An Approach to Adaptive Carbon Taxes in the Presence of Global Warming

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Abstract

The optimal extraction path of fossile fuels and the corresponding corrective tax on extraction are derived when two types of externalities associated with emission of carbondioxide (CO_2) are taken into account. The optimal path is derived as a feedback control, that is, as a function of time and pollution. The tax-path is thereby adaptive to the aggregated level of carbondioxide. The two types of externalities are flow externalities associated with the extraction, defined as the difference between private and social marginal costs, and stock externalities associated with the aggregated level of CO_2.

The total time horizon is divided into two periods: an initial phase with extraction and a terminal phase without extraction. The lengths of these periods are endogeneously determined, as is the scarcity rent of fossile fuels and the shadow cost of CO_2. The mathematical model is completely general in the state variable, CO_2, the decay function included. Furthermore, there are no assumptions about separability in the objective function, which is to maximize social benefit.

INTRODUCTION

The literature on flow and stock externalities is quite extensive. This literature is divided in two parts. On the one hand, there are static models concentrating upon flow externalities (see for example Baumol and Oates 1988, Bovenberg and van der Ploeg 1994). These are the externalities associated with extraction itself, for example, the flow of pollution connected with production or consumption. On the other hand, there are dynamic models concentrating upon stock externalities (Plourde 1972, Keeler et al. 1972, d'Arge and Kogiku 1973, Brito and Intriligator 1987, Ko et al. 1992, Ulph and Ulph 1994, to mention a few). These are the externalities associated with the aggregate level of pollution. Very few attempts so far have been made to include flow externalities in dynamic models. Therefore authors rightly conclude that the optimal corrective tax should equal the shadow cost of the pollutant with the shadow cost defined as the costate variable in the dynamic optimization problem. Here it is shown that such a tax corrects for the stock externality but not for the flow externalities, if any. Wirl (1994a) is one of the few who attempt to take both flow and stock externalities into account simultaneously, but he uses a number of special assumptions such as zero private marginal cost, linear-quadratic formulation of the model and no decay of pollution.

In connection with stock externalities a whole literature has emerged on global warming (Nordhaus 1982, 1991a, 1991b, Peck and Teisberg 1992, Sinclair 1992, 1994, Wirl 1994a, 1994b). A significant contribution to this literature has been made by Sinclair (1992) and Ulph and Ulph (1994) by combining the time path of carbon taxes with the extraction of nonrenewable resources (fossile fuels). They do not, however, take the flow externalities into account. It is impossible to take flow externalities into account without making the damage function a function of production (extraction) of fossile fuels or of the time derivative (flow) of the pollutant.

In the literature on the economics of global warming the question of whether a carbon tax should increase or decrease over time has received considerable attention (Sinclair 1992, Ulph and Ulph 1994). In most of the literature on stock externalities, however, quite a few special assumptions are made. One such assumption, as mentioned above, is that the optimal tax on pollution should always equal the shadow cost of the pollutant. Other common assumptions are that the decay of pollution is linear and that social welfare is separable in consumption and the pollutant stock (Plourde, 1972, Keeler et al., 1972, Ulph and Ulph, 1994, to mention a few). In the case of carbon emissions, this assumption is at best true in the short run. On a longer time scale the assimilation of carbon in the ocean will gradually be reduced due to saturation. Hence, decay as a function of the aggregate level of CO_2 will be decreasing sooner or later. In a time horizon of more than ten years the decay function will certainly be a non-linear function (see Joos et al. 1996). In addition many papers concentrate on analysis of steady states (Forster, 1975, Brito and Intriligator, 1987, Ko et al., 1992).

The aim of the present paper is to derive an optimal corrective tax for a dynamic problem with both flow and stock externalities, and to investigate analytically the time path of the tax. The reason analytical results are emphasized, is that such results do not depend upon particular numerical specifications or assumptions. It makes it possible to perform general parameter analysis. In order to account for numerical considerations, the feedback control law derived for the corrective tax is examplified. A feedback control makes it possible to determine the optimal corrective tax at any point in time as an explicit function of the stock pollutant.

This approach represents a genuine example of adaptive regulation as the optimal carbon taxes change each time new information about the aggregate level of CO_2 is available. In this paper CO_2 represents all of the so-called greenhouse gases. A similar approach to adaptive corrective taxes, but without the nonrenewable resource aspect, can be found in Sandal and Steinshamn (1998).

THE MODEL

The model applied here is a generalisation of the model applied by Ulph and Ulph (1994). The objective is

$$\max_x W(x,a) = \int_0^{\infty} e^{-rt} \left[B(x,a) - \alpha(a) \right] dt \tag{1}$$

where B is the social benefit derived from production and consumption of the good, x, and α is the damage or disutility from the aggregate level of pollution (CO_2), a, which represents the stock externality. The product, x, can be thought of as the flow rate of fossile fuels on the market. The functions B and α are *completely general* in a. The only restriction is that the benefit function is quadratic in x. This is, for example, the case when B is thought of as the sum of consumers' and producers' surplus under linear supply and demand curves, adjusted for externalities. The inclusion of a in B then describes how the level of CO_2 affects the demand and cost structure. The infinite time horizon is divided in two periods, an initial phase up to T with positive extraction and a terminal phase with zero extraction from T to infinity.

Denoting the remaining stock of fossile fuels s, equation (1) has to be maximized subject to the dynamic constraints[1]

$$\dot{s} = -x, \tag{2}$$

and

$$\dot{a} = \delta \left[x - f(a) \right]. \tag{3}$$

The parameter δ is the emission of CO_2 per unit of fossile fuel consumed whereas f is the decay function which is a completely general function. The decay measured in CO_2 units is δf whereas f represents the decay measured in fossile fuel units. A

[1]Dots denote time derivatives.

genereal decay function is useful as the decay of CO_2 through photosynthesis may be a very complex process. Global warming may affect the growth of forests and phytoplankton which again affect the CO_2 level. In addition to this,

$$\begin{aligned} s(0) &= s_0 > 0,\ \underline{s} \in (0, s_0),\ s_0 - \underline{s} = \Delta s = \int_0^T x\,dt, \\ x &\geq 0,\ a \geq 0,\ \lim_{t\to\infty} a = 0. \end{aligned}$$

The initial stock of fossile fuel, s_0, is given. The stock level $\underline{s}$ is also exogeneously given and is the stock level below which the costs of extracting increase to infinity. The level $a = 0$ is defined as the natural, pre-industrial level of CO_2 which is a natural steady state and does not harm the global climate. That is $f(0) = 0$ and $\alpha(0) = 0$.

The benefit function represents the sum of producers and consumers surplus adjusted for any flow externalities. That is

$$B(x, a) = \int_0^x [D(y, a) - C^s(y, a)]\ dy$$

where D is the inverse demand function and C^s is the social marginal cost of extraction. This means that by maximizing B the flow externalities are internalized. The functions D and C^s are assumed to be linear,

$$\begin{aligned} D(x, a) &= p_0(a) - p_1(a)x, \\ C^s(x, a) &= c_{0s}(a) + c_{1s}(a)x, \end{aligned}$$

and B can then be written as a quadratic function in x,

$$B(x, a) = \beta(a)x - \gamma(a)x^2 \tag{4}$$

where

$$\begin{aligned} \beta(a) &= p_0(a) - c_{0s}(a), \\ \gamma(a) &= \frac{1}{2}\left(p_1(a) + c_{1s}(a)\right). \end{aligned}$$

At any point in time market clearing is assumed, implying that the equilibrium level of x is given by $D = C^p$ without any policy measures. The function C^p is the supply function corresponding to the private marginal cost of production,

$$C^p(x,a) = c_{0p}(a) + c_{1p}(a)x.$$

In other words, C^p is the market supply of x in a competitive economy. The flow externality is defined as the difference $C^p - C^s$ which is a function of x for any given level of a. The optimal tax also must include the scarcity rent of the resource. In Ulph and Ulph (1994) the scarcity rent is instead included in the producer price; in other words they assume a monopolist who controls the resource.

Above we have an argument for choosing an objective function that is quadratic in extraction, x. On the other hand, there are no arguments for choosing special functional forms with respect to the pollutant, a. It is quite common in the literature to choose objective functions that are quadratic both in the control variable and the state variable and constraints that are linear in both (so-called linear-quadratic models) with the only justification being mathematical convenience. In the present model both the objective function and the constraints are general in the state variable, a. In other words, it is assumed that demand, as well as social and private marginal costs can be affected arbitrarily by the level of CO_2, due to changes in environmental concern among consumers or changing production costs.

As mentioned, there are two types of externalities in this paper. The stock externality associated with the aggregate level of CO_2, is represented by the damage term α. The flow externality associated with the production of x is represented by the difference between social and private marginal costs for any given level of a. This is defined as a flow externality because production is directly associated with the flow of pollution, the time derivative $\dot{a}$, through Eq. (3). A pure flow externality is the reduction in welfare associated with δx in Eq.(3). The common thing to do in the literature is to focus solely upon flow externalities in static models and solely upon stock externalities in dynamic models. Here both externalities are included simultaneously in a dynamic model.

The externalities referred to above indicate that there is need for some policy instruments in the form of quotas or corrective taxes, and in this paper we will concentrate on two possibilities, namely a tax per unit of fossile fuel extracted, τ, or an ad valorem tax, θ. It is important to keep in mind that these instruments are in effect only during the initial phase with extraction. As a consequence of the market clearing condition, there will be a one-to-one correspondance between the unit tax,

$$\tau(x,a) = D(x,a) - C^p(x,a) \tag{5}$$

and the level of extraction, x, for a given value of a. Here C^p is the producer price and $D = C^p + \tau$ is the consumer price. Note that maximizing the sum of the consumers' surplus as a function of the consumer price, the producers' surplus as a function of the producer price, and the government's surplus which is the tax revenue, is equivalent to maximizing $B - \alpha$.

Given the assumptions above, x can be written as a linear function of τ,

$$x = X_0 - X_1\tau \tag{6}$$

where

$$\begin{aligned} X_0 &= \frac{p_0 - c_{0p}}{p_1 + c_{1p}}, \\ X_1 &= \frac{1}{p_1 + c_{1p}}. \end{aligned}$$

The interpretation of X_0 is the market equilibrium without any policy measures. Similarly, there will be a one-to-one correspondance between the ad valorem tax

$$\theta(x,a) = \frac{D(x,a) - C^p(x,a)}{C^p(x,a)} \tag{7}$$

and x.

A corrective tax based on the shadow cost of pollution defined as the costate variable, which is quite common in the literature, corrects only for the stock externality, whereas a corrective tax $\tau = C^s - C^p$ corrects only for the flow externality (this is the common result from static models, see for example Baumol and Oates, 1988).

As there is a one-to-one correspondance between the instruments, τ or θ, and production, x, it does not matter whether we choose τ, θ or x as the control variable in the mathematical model. The approach taken here is that we first find the optimal extraction level, x, and then substitute this into (5) or (7) to find the unit tax or the ad valorem tax respectively.

The infinite time horizon is divided in two periods: one with extraction, $x > 0$, up to T, and one with zero extraction, $x = 0$, from T to infinity. At what time it is optimal to stop extraction is determined by the alternatives we have. It is assumed that there exists a substitute to fossile fuels which does not emit CO_2 and which yields a constant social benefit $\widehat{B}$. Depending upon the magnitude of $\widehat{B}$, the switching time, T, may vary from zero to infinity. As the corrective tax is on extraction, x, it is not possible to levy any tax when $t > T$ even though the harmful effects, $\alpha(a)$, continue to exist in this period. An optimal tax in the initial phase, therefore, also has to take into account the stock externality in the terminal phase.

Let $\mathcal{H}$ denote the Hamiltonian and let m and n denote the costate variables associated with a and s respectively. The optimization problem and its solution is then outlined in Table 1 below.

Table 1

Description	Initial phase	Terminal phase
Time	$0 \leq t \leq T$	$T < t \leq \infty$
Production	$x > 0,\ x(T) = x_T$	$x = 0$
Social utility	$B(a,x) - \alpha(a)$	$\widehat{B} - \alpha(a)$
Hamiltonian	$\mathcal{H} = B(x,a) - \alpha(a) +$	$\mathcal{H} = \widehat{B} - \alpha(a) - \delta m f(a)$
	$\delta m\,[x - f(a)] - nx$	
Dynamic constraint	$\dot{a} \ = \delta\,[x - f(a)]$	$\dot{a} \ = -\delta f(a)$
Interior solution	$\delta m - n = -B_x$	
Costate equation	$\dot{n} \ = rn - \mathcal{H}_s = rn$	
	$(\Rightarrow n(t) = n_0 e^{rt})$	
Costate equation	$\dot{m} \ = (r + \delta f')m + \alpha' - B_a$	$\dot{m} \ = (r + \delta f')m + \alpha'$

The interpretation of the costate variables is that m is the shadow cost of pollution (CO_2) wheras n is the scarcity rent of the resource and n_0 is a constant to be determined. All variables are in current values.

In addition to this we have in general that

$$\frac{d\mathcal{H}}{dt} = r(m\dot{a} - xn) \tag{8}$$

and we have the requirement that the Hamiltonian and the costate variables are continuous at time T. The state variables in this maximization problem are a and s. As the stock of fossile fuel, s_0, is limited, the system will not settle on a steady state with respect to s but the trivial one, $s = \underline{s}$.

By defining zero as the pre-industrial level of CO_2 which is the natural steady state, we have $f(0) = 0$. Further, by solving $\dot{a} = -\delta\, f(a)$, we can define

$$-(t-T) = \psi(a) \equiv \frac{1}{\delta}\int_{a_T}^{a} \frac{da}{f(a)} \quad \text{for} \quad t > T.$$

With this definition we get

$$\delta m f e^{r\psi} = -\int_0^a \alpha'(a) e^{r\psi(a)} da \tag{9}$$

by solving the equation for $\dot{m}$ in Table 1 for $t > T$. Notice that $e^{r\psi} \to 0$ when $t \to \infty$. As $a_\infty = 0$, we have $mfe^{r\psi} \to 0$ when $t \to \infty$. From (9) it is seen that $m < 0$ when $t > T$.

It is easily seen that $\mathcal{H}(r > 0) < \mathcal{H}(r = 0)$, that $\mathcal{H}(r > 0)$ is increasing in time, and that $\mathcal{H}(r > 0) = \mathcal{H}(r = 0) = \widehat{B}$ when $t \to \infty$.

It is now possible to determine in which cases the optimal corrective tax should correspond to the shadow price of pollution, defined as the costate variable, m, given the assumptions in this model. Let σ denote a corrective tax based on the shadow price. As the shadow price is per unit pollution and the tax is supposed to be per unit production, this tax must be given by

$$\sigma = -\delta m. \tag{10}$$

From Table 1 it is seen that in the initial phase,

$$\begin{aligned} \sigma &= B_x - n = \beta - 2\gamma x - n \\ &= \beta - n - 2\gamma X_0 + 2\gamma X_1 \tau. \end{aligned} \tag{11}$$

In other words, the only case in which $\sigma = \tau$ is when $2\gamma X_1 = 1$ and $\beta - 2\gamma X_0 = 0$. (Remember that n adjusts for the scarcity of the resource in a competitive economy and is therefore neglected in this context). From the definitions of β and γ in (4) it is seen that this will be the case if and only if there is no difference between social and private marginal costs in the model, but that is exactly the case when there is no flow externality and the only externality is the stock externality represented by α.

In the general case it is easily seen that $\tau > \sigma + n$ as $\sigma + n = B_x = D(x,a) - C^s(x,a)$ whereas, by definition, $\tau = D(x,a) - C^p(x,a)$ and $C^s(x,a) > C^p(x,a)$ for all values of x. An alternative way of stating this is to write the optimal corrective tax as $\tau = \sigma + n + (C^s - C^p)$ where σ accounts for the stock externality, n accounts for the scarcity rent and $(C^s - C^p)$ accounts for the flow externality. Remember that a corrective tax implemented as a unit tax or an advalorem tax can only be collected in the initial phase with positive production.

Zero discounting

In order to study the optimal paths of the control variable x and the corresponding per unit tax and ad valorem tax, it is useful to derive the optimal control as an explicit function of the state variable(s); that is, as a feedback control law. A feedback control law represents true adaptive regulation as the optimal tax is directly affected by changes in the environment.

We start with the case of a zero discount rate. The results in this section are less complicated and are of general interest as introduction of a non-zero discount rate only implies small perturbations of the qualitative (and quantitative results). This is in accordance with Nordhaus' (1991a) claim that discounting is of second-order importance and can be ignored. To do a complete dynamic analysis, however, discounting is included in the next section. When it comes to numerical implementations of the model it is more worthwile trying to estimate the correct non-linearity coefficient, γ, than finding the "correct" social discount rate.

In the case of zero discounting, the optimal feedback rule for the control variable, x, is readily derived (see Appendix for details). Let

$$S(a) \equiv B\left(f(a), a\right) - \alpha(a) = \beta(a)f(a) - \gamma(a)f(a)^2 - \alpha(a)$$

be defined as the level of social welfare obtained by fixing the level of CO_2; that is, keep production at the level $x = f$ such that $\dot{a} = 0$. Let M be defined by

$$M(a) = \frac{\widehat{B} + n_0 f(a) - S(a)}{\gamma(a)}. \tag{12}$$

Assuming an interior solution, that is M non-negative, the following theorem can be stated:

Theorem 1 *The feedback control law for the optimal extraction path in the case of zero discounting is given by*

$$x(a) = f(a) \pm \sqrt{M(a)}. \tag{13}$$

The corresponding optimal unit tax and ad valorem tax can be found by substitutuing this into (5) and (7) respectively.

Proof: Using $\delta m - n = -B_x$ from Table 1 the optimal Hamiltonian can be written $\mathcal{H} = S - n_0 f + \gamma(x - f)^2$. Combining $\mathcal{H} = \widehat{B}$ for $t > T$ from (8) with the definition of M then gives (13)■

Eq. (13) represent an explicit solution to a large class of dynamic optimisation problems. Plus is chosen if $a_0 < a_T$ and minus is chosen if $a_0 > a_T$, where a_0 is the exogenously-given level of CO_2 at $t = 0$ and a_T is the endogenously determined level at T. The time derivative, $\dot{a} = \mp\sqrt{M}$, will not change sign along an optimal path for $t < T$.

One question remains, and that is how a_T, x_T, n_0, m_T and T are determined. The requirement that $\mathcal{H}$ and m are continuous at T, yields

$$B(a_T, x_T) - x_T B_x(a_T, x_T) = \gamma(a_T) x_T^2 = \widehat{B}$$

which gives us x_T as a function of a_T :

$$x_T(a_T) = \left(\frac{\widehat{B}}{\gamma(a_T)}\right)^{\frac{1}{2}} \geq 0.$$

The state equation $\dot{a} = \delta(x - f(a))$ yields

$$T(a_0, a_T) = \frac{1}{\delta}\left|\int_{a_0}^{a_T} \frac{da}{x(a) - f(a)}\right|,$$

and the shadow value is given as a function, $m_T(a_T)$, by

$$\delta m_T\, f(a_T) = -\int_0^{a_T} \alpha'(y)dy = -\alpha(a_T)$$

from (9). This value inserted in

$$\delta m_T - n_T = -B_x(a_T, x_T) = -\beta(a_T) + 2\gamma(a_T)x_T,$$

from Table 1, yields

$$n_T(a_T) = \beta(a_T) - \frac{\alpha(a_T)}{f(a_T)} - 2\gamma(a_T)x_T = n_0.$$

Now we have found T, x_T, m_T and n_0 as functions of a_T. The only thing that remains to determine is a_T itself, and this is given by the resource constraint

$$s_0 - \underline{s} \equiv \Delta s = \int_0^T x\, dt = \int_{a_0}^{a_T} \frac{x(a)}{\delta\left[x(a) - f(a)\right]} da$$

when s_0 and $\underline{s}$ are known. More specifically we have

$$\delta\Delta s = a_0 - a_T + \mathrm{sgn}(a_T - a_0)\int_{a_0}^{a_T} \frac{f(a)}{\sqrt{M(a)}} da.$$

From this we see that T, x_T, m_T and n_0 are determined endogeneously by the optimisation. In particular the resource rent, n_0, is determined endogeneously and is not given exogeneously as is often assumed in the literature. The reason this can be done here is that the flow externality (the difference between social and private costs) has been specified from the beginning and has not been disguised in the social benefit function, B. Next we find an explicit feedback control law for the case with positive disconting.

Discounting

Also in the case of a positive discount rate it is possible to derive explicitly analytical asymptotic expressions for the optimal production level as a feedback control law. As discounting is of second-order importance this is only done to establish a complete dynamic analysis. In order to do so, we have to resort to perturbation methods, and the analysis becomes slightly more complicated. The technical details can be found in the Appendix. It is of course always possible to derive the optimal production as a feedback rule numerically both for the case of zero and positive discounting. Numerical analysis cannot, however, yield general conclusions about the optimal time paths of the corrective taxes, so emphasis best placed on analytical expressions.

By defining

$$\widetilde{S} \equiv S + r\int_{a_0}^{a} \mu(z)dz,$$

where $\mu \equiv \frac{\partial B}{\partial x}\Big|_{x=f} = \beta - 2\gamma f$, an explicit expression for the optimal production level is given in the following theorem:

Theorem 2 *The feedback control law for the optimal extraction path in the case of positive discounting is given to the lowest order by*

$$x(a, r, t) = f(a) \pm \sqrt{\frac{\widehat{B}e^{2r\delta t} + n_0 e^{r\delta t} f(a) - \widetilde{S}}{\gamma(a)}}. \tag{14}$$

The corresponding optimal unit tax and ad valorem tax can be found by substitutuing this into (5) and (7) respectively.

The proof for and derivation of (14) is in the Appendix. Note that the difference between $\widetilde{S}$ and S has an interesting interpretation. As μ is the marginal change in benefit when the aggregate level of CO_2 is fixed, that is when $x = f(a)$, the integral $\int \mu\, da$ can be interpreted as the benefit of the change in a when it changes at the rate $\dot{a} = f(a)$, but that is when there is no extraction. Hence the finite integral $\int_{a_0}^{a} \mu\, dz$

represents the benefit of a change in a from a_0 to a with zero extraction, or, in other words, an investment in the environment by stopping extraction. The term $r\int_{a_0}^{a}\mu\, dz$ is simply the alternative rate of return on this investment. The function $\widetilde{S}$ can now be interpreted as the difference between the benefit accruing from fixing the level of CO_2, S, and the alternative rate of return on an investment in the environment accruing from stopping all extraction immediately.

A NUMERICAL EXAMPLE.

The feedback rule derived in the previous sections can be used to determine the optimal extraction path both as a function of the aggregate level of CO_2 and as a pure function of time. The former seems, however, to be more interesting from a regulators point of view.

In this section the model is illustrated by a numerical example using quasi-realistic data. In this example a zero discount rate is assumed as reasonably small discount rates only imply small perturbations to the results. Further, linear decay and a quadratic damage function (stock externality) are assumed. The linear decay function may be of more general interest, as one way of approaching nonlinear decay functions is to use piece-wise linear functions.

A special class of models

Before we go on to the numerical specifications of the model, some general comments can be made on the following special class of models. Assuming linear decay, $f(a) = \phi a$, and quadratic damage, $\alpha(a) = \alpha_0 a^2$, it is relatively easy to show that

$$\left(\frac{da}{d\tau}\right)^2 = (a-\xi)^2 + \zeta$$

where

$$\xi = \frac{\phi(\beta - n)}{2(\alpha_0 + \gamma\phi^2)},$$
$$\zeta = \frac{\widehat{B}}{\alpha_0 + \gamma\phi^2} - \xi^2$$

and $\tau = \delta\sqrt{\alpha_0 + \gamma\phi^2} \cdot t$ is the new time-scale. This is the standard representation of a hyperbola. In other words, with these assumptions the phase-plane for a will always consist of hyperbolae. The optimal path is given by one or multiple hyperbolae for linear and piece-wise linear decay functions respectively. This also makes it possible to derive the optimal development in $a(t)$ as an explicit function:

$$a(t) = \begin{cases} \xi \pm |\zeta|^{1/2} \sinh\left[\tau - \tau^*\right], & \zeta > 0, \\ \xi \pm |\zeta|^{1/2} \cosh\left[\tau - \tau^*\right], & \zeta < 0, \\ \xi + \exp\left[\pm(\tau - \tau^*)\right], & \zeta = 0, \end{cases}$$

where τ^* is a reference time (constant of integration). The saddle-point is characterized by $a = \xi$ and $\dot{a} = 0$. This also explains why the optimal tax is decreasing or increasing over time depending upon whether we start below or above a_T, see Sinclair (1992) and Ulph and Ulph (1994). Now the optimal extraction path, and the corresponding tax, can be found as explicit functions of time by substituting into

$$x(t) = \frac{1}{\delta}\dot{a} + f(a).$$

Numerical results

Next we look at the numerical specifications of the model. The aggregate level as well as emissions of CO_2 are measured in giga tonnes CO_2 (Gt-CO_2). One Gt-CO_2 corresponds to 7.81 parts per million (p.p.m.) which is another common measure of carbon. The year 1997 is used as the base year, $t = 0$, and the meteorological data are given in Table 2. The value $\underline{a}$ is the pre-industrial level which is a natural steady state without extraction. This level is estimated to 2187 and the current level, a_0, is 2812. The parameter ϕ has been calculated according to $\dot{a}_0 = \delta\,[x_0 - f(a)]$ where $f(a) = \phi(a - \underline{a})$.

Table 2. Meteorological data

Parameter	Value	Parameter	Value
$\underline{a}$	2187	a_0	2812
x_0	21.9	$\dot{a}_0$	11.7
δ	1	$\widehat{B}$	75
ϕ	0.016	Δs	3000
		$\underline{s}$	0

The data in Table 2 has been provided by the Nansen Environmental and Remote Sensing Centre in Bergen, Norway, except $\widehat{B}$, which is a guesstimate. The parameter δ has been set to one such that a and x are measured in the same units. Also the stock of fossile fuels, s, are measured in the same units. The economic data given in Table 3 are guesstimates.

Table 3. Economic data

Parameter	Value	Parameter	Value
p_0	11.95	p_1	0.5
c_{0p}	1	c_{1p}	0
c_{0s}	1	c_{1s}	0.05
$\alpha(a)$	$1.5 \cdot 10^{-5} \cdot a^2$		

It is seen from Table 3 that the current private marginal cost of extraction is assumed to be constant and has been normalized to one. Further, a price approximately twelve times higher than the current private marginal cost is assumed to choke all demand. The market equilibrium level with these parameter values is equal to current emissions, and the social marginal cost at this level is about twice the private marginal cost. The stock externality is assumed to be quadratic in a. Due to the normalisation and the constant private marginal cost, there will be no difference between the unit tax and the ad valorem tax in this example.

Some key results conditional on these assumptions are: $T = 193$ and $a(T) \approx 3158$. The optimal extraction path as a function of a from a_0 to a_T is illustrated in Figure 1. This shows that optimal extraction should initially drop from the current level (21.9) to 15.5 and then follow the curve based on continuous measurements of CO_2. Figure 2 illustrates the corresponding time path of the corrective ad valorem tax in a deterministic world. Optimality requires an ad valorem tax that is initially 218 per cent and first increases and then decreases over the period to 169 per cent. Figure 3 illustrates the development in a over time with and without optimal regulation. It is seen that with regulation a increases until T and then decreases towards the pre-industrial steady state level. Without regulation annual emmissions are 21.9, and a increases to 3473 in year 137 and declines thereafter. Again it must be emphasized that the example in this section does not pretend to describe reality but is only meant to illustrate the use of the model.

CONCLUSIONS

In this paper we have derived an explicit feedback control law which can be used to control the production of fossile fuel products in the presence of both flow and stock externalities associated with emissions of greenhouse gases (CO_2). Flow externality refers to the instantaneous externality associated with extraction and consumption which is the difference between social and private marginal costs. Stock externality refers to the disutility associated with the aggregate level of CO_2 in the atmosphere; for example global warming. Production can be controlled directly through quotas or indirectly through corrective taxes such as a unit tax or an ad valorem tax.

The total time horizon in the optimization problem is divided in two periods, one with extraction and emissions and one without extraction after the stock of fossile fuels has been depleted. The feedback law derived here can be used to analyse the time path of the corrective tax in the period with extraction but taking into account the stock externality both in this period and in the remaining period. Such an analysis, however, has at the same time been made somewhat obsolete as the corrective tax can be formulated as an explicit function of the CO_2-level.

Including both flow and stock externalities simultaneously in a dynamic model results in a corrective tax that is different from a tax based on the shadow cost of pollution, defined as the costate variable in the dynamic problem. A tax based on the shadow cost can only correct for the stock externality and not for the flow externality. The tax derived here can be separated in three parts, namely the part that corrects for the flow externality, the part that corrects for the stock externality and the scarcity rent of the resource (fossile fuels). The parts correcting for the flow and stock externalities will be increasing or decreasing over time depending upon the shape of the decay function and other conditions.

In the model presented here the optimal period with extraction, the shadow cost of CO_2 and the scarcity rent of the resource are all determined endogeneously. This is in contrast to most of the existing literature.

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APPENDIX.

Finding the feedback rule

In this appendix the feedback rule with positive discounting, eq. (14), is derived. The method used here is perturbation theory, see for example Nayfeh (1973). To simplify notation time units are rescaled such that $\delta \equiv 1$, that is $t \to \delta t$, and we define $\eta = x - f = \dot{a}$. Together with the definition $\mu = \beta - 2\gamma f$ we have from (8) and Table 1

$$\dot{\mathcal{H}} = r(2\gamma\eta^2 - \mu\eta - nf) \tag{15}$$

For simplicity, let us denote $\overline{\mu} = \int_{a_0}^{a} \mu(z)dz$ and note that $\dot{\overline{\mu}} = \mu\eta$. Let P be defined as $P = \gamma\eta^2 + \widetilde{S} - nf$, and note that P is equal to the optimal Hamiltonian expressed as a function of $\widetilde{S}$ instead of S. Inserted into (15) this yields

$$\dot{P} = r(2\gamma\eta^2 - nf). \tag{16}$$

Next η can be eliminated from the right-hand side of (16) using the definition of P:

$$\dot{P} = r\left[2\left(P - \widetilde{S}\right) + nf\right] \Leftrightarrow \frac{d}{dt}\left(e^{-2rt}P\right) = re^{-2rt}\left(nf - 2\widetilde{S}\right).$$

Using the information in Table 1 it can be seen that $P = \mathcal{H}$ when $r = 0$. From optimal control theory it is well known that the Hamiltonian is constant when $r = 0$. Therefore, P is constant when $r \to 0$. Next we make an asymptotic series expansion of P at this value. Remember that $e^{2rt} = (n/n_0)^2$. This yields

$$P = \left[P_0 + r\omega(t)\right] e^{2rt}, \;\; \omega(t) \leq O(1).$$

The variables n, f, a, and $\widetilde{S}$ are treated as first-order, $O(1)$. Let η be perturbated by the expansion

$$\eta = \eta_0 + r\eta_1 + r^2\eta_2 + \cdot\cdot$$

Then we get

$$\begin{aligned} \gamma\eta_0^2 + \widetilde{S} - nf &= P_0 e^{2rt}, \ \ P_0 \text{ is constant,} \\ \dot{\omega} &= e^{-2rt}\left(nf - 2\widetilde{S}\right) \\ \omega &= \gamma\left[2\eta_0\eta_1 + r\left(\eta_1^2 + 2\eta_0\eta_1\right) + 2r^2\left(\eta_0\eta_3 + \eta_1\eta_2\right) + \cdot\cdot\right] \end{aligned} \tag{17}$$

By solving the first equation ($\eta_0 = \dot{a}$) it is possible to find $a = a(t)$, and this enables us to solve the second equation in (17). Then it is possible to determine η to any order from the last equation, for example by letting the terms in ordinary brackets be zero. This yields

$$\eta_1 = \frac{\omega}{2\gamma\eta_0}, \ \ \eta_2 = -\frac{\eta_1^2}{2\eta_0}, \ \ \eta_3 = -\frac{\eta_1\eta_2}{\eta_0}, \ldots$$

Having determined all the η's we have also determined the optimal control ($x = f(a) + \eta$). The lowest order is sufficient to comprise the main correction. From earlier we remember that $\mathcal{H}(r = 0) = P_0 = \widehat{B}$. The feedback solution is then given by

$$x = f \pm \sqrt{\frac{\widehat{B}e^{2rt} + nf - \widetilde{S}}{\gamma}} + O(r).$$

This is the feedback solution given in Theorem 2. It is also easily seen from the formula that when $r = 0$, we have the feedback rule in Theorem 1.

Matching conditions

In this section the endogeneous variables a_T; x_T; the resource rent, n, and the time horizon, T, is determined. This analysis is quite analogous to the case with zero discounting. Continuity in the Hamiltonian, $\mathcal{H}$, and the shadow cost of CO_2, m, yields $B(x, a) + (\delta m - n)x = \widehat{B}$ at time T. This implies

$$x_T = \sqrt{\frac{\widehat{B}}{\gamma(a_T)}}.$$

In the period after extraction has terminated ($t > T$), we have

$$\dot{a} = -\delta\, f(a), \quad \dot{m} = [r + \delta\, f'(a)]\, m + \alpha'(a).$$

Recall that the function ψ given by

$$\psi(a) = \frac{1}{\delta} \int_{a_T}^{a} \frac{da}{f(a)} = -(t - T)$$

is known. The shadow cost, m, can now be expressed by ψ:

$$mfe^{r\psi} = m_T f_T - \frac{1}{\delta} \int_{a_T}^{a} \alpha'(a) e^{r\psi(a)} da. \tag{18}$$

It is seen that $t \to \infty$ implies $\psi \to -\infty$, and the lefthand-side of (18) approaches zero (assuming that the rest does not approach infinity, which is a reasonable assumption as $a = 0$ is the closest steady state). This implies that $f \to 0$ because $a \to 0$. It is now possible to determine the shadow cost at time T as a function of a_T:

$$\delta m_T f_T = \int_{a_T}^{0} \alpha'(a) e^{r\psi(a)} da.$$

With zero discounting $\delta m_T f_T = -\alpha_T$. In the present case we apply the information that the Hamiltonian is maximized and that the costate variables must be continuous:

$$\delta m_T - n_T = -\frac{\partial}{\partial x} B(x_T, a_T) = 2\gamma_T x_T - \beta_T.$$

This yields the resource rent as a function in a_T through β_T, γ_T, x_T and m_T:

$$n_T = n_0 e^{rT} = \delta m_T + \beta_T - 2\gamma_T x_T.$$

It now remains to determine T and a_T. The evolution of a in the initial phase (with positive extraction) is

$$\delta T = \int_{a_0}^{a_T} \frac{da}{x - f(a)}.$$

In addition, in order to determine a_T, we have the equation for the evolution in fossile fuel:

$$\delta \Delta s = \int_0^T \delta x\, dt = \int_{a_0}^{a_T} \frac{x}{x - f(a)} da = a_T - a_0 + \int_{a_0}^{a_T} \frac{f(a)\, da}{\eta}$$

where Δs is the exogeneously given amount of fossile fuels to be extracted.

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