the parameter $\alpha$ are known from independent studies in the economics of uncertainty to be in the interval $[1/2, 7]$, say. This is the case, at least in one-period settings of choice. The Kelly criterion is obtained if $\alpha = 1$, i.e., for the logarithmic utility. Notice the local mean-variance flavor of equation (10).

Using the estimates 0.0183 and 0.0357 for $\mu_c$ and $\sigma_c$ respectively, estimates that are consistent with the Mehra and Prescott (1985)-study (see e.g. Constantinides (1990), and in addition $\rho = 0.01$, we find the equilibrium, annual interest rate to be around 4.3 per cent, consistent with McGrattan and Prescott (2003).

In the Mehra and Prescott - study $r$ was independently estimated to be around one per cent. Maintaining that approximately $\rho \approx 0.01$, this gives an estimate of 27.81 for $\alpha$, generally considered to be too large. On the other hand, suggesting that $\alpha = 2$, this gives $\rho \approx -0.03$, i.e., $-3$ per cent. This latter observation is related to the "risk-free rate puzzle" (Weil (1989)). □

## 1.3 The Consumption-Based CAPM

Suppose that the real dividends of a risky asset is given by

$$dD_t = \mu_D(t)dt + \sigma_D(t)dB_t, \tag{11}$$

The real price $S_t$ of the asset is supposed to be an Itô-process, and will in general be determined in equilibrium. $G = S + D$ is the adjusted price process of the risky asset. It is also assumed to be an Itô-process, i.e., $dG(t) = \mu_G(t)dt + \sigma_G(t)dB(t)$. We can then show that in equilibrium it must be the case that

$$\mu_R(t) - r_t = \left(-\frac{u''(c_t, t)c_t}{u'(c_t, t)}\right)\frac{\sigma_G(t)}{S_t}\frac{\sigma_c(t)}{c_t}, \tag{12}$$

for any $t \leq T$ with probability one, where $\mu_R(t) =: \frac{\mu_G(t)}{S_t}$.

The quantity $\frac{\sigma_G(t)}{S_t}\frac{\sigma_c(t)}{c_t}$ is interpreted as the covariance rate between the returns of the risky asset and the aggregate growth rate in consumption. Here we also notice the local mean-variance nature of the equilibrium relationship given in (12): The risk premium, a mean rate, is proportional to this covariance rate in equilibrium.

## 1.4 The Equity Premium Puzzle

In this section we briefly explain the equity premium puzzle. Also this puzzle is perhaps best illustrated by an example:

*Example 2.* The situation is as in Example 1 related to aggregated consumption and preferences, and in addition we consider an asset having price dividend process $D$ given by

$$dD_t = S_t \mu_D dt + S_t(\sigma_{D,1} dB_1(t) + \sigma_{D,2} dB_2(t)),$$

where $B_1(t)$ and $B_2(t)$ are two independent Brownian motions. The price process $S$ is only assumed to be an Itô-process. In equilibrium it turns out that the price process $S$ is consistent with the following representation

$$dS_t = S_t \mu_S dt + S_t(\sigma_{S,1} dB_1(t) + \sigma_{S,2} dB_2(t)).$$

In this situation the CCAPM involves the following restriction of the parameters in equilibrium

$$\mu_S + \mu_D - r = \alpha(\sigma_{S,c} + \sigma_{D,c}) \tag{13}$$

The expression for the spot interest rate $r$ is given in equation (10).

In the Mehra and Prescott-study referred to above, the risky asset is represented by the Standard and Poor's composite stock price index during the time period 1889-1978. They estimated $(\mu_S + \mu_D)$ to 0.07, $(\sigma_{S,c} + \sigma_{D,c})$ to 0.0059, and $(\sigma_S + \sigma_D)$ to 0.165. (e.g., Constantinides (1990)). Using their estimate for $r$ of 0.01, we can substitute these estimates, together with those reported in Example 1, into equations (13) and (10) and solve for the two unknowns $\rho$ and $\alpha$. The solution is $\hat{\alpha} = 10.2$ and $\hat{\rho} = -0.10$.

Hence, not only do we get a relatively high (method of moments-) estimate of $\alpha$, but also the estimate for the subjective discount rate is at odds with the assumption that it must be non-negative. The first aspect, that $\hat{\alpha}$ is relatively high is known as the equity premium puzzle as posed by Mehra and Prescott (1985). The second aspect, that $\hat{\rho}$ is negative meaning a negative time value of money, is the riskless rate puzzle suggested by Weil (1989). If $\rho$ is forced to be positive, then the riskless interest rate becomes too high relative to the values reported by Mehra and Prescott (1985), as follows from equation (10) □

McGrattan and Prescott (2003) re-examine the equity premium puzzle, taking into account some factors ignored by the Mehra and Prescott: Taxes, regulatory constraints, and diversification costs - and focusing on long-term rather than short-term savings instruments. Accounting for these factors, the authors find the difference between average equity and debt returns during

peacetime in the last century is less than 1 percent, with the average real equity return somewhat under 5 percent, and the average real debt return almost 4 percent. The latter is more in agreement with the findings of Siegel (1992).

Using the values one per cent for the risk premium, four per cent for the quantity $r_t$ of this model, and maintaining the covariance rate between the equity index and aggregate consumption .0059, we find an estimate of the subjective interest rate $\hat{\rho} = .012$, or 1.2 per cent, and an estimate of the relative risk aversion $\hat{\alpha} = 1.70$. Thus these new interpretation will simply solve both puzzles.

In the rest of the paper we develop an analogous theory to the one presented above using discontinuous dynamics. We claim that this framework has certain advantages. Here we present a relatively simple example where the $\rho$ and the $\alpha$ parameters are calibrated to the above data for various values of the risk premium and the average real debt return. That is, we have retained the historical low values of the covariance between equity and aggregate consumption of the last century, and also the estimates for the variance of the equity index, the variance of aggregate consumption and the estimate of the growth rate in consumption, all values reported above. This should provide the reader some insights into how the model is doing compared to the continuous model. It turns out that the original equity premium puzzle can not be explained by the model allowing jump dynamics, but this model may provide more reasonable values than the continuous one. For the interpretation of McGrattan and Prescott (2003), we can calibrate $\rho = .014$ and $\alpha = 1.59$, which start looking very plausible indeed.

# 2 Discontinuous dynamics

In this section we introduce discontinuous dynamics for the exogenously given processes $c$ of aggregate consumption and $D$ of the cumulative dividends of the risky assets. In doing so we maintain the economic model of the previous section, but allow for a different revelation of uncertainty as time goes. We assume that the aggregate consumption $c$ and the dividend process $D$ of a risky asset are given by:

$$dc(t) = \mu_c(t)dt + \sigma_c(t)dB(t) + \int_Z \gamma_c(t,z)\tilde{N}(dz,dt), \tag{14}$$

and

$$dD(t) = \mu_D(t)dt + \sigma_D(t)dB(t) + \int_Z \gamma_D(t,z)\tilde{N}(dz,dt), \tag{15}$$

where $\tilde{N}(dz, dt) = N(dz, dt) - \nu(dz)dt$. Here $N(dz, dt)$ is a random measure, where the two independent, underlying Levy-processes are assumed to be in $L^2$, i.e., random processes having finite variances, $\nu(dz)$ is the Levy measure and $\tilde{N}(dz, dt)$ is the centered random measure.

The reason we choose our primitive processes to be in $L^2$ is that then state prices will in the dual space, which is also $L^2$. Thus we avoid unnecessary technical complications, in particular with regard to the representation of the underlying jump processes.

The terms $\sigma_D$, $\sigma_c$, and $\gamma_D$, $\gamma_c$ may all be matrices of appropriate dimensions, $B$ and $N$ are vector or scalar processes, depending on the circumstances, where $B$ is again a Brownian motion.

For this type of processes we may perform several kinds of relevant analyses, including: Optimal stopping, stochastic control,the stochastic maximum principle, impulse control, singular control, chaos expansion and Malliavin calculus, the Girsanov theorem, statistical inference, etc.

## 2.1 A General Pricing Formula

Let $(S, D)$ represent any given primitive security with real price process $S$ and accumulated dividends process $D$. In the Aase (2002) it is demonstrated that a security-spot market equilibrium is characterized as follows: The real market value $S$ at each time $t$ satisfies

$$S(t) = \frac{1}{u'(c_t, t)} E\left\{ \int_t^T \left( u'(c_s, s)\, dD(s) + d[D, u'](s) \right) \mid \mathcal{F}_t \right\}. \tag{16}$$

Here $u'$ is the marginal utility of the representative agent, $c_t := \sum_{i=1}^m c^i{}_t$ is the aggregate consumption process in the market, and $[D, u']$ is the square covariance process between accumulated dividends and the marginal utility process.

The additional covariance term, not following from the Lucas (1978)-model, is in fact also important in the continuous model, in the case where the dividend process is assumed to follow an Itô-diffusion with a nonvanishing diffusion term $\sigma_D(t)$. In Aase (2002) the pricing relation (16) was taken as the main starting point in deriving both the equilibrium interest rate and the equilibrium risk premium.

In the following we assume that there exists a solution in the marketed subspace of the representative agent problem, or we may argue directly using a single agent economy. In either case the following can be done. We start with the equilibrium short rate process.

## 2.2 The equilibrium short-term interest rate

In this section we derive a model for the short-term interest rate in equilibrium. This spot rate is given by

$$r(t) = -\frac{\mu_p(t)}{p(t)}, \qquad \text{for } t \leq T, \tag{17}$$

(almost surely) also in this case, where $p_t = u'(c_t, t)$ (see e.g., Aase (1993a-b, 2002)). By Itô's lemma

$$dp_t = \mu_p(t)dt + \sigma_p(t)dB(t) + \int_Z \gamma_p(t,z)\tilde{N}(dt,dz).$$

More precisely, we have that

$$\begin{aligned} dp_t = \Bigg[ & \frac{\partial}{\partial t}u'(c_t,t) + u''(c_t,t)\mu_c + \frac{1}{2}u'''(c_t,t)tr(\sigma_c(t)\sigma_c^T(t)) \\ & - u''(c_t,t)\int_Z \gamma_c(t,z)\nu(dz) + \int_Z \Big(u'(c_{t-} + \gamma_c(t,z),t) - u'(c_{t-},t)\Big)\nu(dz)\Bigg]dt \\ & + u''(c_t,t)\sigma_c dB(t) + \int_Z \Big(u'(c_{t-} + \gamma_c(t,z),t) - u'(c_{t-},t)\Big)\tilde{N}(dt,dz) \end{aligned} \tag{18}$$

Thus the equilibrium interest rate equals

$$\begin{aligned} r_t = & \left(-\frac{\frac{\partial}{\partial t}u'(c_t,t)}{u'(c_t,t)}\right) + \left(-\frac{u''(c_t,t)c_t}{u'(c_t,t)}\right)\left(\frac{\mu_c(t)}{c_t}\right) \\ & - \frac{1}{2}\frac{u'''(c_t,t)}{u'(c_t,t)}tr(\sigma_c(t)\sigma_c^T(t)) \\ & - \left(\left(-\frac{u''(c_t,t)}{u'(c_t,t)}\right)\int_Z \gamma_c(t,z)\nu(dz)\right. \\ & \left. + \int_Z \left(\frac{u'(c_{t-} + \gamma_c(t,z),t) - u'(c_{t-},t)}{u'(c_t,t)}\right)\nu(dz)\right) \end{aligned} \tag{19}$$

The first term is the subjective interest rate $\rho(t)$. The second term is the product of the growth rate in consumption and the reciprocal of the intertemporal elasticity of substitution in consumption, the third term is related to *precautionary savings*. The last two terms stem from the jumps.

Note from the last term that we are no longer in the local mean-variance framework - the entire probability distribution of the jump sizes may be required.

*Example 3.* Let us specialize as follows: Consider a compound Poisson process as a model for for the jump term, where a geometric model is adopted for the aggregate consumption process with constant coefficients: This means that the "mark space" $Z = [-1, \infty) \times [-1, \infty)$, and $\gamma(t, (z_c, z_R)) = z_c \geq -1$, the latter requirement must hold in order to avoid negative consumption, i.e.

$$dc(t) = c_{t-}\left(\mu_c dt + \sigma_c dB(t) + \int_Z z_c \tilde{N}(dz, dt)\right) \tag{20}$$

Notice that the Lévy measure $\nu$ is now the joint probability distribution function of the jump sizes $(Z_c, Z_R)$ multiplied by the frequency of jumps $\lambda$.

The representative agent is the same as in Example 1, the constant relative risk aversion case. Taking this into account we get for the equilibrium interest rate

$$r = \rho + \alpha\mu_c - \frac{1}{2}\alpha(1+\alpha)\sigma_c^2 - \lambda\left(\alpha E(Z_c) + (E(1+Z_c)^{(-\alpha)} - 1)\right). \tag{21}$$

The random variable $Z_c$ signify the *jump sizes* in the aggregate consumption process $c$ (a compound Poisson process). The first of the jump terms helps in explaining the "risk-free-rate" puzzle when $EZ_c > 0$, the second term may not. □

## 2.3 An economic interpretation of the interest rate jump term

In this section we attempt to find an economic interpretation of the jump term in equation (19). The idea is perhaps best illustrated by an example:

*Example 4.* Consider equation (21) in the above example, in the very special case where $\alpha = 2$. The "jump term" is then approximately equal to $-3\lambda E(Z_c^2)$ based on a Taylor series argument, truncating the series after two terms.

One way of increasing the level of consumption uncertainty is to increase $\lambda E(Z_c^2)$, which is the variance rate of the jump process. This has the effect of lowering the interest rates. The implication is that the consumer *saves* in the presence of *increasing* consumption uncertainty, as is typically the case with a prudent representative agent. □

We now extend this idea. To this end we start by expanding the jump term in a Taylor series, which seems valid here since the support of the distribution of $Z_c$ is mostly the interval $(-1, 1)$. The marginal utility difference

can be approximated as follows:

$$(u'(c_{t-} + \gamma_c(t,z), t) - u'(c_{t-}, t)) \approx u''(c_t, t)\gamma_c(t,z) +$$

$$\frac{1}{2}u'''(c_t, t)\gamma_c(t,z)^2 + \frac{1}{6}u^{(4)}(c_t, t)\gamma_c(t,z)^3 + \cdots$$

In the jump term referred to above, the two first terms cancel and this term equals:

$$-\frac{1}{2}\frac{u'''(c_t,t)}{u'(c_t,t)}\int_Z \gamma_c(t,z)^2\nu(dz) + \frac{1}{6}\frac{u^{(4)}(c_t,t)}{u'(c_t,t)}\int_Z \gamma_c(t,z)^3\nu(dz) + \cdots$$

Truncating after three terms we get

$$\begin{aligned} r_t = &\left(-\frac{\frac{\partial}{\partial t}u'(c_t,t)}{u'(c_t,t)}\right) + \left(-\frac{u''(c_t,t)c_t}{u'(c_t,t)}\right)\left(\frac{\mu_c(t)}{c_t}\right) \\ &- \frac{1}{2}\frac{u'''(c_t,t)}{u'(c_t,t)}\left(tr(\sigma_c(t)\sigma_c^T(t)) + \int_Z \gamma_c(t,z)^2\nu(dz)\right) \\ &- \frac{1}{6}\frac{u^{(4)}(c_t,t)}{u'(c_t,t)}\int_Z (\theta(t,z)\gamma_c(t,z))^3\nu(dz), \end{aligned} \tag{22}$$

where $\theta(t,z) \in [0, \gamma_c(t,z)]$ a.s., dictated by by the mean value theorem.

The jump term is now split in two, one part related to precautionary savings, the other part depending on the fourth derivative of the utility function.

The requirement that $\rho_t \geq 0$ can be written as

$$\begin{aligned} el(c_t)^{-1}\frac{\mu_c}{c_t} \leq r_t &+ \frac{1}{2}\frac{u'''(c_t,t)}{u'(c_t,t)}\left(tr(\sigma_c(t)\sigma_c^T(t)) + \int_Z \gamma_c(t,z)^2\nu(dz)\right) \\ &+ \frac{1}{6}\frac{u^{(4)}(c_t,t)}{u'(c_t,t)}\int_Z (\theta(t,z)\gamma_c(t,z))^3\nu(dz) \end{aligned} \tag{23}$$

for any $t \leq T$. Notice that in comparing with the corresponding restriction (8) in the continuous model setting, the above requirement is less demanding if the last term is positive.

As with the continuous model, also here it helps with a prudent consumer. It is likely, however, that the order of magnitude of the three last terms in equation (22) is smaller than that of the second term on the right hand side. It will require an empirical investigation to settle these issues. Continuing our previous example, we have:

*Example 5.* Returning to Example 4, the jump term in (21) is now approximately equal to $-\lambda(3E(Z_c^2)$ - $4E(Z_c^3))$, bringing in one more term in the Taylor series. Here the representative agent is prudent. If the last term is negative as well, this may help in explaining the risk-free rate puzzle. This could happen here if the third central moment of the jump sizes in consumption is negative, which is quite possible, since $Z_c$ is the relative jump size in aggregate consumption.

In the pure jump case ($\sigma_c = 0$), the model in (21) looks like, to the second order approximation

$$r = \rho + \alpha\mu_c - \frac{1}{2}\alpha(\alpha+1)\lambda E(Z_c^2) + \frac{1}{6}\alpha(\alpha+1)(\alpha+2)\lambda E(Z_c^3) \tag{24}$$

Returning to the discussion in Example 1, consider the case where $\alpha = 2$. In this case the continuous model gave $\rho = -0.03$ for the estimates used by Mehra and Prescott (1985). Using the same population estimates, the pure jump model in (24) gives $\rho = 0.01$ for this level of relative risk aversion, if the term $\lambda E(Z_c^3) = -0.0082$. This latter numerical value is, however, too large in absolute value compared to the estimate $\lambda E(Z_c) = 0.0016$, where this latter value follows from the given population estimates. This is in fact the highest absolute value that can be allowed for the term $\lambda E(Z_c^3)$. Using this in the above equation gives $\rho = -0.025$ when $\alpha = 2$. Thus the extra term in equation (24) brings the subjective rate in the right direction, but not quite enough. □

In conclusion, one part of the jump term enters quite naturally as a variance rate, analogues to in the continuous case. This retains the precautionary savings interpretation of this term. In addition, according to this approximation a new term enters, depending on the fourth derivative of the utility function and the third central moment of the aggregate consumption process. If the latter term is positive, it will tend to increase the subjective rate $\rho$, in which case it will help in explaining the risk-free rate puzzle. As example 5 indicates, however, this does not seem to be enough to reconcile the data analyzed by Mehra and Prescott (1985) with the model assumptions.

## 2.4 The Consumption Based Capital Asset Pricing Model (CCAPM)

In this section we derive the CCAPM in the case of jump dynamics. Let $S$ be a price process of a no-dividend paying risky security in this market, or, S may alternatively be interpreted as an adjusted price process $G$, as in section

1.3. We assume it to be of the Itô-Lévy type, i.e.

$$dS_t = S_{t-}\left(\mu_R(t)dt + \sigma_R(t)dB_t + \int_Z \gamma_R(t,z)\tilde{N}(dt,dz)\right), \tag{25}$$

A *state price deflator* is a strictly positive Itô-Lévy process $p$ such that the deflated price process $S^p(t) =: S(t)\cdot p(t)$ is a martingale. We deduce the dynamic equation for the deflated price process $S^p$, using a generalized version of Itô's lemma, here sometimes called the product rule:

$$\begin{aligned} S^p(t) &= S(0)p(0) + \int_0^t S(s)dp(s) + \int_0^t p(s)dS(s) \\ &+ \int_0^t \int_Z \gamma_p(s,z)\gamma_R(s,z)S(s)N(ds,dz) + \int_0^t \sigma_p(s)S(s)\sigma_R(s)ds. \end{aligned}$$

Then we insert the equations for $p$ and $S$ and collect terms; the martingale requirement amounts to a zero drift term, or

$$\begin{aligned} &S(t)\mu_p(t) + p(t)S(t)\mu_R(t) + \sigma_p(t)\sigma_R(t)S(t) \\ &+ \int_Z \gamma_p(t,z)\gamma_R(t,z)S(t)\nu(dz) = 0. \end{aligned}$$

Assuming both $S$ and $p$ strictly positive, we may divide through by the term $S\cdot p$. Using the relation $\mu_p(t)/p(t) = r(t)$ we get:

$$\mu_R(t) - r(t) = -\frac{1}{p_t}\int_Z \gamma_p(t,z)\gamma_R(t,z)\nu(dz) - \frac{\sigma_p(t)}{p_t}\sigma_R(t).$$

Now recall, using again Itô's lemma, the endogenous expressions for $\sigma_p(t)$ and $\gamma_p(t,z)$ in terms of the exogenous consumption process and preferences:

$$\sigma_p(t) = c_t u''(c_t,t)\sigma_c(t)$$

and

$$\gamma_p(t,z) = u'(c_{t-} + \gamma_c(t,z),t) - u'(c_{t-},t).$$

This leads directly to the following relation for the risk premium:

$$\begin{aligned} \mu_R(t) - r(t) &= \left(-\frac{u''(c_t,t)c_t}{u'(c_t,t)}\right)\left(\frac{\sigma_c(t)}{c_t}\sigma_R(t)\right) \\ &- \int_Z \left(\frac{u'(c_{t-}+\gamma_c(t,z),t) - u'(c_{t-},t)}{u'(c_t,t)}\right)\gamma_R(t,z)\nu(dz). \end{aligned} \tag{26}$$

Equation (26) is an extension of the celebrated CCAPM to jump models. A more general expression for the risk premium than (26) is known (see Back (1991)) for special semimartingales when aggregation across agents does not work as we assume it does. The above expression was first appearing in Aase (1993 a-b).

Notice in particular how the last term brings us outside of the the mean-variance type of analysis. This term stems from the jumps in the equity $S$ and the aggregate consumption $c$.

*Example 6.* Let us adopt the same assumptions as in examples 1 and 2. Assuming in addition that the jump sizes $Z_R$ and $Z_c$ are independent, the risk premium can be written:

$$\mu_R - r = \alpha\sigma_R\sigma_c - \lambda E(Z_R)E\{(1 - Z_c)^{-\alpha} - 1\}. \tag{27}$$

Here we have assumed that $\gamma_c(t, z) = c_{t-}Z_c$ so that $Z_c$ signify jumps in percentage in the aggregate consumption, i.e., we have a geometric model for the aggregate consumption. Note that the above independence assumption does not imply that the return $R$ on the stock index is independent of the consumption growth $c$, as is evident from equation (27). This will be further explained below. □

## 2.5 An interpretation of the CCAPM jump term

In this section we attempt to give an economic interpretation of the jump term in the CCAPM-relation (27). To this end, let us return to the general form of the risk premium and rewrite the last term. It can be written

$$-\int_Z \frac{u'(c_{t-} + \gamma_c(t, z), t) - u'(c_{t-}, t)}{u'(c_t, t)}\gamma_R(t, z)\nu(dz) =$$
$$\int_Z \left( - \frac{(u'(c_{t-} + \gamma_c(t, z), t) - u'(c_{t-}, t))c_t}{\gamma_c(t, z)u'(c_t, t)} \right) \left( \frac{\gamma_c(t, z)}{c_t}\gamma_R(t, z) \right) \nu(dz)$$

Written this way, this term somehow corresponds to the first term in (26). This follows since the first term in the last integrand is a "first order" approximation of the intertemporal coefficient of relative risk aversion, the second is the instantaneous covariance (by the Itô-Lévy isometry) between the jumps in the return of the risky asset and the jumps in the growth rate of the aggregate consumption in the market.

A better approximation follows by including one more term:

$$
\begin{aligned}
&-\int_Z \frac{u'(c_{t-}+\gamma_c(t,z),t)-u'(c_{t-},t)}{u'(c_t,t)}\gamma_R(t,z)\nu(dz) = \\
&\left(\left(-\frac{u''(c_t,t)c_t}{u'(c_t,t)}\right)\int_Z\left(\frac{\gamma_c(t,z)}{c_t}\right)\gamma_R(t,z)\nu(dz)\right) \\
&+\left(-\frac{1}{2}\left(\frac{u'''(c_t,t)c_t^2}{u'(c_t,t)}\right)\int_Z\left(\frac{\theta(t,z)\gamma_c(t,z)}{c_t}\right)^2\gamma_R(t,z)\nu(dz)\right)
\end{aligned}
$$

The first term corresponds to the usual one in continuous dynamics. The second term is new, and signify the product of the degree of prudence and a second-first cross moment between the the consumption growth and the the risky asset.

Our expression for the risk premium is, based on this approximation

$$
\begin{aligned}
\mu_R(t)-r(t) = \left(-\frac{u''(c_t,t)c_t}{u'(c_t,t)}\right)&\left\{\left(\frac{\sigma_c(t)}{c_t}\sigma_R(t)\right)\right. \\
&\left.+\int_Z\left(\frac{\gamma_c(t,z)}{c_t}\right)\gamma_R(t,z)\nu(dz)\right\} \qquad (28) \\
+\left(-\frac{1}{2}\left(\frac{u'''(c_t,t)c_t^2}{u'(c_t,t)}\right)\int_Z\left(\frac{\theta(t,z)\gamma_c(t,z)}{c_t}\right)^2\gamma_R(t,z)\nu(dz)\right).&
\end{aligned}
$$

From this expression we notice that if $u''' > 0$, i.e., the consumer is prudent, this will tend to lower the risk premium in the case where the intertemporal cross-moment in the last expression is positive, and to increase the risk premium if this moment is negative. Thus there are two possibilities for this model to give a larger risk premium that is the case for the continuous model. The first is where the representative agent is not prudent, and the cross-moment is positive, the second involves a prudent representative investor and a negative cross-moment. The sign of this cross-moment is an empirical question.

When treating the equilibrium riskless rate $r$, we noticed that a prudent consumer would in fact help in a possible resolution of the "risk-free rate puzzle". This illustrates the potential difficulty to explain both puzzles at once.

*Example 7.* Let us illustrate the use of the expression (28). We consider the pure jump case, where $\sigma_c = \sigma_R = 0$. Using the geometric model for the consumption process of Example 3, the risk premium can be written

$$\mu_R - r = \alpha \int_{-1}^{\infty} \int_{-1}^{\infty} z_R z_c \nu(dz_R, dz_c) - \frac{1}{2}\alpha(\alpha+1) \int_{-1}^{\infty} \int_{-1}^{\infty} z_R z_c^2 \nu(dz_R, dz_c). \tag{29}$$

Notice that we do not assume that $Z_R$ and $Z_c$ are independent here. From Example 2 and the Itô-Lévy isometry (explained in the next section), an estimate of the first term on the right hand side is .0059. The risk premium of the original Mehra and Prescott (1985)-investigation was 0.06. Let us denote the mixed crossmoment appearing in last term on the right hand side in equation (29) by $\kappa$. We then have the equation

$$.06 = \alpha(0.0059) - \frac{1}{2}\alpha(\alpha+1)\kappa. \tag{30}$$

Suppose, for example, that the value of the coefficient of the relative risk aversion $\alpha = 2$. This leads to a corresponding value of $\kappa = -0.016$. Unlike the situation in Example 5, there is no restriction on this mixed crossmoment $\kappa$, so this value is in the permitted range, given the estimates of the other moments. Only an empirical investigation can bring us further on this point. But this illustrates the potential of the expression for the risk premium in equation (28). □

## 2.6 A comparison between the continuous model and a particular simplified version of the pure jump model

In order to better understand the jump components of our model, we here find the conditions when the the pure jump model is a direct analogue of the continuous model. By the pure jump model we simply mean the model resulting from ignoring the Brownian motion part of the dynamics.

We make the same simplifying assumption as in Example 6, that the relative jump sizes $(Z_R, Z_c)$ in the composite stock index and the aggregate consumption are independent. There is still an instantaneous correlation between these quantities, since the respective jumps take place at the same random, time instants $\tau_1, \tau_2, \ldots$. The dynamic equations are then

$$\frac{dc(t)}{c(t-)} = \mu_c dt + \int_Z z_c \tilde{N}(dz, dt),$$

and

$$dR(t) =: \frac{dS(t)}{S(t-)} = \mu_R dt + \int_Z z_R \tilde{N}(dz, dt)$$

This leads to the following two relations for the equilibrium interest $r$ rate and risk premium $(\mu_R - r)$:

$$\mu_R - r = -\lambda E(Z_R) E\{(1 + Z_c)^{-\alpha} - 1)\}, \tag{31}$$

and

$$r = \rho + \alpha\mu_c - \lambda\Big(\alpha E(Z_c) + \Big(E(1 + Z_c)^{-\alpha} - 1\Big)\Big). \tag{32}$$

We modify these equations as follows: Since $(1 + Z_c)^{-\alpha} \approx 1 + -\alpha Z_c$, we get for the equilibrium risk premium

$$\mu_R - r \approx \alpha\lambda E(Z_R)E(Z_c)), \tag{33}$$

Taking one more term in the Taylor series for the interest rate, $(1+Z_c)^{-\alpha} \approx 1 + (-\alpha)Z_c + \frac{1}{2!}(-\alpha)(-\alpha - 1)E(Z_c^2)$, we get for the equilibrium interest rate $r$:

$$r \approx \rho + \alpha\mu_c - \frac{1}{2}\alpha(1 + \alpha)\lambda E(Z_c^2). \tag{34}$$

In order to calibrate this model, we need the estimates of the variances and covariances. These we assume unaffected by the tax rules and other frictions taken into account by McGrattan and Prescott (2003). We thus use the same numerical values of these estimates as in examples 1 and 2.

We now invoke the Itô-Lévy isometry, giving the following estimates: $\lambda E(Z_R^2) = (.1654)^2$, $\lambda E(Z_c^2) = (.03574)^2$ and $\int_Z z_R z_c \nu(dz_R, dz_c) = .0059$. The latter equality can be written $\lambda E(Z_R)E(Z_c) = .0059$, using the assumed independence between $Z_R$ and $Z_c$,

Note that this particular independence in the joint jumps of $S$ and $c$ does not lead to a zero annualized covariance.

In order to illustrate the Itô-Lévy isometry employed in the above, recall that the variance of a compound Poisson process is $\text{var}X(t) = \lambda t E(X_1^2)$, not $\lambda t \text{var}(X_1)$. Likewise, the covariance of two compound Poisson processes making independent jumps at the same time points is $\text{cov}(X(t), Y(t)) = \lambda t E(X_1 Y_1)$, not $\lambda t \text{cov}(X_1, Y_1)$, which would have been zero under independence.

Notice that these equations (33) and (34) have the same interpretations as the corresponding ones provided by the continuous model, using this approximation of the discontinuous model.

Recalling that $\hat{\mu}_R - r = .01$ in the McGrattan and Prescott (2003) interpretation, and $\hat{\mu}_c = 0.0183$, from the above equations (33) and (34) we obtain directly the following calibrated values: $\hat{\alpha} = 1.70$ and $\hat{\rho} = .012$. These are, of course, the same as provided by the continuous model for the this interpretation of the data set.

Regarding the problems that we discuss, with the above interpretation of the frequency parameter $\lambda$ and independence assumption, the continuous model is equivalent to a first order approximation of the jump model regarding the risk premium, and a second order approximation of the jump model for the interest rate.

In the next, and final, section we calibrate the model without using any Taylor series approximations, but we retain the independence assumption of this section.

# 3 Calibration and Numerical Results

In this section we calibrate our jump model to the same data set as indicated above, without using any Taylor series approximations. Naturally, in order to make any progress we need some assumptions regarding the joint probability distributions of the jump sizes in our model. To this end we make the following assumptions: The random variables $(1+Z_c, 1+Z_R)$ are jointly lognormally distributed. As a consequence $\ln(1+Z_c)$, $\ln(1+Z_R)$ is bivariate normal with parameters $\mu_1, \sigma_1^2$ and $\mu_2, \sigma_2^2$ respectively, the correlation coefficient being assumed equal to zero. Since both $c$ and $R$ are aggregated quantities, this assumption is partly supported by the Central Limit Theorem. Note in particular that this gives positive consumption, as our model requires.

In this case we get the following:

$$E(1+Z_c) = e^{(\mu_1 + \frac{1}{2}\sigma_1^2)}, \qquad E(1+Z_R) = e^{(\mu_2 + \frac{1}{2}\sigma_2^2)},$$

$$E(1+Z_c)^2 = e^{(2\mu_1 + 2\sigma_1^2)}, \qquad E(1+Z_R)^2 = e^{(2\mu_2 + 2\sigma_2^2)}$$

and

$$E(1+Z_c)^{-\alpha} = e^{(-\alpha\mu_1 + \frac{1}{2}\sigma_1^2\alpha^2)}.$$

As a consequence of these relations we also get:

$$E(Z_c^2) = e^{(2\mu_1+2\sigma_1^2)} - 2(e^{(\mu_1+\frac{1}{2}\sigma_1^2)} - 1) - 1,$$

with a similar expression for $E(Z_R^2)$. From the previously observed estimates, we can now set up the following five equations in order to calibrate all the parameters. From the sample estimates reported in examples 1 and 2 we get: (i) $\lambda E(Z_c^2) = 0.001277$, (ii) $\lambda E(Z_R^2) = 0.027357$ and (iii) $\lambda E(Z_c)E(Z_R) = .0059$.

The two main equations are the one for the equilibrium risk premium and the equilibrium market interest rate:

$$(iv) \qquad \mu_R - r = -\lambda\left(e^{(\mu_2+\frac{1}{2}\sigma_2^2)} - 1\right)\left(e^{(-\alpha\mu_1+\frac{1}{2}\sigma_1^2\alpha^2)} - 1\right)$$

and

$$(v) \qquad r = \rho + \alpha(.0183) - \lambda\left(\alpha\left(e^{(\mu_1+\frac{1}{2}\sigma_1^2)} - 1\right) + \left(e^{(-\alpha\mu_1+\frac{1}{2}\sigma_1^2\alpha^2)} - 1\right)\right).$$

The idea is now to vary the risk premium $(\mu_R - r)$ and the interest rate $r$ in equations (iv) and (v), and find the corresponding values of the risk aversion $\alpha$ and subjective rate $\rho$. This we do in the next sections.

## 3.1 Varying the frequency $\lambda$ in the pure jump model

We have a system of five equations in seven unknown parameters, and naturally infinitely many solutions may fit the equations. By trial and error we fix $\sigma_1 = .01\sigma_2$, and solve the system for different values of the frequency parameter $\lambda$. This way we get rid of two parameters, and we can solve the five equations in the five unknowns $\alpha$, $\rho$, $\mu_1$, $\mu_2$ and $\sigma_1$ for various values of $\lambda$.

For the Mehra and Prescott (1985) pair $(\mu_R - r, r) = (.06, .01)$ we obtained a relative risk aversion $\alpha = 4.69$ for the parameters $\lambda = .02738$, $\mu_1 = -.2436$, $\mu_2 = -7.8567$, $\sigma_1 = .0163$ and $\sigma_2 = .1634$. This is a rather low value of this coefficient. However, the corresponding subjective rate $\rho = -.04$, i.e., minus four per cent. Although this is relatively small in absolute value, it still has the wrong sign. This is typical for our calibrations using the Mehra and Prescott (1985) values $(\mu_R - r, r) = (.06, .01)$; it seems easier to obtain a reasonable coefficient of the relative risk aversion than a reasonable value for the subjective interest rate.

As the frequency of the jumps $\lambda$ increases, the values $(\alpha, \rho)$ approach the corresponding ones for the continuous model. For example, when $\lambda = 10.000$, then $\alpha = 10.17$ and $\rho = -.1023$, close to the values $\alpha = 10.2$ and $\rho = -.10$ for the continuous model. This seems reasonable, since an increasing frequency implies that the pure jump model will display an ever increasing sequence of smaller and smaller jumps, approaching the continuous model in the limit. The corresponding values for the other parameters are $\mu_1 = -.0001$, $\mu_2 = -.0005$, $\sigma_1 = .48 \cdot 10^{-8}$ and $\sigma_2 = .48 \cdot 10^{-7}$. As we can see, these values confirm that the jump sizes are small on the average.

Turning to the McGrattan and Prescott (2003) pair $(\mu_R - r, r) = (.01, .04)$, both puzzles disappear, as we have seen also for the continuous model. Typical values are $\alpha$ around 2 with $\rho$ varying around .02, both considered as very plausible values. When the frequency gets large, e.g., $\lambda = 1000$, then $\alpha = 1.69$ and $\rho = .01$. The values for the other parameters are $\mu_1 = -.001$, $\mu_2 = .005$, $\sigma_1 = .48 \cdot 10^{-7}$ and $\sigma_2 = .48 \cdot 10^{-6}$. This is close to the values provided by the continuous model.

For the frequency $\lambda = .02738$, $\alpha = 2.3386$, $\rho = .001$ and $\mu_1 = -.2436$, $\mu_2 = -7.8567$, $\sigma_1 = .0163$ and $\sigma_2 = .1634$, all acceptable values for the parameters.

Since the data are annual, it could be of some interest to investigate the case $\lambda = 1$, corresponding to one jump per year on the average. For the pair $(\mu_R - r, r) = (.06, .01)$, we obtain then $\alpha = 8.50$ and $\rho = -.0856$. The other parameters are then $\mu_1 = -.0366$, $\mu_2 = -.1795$, $\sigma_1 = .65 \cdot 10^{-4}$ and $\sigma_2 = .65 \cdot 10^{-3}$. Similarly, for the pair $(\mu_R - r, r) = (.01, .04)$, we obtain then $\alpha = 1.78$ and $\rho = 0.01$. The other parameters are then $\mu_1 = .0353$, $\mu_2 = .1516$, $\sigma_1 = .37 \cdot 10^{-4}$ and $\sigma_2 = .37 \cdot 10^{-3}$.

Table 1 presents values for $\alpha$ for different values of $(\mu_R - r)$, when the frequency $\lambda = .02738$. The interest rate $r$ does not affect the values of $\alpha$, only the difference $(\mu_R - r)$, as can be seen from equation (iv). Note that the values of the relative risk aversion are surprisingly low, even for large risk premiums.

| $(\mu_R - r)$ | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 |
|---|---|---|---|---|---|---|---|---|
| $\alpha$ | 0 | 2.3 | 2.2 | 3.0 | 3.7 | 4.2 | 4.7 | 5.1 |

Table 1: The relative risk aversion $\alpha$ for given variances and covariances, for different values of $(\mu - r)$: $\lambda = .02738$.

Table 2 presents values of the subjective rate $\rho$ for various values of the risk premium $(\mu_R - r)$ and the equilibrium interest rate $r$, also for the frequency value $\lambda = .02738$.

| $(\mu_R - r)\|r$ | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 |
|---|---|---|---|---|---|---|---|---|
| .00 | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 |
| .01 | -.039 | -.029 | -.019 | -.009 | .001 | .011 | .021 | .031 |
| .02 | -.034 | -.024 | -.014 | -.004 | .005 | .0158 | .0259 | .0359 |
| .03 | -.043 | -.033 | -.023 | -.013 | -.003 | .007 | .017 | .027 |
| .04 | -.048 | -.038 | -.028 | -.018 | -.008 | .002 | .012 | .022 |
| .05 | -.052 | -.042 | -.032 | -.022 | -.012 | -.002 | .008 | .018 |
| .06 | -.054 | -.044 | -.034 | -.024 | -.014 | -.004 | .001 | .016 |
| .07 | -.054 | -.044 | -.034 | -.024 | -.014 | -.004 | .006 | .016 |

Table 2: The subjective interest rate $\rho$ for given variances and covariances, as a function of the risk premium and the interest rate. $\lambda = .02738$.

The feasible range of non-negative values for the subjective rate in Table 2 is shown in the upper right corner of the table. The negative values in the table are inconsistent with rational behavior, violating the inequality (23), but notice that their absolute values are relatively small.

The tables allow the reader to fill in their "favorite" pair of the risk premium $(\mu_R - r)$ and the risk-free interest rate $r$ and see what values of $\alpha$ and $\rho$ could be consistent with the low variances and covariances of consumption and equity of the last century.

For the particular case outlined in Table 1 and 2, we have a steady growth in both consumption and the equity index, but about three times during the century, on the average, a drop occurs in both these variables. The interpretation is that there is a growth in both the processes $c$ and $R$, measured by the positive drift terms in the corresponding dynamic equations, while the compensated jump terms, the noise terms, adjust the process parameters to the observed first and second sample moments. Compensating for the continuous upwards drift, jumps occur to align the processes to the the data. The jump sizes are downwards on the average, indicated by the signs of the parameters $\hat{\mu}_1 = -.2436$ and $\hat{\mu}_2 = -7.8567$.

The situation may be partly compared to the model by Rietz (1988). In his model the endowment growth rate is assumed to follow a three-state Markov process, two states are normal growth rate states while the third state is a crash state. Also Ritz was able, in a calibration exercise, to match the unconditional means maintaining the assumptions of time-separable iso-elastic preferences and relatively low values for agents' risk aversion. He did not, however, consider the riskless rate puzzle.

Our model is considerably more flexible than the one considered by Rietz, and our dynamics does not imply any "crash state", only some rare adjust-

ments. The results in our tables are consistent with the findings of Hansen and Jagannathan (1991), but we are able to explain more reasonable values of the relative risk aversion than the values they indicate. Our results are not consistent with those of Rietz (1988), nor with those of Salyer (1988). The latter tried to reconcile the results of the former two investigations, and in doing so he discovered a "new puzzle" in that his crash model, similar to that of Rietz, could not explain the observed volatility of excess returns. Our model is, in contrast, calibrated to the standard deviation of .165 in the sample period, which presents no problem.

Another, but perhaps related, matter is that Salyer did not really address the risk-free rate puzzle the way we do, since he fixed the agent's discount factor at .98, implying a positive value of the subjective rate $\rho = .0202$ in our setting. Fixing $\rho = .0202$, by examining our system of equations (i) - (v) in this situation, we typically get solutions with large risk aversion and/or negative values of some of the parameters that must be positive, like $\lambda$, $\sigma_1$, $\sigma_2$ or even $\alpha$.

## 3.2 Including also diffusion uncertainty

In this section we allow for diffusion uncertainty to enter in addition to the jumps. Below we consider a case where roughly one half of the standard deviations are attributed to the diffusion sample paths, the other half to the jump terms.

Here we fix the values of $\sigma_c = .02$, and $\sigma_R = .10$. In this case $\sigma_R\sigma_c = 0.002$ which is about one third of the total covariance rate 0.0059. The system of equations (i) - (v) is solved to yield $\lambda = .1761$, $\mu_1 = -.074$, $\mu_2 = -.376$, $\sigma_1^2 = .0001$, and $\sigma_2^2 = .002$, the latter two parameters were subject to trial and error. As before, all these parameter values are the same regardless of the values of $r$ and $(\mu_R - r)$. For various values of these latter quantities, we have computed the following two tables of $\alpha$ and $\rho$:

| $(\mu_R - r)$ | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 |
|---|---|---|---|---|---|---|---|---|
| $\alpha$ | 0 | 1.59 | 3.05 | 4.41 | 5.68 | 6.85 | 7.96 | 8.99 |

Table 3: The relative risk aversion $\alpha$ for given variances and covariances, for different values of $(\mu - r)$: $\lambda = .1761$.

In the above the frequency of jumps has risen to about one in every five years on the average.

By comparing these tables with Table 1 and 2, we notice that including the diffusion uncertainty in the present situation does not change the preference parameters very much. From Table 1 and Table 3 we see that smaller

| $(\mu_R - r)\|r$ | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 |
|---|---|---|---|---|---|---|---|---|
| .00 | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 |
| .01 | -.026 | -.016 | -.006 | .004 | .014 | .024 | .034 | .044 |
| .02 | -.047 | -.037 | -.027 | -.017 | -.007 | .003 | .013 | .023 |
| .03 | -.063 | -.053 | -.043 | -.033 | -.023 | -.013 | -.003 | .007 |
| .04 | -.075 | -.065 | -.055 | -.045 | -.035 | -.025 | -.015 | -.005 |
| .05 | -.084 | -.074 | -.064 | -.054 | -.044 | -.034 | -.024 | -.014 |
| .06 | -.090 | -.080 | -.070 | -.060 | -.050 | -.040 | -.030 | -.020 |
| .07 | -.092 | -.082 | -.072 | -.062 | -.052 | -.042 | -.032 | -.022 |

Table 4: The subjective interest rate $\rho$ for given variances and covariances, as a function of the risk premium and the interest rate, when diffusion uncertainty is also included. $\lambda = .1761$.

risk aversion can be explained for low values of the market risk premium in the combined case, while for high risk premiums the situation is reversed. Comparing Table 2 and Table 4, we see that the case of pure jumps gives a wider range of acceptable values of $\rho$ than the combined case, as can be observed from inspecting the upper right corners of the two tables.

The pair for the classical puzzles $(\mu_R - r, r) = (.06, .01)$ gives $\hat{\alpha} = 7.96$ and $\hat{\rho} = -.080$. For the pair $(\mu_R - r, r) = (.01, .04)$ we have $\hat{\alpha} = 1.59$ and $\hat{\rho} = .014$.

This basic trend can be brought further by excluding the jumps from the model entirely, in which case the risk aversion $\alpha$ would increase, and the acceptable range for the subjective rate $\rho$ is further diminished, a demonstration that the jump model is more flexible in this regard that the continuous one. The details can be found in tables 5 and 6.

| $(\mu_R - r)$ | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 |
|---|---|---|---|---|---|---|---|---|
| $\alpha$ | 0 | 1.70 | 3.40 | 5.09 | 6.79 | 8.49 | 10.19 | 11.88 |

Table 5: The relative risk aversion $\alpha$ for given variances and covariances, for different values of $(\mu - r)$: The continuous case.

Generally it is seen that an increase in the equilibrium interest rate $r$ typically increases $\rho$, and a decrease in the equilibrium risk premium $(\mu_R - r)$ similarly decreases the relative risk aversion $\alpha$, ceteris paribus. In the tables there is only one violation of this, as the risk aversion drops slightly from 2.34 to 2.23 as the risk premium increases from .01 to .02 in Table 1. Also an increase in the risk premium $(\mu_R - r)$ generally decreases the subjective rate $\rho$, the only exception in the tables again being in Table 2 when the risk

| $(\mu_R - r)\|r$ | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 |
|---|---|---|---|---|---|---|---|---|
| .00 | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 |
| .01 | -.030 | -.020 | -.010 | .002 | .012 | .022 | .032 | .042 |
| .02 | -.053 | -.043 | -.033 | -.023 | -.013 | -.003 | .007 | .017 |
| .03 | -.073 | -.063 | -.053 | -.043 | -.033 | -.023 | -.013 | -.003 |
| .04 | -.091 | -.081 | -.071 | -.061 | -.051 | -.041 | -.031 | -.021 |
| .05 | -.104 | -.094 | -.084 | -.074 | -.064 | -.054 | -.044 | -.034 |
| .06 | -.114 | -.104 | -.094 | -.084 | -.074 | -.064 | -.054 | -.044 |
| .07 | -.120 | -.110 | -.100 | -.090 | -.080 | -.070 | -.060 | -.050 |

Table 6: The subjective interest rate $\rho$ for given variances and covariances, as a function of the risk premium and the interest rate: The continuous case.

premium goes from .01 to .02.

It seems like the representative investor of the 20. century has been very rational (e.g., $\alpha = 1.59$) if the risk premium is around one per cent, in fact approaching the Kelly criterion of $\alpha = 1$. This is an interesting observation in its own right, since this criterion is known to have certain optimality properties, see i.a., Thorp (1971), Breiman (1960) or Aase and Øksendal (1988).

# 4 Conclusions

We have introduced jump dynamics in the "noise term" of the dynamic stochastic differential equations for the aggregate consumption process and the dividend processes of the risky assets. As a result, the equilibrium relations for the short rate and the risk premium could no longer be fully described by the two first moments only. We demonstrate that this gives some added flexibility in modelling, for example, it brings us outside the local mean-variance framework, permitting us to utilize other properties of a joint probability distribution than merely its two first moments.

The analysis revealed that the jump components in the model open up several possibilities related to the classical equity premium puzzle. For example could we imagine a non-prudent representative agent and a positive mixed cross-moment, or, and perhaps more realistic, we could imagine a prudent representative agent and a negative sign of a certain mixed cross-moment. Only an empirical investigation can, of course, resolve these issues.

In the latter case the model can also be related to the classical risk-free rate puzzle. A better fit could be obtained provided the consumer has a fourth derivative of the utility function satisfying a mixed preference and

moment requirement of the type

$$u^{(4)}(c_t, t) \int_Z (\theta(t, z)\gamma_c(t, z))^3 \nu(dz) > 0$$

and, preferably, not small in absolute value.

We have presented a range of values for the parameters of the representative agent's utility function for different values of risk premia and interest rates. It turns out that the McGrattan and Prescott (2003) pair of values for the difference between average equity and debt returns of the last century, and the average real debt return, fits nicely into the permitted range. If these values can be trusted, both puzzles are resolved at one stroke, both for the continuous model and for the model containing jumps.

In general the model can be used to indicate fairly reasonable values for the relative risk aversions when calibrated to the data of the last century. However, the subjective rates are negative for the Mehra and Prescott (1985) pair of values for the risk premium and the the average real debt return.

We have carried out a rather crude calibration, and many refinements could easily be imagined. It is likely, for example, that one could improve the results by choosing other distributions for the stock index, like the Normal Inverse Gaussian distribution or variants thereof, that are popular in parts of the extant empirical work on stock price returns. Despite these possible shortcomings, we feel we have demonstrated that the Lucas type equilibrium model with a time additive and separable utility function for the representative consumer still represents a viable framework for many types of economic analyses, especially seen in light of the new interpretation of the data of the last century.

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