Pooling in Insurance

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Abstract

Risk sharing resulting in pooling of risk is considered. First pooling is discussed from the perspective of life and pension insurance. Second we take the perspective of Pareto optimal risk sharing, originating from a model introduced by Karl Borch in the late 50ties, and discuss when pooling may result, and also what is needed for pooling not to be optimal. Finally we illustrate by some examples, including a limited liability partnership.

KEYWORDS: Life and Pension Insurance, Reinsurance Exchange, Pareto Optimality, Pooling, Linear Sharing Rules, mutual insurance, limited liability partnership.

Introduction

Pooling of risk may roughly be taken to mean the following: Consider a group of I individuals, each one facing a certain risk represented by a random variable X_i, $i \in \mathcal{I} = \{1, 2, \ldots, I\}$. The individuals then decide to share the risks between themselves, rather than facing them in splendid isolation. Let the risk confronting individual i be denoted by Y_i after the exchange has taken place. Normally Y_i will be a function of $X = (X_1, X_2, \ldots, X_I)$. We call this pooling of risk if the chosen sharing rule is of the form $Y_i(X_1, X_2, \ldots, X_I) = Y_i(X_M)$, where $X_M := \sum_{i=1}^{I} X_i$, i.e., the final portfolio is a function of the individual risks $X_1, X_2, \ldots, X_I$ only through their aggregate X_M.

In this paper we want to identify situations where pooling typically plays an important role.

Life and Pension Insurance

Life and pension insurance dates back several hundred years, and is one important form of risk sharing where pooling is essential. In some of the extant literature, however, where different modern contracts are analyzed such as unit linked contracts, defined benefit based pension plans, defined contribution based pension plans, rate of return guarantees and the like, the simple idea of pooling seems to be more or less forgotten. In much of this literature idiosyncratic risk is ignored, which is surprising, since this is the rationale behind life insurance and pensions.

In order to explain the idea very briefly, consider a group of individuals, all of the same age x, each paying a certain amount to a fund, managed by a clerk. At the age of retirement, a fixed amount is paid back each year from the fund to the members of the group. As time goes, the amount received by the surviving individuals increases year by year, since the same fixed amount will have to be divided by an ever smaller number of individuals.

Here we notice several interesting features: First the members will enjoy a higher "interest rate" than the riskfree rate r_t at time t, because of mortality. Note that this is a simple consequence of pooling, allowing the individuals to gamble on their remaining lifetimes. Second, payments are received when needed, and only then, i.e., only when the member is alive. In conclusion, welfare at large is increased in a society where pooling is allowed, compared to a society where this is not the case, ceteris paribus.

We may of course quantify the addition to the interest rate r_t from pooling in life and pension insurance. Let $\mu_{x+t} > 0$ be the mortality rate at time t of an individual who pooled his risk with others at age x. Then one unit of investment grows to $1 \cdot e^{\int_0^n (r_s + \mu_{x+s})\, ds}$ by time n provided the recipient is alive then, assuming continuous compounding, which is clearly larger than $1 \cdot e^{\int_0^n r_s\, ds}$, the result without pooling.

Thus pooling is simply a good idea in itself, and there should be no need for any additional government support, such as tax crosssubsidization or other incentive mechanism to motivate people to engage in such arrangements.

Early references to the economics of life and pension insurance can be found in e.g., Hakanson (1969) and Yaari (1964-65).

Actually pooling permeates most of insurance, as our next example shows.

Reinsurance

Consider the situation in the introduction: Let the individuals be interpreted as reinsurers, having preferences $\succeq_i$ over a suitable set of random variables.

These preferences are represented by expected utility, meaning that $X \succeq_i Y$ if and only if $Eu_i(X) \geq Eu_i(Y)$. We assume smooth utility functions; here $u_i'(w) > 0, u_i''(w) \leq 0$ for all w in the relevant domains, for all $i \in \mathcal{I}$.

We suppose the agents can negotiate any affordable contracts among themselves, resulting in a new set of random variables $Y_i, i \in \mathcal{I}$, representing the possible final payout to the different members of the group, or the final portfolio.

In characterizing reasonable sharing rules, we concentrate on the concept of Pareto optimality:

Definition 1 *A feasible allocation $Y = (Y_1, Y_2, \ldots, Y_I)$ is called Pareto optimal if there is no feasible allocation $Z = (Z_1, Z_2, \ldots, Z_I)$ with $Eu_i(Z_i) \geq Eu_i(Y_i)$ for all i and with $Eu_j(Z_j) > Eu_j(Y_j)$ for some j.*

Consider for each nonzero vector $\lambda \in R_+^I$ of agent weights the function $u_\lambda(\cdot) : R \to R$ defined by

$$u_\lambda(v) =: \sup_{(z_1,\ldots,z_I)} \sum_{i=1}^I \lambda_i u_i(z_i) \qquad \text{subject to} \qquad \sum_{i=1}^I z_i \leq v. \qquad (1)$$

As the notation indicates, this function depends only on the variable v, meaning that if the supremum is attained at the point $(y_1, \ldots, y_I)$, all these $y_i = y_i(v)$ and $u_\lambda(v) = \sum_{i=1}^I \lambda_i u_i(y_i(v))$. It is a consequence of the *Implicit Function Theorem* that under our assumptions, the function $u_\lambda(\cdot)$ is two times differentiable in v. The function $u_\lambda(v)$ is often called the supconvolution function, and is typically more "well behaved" than the individual functions $u_i(\cdot)$ that make it up.

The following fundamental characterization is proved using the Separating Hyperplane Theorem (see e.g., Aase (2002))

Theorem 1 *Suppose u_i are concave and increasing for all i. Then Y is a Pareto optimal allocation if and only if there exists a nonzero vector of agent weights $\lambda \in R_+^I$ such that $Y = (Y_1, Y_2, \ldots, Y_I)$ solves the problem*

$$Eu_\lambda(X_M) := \sup_{(Z_1,\ldots,Z_I)} \sum_{i=1}^I \lambda_i Eu_i(Z_i) \qquad subject\ to \qquad \sum_{i=1}^I Z_i \leq X_M. \qquad (2)$$

Here the real function $u_\lambda : R \to R$ satisfies $u_\lambda' > 0, u_\lambda'' \leq 0$.

Notice that since the utility functions are strictly increasing, at a Pareto optimum it must be the case that $\sum_{i=1}^I Y_i = X_M$.

We now characterize Pareto optimal allocations under the above conditions. This result is known as Borch's Theorem:

Theorem 2 *A Pareto optimum Y is characterized by the existence of non-negative agent weights $\lambda_1, \lambda_2, \ldots, \lambda_I$ and a real function $\lambda : R \to R$, such that*

$$\lambda_1 u_1'(Y_1) = \lambda_2 u_2'(Y_2) = \ldots = \lambda_I u_I'(Y_I) = u_\lambda'(X_M) \qquad a.s. \qquad (3)$$

In proving the above theorem we have used the Saddle Point Theorem and directional derivatives in function space.

Existence of Pareto optimal contracts has been treated by DuMouchel (1968). The requirements are, as we may expect, very mild.

As a consequence of this theorem *pooling* occurs in a Pareto optimum. Consider agent i: His optimal portfolio is $Y_i = (u_i')^{-1}(u_\lambda'(X_M)/\lambda_i)$, which clearly is a function of $(X_1, X_2, \ldots X_I)$ only through the aggregate X_M. Here $(u_i')^{-1}(\cdot)$ means the inverse function of u_i', which exists by our assumptions. Pooling is simply a direct consequence of (2) in Theorem 1. Let us look at an example.

Example 1: Consider the case with negative exponential utility functions, with marginal utilities $u_i'(z) = e^{-z/a_i}, i \in \mathcal{I}$, where a_i^{-1} is the *absolute risk aversion* of agent i, or a_i is the corresponding *risk tolerance*. Using the above characterization (3), we get

$$\lambda_i e^{-Y_i/a_i} = u_\lambda'(X_M), \qquad a.s., \qquad i \in \mathcal{I}.$$

After taking logarithms in this relation, and summing over i, we get

$$u_\lambda'(X_M) = e^{(K-X_M)/A}, \qquad a.s. \qquad \text{where} \qquad K := \sum_{i=1}^I a_i \ln \lambda_i, \qquad A := \sum_{i=1}^I a_i.$$

Furthermore, we get that the optimal portfolios are

$$Y_i = \frac{a_i}{A} X_M + b_i, \qquad \text{where} \qquad b_i = a_i \ln \lambda_i - a_i \frac{K}{A}, \qquad i \in \mathcal{I}.$$

Note first that the reinsurance contracts involve pooling. Second, we see that the sharing rules are are *affine* in X_M.

The constant of proportionality a_i/A is agent i's risk tolerance a_i relative to the group's risk tolerance A. In order to compensate for the fact that the least risk-averse agent will hold the larger proportion, zero-sum side payments b_i occur between the agents. $\square$

The other example where the optimal sharing rules are affine is the one with power utility with equal coefficient of relative risk aversion throughout the group, or the one with logarithmic utility, which can be considered as a limiting case of the power utility function. Denoting agent i's risk tolerance function by $\rho_i(x_i)$, the reciprocal of the absolute risk aversion function, we can in fact show the following (see e.g., Wilson (1968)):

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Theorem 3 *The Pareto optimal sharing rules are affine if and only if the risk tolerances are affine with identical cautiousness, i.e., $Y_i(x) = A_i + B_i x$ for some constants $A_i, B_i, i \in \mathcal{I}, \sum_j A_j = 0, \sum_j B_j = 1, \Leftrightarrow \rho_i(x_i) = \alpha_i + \beta x_i$, for some constants β and $\alpha_i, i \in \mathcal{I}$.*

No Pooling

Let us address two situations when we can not expect pooling to occur.

(i) If the utility functions of the insurers depend also on the other insurers' wealth levels, i.e., $u = u(x_1, \dots, x_I)$, pooling may not result.

(ii) Suppose the situation does not give affine sharing rules as in the above example, but one would like to *constrain* the sharing rules to be affine, i.e., of the form

$$ Y_i = \sum_{k=1}^{I} b_{i,k} X_k + b_i, \qquad i = 1, 2, \dots I. $$

Here the constants $b_{i,k}$ imply that the rule keeps track of who contributed what to the pool. In general the constrained optimal affine sharing rule will not have the pooling property, which of course complicates matters. However, these constrained affine sharing rules are likely to be both easier to understand, and simpler to work with than nonlinear Pareto optimal sharing rules.

Some examples

The above model is formulated in terms of a reinsurance syndicate. Several such syndicates exist in the US, in Europe the Lloyds of London is the most prominent example.

However, other applications are manifold, since the model is indeed very general. For instance,

- X_i might be the initial portfolio of a member of a *mutual* insurance arrangement. The mutual fire insurance scheme which still can be found in some rural communities may serve as one example. Shipowners have formed their P& I Clubs, and off shore oil operators have devised their own insurance schemes, which may be approximated by the above model.

- X_i might be the randomly varying water endowments of agricultural region (or hydro-electric power station) i;

- X_i might be the initial endowment of individual i in a *limited liability partnership*;

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- X_i could stand for nation i's state-dependent quotas in producing diverse pollutants (or in catching various fish species);
- X_i could account for uncertain quantities of different goods that transportation firm i must bring from various origins to specified destinations;
- X_i could be the initial endowments of shares in a stock market, in units of a consumption good.

Consider an example of a limited liability partnership. A ship is to be built and operated. The Operator orders the ship from the Builder, and borrows from the Banker to pay the Builder. The cautious Banker may require that the operator consults an Insurer to insures his ship. Here we have four parties involved in the partnership, and it is easy to formulate a model of risk sharing that involves pooling, and where the above results seem reasonable. More along these lines can be found in e.g., Borch (1979).

Conclusion

A short review of pooling has been presented. We have given an informal discussion of pooling in life insurance, and also a more concrete discussion of pooling in reinsurance. Since many of the results in reinsurance carry over also to direct insurance, the results are valid in a rather general setting.

Furthermore, we have indicated situations of risk sharing outside the field of insurance, where the model seems appropriate.

The pioneer in the field is Karl Borch, see e.g. Borch (1960a), (1960b),(1962) and many other articles by Borch (he published 150 papers, among them 3 books). See also Bühlmann (1980) for making Borch's results better known to actuaries, and also improving some of his original proofs. Lemaire (1990), (2004) mainly review the theory. Gerber (1978), Wyler (1990) and Aase (1993) review and further expanded the theory of optimal risk sharing as related to pooling. Aase (2002) contains a review of the theory, and also gives some new results on game theory, and moral hazard in the setting of insurance, presenting modern proofs of many of the classical results.

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