A method for numerical and analytical solutions to a class of nonlinear optimal control problems

L. K. Sandal* and G. Berge†

February 11, 2004

ABSTRACT Dynamic optimization problems cover a great class of problems in theoretical and applied economics and technology. In this account the exploitation of a general renewable capital stock is modeled through an alternative formulation to the classical optimal control approach. We propose a very simple alternating iterative algorithm that is shown to converge very fast towards a solution, where the accuracy of the solution can also be determined. The algorithm can also produce approximate closed form (analytical) feedback solutions.

By using a special simple seed in the iteration scheme we reproduce perturbation results (formulae) that are published in the last decade. Often only two or three iteration steps are necessary to produce sufficiently accurate approximations.

JEL classification: C61; C63; Q00

Keywords: Capital Dynamics Modeling; Natural Resource Exploitation; Analytical and Numerical Contraction

Corresponding author:
Professor Gerhard Berge
Department of Mathematics, University of Bergen,
J. Brunsgt. 12, 5008 Bergen, NORWAY
E-mail: gerhard.berge@mi.uib.no
Phone: Work 47 55584855, home 47 55121434, Fax: 47 55589672

*Department of Finance and Management Science, Norwegian School of Economics and Business Administration

†Department of Mathematics, University of Bergen, Norway

1 Introduction

Dynamic optimization problems cover a great class of problems in theoretical and applied economics and technology. The classical formulations being the variational approach as in classical mechanics, like Hamilton's principle and the optimal control theory in economics as the Pontryagin's maximum principle. In this account the exploitation of a general renewable capital stock is modeled through an alternative continuous time formulation rather than the usual classical optimal control approach. We propose a very simple alternating iterative algorithm that is shown to converge rapidly to a solution. A remarkably simple error control is also provided. The algorithm can be used as a tool to obtain a fast numerical feedback solution as well as being an efficient scheme to produce approximate closed form (analytical) feedback solutions. In this respect we point out that an analytic solution is a more powerful tool for parameter search than any numerical solutions can ever be.

The method is well suited for dealing with real world problems involving exploitation of a general renewable capital stock by generating feedback rules (numerically as well as analytically) for a family of problems concerning a single resource. We reproduce perturbation results (formulae) published in the last decade.

The modeling approach and results presented in this paper are, however, of a much more general character than the special field of application we focus on here i. e. ideas from a natural renewable resource management[1]. From a mathematical point of view we can always distinguish between the optimization problem and the problem of finding the separatrix solutions to a set of differential equations.

A summary and discussion of results are presented at the end. Key features are a simple scheme, fast convergence (two or three iterations) and error control.

2 The Model

In management of renewable resources the following class of problems is of interest, find

$$\max_u \int_0^\infty e^{-\delta t}\Pi(x(t), u(t))dt\,, \tag{1}$$

where $x \in X$, $u \in U$ and X, U are closed intervals. Here Π is a twice differentiable and strictly concave function of u. The discount rate δ is nonnegative. To avoid unnecessary technicalities it is sufficient in our context to assume that $x = x(t)$

[1]See e.g., Clark 1990, Arnason et al. 2004, Sandal and Steinshamn 1998 and 2000.

and $u = u(t)$ are *piecewise smooth* and in addition that $x = x(t)$ is continuous. The maximization (Eq. (1)), is to be performed subject to the constraint (state equation),

$$\dot{x} \stackrel{def}{=} \frac{dx}{dt} = f(x) - u \,, \tag{2}$$

where f is continuous.

Here x can be interpreted as a measure of the stock level of a renewable resource, u the harvest rate and $f(x)$ the natural growth rate of the resource.

In the next section we introduce an alternative capital dynamic formulation to this control problem.

2.1 The Feedback Approach

In the modeling approach we consider the space (x, u) (the state/control space) where time t is a redundant parameter. In this space we examine *piecewise smooth functions* (paths) $u = u(x)$, usually called controls/policies in feedback form. We use the following definition.

Definition 1 *Piecewise smooth function*

The notion will be used for functions that are continously differentiable except at possibly a finite number of points.

This means that we allow for discontinuities in the derivative as well as the function itself at these points. The model can in mathematical terms be formulated as an integral equation

$$M(x, u) = N(x, \alpha)[u] \,, \tag{3}$$

where

$$M(x, u) \stackrel{def}{=} \Pi(x, u) + \Pi_u(x, u) \{f(x) - u\} - S(x) \,, \tag{4}$$

and

$$S(x) \stackrel{def}{=} \Pi(x, f(x)) \,. \tag{5}$$

This implies that $M(x, f(x)) \equiv 0$. Furthermore

$$N(x)[u, \alpha] \stackrel{def}{=} K(\alpha) - S(x) + \delta \cdot \int_{\alpha}^{x} \Pi_u(s, u(s)) ds \,, \tag{6}$$

where $K(\alpha)$ is a parameter to be determined. Eq. (3) reads,

$$\Pi(x, u) + \{f(x) - u\} \cdot \Pi_u(x, u) = K(\alpha) + \delta \cdot \int_{\alpha}^{x} \Pi_u(s, u(s)) ds \,, \tag{7}$$

where again $f(x)$ is the natural growth as given through the state equation, Eq. (2). In the present setting Eq. (2) is actually redundant and only serves the purpose of connecting the problem to the time domain. Interpretations of Eq. (7) within a typical economic framework are:

$$\begin{aligned}
\Pi &\sim \text{(rate of) rent}\,, \\
S(x) &\sim \text{(rate of) sustainable rent} \\
(f-u)\cdot\Pi_u &\sim \text{(rate of) saving/investments}\,, \\
K(\alpha) &\sim \text{(rate of) rent in a reference state}\,, \\
\delta\cdot\int_\alpha^x \Pi_u(s,u(s))ds &\sim \text{(rate of) discounted cost in moving from } \alpha \text{ to } x\,.
\end{aligned}$$

We have that $M = N$ is a point of balance in marginal contributions in time. Thus the optimal policy corresponds to the case where we have a decreasing marginal contribution by following an optimal policy in time, i.e. $M_u < 0$, where $M_u = (f-u)\cdot\Pi_{uu}$ (this applies to the region $u < f$ and in the case $u > f$ we have in the time domain that this corresponds to moving towards decreasing values in u, in both cases $|u - f|$ is decreasing).

If we look for a place to "park", then we have to continue until the marginal contribution (consume) is as small as possible i.e. $M_u \to 0 \;\Leftrightarrow\; u \to f$, since $\Pi_{uu} \neq 0$. Then we ask the question: For what value of α can this be realized. We restrict ourselves to an internal maximum point, thus

$$\alpha = \operatorname{argmax}(N) \quad \Rightarrow \begin{cases} K(\alpha) = S(\alpha) = \Pi(\alpha, f(\alpha))\,, \\ K'(\alpha) = \delta\cdot\Pi_u(\alpha, u(\alpha))\,, \\ u(\alpha) = f(\alpha)\,, \end{cases} \tag{8}$$

i.e. $S'(\alpha) = \delta\cdot\mu(\alpha)$[2] which has been shown by Sandal and Steinshamn, 1997c to be the equilibrium condition ($u(\alpha) = f(\alpha)$).

In our context this leads to a uniquely determined point α which we choose to be our point of reference $x^*_\delta \overset{def}{=} \alpha$. From now on we make the replacement $\alpha = x^*_\delta$.

2.1.1 Key definitions and assumptions

The following definitions give the core expressions involved in establishing the main results in this paper.

$$S^*_\delta \overset{def}{=} S(x^*_\delta) \overset{def}{=} \Pi(x^*_\delta, f(x^*_\delta))\,, \tag{9}$$

[2] μ is defined in Eq. (10).

$$\mu(x) \stackrel{def}{=} \Pi_u(x, f(x))\,. \tag{10}$$

$$\psi(x) \stackrel{def}{=} S_\delta^* - S(x) + \delta \int_{x_\delta^*}^{x} \mu(s))ds\,, \tag{11}$$

$$N(x)[u] \stackrel{def}{=} S_\delta^* - S(x) + \delta \cdot \int_{x_\delta^*}^{x} \Pi_u(s, u(s))ds\,, \tag{12}$$

$$A^* \stackrel{def}{=} \{(x,u)|(x - x_\delta^*)(f(x) - u) \le 0\}\,, \tag{13}$$

$$A_L^* \stackrel{def}{=} \{(x,u)|\,(x,u) \in A^* \quad \& \quad x \le x_\delta^*\}\,, \tag{14}$$

$$A_R^* \stackrel{def}{=} \{(x,u)|\,(x,u) \in A^* \quad \& \quad x \ge x_\delta^*\}\,. \tag{15}$$

Notice that $N(x)[u, x_\delta^*] = N(x)[u]$ and that $\psi(x) = N(x)[f(x)]$ has a critical point for $x = x_\delta^*$, which follows from Eqs. (8). Also notice that the global minimum of $\psi(x)$ is the reference point $x = x_\delta^*$.

The following assumptions are expected to hold throughout the rest of the paper:

A1 $\Pi(x, u)$ is twice differentiable in its arguments and $\Pi_u \ge 0, \quad -\infty < \Pi_{uu} < 0$.

A2 $\psi(x) \ge 0 \quad$ for $\quad x \in X$ and $\psi(x)$ has a single zero.[3]

These assumptions are made more for the reasons of convenience than for the reasons of necessity.

Definition 2 *Admissible controles*

Admissible controls (feedback policies) are $u = u(x) \in PS(X)$ where $PS(X)$ is the set of piecewise smooth and bounded functions on X such that $(x, u(x)) \in A^*$.

Solutions are particularly easy to find when $\delta \to 0$. In this limit equation (7) becomes an algebraic equation for the feedback policy u(x). Formally this limit may need to be interpreted as a generalized optimality. That is, even though different policies may create infinite large utilities there are no value of a discount rate that can alter the practical result that "two dollars a day is better than one dollar a day for infinitely long time". Catching-Up optimality (CU-optimality) is a generalization along theses ideas. This an other natural extensions of the notion optimality can be found in e.g., (Seierstad and Sydsæther, 1999). Indeed, the formal solution of this algebraic equation is in fact the separatrix in phase space going through the reference point x_δ^*.

[3]The generalization to the case with a finite number of zeros is straight forward.

2.1.2 Monotonicity properties

The functional Q given by

$$Q(x)[u] \stackrel{def}{=} S(x) + M(x,u) - S^*_\delta - \delta \int_{x^*_\delta}^{x} \Pi_u(s,u(s))ds = M(x,u) - N(x)[u]\,, \quad (16)$$

is well defined on the set of admissible feedback controls.

The problems stated in this paper are transferred to the problem of finding $Q \equiv 0$ such that $(x, u(x)) \in A^*$.

Lemma 1 *Important inequalities*

The following important inequalities hold,

$$\{f(x) - v\}(\Pi_u - \Pi_v) \leq M(x,u) - M(x,v) \leq \{f(x) - u\}(\Pi_u - \Pi_v)\,. \quad (17)$$

Proof: These inequalities follow directly from the definition of M and the concavity of Π. Thus from Eq. (4) we obtain for $u = u(x)$ and $v = v(x)$ that

$$M(x,u) - M(x,v) = \Pi(x,u) - \Pi(x,v) + \Pi_u(x,u)\{f(x) - u\} - \Pi_v(x,v)\{f(x) - v\}\,,$$

and the concavity of Π with respect to u implies

$$\Pi_u(x,u)(u - v) \leq \Pi(x,u) - \Pi(x,v) \leq \Pi_u(x,v)(u - v)$$

and Eq. (17) is obtained.

□

By using

$$M(x,u) - M(x,v) = Q(x)[u] - Q(x)[v] + \delta \int_{x^*_\delta}^{x} \Delta_{uv}(s)ds\,,$$

where

$$\Delta_{uv}(x) \stackrel{def}{=} \Pi_u(x,u(x)) - \Pi_v(x,v(x))\,, \quad (18)$$

the inequalities Eq. (17) can again be rewritten as

$$\begin{aligned} \{f(x) - v(x)\}\Delta_{uv}(x) \quad &- \quad \delta \int_{x^*_\delta}^{x} \Delta_{uv}(s)ds \\ &\leq \quad Q(x)[u] - Q(x)[v] \\ &\leq \quad \{f(x) - u(x)\}\Delta_{uv}(x) - \delta \int_{x^*_\delta}^{x} \Delta_{uv}(s)ds\,. \quad (19) \end{aligned}$$

A key property for the functional Q restricted to A^* is:

Lemma 2 *Monotonicity*

$(x - x^*_\delta)Q(x)[u]$ *is a monotonically increasing function of* u *in the set of admissible controls.*

Proof: Since $\Pi(x, u)$ is concave in u, then $u \geq v \;\Rightarrow\; \Delta \leq 0$. From the inequalities, Eq. (19), we obtain:

1. $Q(x)[u]$ is monotonically decreasing as function of u in A^*_L.
2. $Q(x)[u]$ is monotonically increasing as function of u in A^*_R.

□

2.1.3 An Iteration Scheme

We are studying the relation $Q(x)[u] = 0$, or

$$M(x, u) = N(x)[u]\,. \tag{20}$$

Introducing M as given by Eq. (4) has several advantages. The function $M(x, u)$ measures a potential gain of changing the state or moving in state space. There is a potential benefit associated with changing the situation. It plays a role much the same as the kinetic energy do for a mechanicle system. The motion by itself has the potential of doing a physical work. The functional $N(x)[u]$, Eq. (12), can be associated with potential energy. It has the potential to change the state of motion.

Concavity of $\Pi(x, u)$ with respect to u, implies uniqueness of the solution of Eq.(20). This is the next proposition.

Proposition 1 *Uniqueness*

Eq. (20) has at most one admissible solution.

Proof: The crucial entity in the proof is $\Delta_{uv}(x)$ as defined in Eq. (18). Suppose there are two solutions u, v of the problem. It then follows that $Q(x)[u] \equiv Q(x)[v] \equiv 0$, and the inequality (19) becomes

$$\{f(x) - v(x)\} \cdot \Delta_{uv}(x) \leq \delta \int_{x^*_\delta}^{x} \Delta_{uv}(s)ds \leq \{f(x) - u(x)\} \cdot \Delta_{uv}(x)\,. \tag{21}$$

From Eqs. (13) and (21) it follows that if not $\Delta_{uv}(x) \equiv 0$, we have a contradiction and the result follows trivially from this fact. This can be seen in the following two steps:

(i) Consider the case where we have two *admissible* solutions $u = u(x)$, $v = v(x)$ which are different and where $u \geq v$. Then $\Delta_{uv}(x) > 0$. In inequality (21), the left hand side and the right hand side are positive/negative and the term in the middle is negative/positive according to whether $x < x^*_\delta$ or $x > x^*_\delta$, which gives a contradiction.

(ii) Let u and v have a finite number of crossings in A^*. Let x be a fixed state between x^*_δ and the next crossing on one of the sides of x^*_δ. In this region u and v satisfy (i) and hence must be identical in this region. For a given x in between the next crossing we are still within the scope of (i). Hence u must be identical to v by continuing this reasoning. □

Definition 3 *Super and sub solutions*

Let $B \in \{A^*_L\,,\, A^*_R\}$. *An admissible control* $v = v(x)$ *such that* $v(x^*_\delta) = f(x^*_\delta)$ *is called:*

1. *A formal super solution on* B *provided* $(x - x^*_\delta)\, Q(x)[v] \geq 0$ *on* B *and a sub solution if the inequality is reversed.*

2. *The notion geometrical is here used as a prefix on an ordering relation* $(>, <, \geq, \leq,$ *super and sub*$)$ *if the ordering relation holds for all states considered.*

Remark: Notice that the formal *sub (super)* solution is not necessarily a geometrical *sub (super)* solution, however a geometrical *sub (super)* solution is always a formal *sub (super)* solution. This difference is due to the term in $N(x)[u]$ which depend on an interval, i.e. the integral term. Later on we find it convenient to restrict ourselves to using the notion *sub (super)* solutions meaning geomertical *sub (super)* solutions.

Notice that it follows directly that $(x - x^*_\delta)\, Q(x)[v = f(x)] = -(x - x^*_\delta)\psi(x)$, i.e., $f = f(x)$ is a super solution on A^*_L and a sub solution on A^*_R or in short notation, a *super - sub* solution. It should be clear from the definition of A^* and the notion of admissibilty that $u = f$ must be a geometrical super-sub solution if the problem has any solution in A*. If $v = v(x)$ is both a sub and a super solution on B then it is the unique solution on B according to proposition 1.

We continue by stating some important properties associated with M, Eq. (4), and N, Eq. (12).

Lemma 3 *Monotonicity properties*

The following properties holds for admissible controls:

1. *$M(x,u)$ is non negative and zero only for $u = f(x)$ and*

   $T_M(u) \stackrel{def}{=} (x - x^*_\delta) \cdot M(x,u)$ is an increasing function of u.

2. *$N(x)[u] \leq \psi(x)$.*

3. *$T_N(u) \stackrel{def}{=} (x - x^*_\delta) \cdot N(x)[u]$ is a decreasing functional of u.*

4. *$T(u) \stackrel{def}{=} (x - x^*_\delta) \cdot Q(x)[u]$ is an increasing functional of u.*

Proof:

1. From Eq. (4) we find

$$(x - x^*_\delta) \cdot M_u(x,u) = [(x - x^*_\delta)(f(x) - u)] \cdot \Pi_{uu}(x,u)\,. \tag{22}$$

By assumption $\Pi_{uu}(x,u) < 0$ and the square bracket is non positive by definition of A^*.

2. We observe that from Eqs. (11) and (12) we have

$$\psi(x) - N(x)[u] = \int_{x^*_\delta}^{x} \Pi_u(s,z)|_{z=u(s)}^{z=f(s)} ds \geq 0 \quad \text{for} \quad (x,u) \in A^*\,.$$

3. This point follows directly from the expression

$$T_N(v) - T_N(u) = (x - x^*_\delta)\delta \int_{x^*_\delta}^{x} \Delta_{uv}(s) ds\,.$$

4. This is Lemma 2.

□

We will now consider a particular iteration scheme. To simplify notation we use the following definitions:

$$M_{n+1} \stackrel{def}{=} M(x, u_{n+1})\,, \qquad N_n \stackrel{def}{=} N(x)[u_n] \quad \text{and} \quad Q_n \stackrel{def}{=} Q(x)[u_n]. \tag{23}$$

Proposition 2 *A sub - super sequence*

Let the iteration scheme be

$$M_{n+1} = N_n \geq 0\,. \tag{24}$$

For every given admissible u_n, Eq. (24) has at most one admissible solution, u_{n+1}, and the generated sequence has the following properties:

1. *If there exist an element $u_{n+1} \geq u_n$ (less or equal) then u_{n+1} is a super (sub) solution and u_n is a sub (super) solution.*

2. *If one element in the sequence is a super (sub) solution, then the following members in the sequence will alternate between sub and super solutions.*

Proof: From the definitions and properties of functions involved it is clear that $M(x,v) = N(x)[u]$ generates a function v which is piecewise smooth for any $u \in PS$ (admissible). The following relations are useful

$$T(u_{n+1}) = T_N(u_n) - T_N(u_{n+1})\,, \qquad T(u_n) = T_M(u_n) - T_M(u_{n+1})\,. \tag{25}$$

These relations are derived Appendix A. Moreover we have that $M(x,u)$ is convex in u and has a global minimum for $u = f(x)$. Therefore the equation $M(x,u) = K(x)$ (where $K(x)$ can be any nonnegative function), has at most two solutions for u, but only one of them is in A^*. This can be seen as follows: $M_u = \Pi_{uu} \cdot (f - u)$ has only one zero $u = f(x)$, i.e., the euilibrium point, $(x^*_\delta, u(x^*_\delta))$. Omitting the equilibrium point, either $x < x^*_\delta$ and M is monotonically decreasing, or $x > x^*_\delta$ and M is monotonically increasing, thus only one solution is possible.

1. The first of Eqs. (25) implies $T(u_{n+1}) \geq 0$ since $T_N(u)$ is a monotonically decreasing function of u (lemma 3 item 3), thus u_{n+1} is a super solution. Furthermore the last relation in Eqs. (25), since $T_M(u)$ is a monotonically increasing function of u (lemma 3 item 1), implies $T(u_n) \leq 0$ thus u_n is a sub solution.

2. Let u_n be a super solution, i.e. $T(u_n) \geq 0$. Then the second relation in Eq. (25) implies that $T_M(u_{n+1}) \leq T_M(u_n)$, and since $T_M(u)$ is an increasing function of u (lemma 3 item 1), it follows that $u_{n+1} \leq u_n$. And vice versa for the sub solution case.

□

In Eq. (24) there is a nonnegative restriction on N_n. It is crucial for the iteration process to continue that N_n stays nonnegative throughout. Technically this can be achieved by replaceing N_n by $\max\{N_n, 0\}$. This amounts to replacing a nonreal value with the real value $u = f(x)$. Later on we will show that $u = f(x)$ is a good seed for starting the iteration process given by eq. (24).

Lemma 4 *Property of an iterated admissible solution*

Let v be a strict geometric and admissible super (sub) solution, i.e. $v > u^*$ for $x \neq x^*_\delta$. All admissible iterates of v will have no common points with the unique solution u^* for $x \neq x^*_\delta$. Two admissible iterations from an arbitrary seed are either identical or have only the reference point $x = x^*_\delta$ in common.

Proof: From the assumptions in the lemma we have

$$\begin{aligned} M(u) &= N(x)[v]\,, \\ M(u^*) &= N(x)[u^*]\,. \end{aligned}$$

Suppose contrary to the lemma that there exist a finite number of points $\hat{x}_n\,,\quad n = 1, 2, 3, \ldots$ such that $u(\hat{x}_n) = u^*(\hat{x}_n)$. Then choose the point $\hat{x}_k$ closest to x^*_δ. Since $M(u(\hat{x}_n) = M(u^*(\hat{x}_n))$ it follows that $N(\hat{x}_k)[u] - N(\hat{x}_k)[u^*] = 0$ or

$$\delta \int_{x^*_\delta}^{\hat{x}_k} \Delta_{uu^*}(s) ds = 0\,.$$

We observe that in our case $\Delta_{uu^*}(s)$ is different from zero and have a fixed sign in the region of integration, thus the integral must be different from zero, thus we have arrived at a contradiction. This argument applies equally well for the case $v < u^*$. Notice that the case $\delta = 0$ is trivial since the first iterate gives the unique solution u^*. This proves the first part of the lemma.

With respect to the second part, let v_0 and u_0 be two admissible seeds. Suppose that contrary to the lemma there exist common points and let $\hat{x}$ be the common point closest to the reference point x^*_δ, for the iterated of these seeds. That is $u_{r+1}(\hat{x}) = v_{s+1}(\hat{x})$. As in the first part of the proof we then have $\delta \int_{x^*_\delta}^{\hat{x}} \Delta_{u_r\, v_s}(s) ds = 0$. Since u_r is geometrically greater than v_s or the opposite in the interval between $\hat{x}$ and x^*_δ and $\delta \int_{x^*_\delta}^{\hat{x}} \Delta_{u_r\, v_s}(s) ds \neq 0$. Thus we again arrive at a contradiction.

The case $\delta = 0$ is trivial since all seeds give exact solution (this is a degenerate case where $N(x)[u]$ is independent of u). $\square$

This lemma makes it possible to consider only geometrical *sub (super)* solutions, which from now on will be the case. The following proposition relates the super and sub solutions to the exact solution, and thus makes these names meaningful.

Proposition 3 *Super- sub solution*

Let u_+ be a super solution, u_- a sub solution and u^* the unique solution to $T(u) = 0$. Then $u_- \leq u^* \leq u^+$.

Proof: It follows by definition that we have $T(u_-) \leq T^*(u) = 0 \leq T(u^+)$. Proposition 3 is then a consequence of the monotonicity of $T(u)$ given in lemma 3. $\square$

2.1.4 Convergence of a sub-super sequence

The iteration scheme defined in Eq. (24), $M_{n+1} = N_n$ and the monotonicity properties of M and N as discussed in lemma 3 can be used to to prove the next proposition.

Proposition 4 *Convergence of a special sequence*

The admissible sequence obtained from the iteration algorithm $M(x, u_{k+1}) = N(x)[u_k]$ *with* $u_0 = f(x)$*, converges to the unique admissible solution* u^* *where* $M(x, u^*) = N(x)[u^*]$ *if* $N(x)[u_1] > 0$.

Proof: From item 1 in lemma 3 we have the basic properties: M is a decrasing function of u and N is an incrasing function of u for $x < x^*_\delta$. For $x > x^*_\delta$, M and N behave oppositely. Let $u_0 = f(x)$ as given by Eq. (2), then:

$$\begin{aligned} &M(x, u_0) = M(x, f(x)) = 0\,, \\ &M(x, u_1) = N(x)[u_0] = \psi(x) > 0\,, \\ &M(x, u_2) = N(x)[u_1] > 0 \text{ (by assumption)} \end{aligned} \tag{26}$$

thus $M(x, u_2) > M(x, u_0) \Rightarrow u_2 < u_0$. Then consider the general recurrency relation $u_r > u_s \Rightarrow u_{r+1} > u_{s+1}$ which we prove by using the monotonisity properties shown in items 1 and 3 in lemma3, and the iteration algoritme as follows. Let $u_r > u_s$ then

$$\left.\begin{aligned} M(x, u_{r+1}) = N(x)[u_r] \\ M(x, u_{s+1}) = N(x)[u_s] \end{aligned}\right\} \text{ thus } N(x)[u_r] > N(x)[u_s] \Rightarrow M(x, u_{r+1}) > M(x, u_{s+1}).$$

and it follows that

$$u_r > u_s \Rightarrow u_{s+1} > u_{r+1} \tag{27}$$

Then using $u_0 > u_2$ as a seed in the recurrency relation, Eq. (27), we obtain

$$u_0 > u_2 \Rightarrow u_3 > u_1 \Rightarrow u_2 > u_4 \Rightarrow u_5 > u_3 \Rightarrow \ldots \Rightarrow u_{2n} > u_{2n+2} \Rightarrow u_{2n+3} > u_{2n+1} \ldots \tag{28}$$

Thus we have by this construction a monotonically decreasing sequence: $u_0, u_2, \ldots, u_{2n}$ and a monotonically increasing sequence $u_1, u_3, \ldots, u_{2n+1}$. The first sequense is a a sequence of super solutions and the second a sequence of sub solutions according to definition 3. Thus the sequence of super solutions is monotonically decreasing and the sequence of sub solutions is monotonically increasing, furthermor according to proposition 3 both sequences are bounded by the unique solution x^* satisfying $M(x, u^*) = N(x)[u^*]$. Thus we conclude that both sequences converge. Let the sequence of super solutions converge to $u^*_+ \geq u^*$ and the sequence of sub solutions converge to $u^*_- \leq u^*$. Both these limits must, however, satisfy $M(x, u^*_\pm) = N(x, u^*_\pm)$ and since by proposition 1, the solution is unique, we must have $u^*_- = u^*_+ = u^*$. In conclusion we have that the super solutions are approching u^* from above and the sub solutions are are approching u^* from below, and they meet in the commen limit, the unique solution u^* satisfying $M(x, u^*) = N(x)[u^*]$.

For the case $x > x^*_\delta$ the monotonicity properties of M and N are switched arround and the two sequences generated also switch so that $\{u_{2n}\}$ and $\{u_{2n+1}\}$, $n = 0, 1, 2, \ldots$ become sub and super solutions correspondingly. The proof, however, follows the same pattern and is not reproduced here. $\square$

A straight forward approximation to the solution produced by "alternating" sequences like those above (in fact any sequences produced by a super-sub seed) is

$$u^* = \frac{1}{2}(u_n + u_{n+1}) + \epsilon \quad \text{where} \quad |\epsilon| < \frac{1}{2}|u_n - u_{n+1}|\,, \tag{29}$$

and ϵ is a measure of the error. This result comes from the fact that the exact solution u^* is located somewhere in between the super and sub solutions considered, see lemma 2 and definition 3. We now make a general statement about convergence.

Proposition 5 *General convergence*

Any infinite admissible sequence produced by $T_M(v_{k+1}) = T_N(v_k)$ converge to the unique admissible solution of $T(u) = 0$.

Proof: Either v_2 is identical to v_0 and they are then equal to u^* or we have a solution located over or under v_0 except for $x = x^*_\delta$. This is a consequence of lemma 4.

Then let $v_2 > v_0$ for $x \neq x^*_\delta$. Since Eq. (27) is a general result it follows that v_{2k} is an increasing sequence and v_{2k+1} is decreasing. The increasing sequence is limited by f in A^*_L and the decreasing sequence by f in A^*_R. In other words these sequences converge in A^*_L and A^*_R respectively towards u^*. Since the elements in between (odd and even) are produced by $M(x, v) = N(x)[v] -> N(x)[u^*]$ and M is continuous and the equation has a unique solution, then the sequence in total must converge. The case $v_2 < v_0$ can be proved the same way.

Any infinit admissible sequence produced by $T_M(v_{k+1}) = T_N(v_k)$ converge to the unique admissible solution u^* where $T(u^*) = 0$. $\square$

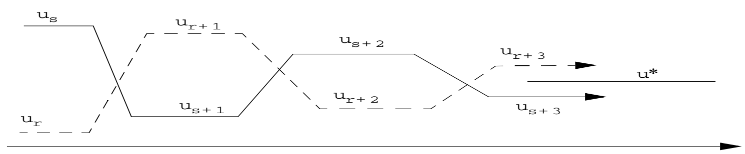

Figure 1: *This figure shows schematically how the iterations work, with new iterations along the horizontal axis. Also notice that the sequence* $\{u_s, u_{s+2}, u_{s+4}, \cdots\}$ *is monotonically decreasing wheras* $\{u_r, u_{r+2}, u_{r+4}, \cdots\}$ *is monotonically increasing, converging to the same limit* u^*.

3 Examples from fisheries management

In this section we exemplify the method derived in this paper by reproducing already published results, mainly in the field of bioeconomics within fishery management and the problem of deciding on total allowable catch (TAC) quotas. The approach is by no means limited to this class of problems. Among others, the papers, Sandal and Steinshamn 1998, 2000, 2001a,b, deal with models that can be suitably handled with the procedure described in this paper.

3.1 A Northern Cod Fishery Model

In Grafton et al. 2000, the collapse of the Canadian Northern Cod Fishery was investigated by using a model defined by the economic relations

$$\Pi(x,u) = P(u)\,u - C(x,u)\,, \quad P(u) = \frac{a\,p_1 + u\,p_0}{a+u}\,, \quad C(x,u) = q\frac{u}{x}\,. \tag{30}$$

The prices p_0 and p_1 are the minimum and the maximum prices and a the flexibility parameter in the inverse demand function, q is the derived cost parameter measuring the cost per unit output per unit biomass. The biomass is assumed to be updated according to

$$\dot{x} = f(x) - u = r\,x\,(1 - \frac{x}{K})^{\alpha} - u \tag{31}$$

where r is the intrinsic growth rate for the biomass and K is the stocks carrying capacity in terms of biomass. The parameter α is measuring how much the maximum sustainable yield (MSY) is skewed to left or right of the MSY in the logistic model ($x = 0.5K$ when $\alpha = 1$). The problem is naturally restricted to the region $X = [0, K]$. Applying the definitions: (4), (5), (9), (10) and (12) we get

$$
\begin{aligned}
M(x,u) &= \frac{a^2(p_1-p_0)}{a+f(x)} \cdot \left[\frac{(f(x)-u)}{(a+u)}\right]^2 , && (32)\\
S(x) &= \Pi(x, f(x)) , && (33)\\
\Pi_u(x,u) &= p_0 - \frac{q}{x} + \frac{(p_1-p_0)a^2}{(a+u)^2} , && (34)\\
\mu(x) &= \Pi_u(x, f(x)) , && (35)\\
N(x)[u] &= S(x^*) - S(x) + \delta\, p_0(x-x^*) - \delta \ln\left(\frac{x}{x^*}\right)\\
+\delta\,(p_1-p_0)\,a^2 \int_{x^*}^{x} \frac{ds}{(a+u(s))^2} &= \hat{S}(x) + \delta\,(p_1-p_0)\,a^2 \int_{x^*}^{x} \frac{ds}{(a+u(s))^2} . && (36)
\end{aligned}
$$

The reference state x^*_δ is the solution of $S'(x) = \delta\,\mu(x)$ (if more than one solution we must choose the one that maximizes the sustainable rent S). It is worth pointing out that we have an exact formula for the feedback policy in the limit of vanishing discounting. In this case we must interpret the optimality in a generalized sense, e.g., the Catching-Up optimality (CU-optimality [4]). This is a generalization of the practical notion that a dollar a day is less optimal than two dollars a day and no discounting can change that fact even the accumulated amount of dollars becomes infinitely large. The iteration scheme $M_{n+1} = N_n$, can be started with the static optimal policy (Bliss policy) making the $\Pi_u(x, u_0) = 0$, which is the same as neglecting the discount rate in the first iteration. Thus it gives the exact discount free solution. We get

$$
\frac{a^2(p_1-p_0)}{a+f(x)} \cdot \left[\frac{f(x)-u_1}{a+u_1}\right]^2 = \hat{S}(x)
$$

$$
\frac{a^2\,(p_1-p_0)}{a+f(x)} \cdot \left[\frac{f(x)-u_{n+1}}{a+u_{n+1}}\right]^2 = \hat{S}(x) + \delta\,(p_1-p_0)\,a^2 \int_{x^*_\delta}^{x} \frac{ds}{(a+u_n(s))^2}
$$

The second iteration is for all practical purposes the solution. This is the same type of solution one would get if one used classical perturbation theory with the discount rate as the perturbational parameter, Sandal and Steinshamn 1997a,b. In our framework it is just a result of a particular choice of seed in the iteration scheme. Notice that the system becomes singular when $p_1 = p_0$, signalling that only the equilibrium point fits the equation. This is in fact what we will expect since we then know that the

[4]See e.g., Seierstad and Sydsæter 1999, for this and other extensions of the notion of optimality.

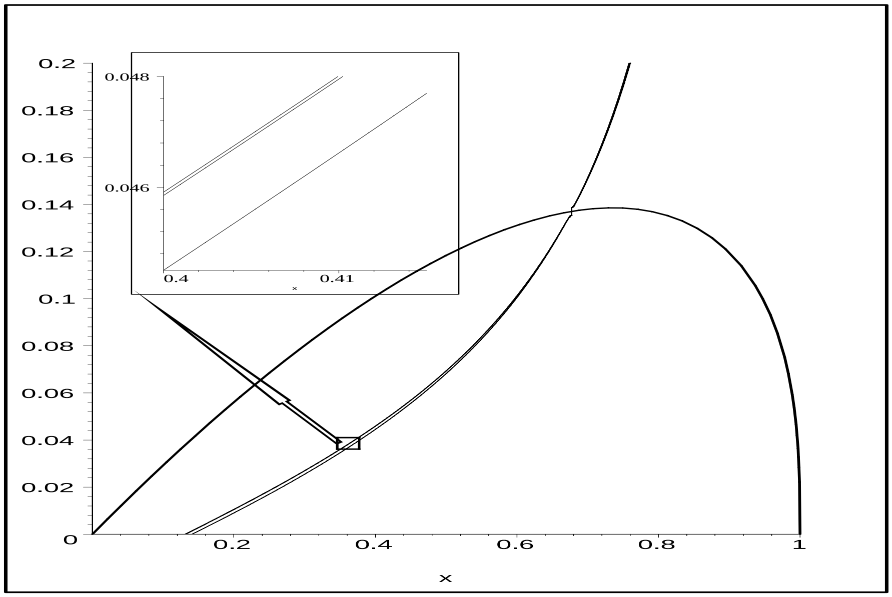

Figure 2: *This figure shows the function* $f(x) = u_0$ *(starts at* $(0, 0)$ *and ends at* $(1, 0)$*, and the approximate optimal harvests* u_1, u_2, u_3*. The commen poin for all curves is* $(x^*_\delta, u(x^*_\delta))$*, the equilibrium point. Absolute error after 1,2,3 iterations:* 0.058, 0.00167, 0.0000577. *Discount rate is 5%. Considering the curves for* $x < x^*_\delta$*, the lower curve is* u_1*,* u_2 *is the upper curve and* u_3 *is the middel curve. This can be seen more clearly in the window where a short portion of these curves ar enlarged. The unique solution* u^* *is located in between* u_2 *and* u_3*. This demostrates that* $u = \frac{1}{2}(u_2 + u_3)$ *is a very good approximation.*

optimal behaviour is a bang-bang policy with the switch at the equilibrium point. The CU-optimal policy when $\delta \to 0$ is given by $u = u_1$.

$$\frac{u_1(x) - f(x)}{a + u_(x)} = \operatorname{sgn}(x - x^*_\delta)\sqrt{\frac{(a + f(x))\hat{S}(x)}{a^2(p_1 - p_0)}} \overset{def}{=} \Lambda(x)$$

$$u_1(x) = \frac{a\Lambda(x) + f(x)}{1 - \Lambda(x)} \tag{37}$$

where the signum function stems from the definition of the region A^*. The above seed in the iterative process is only worth doing when one needs a formula for the solution. In the numerical analysis we can start with any function in the relevant region. However, $u_0 = f(x)$ is an excellent choice since it is a super-sub solution. By applying the parameter values from, Sandal and Steinshamn 1997b: $r = 0.3036$,

$K = 3.2 \times 10^6$, $\alpha = 0.3587$, $p_{min} = 200$, $p_{max} = 1250$, $a = 0.139 \times 10^6$, $q = 2.006 \times 10^8$. We plot for the case $\delta = 5\%$ the iterative solutions $\{u_0,\, u_1,\, u_2,\, u_3\}$.

3.2 Bluefin Tuna

In the article *Implications of a nested Stochastic/Deterministic Bio-Economic Model for a Pelagic Fishery*, McDonald et al. 2001, a bioeconomic model for Bluefin Tuna in the Southeren hemisphere is investigated. We will here illustrate our technique implemented on the model used in the cited paper. We summarize the basics in the following relations

$$\Pi(x,u) = \gamma(x)\, u - \Gamma(x)\, u^2 - \alpha(x)\,, \quad \dot{x} = f(x) - u\,. \tag{38}$$

Our iteration scheme then reads

$$\begin{aligned}
M(x,u) &= \Gamma(x)(u - f(x))^2\,, \\
\Pi_u(x,u) &= \gamma - 2\Gamma(x)\, u\,, \\
\mu(x) &= \gamma(x) - 2\,\Gamma(x)\, f(x)\,, \\
S(x) &= \gamma(x)\, f(x) - \Gamma(x)\, f(x) - \alpha(x)\,, \\
N(x)[u] &= S(x_\delta^*) - S(x) + \delta \int_{x_\delta^*}^{x} \gamma(s) ds - 2\delta \int_{x_\delta^*}^{x} \Gamma(s) u(s) ds \\
&= \hat{S}(x) - 2\delta \int_{x_\delta^*}^{x} \Gamma(s) u(s) ds\,.
\end{aligned}$$

Hence the iterations can be explicitly calculated, giving

$$u_{n+1}(x) = f(x) + \operatorname{sgn}(x - x_\delta^*) \sqrt{\frac{\hat{S}(x)}{\Gamma(x)} - \frac{2\delta}{\Gamma(x)} \int_{x_\delta^*}^{x} \Gamma(s)\, u_n(s) ds}\,. \tag{39}$$

We define the following basic functions for the problem, which can also be found in, McDonald et al, 2001 where a model dealing with Bluefin Tuna is discussed. We consider the case where the Bluefin Tuna is modeled by a surplus growth function with critical dipensation:

$$\Pi = a \cdot u - b \cdot u^2 - \frac{d}{x}\,, \quad f(x) = r \cdot (x - k) \cdot (1 - \frac{x}{K})\,, \tag{40}$$

where the following parameter values were used: $r = 0.2246$, $K = 0.565 \times 10^6$, $a = 88.25$, $b = 9.0 \times 10^{-4}$, $d = 1.63 \times 10^{11}$, $k = 0.716 \times 10^6$. In the following solution plots we have scaled to Carrying capacity.

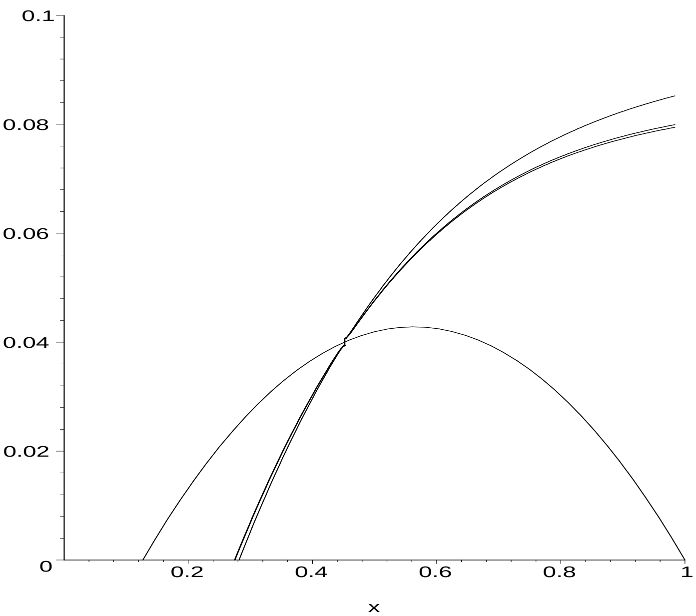

Figure 3: *This figure shows the function* $u_0 = f(x)$ *(ends at* $(1,0)$*), and the approximate optimal harvest* u_1, u_2, u_3. *The commen poin for all curves is* $(x_\delta^*, u(x_\delta^*))$, *the equilibrium point. Absolute error after 1,2,3 iterations:* 0.041, 0.00289, 0.000246. *Discount rate is 5%. Considering the curves for* $x > x_\delta^*$, *the upper curve is* u_1, u_2 *is the lowest curve and* u_3 *is the middel curve. The unique solution* u^* *is located in between* u_2 *and* u_3. *This demonstrates that* $u = \frac{1}{2}(u_2 + u_3)$ *is a very good approximation.*

The optimal solutions produced in the plots presented in these two examples are in full agreement with the cited reference.

4 Summary and discussion

We have presented a method for solving a class of control problems. The exposition in this work is restricted to a single state and single control situation. This kind of modeling approach requires lumped variables for the state and policy. However, we believe that one can be able to catch the main behavior of such systems by this kind of model. Our philosophy is to make the models as simple as possible but still containing the overall behavior that we want to model and investigate. The most important feature in our method of solution is a simple, fast and accurate scheme. The accuracy can be determined and is of course dependent of the number of iterations, and the specific problem investigated.

For the two examples presented here, taken from current literature, two or three iterations are sufficient to obtain an approximate solution with sufficient accuracy. This is visualized in Fig. 2 and Fig. 3. One basic feature in our formulation is that the source term dependency on the solution in the iteration scheme is of order δ where δ is the discounting rate which is normally less than 10% i.e. 0.1 in actual number. This is one of the key features that makes the iteration scheme converge so fast. The convergence rate can actually be investigated analytically and this analysis brings forward this feature. We do not include this analysis in the present account because we do not want to overburden this presentation with too much details also because a numerical check of the accuracy is easily available and it seems to be the rule that only very few iterations are needed. This fact that so few iterations make a sufficiently accurate solution also offers the possibility to analytically determine an approximate solution. This of course is a much more powerful tool for parameter search than any numerical solution can be.

Acknowledgment

Financial support from the Norwegian Research Council is acknowledged.

References

[1] Arnason, R. Sandal, L. K. Steinshamn, S. I. and Vestergaard, N. Optimal Feedback Controls: Comparative Evaluation of the Cod fisheries in Denmark, Iceland and Norway, in press, American Journal of Agricultural Economics,

[2] Clark, C. W., 1990. Mathematical Bioeconomics The Optimal Management of Renewable Resources. New York, John Wiley & Sons,.

[3] Grafton, R.Q. Sandal, L. K. and Steinshamn, S. I., 2000 How to improve the management of renewable resources: The case of Canada's northern cod fishery, American Journal of Agricultural Economics, 82, 570 - 580.

[4] Kamien and Schwartz, 1991. Dynamic Optimization. North-Holland, Amsterdam (second ed.) .

[5] McDonald, D., Sandal, L. K. and Steinshamn, S. I., 2001. Implications of a nested Stochastic/Deterministic Bio-Economic Model for a Pelagic Fishery, Ecological Modelling, 149, 1-2.

[6] Sandal, L. K. and Steinshamn, S. I. 1997a. A Stochastic Feedback Model for Optimal Management of Renewable Resources, Natural Resource Modeling, 10, 31 - 52.

[7] Sandal, L. K. and Steinshamn, S. I. 1997b. A feedback model for optimal management of renewable natural capital stocks, Canadian Journal of Fisheries and Aquatic Sciences, 54, 2475 - 2482.

[8] Sandal, L. K. and Steinshamn, S. I., 1997c. Optimal Steady State and the Effect of Discounting, Marine Resource Economics, 12, 95 - 105.

[9] Sandal, L. K. and Steinshamn, S. I., 1998. Dynamic corrective taxes with flow and stock externalities: A feedback approach, Natural Resource Modeling, 11, 217 - 239.

[10] Sandal, L. K. and Steinshamn, S. I., 2000. The Cost of Underutilization of Labour in Fisheries, Annals of Operations Research, 94, 1 - 13.

[11] Sandal, L. K. and Steinshamn, S. I., 2001a. A simplified feedback approach to optimal resource management, Natural Resource Modeling, 14,3, 419 - 432.

[12] Sandal, L. K. and Steinshamn, S. I., 2001b. A Bio-economic model for Namibian Pilchard, The South African Journal of Economics (Die Suid-Afrikaanse Tydskrif vir Ekonomie), 69, 2, 1 - 24.

[13] Seierstad, A. and Sydsæter, K., 1999. Optimal Control Theory with Economic Applications Second reprint, North-Holland, Amsterdam.

A Details Eqs. 25

From Lemma 3, the definitions given in Eqs. (23) and the iteration scheme, Eq. (24) we find

$$\begin{aligned}
T(u_{n+1}) &= (x - x^*_\delta) \cdot \{M(x, u_{n+1}) - N(x)[u_{n+1}]\} \\
&= (x - x^*_\delta) \cdot \{M(x, u_{n+1} - N(x)[u_n] + N(x)[u_n] - N(x)[u_{n+1}]\} \\
&= (x - x^*_\delta) \cdot \{N(x)[u_n] - N(x)[u_{n+1}]\} \\
&= T_n(u_n) - T_N(u_{n+1})\,.
\end{aligned}$$

By definitinon (Eq. (24)) we have $T_M(u_{n+1}) = T_N(u_n)$ and from the above relation $T(u_n) = T_N(u_{n-1}) - T_N(u_n) = T_M(u_n) - T_M(u_{n+1})$.