Strategic Insider Trading Equilibrium: A Forward Integration Approach.

Knut K. Aase, Terje Bjuland and Bernt Øksendal

Norwegian School of Economics and Business Administration

5045 Bergen, Norway

and

Centre of Mathematics for Applications (CMA),

University of Oslo, Norway.

`Knut.Aase@NHH.NO, oksendal@math.uio.no`

November 1, 2007

Abstract

The continuous-time version of Kyle's (1985) model of asset pricing with asymmetric information is studied, and generalized in various directions, i.e., by allowing time-varying noise trading, and by allowing the orders of the noise traders to be correlated with the insider's signal. From rather simple assumptions we are able to derive the optimal trade for an insider; the trading intensity satisfies a deterministic integral equation, given perfect inside information.

We use a new technique called forward integration in order to find the optimal trading strategy. This is an extension of the stochastic integral which takes account of the informational asymmetry inherent in this problem. The market makers' price response is found by the use of filtering theory. The novelty is our approach, which could be extended in scope.

KEYWORDS: Insider trading, asymmetric information, equilibrium, strategic trade, filtering theory, forward integration

1 Introduction

We take as our starting point the seminal paper of Kyle (1985), where a model of asset pricing with asymmetric information is presented. Traders submit order quantities to risk-neutral market makers, who set prices competitively by taking the opposite position to clear the market. Excluding the market makers, the model has two kinds of traders: a single risk neutral informed trader and noise traders. The informed trader rationally anticipates the effects of his orders on the price, i.e., he acts non-competitively or strategically. In the presence of noise traders it is impossible for the market makers to exactly invert the price and infer the informed trader's signal. Thus markets are semi-strong, but not strong form efficient.

In this model the insider makes positive profits in equilibrium by exploiting his monopoly power optimally in a dynamic context. Noise trading provides camouflage which conceals his trading from market makers. An important issue is to demonstrate that this is possible in equilibrium without destabilizing prices.

Kyle's approach is to first study a one-period auction, then extend the analysis to a model in with auctions take place sequentially, and finally letting the time between the auctions go to zero, in which case a limiting model of continuous trading is obtained. Back (1992) formalize and extend the continuous-time version of the Kyle model, by i.a., the use of dynamic programming.

There is a rich literature on the one period model, as well as on discrete insider trading, e.g., Holden and Subrahmanyam (1992), Admati and Pfleiderer (1988), and others, all adding insights to this class of problems. Glosten and Milgrom (1985) present a different approach, containing similar results to Kyle. Before Kyle (1985) and Glosten and Milgrom (1985) there is also a huge literature on insider trading in which the insider acts competitively, e.g., Grossman and Stiglitz (1980).

The purpose of this article is to study the continuous-time model directly, not as a limiting model of a sequence of auctions, and use certain aspects of the modern methodological machinery in continuous-time modeling to resolve the problem of the informed trader, in a more general setting with time-varying noise trading, where the orders of the noise traders are also allowed to depend upon the insider's private information. Furthermore, we do not assume that the final price p_T equals the insiders signal $\tilde{v}$, but show that this is a consequence of our other model assumptions. The wealth of the

insider can be represented as a stochastic integral of his orders with respect to the changes in the market price. This integral is not of a standard form, since the insider's order is not in the information set generated by the prices. This is precisely where a key part of the problem lies; the insider has more information then reflected in the market prices.

There is, however, an extension of the stochastic integral, called the *forward* integral, in which the usual information constraint of this type of analysis need not be satisfied. This is exactly what we need in the present context of asymmetric information.

The prices set by the market makers are in the form of a conditional expectation, which calls for the use of filtering theory. Combining these two methodologies, we are able to solve the insider's problem in a direct way, leading to a deterministic integral equation for the insider's trading intensity $\beta(t)$ at time t, given his information set with perfect forward information.

We solve the integral equation for the trading intensity $\beta(t)$ by by transforming this equation to a non-linear, separable differential equation, which calls for a simple solution. This we compare to the solution of Kyle (1985) (and also Back (1992)). In the special case of time homogeneous noise trading, we recover the Kyle-solution. For time-varying noise trading we get the result that the market depth is still a constant, and the expected (ex ante) profits of the insider depends on the average volatility process.

2   The Model

At date T there is to be a public release of information that will perfectly reveal the value of an asset; cf. fair value accounting. Trading in this asset and a risk-free asset with interest rate zero is assumed to occur continuously during the interval $[0, T]$. The information to be revealed at time T is represented as a signal $\tilde{v}$, a random variable which we interpret as the price at which the asset will trade after the release of information. This information is already possessed by a single insider at time zero. The unconditional distribution of $\tilde{v}$ is assumed to be *normal* with parameters $\mu_{\tilde{v}}$ and $\sigma_{\tilde{v}}$.

In addition to the insider, there are liquidity traders who have random, price-inelastic demands, and risk neutral market makers. All orders are market orders and the net order flow is observed by all market makers. We denote by z_t the cumulative orders of liquidity traders through time t. The process z is assumed to be a Brownian motion with mean zero and variance rate σ_t^2,

i.e., $dz_t = \sigma_t dB_t$, where $\sigma_t > 0$ is a deterministic continuously differentiable function on $[0, T]$, for a standard Brownian motion B defined on a probability space (Ω, P). Note that we do *not* assume that $\tilde{v}$ is independent of B_s; $s \leq T$. This makes it necessary to use anticipative stochastic calculus. We use *forward integrals* to model this situation. See (2.5) below and Appendix 1. We let x_t be the cumulative orders of the informed trader, and define

$$(2.1) \qquad y_t = x_t + z_t \qquad \text{for all } t \in [0, T]$$

as the total orders accumulated by time t.

Market makers only observe the process y, so they cannot distinguish between informed and uninformed trades. Let $\mathcal{F}_t^y = \sigma(y_s; s \leq t)$ be the information filtration of this process. Since the market makers are assumed to be perfectly competitive and risk neutral, they will set the price p_t at time t as follows

$$(2.2) \qquad p_t = E(\tilde{v} | \mathcal{F}_t^y),$$

which we will call a *rational* pricing rule. We assume that the insider's portfolio is of the form

$$(2.3) \qquad dx_t = (\tilde{v} - p_t)\beta_t dt, \qquad x_0 = 0,$$

where $\beta \geq 0$ is some deterministic function, both assumptions consistent with Kyle (1985).[1] The function β_t is the trading intensity on the information advantage $(v - p_t)$ of the insider.

Denote the insider's wealth by w and the investment in the risk-free asset by b. The budget constraint of the insider can best be understood by considering a discrete time model. At time t the agent submits a market order $x_t - x_{t-1}$ and the price changes from p_{t-1} to p_t. The order is executed at price p_t, in other words, x_t is submitted *before* p_t is set by the market makers. The investment in the risk-free asset changes by $b_t - b_{t-1} = -p_t(x_t - x_{t-1})$, i.e., buying stocks leads to reduced cash with exactly the same amount. Thus, the associated change in wealth is (which was pointed out by Back (1992))

$$(2.4) \qquad b_t - b_{t-1} + x_t p_t - x_{t-1} p_{t-1} = x_{t-1}(p_t - p_{t-1}).$$

---

[1] The finite variation property of x is assumed by Kyle (1985), and an equilibrium where this is the case is found by Back (1992).

4

In other words, the usual accounting identity for the wealth dynamics is of the same type as in the standard price-taking model, except for one important difference; while, in the rational expectations model, the number of stocks in the risky asset at time t is depending only on the information available at this time, so that both the processes x and p are adapted processes with respect to the same filtration, here the order x depends on information available only at time T for the market makers (and the noise traders). As a consequence writing the dynamic equation for the insider' wealth as follows

$$w_t = w_0 + \int_0^t x_s dp_s$$

is not well defined as a stochastic integral in the traditional interpretation, since p_t is $\mathcal{F}_t^y$-adapted, and x_t is not.

Let us define the information filtration of the informed trader as $\mathcal{G}_t = \mathcal{F}_t^y \vee \sigma(\tilde{v})$. Thus the informed trader knows $\tilde{v}$ at time zero and observes y_t at each time t. Obviously the filtration $\mathcal{G}_t \supset \mathcal{F}_t^y$ and this extension is not of a trivial, or technical type, but a significant one. For example, there is information in $\mathcal{G}_t$ for any $t \in [0, T)$ that will only be revealed to the market makers at the future time T. The key point here is that from (2.3) the order x_t depends on $\tilde{v}$ which is not in $\mathcal{F}_t^y$. Since the insider knows the realization of $\tilde{v}$ at time 0, he has long-lived forward-looking information. When z is not assumed to be independent of $\tilde{v}$, the extension of the ordinary stochastic integral to a semimartingale setting is not justified any longer.[2]

In the stochastic integral representing the budget constraints x_t is $\mathcal{G}_t$-measurable, and p_t is $\mathcal{F}_t$-measurable which is the violation of the standard, important requirement of any stochastic integral in the traditional interpretation.

There is, however, a stochastic integration theory based on the so-called *forward integral*, which turns out to be useful under the informational asymmetry that we have. It is a natural extension of the usual stochastic integral, with the informational constraints that we require of the dynamic wealth equation based on the above budget constraints. It is denoted by

$$(2.5) \qquad\qquad w_t = w_0 + \int_0^t x_s d^- p_s,$$

---

[2]It does not help here to extend to a stochastic integral of a predictable process with respect to a semimartingale, as in Back (1992). In his case this procedure was valid, since z was explicitly assumed independent of $\tilde{v}$.

where $d^- p_s$ stands for forward integration. From its very definition, which is given by a limit (in probability) of the usual partial sums of the type appearing in (2.4), it follows that it will have the correct financial interpretation, given that the concept is meaningful. It turns out that it is, and naturally the forward integral will not possess many of the standard properties of the stochastic integral, but there is a version of Itô's formula that still is valid, and which we need in the following (see Appendix I for a definition, Itô's formula, and references).

We can now formulate the problem: The insider wants to solve, for each time point t

$$(2.6) \qquad \max_x E(w_T | \mathcal{G}_t)$$

subject to the price p satisfying the rational pricing rule (2.2), the insider's strategy x satisfying (2.3), and the dynamic forward stochastic differential equation (2.5) holding for all $t \in [0, T]$.

Usually the assumption is made that $p_T = \tilde{v}$ a.s., but as we will show below, this is a consequence of our other model assumptions. This result seems natural, ensuring that all information available has been incorporated in the price at the time T of the public release of the information.

Since there is a tacit understanding that the price process p is continuous in this model, this result also means that the insider must trade continuously throughout the time interval $[0, T]$, and we can expect that the trading intensity β must be large as t approaches T in order for this condition to be satisfied. [3]

An *equilibrium* is a pair (p, x) such that p satisfies (2.2), given x, and x is an optimal trading strategy solving (2.6), given p. We now have the following result:

Theorem 2.1. *Given the linear trading strategy (2.3), the optimal trading intensity $\beta(t)$ is given by*

$$(2.7) \qquad \beta_t = \left(\frac{\int_0^T \sigma_s^2 ds}{S_0} \right)^{\frac{1}{2}} \frac{\sigma_t^2}{\int_t^T \sigma_s^2 ds}; \qquad 0 \le t \le T.$$

---

[3]If the price $p_t \ne \tilde{v}$ for some $t < T$, and the agent did not trade in $[t, T)$, there would have to be a jump in the price at time T, which the results of our model rule out. This would not be rational for the insider to do, as he would miss some profit opportunities by not trading.

The corresponding price p_t set by the market makers is

$$(2.8) \qquad p_t = E(\tilde{v}) + \int_0^t \lambda_s dy_s,$$

where $\tilde{y}_t$ defined by $d\tilde{y}_t = \frac{1}{\sigma_t} dy_t$ is a Brownian motion with respect to the market makers' information, and the price sensitivity λ_t is given by

$$(2.9) \qquad \lambda_t \equiv \lambda = \frac{S_0^{\frac{1}{2}}}{\left(\int_0^T \sigma_s^2 ds\right)^{\frac{1}{2}}}; \qquad a \; constant \; over \; time.$$

At the terminal time T the price p_T corresponding to the optimal insider insider tensity β satisfies

$$(2.10) \qquad p_T = \tilde{v} \quad a.s.$$

Remark 2.2 To summarize, our paper differes from the papers of Kyle (1985) and Back (1992) both with respect to basic assumptions and method:

(i) We do *not* assume that $\tilde{v}$ is independent of $\{z(s); 0 \le s \le T\}$. Because of this, the integral in (2.5) may not exist as a semimartingale integral. Therefore we have to deal with anticipative stochastic calculus, by means of the forward integral.

(ii) We do *not* assume that the volatility $\sigma(t)$ of the noise traders is constant. Nevertheless we prove that the price sensitivity λ_t is constant also in our case, if the optimal strategy is applied.

(iii) We do *not* assume a priori that

$$p_T = \tilde{v} \quad a.s.$$

But this is *proved* to be the case if the optimal strategy is used.

We remark that if we had made this assumption a priori, then our proof could have been simplified as follows: The last term in (4.14) would have been 0. Hence (see (4.16)) we would have $S_{t,T}^{(\beta)} = 0$ for all $t \in [0, T]$ and Problem 1 would automatically reduce to Problem 2.

(iv) We do *not* assume a priori that the strategy x_t is *incospicuous*, i.e. that

$$\frac{1}{\sigma_t} dy_t = \frac{1}{\sigma_t} x_t dt + dz_t$$

is a Brownian motion with respect to its own filtration. However, this is *proved* to hold if x_t is chosen optimally.

(v) We do *not* assume a priori that there exists a function H such that

$$p_t = H(t, y_t).$$

But this is *proved* to be the case if the insider acts optimally.

(vi) Finally, since we are not assuming a Markovian setup we cannot use dynamic programming (the HJB equation) to find the optimal strategy, but we use forward integrals and a perturbation argument instead.

Remark 2.3 It is interesting to note that also in our general setting the total order process y_t becomes a *Brownian bridge* if the optimal insider strategy is used. To see this we proceed as follows:

By (2.7), (2.8), (2.9) we have

$$dy_t = (\tilde{v} - p_t)\beta_t dt + \sigma_t dB_t$$
$$= (\tilde{v} - E[\tilde{v}] - \lambda y_t)\beta_t dt + \sigma_t dB_t$$

(2.11)
$$= \Big[\Big(\frac{\int_0^T \sigma_u^2 du}{S_0}\Big)^{1/2}(\tilde{v} - E[\tilde{v}]) - y_t\Big] \frac{\sigma_t^2 dt}{\int_t^T \sigma_u^2 du} + \sigma_t dB_t\,.$$

Thus y_t is the *bridge* of the process $z_t = \int_0^t \sigma_s dB_s$, conditioned to arrive at the terminal value

$$y_T = \Big(\frac{\int_0^T \sigma_u^2 du}{S_0}\Big)^{1/2}(\tilde{v} - E[\tilde{v}])$$

at time $t = T$.

In particular, if $\sigma_t = \sigma$ is constant we get

(2.12)
$$dy_t = \Big[\sigma\Big(\frac{T}{S_0}\Big)^{1/2}(\tilde{v} - E[\tilde{v}]) - y_t\Big] \frac{dt}{T - t} + \sigma\, dB_t,$$

and hence $\frac{1}{\sigma} dy_t$ is the classical Brownian bridge, conditioned to arrive at

$$\Big(\frac{T}{S_0}\Big)^{1/2}(\tilde{v} - E[\tilde{v}])$$

8

at time $t = T$.

In Section 4 we present a proof of Theorem 2.1. First we discuss the properties of the solution.

3   Properties of the equilibrium.

The generalization relative to Kyle (1985) included in Theorem 2.1 allows for a time varying volatility parameter in the order process of the noise traders. One would, perhaps, expect that as a consequence the market liquidity function λ_t would depend on time, suggested by the expression (4.39) in the next section. The result of Theorem 2.1 is that it does not. The intuition for this can be explained as follows:

The trading intensity β_t will typically increase as t approaches T, since the insider becomes increasingly desperate to utilize his residual information advantage. In particular, from expression (2.7) in Theorem 2.1 we see that β_t / σ_t^2 increases as t increases. It follows from the proof in the next section, equations (4.38) and (4.39), that the price sensitivity λ_t can be written

$$\lambda_t = \frac{\beta_t S_t}{\sigma_t^2}.$$

Here

$$S_t := E[(\tilde{v} - p_t)^2] \qquad \text{and} \qquad S_0 = E[(\tilde{v} - E[\tilde{v}])^2].$$

Furthermore S_t can be shown to have the form

$$S_t = \frac{S_0}{1 + S_0 \int_0^t \tilde{\beta}_s^2 ds}; \qquad t \in [0, T],$$

(see equation (4.7)) where

$$\tilde{\beta}_t = \frac{\beta_t}{\sigma_t}; \qquad 0 \leq t \leq T.$$

The quantity $\int_0^t \tilde{\beta}_s^2 ds$ measures the the "amount" of insider trading to liquidity trading by time t. As this quantity increases over time, the amount of private information S_t remaining at time t is seen, from the above expression, to decrease, where S_t is the (mean square) distance between $\tilde{v}$ and p_t. The function λ_t is seen to depend on two effects:

(i) The quantity β_t/σ_t^2 increases over time, which tends to increase λ_t as time t increases.

(ii) The quantity S_t decreases over time, suggesting that the insider's information advantage is deteriorating, which tends to decrease λ_t as t increases. In equilibrium (i) is offset by (ii) and $\lambda_t = \lambda$ is constant over time.

Notice that the important quantities are β_t/σ_t^2 and $\beta_t/\sigma_t = \tilde{\beta}_t$ in the above arguments. The mere fact that the amount of insider trading represented by $\int_0^t \beta_s^2 ds$ is large, is no guarantee that the market price p_t is close to the fundamental value $\tilde{v}$, i.e., that S_t is small. It could be that the amount of noise trading $\int_0^t \sigma_s ds$ is also large, in which case the insider could hide his trade, and less information about the true value would be revealed to the market makers. Similarly, we do not know that β_t is *monotonically* increasing over time, only that β_t/σ_t^2 is. Notice that the equilibrium value of the price sensitivity λ can be interpreted as the square root of a ratio, where the numerator is the amount of private information, ex ante, and the denominator is the amount of liquidity trading.

From the expressions in Theorem 2.1 we notice that

$$\beta_t = \frac{1}{\lambda} \frac{\sigma_t^2}{\int_t^T \sigma_s^2 ds}$$

so β_t is inversley related to λ for each t. Since the quantity $1/\lambda$ measures the market depth, the insider will naturally trade more intensely, ceteris paribus, when this quantity is large.

From the general discussion in Kyle (1985) it is indicated that if the slope of the residual supply curve λ_t ever decreases (i.e., if the market depth ever increases), then unbounded profits can be generated. This is inconsistent with an equilibrium, so λ_t must be monotonically non-decreasing in any equilibrium. It is argued that this follows since in continuous time, the informed trader can act as a perfectly discriminating monopsonist, moving up or down the residual supply curve (i.e., the market is infinitely tight). Hence, he could exploit predictable shifts in the supply curve. From the analysis of Back (1992) it is known that, more generally, this slope must be a martingale given the market makers' information. Our result that λ_t is indeed a constant is, accordingly, consistent with the literature.

One would, perhaps, expect that the insider, since he can be assumed to know the function σ_t, may use it to further conceal his trade in that he will use a high β_t at a time when σ_t is large. This impression is confirmed by

10

investigating the optimal trading intensity β appearing in expression (2.7) of Theorem 2.1.

However, when σ_t is low the insider must apply a correspondingly lower trading intensity, and it turns out that the expected (ex ante) profits average out. This can be demonstrated as follows: Consider the expected wealth of the insider

$$E[w_T] = w_0 + S_0 \int_0^T \frac{\beta_t dt}{1 + S_0 \int_0^t \tilde{\beta}_s^2 ds},$$

an expression which follows from the results of the next section. Here the last term is the expected (ex ante) profits, which can be shown to be $\sqrt{S_0 \int_0^T \sigma_t^2 dt}$.[4] Thus, trading at a time-varying volatility σ_t corresponds exactly, when it comes to expected profits, to trading at a constant volatility σ determined by $\sigma^2 = \frac{1}{T} \int_0^T \sigma_t^2 dt$, the right comparison in this regard.

When the amount of liquidity trading $\int_0^t \sigma_s^2 ds$ is large, we noticed above that λ is small, in which case the insider's profit is large. However, a small value of λ is, in isolation, no guarantee for a large ex ante profit of the insider, since a large value of S_0 also makes the profit of the insider large, and λ large as well.

This points in one possible direction for extending the present model. Suppose that the private information is connected to quaterly accounting data for the firm, so T stands for one quarter, and let us extend the model beyond T to $2T$, $3T, \cdots$, etc. Let us, as in Admati and Pfleiderer (1988), imagine two types of liquidity traders, discretionary and non-discretionary. Just after each disclosure period of length T, the level of private information relative to the uninformed is at its minimum. It seems reasonable, from the above formula for the ex ante profits of the insider, that the discretionary traders, acting strategically to time their trades, should concentrate their trade to these times in order to loose less to the insider. That this kind behavior is optimal is expected from the conclusions of Admati and Pfleiderer (1988), who noticed that λ is a constant is not in accordance with empirical findings; the bid ask spread 2λ is varying over time.

We also have the following corollary:

Corollary 1. *Suppose $\sigma_t = \sigma > 0$ is a constant. Then the optimal trading*

---

[4]In the case when $\sigma_t = \sigma$ is a constant, we get that the expected profits equal $\sigma\sqrt{S_0 T}$, consistent with Kyle (1985).

intensity for the insider is

(3.1) $$\beta_t = \frac{\sigma\sqrt{T}}{\sqrt{S_0}(T-t)}; \qquad 0 \le t < T.$$

The corresponding price p_t set by the market makers is given by

(3.2) $$dp_t = \lambda_t dy_t,$$

where

(3.3) $$\lambda_t \equiv \lambda = \frac{\sqrt{S_0}}{\sigma} \frac{1}{\sqrt{T}}; \qquad a \; constant \; for \; all \; t \in [0,T).$$

This result follows from Theorem 2.1 by setting $\sigma_s \equiv \sigma$ in (4.42). The results of Corollary 1 are in agreement with Kyle (1985) and Back (1992) (when we set $T = 1$).

Recently, a paper of related interest by Eide (2007) came to our knowledge. Her work, which was done independently of ours, differs from ours in several ways: She focuses on the situation when the price process $\tilde{v}_t$ of the stock is assumed to have a specific dynamics (an Itô diffusion and a martingale with respect to an independent Brownian motion), and its current value $\tilde{v}_t$ (not $\tilde{v}_T$) is known to the insider at time t for all $t \in [0, T]$. She avoids the use of forward integrals by assuming a priori that the processes are semimartingales with respect to the relevant filtrations. Like Back she then assumes that the market makers set the price equal to $p_t = H(t, y_t)$ for some function H and that $H(t, y_t) = E(\tilde{v}_T|\mathcal{F}_t^y)$. These assumptions put the problem of finding a corresponding equilibrium into a Markovian context, which allows her to solve the problem by using dynamic programming. In conclusion, her a priori assumptions are stronger than ours, but they enable her to solve other problems than we do. In particular, the final stock value $\tilde{v} = \tilde{v}_T$ need not be normally distributed in her case.

We now present the proof of Theorem 2.1. It can be noted to be rather different from the corresponding development in Kyle (1985).

4 The solution of the problem

From the requirement that the market makers are able to calculate the correct conditional expectation of $\tilde{v}$ at all times, we are led to consider filtering

theory, which involves the following system of equations:

(4.1) $\qquad d\tilde{v}_t = 0, \qquad \tilde{v}_0 = \tilde{v}, \qquad$ (system equation)

and

(4.2) $\qquad d\hat{y}_t = \tilde{v}\beta_t dt + dz_t, \qquad$ (observation equation).

Let $\mathcal{F}_t^{\hat{y}} = \sigma(\hat{y}_s; s \le t)$ be the information filtration of the process $\hat{y}$. The *innovation process* y is defined by

(4.3) $\qquad\qquad dy_t = (\tilde{v} - E(\tilde{v}|\mathcal{F}_t^{\hat{y}}))\beta_t dt + dz_t$

Let $\mathcal{F}_t^y = \sigma(y_s; s \le t)$ be the information filtration of the process y. Then we have:

Lemma 1. $\mathcal{F}_t^y = \mathcal{F}_t^{\hat{y}}; \quad t \in [0, T].$

Proof. Since, by (4.3)

(4.4) $\qquad\qquad dy_t = d\hat{y}_t - E\big[\tilde{v}|\mathcal{F}_t^{\hat{y}}\big]\beta_t dt$

we see that

(4.5) $\qquad\qquad\qquad \mathcal{F}_t^y \subseteq \mathcal{F}_t^{\hat{y}}.$

To prove the converse we use that by an extension of the Kalman filter we have (see Lipser and Shiryaev (1978), Theorem 12.2)

(4.6) $\qquad p_t := E\big[\tilde{v}|\mathcal{F}_t^{\hat{y}}\big] = \dfrac{p_0 + S_0 \int_0^t \frac{\beta_s}{\sigma_s^2} d\hat{y}_s}{1 + S_0 \int_0^t (\frac{\beta_s}{\sigma_s})^2 ds}; \qquad p_0 = E[\tilde{v}]$

and

(4.7) $\qquad S_t := E[(\tilde{v} - p_t)^2] = \dfrac{S_0}{1 + S_0 \int_0^t (\frac{\beta_s}{\sigma_s})^2 ds}; \qquad S_0 = E[(\tilde{v} - p_0)^2].$

Put

(4.8) $\qquad\qquad\qquad K_t = 1 + S_0 \int_0^t \left(\dfrac{\beta_s}{\sigma_s}\right)^2 ds.$

13

Combining (4.6) and (4.8) with (4.4) we get

$$K_t dy_t = K_t d\hat{y}_t - \left(p_0 + S_0 \int_0^t \frac{\beta_s}{\sigma_s^2} d\hat{y}_s\right)\beta_t dt$$

or

$$\frac{K_t\beta_t}{\sigma_t^2} d\hat{y}_t - \frac{S_0\beta_t^2}{\sigma_t^2}\left(\int_0^t \frac{\beta_s}{\sigma_s^2} d\hat{y}_s\right)dt = \frac{K_t\beta_t}{\sigma_t^2}(dy_t + p_0\beta_t dt).$$

If we define

$$R_t = \int_0^t \frac{\beta_s}{\sigma_s^2} d\hat{y}_s$$

this can be written

$$K_t dR_t - \frac{S_0\beta_t^2}{\sigma_t^2} R_t dt = \frac{K_t\beta_t}{\sigma_t^2}(dy_t + p_0\beta_t dt).$$

If we multiply this equation with $\frac{1}{K_t}\exp(-\gamma_t)$, where

(4.9) $$\gamma_t = \int_0^t \frac{S_0\beta_s^2}{\sigma_s^2 K_s} ds\,,$$

we get

$$\exp(-\gamma_t)dR_t - \frac{S_0\beta_t^2}{\sigma_t^2 K_t}\exp(-\gamma_t)R_t dt = \exp(-\gamma_t)\frac{\beta_t}{\sigma_t^2}(dy_t + p_0\beta_t dt).$$

This can be written

$$d(\exp(-\gamma_t)R_t) = \exp(-\gamma_t)\frac{\beta_t}{\sigma_t^2}(dy_t + p_0\beta_t dt).$$

Integrating this we obtain

$$R_t = \exp\gamma_t \int_0^t \exp(-\gamma_s)\frac{\beta_s}{\sigma_s^2}(dy_s + p_0\beta_s ds).$$

Therefore

$$dR_t = \frac{\beta_t}{\sigma_t^2}d\hat{y}_t = \frac{\beta_t}{\sigma_t^2}dy_t + p_0\beta_t dt$$

(4.10) $$+ \exp\gamma_t\left(\int_0^t \exp(-\gamma_s)\frac{\beta_s}{\sigma_s^2}(dy_s + p_0\beta_s ds)\right)\frac{S_0\beta_t^2}{\sigma_t^2 K_t}dt\,.$$

14

This shows that $\hat{y}_t$ can be expressed in terms of y_s; $s \leq t$ and hence that

$$\mathcal{F}_t^{\hat{y}} \subseteq \mathcal{F}_t^y.$$

Combining this with (4.5) we obtain $\mathcal{F}_t^y = \mathcal{F}_t^{\hat{y}}$ and the proof of Lemma 4.1 is complete. $\qquad\square$

From filtering theory we know that $\tilde{y}$ defined by $d\tilde{y}_t := \frac{1}{\sigma_t} dy_t$ is a Brownian motion with respect to the information filtration $\mathcal{F}_t^y$.[5]

Using (2.2), (2.3) and the definition $y = x + z$, we see that what we have called the innovation process y in the above is equal to the total accumulated order process of the previous section. Returning to the equation (2.5), there is a analog of Itô's formula for forward integration, which says that

(4.11)
$$d^-(x_t p_t) = x_t d^- p_t + p_t d^- x_t + dp_t dx_t,$$

(see formula (5.11) of Appendix I). Since x has finite variation, $dp_t dx_t = 0$ and we get

(4.12)
$$w_T = w_0 + x_T p_T - x_0 p_0 - \int_0^T p_t d^- x_t.$$

Since $(\tilde{v} - p_t) \perp p_t$ in $L^2(P)$, i.e., $E[(\tilde{v} - p_t)p_t] = 0$, we see that

(4.13)
$$E\left[\int_0^T p_t d^- x_t\right] = \int_0^T E[p_t(\tilde{v} - p_t)]\beta_t dt = 0.$$

Therefore we get that

$$E[w_T] = w_0 + E[x_T p_T] = w_0 + E\left[p_T \int_0^T (\tilde{v} - p_t)\beta_t dt\right]$$

$$= w_0 + E\left[(p_T - \tilde{v} + \tilde{v}) \int_0^T (\tilde{v} - p_t)\beta_t dt\right]$$

$$= w_0 + E\left[\int_0^T (\tilde{v} - p_t)\beta_t dt\right] - \int_0^T E[(\tilde{v} - p_T)(\tilde{v} - p_t)]\beta_t dt$$

(4.14)
$$= w_0 + \int_0^T E[(\tilde{v} - p_t)^2)]\beta_t dt - \int_0^T E[(\tilde{v} - p_T)(\tilde{v} - p_t)]\beta_t dt.$$

---

[5] The result that $\frac{1}{\sigma} y$ is a Brownian motion with respect to the market makers' information was assumed by Back (1992).

15

As before let

(4.15) $$S_t = S_t^{(\beta)} := E[(\tilde{v} - p_t)^2]$$

and define

(4.16) $$S_{t,T} = S_{t,T}^{(\beta)} := E[(\tilde{v} - p_t)(\tilde{v} - p_T)]; \qquad 0 \le t \le T.$$

(Note that if we had assumed that

$$p_T = \tilde{v} \quad \text{a.s.}$$

then we would get $S_{t,T} = 0$ and the proof would simplify considerably.)

Then (4.14) can be written

(4.17) $$E[w_T] = w_0 + \int_0^T S_t^{(\beta)} \beta_t dt - \int_0^T S_{t,T}^{(\beta)} \beta_t dt.$$

From (4.7) we see that S_t satisfies the Riccati equation

(4.18) $$S_t' := \frac{dS_t}{dt} = -\frac{\beta_t^2}{\sigma_t^2} S_t^2; \qquad S_0 = E[(\tilde{v} - E[\tilde{v}])^2].$$

By (4.6) we get that

(4.19) $$p_t = E[\tilde{v}|\mathcal{F}_t^y] = E[\tilde{v}|\mathcal{F}_t^{\hat{y}}]$$

satisfies the equation

(4.20) $$dp_t = \frac{\beta_t S_t}{\sigma_t^2} dy_t = \frac{\beta_t S_t}{\sigma_t^2} \left((\tilde{v} - p_t)\beta_t dt + \sigma_t dB_t \right).$$

Hence

$$d(\tilde{v} - p_t) = -\frac{\beta_t^2 S_t}{\sigma_t^2} (\tilde{v} - p_t) dt - \frac{\beta_t S_t}{\sigma_t} dB_t,$$

or

$$d(\tilde{v} - p_t) + \frac{\beta_t^2 S_t}{\sigma_t^2} (\tilde{v} - p_t) dt = -\frac{\beta_t S_t}{\sigma_t} dB_t.$$

This can be written

$$d\left(\exp\left(\int_0^t \frac{\beta_s^2 S_s}{\sigma_s^2} ds \right)(\tilde{v} - p_t) \right) = -\frac{\beta_t S_t}{\sigma_t} \exp\left(\int_0^t \frac{\beta_s^2 S_s}{\sigma_s^2} ds \right) dB_t.$$

16

Integrating this we get

$$\tilde{v} - p_t = (\tilde{v} - p_0) \exp\left(- \int_0^t \frac{\beta_s^2 S_s}{\sigma_s^2} ds\right) - \int_0^t \exp\left(- \int_s^t \frac{\beta_u^2 S_u}{\sigma_u^2} du\right) \frac{\beta_s S_s}{\sigma_s} dB_s\,.$$

This implies that

$$S_{t,T}^{(\beta)} = E[(\tilde{v} - p_t)(\tilde{v} - p_T)]$$

$$= E[(\tilde{v} - p_0)^2] \exp\left(- \int_0^t \frac{\beta_s^2 S_s}{\sigma_s^2} ds - \int_0^T \frac{\beta_s^2 S_s}{\sigma_s^2} ds\right)$$

(4.21)
$$+ \int_0^t \exp\left(- \int_s^t \frac{\beta_u^2 S_u}{\sigma_u^2} du - \int_s^T \frac{\beta_u^2 S_u}{\sigma_u^2} du\right) \frac{\beta_s^2 S_s^2}{\sigma_s^2} ds\,.$$

In particular, note that

(4.22)
$$S_{t,T}^{(\beta)} \geq 0 \qquad \text{for all } t \in [0,T]$$

and

(4.23)
$$S_{t,T}^{(\beta)} = 0 \qquad \text{if } p_T = \tilde{v}.$$

We now return to problem (2.6). By combining (4.17) and (4.7) we see that our original problem can be formulated as the following control problem:

Problem 1. Maximize

(4.24)
$$J_1(\beta) := S_0 \int_0^T \frac{\beta_t dt}{1 + S_0 \int_0^t (\frac{\beta_s}{\sigma_s})^2 ds} - \int_0^T S_{t,T}^{(\beta)} \beta_t dt$$

over all $\beta \in \mathcal{A}$, where $\mathcal{A}$ is the set of all (deterministic) functions $\beta : [0,T] \to \mathbb{R}$ which are continuously differentiable on $(0,T)$.

We will first study the following related problem:

Problem 2. Maximize

(4.25)
$$J(\beta) := S_0 \int_0^T \frac{\beta_t dt}{1 + S_0 \int_0^t (\frac{\beta_s}{\sigma_s})^2 ds}$$

over all $\beta \in \mathcal{A}$.

17

We will find the optimal control $\hat{\beta} \in \mathcal{A}$ for Problem 2 and show that the corresponding terminal price $p_T^{(\hat{\beta})}$ satisfies

(4.26) $$p_T^{(\hat{\beta})} = \tilde{v} \quad \text{a.s.}$$

It follows by (4.16) that $S_{t,T}^{(\hat{\beta})} = 0$ and hence $\hat{\beta}$ is also optimal for Problem 1, because,

$$\sup_{\beta \in \mathcal{A}} J_1(\beta) \leq \sup_{\beta \in \mathcal{A}} J(\beta) = J(\hat{\beta}) = J_1(\hat{\beta}) \leq \sup_{\beta \in \mathcal{A}} J_1(\beta).$$

In view of this we now proceed to solve Problem 2. Since the map

$$\beta \to J(\beta); \qquad \beta \in \mathcal{A}$$

is concave, we can use the following perturbation argument to find the maximizer for $J(\cdot)$:

Suppose $\beta \in \mathcal{A}$ maximizes

$$J(\beta) := S_0 \int_0^T \left(1 + S_0 \int_0^t \left(\frac{\beta_s}{\sigma_s} \right)^2 ds \right)^{-1} \beta_t dt.$$

Choose an arbitrary function $\xi \in \mathcal{A}$ and define the real function g by

(4.27) $$g(y) = J(\beta + y\xi); \qquad y \in \mathbb{R}.$$

Then g is maximal at $y = 0$ and hence

$$0 = g'(0) = \frac{d}{dy} J(\beta + y\xi)|_{y=0} =$$

$$\frac{d}{dy} \left(S_0 \int_0^T \left(1 + S_0 \int_0^t \frac{(\beta_s + y\xi_s)^2}{\sigma_s^2} ds \right)^{-1} (\beta_t + y\xi_t) dt \right) \Big|_{y=0} =$$

$$S_0 \int_0^T \left(1 + S_0 \int_0^t \tilde{\beta}_s^2 ds \right)^{-1} \xi_t dt - S_0^2 \int_0^T \left(1 + S_0 \int_0^t \tilde{\beta}_s^2 ds \right)^{-2} \left(\int_0^t \frac{2\beta_s \xi_s}{\sigma_s^2} ds \right) \beta_t dt$$

$$= \int_0^T S_t \xi_t dt - 2 \int_0^T S_t^2 \left(\int_0^t \frac{\beta_s \xi_s}{\sigma_s^2} ds \right) \beta_t dt.$$

Changing the order of integration in the last term we get

$$\int_0^T S_t \xi_t dt - 2 \int_0^T \left(\int_s^T S_t^2 \beta_t dt \right) \frac{\beta_s \xi_s}{\sigma_s^2} ds = 0,$$

18

or

$$\int_0^T \{S_t - 2(\int_t^T S_s^2 \beta_s ds)\frac{\beta_t}{\sigma_t^2}\}\xi_t dt = 0.$$

Since $\xi \in \mathcal{A}$ was arbitrary, we conclude that an optimal β_t must satisfy the equation

(4.28)
$$\sigma_t^2 S_t = 2\beta_t \int_t^T S_s^2 \beta_s ds$$

where, as before, S_t is given by equation (4.7). This is an integral equation in the unknown function β. Differentiating (4.28) with respect to t we get

$$2\sigma_t \sigma_t' S_t + \sigma_t^2 S_t' = 2\beta_t' \int_t^T S_s^2 \beta_s ds - 2\beta_t^2 S_t^2.$$

Combining this with (4.11) we obtain

(4.29)
$$2\sigma_t \sigma_t' S_t + \beta_t^2 S_t^2 = 2\beta_t' \int_t^T S_s^2 \beta_s ds.$$

We now combine (4.28) and (4.29) to get

$$2\sigma_t \sigma_t' S_t + \beta_t^2 S_t^2 = \frac{\beta_t'}{\beta_t}\sigma_t^2 S_t$$

or

$$\frac{\beta_t'}{\beta_t} = \frac{2\sigma_t'}{\sigma_t} + \frac{\beta_t^2}{\sigma_t^2}\frac{S_0}{(1 + S_0 \int_0^t \frac{\beta_s^2}{\sigma_s^2} ds)}.$$

Integrating this we obtain, with c_i integration constant, $i = 1, 2, \ldots$

$$\log \beta_t = 2\log \sigma_t + \log(1 + S_0 \int_0^t \frac{\beta_s^2}{\sigma_s^2} ds) + c_1$$

or

(4.30)
$$\beta_t = c_2\sigma_t^2(1 + S_0 \int_0^t \frac{\beta_s^2}{\sigma_s^2} ds).$$

Define

(4.31)
$$\alpha_t = \frac{\beta_t}{\sigma_t^2}.$$

19

Then equation (4.30) gives the non-linear, separable differential equation

$$\alpha'_t = c_2 S_0 \sigma_t^2 \alpha_t^2,$$

which has the general solution

$$\alpha_t = \left(c_3 - c_2 S_0 \int_0^t \sigma_s^2 ds\right)^{-1}$$

or

(4.32) $$\beta_t = \sigma_t^2 \left(c_3 - c_2 S_0 \int_0^t \sigma_s^2 ds\right)^{-1}.$$

Substituting (4.32) into the right hand side (RHS) of (4.30) we get

$$RHS = c_2 \sigma_t^2 \left(1 + S_0 \int_0^t \sigma_s^2 (c_3 - c_2 S_0 \int_0^s \sigma_u^2 du)^{-2} ds\right)$$

$$= c_2 \sigma_t^2 \left(1 + \frac{1}{c_2}\Big|_0^t \frac{1}{c_3 - c_2 S_0 \int_0^s \sigma_u^2 du}\right)$$

$$= \sigma_t^2 \left[c_2 + \left(\frac{1}{c_3 - c_2 S_0 \int_0^t \sigma_u^2 du} - \frac{1}{c_3}\right)\right]$$

$$= \frac{\sigma_t^2 \left((\int_0^t \sigma_u^2 du)(c_2 S_0 - c_2^2 c_3 S_0) + c_2 c_3^2\right)}{c_3 (c_3 - c_2 S_0 \int_0^t \sigma_u^2 du)}.$$

Therefore, (4.30) holds if and only if

$$c_2 S_0 - c_2^2 c_3 S_0 = 0,$$

i.e.,

(4.33) $$c_2 c_3 = 1.$$

Substituting this into (4.32) we get

(4.34) $$\beta_t = \frac{\sigma_t^2 c_2}{1 - c_2^2 S_0 \int_0^t \sigma_s^2 ds}.$$

From (4.28) we deduce that

(4.35) $$\lim_{t \to T^-} \beta_t = \infty.$$

20

Using this in (4.34) we deduce that

$$(4.36) \qquad c_2^2 S_0 \int_0^T \sigma_s^2 ds = 1$$

which gives

$$(4.37) \qquad \beta_t = \frac{\sigma_t^2 \left(\int_0^T \sigma_s^2 ds \right)^{\frac{1}{2}}}{S_0^{\frac{1}{2}} \int_t^T \sigma_s^2 ds}.$$

By (4.20) we know that the corresponding conditional expected value $p_t = E(\tilde{v}|\mathcal{F}_t^y)$ is given by

$$(4.38) \qquad dp_t = \frac{\beta_t S_t}{\sigma_t^2} dy_t = \lambda_t dy_t,$$

with

$$(4.39) \qquad \lambda_t = \frac{S_t \left(\int_0^T \sigma_s^2 ds \right)^{\frac{1}{2}}}{S_0^{\frac{1}{2}} \int_t^T \sigma_s^2 ds}; \qquad 0 \leq t < T.$$

Now recall from equation (4.7) that

$$S_t = E[(\tilde{v} - p_t)^2] = \frac{S_0}{1 + S_0 \int_0^t \left(\frac{\beta_s}{\sigma_s} \right)^2 ds}; \qquad S_0 = \text{var}(\tilde{v}) = \sigma_{\tilde{v}}^2.$$

By the use of (4.37) we find that

$$(4.40) \qquad S_t = \frac{S_0}{1 + \left(\int_0^T \sigma_s^2 ds \right) \int_0^t \frac{\sigma_u^2}{(\int_u^T \sigma_s^2 ds)^2} du} = \frac{S_0 \int_t^T \sigma_s^2 ds}{\int_0^T \sigma_s^2 ds}.$$

In particular,

$$(4.41) \qquad S_T = 0 \qquad \text{and hence} \quad p_T = \tilde{v} \text{ a.s.}$$

Inserting the expression (4.40) for S_t into the expression for λ_t in (4.39), we obtain

$$(4.42) \qquad \lambda_t \equiv \lambda = \frac{\sqrt{S_0}}{\sqrt{\int_0^T \sigma_s^2 ds}}; \qquad \text{a constant.}$$

This solves Problem 2 and hence, in view of (4.41), also Problem 1. That completes the proof of Theorem 2.1. $\qquad \square$

5 Conclusions

Under a set of rather natural assumptions we have formulated an insider's problem as maximizing the expected value of future wealth subject to the price of the stock satisfying the rational pricing rule (2.2) and the strategy satisfying (2.3). This latter constraint seems reasonable, since from (4.12) we see that the insiders wealth can be written ($x_0 = 0$)

$$(5.1) \qquad w_T = w_0 + \tilde{v} x_T - \int_0^T p_t d^- x_t = w_0 + \tilde{v} x_T - \int_0^T p_t dx_t,$$

where the equality follows since x has finite variation. As a consequence the final net wealth equals the value of the final position less the cost of acquiring it. The cost formula is analogous to the usual one for the cost of a discriminating monopsonist. It also follows that this final wealth can be written

$$(5.2) \qquad w_T = w_0 + \int_0^T (\tilde{v} - p_t) d^- x_t = w_0 + \int_0^T (\tilde{v} - p_t) dx_t,$$

(assumption (4.2) on p. 1326 in Kyle (1985)).

From our assumptions we derive that the rational pricing rule has the form

$$(5.3) \qquad p_t = E(\tilde{v}) + \int_0^t \lambda_s dy_s$$

(assumption (4.3) p. 1326 of Kyle (1985)). Even in the case of time-varying noise trading we obtain that the price response function $\lambda_t = \lambda$ for all t, a constant.[6]

We had to use an extended stochastic integral to achieve our goal, and given this new concept our approach was rather direct and gave a unique solution to the problem, provided our assumptions. Moreover, this line of attack seems like a natural framework to further investigate some of the problems underlying insider trading and differential information.

---

[6]The results (5.1)-(5.3) follow from our assumptions, which are the same as the ones that Kyle employ, even if he chooses to call them assumptions (Kyle (1985) (4.1)-(4-3) p. 1236).

We emphasize that our paper differs from those of Kyle (1985) and Back (1992) in several ways:

1) We have fewer and weaker assumptions about the model. Several assumptions in the above papers are proved still to hold under the more general setup of our model.

2) Our method of proof is different. Since we do not assume a priori a Markovian setup, and since we allow $\tilde{v}$ to depend on $\{z(s); 0 \le s \le T\}$, we cannot use classical dynamical programming, as Back does. Instead we use forward integration and a perturbation method.

See Remark 2.2 for more details.

Appendix I: The forward integral

Consider a general information filtration $\mathcal{G}_t \supset \mathcal{F}_t$. If B_t is a Brownian motion with respect to $\mathcal{F}_t$, it need not be a semimartingale with respect to a bigger filtration $\mathcal{G}_t \supset \mathcal{F}_t$. A simple example is

$$\mathcal{G}_t = \mathcal{F}_{t+\delta}; \qquad t \ge 0$$

where $\delta > 0$ is a constant.

First we ask the question what integrals of the form $\int_0^t x_s dB_s$ are supposed to mean when x_s is $\mathcal{G}_s$-adapted. In this paper $\mathcal{G}_t$ is the information filtration of the insider, while $\mathcal{F}_t$ is the corresponding information filtration generated by the order process y and thus possessed by the market makers. Below we consider forward integrals of processes driven by Brownian motion.

The *forward* integral $\int_0^t x_s d^- B_s$ is defined by

(5.4)
$$\int_0^T x_t d^- B_t := \lim_{\Delta t_i \to 0} \sum_i x_{t_i}(B_{t_{i+1}} - B_{t_i}),$$

whenever the limit exists in probability, and $0 = t_0 < t_1 < t_2 < \cdots < t_n = T$ is a partition of $[0, T]$. Thus this integral is defined in the intuitive manner as a limit of sums, and it should be clear that when x_t is $\mathcal{F}_t$-adapted, this integral coincides with the ordinary Itô integral over non-anticipating functions. Viewed this way, the forward integral is a direct and very natural extension of the Itô integral to anticipating (non-adapted) functions.

More formally, suppose $x : [0, T] \to R$ is a measurable stochastic process adapted to the filtration $\mathcal{G}_t$ but not necessarily to the filtration $\mathcal{F}_t$. The forward integral of x with respect to B_t was first defined by Russo and Vallois (1993), and was applied to insider trading, in a framework different from the one in the present paper, in Biagini and Øksendal (2005). For our purpose, it is sufficient to consider the case when x is left continuous with right-sided limits (càglàd). Then the original definition simplifies to (5.4).

One can show that if x_t is adapted to some filtration $\mathcal{G}_t$ such that B_t is a $\mathcal{G}_t$-semimartingale, then the forward integral of x coincides with the semimartingale integral of x (if it exists). See Biagini and Øksendal (2005). Thus the forward integral is an extension of the semimartingale integral to (possibly) non-semimartingale contexts.

An Itô formula for the forward integrals was first obtained by Russo and Vallois (1995, 2000). It may be presented as follows: Let $X_t = X_t(\omega)$ be a stochastic process of the form

$$(5.5) \qquad X_t = X_0 + \int_0^t \alpha_s ds + \int_0^t \beta_s d^- B_s; \qquad X_0 \in \mathbb{R}, \qquad \text{a constant,}$$

where α and β are measurable processes, such that

$$\int_0^t \{|\alpha_s| + \beta_s^2\} ds < \infty \qquad \text{a.s. for all } t,$$

and β is forward integrable. A short hand differential notation for (5.5) is

$$(5.6) \qquad d^- X_t = \alpha_t dt + \beta_t d^- B_t; \qquad X_0 \in \mathbb{R}.$$

Such processes X_t are called forward processes.

Theorem 5.1 (The one-dimensional Itô formula for the forward processes). *Let X_t be as above and let $f \in C^{1,2}(\mathbb{R} \times \mathbb{R})$. Define*

$$Y_t = f(t, X_t).$$

Then Y_t is again a forward process and

$$(5.7) \qquad d^- Y_t = \frac{\partial}{\partial t} f(t, X_t) \, dt + \frac{\partial}{\partial x} f(t, X_t) \, d^- X_t + \frac{1}{2} \frac{\partial^2}{\partial x^2} f(t, X_t) \, \beta_t^2 dt.$$

Note the similarity between this and the classical Itô formula. We refer to Russo and Vallois (1995, 2000) for a proof.

The Itô formula extends to several dimensions, as follows:

Theorem 5.2 (The multi-dimensional Itô formula for the forward processes). *Let*

$$(5.8) \qquad d^- X_t^{(i)} = \alpha_t^{(i)} dt + \sum_{k=1}^m \beta_t^{(i,k)} d^- B_t^{(k)}; \qquad 1 \le i \le n$$

be n forward processes, driven by m independent Brownian motions $(B_t^{(1)}, \cdots, B_t^{(m)})$. Let $f \in C^{1,2}(\mathbb{R} \times \mathbb{R}^n)$ and define

$$Y_t = f(t, X_t).$$

Then Y_t is again a forward process and

$$
\begin{aligned}
(5.9) \qquad d^- Y_t = {} & \frac{\partial}{\partial t} f(t, X_t) \, dt + \sum_{i=1}^n \frac{\partial}{\partial x_i} f(t, X_t) \, d^- X_t^{(i)} \\
& + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} f(t, X_t) \, dX_t^{(i)} dX_t^{(j)},
\end{aligned}
$$

where

$$(5.10) \qquad dX_t^{(i)} dX_t^{(j)} = \sum_{k=1}^m \beta_t^{(i,k)} \beta_t^{(j,k)} \, dt.$$

Example 5.3. Suppose $m = 1$ and $n = 2$, i.e.,

$$d^- X_t^{(i)} = \alpha_t^i \, dt + \beta_t^{(i)} \, d^- B_t; \qquad i = 1, 2.$$

Choose $f(t, x_1, x_2) = x_1 x_2$ and define

$$Y_t = f(t, X_t) = X_t^{(1)} X_t^{(2)}.$$

Then by (5.9) and (5.10) we get

$$(5.11) \qquad d^- (X_t^{(1} X_t^{(2)}) = d^- Y_t = X_t^{(1)} d^- X_t^{(2)} + X_t^{(2)} d^- X_t^{(1)} + \beta_t^{(1)} \beta_t^{(2)} \, dt.$$

This is the formula we use in (4.11), and later.

References

[1] Admati, A. R. and P. Pfleiderer (1988). "A Theory of Intraday Patterns: Volume and Price Variability". *The Review of Financial Studies* 1, 1, 3–40.

[2] Allinger, D. F. and S. K. Mitter (1981). "New Results on the Innovations Problem for Non-Linear Filtering". *Stochastics* 4, 339–348.

[3] Back, K. (1992). "Insider Trading in Continuous Time". *The Review of Financial Studies* Vol. 5, No 3, 387-409.

[4] Biagini, F., and B. Øksendal (2005). "A general stochastic calculus approach to insider trading". *Appl. Math. Optim.* 52, 167–181.

[5] Davis, M. H. A. (1977). *Linear Estimation and Stochastic Control.* Chapman and Hall.

[6] Davis, M. H. A. (1984). *Lectures on Stochastic Control and Nonlinear Filtering..* Tata Institute of Fundamental Research, Bombay.

[7] Eide, I. B. (2007). *"An equilibrium model for gradually revealed asymmetric information".* Preprint, University of Oslo 6/2007.

[8] Glosten, L. R., and P. R. Milgrom (1985). "Bid, Ask and Transaction Prices in a Specialist Market with Hetrogeneously Informed Traders". *Journal of Financial Economics*, 14, 71–100.

[9] Grossman, S. J., and J. E. Stiglitz (1980). "On the Impossibility of Informationally Efficient Markets". *American Economic Review*, 70 , 393–408.

[10] Holden, C. W. and A. Subrahmanyam (1992). "Long-Lived Private Information and Imperfect Competition". *The Journal of Finance*, XLVII, 1 247–270.

[11] Kallianpur, G. (1980). *Stochastic Filtering Theory.* Springer.

[12] Kalman, R. E. (1960). "A new approach to linear filtering and prediction problems". *J. Basic Engineering* D 82, 35–45

[13] Kyle, A. S. (1985). "Continuous Auctions and Insider Trading". *Econometrica* Vol.53, No. 6, 1315–1336.

[14] R. S. Liptser and A. N. Shiryaev: Statistics of Random Processes II. Springer 1978.

[15] Øksendal, B. (2003). *Stochastic Differential Equations.* 6th Edition. Springer.

[16] Russo, F., and P. Vallois (1993). "Forward, backward and symmetric stochastic integration". *Probab. Theory Related Fields* 97, 403–421.

[17] Russo, F., and P. Vallois (1995). "The generalized covariation process and Itô formula". *Stoch. Process. Appl.* 59, 81–104.

[18] Russo, F., and P. Vallois (2000). "Stochastic calculus with respect to continuous finite quadratic variation processes". *Stoch. Stoch. Rep.* 70, 1–40.