Wealth Effects on Demand for Insurance.

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Abstract

A standard result states that under decreasing absolute risk aversion the indifference premium of the insured is a decreasing function of wealth. This has been interpreted to mean that insurance is an inferior good, which has been considered as a puzzle in insurance theory, in particular since the result does not seem to explain observed behavior in insurance markets.

We reformulate the standard model of risk sharing to incorporate the amount invested in the insurable asset. From this we identify two wealth effects, one direct and one indirect. The direct one is explained by the classical result, and is negative when risk aversion is decreasing. The indirect effect is positive when the insurable asset is a normal good, and we find conditions when insurance is a normal good, and when it is not.

The analysis is extended to Pareto optimal risk sharing, where we also analyze the joint problem of finding an optimal amount in the insurable asset, as well as a Pareto optimal insurance contract. In this latter case insurance turns out to be inelastic to changes in wealth of the insurance customer, provided the insurer's reserves are held fixed, but a normal good if this assumption is relaxed.

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KEYWORDS: The wealth effect in insurance, decreasing absolute risk aversion, inferior good, normal good, deductible, Pareto optimal risk exchange.

I Introduction

In a seminal paper Mossin (1968) analyzed several aspects of insurance purchasing. Among other things he showed that under decreasing absolute risk aversion (DARA), the indifference premium of the insured is a decreasing function of wealth. This has been later interpreted as insurance is an inferior good. This result became well known for various reasons. The technique of proof was elegant, although he only considered risks that could take on two values, and it was mimicked in many subsequent papers dealing with the microeconomics of risk theory. The result also had an intuitive appeal: The wealthier a person is, the greater his ability to bear the risk on own accounts. In financial economics there is a well known result in portfolio theory (Arrow (1970)) involving one risky and one risk-free asset stating that, if the investor has DARA, the demand for the risky asset increases as the investor's wealth level w increases. Turning this result around to insurance, the risky asset can be interpreted as accepting the "perils of the sea" (without insurance), and demanding more of this "asset" is interpreted as demanding less insurance. Thus Mossin's result seems well in accordance with this fact as well. Finally, the assumption of DARA seems so natural that the implications for insurance demand looked unquestionable. Mossin's results are widely taught and used in academic circles. Since inferior goods may also be Giffen goods, Hoy and Robson (1981), Briys, Dionne, and Eeckhoudt (1988), Borch (1990) i.a., have analyzed the conditions under which insurance might belong to this category as well.

However, skepticism has been expressed, both in the insurance profession and among researchers, about the validity of this result. Insurers, for example, typically prefer to target their products to high income groups on the ground that they have a greater capacity of paying the premiums. Presumably these individuals also have more assets and other belongings subjected to risk. In the same vein, empirical studies indicate that the elasticity of insurance demand with respect to wealth is positive, whether the problem is studied cross-sectionally as by Beenstock et.al. (1988), or as a time series, as in Szpiro (1986).

Researchers have developed some alternative models in which insurance may not be inferior. For example, while remaining in the single period context of Mossin, a distinction can be made between two sources of total wealth;

a safe (hence non-insurable) component and a risky, insurable one. Under those conditions Beenstock, Dickinson and Khajuria (1988) showed that for a particular utility function the demand for insurance varies directly with the value of the property at risk, so that insurance is a normal good with regard to the relevant component of wealth. Doherty and Schlesinger (1990) use a model very similar to that of Mossin, but assume the insured has doubts about the insurer's ability to pay valid claims in full. In that context they show that insurance might be a normal good with respect to the amount of sure wealth. Dionne and Eeckhoudt (1984) have shown that in a two period model, even with temporal DARA, the normality of insurance cannot be excluded a priori. Eeckhoudt and Venezian (1990) consider the simultaneous decision of the level of investment in a risky asset and the level of the indemnity against destruction of the asset. Under certain credit market and insurance market restrictions they find that insurance is a normal good. This is the approach that is closest in spirit to ours. Section 5 has a model that resembles the one by these authors, but our perspective is somewhat different. Dreze (1987) considers a simple term insurance where where a compensation is paid only in the case of death, in a model with state dependent utility. The demand for insurance is found to be an increasing function of wealth in the case of life, and a decreasing function wealth in the case of death. This result depends on the fact that the wealth parameter is unrelated in the two different states.

We reexamine the conditions behind Mossin's result, and find that a critical assumption is that the risk under consideration does not depend upon the wealth. Thus, increasing the wealth amounts to increasing only the safe component, leaving the risky part unaltered. This assumption turns out to be indispensable for the derived result, a feature we do not find entirely satisfactory. Increasing individual wealth normally means that both the safe and the risky component increases. A well-off person typically has more belongings subject to potential losses than a less wealthy individual. An increase in wealth may, for example, be partly invested in real estate, furniture, cars, art, etc., and this could lead to more demand for insurance, not less.

In this paper we reformulate the standard model of risk sharing to incorporate the amount invested in the insurable asset. From this we identify two wealth effects, one direct and one indirect. The direct one is explained by the classical result, and is negative when risk aversion is decreasing. The indirect effect is positive when the insurable asset is a normal good, and we find conditions when insurance is a normal good, and when it is not.

The paper is organized as follows: First we recall the standard result that the indifference premium is a decreasing function of wealth, if the insurance customer has a decreasing absolute risk aversion function. This is the direct effect, and here we generalize the originally published proof from the special case with a random loss that can only take two values, to the situation of any non-negative random loss $Y \geq 0$. Since the concept of an inferior/normal good established via the indifference premium is not the standard in economics, in Section 3 we present an alternative model where the optimal indemnity function is derived, when the insurance premium is defined in the standard way by a loading of the expected loss. By keeping the premium per unit of risk fixed, we show that the demand for insurance is a decreasing function of wealth when the absolute risk aversion is decreasing over its entire domain, a different and independent derivation of the direct wealth effect of Mossin.

We then modify the standard model of insurance to include the amount invested in the insurable asset. This allows us to also derive an indirect wealth effect which is shown to be positive if the insurable asset is a normal good. Conditions when insurance is a normal good are found for both the models mentioned above. We investigate two different scenarios throughout, one where the amount in the insurable asset is taken as given, and one when this amount is a decision variable together with the amount of insurance. In the latter case both insurance and the risky asset are typically normal goods.

We also incorporate the supply side of insurance, and identify the two wealth effects for Pareto optimal insurance contracts. Here we solve the problem of finding both the optimal amount in the insurable asset, as well as the Pareto optimal insurance indemnity, and show that insurance is inelastic to changes in wealth provided the reserves of the insurer is kept fixed. When this assumption is relaxed, both a Pareto optimal insurance contract and the optimal amount in the risky asset are normal goods.

II The standard result for any random loss

In order to better be able to put the standard result under scrutiny, we now reexamine the proof of this result. By doing so, we are able to generalize Mossin's proof to an arbitrary, non-negative random variable Y representing the loss of the insurance customer.

Under risk aversion the agent has a decreasing marginal utility in wealth. Suppose that his absolute risk aversion function $R(w)$ decreases with wealth w, which as been argued to be a natural property of behavior in the presence of risk: As the consumer becomes wealthier, he becomes less and less risk averse (DARA). Under this assumption Mossin (1968) showed that the reservation premium is a decreasing function of wealth. He showed his result for a risk Y that could take two different values; $Y = 0$ with probability $(1 - q)$

and $Y = y$ with probability q, where $w \geq y > 0$. Full insurance is available, and the indifference premium p_r satisfies

$$u(w-y)q + u(w)(1-q) = u(w - p_r)$$

where w is the wealth of the insurance purchaser. The proof presented by Mossin was largely graphical in flavor.

Below we present a simple proof, maintaining his assumptions, but for any non-negative random loss Y having cumulative distribution function F. By EY we thus mean $\int_0^M ydF(y)$, for some $M < \infty$.

The reservation premium $p_r(w, F) := p_r(w)$ defined for any non-random initial wealth w by

$$Eu(w - Y) = u(w - p_r(w)). \tag{1}$$

Theorem 1 *Consider the following assumptions: (i) $R(x)$ is a strictly decreasing function in x. (ii) The distribution F does not depend upon the initial wealth w. (iii) the random loss Y has a non-degenerate distribution. (iv) the utility function u satisfies $u' > 0$, $u'' < 0$.*
Then the reservation premium is a strictly decreasing function of wealth, or

$$\frac{\partial p_r(w)}{\partial w} < 0.$$

Proof: Differentiating the equation (1) with respect to w yields

$$Eu'(w-Y) = u'\big(w - p_r(w)\big)\Big(1 - \frac{\partial p_r(w)}{\partial w}\Big),$$

which gives that

$$\frac{\partial p_r(w)}{\partial w} = -\frac{Eu'(w-Y) - u'\big(w - p_r(w)\big)}{u'\big(w - p_r(w)\big)}.$$

Since the marginal utility $u' > 0$, the conclusion will follow if we can show that

$$\theta := Eu'(w-Y) - u'\big(w - p_r(w)\big) > 0.$$

To this end, define the random variable $Z := u(w - Y)$. Clearly Z is non-degenerate if Y is. Since $(w - Y) = u^{-1}(Z)$, the constant θ can be written, using the definition (1)

$$\theta = Eu'\big(u^{-1}(Z)\big) - u'\Big(u^{-1}\big(Eu(w-Y)\big)\Big) =$$

$$Eu'\big(u^{-1}(Z)\big) - u'\big(u^{-1}(EZ)\big).$$

The conclusion of the theorem will follow if we can show that the composite function $f(z) := u'(u^{-1}(z))$ is strictly convex, since by Jensen's inequality we then get that

$$Eu'\big(u^{-1}(Z)\big) > u'\big(u^{-1}(EZ)\big)$$

with strict inequality for any non-degenerate random variable Z, which means that $\theta > 0$.

So it remains to prove that f is strictly convex. To this end, consider $f'(z) = \frac{d}{dy}u'(u^{-1}(z))$. By virtue of the rule for differentiating the inverse of a function, we get

$$f'(z) = \frac{u''(u^{-1}(z))}{u'(u^{-1}(z))}.$$

Since the marginal utility $u' > 0$, the inverse function $u^{-1}(z)$ is a strictly increasing function in z, since

$$\frac{d}{dz}u^{-1}(z) = \frac{1}{u'(u^{-1}(z))}.$$

Denote by $u^{-1}(z) := v$. The function

$$\frac{u''(v)}{u'(v)} = -R(v),$$

and from our assumptions $R(v)$ decreases as v increases. Thus $u''(v)/u'(v)$ increases as v increases, or

$$\frac{u''(u^{-1}(z))}{u'(u^{-1}(z))} \uparrow \qquad \text{as} \qquad z \uparrow$$

by the above observation. In other words, the function $f(z) = u'(u^{-1}(z))$ has a derivative which is strictly increasing in z, so f is accordingly strictly convex. This finishes our proof. $\square$

Notice that the assumption (ii) is utilized already in the first line of the proof; the differentiation of (1) with respect to w leaves the distribution function of Y unaltered.

If we do not require that the risk aversion function is strictly decreasing, the conclusion would be that the function $p_r(w)$ would only be decreasing, not necessarily strictly decreasing. This is, perhaps, the most natural formulation of this theorem.

In investment theory a well known result tells us that the optimal amount invested in the risky asset is an increasing function in wealth when the investor's absolute risk aversion function is decreasing (in wealth), assuming there is only one risky asset and one risk-less one. If two fund separation applies, the result is true also if there are more than one risky assets. Thus the demand for the risky asset is, under these special circumstances, an increasing function of wealth, and hence a *normal good* see e.g., Arrow (1970). By the same reasoning the insurance contract, under the above conditions of Theorem 1, is often termed an *inferior good*, since the "demand" for insurance, as reflected in the reservation premium $p_r(w)$, is decreasing in the customer's wealth.

In view of the property of demand for a risky security in a market for securities, the interpretation of an insurance contract as an "inferior good" may not appear all that odd, at least at first sight, since it could, perhaps, be interpreted as just the other side of the financial result. As the insurance customer gets more wealthy, he will demand more of the "risky asset", here interpreted as just nature itself, hence less insurance. There seems to be nothing pathological about the insurance contract derived from Theorem 1, in light of this interpretation. In fact, this seems like every insurer or reinsurer's dream, to be able to retain more of the risk on own accounts. As it turns out, this interpretation is still valid for insurers. For insurance customers, however, who hold insurable assets as consumption goods, this turns out to be only one side of the issue.

III An alternative model

The standard definition of an inferior good in microeconomics comes from holding prices fixed and allowing income to vary. In the interpretation in last section we have in mind a unit price of insurance at level of risk Y, so that the reservation premium is somehow interpreted as this unit price times "quantum of risk". Keeping this unit price fixed, the quantum demanded must be decreasing in the wealth parameter w, allowing the result of the last section to be consistent with this standard formulation.

Since it is not transparent how the premium p_r may decomposed into a unit price times quantum of risk, we now check if the above result is sensitive to the special interpretation of considering the wealth effect on the indifference premium. To this end consider the following simple two-state model. The individual's loss Y can take two values, 0 and y with probabilities $(1-q)$ and q respectively. The individual can purchase insurance that pays indemnity I if a loss has materialized, and nothing in the good state, against a premium $p = (1+\lambda)qI$. Here the factor $(1+\lambda)$ is the traditional loading, $\lambda \geq 0$, so the premium is actuarially fair if $\lambda = 0$. We assume $(1-(1+\lambda)q) > 0$.

The individual has a strictly increasing and strictly concave utility function u. His objective is to select the amount of indemnity I so as to maximize the expected utility of his income. Thus he seeks the optimal indemnity level I^*, where

$$U(I^*) = \max_{I \geq 0} U(I)$$

and

$$U(I) = (1-q)u(w-p) + qu(w-y+I-p).$$

The strict concavity of u induces U to be strictly concave. Consequently, insurance will be purchased if and only if $U'(0) > 0$. In this case the optimal indemnity I^* is a solution of the first order condition

$$\frac{u'(w-y+I-p)}{u'(w-p)} = \frac{(1+\lambda)(1-q)}{1-(1+\lambda)q}. \tag{2}$$

Let us consider the wealth effect by keeping the premium fixed. By this we mean that the right-hand side of (2) is constant. Since the loss y is exogenous, this relation defines the optimal indemnity $I^* := I^*(w)$ as a function of wealth w. By differentiating (2) in w along the optimal contract, and then dividing the result by the first order condition, by the implicit function theorem we then obtain

$$\frac{d}{dw}I^*(w) = \frac{R(w-p) - R(w-y+I^*(w)-p)}{A}, \tag{3}$$

where

$$A = (1+\lambda)q\big(R(w-p) + (1-(1+\lambda)q)R(w-y+I^*(w)-p)\big).$$

Since $A > 0$, it follows that the optimal demand for insurance I^* is increasing in wealth parameter w, or decreasing, depending upon the sign of the numerator on the right-hand side of (3). If the absolute risk aversion function $R(\cdot)$ is decreasing in its entire domain, it follows that insurance is an inferior good, and if the absolute risk aversion function is an increasing function in its entire domain, insurance is a normal good.

These conclusions follow from our assumptions, since the optimal insurance is either full (i.e., $I^* = y$) if the loading $(1+\lambda) = 1$, and less than full (i.e., $I^* < y$) if $\lambda > 0$. Consequently, $(w - p(w)) > (w - y + I^*(w) - p(w))$ in the latter case. We have shown

Theorem 2 *If less than full insurance is optimal, then* $\frac{d}{dw}I^*(w) < 0$ *under strictly decreasing risk aversion, and* $\frac{d}{dw}I^*(w) > 0$ *under strictly increasing risk aversion.*

If full insurance is optimal, then $\frac{d}{dw}I^*(w) = 0$*, so insurance demand is inelastic to changes in wealth.*

As a consequence of this, Mossin's (1968)-result is not sensitive to his special formulation in terms of the indifference premium $p_r(w)$. Notice that in his model, full insurance is the only alternative, while in the present model, under full insurance the wealth effect turns out to be zero, which is the main difference between these two models. This is not an important difference, since the case where the loading $\lambda > 0$ is the one of practical interest in the present formulation, in which less than full insurance is optimal.

Returning to the financial result mentioned in the last section, the amount invested in the risky asset becomes a function of w, which is increasing if $R(\cdot)$ is decreasing. In the insurance model considered so far we do not have any analogue to this amount in the insurable asset. The above insurance results only say that if the wealth increases, and the individual keeps the wealth in insurable assets unaffected by this change, then the demand for insurance decreases. Thus the insurance model is not really capturing the possibility that the insurable good is itself a normal good. In the next section we reformulate the insurance model such that this aspect can be taken into account. First we study the situation with an indifference premium, and later we return to the less than full insurance scenario of this section.

IV The amount in the insurable asset depends on wealth

IV-A Introduction

In our first attempt at explaining the wealth puzzle, consider a simple model where an increase in wealth means that the amount in the insurable asset also changes. In order to analyze this, we need to introduce the amount in the insurable asset in the model.

Recalling assumption (ii) in Theorem 1, it clearly ignores the effect of increasing wealth on the risky part of the customer's wealth. Casual observations suggest that the more wealth an insurance customer possesses, the larger is the potential loss that he may suffer. Thus an increase in w should typically go along with an increase the insurable asset, at least if this is a normal good. On one side the customer has become wealthier and thus better equipped to carry a potential loss, but on the other side his or her property that is subject to damage has also increased in value and may require a larger insurance cover. Under these circumstances it is far from obvious that the insured's reservation premium is decreasing in wealth, even if the person has a decreasing risk aversion, or that the optimal insurance coverage is a decreasing function of wealth.

The insurance customer has wealth w as before, where the amount $(w-a)$ is invested in a safe asset having return $(1+r_f)$ and the remaining amount $a > 0$ is invested in a risky asset having return $(1+X)$, where X is a random variable with support $[-1, \infty)$. In the present context the risky asset can be thought of as a house or another insurable asset, in which case the rate of return is

$$X = \begin{cases} x, & \text{with probability } (1-q) \\ -1, & \text{with probability } q, \end{cases}$$

where $x > -1$. We may interpret the constant x as the conditional expected rate of return on the insurable asset, given that the good state prevails. [1] For our first result we shall assume that

$$E(X) = (1-q)x - q > 0, \tag{4}$$

or that $x > \frac{q}{1-q}$, the fair-odds-ratio.

The insurable event, or the bad state, has probability q, the good state has probability $(1-q)$. The insurance customer's uncertain end of period wealth

$$W = (w-a)(1+r_f) + a(1+X) = w(1+r_f) + a(X-r_f)$$

and his expected utility is

$$Eu(X) = u(w(1+r_f) + a(x-r_f))(1-q) + u((w-a)(1+r_f))q.$$

Since the risk-less rate will play no role in what follows, we set $r_f = 0$ without loss of generality. The associated loss Y has accordingly two values

$$Y = \begin{cases} 0, & \text{with probability } (1-q) \\ y, & \text{with probability } q, \end{cases}$$

where $y := a(x+1)$.

First let us demonstrate that the optimal amount a^* invested in the risky asset is an increasing function of w under decreasing risk aversion.

Theorem 3 *Assume that (4) holds. Then the optimal amount $a^*(w)$ held of the insurable asset as a function of wealth w is an increasing function if the absolute risk aversion function $R(\cdot)$ is decreasing over its entire domain. On the other hand, $a^*(w)$ is decreasing in wealth if $R(\cdot)$ is increasing over its entire domain.*

[1] This conditional rate of return may be subject to a probability distribution of its own, determined e.g., in the market for real estate if the risky asset is a house, etc., uncertainty which we ignore here for reasons of parsimony.

Proof: First we find the optimal amount invested in the risky asset. The first order condition for this is

$$\frac{dEu(W)}{da} = u'(w+ax)(1-q)x - u'(w-a)q = 0. \tag{5}$$

The equation (5) defines the amount a as a differentiable function of wealth w in a suitable neighborhood. Differentiating the equation with respct to w along the optimal a^* we get (with a slight abuse of notation)

$$(1-q)u''(w+ax)x(1+xa'(w)) - qu''(w-a)(1-a'(w)) = 0$$

for all w in this neighborhood. Dividing by the first order condition we get

$$R(w+ax)(1+xa'(w)) = R(w-a)(1-a'(w))$$

which can be solved for $a'(w)$ to give

$$\frac{da^*(w)}{dw} = \frac{R(w-a^*) - R(w+a^*x)}{R(w-a^*) + xR(w+a^*x)}. \tag{6}$$

The denominator of the right-hand side of (6) is positive, so the sign of the derivative of a^* is determined by the numerator. Since the optimal $a^* > 0$ when $x > \frac{q}{1-q}$, we see that $(w - a^*) < (w + a^*x)$, and the conclusions of the theorem follow from our assumptions about the absolute risk aversion function $R(\cdot)$. □

The result of this theorem seems rather plausible, and corresponds well with how we interpret decreasing risk aversion. However, some care is called for, since the name of R may be too well chosen.

Notice that the standard model tells us that $a' > 0$ and $p' < 0$ or $a' > 0$ and $I' < 0$ in the decreasing risk aversion case, with all inequalities reversed when R increases over its entire domain.

Example 1. Suppose the insurer has power utility, i.e., $u(w) = \frac{1}{1-\rho}w^{1-\rho}$. The parameter $\rho > 0, \rho \neq 1$ signifies the relative risk aversion of the insurance customer, here a constant (if $\rho = 1$, use $u(w) = \ln(w)$). Here $R(w) = \frac{\rho}{w}$, which is decreasing in wealth. From the first order condition we readily derive the expression for the optimal amount a^*. It is

$$a^*(w) = \frac{w\Big(\big(\frac{q}{(1-q)x}\big)^{-\frac{1}{\rho}} - 1\Big)}{x + \big(\frac{q}{(1-q)x}\big)^{-\frac{1}{\rho}}}. \tag{7}$$

It is seen that the optimal amount in the risky asset increases in w, consistent with the above theorem. Furthermore (i) a^*; decreases when q increases, (ii) increases when x increases, (iii) decreases when ρ increases, all according to intuition. We notice $a^* > 0$ only if $x > q/(1-q)$. $\square$

When the insurable good is both a durable consumption good and at the same time can be considered as an investment, then a condition like (4) seems reasonable. This could include real estate, art, or any other items that could be expected to increase in price. In this case Theorem 2 tells us that the insurance item is a normal good. However, there exist many insurance items that are durable consumption goods which can not be expected to increase in price. For these items $x < 0$ and yet, they may very well be normal goods. An example could be cars or furniture; a car, for example, would normally decrease in value, but affluent people typically have expensive cars. In our model we allow the agent to hold the risky asset even if (4) is not satisfied, for reasons of consumption. For the discussion to follow we therefore do not require that (4) holds in general, only when a is a decision variable.

IV-B Insurance as a normal good

We now assume that full insurance is available, and want to investigate how the indifference premium p depends on the wealth parameter w in the model of this section. Let us start by defining the indifference premium p in our model.

$$Eu(W) = (1-q)u(w+ax) + qu(w-a) := u(w+ax-p). \tag{8}$$

Thus insurance will leave the customer indifferent between expected utility without insurance and and utility of wealth in the good state less the insurance premium, providing full insurance against the consequences in the bad state. Proceeding as in the proof of Theorem 1, we find by the implicit function theorem that

$$\frac{dp(w)}{dw} = -\frac{\theta_1 + \theta}{u'(w+ax-p)}, \tag{9}$$

where

$$\theta = (1-q)u'(w+ax) + qu'(w-a) - u'(w+ax-p),$$

and

$$\theta_1 = (1-q)xa'(w)u'(w+ax) - qa'(w)u'(w-a) - a'(w)xu'(w+ax-p).$$

With these preparations we can now show the following.

Theorem 4 *Consider the following assumptions in the model of this section: (i) $u'(\cdot)$ is a strictly decreasing function in its entire domain, (ii) $u'(\cdot) > 0$*

in its entire domain, (iii) the absolute risk aversion function is a decreasing function in its entire domain, and (iv)

$$\frac{da(w)}{dw} > \frac{\theta}{\big(qu'(w-a)-\theta\big)x + qu'(w-a)}. \tag{10}$$

Then full insurance is a normal good, or $\frac{dp(w)}{dw} > 0$.

Proof. Starting from the expression (9) for the derivative of p with respect w, notice that

$$\theta + \theta_1 = \theta(1 + xa'(w)) - qa'(w)(1+x)u'(w-a).$$

The right-hand side above can can also be written

$$(1 + xa'(w))\big(\theta - qu'(w-a)\big) + q(1-a'(w))u'(w-a).$$

If insurance is to be a normal good, $(\theta + \theta_1) < 0$, which is seen to hold if

$$(1 + xa'(w))\big(\theta - qu'(w-a)\big) + q(1-a'(w))u'(w-a) < 0,$$

or, if

$$a'(w)\big(\big(qu'(w-a) - \theta\big)x + qu'(w-a)\big) > \theta.$$

The term

$$\theta - qu'(w-a) = \Big(\big(u'(w+ax) - u'(w+ax-p)\big) - qu'(w+ax)\Big) < 0,$$

by our assumptions of risk aversion and increasing utility, since the first parenthesis on the right-hand side is negative as the marginal utility function $u'(\cdot)$ is a strictly decreasing function over its entire domain, and the second term on the right-hand side is negative since $u' > 0$. From this it follows that, as long as $\theta > 0$, the term

$$\big(qu'(w-a) - \theta\big)x + qu'(w-a)a'(w) > 0 \tag{11}$$

as long as $x > -1$, the latter following from the definition of x. The condition $\theta > 0$ follows from our assumption of decreasing risk aversion by the result of Theorem 1. Accordingly, (11) implies the inequality (10) if $(\theta + \theta_1) < 0$ is to hold. □

In the interesting case when the risk aversion $R(\cdot)$ is a decreasing function in wealth, the quantity $\theta > 0$ by Theorem 1, and the denominator on the right-hand side of (10) is also positive as shown in the above proof, so a positive lower bound on $a'(w)$ is implied. This theorem simply says that depending on to what extent the insurable asset is a normal good, insurance may or may not be a normal good. If an increase in wealth implies an increase spent on the insurable asset a' which is larger than the right-hand side of (10), then insurance is a normal good.

Notice that the theorem gives inequalities for a' and p' that point in the same direction, contrary to the standard model.

When the absolute risk aversion function is a constant, as e.g., for exponential utility, then $\theta = 0$, and it follows from modifying the above proof that insurance is a normal good if the insurable asset is.

If the absolute risk aversion is an increasing function, then the right-hand side of (10) is negative, but the inequality (10) is still valid if

$$x > -\frac{qu'(w-a)}{qu'(w-a)-\theta}, \tag{12}$$

which means that the expected return of the insurable asset x has a lower bound larger than -1. When this is satisfied insurance will be a normal good even if the insurable asset is an inferior good as long as (12) holds. The inequalities for a' and p' are still in the same direction. In the case when $-1 < x < -\frac{qu'(w-a)}{qu'(w-a)-\theta}$ the inequality (10) is reversed, and the right-hand side becomes positive. In this case there is an upper bound on $a'(w)$ in order for insurance to be a normal good, but this case is of less interest.

Returning to the situation of decreasing risk aversion, the theorem naturally becomes "stronger" the smaller the right-hand side of (10) is. Furthermore, the inequality (10) is consistent with Theorem 2 in the following sense:

Corollary 1 *Suppose that (4) is satisfied, and the absolute risk aversion is decreasing in its entire domain. If the insurance customer holds the optimal amount a^* of the insurable asset before insurance, and if $\frac{da^*(w)}{dw}$ is larger that the right-hand side of (10), then full insurance is a normal good.*

Proof. We see from the definition of the indifference premium

$$Eu(W) = (1-q)u(w+ax) + qu(w-a) := u(w+ax-p)$$

that if $a^* > 0$ optimizes the left hand side, it will satisfy

$$\frac{da^*(w)}{dw} = \frac{R(w-a^*) - R(w+a^*x)}{R(w-a^*) + xR(w+a^*x)} > 0$$

by Theorem 2. By our assumptions condition (iv) of Theorem 3 is satisfied, and the conclusion follows. □

To sum up the conclusions of this section, consider the following interpretation of our results. Notice first that the loss y in the bad state can be written $y = a(x+1)$, so that θ_1 can alternatively be written as a function of y. As a consequence we may consider the indifference premium as a function of y and w, where y is a function of w (since a is), or $p(w) = p_r(y(w), w)$. From (9) we see that

$$\frac{dp(w)}{dw} = \frac{\partial p_r(y,w)}{\partial y}\frac{dy(w)}{dw} + \frac{\partial p_r(y,w)}{\partial w}, \tag{13}$$

where

$$\frac{\partial p_r(y,w)}{\partial w} = -\frac{\theta}{u'(w+ax-p)},$$

and

$$\frac{\partial p_r(y,w)}{\partial y}\frac{dy(w)}{dw} = -\frac{\theta_1}{u'(w+ax-p)}.$$

The wealth effect can accordingly be separated into two parts, one direct and one indirect. The direct part is represented by the last term in the (13), and is negative under decreasing risk aversion, since then $\theta > 0$. This is the effect described in Theorem 1, the one found by Mossin. The indirect effect is represented by the first term in (13) and is composed of two factors. One is the effect from increasing the loss, the other is the effect on the loss from an increase in the wealth.

Since $y = a(x+1)$, the latter factor $y'(w) = a'(w)(x+1) > 0$ if the insurable asset is a normal good. Also the first factor is positive, since the indifference premium must increase if the potential loss increases. This is seen by observing that

$$\theta_1 = \big((1-q)xu'(w+ax) - qu'(w-a) - xu'(w+ax-p)\big)\frac{\frac{dy(w)}{dw}}{x+1}.$$

The term in parenthesis is negative because of risk aversion, since then $u'(w+ax-p) > u'(w+ax)$, and $(1-q) < 1$ is always true. Furthermore $u'(w-a) > 0$ because of increasing utility.

As a consequence, the indirect effect is positive when the insurable good is a normal good. Weather or not *insurance* is a normal good, thus depends on which one of these two terms has the largest absolute value. The conclusion of Theorem 4 is that so long as condition (iv) is satisfied, the indirect wealth effect dominates. We have shown

Corollary 2 *Suppose that either (4) holds, or that the insurable asset is held for consumption purposes and is a normal good. In each case insurance is a normal good provided* $|\theta_1| > \theta$*, i.e., provided the indirect wealth effect dominates the direct one.*

With this interpretation, Mossin's result is not "wrong", but only provides us with "half the story".

V The alternative model and variable insurable wealth

Instead of investigating the wealth effect on the indifference premium, we now consider the model of Section III, where we can study the wealth effect on the demand of insurance directly. With the same assumptions as in the previous section, the end of period wealth W can be written

$$W = \begin{cases} w(1+r_f) + a(x - r_f), & \text{with probability } (1-q) \\ (w-a)(1+r_f), & \text{with probability } q. \end{cases}$$

As before we set $r_f = 0$ without loss of generality, and consider the utility $U(I, a)$ of the customer after insurance cover I at premium $p = (1+\lambda)qI$, where the factor $(1+\lambda)$ is the usual insurance loading, for some $\lambda \geq 0$. With these assumptions it follows that

$$U(I,a) = u(w + ax - (1+\lambda)qI)(1-q) + u(w - a + I - (1+\lambda)qI)q,$$

where we consider $a(w)$ as a given function of w, but not as a decision variable. The first order condition for an optimal indemnity I^*, holding a fixed, is

$$\frac{\partial U(I,a)}{\partial I} = 0,$$

which is equivalent to the following condition

$$\frac{u'\big(w - a + I(1-(1+\lambda)q)\big)}{u'\big(w + ax - (1+\lambda)qI\big)} = \frac{(1+\lambda)(1-q)}{1-(1+\lambda)q}. \tag{14}$$

The relation (14) defines a and I as differentiable functions of w is a suitable neighborhood. Assuming that $u''(\cdot)$ exists, we may differentiate the relation (14) in this w-neighborhood. After dividing by the first order condition, we obtain the following relationship

$$\begin{aligned} R\big(w + ax - (1+\lambda)qI\big)\big(1 + xa'(w) - (1+\lambda)qI'(w)\big) = \\ R\big(w - a + I(1-(1+\lambda)q)\big)\big(1 - a'(w) + I'(w)(1-(1+\lambda)q)\big). \end{aligned} \tag{15}$$

From equation (15) we find that

$$\frac{dI^*(w)}{dw} = \frac{R_2\big(1 + xa'(w)\big) - R_1\big(1 - a'(w)\big)}{R_2\big((1+\lambda)q\big) + R_1\big(1-(1+\lambda)q\big)}, \tag{16}$$

where $R_1 := R\big(w - a + I^*(1-(1+\lambda)q)\big)$ and $R_2 := R\big(w + xa - (1+\lambda)qI^*\big)$. Let us denote by $b = \frac{R_2}{R_1}$. Below we consider the case where the absolute risk aversion is a decreasing function in its entire domain, so that $0 < b < 1$. We are then in position to prove the following.

Theorem 5 *(i) Suppose that full insurance is optimal* $(\lambda = 0)$*. If* $a'(w) > 0$*, then*

$$\frac{dI^*(w)}{dw} > 0, \tag{17}$$

or, full insurance is a normal good.
(ii) Suppose that less than full insurance is optimal $(\lambda > 0)$*. If* $a'(w) > \frac{1-b}{1+bx}$*, then the inequality (17) holds as well, and less than full insurance is a normal good.*

Proof. We first observe that the denominator on the right-hand side of equation (16) is positive since $1 > (1+\lambda)q$ (if this were not the case, the premium would exceed the compensation in the bad case). Also, since

$$w + xa - qI^* = w - a + I^* - qI^*$$

when full insurance is optimal, $R_1 = R_2$ and we see that in the latter case the numerator in the right-hand side of equation (16) is strictly positive when $(1 + a'(w)x) > (1 - a'(w))$, which holds if and only if $a'(w) > 0$, proving the first part of the theorem.

The numerator on the right-hand side of equation (16) is positive if and only if $b(1+xa'(w)) > (1-a'(w))$ which is equivalent to $a'(w)(1+bx) > (1-b)$. By our assumption that the risk aversion is a decreasing function, $(1+xb) > 0$ for any $x > -1$, in which case it follows that

$$a'(w) > \frac{1-b}{1+bx},$$

verifying the second part of the theorem. □

Also this theorem says that depending on to what extent an increase in wealth is spent on the insurable asset, insurance may or may not be a normal good. If an increase in wealth implies an increase spent on the insurable asset $a'(w) > \frac{1-b}{1+bx}$, then insurance is a normal good.

Again we notice that the inequalities for a' and I' point in the same direction. The observation in part (ii) that if the customer has a decreasing absolute risk aversion in its entire domain then $0 < b < 1$, and if R is increasing in its entire domain $b > 1$, follow since

$$w + xa - (1+\lambda)qI^* > w - a + I^*((1-(1+\lambda)q)$$

when the optimal insurance is less than full. Thus the lower bound $\frac{1-b}{1+bx}$ is smaller in the latter case than in the former provided x is such that $(1+bx) > 0$. In this case the lower bound is negative, and the inequalities for a' and I' are still in the same direction. However, if $-1 < x < -\frac{1}{b}$ then the reversed inequality $a'(w) < \frac{1-b}{1+bx}$ is the relevant sufficient condition for less than full insurance to be a normal good. In this situation the upper bound on $a'(w)$ is strictly positive, similar to the observations of the previous section. This case has of course less interest. If the risk aversion is constant, $b = 1$, and we obtain that $a'(w) > 0$ is the sufficient condition for less than full insurance to be a normal good, the same condition as in part (i).

The result in (i) is quite intuitive, since when full insurance is optimal and the insurable asset is a normal good, then insurance should also be a normal good.

Notice that the lower bound $\frac{1-b}{1+bx}$ is strictly positive in the decreasing risk aversion case. If the utility in the bad state after insurance is sufficiently low compared to the good state, for insurance to be a normal good $a'(w)$ must be bounded below by some positive fraction, which also seems intuitive. The result is of course stronger the smaller this fraction is.

Regarding case (i) of Theorem 5, note that if full insurance is optimal, then $\lambda = 0$ and

$$U(I, a) = U(a(x+1), a) = u(EW) = u\big(w + a(x(1-q) - q)\big).$$

The associated optimal a is here either $a^* = w$ if $x > \frac{q}{1-q}$ or $a^* = 0$ if $x < \frac{q}{1-q}$ assuming no borrowing possibilities.

Also for the model of this section we can interpret our results in light of a direct and an indirect wealth effect. We can write

$$\frac{dI^*(y;w)}{dw} = \frac{\partial I^*(y;w)}{\partial y}\frac{dy}{dw} + \frac{\partial I^*(y;w)}{\partial w}. \tag{18}$$

The last term in this equation gives the direct effect, holding the loss y fixed. This corresponds to our result in Theorem 2, and can here be written

$$\frac{\partial I^*(y;w)}{\partial w} = \frac{R_2 - R_1}{R_2((1+\lambda)q) + R_1(1-(1+\lambda)q)},$$

which is negative since $R_2 < R_1$ under decreasing absolute risk aversion. The direct wealth effect can be written

$$\frac{\partial I^*(y;w)}{\partial y}\frac{dy}{dw} = \frac{R_2\frac{x}{x+1} + R_1\frac{1}{x+1}}{R_2((1+\lambda)q) + R_1(1-(1+\lambda)q)} \cdot \frac{dy(w)}{dw},$$

where $\frac{dy(w)}{dw} = (x+1)\frac{da(w)}{dw}$. We notice that $\frac{\partial I^*(y;w)}{\partial y} > 0$, so the indirect effect is positive if the insurable asset is a normal good. Insurance it then a normal good provided the indirect effect is larger than the direct effect, which is the contents of part (ii) of Theorem 5. Part (i) is explained by Theorem 2, since when full insurance is optimal in this model, the direct wealth effect is zero, and only the indirect effect remains, which is strictly positive when the insurable asset is a normal good.

One may wonder if there is an analogue of Corollary 1 in the present model. Considering both a let I as decision variables, the second of the first order conditions is

$$\frac{\partial U(I,a)}{\partial a} = (1-q)u'(w+ax-(1+\lambda)qI)x+qu'(w-a+I-(1+\lambda)qI)(-1) = 0.$$

The two first order conditions admit a unique solution (a^*, I^*) only if

$$\lambda \neq \frac{x(1-q)-q}{q(1+x)}. \tag{19}$$

Differentiating the equation $\frac{\partial U(I,a)}{\partial a} = 0$ in w along the optimal a^* and I^* however, we again recover the relation (14), suggesting that this model is not a straight-forward one to analyze. [2] Here we can show the following:

Corollary 3 *We consider the decreasing risk aversion case. If*

$$\frac{dI^*(w)}{dw} > \frac{b-1}{1-(1+\lambda)q(1-b)}, \tag{20}$$

then $\frac{da^*}{dw} > 0$, *so that the insurable asset is a normal good at the optimum.*

Proof. Let $R_1 := R\big(w-a+I^*(1-(1+\lambda)q)\big)$ and $R_2 := R\big(w+xa-(1+\lambda)qI^*\big)$, where $b = \frac{R_2}{R_1}$ as before. From (14) we find that

$$\frac{da^*}{dw} = \frac{R_1\big(1+\frac{dI^*(w)}{dw}(1-(1+\lambda)q)\big) - R_2\big(1-(1+\lambda)q\frac{dI^*(w)}{dw}\big)}{xR_2+R_1}.$$

By our assumption, $0 < b < 1$, so the denominator $(R_2x+R_1) = (bx+1)/R_1$ is seen to be positive since $x > -1$ (recall that $x > \frac{q}{1-q} > 0$). It remains to check the sign of the above numerator.

$$R_1\big(1+\frac{dI^*(w)}{dw}(1-(1+\lambda)q)\big) - R_2\big(1-(1+\lambda)q\frac{dI^*(w)}{dw}\big) > 0$$

is equivalent to

$$\frac{dI^*}{dw}\big(1-(1+\lambda)q(1-b)\big) > b-1.$$

[2] This is the model analyzed in detail by Eeckhoudt and Venezian (1990), which we refer the interested reader to for further details.

Since $(1-(1+\lambda)q) > 0$ by assumption, otherwise the insurance benefit would be exceeded by the premium payment in the bad state, it follows that $\big(1-(1+\lambda)q(1-b)\big) > 0$, since risk aversion is decreasing. Thus the inequality (20) is sufficient for $da^*(w)/dw > 0$. $\square$

This result says that if an increase in wealth implies a change in the optimal demand for insurance that is bounded below by the right-hand side of (20), then the optimally held insurable asset is a normal good.

Although the right-hand side of the inequality (20) is negative, the inequalities for a' and I' still point in the same direction, contrary to the standard model, and the optimally held insurable asset may be a normal good. However, if at the optimum

$$\frac{da^*}{dw} > \frac{1-b}{1+bx} > 0,$$

then (less than full) insurance *is* a normal good by Theorem 5, in our new interpretation. We provide an example indicating that more can be said:

Example 2. Suppose the insurance customer has power utility, i.e., $u(w) = \frac{1}{1-\rho}w^{1-\rho}$, $u(w) = \ln(w)$ when $\rho = 1$. Here we can solve the two FOC when there is a unique solution, i.e., provided the condition (19) is satisfied. This solution is

$$I^*(w) = \frac{(d-1)(x+c)-(c-1)(x+d)}{(1+\lambda)q(d-c)+(1-(1+\lambda)q)x(c-d)}w$$

and

$$a^*(w) = \frac{d(1-(1+\lambda)q)+(1+\lambda)q}{x+d}I^* + \frac{d-1}{x+d}w,$$

where

$$c = \Big(\frac{(1-q)x}{q}\Big)^{1/\rho} \qquad \text{and} \qquad d = \Big(\frac{(1+\lambda)(1-q)}{1-(1+\lambda)q}\Big)^{1/\rho}.$$

It is easy to see that when $d > c$, then $x < \frac{1+\lambda)q}{1-(1+\lambda)q}$ and both the numerator and the denominator in the expression for I^* are positive. Similarly, when $d < c$, then $x > \frac{1+\lambda)q}{1-(1+\lambda)q}$ and both the numerator and the denominator are negative. Consequently $I^*(w)$ is an increasing function of wealth in either case, so insurance is invariably a normal good. Since $d > 1$, this is also the case for the insurable asset provided $x > q/(1-q)$.

In the case when (19) does not hold, $c = d$ and the above solution breaks down. In this particular case the unit price of insurance is determined by

x and q and given by $\lambda = \frac{x(1-q)-q}{q(1+x)}$, and $\lambda > 0$ if $x > \frac{q}{1-q}$. [3] Second order considerations now suggest that the optimal solution is located along a straight line in (a, I)-space. If no borrowing is allowed, $a^* \in (\frac{(d-1)w}{x+d}, w)$ ensures that $I^* \in (0, \frac{(x+1)w}{(1+\lambda)q(1-d)+d})$, which follows from the fact that the line is given by

$$I^*(a^*) = \frac{x+d}{(1+\lambda)q(1-d)+d}a^* + \frac{1-d}{(1+\lambda)q(1-d)+d}w,$$

and the fact that we do not accept a negative indemnity. This gives rise to a *deductible*, meaning that when $y \in [0, \frac{(d-1)w}{(x+d)(1+x)})$, the optimal indemnity $I^*(y) = 0$. Since less that full insurance is optimal, it is clear that $I^*(a^*) < a^*(x+1)$ for all a^*. Consider e.g., the particular case of $\rho = 1$. Then $I^*(w) = \frac{1}{1-q-\lambda q-\lambda^2}w > w$ and $w < I^*(w) < w(x+1)$. This means that when w increases, this line gets tilted, such that for values of a^* to the left (close to the no-insurance point), $I^*(a^*)$ decreases, and for values of a^* close to the upper value w, the optimal insurance $I^*(a^*)$ increases. Thus depending upon the optimal value of the insurable asset, insurance may be an inferior good, an indifferent good or a normal good. □

From this example we may conjecture that in the situation that the two first order conditions have a unique solution, insurance is a normal good, and when the solution is not unique, but located along some locus in the (a, I)-space, the all three situations may occur.[4]

The above example shows a situation where deductibles occur in insurance without resorting to asymmetric information costs, or other frictions (see e.g., Holmström (1979), Raviv (1979), Rothschild and Stiglitz (1976)). When the optimal amount in the insurable asset is below a certain limit, the customer would like to short the insurance contract, but since this is not allowed, a deductible arises, which here means no insurance. As the optimal value a^* varies, the value $\frac{(d-1)w}{(x+d)(1+x)}$ may be considered as a deductible across different insurance contracts. If the range of the different losses that can happen is enlarged from one value to several, a deductible per contract, i.e., in its usual meaning, is likely to result.

Finally notice in this example how the wealth effect can be spit into the negative direct effect, and the positive indirect effect, as explained earlier.

[3] Normally this is not enough to determine a market premium, as it should also depend on marginal utility, but since the insurance industry avoids utility considerations in general, this information may be considered enough.

[4] Eeckhoudt and Venezian (1990) do not seem aware of the unique solution case analyzed in this example. They claim that unless $\lambda = \frac{x(1-q)-q}{q(1+x)}$ in our terminology, "the two first order conditions cannot hold simultaneously and the solution will therefore be an edge or corner solution".

In the next section we study Pareto optimal indemnity functions when the supplier of insurance is also included in the model.

VI Pareto optimal contracts

VI-A Introduction

So far only the demand side of insurance has been considered. Results are sometimes altered when also the supply side is brought into the model. In this section we discuss the situation with one insurer and one insurance customer.

Consider a policy holder having initial capital w, a positive real number, and facing a risk Y, a non-negative random variable. The insured has utility function u, where $u' > 0$, $u'' < 0$. The insurer has utility function u_0, $u_0' > 0$, $u_0'' \leq 0$, and initial reserves w_0, also a positive real number. These parties can negotiate an insurance contract, stating that the indemnity $I(y)$ is to be paid by the insurer to the insured if claims amount to $y \geq 0$. It seems reasonable to require that $0 \leq I(y) \leq y$ for any $y \geq 0$. Notice that this implies that no payments should be made if there are no claims, i.e., $I(0) = 0$. The premium p for this contract is payable when the contract is initialized.

This model can be analyzed by the general theory developed by Borch (1960-62). There is no simple result saying that insurance is an inferior good when the absolute risk aversions of the participants are decreasing. In this section we want to establish that this is indeed the case for a standard Pareto optimal sharing rule.

It follows from the first order conditions of Pareto optimality that the real indemnity function I: $R_+ \to R_+$, satisfies the following nonlinear, differential equation

$$\frac{\partial I(y)}{\partial y} = \frac{R(w-p-y+I(y))}{R(w-p-y+I(y)) + R_0(w_0+p-I(y))}, \tag{21}$$

where the functions $R = -\frac{u''}{u'}$, and $R_0 = -\frac{u_0''}{u_0'}$ are the absolute risk aversion functions of the insured and the insurer, respectively. This analysis is conditional upon the premium p being taken as some given constant. In the case where the insurer is risk neutral, we notice that the contract $I(y) = y$ for all $y \geq 0$ results, which means full insurance is Pareto optimal. Let us consider the case where less than full insurance becomes Pareto optimal. In the example presented below both parties have power utility functions, but unequal initial values of wealth.

Example 3. Assume that $u_1(y) = u_2(y) = \frac{1}{1-\rho}x^{1-\rho}$, where $\rho > 0; \rho \neq 1$. In this case the solution of the differential equation (21) is given by (see e.g., Aase (1993, 2002))

$$I(y) = \frac{w_0 + p}{w + w_0} y, \tag{22}$$

i.e., full coverage is Pareto optimal only if $p = w$. Since it is highly unlikely that the customer is willing to pay all his fortune in premium, we may safely conclude that less than full coverage is here Pareto optimal, or, since $p < w$ would normally hold, *coinsurance* typically results. Notice that normally w_0 is much larger than w. The problem is well-posed for these utility functions only if the loss $Y \leq \min\{w, w_0\}$ with probability one.

So far we have said nothing about the premium p. Suppose we would like to employ the premium $p = (1 + \lambda)qI$ as previously. Then, mechanically inserting this in the above equation for I yields an expression which is seen to be a decreasing function of w for each value of the loss y. Thus, it seems as if insurance becomes an inferior good under these assumptions. However, this is not really a valid procedure, since it violates the premises upon which the equations (21) and (22) have been derived. □

As the last example shows, we have to start afresh on the problems of this section. To this end, consider the assumptions of sections 4 and 5. In this case the loss Y can take the two values; $y = a(x + 1)$ with probability q and $y = 0$ with probability $(1 - q)$. The insurance customer's expected utility is given by

$$U(I, a) = u(w + ax - y - p + I)q + u(w + ax - p)(1 - q)$$

and the insurer's utility is similarly given by

$$U_0(I) = u_0(w_0 + p - I)q + u_0(w_0 + p)(1 - q),$$

where we assume that $p = (1 + \lambda)qI$. From Borch's Theorem it does indeed follow that the first order conditions for Pareto optimal risk exchange are given by

$$u'(w + ax - y - p + I) = ku_0'(w_0 + p - I) \tag{23}$$

$$u'(w + ax - p) = ku_0'(w_0 + p), \tag{24}$$

where k is some positive constant. Notice that the marginal utilities of the two agents are are equalized, except for a constant, at each state. This fact is very useful for us in what follows.

First we assume that the risky asset is unaffected by w. We then differentiate the first order conditions in w along the optimal I, keeping the premium per unit of risk fixed, divide by the first order conditions and add the resulting equations. This gives

$$\frac{dI(w)}{dw} = \frac{R(w + ax - p) - R(w - a - p + I)}{S}, \tag{25}$$

where

$$S = (1-(1+\lambda)q)\big(R(w-a-p+I)+R_0(w_0+p-I)\big)$$
$$+(1+\lambda)q\big(R(w+ax-p)+R_0(w_0+p)\big).$$

Since $S>0$ we see that $I'(w)<0$ if the absolute risk aversion function $R(\cdot)$ is strictly decreasing over its entire domain, which is analogous to the classical result of sections 2 and 3. A Pareto optimal insurance contract is accordingly seen to be an inferior good in the standard model as well.

VI-B When is a Pareto optimal contract a normal good?

In this part we assume that the insurable asset depends upon wealth w so that $a(w)$ is a function of w, but a is not a decision variable yet. For example, we may assume that the risky asset is a normal good.

We now differentiate the first order conditions in w along a Pareto optimal I, keeping the price per unit of risk fixed. This gives us the following two equations.

$$R(w-a-p+I)\big(1-\frac{da(w)}{dw}+\frac{dI(w)}{dw}(1-(1+\lambda)q)\big) \tag{26}$$
$$= R_0(w_0+p-I)\big((1+\lambda)q-1\big)\frac{dI(w)}{dw},$$

and

$$R(w+ax-p)\big(1+x\frac{da(w)}{dw}-(1+\lambda)q\frac{dI(w)}{dw}\big) \tag{27}$$
$$= R_0(w_0+p)(1+\lambda)q\frac{dI(w)}{dw}.$$

Then we add these equations, and solve for $I'(w)$ in terms of $a'(w)$, the latter being taken as given. The result is

$$\frac{dI(w)}{dw}=\frac{\frac{da(w)}{dw}(A+\alpha)-(C-\gamma)}{B+\beta}, \tag{28}$$

where

$$A=R(w-a-p+I), \qquad B=(1-(1+\lambda)q)\big(R(w-a-p+I)+R_0(w_0+p-I)\big),$$
$$C=R(w-a-p+I), \qquad \alpha=xR(w+ax-p),$$
$$\beta=(1+\lambda)q\big(R(w+ax-p)+R_0(w_0+p)\big), \qquad \text{and} \qquad \gamma=R(w+ax-p).$$

Here A, B, C, β and γ are all positive, and the sign of α is the same as the sign of x. We then have the following.

Theorem 6 *Suppose that the absolute risk aversion function $R(\cdot)$ is strictly decreasing over its entire support. Then a Pareto optimal insurance contract is a normal good, or $\frac{dI(w)}{dw} > 0$, provided*

$$\frac{da(w)}{dw} > \frac{R(w-a-p+I) - R(w+ax-p)}{R(w-a-p+I) + xR(w+ax-p)}. \tag{29}$$

The expression on the right-hand side of this inequality is strictly positive.

Proof. From equation (28) we notice that $I'(w) > 0$ if $(\frac{da(w)}{dw}(A+\alpha) - (C-\gamma)) > 0$, since $(B+\beta) > 0$. Since $(A+\alpha) = R(w-a-p+I)+xR(w+ax-p) > 0$ follows from our assumption of decreasing risk aversion since $x > -1$, we obtain the inequality of the theorem, because $(C-\gamma) = R(w-a-p+I) - R(w+ax-p)$. That this latter quantity is strictly positive, also follows from the decreasing risk aversion of the insurance customer. $\square$

The theorem says that for a Pareto optimal insurance contract I, insurance is a normal good as long as the risky asset is a normal good with $a'(w)$ bounded below by the strictly positive quantity given on the right-hand side of (29).

We may split the wealth effect into a direct and an indirect part also for Pareto optimal contracts. In order to do this, we need to study the indemnity function $I(y;w)$ in some more detail.

VI-C The indemnity as a function of the loss

In the present model the equation (21), describing how $I(y)$ varies with the loss y, does not hold, since its derivation takes as given that the premium p is constant as w varies. In addition we have the amount a in the risky asset in our model, that is connected to the loss y through $y = a(x+1)$. Starting with the first order conditions (23) and (24), we differentiate these in y along a Pareto optimal contract I. Using that $y = a(x+1)$, we may carry out this differentiation in a at first. This gives the two equations

$$u''(w+ax-y-p+I)(-1-((1+\lambda)q-1)\frac{\partial I}{\partial a}) = ku_0''(w_0+p-I)((1+\lambda)q-1)\frac{\partial I}{\partial a},$$

and

$$u''(w+ax-p)(x-(1+\lambda)q\frac{\partial I}{\partial a}) = ku_0''(w_0+p)(1+\lambda)q\frac{\partial I}{\partial a}.$$

Dividing these two equations by (23) and (24) respectively, we get

$$R(w-a+I-p)(-1+\frac{\partial I}{\partial a}(1-(1+\lambda)q) = R_0(w_0+p-I)((1+\lambda)q-1)\frac{\partial I}{\partial a},$$

and

$$R(w+ax-p)(x-(1+\lambda)q\frac{\partial I}{\partial a}) = R_0(w_0+p)(1+\lambda)q\frac{\partial I}{\partial a}.$$

By adding these latter two equations, we find the required differential equation for I.

$$\frac{\partial I(y)}{\partial y} = \frac{1}{1+x}\frac{R(w-a+I-p)+xR(w+ax-p)}{D}, \qquad (30)$$

where

$$D = (1-(1+\lambda)q)\big(R(w-a+I-p)+R_0(w_0+p-I)\big)$$
$$+(1+\lambda)q\big(R(w+ax-p)+R_0(w_0+p)\big) = B+\beta.$$

The equation (30) is seen to differ somewhat from the standard one given in (21), but still describes some of the same basic properties of Pareto optimal risk sharing, which include; (i) under strict risk aversion ($R > 0$) of the insurance customer the indemnity is an increasing function of the loss, (ii) when the insurance customer is risk neutral ($R \equiv 0$), no insurance is Pareto optimal, and (iii) when the insurer is risk neutral ($R_0 \equiv 0$), full insurance is Pareto optimal is consistent with the equation (30).

In order to check the claim (iii), note first that the differential equation (30) becomes

$$\frac{\partial I(y)}{\partial y} = \frac{1}{1+x}\cdot\frac{R(w-a+I-p)+xR(w+ax-p)}{(1-(1+\lambda)q)R(w-a+I-p)+(1+\lambda)qR(w+ax-p)},$$

when the insurer is risk neutral. In this case full insurance is known to be Pareto optimal, implying that $(w-a+I-p) = (w+ax-p)$. To check wether this is consistent with the above result, notice that the above version of (30) now becomes

$$\frac{\partial I(y)}{\partial y} = \frac{1}{1+x}\cdot\frac{R(w-a+I-p)(1+x)}{R(w-a+I-p)} \equiv 1,$$

which, together with $I(0) = 0$ implies that $I(y) = y$ for all $y \geq 0$, or, full insurance is Pareto optimal is indeed consistent with the equation (30) in this situation. Notice that we need no requirement about the loading λ for this result to hold.

However, we have no guarantee that $\frac{\partial I(y)}{\partial y} < 1$ as in the standard case, when the insurance customer has decreasing risk aversion and when less that full insurance is optimal. Accordingly, when the customer increases his position in the risky asset by a certain amount, it could be optimal to increase the insurance coverage by more than this amount.

By inspection of the equation (28), and comparing this to equation (25), we notice that the direct wealth effect is given by the term

$$\frac{\partial I(y;w)}{\partial w} = -\frac{C-\gamma}{B+\beta},$$

where $S = B + \beta$, and the indirect wealth effect is accordingly given by

$$\frac{\partial I(y;w)}{\partial y}\frac{dy(w)}{dw} = \frac{1}{x+1}\cdot\frac{A+\alpha}{B+\beta}\cdot\frac{dy(w)}{dw},$$

since $y = a(x+1)$ as before. We notice that these two latter expressions are also consistent with equation (30), since $D = B + \beta$.

From our result in part A of this section, the direct wealth effect is negative, and from the above analysis the indemnity $I(y;w)$ is an increasing function in the loss y. Thus, if the insurable asset is a normal good, the indirect effect is strictly positive and the interpretation of Theorem 6 is that when the indirect wealth effect dominates the direct one, insurance is a normal good.

VI-D Both a and I are decision variables

In order to properly study the problem where the insurance customer optimally selects the amount in the risky, insurable asset, and jointly determines, together with the insurer, a Pareto optimal insurance contract I, we consider the following problem.

$$\sup_{a,I}\left\{u(w-a-p+I)q + u(w+ax-p)(1-q)\right\} \tag{31}$$

subject to

$$u_0(w_0+p-I)q + u_0(w_0+p)(1-q) \geq \bar{u}_0, \tag{32}$$

where $\bar{u}_0$ is the alternative utility of the insurer. The Lagrangian $L(a,I;k)$ for the problem is

$$\begin{aligned} L(a,I;k) = {} & u(w-a-p+I)q + u(w+ax-p)(1-q) \\ & + k(u_0(w_0+p-I)q + u_0(w_0+p)(1-q) - \bar{u}_0), \end{aligned} \tag{33}$$

where k is the Lagrange multiplier. In order to solve our problem we have to find $\frac{\partial L(a,I;k)}{\partial a} = 0$ and $\frac{\partial L(a,I;k)}{\partial I} = 0$. This gives us the following two equations

$$u'(w-a+I-p)q = u'(w+ax-p)(1-q)x, \tag{34}$$

and

$$u'(w-a+I-p)q(1-(1+\lambda)q)+u'(w+ax-p)(1-q)(-(1+\lambda)q)$$
$$+k(u_0'(w_0+p-I)q((1+\lambda)q-1)+u_0'(w_0+p)(1+\lambda)q(1-q)=0. \quad (35)$$

From equation (34) we see that under strict risk aversion of both agents the optimal $a^*>0$ only if $x>\frac{q}{1-q}$, which we assume to hold from now on.

From equation (34) we derive the following equation in $\frac{da^*(w)}{dw}$ and $\frac{dI^*(w)}{dw}$ by differentiation in w along the optimal a^* and I^*, and then dividing through by (34)

$$\frac{da^*(w)}{dw}\big(R(w-a^*+I^*-p)+xR(w+a^*x-p)\big)$$
$$-\frac{dI^*(w)}{dw}\big((1+\lambda)qR(w+a^*x-p)+(1-(1+\lambda)q)R(w-a^*+I^*-p)\big)$$
$$=R(w-a^*+I^*-p)-R(w+a^*x-p). \quad (36)$$

Equation (35) obviously holds true if the two equations (23) and (24) are satisfied. It is the contents of Borch's theorem that the other implication is also true under our assumptions. From our previous work we conclude from equation (28) that the first order condition (35) implies that

$$\frac{dI^*(w)}{dw}(B+\beta)-\frac{da^*(w)}{dw}(A+\alpha)=-(C-\gamma). \quad (37)$$

Since the two equations (36) and (37) must hold simultaneously, we solve these in order to find $\frac{da^*(w)}{dw}$ and $\frac{dI^*(w)}{dw}$. By adding these two equations we first obtain

$$\frac{dI^*(w)}{dw}\big((1+\lambda)qR_0(w_0+p)+(1-(1+\lambda)q)R_0(w_0+p-I^*)\big)=0. \quad (38)$$

From this we conclude that unless $R_0\equiv 0$, it must be the case that $\frac{dI^*(w)}{dw}\equiv 0$. Second, it follows that

$$\frac{da^*(w)}{dw}=\frac{R(w-a^*+I^*-p)-R(w+a^*x-p)}{R(w-a^*+I^*-p)+xR(w+a^*x-p)}. \quad (39)$$

We have then shown the following

Theorem 7 *Suppose that both agents are risk averse and the customer has decreasing risk aversion. When the customer chooses a^* optimally, then a Pareto optimal insurance indemnity is inelastic to changes in wealth, provided the insurer's reserves are kept fixed. The insurable asset is a normal good in a Pareto optimum.*

We remark that from equations (28) and (39) it alternatively follows that $\frac{dI^*(w)}{dw} = 0$.

In order to interpret the result of this theorem, consider again the direct and the indirect wealth effects: The direct effect, holding the loss y fixed, is here negative. This stems from the decreasing risk aversion of the insurance customer as shown in part A of this section.

The indirect effect is under our assumptions positive, since $\frac{\partial I(y;w)}{\partial y} > 0$ follows from risk aversion as explained in remark (i) of the previous section, and $\frac{dy}{dw} > 0$ since the insurable asset is a normal good under decreasing risk aversion, as can be seen from equation (39), recalling that $y = a^*(x+1)$. The conclusion of our last theorem is that these two effects, the indirect and the direct one, exactly cancel out under strict risk aversion of both agents, in the case when both I and a are decision variables.

In contrast to the model of Section 5, we here obtained two equations after differentiation in wealth w. The inclusion of the insurer in the model made this possible. When a contract (I^*, a^*) is Pareto optimal at a certain level of wealth w of the insurance customer, it follows that an increase in wealth does not lead to an increase in the Pareto optimal insurance indemnity I^*, even if such an increase would follow for the optimal indemnity in the model of the previous section. The reason is that the insurer's reserves are not increased, and an increase in $I^*(w)$ would simply leave the insurer worse off at a given level of w_0, hence not satisfy the criterion of being Pareto optimal. In the final paragraph we return to this issue.

Finally we investigate what happens when the insurer is risk neutral. Let us assume that the insurance customer can not borrow, so the maximal amount a in the risky asset equals w.

Theorem 8 *When the insurer is risk neutral and the insurance customer is risk averse, a Pareto optimal insurance contract is a normal good provided* $x > \frac{(1+\lambda)q}{1-(1+\lambda)q}$.

Proof. We know that when the insurer is risk neutral, full insurance is Pareto optimal, so $I^*(y;w) = y = a^*(w)(1+x)$. In this situation the first order condition (34) is not satisfied unless $x = \frac{q}{1-q}$, so a direct argument is required. To find the optimal a in this situation, the customer maximizes

$$U(a) = u(w - a + I - p)q + u(w + ax - p)(1-q),$$

which in this case can be written

$$U(a) = u\big(w + a\big(x(1-(1+\lambda)q) - (1+\lambda)q\big)\big).$$

Clearly, if $\big(x(1-(1+\lambda)q)-(1+\lambda)q\big)\big) > 0$ the function $U(a)$ has its maximum in a at $a^* = w$, and when $x < \frac{(1+\lambda)q}{1-(1+\lambda)q}$ then $a^*(w) = 0$ for all w, implying that $I^*(y) = 0$. In the first case $\frac{dI^*(w)}{dw} = (1 + x) > 0$, which gives the conclusion of the theorem. $\square$

For this corner solution $\frac{\partial I^*(y;w)}{\partial w} = 0$, so the direct wealth effect is here zero.

VI-E A simultaneous wealth effect

Finally we turn to the situation of simultaneously changing the wealth of both the customer and the insurer. We denote the total wealth by $w_1 = w + w_0$, and investigate how a unit change in both w and w_0 affect a Pareto optimal contract $(I^*(w, w_0), a^*(w, w_0))$. In the following $\frac{\partial I^*}{\partial w_1} = \frac{\partial I^*}{\partial w} + \frac{\partial I^*}{\partial w_0}$ and similarly for $\frac{\partial a^*}{\partial w_1}$. We then have the following.

Theorem 9 *Suppose that both the insurer and the insurance customer are risk averse and have decreasing risk aversions. When the insurance customer chooses a^* optimally, then a Pareto optimal insurance indemnity is a normal good, provided the insurer's wealth is increased accordingly. Similarly, the insurable asset is a normal good in a Pareto optimum.*

Proof. Proceeding as in the previous section, we now get the following two equations, stemming from the two first order conditions:

$$\frac{\partial I^*(w_1)}{\partial w_1}(B+\beta) - \frac{\partial a^*(w_1)}{\partial w_1}(A+\alpha) = \gamma - C + R_0(w_0+p-I^*) - R_0(w_0+p) \quad (40)$$

and

$$\frac{\partial a^*(w_1)}{\partial w_1}\big(R(w - a^* + I^* - p) + xR(w + a^*x - p)\big)$$
$$- \frac{\partial I^*(w_1)}{\partial w_1}\big((1 + \lambda)qR(w + a^*x - p) + (1 - (1 + \lambda)q)R(w - a^* + I^* - p)\big)$$
$$= R(w - a^* + I^* - p) - R(w + a^*x - p). \quad (41)$$

Adding these two equations gives

$$\frac{\partial I^*(w_1)}{\partial w_1}\big((1 + \lambda)qR_0(w_0 + p) + (1 - (1 + \lambda)q)R_0(w_0 + p - I^*)\big)$$
$$= R_0(w_0 + p - I^*) - R_0(w_0 + p),$$

or

$$\frac{\partial I^*(w_1)}{\partial w_1} = \frac{R_0(w_0 + p - I^*) - R_0(w_0 + p)}{(1+\lambda)qR_0(w_0 + p) + (1-(1+\lambda)q)R_0(w_0 + p - I^*)}. \tag{42}$$

From equation (42) it follows that $\frac{\partial I^*(w_1)}{\partial w_1} > 0$ since the insurer has decreasing risk aversion, and from equation (41) it follows that $\frac{\partial a^*(w_1)}{\partial w_1} > 0$ from the previous result and the assumption that the insurance customer has decreasing risk aversion. Thus the conclusion of the theorem folows. □

The explanation of this result follows from the following decomposition:

$$\frac{\partial I^*(w_1)}{\partial w_1} = \frac{\partial I^*(y; w, w_0)}{\partial w} + \frac{\partial I^*(y; w, w_0)}{\partial w_0} = \frac{\partial I^*(y; w, w_0)}{\partial y}\frac{\partial y}{\partial w} + \frac{\partial I^*(y; w, w_0)}{\partial w} + \frac{\partial I^*(y; w, w_0)}{\partial w_0}.$$

The two first terms on the right-hand side follow from Theorem 7, the first stemming from the indirect wealth effect, the second from the direct effect of increasing the wealth of the insurance customer. As before this latter term is negative, and together these two terms exactly cancel. The last term also stems from a direct effect, this time of the insurer, and it has the opposite sign of the customer term. This is because the insurer has a decreasing risk aversion, so an increase in the reserves w_0 makes the insurer less risk averse, so he will be inclined to take more risk. As a consequence it will be Pareto optimal that $I^*(w, w_0)$ increases when both w and w_0 increase.

VII Conclusions

We have questioned the famous wealth effect on insurance purchasing. In the first part of the paper we considered two different models, both of which conclude that insurance is an inferior good in the standard formulation. By allowing the amount in the insured asset to depend upon wealth, we have demonstrated that insurance can be a normal good under plausible assumptions in both these models. This is true if the amount invested in the insurable asset is optimally determined, and it is also true if the insurable asset is held for consumption purposes only, provided it is a normal good.

In the second part we considered Pareto optimal insurance contracts by also taking the insurer into account. First, we found conditions when a Pareto optimal indemnity is a normal good, provided the amount in the insurable asset is taken as given. Second, we showed that a Pareto optimal indemnity is inelastic to changes in the wealth of the insurance customer, when both the indemnity and the amount in the insurable asset are decision variables. The crucial assumption behind this result is that the insurer's reserves do not change.

When this latter assumption is relaxed, we finally demonstrated that a Pareto optimal insurance indemnity is a normal good, provided the optimal amount in the insurable asset is also a normal good.

Two wealth effects were identified, one direct and one indirect. When the amount in the risky asset is an increasing function in the customer's wealth, these two effects have the opposite sign. Therefore the classical effect, the direct one, can be counterbalanced by the indirect one, and insurance is, more often than not, a normal good.

Finally, for an insurer or reinsurer only the direct wealth effect is of relevance, so, for example, the classical result that an insurer can retain more risk on own accounts as the reserves increase, is still valid.

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