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Discussion paper

Stackelberg equilibria in a multiperiod vertical contracting model with uncertain and price-dependent demand

BY
Leif K. Sandal AND **Jan Ubøe**

Norges
Handelshøyskole
NORWEGIAN SCHOOL OF ECONOMICS

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Leif Sandal* and Jan Ubøe

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Abstract

In this paper, we consider Stackelberg games in a multiperiod vertical contracting model with uncertain demand. Demand has a distribution with a mean and variance that depend on the current retail price, and this dependence may vary from period to period. We focus on a class of problems in which the market has a memory-based scaling of demand, and the mean scaling is a function of previous retail prices. This leads to a strategic game in which the parties must balance high immediate profits with reduced future earnings. We propose a complete solution to this multiperiod Stackelberg game, covering cases with finite and infinite horizons. The theory is illustrated by using a Cobb–Douglas demand function with an additive, normally distributed random term, but the theory applies to more general settings.

Keywords: Stackelberg game, multiperiod vertical contracting model, price-dependent demand
JEL codes: C61, C73, D81

Leif Sandal: Norwegian School of Economics, Helleveien 30, 5045 Bergen, Norway,
e-mail: leif.sandal@nhh.no
Jan Ubøe: Norwegian School of Economics, Helleveien 30, 5045 Bergen, Norway,
e-mail: jan.uboe@nhh.no

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Main symbols:

$$
\begin{aligned}
W &= \text{ wholesale price per unit (chosen by the manufacturer)} \\
q &= \text{ order quantity (chosen by the retailer)} \\
R &= \text{ retail price per unit (chosen by the retailer)} \\
D &= \text{ demand (random)} \\
M &= \text{ production cost per unit (fixed)} \\
S &= \text{ salvage price per unit (fixed)} \\
\Pi^r &= \text{ profit for the retailer} \\
\Pi^m &= \text{ profit for the manufacturer}
\end{aligned}
$$

1 Introduction

In a vertical contracting problem, a retailer wants to order a quantity q from a manufacturer. Demand D is a random variable, and the retailer wishes to select an order quantity q to maximize his or her expected profit $\mathrm{E}[\Pi^r[q, D]]$. When the distribution of D is known, this problem is easily solved, and is commonly referred to as the "newsvendor problem". The basic problem is simple, but appears to have a never-ending number of variations. There is now a large literature on such problems, which is surveyed by Cachón (2003) and Qin et al. (2011). (See also the numerous references therein.)

In this paper, we consider multiperiod Stackelberg games between a manufacturer and a retailer, and study cases where future demand depends on past as well as future prices. The players must then take into account actions from a third party, the customers. Aggressive pricing may lead to short-term profits, but may be harmful to demand in the long run.

The one-period newsvendor problem with price-dependent demand is classical (see Whitin 1955). Mills (1959) refined the construction by considering the case in which demand uncertainty is added to the price–demand curve, and Karlin and Carr (1962) considered the case in which demand uncertainty is multiplied by the price–demand curve. For a useful review of the problem with extensions, see Petruzzi and Dada (1999). See also Xu et al. (2010) and Xu et al. (2011) for recent contributions to the price-dependent case.

In our Stackelberg game, the manufacturer is the leader and offers the retailer a wholesale price W for items delivered in the next period. The retailer is the follower and he or she tries to select an order quantity q and a retail price R to maximize future expected profits. Stackelberg games for the one-period case with fixed R have been studied extensively by Lariviere and Porteus (2001), who provide quite general conditions under which unique equilibria can be found. Petruzzi and Dada (1999) consider multiperiod cases with price-dependent demand, and show how to adapt such models to include backorders. They do not, however, address Stackelberg games. In this paper, we extend the theory to Stackelberg equilibria in multiperiod cases in which demand in the future is a function of prices and demand in the past. We assume that unmet demand is lost, and hence ignore cases with backordering. Although this is a serious limitation, our theory can be applied to important cases such as electricity markets and markets for fresh foods.

Multiperiod Stackelberg games of the type we discuss in this paper are generally difficult to solve. Some types may admit numerical solutions, but the general problem is difficult to compute or analyze even in the two-period case. By comparison, the model we present in this paper yields an explicit solution that is easily computed for any number of periods. Our model retains the main essence of the problem itself, while simultaneously providing a solution that can be analyzed without the need for advanced optimizing techniques.

Øksendal et al. (2011) consider continuous time Stackelberg games for Itô–Lévy processes with price-dependent demand. They prove that equilibria can be found by solving a coupled system of stochastic differential equations. In principle, such systems can be solved, but even simple cases lead to equations that cannot be solved by conventional means. Solutions appear to require mathematical optimization techniques not yet discussed in the literature. By comparison, the discrete version we consider in this paper is transparent. Our scaling approach decouples a multiperiod problem into a sequence of one-period problems, each of which is fairly easy to solve.

This paper is organized as follows. In Section 2, we introduce basic notation and review classical formulas for the one-period case. In Section 3, we formulate general Stackelberg games for the two-period case. Although general problems of this kind can be solved numerically, the problem is so deeply nested that one cannot expect to find an analytical solution. However, if demand in the next period is scaled by a factor that depends on the current demand and retail price, the system decouples into two separate cases. This decoupling carries over to the multiperiod case, and we can obtain a complete solution by backward iteration; i.e., we first solve the problem for the final period, feed the solution into a similar problem for the previous period, and continue backwards until we reach the first period. Our main result is stated in Theorem 3.1. In Section 4, we discuss problems related to existence and uniqueness, and also show how to address problems that have an infinite horizon. In Section 5, we illustrate the theory developed in Sections 3 and 4 by using explicit numerical examples. In Section 6, we offer concluding remarks.

2 The newsvendor model with price-dependent demand

In this section, we review well-known facts about the one-period newsvendor model. These facts explain the formula in (10), which is used throughout the paper.

In the classical newsvendor model, a retailer plans to sell a commodity in a market with uncertain demand D. The retailer orders a number of units of the commodity from a manufacturer, and expects to sell a sufficient number of these units to make a profit. The manufacturer decides the wholesale price W, while the retailer decides the selling price (revenue) R and the order quantity q. Any unsold items can be salvaged at the price S.

The retailer's profit is denoted by Π^r. Profits in the newsvendor model can be written in several different ways. For our analysis, it is convenient to express everything in terms of the random variable $\min[D,q]$. By using the relation $(q-D)^+ = q - \min[D,q]$, we obtain:

$$\begin{aligned}\Pi^r &= R\min[D,q] + S(q-D)^+ - Wq \\ &= R\min[D,q] + S(q-\min[D,q]) - Wq \\ &= (R-S)\min[D,q] - (W-S)q.\end{aligned} \tag{1}$$

From this expression, we obtain:

$$\mathrm{E}[\Pi^r] = (R-S)\mathrm{E}[\min[D,q]] - (W-S)q. \tag{2}$$

To consider situations with price-dependent demand, we consider cases with D of the form:

$$D = \mu[R] + \sigma[R]\mathcal{E} \tag{3}$$

where $\mu[R]$ and $\sigma[R]$ are given deterministic functions, and $\mathcal{E}$ is an arbitrary distribution satisfying $\mathrm{E}[\mathcal{E}] = 0, \mathrm{Var}[\mathcal{E}] = 1$. Note that multiplicative cases with $\mathrm{E}[\mathcal{E}] \neq 0$ and $\mathrm{Var}[\mathcal{E}] \neq 1$ are easily transformed into the format in (3). When R is given, it is well known that maximum expected profit is obtained when:

$$P(D \leq q) = \frac{R-W}{R-S}. \tag{4}$$

Let $F_{\mathcal{E}}$ denote the cumulative distribution of $\mathcal{E}$. We assume that $\mathcal{E}$ is continuous, supported on an interval, with density $f_{\mathcal{E}} > 0$ a.e. on its support. Under these conditions, the expected profit $\overline{\Pi}^r$ is strictly concave in q on the support of D, and the order quantity q from (4) is unique. It is clear that

$$q = \mu[R] + \sigma[R] \cdot F_{\mathcal{E}}^{-1}\left[\frac{R-W}{R-S}.\right] \tag{5}$$

By using (3) and (4), we obtain:

$$\begin{aligned}\mathrm{E}[\Pi^r] &= (R-S)\mathrm{E}[\min[D,q]] - (W-S)q \\ &= (R-S)\left(\mu[R] + \mathrm{E}\left[\min\left[\sigma[R]\mathcal{E}, \sigma[R]\,F_{\mathcal{E}}^{-1}\left[\frac{R-W}{R-S}\right]\right]\right]\right) \\ &\quad - (W-S)\left(\mu[R] + \sigma[R]F_{\mathcal{E}}^{-1}\left[\frac{R-W}{R-S}\right]\right).\end{aligned} \tag{6}$$

Equations (3)–(5) indicate that:

$$\mathrm{E}\left[\min\left[\mathcal{E},\, F^{-1}\left[\frac{R-W}{R-S}\right]\right]\right] \tag{7}$$
$$= \int_{-\infty}^{F_{\mathcal{E}}^{-1}\left[\frac{R-W}{R-S}\right]} x f_{\mathcal{E}}[x]dx + \mathcal{F}_{\mathcal{E}}^{-1}\left[\frac{R-W}{R-S}\right] \cdot P\left(\mathcal{E} \geq F_{\mathcal{E}}^{-1}\left[\frac{R-W}{R-S}\right]\right) \tag{8}$$
$$= \int_{-\infty}^{F_{\mathcal{E}}^{-1}\left[\frac{R-W}{R-S}\right]} x f_{\mathcal{E}}[x]dx + \mathcal{F}_{\mathcal{E}}^{-1}\left[\frac{R-W}{R-S}\right] \cdot \left(1 - \frac{R-W}{R-S}\right). \tag{9}$$

Inserting the expression in (9) into (6) and simplifying the resulting expression yields:

$$\overline{\Pi}^r = \mathrm{E}[\Pi^r] = (R-W)\mu[R] + L_{\mathcal{E}}[R,W]\sigma[R] \tag{10}$$

where $L_{\mathcal{E}}$ is defined by:

$$L_{\mathcal{E}}[R,W] = (R-S)\int_{-\infty}^{z} x f_{\mathcal{E}}[x]dx \qquad z = F_{\mathcal{E}}^{-1}\left[\frac{R-W}{R-S}\right]. \tag{11}$$

By assumption $\mathrm{E}[\mathcal{E}] = \int_{-\infty}^{\infty} x f_{\mathcal{E}}[x]dx = 0$, and hence $L_{\mathcal{E}}[R,W] \leq 0$. In the literature, the term $L_{\mathcal{E}} \cdot \sigma$ is often referred to as loss due to randomness. Note that $L_{\mathcal{E}}$ does not depend on the choice of the function $\sigma[R]$. For the construction used to solve multiperiod Stackelberg games in this paper, it is important that the deterministic function $\sigma[R]$ enters as a multiplicative factor in (10). Thus, it is essential that the σ dependence is handled through the format we use in (3).

3 The multiperiod newsvendor game

In this section, we provide a theoretical discussion of the multiperiod newsvendor game. Starting with a brief description of the classical one-period game, we discuss the structure of the multiperiod case. In particular, we focus on the case in which demand in the next period is scaled by a factor that depends on prices and demand in the current period. This is a type of Markov assumption in that it only requires knowledge of the current state, not of how prices and demand arrived at that state.

In the multiperiod game, we assume that the parties are risk neutral and try to maximize discounted expected profit:

$$J_r = \overline{\Pi}_1^r + \alpha\overline{\Pi}_2^r + \alpha^2\overline{\Pi}_3^r + \cdots + \alpha^{n-1}\overline{\Pi}_n^r \tag{12}$$

$$J_m = \overline{\Pi}_1^m + \alpha\overline{\Pi}_2^m + \alpha^2\overline{\Pi}_3^m + \cdots + \alpha^{n-1}\overline{\Pi}_n^m \tag{13}$$

where n is the number of periods, α is a discounting factor, and barred symbols indicate expected values.

3.1 The one-period game

In the one-period newsvendor model, to formulate a Stackelberg game, we assume that both parties are risk neutral. The manufacturer (leader) offers a wholesale price W. We ignore the possibility that the retailer can negotiate the wholesale price. Given W, the retailer (follower) then chooses the retail price R and the order quantity q to maximize expected profit as given by (10). The manufacturer knows that the retailer will choose q to maximize expected profit. Given each possible value of W, the manufacturer can hence anticipate the resulting order quantity $q = q[W]$, and so chooses W to maximize expected profit (which happens to be deterministic in this case). The manufacturer's profit is given by:

$$\Pi^m = (W - M)q \tag{14}$$

where M is the production cost per unit.

3.2 General two-period games

For a two-period Stackelberg game, demand in the first period is given by:

$$D_1 = \mu_1[R_1] + \sigma_1[R_1]\mathcal{E}_1. \tag{15}$$

In the second period, we have:

$$D_2 = \mu_2[R_1, R_2, D_1] + \sigma_2[R_1, R_2, D_1]\mathcal{E}_2. \tag{16}$$

We assume that μ_1, μ_2, σ_1, and σ_2 are deterministic functions, and that the random variables $\mathcal{E}_1$ and $\mathcal{E}_2$ are centered and normalized. The idea is that the level of demand in the first period can (to some extent) carry over to the second period. Moreover, a high price in the first period can lead to reduced demand in the second period, whereas a low initial price can have the opposite effect by stimulating demand.

We let Π_1^r and Π_2^r denote the profits for the retailer in the two periods. Π_1^m and Π_2^m denote the corresponding profits for the manufacturer. We assume that both parties try to maximize discounted expected profits:

$$J_r = \mathrm{E}[\Pi_1^r + \alpha \cdot \Pi_2^r] \tag{17}$$

$$J_m = \mathrm{E}[\Pi_1^m + \alpha \cdot \Pi_2^m] \tag{18}$$

where $0 \leq \alpha \leq 1$ is a fixed discounting factor. When decisions are taken for the second period, we assume that the values R_1, W_1, and D_1 are common knowledge. Conditional on D_1, and given values for R_1, R_2, and W_2, it follows from (5) and (10) that:

$$q_2 = \mu_2[R_1, R_2, D_1] + \sigma_2[R_1, R_2, D_1] \cdot F_{\mathcal{E}_2}^{-1}\left[\frac{R_2 - W_2}{R_2 - S}\right] \tag{19}$$

$$\mathrm{E}[\Pi_2^r | D_1] = (R_2 - W_2)\mu_2[R_1, R_2, D_1] + L_{\mathcal{E}_2}[R_2, W_2]\sigma_2[R_1, R_2, D_1]. \tag{20}$$

In the second and final period, there is no need to worry about future demand. Hence, given R_1, D_1, and W_2, the retailer chooses R_2 to maximize $\mathrm{E}[\Pi_r^2 | D_1]$. By assuming that the mapping $R_2 \mapsto \mathrm{E}[\Pi_2^r | D_1]$ has a unique maximum, we can hence construct a function $R_2 = R_2[R_1, D_1, W_2]$ that maximizes this conditional expected value. At the time when the manufacturer chooses W_2, the values of R_1 and D_1 are common knowledge. Hence, the manufacturer chooses W_2 to maximize conditional profit:

$$\mathrm{E}[\Pi_2^m | D_1] = (W_2 - M_2)q_2 \tag{21}$$

where q_2 is given by (19) and $R_2 = R_2[R_1, D_1, W_2]$. Given values for R_1 and D_1, it follows that

$\mathrm{E}[\Pi_2^m|D_1]$ is a function of only W_2. Assuming that this function has a unique maximum, we can then construct a function $W_2 = W_2[R_1, D_1]$ that maximizes the manufacturers conditional expected profit.

By the law of double expectation, we have:

$$J_r = \mathrm{E}[\Pi_1^r] + \alpha \cdot \mathrm{E}[\mathrm{E}[\Pi_2^r|D_1]] \tag{22}$$

$$J_m = \mathrm{E}[\Pi_1^m] + \alpha \cdot \mathrm{E}[\mathrm{E}[\Pi_2^m|D_1]]. \tag{23}$$

Given a value for W_1, the retailer, knowing that the manufacturer is a Stackelberg optimizer, can anticipate that the manufacturer will offer the price $W_2 = W_2[R_1, D_1]$ in the second period. By (10), we have:

$$\mathrm{E}[\Pi_1^r] = (R_1 - W_2)\mu_1[R_1] + L_{\mathcal{E}_1}[R_1, W_1]\sigma_1[R_1]. \tag{24}$$

Given R_1, the distribution of D_1 is known. Equation (22), together with (20) and (24), enables us to compute the final value of J_r given this particular choice of R_1. The retailer chooses R_1 to maximize this value. From this choice, the manufacturer obtains (deterministic) profit of:

$$\Pi_1^m = (W_1 - M)\left(\mu_1[R_1] + \sigma_1|R_1] \cdot F_{\mathcal{E}_1}^{-1}\left[\frac{R_1 - W_1}{R_1 - S}\right]\right). \tag{25}$$

The manufacturer's (possibly) random profit is:

$$\Pi_2^m = (W_2 - M)q_2[R_1, D_1, W_2]. \tag{26}$$

Knowing that the retailer will choose R_1 as above, the manufacturer can hence choose W_1 to maximize his or her total expected profit.

3.3 Two-period games with multiplicative scaling

The general construction in Section 3.2 is sufficiently explicit to enable numerical solutions of the problem for most choices of the functions μ_1, μ_2, σ_1, and σ_2. However, the problem is so deeply nested that one cannot expect to find an analytical solution. To search for cases that can be studied in more detail, we consider the following special case:

$$\mu_2[R_1, R_2, D_2] = \tilde{\mu}_2[R_2] \cdot g_1[R_1, D_1] \qquad \sigma_2[R_1, R_2, D_2] = \tilde{\sigma}_2[R_2] \cdot g_1[R_1, D_1] \tag{27}$$

with g_1 being a common scaling factor. In this case, (20) takes the form:

$$\mathrm{E}[\Pi_2^r | D_1] = (R_2 - W_2)\mu_2[R_1, R_2, D_1] + \sigma_2[R_1, R_2, D_2](R_2 - S) \int_{-\infty}^{F_{\mathcal{E}_2}^{-1}\left[\frac{R_2 - W_2}{R_2 - S}\right]} x f_{\mathcal{E}_2}[x] dx \tag{28}$$

$$= g_1[R_1, D_1] \left((R_2 - W_2)\tilde{\mu}_2[R_2] + \tilde{\sigma}_2[D_2](R_2 - S) \int_{-\infty}^{F_{\mathcal{E}_2}^{-1}\left[\frac{R_2 - W_2}{R_2 - S}\right]} x f_{\mathcal{E}_2}[x] dx \right) \tag{29}$$

and the optimal values of R_2 and W_2 are then independent of R_1 and D_1. The retailer's optimization problem reduces to the problem of maximizing:

$$J_r[R_1] = (R_1 - W_1)\mu_1[R_1] + L_{\mathcal{E}_1}[R_1, W_1]\sigma_1[R_1] + \mathrm{E}[\alpha \cdot g_1[R_1, D_1] \cdot \overline{\Pi}_2] \tag{30}$$

where $\overline{\Pi}_2$ is the expected profit the retailer would have obtained in the final period had the scaling factor been 1. This simplification separates our original problem into two separate subproblems, which are both easily solved. The problem for the final period is a standard one-period problem with price-dependent demand. The second problem is quite similar, the only difference being the extra term $\mathrm{E}[\alpha \cdot g_1[R_1, D_1] \cdot \overline{\Pi}_2]$.

3.4 Multiperiod games with multiplicative scaling

Whereas it is straightforward to formulate an n-period game in the general case, numerical solutions are difficult to obtain even if n is moderately large. The nonlinear structure of the problem branching into separate cases for each particular choice made on every level quickly renders the problem infeasible.

We show how to generalize the scaling approach described in the previous section to multiperiod problems. First, we discuss an important technical issue. Consider the three-period problem:

$$D_1 = \mu_1[R_1] + \sigma_1[R_1]\mathcal{E}_1 \tag{31}$$

$$D_2 = g_1[R_1, D_1](\mu_2[R_2] + \sigma_2[R_2]\mathcal{E}_2) \tag{32}$$

$$D_3 = g_1[R_1, D_1]g_2[R_2, D_2](\mu_3[R_3] + \sigma_3[R_3]\mathcal{E}_3) \tag{33}$$

where $\mathcal{E}_1$, $\mathcal{E}_2$, and $\mathcal{E}_3$ are independent and g_1 and g_2 are scaling factors. The scaling factors $g_1[R_1, D_1]$ and $g_2[R_2, D_2]$ are typically not independent, and it is easy to check that the problem does not in general separate into independent subproblems. However, this issue does not arise if the scaling factors depend only on price. Generally, one can extend the construction to cases in which the scaling factors are functions of $\mathcal{E}$, because these are independent. When all the scaling factors are independent, the system can be solved by repeated use of backward iteration. This always applies if $n = 2$ because there is only one scaling factor involved.

To simplify notation, we define:

$$\overline{g}[R] = \mathrm{E}[g[R, \mathcal{E}]]. \tag{34}$$

First, we solve for the final period to obtain expected profits $\overline{\Pi}^r_n$ and $\overline{\Pi}^m_n$. Once these values are known, the previous level can be computed as shown in Section 3.3. That produces numerical values of $\overline{\Pi}^r_{n-1,n}$ and $\overline{\Pi}^m_{n-1,n}$ (total discounted expected profits in the two periods). To determine the strategy for the $(n-2)$nd level, we consider the problem:

$$J_r[R_{n-2}] = (R_{n-2} - W_{n-2})\mu_{n-2}[R_{n-2}] + L_{\mathcal{E}_{n-2}}[R_{n-2}, W_{n-2}]\sigma_{n-2}[R_{n-2}] \tag{35}$$

$$+ \alpha \cdot \overline{\Pi}^r_{n-1,n} \cdot \overline{g}_{n-2}[R_{n-2}] \tag{36}$$

$$J_m[W_{n-2}] = (W_{n-2} - R_{n-2}) \left(\mu_{n-2}[R_{n-2}] + \sigma_{n-2}[R_{n-2}] \cdot F^{-1}_{\mathcal{E}_{n-2}} \left[\frac{R_{n-2} - W_{n-2}}{R_{n-2} - S} \right] \right) \tag{37}$$

$$+ \alpha \cdot \overline{\Pi}^m_{n-1,n} \cdot \overline{g}_{n-2}[R_{n-2}]. \tag{38}$$

To simplify notation, we have suppressed dependence on arguments that are not yet active; μ_{n-2} and σ_{n-2} are in general functions of $(R_{n-3}, \mathcal{E}_{n-3})$ but according to our assumptions, this dependence enters as an independent multiplicative factor and can hence be factored out of the optimization problem.

By using the argument above repeatedly, it is clear that we can solve this problem for any value of n. We state the final result in the form of a theorem.

Theorem 3.1

Let n be the number of periods and assume that demand in period k is given by:

$$D_k = (\mu_k[R_k] + \sigma_k[R_k]\mathcal{E}_k) \cdot \prod_{i=1}^{k-1} g_i[R_i, \mathcal{E}_i] \tag{39}$$

where $\mathcal{E}_1, \ldots, \mathcal{E}_n$ are independent and continuously distributed with $\mathrm{E}[\mathcal{E}_k] = 0$ and $\mathrm{Var}[\mathcal{E}_k] = 1$ for all k, and supported on intervals with $f_{\mathcal{E}_k} > 0$ a.e. on their supports. If, for each k, the following one-period Stackelberg problem has a unique equilibrium at $R_k = \hat{R}_k, W_k = \hat{W}_k$:

$$\begin{aligned} J_r^{(k)}[R_k] &= (R_k - W_k)\mu_k[R_k] + L_{\mathcal{E}_k}[R_k, W_k]\sigma_k[R_k] + \alpha \cdot \overline{\Pi}_k^r \cdot \overline{g}_k[R_k] \\ J_m^{(k)}[W_k] &= (W_k - R_k)\left(\mu_{n-2}[R_k] + \sigma_k[R_k] \cdot F_{\mathcal{E}_k}^{-1}\left[\frac{R_k - W_k}{R_k - S}\right]\right) + \alpha \cdot \overline{\Pi}_k^m \cdot \overline{g}_k[R_k] \end{aligned} \tag{40}$$

where $\overline{\Pi}_k^r$ and $\overline{\Pi}_k^m$ are found recursively from:

$$\overline{\Pi}_n^r = 0 \qquad \overline{\Pi}_n^m = 0 \tag{41}$$

$$\overline{\Pi}_k^r = J_r^{(k+1)}[\hat{R}_{k+1}] \qquad \overline{\Pi}_k^m = J_m^{(k+1)}[\hat{W}_{k+1}] \qquad k = 1, 2, \ldots, n-1, \tag{42}$$

then the multiperiod problem:

$$J_r = \overline{\Pi}_1^r + \alpha\overline{\Pi}_2^r + \alpha^2\overline{\Pi}_3^r + \cdots + \alpha^{n-1}\overline{\Pi}_n^r \tag{43}$$

$$J_m = \overline{\Pi}_1^m + \alpha\overline{\Pi}_2^m + \alpha^2\overline{\Pi}_3^m + \cdots + \alpha^{n-1}\overline{\Pi}_n^m \tag{44}$$

has a unique equilibrium at $\hat{\mathbf{R}} = (\hat{R}_1, \ldots, \hat{R}_n), \hat{\mathbf{W}} = (\hat{W}_1, \ldots, \hat{W}_n)$.

Remarks

The multiplicative factor $\prod_{i=1}^{k-1} g_i[R_i, \mathcal{E}_i]$ controls the memory of the process. If all the scaling factors are equal to 1, there is no memory, and the problem decouples into independent one-period problems. Note that given R_i, the value of $\mathcal{E}_i$ is known if and only if the value of D_i is known.

The condition that $\mathcal{E}_k$ is supported on an interval with $f_{\mathcal{E}_k} > 0$ a.e. on its support is needed to ensure that $F_{\mathcal{E}_k}$ is invertible. If $F_{\mathcal{E}_k}$ is not invertible, it is possible that the retailer's expected profit is maximized at several order quantities between which the retailer is indifferent. Different order quantities lead to different profits for the manufacturer, but the manufacturer lacks an instrument to ensure that the retailer chooses order quantities that are optimal for the manufacturer.

Uniqueness of the equilibria is important. However, given suitable scaling factors, the problems associated with existence and uniqueness are no more complicated than those in the corresponding one-period problem. Thus, problems of uniqueness arise in the one-period case. Our main issue is solving the multiperiod problem with multiplicative scaling, and uniqueness is clearly not restricted to those classical settings that have already been examined in the context of the one-period case.

4 Theoretical aspects of the model

4.1 Optimization for the retailer

In this section, we consider theoretical issues related to the multiperiod newsvendor problem with scaling. As explained in Section 3.4, this problem can be solved by backward iteration. At each stage of the iteration, we must solve a Stackelberg problem of the form (40). Given W, the retailer should try to maximize $J_r : [W, \infty) \to \mathbb{R}$ given by:

$$J_r[R] = (R - W)\mu[R] + L_{\mathcal{E}}[R, W]\sigma[R] + \alpha \cdot \overline{\Pi}^r \cdot \overline{g}[R]. \tag{45}$$

If $\sigma[R] = 0$, demand is deterministic. To isolate the effect due to randomness we split the function into two parts:

$$J_r[R] = J_r^{\det}[R] + J_r^{\sigma}[R] \tag{46}$$

where

$$J_r^{\det}[R] = (R - W)\mu[R] + \alpha \cdot \overline{\Pi}^r \cdot \overline{g}[R] \tag{47}$$

$$J_r^{\sigma}[R] = L_{\mathcal{E}}[R, W]\sigma[R]. \tag{48}$$

It is reasonable to assume that the deterministic part, i.e., $J_r^{\det}[R]$, is quasiconcave, increasing at $R = W$ with $\lim_{R\to\infty} J_r^{\det}[R] = 0$. These conditions are quite mild and are satisfied for fairly wide classes of demand functions and scaling factors. The special case in which Cobb–Douglas functions and linear expected scaling factors are used is discussed subsequently. Under such conditions, the deterministic part has a unique maximum at $R = R^*$. If σ is sufficiently small, the deterministic part is dominant and uniqueness is transferred to the stochastic case. The behavior is complicated when σ is not small. The first extreme is generally a minimum, and hence uniqueness depends on a delicate balance between the deterministic and stochastic parts on the interval starting from the second extreme.

Noting that $\frac{dz}{dR} = \frac{W-S}{f_{\mathcal{E}}(z)(R-S)^2}$, it is straightforward to check that:

$$\frac{dL_{\mathcal{E}}}{dR} = \int_{-\infty}^{z} x f_{\mathcal{E}}(x)dx + \int_{z}^{\infty} z f_{\mathcal{E}}(x)dx < 0 \qquad \frac{d^2L_{\mathcal{E}}}{dR^2} = (1 - F_{\mathcal{E}}[z])\frac{dz}{dR} > 0. \tag{49}$$

Clearly, $\lim_{R\to W} L_{\mathcal{E}}[R, W] = 0$, and a simple application of L'Hôpital's rule gives:

$$\lim_{R\to\infty} L_{\mathcal{E}}[R, W] = -\infty.$$

Thus, the function $L_{\mathcal{E}}$ is negative, globally decreasing and convex in R. If $\sigma[R]$ is constant (as in the additive case examined by Mills (1959)), or more generally, if $\sigma'[R] \geq 0$ and $\sigma''[R] \leq 0$, the same is true for J_r^{σ}. Clearly, $\lim_{R\to W} \frac{dJ_r^{\sigma}}{dR} = -\infty$ and $\lim_{R\to W} \frac{d^2J_r^{\sigma}}{dR^2} = +\infty$. If $\mu, \sigma : (0, \infty) \to \mathbb{R}$ are continuously differentiable, then $J_r[R]$ is almost never quasiconcave, the only exception being when J_r is globally decreasing.

4.2 Cobb–Douglas demand with linear scaling

In Section 5 we discuss the case in which $\mu[R] = CR^{-a}$, where $C > 0$ and $a > 1$ are given constants. Moreover, we consider scaling factors where:

$$\overline{g}[R] = \mathrm{E}[g[R, \mathcal{E}]] = 1 + \gamma(K - R) \tag{50}$$

where $\gamma > 0$ and $K > 0$ are given constants. With these particular choices, we obtain:

$$\frac{dJ_r^{\det}}{dR} = C\,aR^{-a}\left(\frac{W}{R} - \frac{a-1}{a}\right) - \alpha\gamma\Pi \tag{51}$$

$$\frac{d^2J_r^{\det}}{dR^2} = -(a+1)R^{-1}CaR^{-a}\left(\frac{W}{R} - \frac{a-1}{a}\right) - (a-1)R^{-a-1}. \tag{52}$$

If $\frac{dJ_r^{\det}}{dR}[R^*] = 0$, it follows that:

$$\frac{d^2J_r^{\det}}{dR^2}[R^*] = -(a+1)R^{-1}\alpha\gamma\Pi - (a-1)R^{-a-1} < 0. \tag{53}$$

It follows that, for any choice of C, a, γ, K, and Π, then $R \mapsto J_r^{\det}[R]$ is quasiconcave and the value $R^* = \mathrm{Argmax}[J_r^{\det}[R]]$ is unique. If $R > R^*$, then $\frac{dJ_r^{\det}}{dR}[R^*] < 0$. If $\sigma'[R] \geq 0$ for all $R \geq R^*$, the stochastic part is decreasing on $[R^*, \infty)$. Hence, a numerical search for $\mathrm{Argmax}[J_r[R]]$ can be restricted to the interval $[W, R^*]$. We summarize some of the findings from Sections 4.1 and 4.2 in the following proposition.

Proposition 4.1

Assume that $\mu_k[R] = C_kR^{-a_k}$, and that the scaling factor for demand satisfies:

$$\overline{g}_k[R_k] = (1 + \gamma_k(K_k - R_k)),$$

where C_k, a_k, γ_k and K_k are constants. Then, if $\sigma_k'[R_k] \geq 0$ for $R_k \geq R_k^*$, the function:

$$J_r^{(k)}[R_k] = (R_k - W_k)\mu_k[R_k] + L_{\mathcal{E}_k}[R, W_k]\sigma_k[R_k] + \alpha \cdot \overline{\Pi}_k^r \cdot \overline{g}_k[R_k] \tag{54}$$

has a global maximum in the interval $[W, R_k^*]$.

Remark

Note that R_k^* is fixed, i.e., independent of any particular choice of the function $\sigma_k[R]$. The conditions are trivially satisfied if σ_k is constant, which is the case we consider in Section 5.

4.3 The infinite-period case

For given values of $\overline{\Pi}_k^r$ and $\overline{\Pi}_k^m$, the parties try to optimize:

$$J_r^{(k)}[R_k] = (R_k - W_k)\mu_k[R_k] + L_{\mathcal{E}_k}[R_k, W_k]\sigma_k[R_k] + \alpha \cdot \overline{\Pi}_k^r \cdot \overline{g}_k[R_k] \tag{55}$$

$$J_m^{(k)}[W_k] = (W_k - R_k)\left(\mu_{n-2}[R_k] + \sigma_k[R_k] \cdot F_{\mathcal{E}_k}^{-1}\left[\frac{R_k - W_k}{R_k - S}\right]\right) \tag{56}$$

$$+ \alpha \cdot \overline{\Pi}_k^m \cdot \overline{g}_k[R_k].$$

The first-order conditions for this problem yield two equations for the two unknowns R_k and W_k. In the multiperiod case, we start by using $\overline{\Pi}_n^r = 0$ and $\overline{\Pi}_n^m = 0$ and iterate backwards until we reach the starting period. However, if the horizon is infinite, this approach fails because an infinite number of iterations is needed to reach the start.

If $\mu[R]$, $\sigma[R]$, $g[R, \mathcal{E}]$, and $\mathcal{E}$ do not depend on k, i.e., the same functions are used for any k, then cases with an infinite horizon can be solved. To do so, one needs a steady state for the system; i.e., we must find $\overline{\Pi}^r$ and $\overline{\Pi}^m$ such that:

$$\overline{\Pi}^r = (R - W)\mu[R] + L_{\mathcal{E}}[R, W] + \alpha \cdot \overline{\Pi}^r \cdot \overline{g}[R] \tag{57}$$

$$\overline{\Pi}^m = (W - R)\left(\mu[R] + \sigma[R] \cdot F_{\mathcal{E}}^{-1}\left[\frac{R - W}{R - S}\right]\right) + \alpha \cdot \overline{\Pi}^m \cdot \overline{g}[R]. \tag{58}$$

The first-order conditions from (55)–(56), together with (57)–(58), yield four equations in the four unknowns, R, W, $\overline{\Pi}^r$, and $\overline{\Pi}^m$. If this system has a unique solution, we have a unique candidate for the infinite-horizon case.

5 Numerical results

In this section, we illustrate the theory in Sections 3 and 4 by using numerical examples. In these examples, we use a Cobb–Douglas demand function with a normally distributed random term. The problem is as easily solved when using other functional forms. The problem (given W) is reduced to finding maxima for a function of one variable, which is straightforward for almost any $\mu_k[R]$, $\sigma_k[R]$, $\mathcal{E}_k$, and $g_k[R, \mathcal{E}_k]$.

For simplicity, we consider only the case in which the same functions are used for all periods, although using varying functions does not make the problem any harder to solve. We start with the one-period case, and gradually increase the number of periods, n, until we reach the infinite-horizon case.

5.1 The one-period case

We consider the demand function:

$$D = 1000 \cdot R^{-2} + 10 \cdot \mathcal{E} \tag{59}$$

where $\mathcal{E}$ is $N(0, 1)$. Because a normally distributed variable can take negative values, we must impose restrictions to exclude artificial cases. If q, as given by (5), is negative, we set $q = 0$. Moreover, if the expected profit in (10) is negative, we also assume $q = 0$. We choose:

$$M = 2 \qquad\qquad S = 1. \tag{60}$$

By using the formulas in (5) and (10) we can compute the manufacturer's profit as a function of W. This function is illustrated on the left side of Figure 1.

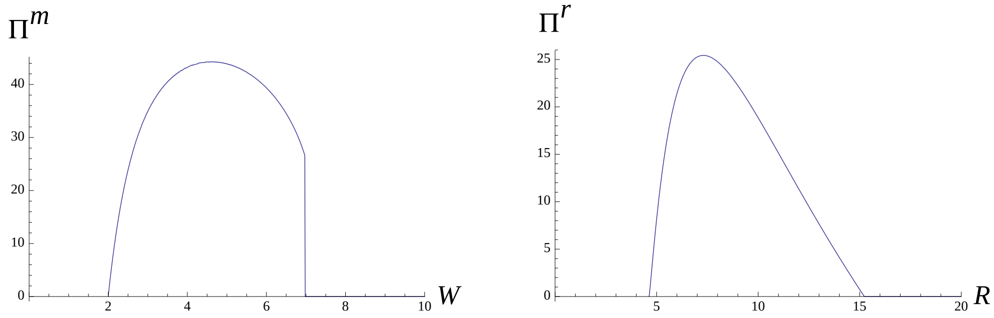

Figure 1: Expected profits for the manufacturer (left) and the retailer (right)

The manufacturer obtains maximum profit at the unique value $W = 4.61$. Given $W = 4.61$, the retailer's profit in (10) is a function of R only. This function is shown on the right side of Figure 1. The retailer's best response is to choose $R = 7.29$, which, from (5), leads to an order quantity $q = 16.96$. To summarize, our particular Stackelberg game has a unique equilibrium at:

$$(W, \overline{\Pi}^m) = (4.61, 44.27) \tag{61}$$

$$(R, q, \overline{\Pi}^r) = (7.29, 16.96, 25.79). \tag{62}$$

5.2 The two-period case

In this section, we extend the discussion in Section 5.1 to a two-period Stackelberg game. We assume that:

$$D_1 = \mu_1[R_1] + 10 \cdot \mathcal{E}_1 \tag{63}$$

where $\mu_1[R] = 1000 \cdot R^{-2}$ and $\mathcal{E}_1$ is $N(0,1)$; i.e., we use the same demand function used in Section 5.1. We further assume that $M = 2$ and $S = 1$ (as before). Now, let:

$$D_2 = g[R_1, D_1](\mu_1[R_2] + 10 \cdot \tilde{\mathcal{E}}_1) \tag{64}$$

where $\tilde{\mathcal{E}}_1$ is an independent copy of $\mathcal{E}_1$ and $g[R_1, D_1]$ is a scaling factor. Regardless of the choice of $g[R_1, D_1]$, it follows from (29) and the results in Section 3 that the second stage of the game will have a unique equilibrium at:

$$(W_2, \Pi_2^m) = (4.61, 44.27 \cdot \mathrm{E}[g[R_1, D_1]]) \tag{65}$$

$$(R_2, q_2, \Pi_2^r) = (7.29, 16.96, 25.79 \cdot \mathrm{E}[g[R_1, D_1]]). \tag{66}$$

Hence, the arguments W_2, R_2, and q_2 are independent of the scaling factor. However, the maximal values depend on the scaling factor, and both parties must take this into account when considering their first-period strategies. We now consider the scaling factors:

$$\overline{g}[R_1] = \mathrm{E}[g[R_1, D_1]] = 1 + \gamma(K - R_1) \tag{67}$$

where $\gamma \geq 0$ and $K \geq 0$ are given constants. The constant K can be interpreted as a "fair" price; i.e., any initial price above K reduces demand, whereas demand is more likely to increase if $R_1 < K$. If the scaling factor is negative, maxima are turned into minima. Hence, if $\mathrm{E}[g[R_1, D_1]] \leq 0$, the optimal order q_2 is zero. To avoid this problem, we consider cases where:

$$\overline{g}[R_1] = \max[1 + \gamma(K - R_1), 0]. \tag{68}$$

Typically, $M \leq W_1 \leq R_1$ is expected. Ruling out short selling implies $W_1 \geq 0$ and $R_1 \geq 0$. If $R_1 < W_1$, the optimal order quantity q_1 is zero. However, $R_1 < W_1$ might represent an optimal strategy. If $R_1 < W_1$, the retailer orders nothing in the first period. Then, he or she might just as well choose $R_1 = 0$ because this is the most efficient way to increase demand in period 2. A strategy of this type makes good sense economically; it corresponds to a situation in which a small number of items ($q_1 \approx 0$) are given away for free ($R_1 = 0$) in the first period to create increased interest for the product in the second period. In our optimization problem, given W_1, the retailer should find the maximum over all R_1 with $R_1 \geq W_1$. The retailer should then compare this maximum value with the value he or she could obtain by using the alternative $R_1 = 0, q_1 = 0$, and then choose the best alternative.

We consider the case:

$$\mu_1[R_1] = 1000 \cdot R_1^{-2} \qquad \sigma_1[R_1] = 10 \tag{69}$$

$$\mu_2[R_1, R_2, D_1] = g[R_1, D_1]\mu_1[R_2] \qquad \sigma_2[R_1, R_2, D_1] = g[R_1, D_1]\sigma_1[R_2] \tag{70}$$

$$\overline{g}[R_1] = \max[(1 + \gamma(K - R_1), 0]. \tag{71}$$

$$S = 1, M = 2 \tag{72}$$

Values for the parameters γ and K are specified below. We investigate how different values of these parameters affect the solutions. Given the choices described above, the strategies and expected profits in the second period are given by (65) and (66). Hence, the retailer's total expected profit, given W_1, is:

$$J_r[R_1] = (R_1 - W_1)\mu_1[R_1] + L_{\mathcal{E}_1}[R_1, W_1]\sigma_1[R_1] \tag{73}$$
$$+ \alpha \cdot 25.79 \cdot \max[(1 + \gamma(K - R_1), 0].$$

The manufacturer's total expected profit is:

$$J_m[W_1] = (W_1 - R_1)\left(\mu_1[R_1] + \sigma_1[R_1] \cdot F_{\mathcal{E}_1}^{-1}\left[\frac{R_1 - W_1}{R_1 - S}\right]\right) \tag{74}$$
$$+ \alpha \cdot 44.27 \cdot \max[(1 + \gamma(K - R_1), 0].$$

The manufacturer knows that, given W_1, the retailer will choose R_1 to maximize $J_r[R_1]$. Given $R_1 = \text{Argmax}[J_r[R_1]]$ in (74), $J_m[W_1]$ is a function of W_1 only.

- Case 1: $\alpha = 1$ (no discounting), $\gamma = 0$ (no dependence on R_1).

In this particular case, the system is effectively decoupled into two identical one-period problems. The equilibrium prices are:

$$W_1 = W_2 = 4.61 \qquad J_m = 88.53 \tag{75}$$

$$R_1 = R_2 = 7.29 \qquad q_1 = q_2 = 16.96 \qquad J_r = 51.57. \tag{76}$$

- Case 2: $\alpha = 1$ (no discounting), $\gamma = 0.1, K = 5$.

Now the pricing effect of R_1 is active. The system has a unique equilibrium state at:

$$W_1 = 4.05, W_2 = 4.61 \qquad J_m = 87.26 \tag{77}$$

$$R_1 = 6.24, R_2 = 7.29 \qquad q_1 = 23.60, q_2 = 16.96 \qquad J_r = 64.19. \tag{78}$$

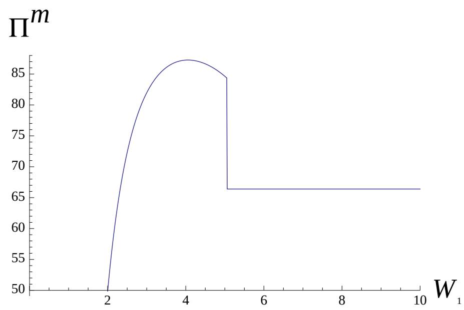

Figure 2: Expected total profit for the manufacturer as a function of W_1

The graph of the function $J_m[W_1]$ is shown in Figure 2. The section to the right is flat because, when W_1 is too high, the retailer's best choice is to order $q_1 = 0$, in which case he or she chooses $R_1 = 0$ to get the benefit of higher demand in period 2. The graph reveals that this does not correspond to the manufacturer's best choice.

In equilibrium, the discounted scaling factor is $\alpha(1 + \gamma(K - R_1)) < 1$. Hence, demand is lower than in Case 1. Although profits are higher than in Case 1, the profit margin is lower, but this is offset by a higher order quantity. The manufacturer has less control and makes a smaller expected profit.

- Case 3: $\alpha = 1$ (no discounting), $\gamma = 0.25, K = 5$.

The system has equilibrium states at:

$$W_1 \geq 3.93, W_2 = 4.61 \qquad J_m = 99.60 \tag{79}$$

$$R_1 = 0, R_2 = 7.29 \qquad q_1 = 0, q_2 = 16.97 \qquad J_r = 58.02. \tag{80}$$

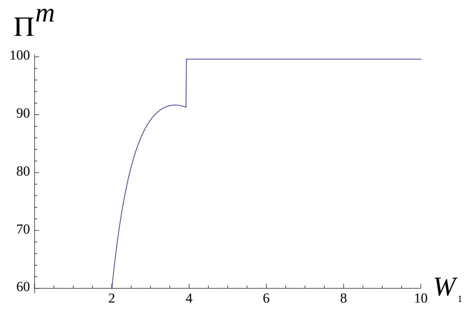

Figure 3: Expected total profit for the manufacturer

The graph of the function $J_m[W_1]$ is shown in Figure 3. The section to the right is flat because, when $W_1 \geq 3.93$, the retailer's best choice is to order $q_1 = 0$, in which case he or she chooses $R_1 = 0$ to get the benefit of higher demand in period 2. Unlike in Case 2, it is in the manufacturer's best interest to provoke a strategy of this sort because it maximizes expected profit.

• Case 4: $\alpha = 0.8$, $\gamma = 0.2, K = 5$.

The system has a unique equilibrium state at:

$$W_1 = 3.75, W_2 = 4.61 \qquad J_m = 80.81 \tag{81}$$

$$R_1 = 5.63, R_2 = 7.29 \qquad q_1 = 29.18, q_2 = 16.97 \qquad J_r = 58.80. \tag{82}$$

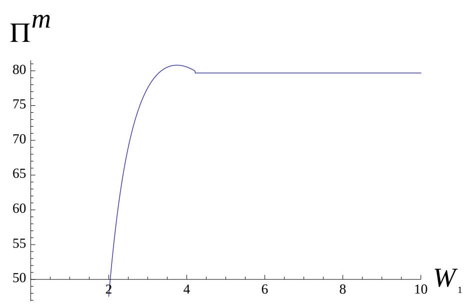

Figure 4: Expected total profit for the manufacturer

The graph of the function $J_m[W_1]$ is shown in Figure 4. Because of the discounting, it no longer pays to have zero sales in the first period; profits in the second period are less valuable because of discounting.

5.3 The multiperiod case

As explained in the theoretical section, once we have a code that solves the two-period case, the same code can be used repeatedly to solve n-period problems. We merely have to update the remaining profits as the construction progresses backwards. Starting with a given demand distribution D_1 in period 1, we consider the case in which demand in period k is given by:

$$D_k = \widetilde{D_1} \cdot \prod_{i=1}^{k-1} g_i[R_i, \mathcal{E}_i] \tag{83}$$

where the tilde signifies that, at each step, an independent draw is made from the original distribution D_1. When setting D_k, the values of R_1, $R_2, \ldots, R_{k-1}$, and D_1, $D_2, \ldots, D_{k-1}$ are all known, in which case the values $\mathcal{E}_1$, $\mathcal{E}_2, \ldots, \mathcal{E}_{k-1}$ are also known. In principle, the scaling factor g_k can change with k. For illustration purposes, we only consider cases in which the expected scaling factors satisfy the following:

$$\overline{g}_k[R_k] = \max[1 + \gamma(K - R_k), 0] \tag{84}$$

where $\gamma > 0$ and $K > 0$ are given constants. As mentioned above, more complicated expressions can be computed without problems.

• Case 1: $\alpha = 1$ (no discounting), $\gamma = 0.02, K = 5, n = 25$.

The equilibrium prices in each period are shown in Figure 5. In this figure, $R_1, \ldots, R_{25}$ are marked with bold dots, and $W_1, \ldots, W_{25}$ are marked with light dots.

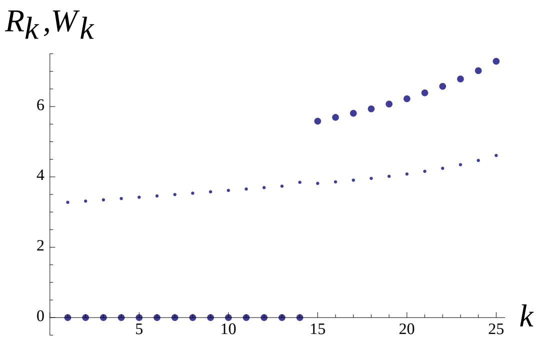

Figure 5: Equilibrium prices in period k

The optimal strategy is to increase demand by letting $R_1 = \cdots = R_{14} = 0$, then start selling in period 15, and gradually increase prices in the remaining periods. The values of R_k and W_k increase with k, as does the profit margin $R_k - W_k$. (In the sales period, $k \geq 15$.) While these properties might apply in general, examination of this issue is beyond the scope of this paper.

- Case 2: $\alpha = 0.9$, $\gamma = 0.02, K = 5, n = 25$.

In this case, $\alpha \cdot \max g = \alpha(1 + \gamma K) = 0.99$. To obtain increased profits from an initial strategy in which $R_1 = 0$, it is clearly necessary that $\alpha \cdot \max g > 1$. Because this is not the case in this example, sales take place in all periods. Moreover, prices hardly vary initially. This suggests that there is an equilibrium strategy for the infinite-horizon case.

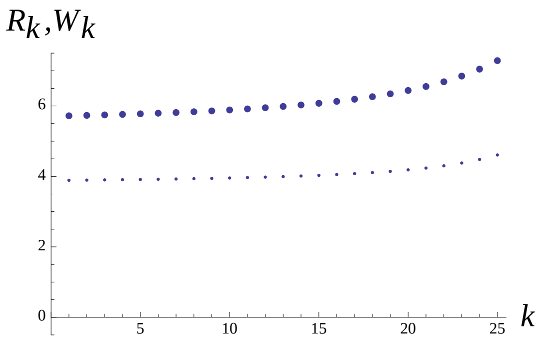

Figure 6: Equilibrium prices in period k

5.4 The infinite-horizon case

If we continue to iterate the system illustrated in Figure 6 backwards, equilibrium prices will progressively stabilize. This is to be expected as we approach an infinite-horizon problem with a well-defined equilibrium state. If $\alpha(1 + \gamma K) > 1$, it is clear that an arbitrarily large total profit can be obtained, and that there is no nondegenerate equilibrium strategy. Hence, we need $\alpha(1 + \gamma K) \leq 1$, in which case we can try to solve a fixed-point problem by using the first-order conditions from (55)–(56) together with (57)–(58).

By using the values $\alpha = 0.9$, $\gamma = 0.02$, and $K = 5$, this fixed-point problem is straightforward to solve, and we find:

$$R = 5.59, q = 29.0, \overline{\Pi}^r = 348.2 \tag{85}$$

$$W = 3.84, \overline{\Pi}^m = 481.6. \tag{86}$$

It is straightforward to check that this is indeed a steady state for the system.

The situation becomes more interesting if, e.g., γ is allowed to change values. To illustrate this point of view, consider the case where

$$\gamma_k = \begin{cases} 0.01 & k \leq 20 \\ 0.02 & k > 20 \end{cases} \tag{87}$$

To solve this problem, we feed the expected remaining profits from (85)–(86) into (40) and iterate this backwards to the start. The solution is shown in Figure 7. Clearly equilibrium prices are constant when $k > 20$.

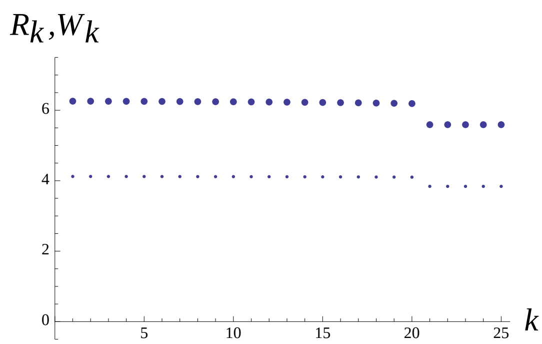

Figure 7: Equilibrium prices in period k

6 Concluding remarks

In this paper, we considered multiperiod newsvendor problems with price-dependent demand. In particular, we studied the case in which demand in one period is a function of prices and demand in the previous period. A problem of this type leads to time dependent pricing strategies. Increasing prices in one period can lead to short-term improvements, but as a consequence of the coupling, long-term demand can be reduced, as, thereby, can overall profits. The parties must then find an optimal balance between current profits and discounted future profits.

We showed how to obtain a complete solution to such problems when demand in one period is scaled by a multiplicative factor that depends on prices in the previous period. The multi-period problem can then be separated into a sequence of one-period problems, and the solution can be found by backward iteration starting from the final period. The problems are linked in that each stage needs the total profit from the previous stage as an input, but otherwise there is no coupling between the stages. When the scaling function is fixed, it is possible to consider infinite-horizon problems of this type, and Stackelberg equilibria can then be found solving an explicit fixed-point problem.

Although the main focus of the paper is on theory, to demonstrate that such problems can be solved, we provided numerical solutions to some special cases. Note, however, that our framework is not limited to such special cases. The numerical illustrations raise questions of interest for future research. We found that prices and profit margins increase with time. This is something that might be true in general, but we leave this and similar problems for future research.

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