Dynamic General Equilibrium and T-Period Fund Separation[*]

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Abstract

We consider a dynamic general equilibrium model with incomplete markets in which we derive conditions for separating the savings decision from the asset allocation decision. It is shown that with logarithmic utility functions this separation holds for any heterogeneity of discount factors while the generalization to constant relative risk aversion only holds for homogeneous discount factors. Our results have simple asset pricing implications for the time series and also the cross section of asset prices. It is found that on data from the DJIA a risk aversion weaker than in the logarithmic case fits best.

1   Introduction

Ever since Tobin (1958) financial economists have been interested in conditions that help to simplify portfolio allocation problems. A great simplification is achieved by those conditions that allow to structure portfolio decisions in two stages: First, deciding how to split one's wealth between a risk-free and a mutual fund of risky assets, and then to allocate among the risky assets within the mutual fund. This separation property is known as two-fund separation, or more

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1

specifically, since one of the funds is assumed to be risk-free, as monetary separation. By now the conditions for two-fund separation are well-known. The seminal paper in this area is Cass and Stiglitz (1970) out of which an impressive literature developed that is too large to be reviewed here in detail. Instead, we refer to Russel (1980) and standard textbooks like Gollier (2001), Huang and Litzenberger (1988), Ingersoll (1987), and Magill and Quinzii (1996). While the finance literature on two-fund separation considers asset returns as exogenously given the general equilibrium literature derives two-fund separation with endogenously determined returns. For example Detemple and Gottardi (1998) derive two-fund separation in a two-period general equilibrium models and Judd, Kubler, and Schmedders (2004) have recently extended the two-fund separation literature to dynamic general equilibrium models.

While in this paper we use the same dynamic general equilibrium methodology as in Judd et al. (2004), we are interested in a different separation property that also simplifies intertemporal asset allocation problems. In a T-period model we study the conditions for separating between consumption and investments in (risky) assets, which we call *T-period fund separation*. One may argue that this separation is even more fundamental than the monetary separation, because before one can decide on how to allocate wealth among (risky) assets one has to decide how much to invest and how much to consume.

Separation properties can be derived from conditions on agent's preferences (Cass and Stiglitz (1970), Detemple and Gottardi (1998), and Judd et al. (2004) and others), or from conditions on the assets' return distributions. As in Cass and Stiglitz (1970), Detemple and Gottardi (1998), and Judd et al. (2004) and others, we do not restrict return distributions but seek for conditions on agents' preferences. Moreover, as it is also standard in this literature, we assume that all agents are discounted expected utility maximizers sharing the same beliefs on the assets' return. Given these assumptions conditions for fund separation do restrict the heterogeneity of the agents' type of risk aversion and possibly also the heterogeneity of their discount rates. Our first result shows that T-period fund separation holds for any heterogneous discount factors if all agents have logarithmic utility functions. In the case of non-unit constant relative risk aversion (CRRA), T-period fund separation is shown to hold if and only if agent's

discount factors are identical. These results generalize Hens, Reimann, and Vogt (2004) to more than two periods. Moreover, they show that in contrast to the two-period case, with T-periods fund separation fails for non-unit CRRA with heterogeneous discount factors. An intuition for this new finding comes from the observation that only in the case of logarithmic utility dynamic optimization reduces to two-period optimization (see e.g. Hakansson (1970)).

Besides giving conditions for intertemporal fund separation our results are also interesting because they relate to various strands of literature. Our results for the logarithmic case give a general equilibrium foundation to the literature on growth-optimal portfolios. See for example Kelly (1956), Breiman (1961), Thorp (1971), Algoet and Cover (1988), Hakansson and Ziemba (1995) and references therein. That is to say in contrast to the standard optimal growth literature, in our model asset prices and hence market values and returns are endogeneized and explained by the exogenous dividend process of the assets. Moreover, under stationarity assumptions on the dividend process we derive the well-known "fix-mix" portfolio rule, giving also a general equilibrium foundation to the literature as for example, Perold and Sharpe (1988), Mulvey and Ziemba (1998), Browne (1998), and Dempster (2002), Dempster, Germano, Medova, and Villaverde (2003).

Our result for the logarithmic case connects nicely to the asset pricing literature which is one of the most important applications of fund separation. The literature on asset pricing is also quite impressive and too large to be reviewed here in detail. The interested reader may consult Campbell (2000) and Hirshleifer (2001) for two recent surveys. From a dynamic general equilibrium point of view the art of constructing asset pricing models is to find an optimal balance between very general models without well structured preferences and a large degree of heterogeneity on the one hand and very specific models with overly simplified preferences and homogeneity of consumers on the other hand. In the first case anything can happen while in the second case asset pricing puzzles arise. Fund separation is an important tool in this respect since it allows for heterogeneity of consumers while keeping the aggregate simple. Indeed, two-fund separation builds the foundation of the capital asset pricing model and T-period fund separation is important for the time series and cross section properties of relative asset prices. Our result shows that in a dynamic general equilibrium, relative

market values of assets are determined by relative dividends of assets. Valuation formulas for economies with CRRA are well known in the finance literature. See Roll (1973), Kraus and Litzenberger (1975) and Rubinstein (1976). Note however, that in contrast to the standard finance literature our valuation formulas are expressed solely in terms of exogenous characteristics of the economy like the dividend process, the degree of risk aversion and the time preference. While our asset pricing implication of logarithmic preferences has recently also been derived by Evstigneev, Hens, and Schenk-Hoppé (1998) based on an evolutionary portfolio selection model, our result for the case of non-logarithmic utility generalizes this asset pricing implication to any degree of constant relative risk aversion. This generalization allows us to test the log versus the non-log CRRA-case on stock market data. For quarterly data from 1992 to 2004 on dividends and market values of stocks from the DJIA we find that a coefficient of relative risk aversion around 0.65 fits best, i.e., asset prices would suggest a weaker degree of risk aversion than in the logarithmic case (CRRA=1). This finding contrasts with the asset pricing literature working on aggregate data instead of individual stocks (cf. Mehra and Prescott (1985) and Kocherlakota (1996), for example) which finds a much stronger degree of risk aversion than in the logarithmic case.

Finally, we show that our heterogenous agent economy can equivalently be described by a single representative consumer whose demand function determines equilibrium asset prices for any exogenously given future dividend process. This aggregation property is weaker than full demand aggregation but far stronger than than the usual notion of a representative consumer whose portfolio decision problem generates asset prices for any given dividend process, but whose optimization problem fails to explain how asset prices change on changing the exogenous characteristics of the economy (here, the dividend process).

The rest of the paper is organized as follows. In Section 2 we set up the dynamic general equilibrium model. Section 3 provides an analysis of T-period fund separation under constant relative risk aversion and in Section 4 we present the results from an empirical test of our model.

2 The Model

We consider a standard multiperiod finance economy. There are $T + 1$ periods $t = 0, \ldots, T$, and S states of nature, where S is finite.[1] Uncertainty is modelled by an *information filtration*

$$\mathcal{F} = (F_0, F_1, \ldots, F_T),$$

where each F_t is a partition of the set of states $\{1, \ldots, S\}$ and

(i) $F_0 = \{\{1, \ldots, S\}\}$,

(ii) $F_T = \{\{1\}, \ldots, \{S\}\}$,

(iii) F_{t+1} is finer than F_t for all $t = 0 \ldots, T - 1$, i.e.

$$\xi_t \in F_t \text{ and } \xi_{t+1} \in F_{t+1} \implies \xi_{t+1} \subset \xi_t \text{ or } \xi_{t+1} \cap \xi_t = \emptyset.$$

Each element ξ_t of F_t is a **date-t event**. Let

$$\mathbb{D} = \{\xi_t \mid \xi_t \in F_t \text{ for some } t = 0, 1, \ldots T\}$$

be the set of all events and let $d = \#\mathbb{D}$. By $\mathbb{D}^+$ we denote the set of non-initial events, i.e.

$$\mathbb{D}^+ = \mathbb{D} \setminus \xi_0,$$

and by $\mathbb{D}^-$ we denote the set of non-terminal events, i.e.

$$\mathbb{D}^- = \mathbb{D} \setminus F_T.$$

The unique $\xi_t \in F_t$ with $\xi_t \supset \xi_{t+1}$ is called the *immediate predecessor* of $\xi_{t+1} \in F_{t+1}, t \leq T - 1$. The immediate predecessor of $\xi \in \mathbb{D}^+$ is denoted by ξ^-. Let $\pi(\xi_T) > 0$ be the probability of $\xi_T \in F_T$. Then, for all $t = 0, \ldots, T$, π defines a probability measure on F_t, which we also denote by π, via

$$\pi(\xi_t) = \sum_{\xi_T \subset \xi_t} \pi(\xi_T).$$

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[1]Since there is asset trade in all but the last period only, we call the model a "T-period model."

For $x \in \mathbb{R}^d$ and any $t \in \{0, \ldots, T\}$ we denote by x_t the vector in $\mathbb{R}^{\#F_t}$ that takes values $x(\xi_t), \xi_t \in F_t$.[2]

There are K assets $k = 1, \ldots, K$, which pay off a dividend per share at the beginning of every period before trade takes place in this period. $D^k(\xi) \geq 0$ is the dividend paid by asset k in event $\xi \in \mathbb{D}$. By $D(\xi) = (D^1(\xi), \ldots, D^K(\xi))$ we denote the vector of dividend payments of all assets in event ξ. We assume that aggregate dividends are strictly positive, i.e.

$$\overline{D}(\xi) := \sum_{k=1}^K D^k(\xi) > 0 \quad \text{for all } \xi \in \mathbb{D}.$$

There are I investors $i = 1, \ldots, I$. Each investor is characterized by her initial endowment of assets $\bar{\theta}^i \in \mathbb{R}^K$ and by her utility function $U^i : \mathbb{R}_+^d \to \mathbb{R}$, respectively $U^i : \mathbb{R}_{++}^d \to \mathbb{R}$. We assume that asset endowments are collinear, i.e. there exists $\bar{\theta} \in \mathbb{R}^K$ such that $\bar{\theta}^i = \delta^i \bar{\theta}$ for all i, where $\delta^i > 0$ for all i and $\sum_i \delta^i = 1$. The aggregate endowment $\bar{\theta}$ is normalized so that $\bar{\theta}_k = 1$ for all k. Moreover, we assume that U^i has expected utility form, i.e. there exist von Neumann-Morgenstern utility functions $u_t^i : \mathbb{R}_+ \to \mathbb{R}$, respectively $u_t^i : \mathbb{R}_{++} \to \mathbb{R}$, for all $t = 0, \ldots, T$, such that

$$U^i(c) = \mathbb{E}\left[\sum_{t=0}^T u_t^i(c_t)\right], \quad \text{for all } c \in \mathbb{R}_+^d \quad (\text{resp. } c \in \mathbb{R}_{++}^d),$$

where the expectation is taken with respect to the probability measure π. Investors have no endowment in periods $t > 0$. Hence, any positive consumption in periods $t > 0$ is generated by an intertemporal transfer of wealth through trade on the asset market.

Investors can trade in the K assets in each non-terminal event. For each $\xi \in \mathbb{D}^-$ let $\lambda_k^i(\xi)$ be the proportion of wealth agent i invests in asset $k \in \{1, \ldots, K\}$ in event ξ, and let $\lambda_0^i(\xi)$ denote the proportion of wealth i consumes in ξ. We assume that $\sum_{k=0}^K \lambda_k^i(\xi) = 1$ for all $\xi \in \mathbb{D}^-$. The investment strategy of agent i then is given by $\lambda^i = (\lambda_k^i(\xi))_{\substack{\xi \in \mathbb{D}^- \\ k=0,\ldots,K}}$.

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[2]Alternatively, we can interpret x_t as an F_t-measurable function $x_t : \{1, \ldots, S\} \to \mathbb{R}$, i.e. $x_t(s) = x_t(s')$ whenever $s, s' \in \xi_t$ for some $\xi_t \in F_t$.

3   T-Period Fund Separation

Let $q^k(\xi) > 0$ denote the price of asset k in event $\xi \in \mathbb{D}^-$. It is convenient to define $q^k(\xi_T) := 0$ for all terminal events $\xi_T \in F_T$ and all k. Let $w^i(\xi)$ be investor i's wealth in event $\xi \in \mathbb{D}$. Then $w_0^i := w^i(\xi_0) = (D_0 + q_0)\bar{\theta}^i$, and for all $\xi_{t+1} \in F_{t+1}$ and $t = 0, \ldots, T-1$,

$$
\begin{aligned}
w^i(\xi_{t+1}) &= w^i(\xi_t) \sum_{k=1}^{K} \frac{D^k(\xi_{t+1}) + q^k(\xi_{t+1})}{q^k(\xi_t)} \lambda_k^i(\xi_t), \\
&= \ldots \\
&= w_0^i \prod_{\tau=0}^{t} \left[\sum_{k=1}^{K} \frac{D^k(\xi_{\tau+1}) + q^k(\xi_{\tau+1})}{q^k(\xi_\tau)} \lambda_k^i(\xi_\tau) \right],
\end{aligned}
$$

where ξ_τ is the unique predecessor of ξ_{t+1} at period τ. Investor i's consumption $c^i \in \mathbb{R}_+^d$ is a function of her investment strategy λ^i and asset prices q and is given by

$$
c^i(\lambda^i, q)(\xi_t) = \lambda_0^i(\xi_t) w^i(\xi_t)
$$

for all $\xi_t \in F_t$ and for all $t = 0, \ldots, T$, where we define $\lambda_0^i(\xi_T) := 1$ for all $\xi_T \in F_T$.

For given asset prices q, investor i solves

$$
\begin{aligned}
\max \quad & U^i(c^i(\lambda^i, q)) \\
\text{s.t.} \quad & \sum_{k=0}^{K} \lambda_{tk}^i = 1 \text{ for all } t = 0, \ldots, T-1.
\end{aligned} \tag{1}
$$

Since assets are in unit supply market clearing requires that for all $k = 1, \ldots, K$, and for all $t = 0, \ldots, T-1$,

$$
q_t^k = \sum_{i=1}^{I} \lambda_{tk}^i w_t^i.
$$

A competitive equilibrium is defined as follows:

Definition 3.1 *A profile of investment strategies* $\lambda = (\lambda^i)_i$ *together with asset prices q is a **competitive equilibrium**, if*

1. *$U^i(c^i(\lambda^i, q)) \geq U^i(c^i(\tilde{\lambda}^i, q))$ for all investment strategies $\tilde{\lambda}^i$ and all $i = 1, \ldots, I$, and*

2. *(**Market clearing**) $q_t^k = \sum_{i=1}^{I} \lambda_{tk}^i w_t^i$ for all $k = 1, \ldots, K$, and for all $t = 0, \ldots, T-1$.*

From now on we assume that u_t^i exhibits constant relative risk aversion (CRRA) $\eta > 0$, i.e. $u_t^i = \beta_i^t u_\eta$ for all $t = 0, \ldots, T$, where $\beta_i, 0 < \beta_i \leq 1$, is a discount factor, and $u_\eta : \mathbb{R}_+ \to \mathbb{R}$ (respectively $u_\eta : \mathbb{R}_{++} \to \mathbb{R}$ for $\eta = 1$) is given by

$$
u_\eta(c) = \begin{cases} \dfrac{1}{1-\eta} c^{1-\eta} & , \eta \neq 1 \\ \ln(c) & , \eta = 1 \end{cases}.
$$

Given the properties of u_η the optimization problem (1) has a unique interior solution. Moreover, the first order condition is necessary and sufficient for a solution and it is given by

$$
q_t^k = \sum_{\tau=t+1}^{T} \beta_i^{\tau-t} \mathbb{E}_t \left[\left(\frac{c_t^i}{c_\tau^i} \right)^\eta \frac{c_\tau^i}{c_{t+1}^i} \lambda_{t+1,0}^i \left(D_{t+1}^k + q_{t+1}^k \right) \right], \tag{2}
$$

for all $t = 0, \ldots, T-1$ and all $k = 1, \ldots, K$, where $\mathbb{E}_t[\cdot]$ denotes the expectation conditional on the sigma-algebra induced by the partition F_t (see Appendix A for a derivation of (2)).

We are interested in the question, whether in equilibrium all investors invest in the same mutual fund, whenever they have the same constant relative risk aversion but differ with respect to their time preference and asset endowment.

Definition 3.2 *A competitive equilibrium* (λ, q) *is an* **equilibrium with T-period fund separation***, if there exists* $(\bar{\lambda}_{tk})_{\substack{k=1,\ldots,K \\ t=0,\ldots,T-1}}$ *with* $\sum_{k=1}^{K} \bar{\lambda}_{tk} = 1$ *for all* $t = 0, \ldots, T-1$*, such that for all* i*,*

$$\lambda_{tk}^i = \left(1 - \lambda_{t0}^i\right) \bar{\lambda}_{tk} \quad \text{for all } k = 1, \ldots, K.$$

Hence, in an equilibrium with T-period fund separation the proportion of non-consumed wealth invested into any asset k is the same across all investors. For unit CRRA we obtain the following result:

Theorem 3.1 *If all investors have constant relative risk aversion equal to 1, then there exists a unique equilibrium with T-period fund separation* (λ, q)*, which is given by*

$$\lambda_{t0}^i = \frac{1 - \beta_i}{1 - \beta_i^{T+1-t}},$$

$$\bar{\lambda}_{tk} = \frac{1}{\sum_j \left(\frac{\beta_j^{t+1} - \beta_j^{T+1}}{1 - \beta_j^{T+1}} \delta^j\right)} \sum_{\tau=t+1}^{T} \left(\sum_j \frac{\beta_j^{\tau} - \beta_j^{\tau+1}}{1 - \beta_j^{T+1}} \delta^j\right) \mathbb{E}_t[d_\tau^k],$$

for all $t = 0, \ldots, T-1$*, for all* $k = 1, \ldots, K$ *and for all* i*, where*

$$d_t^k := \frac{D_t^k}{\overline{D}_t}$$

denotes the relative dividend paid by asset k *in period* t*. Equilibrium prices* q *are given by*

$$q_t^k = \overline{D}_t \frac{1}{\sum_j \left(\frac{\beta_j^t - \beta_j^{t+1}}{1 - \beta_j^{T+1}} \delta^j\right)} \sum_{\tau=t+1}^{T} \left(\sum_j \frac{\beta_j^{\tau} - \beta_j^{\tau+1}}{1 - \beta_j^{T+1}} \delta^j\right) \mathbb{E}_t[d_\tau^k],$$

for all $k = 1, \ldots, K$*, and all* $t = 0, \ldots, T-1$*.*

The proof is in Appendix A. By Theorem 3.1 under logarithmic utility all agents hold the same portfolio of assets and the proportion of wealth each agent invests

9

into asset k is given by some weighted sum of the expected relative dividend paid by this asset in the future. Observe, however, that agents have different consumption rates which increase over time. Moreover, as expected, consumption rates are increasing in the agent's impatience: the smaller an agent's discount factor, the higher the proportion of wealth she consumes in each period.

The following corollaries immediately follow from Theorem 3.1:

Corollary 3.1 (Homogeneity) *If all consumers have the same discount factor, i.e. $\beta_i = \beta$ for all i, then*

$$\bar{\lambda}_{tk} = \frac{1}{\beta^{t+1} - \beta^{T+1}} \sum_{\tau=t+1}^{T} \left(\beta^\tau - \beta^{\tau+1}\right) \mathbb{E}_t[d_\tau^k],$$

$$q_t^k = \overline{D}_t \sum_{\tau=t+1}^{T} \beta^{\tau-t} \mathbb{E}_t[d_\tau^k]$$

for all $t = 0, \ldots, T-1$, and for all $k = 1, \ldots, K$.

Corollary 3.2 (Fix-Mix) *If the conditional expected relative dividends of all assets are event- and time-independent, i.e. if there exists a constant d^k such that*

$$\mathbb{E}_t[d_{t+1}^k] \equiv d^k$$

for all $k = 1, \ldots, K, t = 0, \ldots, T-1$, then

$$\bar{\lambda}_{tk} = d^k$$

for all $k = 1, \ldots, K$ and all $t = 0, \ldots, T-1$.

Hence, if the expected relative dividends of all assets are event- and time-independent, then in equilibrium all agents use the same stationary strategy for their investment in the assets. That is, in each period t the proportion of wealth invested into any asset k is the same, independent of the event at t and thus independent of the investor's wealth that is realized in t. This "fix-mix" strategy is a generalization of Kelly's (1956) "rule of betting" to multiple assets.

For constant relative risk aversion different from 1 we obtain the following result:

Theorem 3.2 *If all investors have constant relative risk aversion $\eta \neq 1$ and if they all have the same discount factor, i.e. $\beta^i = \beta$ for all i, then there exists a unique competitive equilibrium (λ, q). This is an equilibrium with T-period fund separation and it is given by*

$$\bar{\lambda}_{tk} = \frac{\sum_{\tau=t+1}^{T} \beta^{\tau-t} \mathbb{E}_t\left[\frac{D_\tau^k}{(\overline{D}_\tau)^\eta}\right]}{\sum_{\tau=t+1}^{T} \beta^{\tau-t} \mathbb{E}_t\left[(\overline{D}_\tau)^{1-\eta}\right]},$$

for all $t = 0, \ldots, T-1, k = 1, \ldots, K$, and

$$\lambda_{t0} = \frac{\left(\overline{D}_t\right)^{1-\eta}}{\left(\overline{D}_t\right)^{1-\eta} + \sum_{\tau=t+1}^{T} \beta^{\tau-t} \mathbb{E}_t\left[(\overline{D}_\tau)^{1-\eta}\right]}, \tag{3}$$

for all $t = 0, \ldots, T-1$.
Equilibrium prices q are given by

$$q_t^k = (\overline{D}_t)^\eta \sum_{\tau=t+1}^{T} \beta^{\tau-t} \mathbb{E}_t\left[\frac{D_\tau^k}{(\overline{D}_\tau)^\eta}\right].$$

for all $k = 1, \ldots, K$, and all $t = 0, \ldots, T-1$.

The proof can again be found in Appendix A. We have the following corollary:

Corollary 3.3 (Fix-Mix) *If there exists a constant d^k such that*

$$\frac{\mathbb{E}_t\left[\frac{D_{t+1}^k}{(\overline{D}_{t+1})^\eta}\right]}{\mathbb{E}_t\left[(\overline{D}_{t+1})^{1-\eta}\right]} \equiv d^k$$

for all $k = 1, \ldots, K, t = 0, \ldots, T-1$, then

$$\bar{\lambda}_{tk} = d^k$$

for all $k = 1, \ldots, K$ and all $t = 0, \ldots, T-1$.

11

Hence, as in the case of unit CRRA all agents invest according to a fix-mix strategy in equilibrium, if dividends satisfy a certain stationarity requirement. A particular case, where the condition of Corollary 3.3 is satisfied, is the one where the dividend process is i.i.d. Corollaries 3.2 and 3.3 show that a basic insight from portfolio choice theory, namely that CRRA implies a fix-mix investment strategy, carries over to the case where asset returns are determined endogenously. This result is surprising since asset returns need not be stationary in equilibrium[3] and hence it is not clear that a fix-mix strategy is optimal as it is in case of exogenous asset returns.

Under logarithmic utility we have seen that there exists an equilibrium with T-period fund separation even if agents have heterogenous time preferences. This is not true for $T \geq 2$ and CRRA different from 1, i.e. Theorem 3.2 does not carry over to the case of heterogenous discount factors as it is shown by the following example.

Example 3.1 Let $I = K = T = 2$ and let dividends be given by

$$
\begin{array}{rclcrcrcl}
D^1(\xi_0) & = & D^2(\xi_0) & = & 0.5, \\
D^1(\xi_u) & = & D^2(\xi_d) & = & 1, \\
D^1(\xi_d) & = & D^2(\xi_u) & = & 0, \\
D^1(\xi_{uu}) & = & D^1(\xi_{du}) & = & D^2(\xi_{ud}) & = & D^2(\xi_{dd}) & = & 1, \\
D^1(\xi_{ud}) & = & D^1(\xi_{dd}) & = & D^2(\xi_{du}) & = & D^2(\xi_{uu}) & = & 0,
\end{array}
$$

where $\xi_0 = \{uu, ud, du, dd\}$, $\xi_u = \{uu, ud\}$, $\xi_d = \{du, dd\}$, $\xi_{uu} = \{uu\}$, $\xi_{ud} = \{ud\}$, $\xi_{du} = \{du\}$, $\xi_{dd} = \{dd\}$ and $F_0 = \{\xi_0\}$, $F_1 = \{\xi_u, \xi_d\}$, $F_2 = \{\xi_{uu}, \xi_{ud}, \xi_{du}, \xi_{dd}\}$. Let

$$
\begin{array}{rclcrcl}
\pi(\xi_{uu}) & = & p_1 p_2, & & \pi(\xi_{ud}) & = & p_1(1 - p_2), \\
\pi(\xi_{du}) & = & (1 - p_1)p_2, & & \pi(\xi_{dd}) & = & (1 - p_1)(1 - p_2),
\end{array}
$$

where $0 < p_1 < 1$ and $0 < p_2 < 1$. If $p_1 \neq p_2$, i.e. if the dividends are not identically distributed over time, then there does not exist an equilibrium with T-period fund separation. To see this consider the case where $p_1 = 0.9$ and $p_2 = 0.1$ and let $\eta = 2$, $\delta^1 = \delta^2 = 0.5, \beta_1 = 0.1, \beta_2 = 1$. Assume by way of

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[3]This is due to the fact that aggregate dividends $\overline{D}_t$, which enter asset prices, need not be stationary under the conditions of Corollaries 3.2 and 3.3.

contradiction that there exists an equilibrium with T-period fund separation and let λ^i be agent i's investment strategy in this equilibrium. Then, for $k = 1, 2$, there exists $\bar{\lambda}_{0k}$ such that $\lambda^i_{0k} = (1 - \lambda^i_{00})\bar{\lambda}_{0k}$ for $i = 1, 2$. Substituting this into the first order condition (2) for agent $i = 1$ and solving for λ^1 (using the market clearing condition) we obtain the numeric solution $\bar{\lambda}_{01} \approx 0.51$. However, solving agent 2's first order condition gives $\bar{\lambda}_{01} \approx 0.39$ which is a contradiction. Hence, in this example there does not exist an equilibrium with T-period fund separation.

$$\Diamond$$

From the two-period case it is well known that equilibrium allocations are Pareto efficient if the agents' endowments are spanned and if agents have HARA (hyperbolic absolute risk aversion) utility functions, such that each agent's risk tolerance exhibits the same slope. This result carries over to the multiperiod model studied in this paper:

Theorem 3.3 (Effective Completeness) *The consumption allocation $(c^{*i})_i$ corresponding to the equilibrium with T-period fund separation (λ^*, q^*) in Theorem 3.1 and Theorem 3.2 is Pareto efficient.*

The proof, given in Appendix A, is a simple computation showing that all agents' utility gradients are collinear at the consumption allocation corresponding to the equilibrium with T-period fund separation. Effective completeness of the asset market implies the existence of a representative investor whose portfolio decision problem generates the equilibrium asset prices for the heterogenous agents economy.

Theorem 3.4 (Representative Agent Equilibrium) *Assume that the conditions of Theorem 3.1, resp. Theorem 3.2, are satisfied and let (λ^*, q^*) be the corresponding equilibrium with T-period fund separation. Then there exists a representative investor with expected utility function $\hat{U} : \mathbb{R}^d_{++} \to \mathbb{R}$, resp. $\hat{U} : \mathbb{R}^d_+ \to \mathbb{R}$, and endowment $\bar{e} \in \mathbb{R}^d_{++}$, where $\bar{e}_t = \overline{D}_t$ for all $t = 0, \dots, T$, such that equilibrium asset prices in the representative agent economy are given by q^*.*

If investors in the heterogenous agent economy have constant relative risk aversion η, then $\hat{U}$ can be chosen to have expected utility form with the same con-

stant relative risk aversion η. Moreover, $\hat{U}$ is independent of the future dividend process $(D_t)_{t=1,\ldots,T}$.

Out of equilibrium the demand function of the representative agent is not equal to the aggregate demand in the heterogenous agent economy, i.e. we do not have demand aggregation in a strong sense. Nevertheless, by Theorem 3.4 the demand function of the representative agent determines equilibrium asset prices for any given future dividend process. Hence, we have demand aggregation in a sense that is most relevant for asset pricing theory.

4 Empirical Results

In this section we provide an empirical test of the theoretical results derived above. In particular, we test, whether stock prices indeed can be explained by relative dividends as it is predicted by our model. Our empirical analysis differs from previous studies in the literature on empirical dynamic asset pricing which has concentrated on aggregate data instead of individual stocks. As part of our empirical analysis we also estimate the consumers' coefficient of risk aversion. Interestingly, our estimated coefficient is much closer to the risk aversion observed in experimental studies (which is below 1) than to the risk aversion that was found in tests of Lucas' (1978) asset pricing model (which is at least 10).[4]

First, we give an outline of the estimation procedure. First–order conditions of dynamic optimization problems with structural (deep) parameters θ usually are formalized by expectations of a functional f of actual outcomes of state variables and future instances of control variables x_t,

$$x_t = \mathbb{E}_t \left[f(x_{t+1}, x_{t+2}, \ldots; \theta) \right]. \tag{4}$$

To solve dynamic optimization problems numerically, Den Haan and Marcet (1994) suggest to parameterize expectations by a linear or preferably nonlinear function ψ parameterizing expectations by ω based on an information set Ω_t,

$$\mathbb{E}_t \left[f(x_{t+1}, x_{t+2}, \ldots; \theta) \right] = \mathbb{E} \left[f(x_{t+1}, x_{t+2}, \ldots), \theta | \Omega_t \right] = \psi(\Omega_t; \omega). \tag{5}$$

---

[4]See for instance Kocherlakota (1996).

Hence, determining expectations given the trajectories of the control and state variables is simply a stochastic approximation problem,

$$\min_{\omega} \Sigma(x, \omega) = \|f(\cdot; \theta) - \psi(\cdot; \omega)\|, \tag{6}$$

where $\|\cdot\|$ denotes the euclidean norm which is calculated in data samples as mean squared error. The solution to the dynamic problem (4) based on parameterized expectations (5) and (6) is the fixed–point $\omega^{i-1} = \omega^i = \bar{\omega}$ for large i of the iterative map

$$\omega^i = (1 - \lambda)\omega^{i-1} + \lambda \, \mathrm{argmin}_{\omega} \Sigma(x^{i-1}, \omega), \quad i = 1, 2 \ldots, \quad \omega^0 \in \mathbb{R},$$

and

$$x^i = \psi(x^{i-1}; \omega^i),$$

where $\lambda \in (0, 1]$ describes the rate of convergence. Den Haan and Marcet (1994) find numerically that convergence is reached in models such as the neoclassical growth model. To justify numerical convergence, we suggest to consider the p–value associated with the null hypothesis $H_0 : \omega^i(\psi(\Omega_t; \omega^{i-1})) = \omega^{i-1}$. Although the iteration only describes local convergence, Den Haan and Marcet (1994) claim for many stochastic dynamic models that transversality conditions or the assumption of time–invariant solutions ensure a unique solution in the above iterative map.

Assuming that observed real world data is the outcome of the solution to the dynamic model, i.e. the observed sample data of x_t and $f(\cdot; \theta)$ imply $\hat{\omega} = \bar{\omega}$, we estimate the structural parameters of the latter as

$$\hat{\theta} = \mathrm{argmin}_{\theta} \|x - \psi(\cdot; \hat{\omega})\| \quad s.t. \quad \hat{\omega} = \mathrm{argmin}_{\omega} \Sigma(\omega).$$

To put it in another way, we start the numerical solution problem with observed time series, and are searching for the structural parameters of the dynamic model that do not change the time series for the parameters given above.[5]

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[5]In Woehrmann (2005) it is shown by simulations of the neoclassical stochastic growth model of Kydland and Prescott (1982) that this inference approach to dynamic models is unbiased and efficient.

In our model deep parameters $\theta = (\beta, \eta)$ have to be estimated. Furthermore we suppose dividends to follow a random walk implying actual dividends to be best predictors of future dividends.[6] Applying the inference scheme above to the first order conditions of our dynamic model provided in Theorem 3.2, we solve

$$\hat{\theta} = \mathrm{argmin}_\theta \|q - \hat{q}\|$$

s.t.

$$\hat{q}_t = \bar{\lambda}_{tk}(\overline{D}_t)^\eta \sum_{\tau=t+1}^T \beta^{\tau-t}(\overline{D}_t)^{1-\eta}$$

$$\bar{\lambda}_{tk} = \frac{1}{\sum_{\tau=t+1}^T \beta^{\tau-t}(\overline{D}_t)^{1-\eta}} \sum_{\tau=t+1}^T \beta^{\tau-t}\psi(\omega)$$

$$\psi(\omega) = \text{polynomial conditional on } d_{k,t} \text{ and } \bar{\lambda}_{k,t}$$

$$\hat{\omega} = \mathrm{argmin}_\omega \|(\overline{D}_\tau)^{1-\eta}\left(\lambda_{t+1,0}d_{t+1}^k + (1 - \lambda_{t+1,0})\bar{\lambda}_{t+1,k}\right) - \psi(\omega)\|$$

$$\lambda_{t0} = \frac{\overline{D}_t^{1-\eta}}{\overline{D}_t^{1-\eta} + \sum_{\tau=t+1}^T \beta^{\tau-t}(\overline{D}_t)^{1-\eta}},$$

where polynomials are estimated by ordinary least squares as in Den Haan and Marcet (1994). Note, that q_t and $\hat{q}_t$ denote observed and estimated prices, respectively. Estimations are conducted with quarterly data on the stocks of the companies listed in Table 1. Among the 100 largest stocks with respect to market capitalization in 2004 we have chosen those from the FAME data base, which provide histories of at least 50 consecutive quarters of dividend payments. Different from the large body of studies on dynamic asset pricing models based on aggregate data, our model explains the stock market by relative dividends. Hence, we report basic summary statistics of the latter in Table 1. Bottom line, relative dividends of many stocks are normally distributed, but they are mostly not stationary. Note that this is not assumed in our estimation scheme.

Results for the estimation of deep parameters θ are reported in Table 2. Note, that convergence with regard to parameters ω^i of the polynomial function as described above is reached. We find that a coefficient of relative risk aversion

---

[6]This is verified — as in numerous papers — based on the ADF–test for unit roots.

Table 1: *Summary statistics for relative dividends.*

JB and ADF denote the Jarque–Bera test for normality and the augmented
Dickey–Fuller test for unit roots, respectively.

Company	Mean	Std. Dev.	Skewness	Kurtosis	JB p–value	ADF p–value
3M	0.1320	0.0527	1.0383	3.5838	0.0079	0.5222
Altria	0.0832	0.0357	1.3748	5.2152	0.0000	0.4904
American Express	0.0275	0.0099	0.4960	2.2252	0.1920	0.2609
Bank New York	0.0225	0.0105	0.7649	2.7828	0.0831	0.0013
General Electric	0.0162	0.0082	1.4912	6.2053	0.0000	0.3815
General Motors	0.2442	0.1416	0.5259	2.4241	0.2236	0.0924
Hewlett Packard	0.0216	0.0104	0.1991	1.7377	0.1612	0.8243
Intel	0.0092	0.0079	2.0340	8.0703	0.0000	0.0925
IBM	0.1148	0.0637	-0.0605	2.0475	0.3828	0.7228
J. P. Morgan Chase	0.0551	0.0162	0.0284	2.2113	0.5213	0.5807
Johnson & Johnson	0.0335	0.0169	1.0705	3.0277	0.0084	0.2015
McDonalds	0.0190	0.0058	0.0588	2.1375	0.4542	0.3722
Merrill Lynch	0.0429	0.0261	0.6716	2.6040	0.1297	0.3116
Microsoft	0.0071	0.0057	0.2325	1.5908	0.1009	0.5072
Pfizer	0.0137	0.0096	0.6306	2.1112	0.0838	0.1695
United Technologies	0.0756	0.0415	0.6990	2.1477	0.0613	0.1880
Wachovia	0.0819	0.0352	0.3440	2.5655	0.5018	0.4982

around 0.63 fits best, i.e., asset prices would suggest a weaker degree of risk aversion than in the logarithmic case (CRRA=1). This is robust with respect to the degree of the polynomial choosen in the estimation procedure. The null hypothesis $H_0 : \eta = 1$ can be rejected with low p-values of the Wald test. Given that $\eta \sim \mathcal{N}(0, \sigma)$, $\sigma > 0$, results of asymptotic theory give us $\frac{(\eta-1)^2}{\hat{\sigma}^2} \sim F(1, T-1)$, which leads directly to a t–test for η frequently termed Wald–test. $\hat{\sigma}$ is obtained by omitting once each data point. This finding contrasts with the asset pricing literature working on aggregate data instead of individual stocks (cf. Mehra and Prescott (1985) and Kocherlakota (1996), for example) which finds a much stronger degree of risk aversion than in the logarithmic case. However, here we focus on relative stock prices explained by relative dividends rather than considering the equity premium puzzle here.

Table 2: *Estimation results of the structural parameters.*

The degree of polynomial refers to $\psi(\cdot)$. RMSE stands for the root mean squares error of q and $\hat{q}$. The null hypothesis of the Wald test is $H_0 : \eta = 1$.

Polynomial	$\hat{\beta}$	$\hat{\eta}$	RMSE	Wald p–value
2nd order	0.95	0.633	4.6887	0.000
3rd order	0.95	0.631	4.3445	0.000

Estimated time series of relative stock prices, $q_t^k/\bar{q}_t$ for the 17 companies in Table 1 are illustrated in the figures in Appendix B. The average R squared of estimated relative stock prices is 70.65%, its standard deviation is 14.43%, and they range between 42.77% and 96.88% indicating a good fit with the data.

Appendix A: Proofs

General Considerations: In the following we derive the first order condition (2) for the optimization problem (1). The first order condition for an interior solution λ^i to (1) is given by

$$\frac{\partial U^i(c^i(\lambda^i, q))}{\partial \lambda_k^i(\xi_t)} = \alpha^i(\xi_t),$$

for all $k = 0, \ldots, K$, all $\xi_t \in F_t$ and all $t = 0, \ldots, T-1$, where $\alpha^i(\xi_t)$ is a Lagrange multiplier. Hence, for all $k = 1, \ldots, K$, and all $\xi_t \in F_t$,

$$\frac{\partial U^i(c^i(\lambda^i, q))}{\partial \lambda_0^i(\xi_t)} = \frac{\partial U^i(c^i(\lambda^i, q))}{\partial \lambda_k^i(\xi_t)}.$$

$$\frac{\partial U^i(c^i(\lambda^i, q))}{\partial \lambda_0^i(\xi_t)} = w^i(\xi_t)\partial_{\xi_t}U^i = \beta_i^t\pi(\xi_t)u_\eta'(c^i(\xi_t))w^i(\xi_t),^{[7]}$$

where

$$\partial_{\xi_t}U^i := \frac{\partial U^i(c^i)}{\partial c^i(\xi_t)}.$$

Moreover,

$$
\begin{aligned}
\frac{\partial U^i}{\partial \lambda_k^i(\xi_t)} &= \sum_{\tau=t+1}^{T}\sum_{\xi_\tau \subset \xi_t} \partial_{\xi_\tau}U^i \lambda_0^i(\xi_\tau)\frac{\partial w^i(\xi_\tau)}{\partial \lambda_k^i(\xi_t)} \\
&= \sum_{\tau=t+1}^{T}\sum_{\xi_\tau \subset \xi_t} \beta_i^\tau\pi(\xi_\tau)u_\eta'(c^i(\xi_\tau))\lambda_0^i(\xi_\tau)\frac{\partial w^i(\xi_\tau)}{\partial \lambda_k^i(\xi_t)}
\end{aligned}
$$

Let $\xi_\tau \subset \xi_t$ and let $\xi_{t+1}, \ldots, \xi_{\tau-1}$ be the unique predecessors of ξ_τ in periods $t+1, \ldots, \tau - 1$. Then

$$
\begin{aligned}
\frac{\partial w^i(\xi_\tau)}{\partial \lambda_k^i(\xi_t)} &= w_0^i\frac{D^k(\xi_{t+1}) + q^k(\xi_{t+1})}{q^k(\xi_t)}\prod_{\substack{s=0 \\ s\neq t}}^{\tau-1}\left[\sum_{k=1}^{K}\frac{D^k(\xi_{s+1}) + q^k(\xi_{s+1})}{q^k(\xi_s)}\lambda_k^i(\xi_s)\right] \\
&= w^i(\xi_t)\frac{D^k(\xi_{t+1}) + q^k(\xi_{t+1})}{q^k(\xi_t)}\prod_{s=t+1}^{\tau-1}\left[\sum_{k=1}^{K}\frac{D^k(\xi_{s+1}) + q^k(\xi_{s+1})}{q^k(\xi_s)}\lambda_k^i(\xi_s)\right] \\
&= \frac{w^i(\xi_t)}{w^i(\xi_{t+1})}w^i(\xi_\tau)\frac{D^k(\xi_{t+1}) + q^k(\xi_{t+1})}{q^k(\xi_t)}.
\end{aligned}
$$

---

[7] Here and in the following we shortly write $c^i(\xi_t)$ instead of $c^i(\lambda^i, q)(\xi_t)$.

Hence, the first order condition becomes

$$
\begin{aligned}
q_t^k &= \sum_{\tau=t+1}^{T} \beta_i^{\tau-t} \mathbb{E}_t \left[\frac{u_\eta'(c_\tau^i)}{u_\eta'(c_t^i)} \frac{w_\tau^i}{w_{t+1}^i} \lambda_{\tau 0}^i \left(D_{t+1}^k + q_{t+1}^k \right) \right] \\
&= \sum_{\tau=t+1}^{T} \beta_i^{\tau-t} \mathbb{E}_t \left[\frac{u_\eta'(c_\tau^i)}{u_\eta'(c_t^i)} \frac{c_\tau^i}{c_{t+1}^i} \lambda_{t+1,0}^i \left(D_{t+1}^k + q_{t+1}^k \right) \right] \\
&= \sum_{\tau=t+1}^{T} \beta_i^{\tau-t} \mathbb{E}_t \left[\left(\frac{c_t^i}{c_\tau^i} \right)^\eta \frac{c_\tau^i}{c_{t+1}^i} \lambda_{t+1,0}^i \left(D_{t+1}^k + q_{t+1}^k \right) \right],
\end{aligned}
$$

for all $t = 0, \ldots, T-1$ and all $k = 1, \ldots, K$, where $\mathbb{E}_t[\cdot]$ denotes the expectation conditional on the sigma-algebra induced by the partition F_t. This proves (2). $\qquad\square$

Proof of Theorem 3.1: The first order condition (2) for $\eta = 1$ reads

$$
q_t^k = \sum_{\tau=t+1}^{T} \beta_i^{\tau-t} \mathbb{E}_t \left[\frac{c_t^i}{c_{t+1}^i} \lambda_{t+1,0}^i \left(D_{t+1}^k + q_{t+1}^k \right) \right]. \tag{A.7}
$$

If there exists an equilibrium with T-period fund separation, then, for all l and all t there exists $\bar{\lambda}_{tl}$ such that $\lambda_{tl}^i = (1 - \lambda_{t0}^i)\bar{\lambda}_{tl}$ for all i, which implies

$$
q_t^l = \sum_j \lambda_{tl}^j w_t^j = \bar{\lambda}_{tl} \sum_j (1 - \lambda_{t0}^j) w_t^j
$$

and hence

$$
\begin{aligned}
\frac{c_t^i}{c_{t+1}^i} \lambda_{t+1,0}^i &= \frac{\lambda_{t0}^i w_t^i}{w_{t+1}^i} \\
&= \frac{\lambda_{t0}^i}{\sum_l \frac{D_{t+1}^l + q_{t+1}^l}{q_t^l} \lambda_{tl}^i} \\
&= \frac{\lambda_{t0}^i}{1 - \lambda_{t0}^i} \sum_j (1 - \lambda_{t0}^j) w_t^j \frac{1}{\overline{D}_{t+1} + \bar{q}_{t+1}},
\end{aligned}
$$

where

$$
\bar{q}_t := \sum_{l=1}^{K} q_t^l \quad \text{for all } t = 0, \ldots, T-1.
$$

Substituting this into (A.7) gives

$$\bar{\lambda}_{tk} = \sum_{\tau=t+1}^{T} \beta_i^{\tau-t} \frac{\lambda_{t0}^i}{1 - \lambda_{t0}^i} \mathbb{E}_t \left[\frac{D_{t+1}^k + q_{t+1}^k}{\overline{D}_{t+1} + \bar{q}_{t+1}} \right].$$

Since $\sum_{k=1}^{K} \bar{\lambda}_{tk} = 1$ it follows that

$$\lambda_{t0}^i = \frac{1}{1 + \sum_{\tau=t+1}^{T} \beta_i^{\tau-t}} = \frac{1 - \beta_i}{1 - \beta_i^{T+1-t}} \tag{A.8}$$

$$\text{and } \bar{\lambda}_{tk} = \mathbb{E}_t \left[\frac{D_{t+1}^k + q_{t+1}^k}{\overline{D}_{t+1} + \bar{q}_{t+1}} \right],$$

for all i, for all $k = 1, \ldots, K$, and all $t = 0, \ldots, T-1$. From

$$\bar{q}_{t+1} = \sum_j (1 - \lambda_{t+1,0}^j) w_{t+1}^j,$$

it follows that

$$q_{t+1}^k = \bar{\lambda}_{t+1,k} \bar{q}_{t+1}$$

and hence

$$\bar{\lambda}_{tk} = \mathbb{E}_t \left[\frac{D_{t+1}^k + \bar{\lambda}_{t+1,k} \bar{q}_{t+1}}{\overline{D}_{t+1} + \bar{q}_{t+1}} \right].$$

Therefore, it remains to solve for $\bar{q}_t$ for all $t = 0, \ldots, T-1$. For $t = 0$ we have

$$\bar{q}_0 = \sum_j (1 - \lambda_{00}^j) w_0^j = \sum_j (1 - \lambda_{00}^j)(D_0 + q_0) \delta^j \bar{\theta} = (\overline{D}_0 + \bar{q}_0) \sum_j (1 - \lambda_{00}^j) \delta^j.$$

This implies

$$\bar{q}_0 = \overline{D}_0 \frac{\sum_j (1 - \lambda_{00}^j) \delta^j}{\sum_j \lambda_{00}^j \delta^j}. \tag{A.9}$$

Since

$$w_{t+1}^j = (1 - \lambda_{t0}^j) w_t^j \sum_k \bar{\lambda}_{tk} \frac{D_{t+1}^k + q_{t+1}^k}{q_t^k}$$

$$= (1 - \lambda_{t0}^j) w_t^j \frac{\overline{D}_{t+1} + \bar{q}_{t+1}}{\bar{q}_t}$$

$$= \ldots$$

$$= w_0^j \prod_{\tau=0}^{t} \left((1 - \lambda_{\tau 0}^j) \frac{\overline{D}_{\tau+1} + \bar{q}_{\tau+1}}{\bar{q}_\tau} \right),$$

it follows that

$$\bar{q}_{t+1} = \sum_j (1 - \lambda^j_{t+1,0}) w^j_{t+1}$$

$$= \prod_{\tau=0}^{t} \left(\frac{\overline{D}_{\tau+1} + \bar{q}_{\tau+1}}{\bar{q}_\tau} \right) \sum_j \left(w^j_0 \prod_{\tau=0}^{t+1} (1 - \lambda^j_{\tau,0}) \right)$$

$$= (\overline{D}_0 + \bar{q}_0) \prod_{\tau=0}^{t} \left(\frac{\overline{D}_{\tau+1} + \bar{q}_{\tau+1}}{\bar{q}_\tau} \right) \sum_j \left(\delta^j \prod_{\tau=0}^{t+1} (1 - \lambda^j_{\tau,0}) \right).$$

From

$$1 - \lambda^i_{t0} = \beta_i \frac{1 - \beta_i^{T-t}}{1 - \beta_i^{T+1-t}}$$

we compute

$$\prod_{\tau=0}^{t+1} (1 - \lambda^i_{\tau,0}) = \frac{\beta_i^{t+2} - \beta_i^{T+1}}{1 - \beta_i^{T+1}},$$

and hence

$$\bar{q}_{t+1} = (\overline{D}_0 + \bar{q}_0) \prod_{\tau=0}^{t} \left(\frac{\overline{D}_{\tau+1} + \bar{q}_{\tau+1}}{\bar{q}_\tau} \right) \sum_j \left(\frac{\beta_j^{t+2} - \beta_j^{T+1}}{1 - \beta_j^{T+1}} \delta^j \right), \qquad (A.10)$$

for $t = 0, \ldots, T-1$. From (A.10) we can solve for $\bar{q}_t$ for all t and it is straight-forward to verify that

$$\bar{q}_t = \overline{D}_t \frac{\sum_j \left(\frac{\beta_j^{t+1} - \beta_j^{T+1}}{1 - \beta_j^{T+1}} \delta^j \right)}{\sum_j \left(\frac{\beta_j^{t} - \beta_j^{t+1}}{1 - \beta_j^{T+1}} \delta^j \right)}, t = 0, \ldots, T-1,$$

solves (A.10) for all $t = 0, \ldots, T-1$.

Given $\bar{q}_t$ it follows that

$$\bar{\lambda}_{tk} = \mathbb{E}_t \left[\frac{D^k_{t+1} + \bar{\lambda}_{t+1,k} \bar{q}_{t+1}}{\overline{D}_{t+1} + \bar{q}_{t+1}} \right]$$

$$= \frac{\sum_j \left(\frac{\beta_j^{t+1} - \beta_j^{t+2}}{1 - \beta_j^{T+1}} \delta^j \right)}{\sum_j \left(\frac{\beta_j^{t+1} - \beta_j^{T+1}}{1 - \beta_j^{T+1}} \delta^j \right)} \mathbb{E}_t[d^k_{t+1}] + \frac{\sum_j \left(\frac{\beta_j^{t+2} - \beta_j^{T+1}}{1 - \beta_j^{T+1}} \delta^j \right)}{\sum_j \left(\frac{\beta_j^{t+1} - \beta_j^{T+1}}{1 - \beta_j^{T+1}} \delta^j \right)} \mathbb{E}_t[\bar{\lambda}_{t+1,k}],$$

for all $k = 1, \ldots, K$, and all $t = 0, \ldots, T - 1$. Solving for $\bar{\lambda}_{tk}$ recursively, we obtain

$$\bar{\lambda}_{tk} = \frac{1}{\sum_j \left(\frac{\beta_j^{t+1} - \beta_j^{T+1}}{1 - \beta_j^{T+1}} \delta^j \right)} \sum_{\tau=t+1}^{T} \left(\sum_j \frac{\beta_j^\tau - \beta_j^{\tau+1}}{1 - \beta_j^{T+1}} \delta^j \right) \mathbb{E}_t[d_\tau^k],$$

and $q_t^k = \bar{\lambda}_{tk} \bar{q}_t = \overline{D}_t \frac{1}{\sum_j \left(\frac{\beta_j^t - \beta_j^{t+1}}{1 - \beta_j^{T+1}} \delta^j \right)} \sum_{\tau=t+1}^{T} \left(\sum_j \frac{\beta_j^\tau - \beta_j^{\tau+1}}{1 - \beta_j^{T+1}} \delta^j \right) \mathbb{E}_t[d_\tau^k]$

for all $k = 1, \ldots, K$, and all $t = 0, \ldots, T - 1$. This proves the theorem.

$\square$

Proof of Theorem 3.2: Consider the first order condition (2) for the case where $\beta_i = \beta$ for all i and $\eta \neq 1$. Let $t = T - 1$. Then, since $\lambda_{T0}^i = 1$, the first order condition is identical for all investors i. Hence, $\lambda_{T-1,k}^i = \lambda_{T-1,k}$ for all $k = 0, \ldots, K$ and for all i. By induction it follows that $\lambda_{tk}^i = \lambda_{tk}$ for all $k = 0, \ldots, K$, for all i and all $0 \leq t \leq T - 1$. For all $t = 0, \ldots, T - 1$, and all $k = 1, \ldots, K$, define

$$\bar{\lambda}_{tk} = \lambda_{tk}/(1 - \lambda_{t0}).$$

From the market clearing condition we get

$$q_t^l = (1 - \lambda_{t0}) \bar{\lambda}_{tl} \sum_j w_t^j = (1 - \lambda_{t0}) \bar{\lambda}_{tl} \left(\overline{D}_t + \bar{q}_t \right), t = 0, \ldots, T - 1.$$

Substituting this into the first order condition (2) we get

$$\bar{\lambda}_{tk} = \sum_{\tau=t+1}^{T} \beta^{\tau-t} \frac{(\lambda_{t0})^\eta}{1 - \lambda_{t0}} \mathbb{E}_t \left[\left(\frac{c_\tau^i}{w_t^i} \right)^{1-\eta} \frac{D_{t+1}^k + q_{t+1}^k}{\overline{D}_{t+1} + \bar{q}_{t+1}} \right].$$

For $t = 0, \ldots, T - 1$, and $\tau = t + 1, \ldots, T$,

$$w_\tau^i = w_t^i \prod_{s=t}^{\tau-1} \left[\sum_{k=1}^{K} \frac{D_{s+1}^k + q_{s+1}^k}{q_s^k} \lambda_{sk}^i \right]$$

$$= w_t^i \prod_{s=t}^{\tau-1} \frac{\overline{D}_{s+1} + \bar{q}_{s+1}}{\overline{D}_s + \bar{q}_s}$$

$$= w_t^i \frac{\overline{D}_\tau + \bar{q}_\tau}{\overline{D}_t + \bar{q}_t}.$$

23

This implies

$$\bar{\lambda}_{tk} = \frac{(\lambda_{t0})^{\eta}}{1 - \lambda_{t0}} \sum_{\tau=t+1}^{T} \beta^{\tau-t} \mathbb{E}_t \left[(\lambda_{\tau,0})^{1-\eta} \left(\frac{\overline{D}_{\tau} + \bar{q}_{\tau}}{\overline{D}_t + \bar{q}_t} \right)^{1-\eta} \frac{D_{t+1}^k + q_{t+1}^k}{\overline{D}_{t+1} + \bar{q}_{t+1}} \right] \tag{A.11}$$

for $t = 0, \ldots, T - 1$.

We now solve for the equilibrium price q. We have already seen that

$$q_t^k = \bar{\lambda}_{tk}(1 - \lambda_{t0}) \left(\overline{D}_t + \bar{q}_t \right),$$

for $t = 0, \ldots, T - 1$. Summing over all k we get

$$\bar{q}_t = (1 - \lambda_{t0}) \left(\overline{D}_t + \bar{q}_t \right)$$

$$\Longleftrightarrow \bar{q}_t = \frac{1 - \lambda_{t0}}{\lambda_{t0}} \overline{D}_t, \ t = 0, \ldots, T - 1.$$

Substituting this into (A.11) gives

$$\bar{\lambda}_{tk} = \frac{\lambda_{t0}}{1 - \lambda_{t0}} \sum_{\tau=t+1}^{T} \beta^{\tau-t} \mathbb{E}_t \left[\left(\frac{\overline{D}_{\tau}}{\overline{D}_t} \right)^{1-\eta} \frac{D_{t+1}^k + q_{t+1}^k}{\overline{D}_{t+1} + \bar{q}_{t+1}} \right], \ \text{for } t = 0, \ldots, T - 1.$$

Summing over all k and solving for λ_{t0} we obtain that

$$\lambda_{t0} = \frac{(\overline{D}_t)^{1-\eta}}{(\overline{D}_t)^{1-\eta} + \sum_{\tau=t+1}^{T} \beta^{\tau-t} \mathbb{E}_t \left[(\overline{D}_{\tau})^{1-\eta} \right]}, \ t = 0, \ldots, T - 1.$$

Hence,

$$\bar{\lambda}_{tk} = \frac{1}{\sum_{\tau=t+1}^{T} \beta^{\tau-t} \mathbb{E}_t \left[(\overline{D}_{\tau})^{1-\eta} \right]} \sum_{\tau=t+1}^{T} \beta^{\tau-t} \mathbb{E}_t \left[(\overline{D}_{\tau})^{1-\eta} \frac{D_{t+1}^k + q_{t+1}^k}{\overline{D}_{t+1} + \bar{q}_{t+1}} \right]$$

$$= \frac{1}{\sum_{\tau=t+1}^{T} \beta^{\tau-t} \mathbb{E}_t \left[(\overline{D}_{\tau})^{1-\eta} \right]} \sum_{\tau=t+1}^{T} \beta^{\tau-t} \mathbb{E}_t \left[(\overline{D}_{\tau})^{1-\eta} \left(\lambda_{t+1,0} d_{t+1}^k + (1 - \lambda_{t+1,0}) \bar{\lambda}_{t+1,k} \right) \right].$$

Solving backwards for $\bar{\lambda}_{tk}$ gives

$$\bar{\lambda}_{tk} = \frac{\sum_{\tau=t+1}^{T} \beta^{\tau-t} \mathbb{E}_t \left[\frac{D_{\tau}^k}{(\overline{D}_{\tau})^{\eta}} \right]}{\sum_{\tau=t+1}^{T} \beta^{\tau-t} \mathbb{E}_t \left[(\overline{D}_{\tau})^{1-\eta} \right]},$$

for all $t = 0, \ldots, T-1, k = 1, \ldots, K$. This implies that

$$
\begin{aligned}
q_t^k &= \bar{\lambda}_{tk} \bar{q}_t \\
&= \bar{\lambda}_{tk} (\overline{D}_t)^\eta \sum_{\tau=t+1}^T \beta^{\tau-t} \mathbb{E}_t \left[(\overline{D}_\tau)^{1-\eta} \right] \\
&= (\overline{D}_t)^\eta \sum_{\tau=t+1}^T \beta^{\tau-t} \mathbb{E}_t \left[\frac{D_\tau^k}{(\overline{D}_\tau)^\eta} \right],
\end{aligned}
$$

for $t = 0, \ldots, T-1$, which proves the theorem.

$\square$

Proof of Theorem 3.3: Let c^{*i} be investor i's consumption in the equilibrium with T-period fund separation, (λ^*, q^*), as characterized in Theorem 3.1, resp. Theorem 3.2. Then, for all $t = 0, \ldots, T-1$ and all i,

$$
w_{t+1}^i = \frac{\delta^i \prod_{\tau=0}^t (1 - \lambda_{\tau 0}^{*i})}{\sum_j \left(\delta^j \prod_{\tau=0}^t (1 - \lambda_{\tau 0}^{*i}) \right)} \left(\overline{D}_{t+1} + \bar{q}_{t+1} \right) = \delta^i \prod_{\tau=0}^t (1 - \lambda_{\tau 0}^{*i}) Z_{t+1},
$$

where Z_{t+1} is independent of i. This implies

$$
\frac{c_0^{*i}}{c_{t+1}^{*i}} = \frac{\lambda_{00}^{*i} w_0^i}{\lambda_{t+1,0}^{*i} w_{t+1}^i} = \frac{\lambda_{00}^{*i} (\overline{D}_0 + \bar{q}_0)}{\lambda_{t+1,0}^{*i} \prod_{\tau=0}^t (1 - \lambda_{\tau 0}^{*i}) Z_{t+1}}.
$$

By Theorem 3.1, if $\eta = 1$, then $\lambda_{t0}^{*i} = (1 - \beta_i)/(1 - \beta_i^{T+1-t})$ for all t. Hence,

$$
\frac{\lambda_{00}^{*i}}{\lambda_{t+1,0}^{*i} \prod_{\tau=0}^t (1 - \lambda_{\tau 0}^{*i})} = \frac{1}{\beta_i^{t+1}},
$$

which implies

$$
\beta_i^{t+1} \frac{c_0^{*i}}{c_{t+1}^{*i}} = \frac{\overline{D}_0 + \bar{q}_0}{Z_{t+1}},
$$

and

$$
\frac{\partial_{\xi_{t+1}} U^i(c^{*i})}{\partial_{\xi_0} U^i(c^{*i})} = \beta_i^{t+1} \pi(\xi_{t+1}) \frac{c_0^{*i}}{c^{*i}(\xi_{t+1})} = \frac{\pi(\xi_{t+1})(\overline{D}_0 + \bar{q}_0)}{Z(\xi_{t+1})},
$$

which is independent of i.

Let $\eta \neq 1$ and $\beta_i = \beta$ for all i. Then, by Theorem 3.2, $\lambda_{t0}^{*i} = \lambda_{t0}^*$ is independent of i for all t. This implies $w_{t+1}^i = \delta^i (\overline{D}_{t+1} + \bar{q}_{t+1})$. Hence,

$$
\frac{c_0^{*i}}{c_{t+1}^{*i}} = \frac{\lambda_{00}^* (\overline{D}_0 + \bar{q}_0)}{\lambda_{t+1,0}^* (\overline{D}_{t+1} + \bar{q}_{t+1})}
$$

is independent of i. Therefore,

$$\frac{\partial_{\xi_{t+1}} U^i(c^{*i})}{\partial_{\xi_0} U^i(c^{*i})} = \beta^{t+1} \pi(\xi_{t+1}) \left(\frac{c_0^{*i}}{c^{*i}(\xi_{t+1})}\right)^\eta = \beta^{t+1} \pi(\xi_{t+1}) \left(\frac{\lambda_{00}^*(\overline{D}_0 + \bar{q}_0)}{\lambda_{t+1,0}^*(\overline{D}_{t+1} + \bar{q}_{t+1})}\right)^\eta,$$

which is independent of i. Thus, in both cases the agent's utility gradients are collinear in equilibrium, $\nabla U^i(c^{*i}) \parallel \nabla U^j(c^{*j})$ for all $i \neq j$, which implies the Pareto efficiency of the equilibrium allocation $(c^{*i})_i$.

$\square$

Proof of Theorem 3.4: Let (λ^*, q^*) be an equilibrium with T-period fund separation for the economy. Then, by Theorem 3.3 the corresponding consumption allocation (c^{*i}) is Pareto efficient. Hence, the agents' utility gradients $\nabla U^i(c^{*i})$ are collinear, $\nabla U^i(c^{*i}) \parallel \nabla U^j(c^{*j})$ for all $i \neq j$. For all $i = 1, \ldots, I$, define

$$\gamma_i := \frac{1}{\partial_{\xi_0} U^i(c^{*i})} = \left(c_0^{*i}\right)^\eta.$$

If $\eta = 1$ define $\hat{U} : \mathbb{R}_{++}^d \to \mathbb{R}$ by

$$\hat{U}(c) := \sup\left\{\sum_i \gamma_i U^i(c^i) \,\middle|\, \sum_i c^i = c, c^i \in \mathbb{R}_{++}^d \text{ for all } i\right\}, \quad c \in \mathbb{R}_{++}^d.$$

If $\eta \neq 1$ define $\hat{U} : \mathbb{R}_+^d \to \mathbb{R}$ accordingly. Then $\hat{U}(\bar{c}) = \sum_i \gamma_i U^i(\bar{c}^i)$ if and only if

$$\gamma_i \nabla U^i(\bar{c}^i) = \gamma_j \nabla U^j(\bar{c}^j) \text{ for all } i \neq j.$$

Moreover,

$$\nabla \hat{U}(\bar{c}) = \gamma_i \nabla U^i(\bar{c}^i) \text{ for all } i. \tag{A.12}$$

Let $\bar{e} \in \mathbb{R}_{++}^d$ be given by $\bar{e}_t = \overline{D}_t$ for all t. Then, $\bar{e} = \sum_i c^{*i}$ and by definition of $\hat{U}$ it follows that

$$\hat{U}(\bar{e}) = \sum_i \gamma_i U^i(c^{*i}).$$

Hence, by (A.12) q^* is an equilibrium price vector in the representative agent economy, where the agent has utility function $\hat{U}$ and endowment $\bar{e}$.

Since all U^i are in expected utility form, $\hat{U}$ has expected utility form as well. Consider first the case where all investors in the heterogenous agent economy

26

have unit constant relative risk aversion. Then it is straightforward to show that $\hat{U}$ is given by

$$\hat{U}(c) = \mathbb{E}\left[\sum_{t=0}^{T}(A_t + B_t\ln(c_t))\right], \quad \text{for all } c \in \mathbb{R}_{++}^d$$

where

$$A_t = \sum_i \gamma_i\beta_i^t\ln(\alpha_t^i) \quad \text{and} \quad B_t = \sum_i \gamma_i\beta_i^t,$$

and $\alpha_t^i = \gamma_i\beta_i^t/\sum_j\gamma_j\beta_j^t$ for all i and $t = 0,\ldots,T$. Hence, a monotone transformation of $\hat{U}$ has expected logarithmic utility form and therefore, the representative agent has unit CRRA.

Similarly, if all investors in the heterogenous agent economy have CRRA $\eta \neq 1$ and the same discount factor β, then $\hat{U}$ is given by

$$\hat{U}(c) = \mathbb{E}\left[\sum_{t=0}^{T}G_t\frac{1}{1-\eta}(c_t)^{1-\eta}\right], \quad \text{for all } c \in \mathbb{R}_+^d,$$

where $G_t = \beta^t\left(\sum_i(\gamma_i)^{\frac{1}{\eta}}\right)^{\eta}$ for all $t = 0,\ldots,T$. Hence, the representative agent has CRRA equal to η.

Finally, observe that γ_i only depends on $(\beta_j,)_j$ $(\delta^j)_j$ and $\overline{D}_0$ for all i:

$$c_0^{*i} = \lambda_{00}^{*i}w_0^i = \lambda_{00}^{*i}\delta^i(\overline{D}_0 + \bar{q}_0) = \frac{\lambda_{00}^{*i}\delta^i}{\sum_j\lambda_{00}^{*j}\delta^j}\overline{D}_0.$$

If $\eta \neq 1$, then

$$\gamma_i = \left(c_0^{*i}\right)^{\eta} = \left(\delta^i\overline{D}_0\right)^{\eta}.$$

If $\eta = 1$, then

$$\gamma_i = c_0^{*i} = \frac{1-\beta_i}{1-\beta_i^{T+1}}\delta^i\left(\sum_j\frac{1-\beta_j}{1-\beta_j^{T+1}}\delta^j\right)^{-1}\overline{D}_0,$$

Hence, $\hat{U}$ is independent of the future dividend process $(D_t)_{t=1,\ldots,T}$.

$\square$

Appendix B: Figures

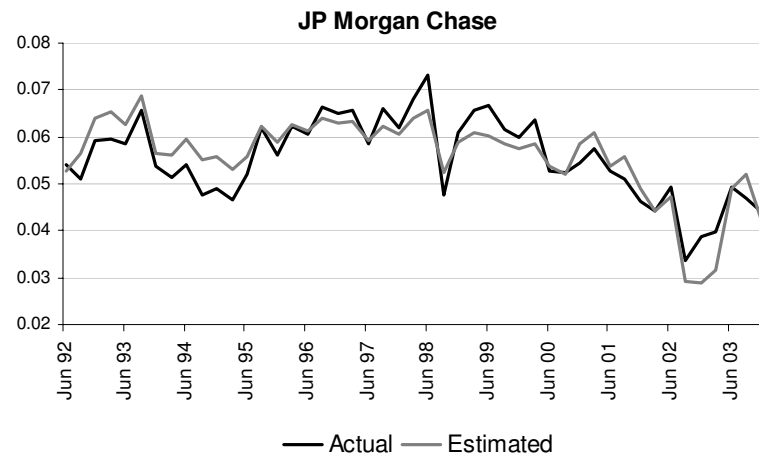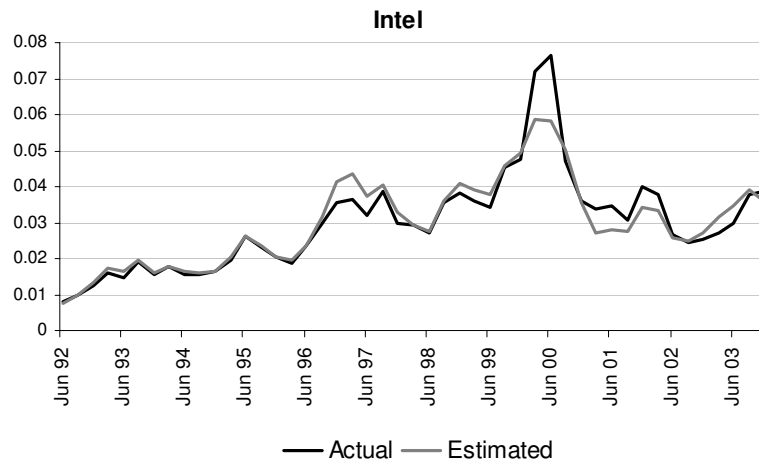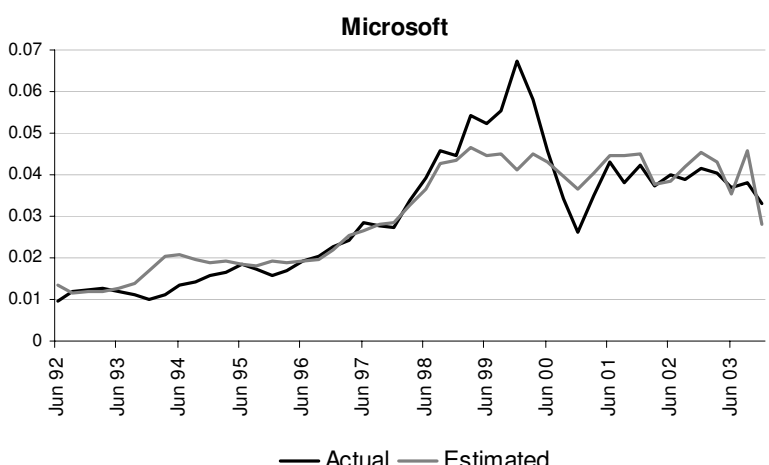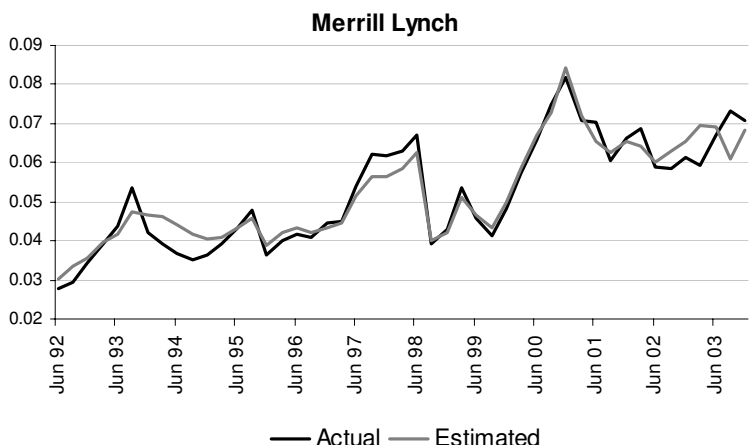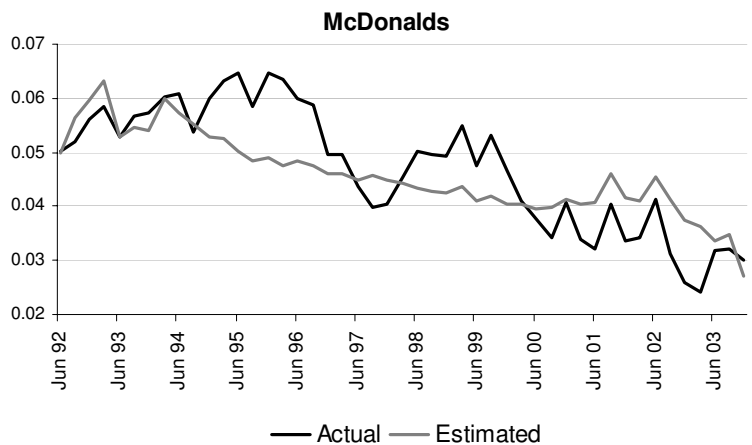The following figures show the estimated time series of relative stock prices for the 17 companies in Table 1.

3M

Altria Group

American Express
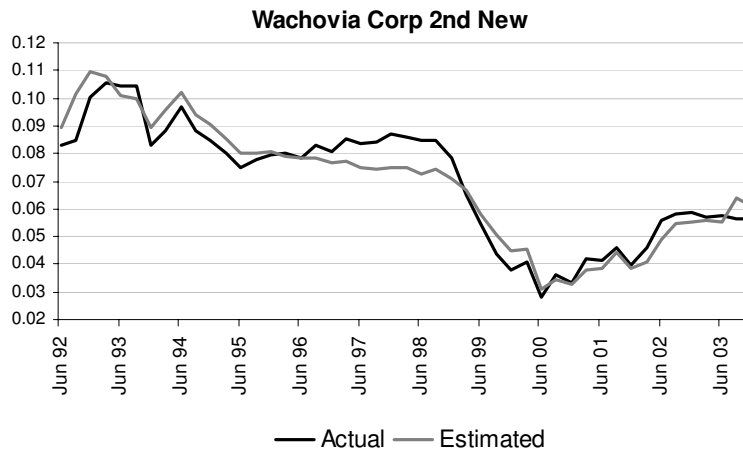

Bank New York

General Electric

29

General Motors

Hewlett Packard

IBM

Intel

Actual — Estimated

Johnson&Johnson

Actual — Estimated

JP Morgan Chase

Actual — Estimated

31

McDonalds

Actual — Estimated

Merrill Lynch

Actual — Estimated

Microsoft

Actual — Estimated

32

Pfizer
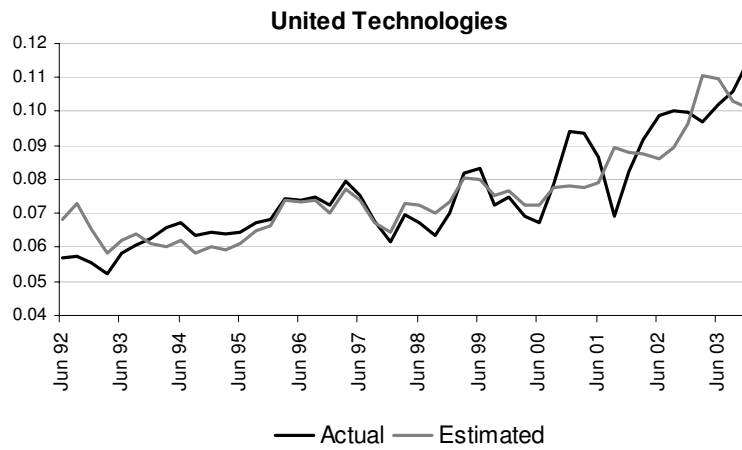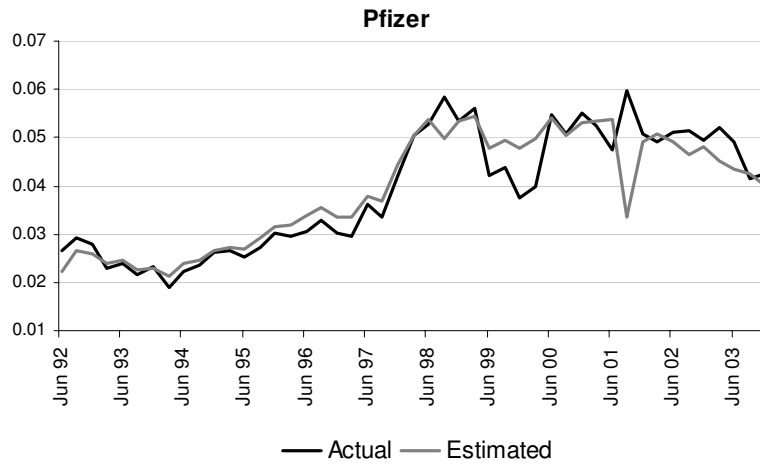

United Technologies

Wachovia Corp 2nd New
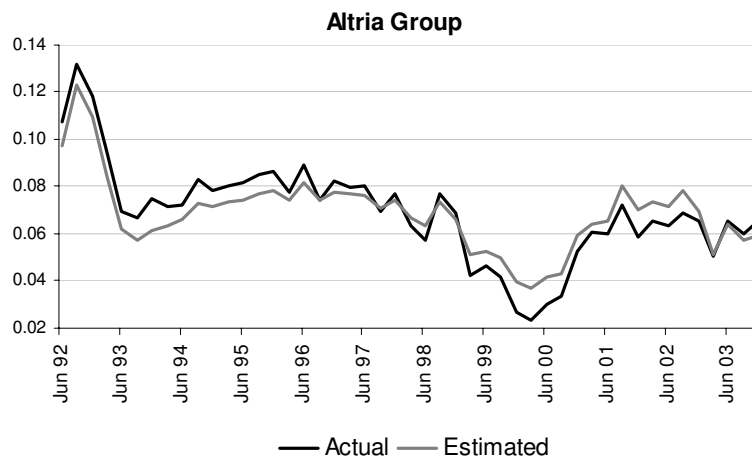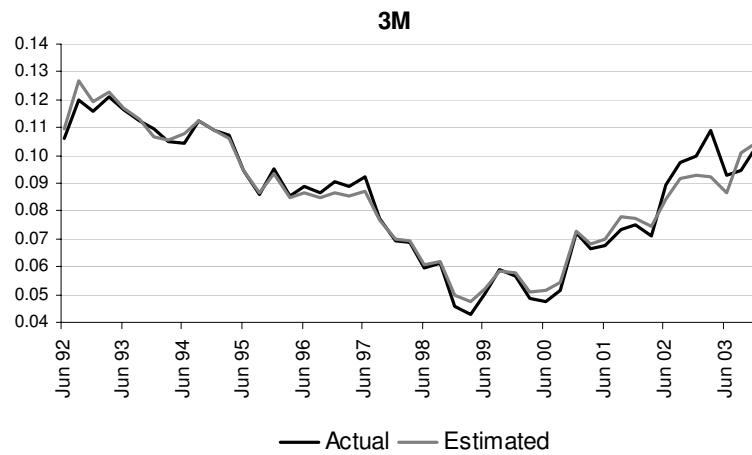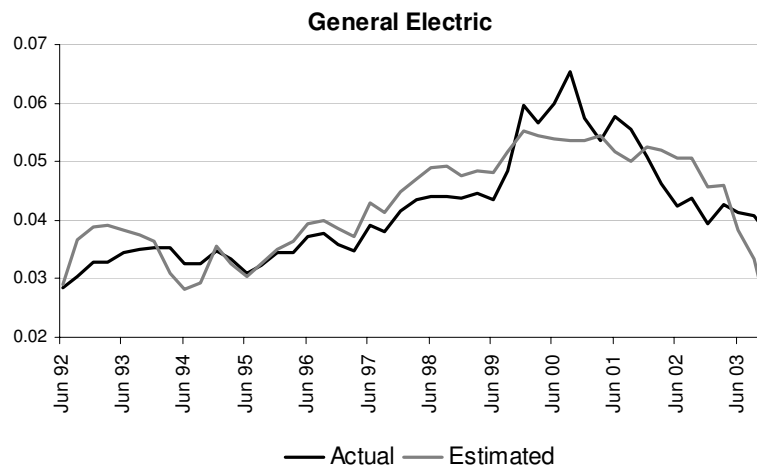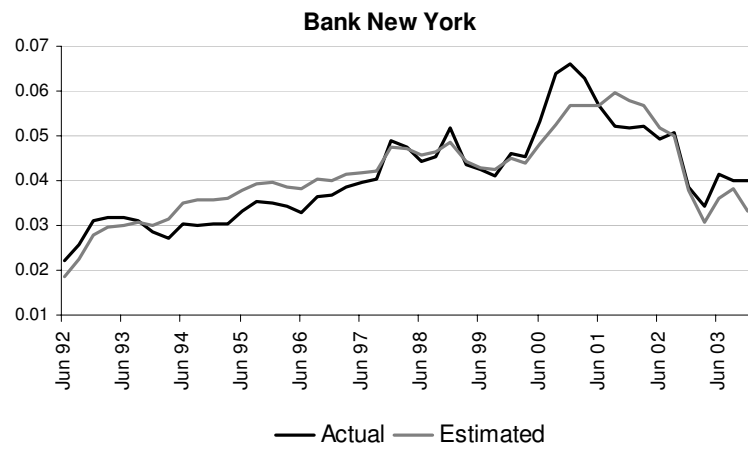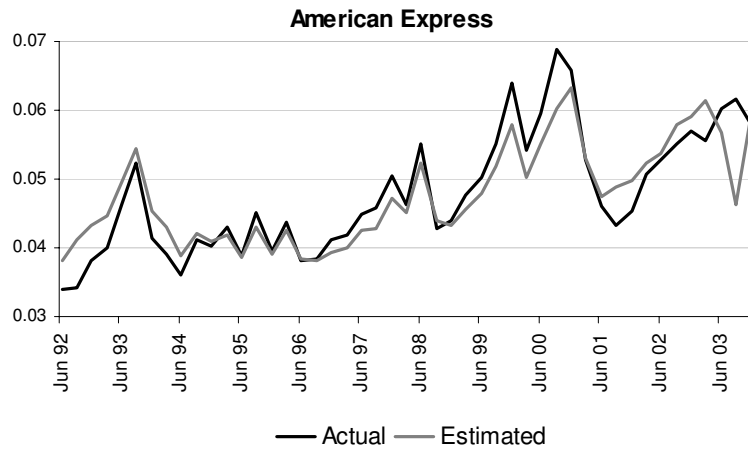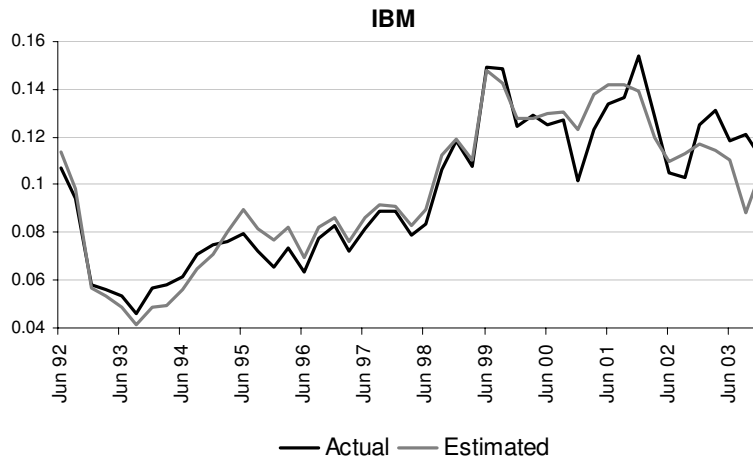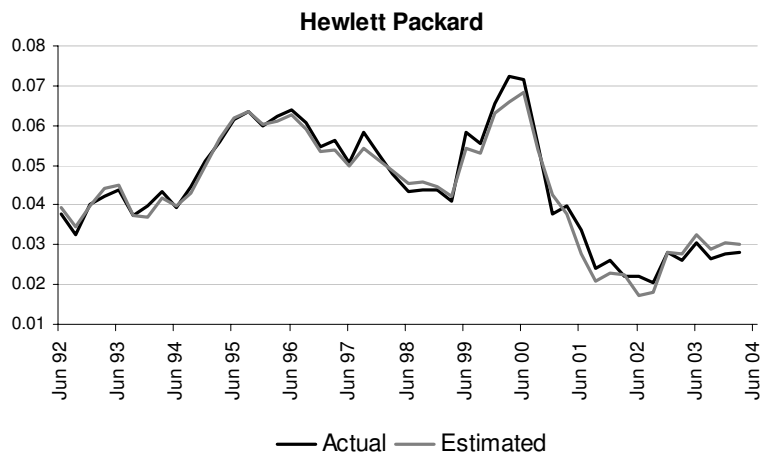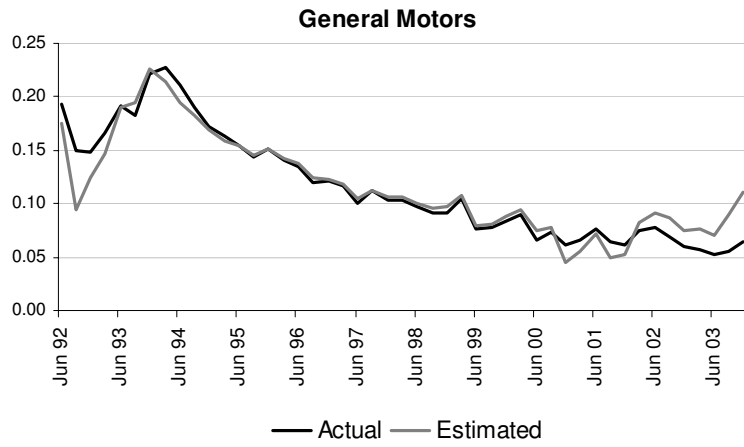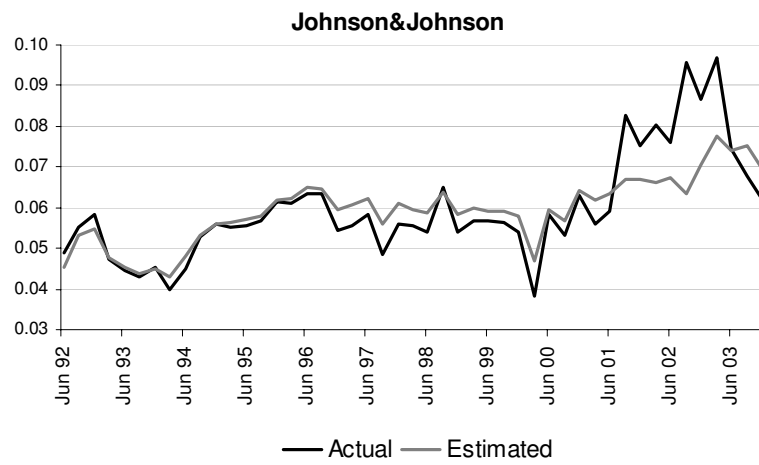
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