Financial Economics

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Abstract

We consider a one period (two time points-) model of efficient risk sharing, when the set of possible sharing rules are constrained to be linear. This can be interpreted as a model of a market for common stocks. Here we study the properties of a competitive equilibrium in an incomplete market.

The lack of Pareto optimality is then the typical case. We do characterize, however, the situations where the competitive financial equilibrium is also Pareto optimal, and illustrate by examples.

Since the marketed subspace M is a closed, linear subspace of L^2, we employ Hilbert space techniques in finding the first order conditions.

We conclude with a discussion of the different features of idiosyncratic risks in insurance, and risks in financial markets, where a common ground is suggested.

KEYWORDS: Incomplete Financial Market, Competitive Equilibrium, Pareto Optimality, Representative Agent, Marketed Subspace

Introduction

Much of the theory of optimal allocation of risks in a reinsurance market can be directly applied to a stock market. The principal difference from the insurance risk exchange model is that only linear risk sharing is allowed in a market for common stocks. In certain situations this may also be Pareto optimal, but by and large this type of risk sharing is not. Still, it is quite plausible that a competitive equilibrium may exist.

Today the modelling framework in continuous time seems to be changing from Ito price processes to price paths containing unpredictable jumps, in which case the model typically becomes incomplete. One could, perhaps, call this a change form "linear" modelling of uncertainty to "nonlinear" uncertainty revelation. What I have in mind here is the much more involved nature of the corresponding random measure behind the jump process term, than the corresponding diffusion term, arising in the stochastic differential equation. Being much more complex, including a random measure facilitates possibilities for far better fits to real observations than does a mere diffusion term. On the more challenging side is the resulting incompleteness of the financial model. Many of the issues of the present paper then inevitably arise.

Classical economics sought to explain the way markets coordinate-ordinate the activities of many distinct individuals each acting in their own self-interest. An elegant synthesis of two hundred years of classical thought was achieved by the general equilibrium theory. The essential message of this theory is that when there are markets and associated prices for all goods and services in the economy, no externalities or public goods and no informational asymmetries or market power, then competitive markets allocate resources efficiently.

The focus of the paper is on understanding the role and functioning of the financial markets, and the analysis is confined to the one period model. The key to the simplicity of this model is that it abstracts from all the complicating elements of the general model except two, which are taken as primitive for each agent, namely his preference ordering and an exogenously given future income. The preference ordering represents the agent's attitude towards the variability of an uncertain consumption in the future (his risk aversion). The characteristics of the incomes is that they are typically not evenly distributed across the uncertain states of nature. A financial contract is a claim to a future income - hence the logic of the financial markets: by exchanging such claims agents change the shape of their future income, obtaining a more even consumption across the uncertain contingencies. Thus the financial markets enable the agents to move from their given income streams to income streams that are more desired by them, according to their preferences. The reason that they could not do this transfer directly is simply that there are no markets for direct exchange of contingent consumption goods.

We start by giving the relevant definitions of the financial model to be studied. Then we refer to the ideal or reference model (the Arrow - Debreu model) in which, for each state $\omega \in \Omega$, there is a claim which promises to pay one unit of account in the specified state. Trading in these primitive claims leads to equilibrium prices $(\xi(\omega))$, which are present values at date 0 of one unit of income in each state at date 1. Since agents in solving their optimum problems are led to equalize their marginal rates of substitution with these prices, the equilibrium allocation is Pareto optimal.

However, Arrow-Debreu securities does not exist in the real world, but common stocks do, together with other financial instruments. The purpose of these various instruments is thus to transform the real market as close to the ideal one as possible.

We introduce a class of financial contracts (common stocks), in which each contract promises to deliver income in several states at date 1, and where there may not be enough securities to span all the states at this date. Two ideas are studied which are crucial to the analysis that follows:

(i) the characterization and consequences of no arbitrage

(ii) the definition and consequences of incomplete financial markets.

We demonstrate in particular how security prices are determined in equilibrium such that agents, in solving their optimum problems, are led to equalize the *projections* of their marginal rates of substitution in the subset where trade of common stocks takes place.

The Financial Model

Consider the following model. We are given I individuals having preferences for period one consumption represented by expected utility, where the utility indices are given by u_i, where $u_i' > 0$, $u_i'' \leq 0$ for all $i \in \mathcal{I} =: \{1, 2, \ldots, I\}$. There are N securities, where Z_n is the payoff at time 1 of security n, $n = 1, 2, \ldots, N$. Let $Z = (Z_1, Z_2, \ldots, Z_N)'$ where prime denotes the transpose of a vector, i.e., Z is a random (column) vector. We use the notation $\sum_{n=1}^{N} Z_n =: Z_M$ for the "market portfolio". We consider a one period model with two time points 0 and 1, one consumption good, and consumption only at the final time point 1.

We suppose individual i is initially endowed with shares of the different securities, so his pay-off at date 1 of his initial endowment is

$$X_i = \sum_{n=1}^{N} \bar{\theta}_n^{(i)} Z_n,$$

where $\bar{\theta}_n^{(i)}$ is the proportion of firm n held by individual i. In other words, the total supply of a security is one share, and the number of shares held by an individual can be interpreted as the proportion of the total supply held. Denote by p_n the price of the security n, $n = 1, \ldots, N$, where $p = (p_1, p_2, \ldots, p_N)'$. We are given the space $L^2 = L^2(\Omega, \mathcal{F}, P)$ where L_+^2 is the non-negative part (the positive cone) of L^2, Ω is the set of states of the world, $\mathcal{F}$ is the set of events, a σ-algebra, and $P : \mathcal{F} \to [0,1]$ is the probability measure common to all the agents.

Consider the following budget set of agent i:

$$B_i^F(p;\bar{\theta}) = \{Y_i \in L_+^2 : Y_i = \sum_{n=1}^{N} \theta_n^{(i)} Z_n, \text{ and } \sum_{n=1}^{N} \theta_n p_n = \sum_{n=1}^{N} \bar{\theta}_n^{(i)} p_n\} \tag{1}$$

Here $\theta_n^{(i)} \in R$, so from the range of these parameters we notice that negative values, i.e. short selling, is allowed.

An *equilibrium* for the economy $[(u_i, X_i), Z]$ is a collection $(\theta^1, \theta^2, \ldots, \theta^I;$ $p)$ such that given the security prices p, for each individual i, θ^i solves

$$\sup_{Y_i \in B_i^F(p;\bar{\theta})} Eu_i(Y_i) \tag{2}$$

and markets clear: $\sum_{n=1}^{N} \theta_n^{(i)} = 1$.

Denote by $M = \text{span}(Z_1, \ldots, Z_N) =: \{\sum_{n=1}^{N} \theta_n Z_n;\ \sum_{n=1}^{N} \theta_n \leq 1\}$, the set of all possible portfolio payoffs. We call M the *marketed subspace* of L^2. Here $\mathcal{F} = \mathcal{F}^Z =: \sigma\{Z_1, Z_2, \ldots, Z_I\}$ (all the null sets are included). The markets are *complete* if $M = L^2$ and are otherwise *incomplete*.

Here we remark that a common alternative formulation of this model starts out with pay-off at date 1 of the initial endowments X_i measured in units of the consumption good, but there are no outstanding shares, so that the clearing condition is $\sum_{i=1}^{I} \theta_n^{(i)} = 0$ for all n. In this case we would have $\mathcal{F} = \mathcal{F}^X$. More generally we could let the initial endowments consist of shares and other types of wealth, in which case $\mathcal{F} = \mathcal{F}^{X,Z}$.

If there is uncertainty in the model not directly reflected in the prices and initial endowments, $\mathcal{F} \supset \mathcal{F}^{X,Z}$. Then we ought to specify these sources of uncertainty in the model.

Arrow securities and complete markets

Let us consider the ideal model of Arrow and Debreu (1954), and assume for expository reasons that there is a finite number of states: $\Omega = \{\omega_1, \omega_2, \ldots, \omega_S\}$. Denote the $N \times S$ payout matrix of the stocks by $\mathbf{Z}$, where

$$\mathbf{Z} = \begin{pmatrix} z_{1,\omega_1} & z_{1,\omega_2} & \cdots & z_{1,\omega_S} \\ z_{2,\omega_1} & z_{2,\omega_2} & \cdots & z_{2,\omega_S} \\ \vdots & \vdots & \ddots & \vdots \\ z_{N,\omega_1} & z_{N,\omega_2} & \cdots & z_{N,\omega_S} \end{pmatrix}$$

and z_{n,ω_s} is the payout of common stock n in state ω_s. If $N = S$ and $\mathbf{Z}$ is nonsingular, then markets are complete. It is sufficient to show that Arrow securities can be constructed by forming portfolios of common stocks. Since $\mathbf{Z}$ is nonsingular we can define

$$\theta^{(\omega_s)} = e^{(\omega_s)}\mathbf{Z}^{-1}$$

where $e^{(\omega_s)} = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 at the s-th place. Then $\theta^{(\omega_s)}\mathbf{Z} = e^{(\omega_s)}$ by construction. The portfolio $\theta^{(\omega_s)}$ tells us how many shares of each common stock to hold in order to create an Arrow security that pays "one unit of account" in state ω_s. It is obvious that as long as $\mathbf{Z}$ is nonsingular, we can do this for each $\omega_s \in \Omega$. Hence a complete set of Arrow securities can be constructed, and then we know that the market structure is complete.

In the one period case markets can not be complete if the random payoffs Z have continuous distributions, or if there is an infinite and countable number of states, cases that interest us. In the finite case, the market can not be complete if the rank of $\mathbf{Z}$ is strictly less than S, the number of states. It is easy to find examples in the finite case where options can complete an otherwise incomplete model (see e.g. Ross (1976), Aase (2002)).

In continuous time models with a finite set of long-lived securities, a redefinition of the concept of Arrow-securities may lead to *dynamically* complete markets, even if the payoffs are continuously distributed, as is the case for e.g., the Black and Scholes model.

<u>Example 1.</u> Suppose $S = 3$ and $N = 2$, and let the payoff matrix Z be given by

$$\mathbf{Z} = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

Here $\text{rank}(Z) = 2 < 3$, and the market is not complete. The payoff of the market portfolio is $(3, 4, 5)$. Let $c_M(k)$ denote the price at date 0 of a European call option on the market portfolio expiring at date 1 with an exercise price k. The payoffs for $c_M(3)$ and $c_M(4)$ are $(0, 1, 2)$ and $(0, 0, 1)$. Putting these payoffs together with the market portfolio, we have the payoff structure

$$\begin{pmatrix} 3 & 4 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

which is a nonsingular matrix. Arrow securities can then be constructed by forming portfolios of the market portfolio and the two call options, so this market structure is complete. $\square$

This example shows a situation where options can play an allocative role, and thus be welfare improving. More generally one can show the following: In an economy where options can freely be created on portfolios of common stocks, the market is Arrow complete if and only if there exists a portfolio of common stocks where payoffs are different in each state, or where payoffs separate (Ross (1976)).

Some general pricing principles

We now consider some general pricing principles. Let there be a stock market in a single good, single period economy. Agents have von Neumann-Morgenstern strictly concave and strictly increasing utility functions. Returning to the problem (2), we substitute the first constraint into the objective function and form the Lagrangian of each individual's optimization problem:

$$\mathcal{L}_i(\theta) = E\{u_i(\sum_{n=1}^{N} \theta_n^{(i)} Z_n) - \alpha_i(\sum_{n=1}^{N} p_n(\theta_n^{(i)} - \bar{\theta}_n^{(i)})\}.$$

The first order conditions are

$$\frac{\partial \mathcal{L}_i(\theta)}{\partial \theta_n^{(i)}} = E(u_i'(Y_i)Z_n) - \alpha_i p_n = 0,$$

implying that

$$p_n = \frac{1}{\alpha_i} E(u_i'(Y_i)Z_n), \qquad n = 0, 1, \ldots, N.$$

Defining $R_n = Z_n/p_n$, the return of asset n, we have that for each $i \in \mathcal{I}$

$$\frac{1}{\alpha_i} E\left(u_i'(Y_i)(R_n - R_m)\right) = 0, \qquad \forall n, m,$$

or, by the definition of covariance,

$$\frac{1}{\alpha_i} E(u_i'(Y_i))E(R_n - R_m) + \frac{1}{\alpha_i}\mathrm{cov}(u_i'(Y_i), R_n - R_m) = 0 \qquad \forall\, n, m, \tag{3}$$

hold for each $i \in \mathcal{I}$.

Suppose there exists a riskless asset, the 0-th asset, that promises to pay one unit of the consumption good at date 1 in all states $\omega \in \Omega$. This asset is assumed to be in zero net supply. Thus

$$p_0 = \frac{1}{\alpha_i} E(u_i'(Y_i) \cdot 1) =: \frac{1}{R_0} =: \frac{1}{1 + r_f} \qquad \text{for all} \qquad i \in \mathcal{I},$$

where r_f denotes the riskfree interest rate. Combining this with equations (3) gives

$$\frac{1}{1+r_f}E(R_n - R_m) + \frac{1}{\alpha_i}\mathrm{cov}(u_i'(Y_i), R_n - R_m) = 0 \qquad \forall\, n, m, \tag{4}$$

for all $i \in \mathcal{I}$. Set $m = 0$ in this relationship. Then (4) becomes

$$E(R_n) - (1+r_f) = -(1+r_f)\mathrm{cov}\left(\frac{u_i'(Y_i)}{\alpha_i}, R_n\right), \qquad \forall\, n, \tag{5}$$

saying that the risk premium of any asset in equilibrium is proportional to the covariance between the return of the asset and the normalized, marginal utility of the equilibrium allocation Y_i for any i of the individuals. This latter quantity one may conjecture to be equal on M across all the individuals in equilibrium. We shall look into this conjecture below, but first we may utilize the relation (5) to derive the Capital Asset Pricing Model (CAPM).

CAPM derived under multinormality

The results of the previous section can now be utilized to derive the standard CAPM. Two avenues could be chosen: One is to assume that all the individuals possess quadratic utility functions. This we do not find plausible in financial economics, where the utility is taken over final consumption, which in a one period model equals final wealth. It is highly unlikely to have a satiation point when it comes to wealth.

The other is to assume that returns of common stocks are multinormally distributed. This means that the model becomes infinite dimensional, and consequently incomplete. Fama (1976) in his book "Foundations of Finance", and numerous other authors have repeatedly tested out this hypothesis on US stocks, and found the assumption acceptable under certain conditions. This assumption is often employed in theoretical models in finance, such as e.g., the Black and Scholes model, but is frequently refuted in empirical studies. For the moment, let us assume that R is multivariate normal, and thus that Z is multivariate normal, since the prices p of the common stocks are all constants at time 0. Using Stein's lemma, from (5) we get that

$$E(R_n) - (1+r_f) = -(1+r_f)E\left(\frac{u_i''(Y_i)}{\alpha_i}\right)\mathrm{cov}(R_n, Y_i), \qquad \forall\, n, i. \tag{6}$$

Let $Z_M =: \sum_{n=1}^N Z_n$ and $p_M =: \sum_{n=1}^N p_n$ and consider the weights $w_n =: p_n/p_M$ for $n = 1, 2, \ldots, N$. Clearly $\sum_{n=1}^N w_n = 1$. By the definition of return, $R_M =: Z_M/p_M$ signifies the return on the market portfolio, and it follows that this can be written $R_M = \sum_{n=1}^N w_n R_n$, i.e., R_M is the return on the value-weighted market portfolio. Multiplying (6) by w_n and summing over the stocks n we get

$$E(R_M) - (1 + r_f) = -(1 + r_f)E\left(\frac{u_i''(Y_i)}{\alpha_i}\right) \operatorname{cov}(R_M, Y_i), \qquad \forall\, i.$$

Rearranging this equation, summing over the individuals i, and noticing that $\operatorname{cov}(R_M, Z_M) = p_M \operatorname{var}(R_M)$, we obtain using the market clearing condition

$$(E(R_M) - (1 + r_f)) \sum_{i \in \mathcal{I}} \frac{\alpha_i}{Eu_i''(Y_i)} = -(1 + r_f)p_M \operatorname{var}(R_M). \tag{7}$$

Returning to equation (6), rearranging and summing over the individuals, using again the market clearing condition, we get

$$(E(R_n) - (1 + r_f)) \sum_{i \in \mathcal{I}} \frac{\alpha_i}{Eu_i''(Y_i)} = -(1 + r_f)p_M \operatorname{cov}(R_n, R_M). \tag{8}$$

Finally, we substitute the term $\sum_{i \in \mathcal{I}} \frac{\alpha_i}{Eu_i''(Y_i)}$ from equation (7) into equation (8), and the result is:

$$E(R_n) - (1 + r_f) = \frac{\operatorname{cov}(R_n, R_M)}{\operatorname{var}(R_M)}(E(R_M) - (1 + r_f)), \qquad \forall\, n. \tag{9}$$

The risk premium of any of the given common stocks, $(E(R_n) - (1 + r_f))$, is proportional to the corresponding risk premium of the market, $(E(R_M) - (1 + r_f))$, where the constant of proportionality $\beta_n := \operatorname{cov}(R_n, R_M)/\operatorname{var}(R_M)$ is called the stock's beta. This is the traditional version of the CAPM due to Mossin, Lintner and Sharp. Note that we needed no completeness assumption for this relationship to hold.

Let $R_\theta = \sum_{n=1}^N \theta_n R_n$ be the return on any portfolio of common stocks, where the portfolio weights satisfy $\sum_{n=1}^N \theta_n = 1$. Then, from the above it is trivial to see that

$$E(R_\theta) - (1 + r_f) = \beta_\theta(E(R_M) - (1 + r_f)), \tag{10}$$

where $\beta_\theta := \operatorname{cov}(R_\theta, R_M)/\operatorname{var}(R_M)$ is the portfolio's beta. Since only portfolio formation can be made in this market, we here see a difference between this version and the corresponding insurance version (Aase (2002)). Note that in this model the budget set of agent i is

$$B_i^{FU}(p; \bar\theta) = \{Y_i \in L^2 : Y_i = \sum_{n=1}^N \theta_n Z_n, \sum_{n=1}^N \theta_n p_n = \sum_{n=1}^N \bar\theta_n^{(i)} p_n\}, \tag{11}$$

instead of the more common $B_i^F(p;\bar{\theta})$ given in equation (1).

This means that the utility functions $u_i(\cdot)$ must be defined over all of R, not only on R_+, and the resulting situation allows for bankruptcy.

Existence of mean variance equilibrium

The problem of existence of equilibrium is, perhaps surprisingly, only dealt with fairly recently (Nielsen (1987, 1988, 1990a,b), Allingham (1991), Dana (1999)). Instead of assuming multinormality as we did in the above, a common assumption in this literature is that the preferences of the investors only depend on the mean and the variance, in other words, if $Z \in M$, then a utility function $u_i : M \to R$ is mean variance if there exists $U_i : R \times R \to R$ s.t.,

$$u_i(Z) = U_i(E(Z), \mathrm{var}(Z)) \qquad \text{for all } Z \in M.$$

The function U_i is assumed strictly concave and C^2, increasing in its first argument and decreasing in the second.

We then have the following result (Dana (1999):

Theorem 1 *Assume that $E(X_i) > 0$ for every $i = 1, 2, \ldots, I$ and Z_M is a non-trivial random variable (i.e., not equal to a constant a.s.). Then there exists an equilibrium.*

When utilities are linear in mean and variance, we talk about quadratic utility, i.e., $U_i(x, y) = x - a_i y$, $a_i > 0$ for every i. If this is the case, equilibrium both exists and is unique. In the above it was assumed that utilities were strictly concave, so quadratic utility only fits into the above framework as a limiting case.

Let us recall one definition of risk aversion: A preference relation $\succeq$ on a subset M of L^2 is called risk averse if $X \succeq X + Y$ for any $X \in M$ and non-zero Y in L^2 satisfying $X + Y \in M$ and $E(Y \mid X) = 0$. This means that an agent is risk averse if the addition of a random prospect that has no incremental effect on expected value is undesirable.

A related concept is the following: A preference relation $\succeq$ on a subset M of L^2 is variance averse if $X \succeq X + Y$ whenever X and $X + Y$ are in M and $EY = \mathrm{cov}(X, Y) = 0$. This means that an increase in variance is disliked if it does not affect expected value. In this case quadratic utility *is* a special case of a variance averse preference relation.

Suppose that the vector space M has a Hamel basis of jointly normally distributed random variables. If $\succeq$ is a risk averse preference relation on M, it follows that $\succeq$ is variance averse. In verifying this, we notice that if X and

Y are bivariate normally distributed, then $E(XY) = EX = 0$ implies that $E(Y \mid X) = 0$.

In these two examples variance aversion applies because the agent's preferences are given only in terms of means and variances of an asset, and for a given mean, more variance is worse. However, nothing in the definition of variance aversion requires that preferences depend only on mean and variance.

No arbitrage restrictions on expected returns.

Instead of relying on the rather restrictive assumptions behind the CAPM, we now indicate a similar relationship assuming only the existence of a state price deflator. For a finite version of the following, see Duffie (2001). First we recall some facts.

The principle of no-arbitrage may be used as the motivation behind a linear pricing functional, since any insurance contract can be perfectly hedged in the reinsurance market. In the standard reinsurance model there is an assumption of arbitrary contract formation. We use the following notation. Let X be any random variable. Then by $X > 0$ a.s. we now mean that $P[X \geq 0] = 1$ and the event $\{\omega : X(\omega) > 0\}$ has strictly positive probability. In the present setting, by an *arbitrage* we mean a portfolio θ with $p \cdot \theta \leq 0$ and $\theta \cdot Z > 0$, or $p \cdot \theta < 0$ and $\theta \cdot Z \geq 0$ a.s. Then we have the following version of "The Fundamental Theorem of Asset Pricing": *There is no arbitrage if and only if there exists a state price deflator.* This means that if there exists a strictly positive random variable $\xi \in L^2_{++}$, i.e., $P[\xi > 0] = 1$, such that the market price $p_\theta := \sum_{n=1}^N \theta_n p_n$ of any portfolio θ can be written

$$p_\theta = \sum_{n=1}^{N} \theta_n E(\xi \cdot Z_n),$$

there can be no arbitrage, and conversely (see e.g., Dalang, Morton and Willinger (1990)).

The extension of this theorem to a discrete time setting is true and can be found in standard texts (see e.g., Duffie (2001)). In continuous time the situation is more complicated, see e.g., Kreps (1981) or Schachermayer (1992). If we assume that the pricing functional π is linear, and in addition positive, i.e., $\pi(Z) \geq 0$ if $Z \geq 0$ a.s., both properties being a consequence of no arbitrage, then we can use the Riesz' representation theorem, since a positive linear functional on an L^2-space is continuous, in which case we obtain the above representation. If we add the assumption of strict positivity of π, also a direct consequence of no arbitrage possibilities, the result is a strictly positive Riesz' representation ξ.

The following result is also useful: *If there exists a solution to at least one of the optimization problems (2) of the agents, then there is no arbitrage.* (Ross (1978)). The conditions on the utility functional may be relaxed considerably for this result to hold. Consider a strictly increasing utility function U: $L^2 \rightarrow R$. If there is a solution to (2) for at least one such U, then there is no arbitrage. The utility function $U : L^2 \rightarrow R$ we use is $U(X) = Eu(X)$. Also if U is continuous and there is no arbitrage, then there is a solution to the corresponding optimization problem.

Clearly, the no-arbitrage condition is a weaker requirement than the existence of a competitive equilibrium, so if an equilibrium exists, there can be no arbitrage.

For any portfolio θ, let the return be $R_\theta = Z_\theta / p_\theta$, where $Z_\theta = \sum_{n=1}^N \theta_n Z_n$, and $p_\theta = \sum_{n=1}^N \theta_n p_n$. We suppose there is no arbitrage, and that the linear pricing functional π is strictly positive. Then there is, by Riesz' representation theorem, a state price deflator $\xi \in L_{++}$ (by strict positivity). We easily verify that

$$E(\xi R_\theta) = \frac{1}{p_\theta} E(\xi \sum_{n=1}^N \theta_n Z_n) = 1 \qquad (12)$$

Suppose as above that there is there is a riskfree asset. It is then the case that

$$E(R_\theta) - R_0 = \beta_\theta (E(R_{\theta^*}) - R_0), \qquad (13)$$

where

$$\beta_\theta = \frac{\mathrm{cov}(R_\theta, R_{\theta^*})}{\mathrm{var}(R_{\theta^*})},$$

and where the portfolio θ^* solves the following problem

$$\sup_\theta \rho(\xi, Z_\theta), \qquad (14)$$

where ρ is the correlation coefficient. Indeed, here $\rho(\xi, Z_{\theta^*}) = 1$. The existence of such a θ^* follows as in Duffie (2001).

We notice that the portfolio θ^* having maximal correlation with the state price deflator ξ plays the same role in the relation (13) as the market portfolio plays in the CAPM of relation (10). The right hand side of (13) can be thought of as the risk adjustment in the expected return of the portfolio θ.

The advantage with the present representation is that it does not require the rather restrictive assumptions underlying the CAPM.

In order to price any portfolio, or security, we get by definition that $E(R_\theta) = E(Z_\theta)/p_\theta$, or

$$p_\theta = \frac{E(Z_\theta)}{E(R_\theta)} \tag{15}$$

In order to find the market value of the portfolio θ, one can compute the ratio on the right hand side of (15). The numerator requires the expected payout, the denominator the expected return of the portfolio. In computing the latter, (13) may be used. It amounts to find the expected, risk adjusted return of the portfolio (security), which one has been accustomed to in finance since the mid 1960's. The method is still widely used in practice, and can find further theoretical support in the above derivation (beyond that of the CAPM).

This in contrast to the more modern contingent claims valuation theory, where one instead risk adjusts the numerator in (15) $E^Q(Z_\theta)$, through a risk adjusted probability measure Q, equivalent to the given probability measure P, and then use the riskfree interest rate R_0 in the denominator, i.e., $p_\theta = E^Q(Z_\theta)/R_0$. Here $dQ/dP = \eta$ and $\eta = \xi R_0$. Both methods require the absence of arbitrage, and the existence of a state price deflator. Which method is the simplest to apply in practice, depends on the situation.

Incomplete models and allocation efficiency

In this section we elaborate on the incomplete case. Consider a model where an equilibrium exists, so that there is no arbitrage, and hence there is a strictly positive state price deflator $\xi \in L^2_{++}$. Recall the optimization problem of the standard risk sharing model in insurance. If $(\pi; Y_1, \dots, Y_I)$ is a competitive equilibrium in the reinsurance model, where $\pi(V) = E(V \cdot \xi)$ for any $V \in L^2$, then there exists a nonzero vector of agent weights $\lambda = (\lambda_1, \dots, \lambda_I)$, $\lambda_i \geq 0$ for all i such that the equilibrium allocation $(Y_1, \dots, Y_I)$ solves the problem

$$Eu_\lambda(Z_M) =: \sup_{(V_1,\dots,V_I)} \sum_{i=1}^{I} \lambda_i E u_i(V_i) \qquad \text{subject to} \qquad \sum_{i=1}^{I} V_i \leq Z_M,$$

where $V_i \in L^2$, $i \in \mathcal{I}$. Here $\lambda_i = \frac{1}{\alpha_i}$, where α_i are the Lagrangian multipliers of the individual optimization problems of the agents. For u_i concave and increasing for all i, we know that solutions to this problem also characterizes the Pareto optimal allocations as $\lambda \geq 0$ varies.

Suppose now that a competitive financial equilibrium exists in M. Then there exists a nonzero vector of agent weights $\lambda = (\lambda_1, \ldots, \lambda_I)$, $\lambda_i \geq 0$ for all i such that the equilibrium allocation $(Y_1, \ldots, Y_I)$ solves the problem

$$E\tilde{u}_\lambda(Z_M) := \sup_{(V_1,\ldots,V_I)} \sum_{i=1}^{I} \lambda_i E u_i(V_i) \qquad \text{subject to} \qquad \sum_{i=1}^{I} V_i \leq Z_M, \quad (16)$$

where $V_i \in M$, $i \in \mathcal{I}$. The relation between the λ_i and α_i is the same as in the above. The first order conditions are

$$E\{(\tilde{u}'_\lambda(Z_M) - \alpha\xi)Z\} = 0 \qquad \text{for all} \qquad Z \in M,$$

where $\alpha > 0$ is a Lagrangian multiplier. This rives rise to the pricing rule

$$\pi(Z) = \frac{1}{\alpha} E(\tilde{u}'_\lambda(Z_M) \cdot Z) = E(\xi \cdot Z) \qquad \text{for all} \qquad Z \in M.$$

Similarly, for the problem in (2) the first order conditions can be written

$$E\{(u'_i(Y_i) - \alpha_i \xi)Z\} = 0 \qquad \text{for all} \qquad Z \in M, \qquad i = 1, 2, \ldots, I,$$

where Y_i are the optimal portfolios in M for agent i, $i = 1, 2, \ldots, I$, giving rise to the market value

$$\pi(Z) = \frac{1}{\alpha_i} E(u'_i(Y_i) \cdot Z) = E(\xi \cdot Z) \qquad \text{for any} \qquad Z \in M, \, i \in \mathcal{I}.$$

Let us use the notation

$$\tilde{\xi} = \frac{\tilde{u}'_\lambda(Z_M)}{\alpha}, \qquad \xi_i = \frac{u'_i(Y_i)}{\alpha_i}, \qquad i = 1, 2, \ldots, I.$$

Since M is a closed, linear subspace of the Hilbert space L^2, if $M \neq L^2$ then the model is incomplete. In this case there exists an X in L^2, $X \neq 0$, such that $E(X \cdot Z) = 0$ for all $Z \in M$. We use the notation $X \perp Z$ to signify $E(X \cdot Z) = 0$, and say that X in orthogonal to Z. Also let $M^\perp$ be the set of all X in L^2 which are orthogonal to all elements Z in M. There exists a unique pair of linear mappings T and Q such that T maps L^2 into M, Q maps L^2 into $M^\perp$, and

$$X = TX + QX$$

for all $X \in L^2$. The orthogonal projection TX of X in M is the unique point in M closest (in L^2-norm) to X. If $X \in M$ then $TX = X$, $QX = 0$;

if $X \in M^{\perp}$, then $TX = 0$, $QX = X$. We now simplify the notation to $TX = X^T$ and $QX = X^Q$ for any $X \in L^2$.

Using this notation, from the above first order conditions we have that

$$(\xi - \tilde{\xi}) \perp M \qquad \text{and} \qquad (\xi - \xi_i) \perp M, \qquad i = 1, 2, \ldots, I.$$

In other words $(\xi - \tilde{\xi}) \in M^{\perp}$ and $(\xi - \xi_i) \in M^{\perp}$ for all i and accordingly $(\xi - \tilde{\xi})^T = 0$ and $(\xi - \xi_i)^T = 0$ for all i, so the orthogonal projections of ξ, $\tilde{\xi}$ and ξ_i, $i = 1, 2, \ldots, I$ on the marketed subspace M are all the same, i.e.,

$$\xi^T = \tilde{\xi}^T = \xi_i^T, \qquad i = 1, 2, \ldots, I. \tag{17}$$

Thus we have shown the following

Theorem 2 *Suppose an equilibrium exists in the incomplete financial model. Then security prices are determined in equilibrium such that agents, in solving their optimum problems, are led to equalize the projections of their marginal rates of substitution in the marketed subspace M of L^2, the projections being given by the equations (17).*

The conditions $\xi^T = \xi_i^T$ for all i correspond to the first order necessary conditions $\xi = \xi_i$ for all i of an equilibrium in the standard reinsurance model, when trade in all of L^2 is unrestricted, and similarly the condition $\xi^T = \tilde{\xi}^T$ corresponds to the first order necessary condition $\xi = \frac{1}{\alpha} u_\lambda(Z_M)$ of the corresponding unrestricted, representative agent equilibrium.

Notice that there is an analogue to the above in the finite dimensional case, saying that if a financial market equilibrium exists, then the equilibrium allocation is *constrained* Pareto optimal (i.e., the optimal allocations are constrained to be in the marketed subspace M) (see Magill and Quinzii (1996), Theorem 12.3).

The law of demand

One may formulate the "law of demand" in the complete model as saying something like: "As the total abundance goes up, the price goes down". What we mean here is that as the "abundance" $Z_M(\omega) = z_M$ increases, the state price $u'_\lambda(z_M)$ decreases. Here we may think of z_M as being a real variable, or one could perhaps take the expectation of the state price deflator, and think in terms of first degree stochastic dominance. In either case, the result follows since the real function $u'_\lambda(\cdot)$ is strictly decreasing. Will the same hold true in the incomplete world? Well, take a look at the optimization problem (16). It is still the result of a sup-convolution, and since the individual utility functions are assumed to possess decreasing marginal utility, the same will be the case for the real function $\tilde{u}'_\lambda(\cdot)$. When defined, this real function will simply coincide with the real function $u'_\lambda(\cdot)$. The reason that the expected value $E\tilde{u}_\lambda(Z_M)$ may be smaller that the corresponding expected value $Eu_\lambda(Z_M)$ is that $M \subset L^2$, i.e., the domain of the optimization problem is now a smaller set as compared to the domain of the unconstrained problem. We thus have the following result:

Theorem 3 *As the random variable Z_M increases in first degree stochastic dominance, the expected state price $E\tilde{u}'_\lambda(Z_M)$ decreases.*

This can be related to a more serious result about monotonic demand in economics: By monotonic demand we mean that the demand function $f(p, w)$ of a price vector p (of l commodities, say) and income w satisfies

$$(p - p') \cdot (f(p, w) - f(p', w)) < 0.$$

whenever $p \neq p'$. In this setting both the income effect and the substitution effect of a price increase is taken into account, and monotonicity in demand imply that the latter is the dominating effect (e.g., Quah (2003)).

Applied to the economics of uncertainty, there is a result saying that the corresponding demand is monotonic if the coefficient of relative risk aversion $\rho_i =: -x_i u''_i(x_i)/u'_i(x_i)$ is between zero and 4 for all i. This holds in the infinite dimensional case, and in incomplete markets, see e.g., Dana (1995) and Bettzuge (1998).

Pareto optimality

If an equilibrium exists and $M = L^2$, then $\xi = \tilde{\xi}$ and the equilibrium allocation $(Y_1, \dots, Y_I)$ is Pareto optimal. In this situation contingent claims in zero net supply would not have any allocative effects, in other words, such financial instruments would not be welfare improving.

If $M \neq L^2$ the market is incomplete, and two situations can arise:
(a) $E\tilde{u}_\lambda(Z_M) = Eu_\lambda(Z_M)$ or (b) $E\tilde{u}_\lambda(Z_M) < Eu_\lambda(Z_M)$ [1].

In situation (b) the equilibrium allocation is not Pareto optimal, but is *constrained Pareto optimal*, which is the typical case. Welfare could hence be improved by allowing trade in non-linear financial instruments (in zero net supply). The difference

$$(Eu_\lambda(Z_M) - E\tilde{u}_\lambda(Z_M)) \geq 0,$$

[1] The λ's in these two functions may not be the same.

can be considered as the welfare loss due to the incompleteness of the market. In case (a) this loss is simply zero. In this situation the "welfare function" $E\tilde{u}_\lambda(Z_M)$ is equal to its maximal value, the value it would obtain if trade in all of L^2 was permitted (or possible). By standard, neoclassical economics the equilibrium allocation is then Pareto optimal. Thus, even if the market is incomplete, there is no loss of welfare in restricting attention to the marketed subspace M. If this is the case we call the market *allocation efficient* (e.g., Rubinstein (1974)). Here we face the same situation as for a complete market: Contingent claims in zero net supply would not improve welfare.

One interesting issue would be to design the minimum set of derivatives required in order to complete the model in case (b). We know from Hart's (1975) investigation that it is simply not enough to introduce more assets. If this does not result in a complete model, welfare may indeed *decrease* after such an introduction. Although this may not be the typical case, Hart was able to construct examples of this, in the finite dimensional case.

There is an interesting, general result on risk sharing characterizing the situation where all the agents have affine risk tolerances $\rho_i(x_i) = \alpha_i + \beta_i x_i$ for all $i \in \mathcal{I}$, where α_i, β_i are constants. In the special case that $\beta_1 = \beta_2 = \ldots = \beta$, we refer to this situation as the one with *identical cautiousness* across the population. Consider the class of negative exponential utility functions, where the marginal utilities $u_i'(x_i) = e^{-x_i/c_i}$, in which case $\rho_i(x_i) = c_i$. For the situation where the utility functions are given by $u_i(x_i) = x_i^{(1-a_i)}/(1-a_i)$, it follows that $\rho_i(x_i) = x_i/a_i$. In the former case we have identical cautiousness since the corresponding β's are all zero, in the latter example we have equal cautiousness across the population only if $a_1 = a_2 = \ldots = a$ for some constant a. Finally, when $u_i(x_i) = \ln(x_i)$, it follows that $\rho_i(x_i) = x_i$. Below we assume $u_i' > 0$ and $u_i'' < 0$ for all $i = 1, 2, \ldots, I$. We can then show the following (e.g., Wilson (1968)):

Theorem 4 *The Pareto optimal sharing rules are affine if and only if the risk tolerances are affine with identical cautiousness, i.e.,* $Y_i(x) = A_i + B_i x$ *for some constants* A_i, B_i, $i \in \mathcal{I}$, $\sum_j A_j = 0$, $\sum_j B_j = 1$, $\Leftrightarrow$ $\rho_i(x_i) = \alpha_i + \beta x_i$, *for some constants* β *and* α_i, $i \in \mathcal{I}$.

In some sense this theorem presents a nice result, since it characterizes the preferences of the agents when the sharing rules are affine, and it also tells us precisely when we can expect sharing rules to be of the affine type. In another sense, however, this result is rather negative. It simply states that in most stock markets one can not hope to obtain efficient risk sharing in common stocks only. When payoffs are continuously distributed, the corresponding stock market model is incomplete. Pareto efficient risk sharing would still be possible if the optimal sharing rules were of the affine type. The above theorem states that this can only take place for very specific cases of preferences, essentially the class of negative exponential, or of power (logarithmic) utility functions with identical coefficients of relative risk aversion.

Let us present two examples of the situation (a). According to Theorem 4 these two are about the only ones that can be found. In the first example the individuals have constant absolute risk aversions.

Example 2. Consider the case of negative exponential utility functions, with marginal utilities $u_i'(z) = e^{-z/a_i}, i \in \mathcal{I}$, where a_i^{-1} is the *absolute risk aversion* of agent i, or a_i is the corresponding *risk tolerance.* We assume that the payouts of the stocks Z_i are continuously distributed random variables, so that the market is incomplete, and let us assume that an unconstrained equilibrium exists in L^2.[2] We know from the standard reinsurance model that the equilibrium allocations are given by

$$Y_i = \frac{a_i}{A} Z_M + b_i, \qquad \text{where} \qquad b_i = a_i \ln \lambda_i - a_i \frac{K}{A}, \qquad i \in \mathcal{I}.$$

where $\lambda_i = \alpha_i^{-1}$ are the agent weights in the representative agent utility function, the reciprocals of the Lagrangian multiplier α_i of agent i's individual optimization problem, and where the constants K and A are given by

$$K = \sum_{i=1}^{I} a_i \ln \lambda_i, \qquad A = \sum_{i=1}^{I} a_i.$$

The constants b_i represented the zero-sum side-payments in the reinsurance application, i.e., $\sum_{i \in \mathcal{I}} b_i = 0$.

The question is now if these allocations can also result in the marketed subspace $M \subset L^2$. Consider the case where a riskless asset exists, denoted the zeroth security. Then we may write

$$Y_i = \sum_{n=0}^{N} \theta_n^{(i)} Z_n = b_i \cdot 1 + \sum_{n=1}^{N} \frac{a_i}{A} Z_n$$

Thus, if individual i puts the same weight a_i/A on each of the common stocks $n = 1, 2, \dots, N$ and invests $\theta_0^{(i)} = b_i$ in the riskless security, he will obtain his unconstrained Pareto optimal equilibrium allocation Y_i. Notice that the more risk tolerant an individual is, the more he holds of each of the risky assets. In order for this to be possible he may borrow or lend the riskfree asset. If, say, a more risk tolerant investor has a "low" initial endowment

[2] Conditions are given in the next section.

X_i, he will finance his optimal portfolio by borrowing, whereas a more risk averse investor will hold less of the risky assets and more of the riskless, i.e., he may be a lender, at least if he is initially well endowed. In equilibrium this just adds up, since $\sum_{i \in \mathcal{I}} \theta_0^{(i)} = \sum_{i \in \mathcal{I}} b_i = 0$.

We notice that the individuals hold varying fractions of the market portfolio Z_M and the riskless asset in equilibrium, called *two fund separation.* $\square$

In the above example, even if the model is incomplete, the individuals obtain their Pareto optimal allocations by an exchange of common stocks only, so long as riskfree borrowing and lending is unrestricted. We notice that this could lead a more risk tolerant, poorly endowed investor to assume a rather risky position (despite the fact that he is, of course, risk averse in the above example).

In the next example we consider the case of constant relative risk aversion. Here it turns out that risk tolerant and poorly endowed individuals may not engage in quite so "risky" positions as in the previous example, and they will do just fine without a riskfree asset:

Example 3. Here we consider the case of power utility, where $u_i(x) = (x^{1-a} - 1)/(1-a)$ for $x > 0$, $a \neq 1$, $u_i(x) = \ln(x)$ if $a = 1$. The parameter $a > 0$ is the *relative risk aversion* of the agents, here assumed equal for all the individuals. The investors are not equal because their initial endowments X_i may be different. Again we consider continuous distributions so the model is incomplete, and we assume that an unconstrained equilibrium exists in L^2. [3] Then we know from the standard reinsurance model that the unconstrained equilibrium allocations are given by

$$Y_i = \frac{\lambda_i^{1/a}}{\sum_{j \in \mathcal{I}} \lambda_j^{1/a}} Z_M \qquad a.s. \qquad \text{for all } i.$$

where again $\lambda_i = 1/\alpha_i$, and the investor weights λ_i are determined by the budget constraints, implying that

$$\lambda_i = k \left(\frac{E(X_i Z_M^{-a})}{E(Z_M^{1-a})} \right)^a, \qquad i \in \mathcal{I},$$

or, λ_i is determined modulo the proportionality constant $k = (\sum_{j \in \mathcal{I}} \lambda_j^{1/a})^a$ for each i. The question is again whether these Pareto optimal equilibrium allocations can be obtained in $M \subset L^2$. Also now the answer is yes. Here

[3] Conditions can again be found in the next section.

agent i may choose the portfolio weights $\theta_n^{(i)}$ such that

$$Y_i = \sum_{n=1}^{N} \theta_n^{(i)} Z_n = \sum_{n=1}^{N} \frac{\lambda_i^{1/a}}{\sum_{j\in\mathcal{I}} \lambda_j^{1/a}} Z_n,$$

which means that

$$\theta_n^{(i)} = \frac{\lambda_i^{1/a}}{\sum_{j\in\mathcal{I}} \lambda_j^{1/a}}, \qquad n = 1, 2, \ldots, N, \qquad \theta_0^{(i)} = 0, \qquad i \in \mathcal{I}.$$

We see that this equilibrium can be obtained in a market for common stocks only, where riskfree lending or borrowing is not necessary. [4] Again the individuals choose the same percentage of each of the stocks, but this time the percentage is a positive linear functional of the initial endowment X_i of each individual i, meaning that someone with a "high" initial endowment will quite naturally hold more stocks in equilibrium than someone with a lower endowment.

Here we notice that each individual holds a fraction of the market portfolio Z_M in equilibrium. $\square$

It should be noticed that when the risk aversion parameter varies across individuals, the optimal sharing rules are no longer linear, and the results of this example no longer apply.

Existence of Equilibrium

In this section we address the issue of existence of equilibrium. It turns out that we will have to relate to three different concepts of equilibrium: First an equilibrium in the reinsurance market, second a financial economics equilibrium, and third a "no arbitrage equilibrium".

Let us start by summing up some of our findings so far. Suppose there is no arbitrage, so that the pricing functional $\pi : L^2 \to R$ is linear and strictly positive, i.e., $\pi(Z) > 0$ for any $Z > 0$. Then, from The Riesz Representation Theorem for L^2, we know that there exists a random variable, the state price deflator $\xi \in L^2_{++}$, such that any $X \in L^2$ has market price

$$\pi(X) = E(\xi \cdot X).$$

If there exists an equilibrium in L^2, we can characterize the state price deflator as $\xi = u'_\lambda(Z_M)$. If the model is not complete and there exists an

[4] There could of course still be a risk free asset, if say $Z_1 = 1$ a.s.

equilibrium in the marketed subspace M, we know that $\xi^T = \tilde{u}'_\lambda(Z_M)^T$. In this case

$$\pi(X) = E\left(X^T \cdot \tilde{u}'_\lambda(Z_M)^T\right) + E\left(X^Q \cdot \xi^Q\right).$$

If $X \in M$, then $X = X^T$ and $X^Q = 0$ so the last term in the above pricing formula disappears. Under this pricing rule, in case (a), if a new financial asset in zero net supply is introduced for trade, the original equilibrium in M will not be upset, and no individual will demand this asset. In case (b) the introduction of new financial instruments may change the equilibrium. Consider e.g., the polar (in the finite dimensional case) where the resulting market becomes complete. Then we know that the final equilibrium allocations must have changed, since the equilibrium allocations are now Pareto optimal unlike the original equilibrium allocations. Some agents will hold other assets than those in the original stock market economy, and pricing is now under the first rule above, i.e., ξ on M has changed from $\tilde{u}'_\lambda(Z_M)$ to $u'_\lambda(Z_M)$.

In studying the existence issue, several approaches are possible. We indicate one which may be extended to the multiperiod case. It involves transforming the concept of a financial market equilibrium into the concept of a "no-arbitrage equilibrium", which is simply a constrained reinsurance equilibrium. This transformation permits techniques developed for analyzing the traditional reinsurance equilibrium to be transferred to the model with incomplete markets.

Let us recall the budget set of the i'th individual in the financial market economy $B_i^F(p;\bar{\theta})$ in equation (1), while the budget set in the reinsurance economy is

$$B_i^R(\xi; X_i) = \left\{Y_i \in L^2_+ : E(\xi \cdot Y_i) = E(\xi \cdot X_i)\right\}. \tag{18}$$

The no-arbitrage equation is $p = E(\xi \cdot Z)$ where $p = (p_1, \dots, p_n)'$ and $Z = (Z_1, \dots, Z_N)'$. The idea is to reformulate the concept of a financial market equilibrium in terms of the variable ξ. Then the demand functions for securities as functions of p are replaced by demand functions for the good as functions of the state price deflator ξ.

Whenever $p = E(\xi \cdot Z)$, the budget set $B_i^F(p;\bar{\theta})$ can be reformulated as

$$B_i^{NA}(\xi; X_i) = \left\{Y_i \in L^2_+ : E(\xi \cdot Y_i) = E(\xi \cdot X_i), Y_i - X_i \in M\right\}. \tag{19}$$

We notice that this budget set is a constrained version of the budget set $B_i^R(\xi; X_i)$.

A *no-arbitrage equilibrium* is a pair consisting of an allocation Y and a state price deflator ξ such that

(i) $Y_i \in \text{argmax}\{Eu_i(V) : V \in B_i^{NA}(\xi; X_i)\}$
(ii) $\sum_{i=1}^{I}(Y_i - X_i) = 0$.

It may then be shown that a financial market equilibrium exists whenever a no-arbitrage equilibrium exists. A proof of this result can be found, in the finite dimensional case, in Magill and Quinzii (1996). Furthermore the existence of a no-arbitrage equilibrium is closely connected to the existence of a reinsurance equilibrium. Again a finite dimensional demonstration can be found in the above reference.

Therefore we now restrict attention to the existence of a reinsurance market equilibrium in the infinite dimensional setting of this paper. It is defined as follows:

A *reinsurance market equilibrium* is a pair consisting of an allocation Y and a state price deflator ξ such that

(i) $Y_i \in \text{argmax}\{Eu_i(V) : V \in B_i^{R}(\xi; X_i)\}$
(ii) $\sum_{i=1}^{I}(Y_i - X_i) = 0$.

One main difficulty is that the positive cone L^2_+ has an empty interior, so that we can not use standard separation arguments to obtain price supportability. One alternative is to make assumptions directly on preferences that guarantee supportability of preferred sets. The key concept here is properness introduced in Mas-Colell (1986), see also Mas-Colell and Zame (1991).

It should be noted that we do not face this difficulty if we allow all of L^2 as our "commodity" space. In a one period model final consumption is equal to final wealth, and if we allow this to be negative, we avoid this particular difficulty.

A preference relation $\succeq$ defined on the set L^2 is proper at an element $X \in L^2$ with respect to another element $V \in L^2$, if there is an open cone K_X at 0, containing V, such that $X - K_X$ does not intersect the preferred set $\{\hat{X} \in L^2 : \hat{X} \succeq X\}$; i.e., if $\hat{X} \succeq X$ then $X - \hat{X} \notin K_X$.

We say that $\succeq$ is uniformly proper with respect to V on a subset $M \subseteq L^2$ if it is proper at every $X \in M$, and we can choose the cone K independently of $X \in M$.

A pair (Y, ξ) is a quasi-equilibrium if $E(\xi \cdot X_M) \neq 0$ and for each i, $E(\xi \cdot \hat{Y}_i) \geq E(\xi \cdot X_i)$ whenever $U_i(\hat{Y}_i) > U_i(Y_i)$. A quasi-equilibrium is an equilibrium if $U_i(\hat{Y}_i) > U_i(Y_i)$ implies that $E(\xi \cdot \hat{Y}_i) > E(\xi \cdot Y_i)$ for all i. The latter property holds at a quasi-equilibrium if $E(\xi \cdot X_i) > 0$ for all i.

We also remark the following: Suppose for every i there is some Z_i with $E(\xi \cdot Z_i) < E(\xi \cdot X_i)$. If (Y, ξ) is a quasi-equilibrium and all the U_i functions are continuous, then (Y, ξ) is also an equilibrium.

When preferences are convex, as in our exposition, properness of a preference relation $\succeq$ at X with respect to V is equivalent to the existence of a price $\xi \in L^2$ which supports the preferred set $\{\hat{X} \in L^2 : \hat{X} \succeq X\}$ at X and has the additional property that $E(\xi \cdot V) > 0$. Indeed, if such a ξ exists, we can simply take $K_X = \{Z : E(\xi \cdot Z) > 0\}$. Conversely, if $\succeq$ is proper at X with respect to V, then $\{\hat{X} \in L^2 : \hat{X} \succeq X\}$ and $X - K_X$ are disjoint *convex* sets, and the latter has non-empty interior ($V \in K_X$), so The Separating Hyperplane Theorem provides a continuous linear functional $\xi \in L^2$ that separates them; i.e., $E(\xi \cdot Z) \leq E(\xi \cdot \hat{X})$ for each $Z \in X - K_X$ and $\hat{X} \succeq X$. Because K_X is an open cone at 0, containing V, it follows that $E(\xi \cdot Z) > 0$ for all $X \in K_X$, and hence that $E(\xi \cdot V) > 0$ and $E(\xi \cdot \hat{X}) \geq E(\xi \cdot X)$ for all $\hat{X} \succeq X$ as asserted.

Let us denote by $U_i(X) = E(u_i(X))$. In the present setting properness of U_i at X with respect to X_M is equivalent to the assertion that the random variable $u_i'(X)$ is in L^2. In this case $u_i'(X)$ represents the supporting linear functional at X (see Araujo and Monteiro (1989)).

Following Mas-Colell and Zame (1991), we have the following:

Lemma 1 *Suppose that $X_M \in L^2_{++}$ and there is any allocation $V \geq 0$ a.s. with $\sum_{i=1}^I V_i = X_M$ a.s., and such that U_i is proper at V_i for each i, then there exists a quasi-equilibrium.*

Using the above result about properness, this lemma can be reformulated as follows (e.g., Aase (1993), (2002)):

Theorem 5 *Assume $u_i(\cdot)$ continuously differentiable for all i. Suppose that $X_M \in L^2_{++}$ and there is any allocation $V \geq 0$ a.s. with $\sum_{i=1}^I V_i = X_M$ a.s., and such that $E\{(u_i'(V_i))^2\} < \infty$ for all i, then there exists a quasi-equilibrium.*

If every agent i brings something of value to the market, in that $E(\xi \cdot X_i) > 0$ for all i, which seems like a reasonable assumption in most cases of interest, we have that an *equilibrium exists* under the above stipulated conditions. We notice that these requirements put joint restrictions on both preferences and probability distributions.

This theorem can be used on Example 3, where the equilibrium was an unconstrained one in L^2. Note that this example does not satisfy uniform properness. The above condition is, for $V = X$, the initial allocation, $E(X_i^{-a}) < \infty$ for all i. In order to compute this equilibrium in detail and calculate all the investor weights λ_i, moments of the above kind appear, and obviously these must exist in order for an equilibrium to exist. The above theorem says that this is, basically, all that is required.

Let us also consider Example 2. The requirement is then

$$E\{\exp(-2X_i/a_i)\} < \infty, \qquad \text{for all} \qquad i.$$

Again these moments appear when calculating the equilibrium, where the zero sum side payments depend on moments of this kind.

Let us now return the incompleteness issue, i.e,. the existence of a financial market equilibrium. We require smoothness of the utility functions, e.g., $u_i' > 0$, $u_i'' \leq 0$ for all i. In addition $X_i \in L^2_{++}$ for each $i \in \mathcal{I}$, and the allocation $V \in M$. Suppose that a reinsurance market equilibrium exists where $E(\xi \cdot X_i) > 0$ for all i. We then conjecture that a financial market equilibrium exists.

Theorem 6 *Assume $u_i(\cdot)$ continuously differentiable for all i, and that a reinsurance market equilibrium exists, such that $E(\xi \cdot X_i) > 0$ for all i. Suppose that $X_i \in L^2_{++}$ and there is any allocation $V \geq 0$ a.s. with $V \in M$ and $\sum_{i=1}^{I} V_i = Z_M$ a.s., such that $E\{(u_i'(V_i))^2\} < \infty$ for all i. Then there exists a financial market equilibrium.*

Proof: (Sketch) The proof that a no arbitrage equilibrium is equivalent to a financial market equilibrium (Proposition 10.3 in Magill and Quinzii (1996)) does not depend on the dimension of Ω, neither does the proof that a no arbitrage equilibrium exists when a reinsurance market equilibrium exists (Proposition 10.4 in Magill and Quinzii (1996)). The problem is then essentially if a reinsurance equilibrium exists, which is directly assumed to be the case in the above. $\square$

If a reinsurance market equilibrium exists, the projections in M of the marginal rates of substitution will be equalized, since now the agents, in solving their optimal problems, are led to equalize the marginal rates of substitution (in L^2). Thus it is obvious that the first order conditions (17) are satisfied.

On the other hand, if the first order conditions (17) hold, by the Hahn-Banach Theorem the resulting linear, positive functional may be extended to a continuous linear functional in all of L^2, although this extension may not be unique. Using the Riesz Representation Theorem there is a linear, continuous pricing functional represented by $\xi \in L^2$, valid in all of L^2.

The following result in fine print should be observed. Suppose there is no arbitrage in the marketed subspace M. Then there is a strictly positive, linear functional in M representing the prices. By a variant of the Hahn-Banach Theorem, sometimes called the Kreps-Yan Theorem, if M is *closed*, this functional can be extended to a linear and *strictly positive* functional on all of L^2. Thus there is no arbitrage in L^2 under the stated conditions.

Thus, if a finance market equilibrium exists, there is a close connection to an equilibrium in L^2 in the corresponding reinsurance market.

When the function $E\tilde{u}_\lambda(\cdot)$ is concave and M is closed, there must exist a solution to the problem (16). However, we do not know that this corresponds to an equilibrium in the single agent economy unless there is a financial market equilibrium in the original economy.

Idiosyncratic risk and stock market risk

A natural interpretation of the foregoing model may be as follows: Consider some consumers having initial endowments X_i measured in units of the consumption good. The uncertainty they face is partly handled by forming a stock market as explained above, but still there may be important risks that can not be hedged in a stock market: Property damage, including house fires, car thefts/crashes etc., labor income uncertainty, and life length uncertainty. In order to deal with *idiosyncratic risk*, we may assume there exists an insurance market where the consumer can, against the payment of a premium, get rid of some of the economic consequences of this type of uncertainty, and also a social security system, which together with unemployment insurance will partly smooth income form labor. The corresponding uncertainties are assumed external.

We are then in situation (b) described above regarding the stock market, but we assume the overall market facing the consumers is complete, just as the reinsurance market is complete by construction. Suppose there exists a unique equilibrium in this overall market. We may then use the results from the standard reinsurance model. Despite the fact that the stock market model is not complete, and indeed also inefficient, consumers can still be able to obtain Pareto optimal allocations in this world, and the state price deflator is ξ, not $\tilde{\xi}$. The optimal allocations in the stock market must hence be supplemented by insurance in order to obtain the final equilibrium allocations Y_i of the consumers.

This way we see that the principles governing the risks are valid in the stock market as well as in the insurance markets, since the state price deflator is the same across all markets, or, a risk is a risk is a risk... The reason is that *the different markets have the same purpose*, namely to enable the consumers to obtain their most preferred outcomes among those that are feasible.

A detailed study of a model based on these principles is beyond the scope of this presentation. The inclusion of idiosyncratic risk together with market risk would presumably complicate matters. Asymmetric information may typically play a role. Suffice it is to note that much of the focus these days in studying incomplete markets seems to be centered on the stock market alone, not seeming to realize that very important aspects of economic uncertainty facing most individuals can not be managed in the financial markets for stocks and options alone.

Conclusions

We have argued that many results in finance can be seen as consequences of the classical theory of reinsurance.

Karl H. Borch both contributed to, and borrowed from, the economics of uncertainty developed during the 1940's and 1950's (e.g. Borch (1960-62)). While he reformulation of the general theory of equilibrium, formulated by Arrow and Debreu (Arrow and Debreu (1954), Arrow (1970)), was perceived as too remote from any really interesting practical economic situation by most economists at the time, Borch found, on the other hand, that the model they considered gave a fairly accurate description of a reinsurance market.

In this paper we have tried to demonstrate the usefulness of taking the reinsurance model as the starting point for the study of financial market equilibrium in incomplete markets. This as a modest counterbalance to the standard point of view, that the influence has mainly gone in the opposite direction.

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