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What Puzzles? New insights in asset pricing

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Abstract

Motivated by the problems of the conventional model in rationalizing market data, we derive the equilibrium interest rate and risk premiums using recursive utility in continuous time. In a representative-agent framework our model allows for the separation of risk aversion from the time preference. We demonstrate how this separation gives new insights in asset pricing: The expressions for risk premiums combine the market-based CAPM with the consumption-based CAPM. The equilibrium real interest rate now combines characterizations of preferences and market returns. This model explains both the Equity Premium Puzzle and the Risk-Free Rate Puzzle with good margin, and give solutions consistent with early resolution of uncertainty.

KEYWORDS: *The equity premium puzzle, the risk-free rate puzzle, recursive utility, early resolution, utility gradients, dynamic programming, The Stern Review*

JEL-Code: G10, G12, D9, D51, D53, D90, E21.

1 Introduction

Rational expectations, a cornerstone of modern economics and finance, has been under attack for quite some time. Are asset prices too volatile relative to the information arriving in the market? Is the mean risk premium on equities over the riskless rate too large? Is the real interest rate too low? Is the market's risk aversion too high?

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Mehra and Prescott (1985) raised some of these questions in their well-known paper, using a variation of Lucas's (1978) pure exchange economy with a Kydland and Prescott (1982) "calibration" exercise. They chose the parameters of the endowment process to match the sample mean, variance and the annual growth rate of per capita consumption in the years 1889 - 1978. The puzzle is that they were unable to find a plausible parameter pair of the utility discount rate and the relative risk aversion to match the sample mean of the annual real rate of interest and of the equity premium over the 90-year period.

The puzzle has been verified by many others, e.g., Hansen and Singleton (1983), Ferson (1983), Grossman, Melino, and Shiller (1987). Many theories have been suggested during the years to explain the puzzle, but to date there does not seem to be any consensus that the puzzles have been fully resolved by any single of the proposed explanations ¹.

We utilize a continuous time setting, to take full advantage of the analytic power of infinite dimensional analysis. We use the framework established by Duffie and Epstein (1992a-b) and Duffie and Skiadas (1994) which elaborates the foundational work by Kreps and Porteus (1978) of recursive utility in dynamic models. This model is extended to a continuous time setting, where future utility is a conditional expected time integral of a felicity index minus a measure of Arrow-Pratt absolute risk aversion multiplied by the variance rate of utility. When there is no uncertainty, the felicity index does not depend upon risk aversion.

As is well know recursive utility leads to the separation of risk aversion from the elasticity of intertemporal substitution in consumption, within a time-consistent model framework. We demonstrate that this gives risk premiums which combine the market-based CAPM with the consumption-based CAPM. The volatility of the market portfolio enters the expression for the risk premium. The equilibrium interest rate now combines characterizations of preferences and market returns. It contains two new terms connecting the risk free asset to the risky securities. In these new terms the risk aversion enters, while all the conventional terms are only affected by consumption substitution. The new feature is the form of the endogenous coefficients of

¹Constantinides (1990) introduced habit persistence in the preferences of the agents. Also Campbell and Cochrane (1999) used habit formation. Rietz (1988) introduced financial catastrophes, Barro (2005) developed this further, Weil (1992) introduced non-diversifiable background risk, and Heaton and Lucas (1996) introduce transaction costs. There is a rather long list of other approaches aimed to solve the puzzles, among them are borrowing constraints (Constantinides et al. (2001)), taxes (Mc Grattan and Prescott (2003)), loss aversion (Benartzi and Thaler (1995)), survivorship bias (Brown, Goetzmann and Ross (1995)), and heavy tails and parameter uncertainty (Weitzmann (2007)).

these factors in terms of the parameters of the problem.

We calibrate our model to the data of Mehra and Prescott (1985), and the model can explain these data with reasonable values for the parameters of the utility function. In addition to giving new insights about these interconnected puzzles, the model is likely to give many other results that are difficult, or impossible, to obtain using the conventional model.

Epstein and Zin (1989) and Weil (1989-90) study recursive intertemporal utility in discrete-time. This utility functions permit a certain degree of separation between substitution and risk aversion and has been used by several authors, like Campbell (1993) and Epstein and Zin (1991). To our knowledge this approach has not solved the problems that this paper addresses. Our analysis also gives a clear indication why this is so, including why Weil (1989) found that recursive utility leads to even larger values for the risk free interest rate than the conventional model (the so-called Risk-Free Rate Puzzle).

We rely on the recursive utility approach in Duffie and Epstein (1992a-b), a development that came after the paper by Weil (1989). While Duffie and Epstein (1992a) used dynamic programming to find risk premiums, we employ first principles (directional derivatives and utility gradients). It turns out that our solution, when calibrated to market data, is consistent with *early* resolution of uncertainty for very plausible values of the parameters.

Most developments that calibrate to market data use dynamic programming, in discrete time, or in continuous time. These models typically fit market data best at late resolution of uncertainty, and then only for relatively large values of the model parameters. It does not seem reasonable that the typical investor in the US stock market, for the 90-year period covered by the data of Mehra and Prescott (1985), prefers late to early resolution of uncertainty, is very risk averse, and requires a large compensation for consumption substitution.

This indicates that for recursive utility, where uncertainty is "dated" by the time of its resolution, and where the individual regards uncertainties resolving at different times as being different, the dynamic programming approach may be too restrictive, at least for the application to the market data that we have in mind.

Also we make use of dynamic programming, although in a cursory sense, namely to connect the volatility of the indirect utility to the volatility of the market portfolio. That our approach gives different results from those of Duffie and Epstein (1992a) is apparent, and emphasized by the following statement of the authors, who write: ..."It is a still unresolved empirical question whether recursive utility or one of the above alternatives, or perhaps a suitable composite, best explains and helps to organize the observed

behavior of consumption and asset returns”... Also, see Kocherlakota (1996), who maintains that the equity premium is still a puzzle.

There is by now a long standing literature that has been utilizing recursive preferences. We mention Avramov and Hore (2007), Avramov et. al. (2010), Eraker and Shaliastovich (2009), Hansen, Heaton, Lee, Roussanov (2007), Hansen and Scheinkman (2009), Wachter (2012), Bansal and Yaron (2004), Campbell (1996) and Bansal and Yaron (2004) to name some important contributions. Related work is also in Browning et. al. (1999), and on consumption see Attanasio (1999).

The paper is organized as follows: In Section 2 we explain the problems with the conventional model, and give a preview of the results. In Section 3 we present a brief introduction to recursive utility along the lines of Duffie and Epstein (1992a-b) and Duffie and Skiadas (1994), and set up the first order conditions of optimal consumption. In Section 4 we derive risk premiums for risky assets and in Section 5 we find the equilibrium interest rate. In Section 6 we connect the volatility of the future utility, a quantity in the representation of preferences in our approach, to the volatility of the market portfolio. In this section we offer an explanation why our results deviate from the rest of the extant literature. In Section 7 we tie the various pieces together, present our main results, and show how both the Equity Premium Puzzle and the Risk-Free Rate Puzzle can be resolved with good margin, for a representative agent who favors early to late resolution of uncertainty.

Section 8 provides a discussion of the implications of our results. Just to give an illustration of the variety of economic consequences of our findings, in this section we present an application to the economics of climate change. Section 9 concludes.

2 The problems with the standard model

2.1 The additive and separable Eu-model

The conventional asset pricing model in financial economics, the consumption-based capital asset pricing model (CCAPM) of Lucas (1978) and Breeden (1979), assumes a representative agent with a utility function of consumption that is the expectation of a sum, or a time integral, of future discounted utility functions. The model has been criticized for several reasons. First, it does not perform well empirically. Second, the standard specification of utility can not separate the risk aversion from the elasticity of intertemporal substitution, while it would clearly be advantageous to disentangle these two conceptually different aspects of preference. Third, while this representation

seems to function well in deterministic settings, and for *timeless* situations, it is not well founded for *temporal* problems (e.g., derived preferences may not satisfy the substitution axiom (Mossin (1969))).

In the conventional model the utility $U(c)$ of a consumption stream c_t is given by

$$U(c) = E \left\{ \int_0^T u(c_t, t) dt \right\} \quad (1)$$

where the felicity index u has the separable form

$$u(c, t) = \frac{1}{1 - \gamma} c^{1 - \gamma} e^{-\beta t}. \quad (2)$$

The parameter γ is the representative agent's relative risk aversion and β is the utility discount rate, or the impatience rate, and T is the time horizon. These parameters are assumed to satisfy $\gamma > 0$, $\beta \geq 0$, and $T \leq \infty$.

In this model the risk premium ($\mu_R - r$) of any risky security can be shown to have the simple form

$$\mu_R(t) - r_t = \gamma \sigma_{Rc}(t) \quad (3)$$

where r_t is the equilibrium real interest rate at time t , and the term $\sigma_{Rc}(t) = \sum_{i=1}^d \sigma_{R,i}(t) \sigma_{c,i}(t)$ is, by the Ito-isometry, the covariance rate between returns of the risky asset and the growth rate of aggregate consumption at time t , a measurable and adaptive process satisfying standard conditions. The dimension of the Brownian motion is $d > 1$. This is the continuous-time version of Breeden's consumption-based CAPM. Similarly, the expression for the risk-free real interest rate is

$$r_t = \beta + \gamma \mu_c(t) - \frac{1}{2} \gamma (\gamma + 1) \sigma'_c(t) \sigma_c(t). \quad (4)$$

The process $\mu_c(t)$ is the annual growth rate of aggregate consumption and $(\sigma'_c(t) \sigma_c(t))$ is the annual variance rate of consumption growths, both at time t , again dictated by the Ito-isometry. Both these quantities are measurable and adaptive stochastic processes, satisfying standard conditions. The return processes as well as the consumption growth rate process in this paper are also assumed to be ergodic processes, implying that statistical estimation makes sense.

Notice that in the model is the instantaneous correlation coefficient between returns and the consumption growth rate given by

$$\kappa_{Rc}(t) = \frac{\sigma_{Rc}(t)}{\|\sigma_R(t)\| \cdot \|\sigma_c(t)\|} = \frac{\sum_{i=1}^d \sigma_{R,i}(t) \sigma_{c,i}(t)}{\sqrt{\sum_{i=1}^d \sigma_{R,i}(t)^2} \sqrt{\sum_{i=1}^d \sigma_{c,i}(t)^2}},$$

	Expectation	Standard deviation
Consumption growth	1.83%	3.57%
Return S&P-500	6.98%	16.67%
Government bonds	0.80%	5.67%
Equity premium	6.18%	16.54%

Table 1: Key US-data for the time period 1889 -1978. $\hat{\sigma}_{Mc} = .003$

and similarly for other correlations given in this model. Here $-1 \leq \kappa_{Rc}(t) \leq 1$ for all t .

In Table 1 we reproduce from Mehra and Prescott (1985) the key summary statistics of the real annual return data related to the S&P-500, denoted by M , as well as for the annualized consumption data ². We have estimated the covariance $\sigma_{Mc}(t)$ directly from the data set to be $\hat{\sigma}_{Mc} = .003$.³ This gives the estimate $\hat{\kappa}_{Mc} = .5$ for the instantaneous correlation coefficient $\kappa(t)$.

Interpreting the risky asset as the value weighted market portfolio M corresponding to the S&P-500 index, we have two equation in two unknowns to provide estimates for the preference parameters by the method of moments. The result is

$$\gamma = 20.91 \quad \beta = -.08,$$

i.e., a relative risk aversion of about 20 and an impatience rate of *minus* 8%.

If we insist on a nonnegative impatience rate (as we should, but see Kocherlakota (1990)), this means that the real interest rate produced by the model is larger than than 9% (when $\beta = .01$, say) for the period considered, but it is estimated, as we see from Table 1, to be less than one per cent.

We denote the elasticity of intertemporal substitution in consumption by ψ , and refer to it as the EIS-parameter. In the standard model $\psi = 1/\gamma$, so if the risk aversion is as large as indicated in the above, it means that $\psi = .048$, which is too low for the average individual.

To better understand the problems with contemporary asset pricing, we propose to consider recursive utility along the lines of Duffie and Epstein (1992a-b), where these two quantities can be separated. This will have clear implications for risk premiums and the equilibrium interest rate, as we shall demonstrate.

²There are of course newer data by now, but these retain the same basic features. If we can explain the data in Table 1, we can explain any of the newer sets as well.

³The full data set was provided by Professor Rajnish Mehra.

2.2 Preview of our results

Let ρ the *time preference* of the individual. Our approach allows $\rho = 1/\psi$ to be different from risk aversion γ . Based on the analysis to follow, the two relationships corresponding to (3) and (4) are, with the same notation as above, given as follows:

$$\mu_R(t) - r_t = \rho \sigma_{Rc}(t) + (\gamma - \rho) \sigma_{RM}(t) \quad (5)$$

and

$$r_t = \beta + \rho \mu_c(t) - \frac{1}{2} \rho (\rho + 1) \sigma'_c(t) \sigma_c(t) + \rho (\rho - \gamma) \sigma_{cM}(t) + \frac{1}{2} (\rho - \gamma) (1 - \rho) \sigma'_M(t) \sigma_M(t). \quad (6)$$

respectively.

The risk premiums in (5) are endogenously derived and the same is true for the expression for the equilibrium interest rate. Here $\sigma_M(t)$ signifies the volatility of the return on the market portfolio of the risky securities, and $\sigma_{RM}(t)$ is the instantaneous covariance of the returns on the risky asset with the return of the market portfolio, and $\sigma_{cM}(t)$ is the instantaneous covariance between consumption growths and the return on the market portfolio. In the model all these quantities are measurable, adaptive stochastic processes satisfying standard conditions.

The risk premium of any risky asset in (5) is seen to be a linear combination of the market-based CAPM of Mossin (1966) and the consumption-based CAPM of Breeden (1979). If $\gamma = \rho$ risk premiums reduce to those of the latter.

In order to demonstrate the rich structure of our model, we fix the impatience rate $\beta = .01$, and solve the two non-linear equations (5) and (6) using the data of Table 1, when $R = M$. The results are

$$\gamma = 2.83, \quad \rho = .68, \quad \text{and} \quad EIS = 1.47.$$

as the method of moment estimates for the remaining parameters in the recursive utility function. The low value of the time preference ρ (and corresponding high value of $\psi = 1/\rho$) indicates a representative agent who does not require too much compensation for consumption substitution in a deterministic world. The value 2.83 must be considered as reasonable for the relative risk aversion γ , in particular compared to the value of 20.

In other words, with these values of the preference parameters of the recursive-utility-representative-agent, the model can explain an equilibrium

interest rate and equity premium estimated to, respectively

$$\hat{r} = .0080 \quad \text{and} \quad (\hat{\mu}_M - \hat{r}) = .0618$$

for the consumption/market data used by Mehra and Prescott (1985), presented in Table 1. This is a solution of both the Equity Premium Puzzle of Mehra and Prescott (1985) as well as the Risk-Free Rate Puzzle of Weil (1989).

That the the risk premium can be large in our model is illustrated by the market based CAPM term, when $\gamma > \rho$, explaining the "missing link" of the risk premium in the CCAPM-specification (that has puzzled economists for more that 27 years now). The richer model allows a reconciliation of the data in Table 1. For this data set the CAPM-term is 0.0598 and the CCAPM-term is .00204 adding to the total of .0618, i.e., the CAPM accounts for 96.7% of the equity premium.

One challenge with the conventional model is that the interest rate is too high. It is lower in our model for several intuitive reasons: The second term in (6) containing μ_c contributes only with 1.4%, while it is of the order of 38% in the standard model (4). The precautionary savings term contributes with 29%, resulting in a difference of 9%. In our model this difference is 1.1%. In addition are the two last terms in (6) negative if $\gamma > \rho$ and $\rho < 1$, both of which we find plausible. This explains an equilibrium real interest rate of less than one per cent for the data of Table 1 with very reasonable parameters.

If $\rho = 0$ the model reduces to

$$\mu_R(t) - r_t = \gamma \sigma_{RM}(t), \quad r_t = \beta - \frac{\gamma}{2} \sigma'_M(t) \sigma_M(t).$$

The risk premium is that of the ordinary CAPM-type, while the interest rate is new. This version of the model corresponds "neutrality" of consumption transfers in some sense, to be explained later. Solving the two non-linear equations consistent with the data of Table 1, we obtain

$$\gamma = 2.22 \quad \text{and} \quad \beta = .039.$$

In the conventional model this simply gives risk neutrality, i.e., $\gamma = 0$, so this model gives a risk premium of zero, and a short rate of $r = \beta$.

When the instantaneous correlation coefficient $\kappa_{Mc}(t)$ between returns and the aggregate consumption growth rate is small, our model handles this situation much better than the conventional one. The extreme case when $\kappa_{Mc}(t) = 0$ is, for example, consistent with the solution presented above for $\rho = 0$, which is a reasonable one. If this is the case, the discrepancy between

the standard model and the present one is even more striking than when $\hat{\kappa}_{Mc} = .5$, as it is for the data. For example, in the hypothetical situation that $\kappa_{Mc}(t) = .01$, but the rest of the summary statistics are as in Table 1, the conventional model gives $\gamma = 1045$ and $\beta = 675$, while our model provides the solution $\beta = .01$, $\gamma = 2.82$, $\rho = .60$ corresponding to $EIS = 1.67$. The reason for this is that the second term in the equity premium is unaffected by $\kappa_{Mc}(t)$, and a decrease in $\kappa_{Mc}(t)$ only leads to a slight increase in the difference $(\gamma - \rho)$, while the expression for the interest rate merely depends on $\kappa_{Mc}(t)$ in the fourth term on the right-hand side, and otherwise changes slightly with ρ and $(\rho - \gamma)$.

Figure 1 illustrates the the feasible region in (ρ, γ) -space. For the conventional model it is the 45°-line shown ($\rho = \gamma$). For the recursive utility model it is all of the first quadrant, including the axes. The points above the 45°-line represent late resolution of uncertainty, the points below correspond to early resolution. As can be seen, both the calibration point ($\rho = .68, \gamma = 2.83, \beta = .01$) reported above and the point corresponding to the market based CAPM, called CAPM++ in the figure, ($\rho = 0, \gamma = 2.22, \beta = .039$), are in the *early resolution* part.

Estimates of the EIS-parameter seem difficult to obtain for several reasons, and the results will naturally depend on circumstances. In e.g., Dagsvik et. al. (2006) an estimate of this parameter is suggested to be in the range from 1 to 1.5.

The larger region for the (ρ, γ) -combinations permitted by our model is not a frivolous generalization of the conventional model. Numerous generalizations have been presented during the last 27 years without achieving any acceptable resolution. That the richer structure of the recursive model is a modest extension is demonstrated by the interpretations and plausible results yielded in our simple expressions. It is based on fundamental assumptions and axioms of rational behavior.

The risk premium of any risky asset is seen to depend on other risky assets through the volatility of the market portfolio, and the return rate on government bonds depends both on how aggregate consumption covariates with the stock market as well as the size the variance of the market portfolio.

Initially one would think that these features should be reflected also in the corresponding formulas in the conventional model, but at the outset it is hard to say if these aspects are internalized or not.

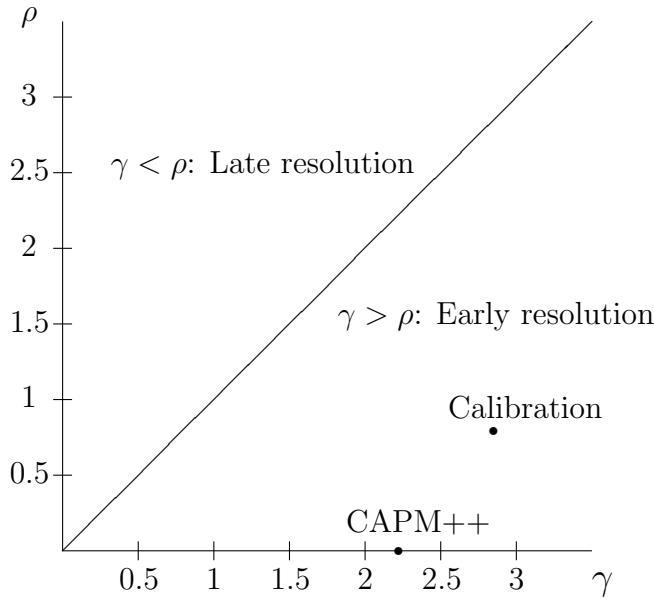


Figure 1: Time preference ρ as a function of risk aversion γ

3 Recursive Stochastic Differentiable Utility

3.1 Specification of the utility

In this section we give a brief introduction to recursive, stochastic, differentiable utility along the lines of Duffie and Epstein (1992a-b) and Duffie and Skiadas (1994).

Despite the fact that the analysis naturally becomes more technically involved once we depart from the additive and separable framework of the expected utility representation, we obtain surprisingly simple and transparent results when we use the Kreps-Porteus specification for the felicity index. The issue of when uncertainty is resolved is an important one in this theory, as Figure 1 illustrates.

We are given a probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, t \in [0, T], P)$ satisfying the usual conditions, and a standard model for the stock market with Brownian motion driven uncertainty, N risky securities and one riskless asset. Consumption processes are chosen from the space L of square integrable

progressively measurable processes with values in R_+ .

The stochastic differential utility $U : L \rightarrow R$ is defined as follows by two primitive functions: $f : L \times R \rightarrow R$ and $A : R \rightarrow R$.

The function $f(c_t, V_t)$ is a felicity index at time t , and A is a measure of absolute risk aversion of the Arrow-Pratt type for the agent. In addition to current consumption c_t , the felicity index also depends on future utility.

The utility process V for a given consumption process c , satisfying $V_T = 0$, is given by the representation

$$V_t = E_t \left\{ \int_t^T (f(c_s, V_s) - \frac{1}{2} A(V_s) \sigma_V(s)' \sigma_V(s)) ds \right\}, \quad t \in [0, T] \quad (7)$$

where E_t denotes conditional expectation given \mathcal{F}_t and $\sigma_V(t)$ is an R^d -valued square-integrable progressively measurable volatility process. Here d is the dimension of the Brownian motion B_t . We think of V_t as the remaining utility for c at time t , conditional on current information \mathcal{F}_t , and $A(V_t)$ is penalizing for risk.

If, for each consumption process c_t , there is a well-defined utility process V , the stochastic differential utility U is defined by $U(c) = V_0$, the initial utility. The pair (f, A) generating V is called an aggregator.

Since $V_T = 0$ and $\int \sigma_V(t) dB_t$ is a martingale, (7) has the stochastic differential equation representation

$$dV_t = \left(-f(c_t, V_t) + \frac{1}{2} A(V_t) \sigma_V(t)' \sigma_V(t) \right) dt + \sigma_V(t) dB_t \quad (8)$$

If terminal utility different from zero is of interest, like for life insurance, then V_T may be different from zero.

We think of A as associated with a function $h : R \rightarrow R$ such that $A(v) = -\frac{h''(v)}{h'(v)}$, where h is two times continuously differentiable. U is monotonic and risk averse if $A(\cdot) \geq 0$ and f is jointly concave and increasing in consumption. Bellman's characterization of optimality can be applied in such a way that state variables reflecting past consumption are unnecessary. The fact that past consumption does not matter, in the sense that the continuation utility V_t is independent of consumption prior to t , is a consequence of the prospective nature of (7).⁴

Stochastic differential utility disentangles intertemporal substitution from risk aversion: In the case of deterministic consumption, $\sigma_V(t) = 0$ for all t . Hence risk aversion A is then irrelevant, since it multiplies a zero variance.

⁴In the general case $A(\cdot)$ is associated with a local gradient representation (LGR) $M(v, x)$ of a certainty equivalent m . When $m = h^{-1}(E[h(V)])$ for h a von Neumann-Morgenstern index, then $A(V_t) = -M_{1,1}(V_t, V_t)$ where $M_{1,1}(v, x) = \partial^2 M(v, x) / \partial v^2$.

Thus certainty preferences, including the willingness to substitute consumption across time, are determined by f alone. Only risk attitudes are affected by changes in A for f fixed. In particular, if

$$\tilde{A}(\cdot) \geq A(\cdot)$$

where U and \tilde{U} are utility functions corresponding to (f, A) and (f, \tilde{A}) respectively, then \tilde{U} is more risk averse than U in the sense that any consumption process c rejected by U in favor of some deterministic process \bar{c} would also be rejected by \tilde{U} . Thus

$$U(c) \leq U(\bar{c}) \Rightarrow \tilde{U}(c) \leq \tilde{U}(\bar{c}). \quad (9)$$

Here it is important that $f(c_t, V_t)$ at the outset does not depend on risk aversion, only on time substitution. In contrast, assuming a CRRA-felicity index in the conventional model, the corresponding remaining utility at t is given by

$$V_t = E_t \left\{ \int_t^T \frac{1}{1-\gamma} c_s e^{-\beta(s-t)} c_s^{-\gamma} ds \right\}, \quad t \in [0, T].$$

When $\gamma \geq 1$ there is a reward for risk, and only when γ is small there will be a penalty for risk. However, it is not clear if these effects are caused by risk aversion or EIS. When c is deterministic, the only sensible interpretation is that of pure time substitution.

Examples.

1) The standard additive and separable utility has aggregator

$$\tilde{f}(c, v) = u(c) - \beta v, \quad \tilde{A} = 0$$

This can be shown by demonstrating that the standard utility function is a solution of the differential equation (8) for this specification of the aggregator. If c is a stochastic process, so is V_t .

If u has the usual properties, then we can also define the aggregator

$$f(c, v) = \beta \frac{u(c) - u(v)}{u'(v)}, \quad A(v) = -\frac{u''(v)}{u'(v)}. \quad (10)$$

By using Ito's lemma on $u(V_t)$ we obtain that the corresponding utility process V satisfies

$$V_t = u^{-1} \left(E_t \left\{ \beta \int_t^T u(c_s) e^{-\beta(s-t)} ds \right\} \right).$$

In particular the utility function \tilde{U} defined by (\tilde{f}, \tilde{A}) (the standard additive and separable one) and U defined by (f, A) are ordinally equivalent since

$u^{-1}(\cdot)$ is increasing, and thus represents the same preference ordering of consumption processes.

2) The Kreps-Porteus utility corresponds to the aggregator in (10) with the CES specification

$$f(c, v) = \frac{\beta}{1 - \rho} \frac{c^{1-\rho} - v^{1-\rho}}{v^{-\rho}} \quad \text{and} \quad A(v) = \frac{\gamma}{v} \quad (11)$$

so that $u(c) = \frac{c^{1-\rho}}{1-\rho}$ and $h(v) = \frac{v^{1-\gamma}}{1-\gamma}$. If, for example, $A(v) = 0$ for all v , this means that the recursive utility agent is risk neutral, but this situation is different from having $A = 0$ in 1).

Here $\rho \geq 0, \rho \neq 1, \beta \geq 0, \gamma \geq 0, \gamma \neq 1$ (when $\rho = 1$ or $\gamma = 1$ it is the logarithms that apply). The elasticity of intertemporal substitution in consumption $\psi = 1/\rho$. The parameter ρ is the time preference parameter referred to in Section 2.2. Here $u(\cdot)$ and $h(\cdot)$ are different functions, resulting in the desired disentangling of γ from ρ .

As for the standard additive utility, this utility function has an ordinally equivalent specification. When the aggregator (f, A) is given corresponding to the utility function U , there exists a strictly increasing and smooth function $\varphi(\cdot)$ such that the ordinally equivalent $\tilde{U} = \varphi \circ U$ has the aggregator (\tilde{f}, \tilde{A}) where

$$\tilde{f}(c, v) = ((1 - \gamma)v)^{-\frac{\gamma}{1-\gamma}} f(c, ((1 - \gamma)v)^{\frac{1}{1-\gamma}}), \quad \tilde{A} = 0.$$

Thus

$$\tilde{U} = \frac{1}{1 - \gamma} U^{1-\gamma}.$$

This is the specification we work with, where \tilde{f} has the CES-form

$$\tilde{f}(c, v) = \frac{\beta}{1 - \rho} \frac{c^{1-\rho} - ((1 - \gamma)v)^{\frac{1-\rho}{1-\gamma}}}{((1 - \gamma)v)^{\frac{1-\rho}{1-\gamma}-1}}, \quad \tilde{A}(v) = 0. \quad (12)$$

The corresponding utility \tilde{U} retains the essential features, namely that of disentangling intertemporal elasticity of substitution from risk aversion. The primary reason for working with the transformed version is that it leads to a manageable form for the Hamilton-Jacobi-Bellman equation.

From now on we use the simpler notation $(f, 0)$ (instead of the tildes) for the representation given in (12), and U for the corresponding utility function. An analysis based directly on (11) is carried out elsewhere.

3.2 The First Order Condition

The representative agent's problem is to solve

$$\sup_{\tilde{c} \in L} U(\tilde{c})$$

subject to

$$E\left\{ \int_0^T \tilde{c}_t \pi_t dt \right\} \leq E\left\{ \int_0^T c_t \pi_t dt \right\}.$$

The Lagrangian for the problem is given by

$$\mathcal{L}(\tilde{c}; \lambda) = U(\tilde{c}) - \lambda E\left(\int_0^T \pi_t (\tilde{c}_t - c_t) dt \right)$$

In order to find the first order condition for the representative consumer's problem, we use Kuhn-Tucker and directional derivatives in function space. The problem is well posed since U is increasing and concave and the constraint is convex. In maximizing the Lagrangian of the problem, we calculate the directional derivative $\nabla U(c; h)$, which equals $(\nabla U(c))(h_t)$ where $\nabla U(c)$ is the gradient of U at c .

Since U is continuously differentiable, this gradient is a linear and continuous functional, and thus, by the Riesz representation theorem, it is given by an inner product. By Duffie and Skiadas (1994) this utility gradient is

$$(\nabla U(c))(h_t) = E\left(\int_0^T Y_t \frac{\partial f}{\partial c}(c_t, V_t) h_t dt \right). \quad (13)$$

where

$$Y_t = \exp\left(\int_0^t \frac{\partial f}{\partial v}(c_s, V_s) ds \right) \quad a.s. \quad (14)$$

The first order condition is that the directional derivative of the Lagrangian is zero at the optimal c_t in all directions $h \in L$:

$$\nabla \mathcal{L}(c, \lambda; h) = 0 \quad \text{for all } h \in L$$

This is equivalent to

$$E\left\{ \int_0^T (Y_t \frac{\partial f}{\partial c}(c_t, V_t) - \lambda \pi_t) h_t dt \right\} = 0 \quad \text{for all } h \in L. \quad (15)$$

The result is that for the Riesz-representation of the gradient of U to be equal to the state price deflator π_t it is necessary and sufficient that

$$\lambda \pi_t = Y_t \frac{\partial f}{\partial c}(c_t, V_t) \quad a.s. \quad (16)$$

For the representative agent the optimal consumption process is the given aggregate consumption c in society, and for this c the remaining utility V_t at time t is optimal. We now turn to risk premiums using this result.

4 Risk Premiums for Recursive Utility

We start by observing that the Riesz-representation for stochastic differential utility for the representative agent is equal to the state price π_t provided

$$\pi_t = Y_t f_c(c_t, V_t). \quad (17)$$

Aggregate consumption is exogenous, with dynamics on of the form

$$\frac{dc_t}{c_t} = \mu_c(t) dt + \sigma_c(t) dB_t, \quad (18)$$

where $\mu_c(t)$ and $\sigma_c(t)$ are measurable, \mathcal{F}_t adapted stochastic processes, satisfying appropriate integrability properties. This is also assumed for processes representing returns. In addition we assume these processes to be ergodic, so that we may replace time averages by state averages.

Similarly the process V_t follows the dynamics

$$\frac{dV_t}{(1-\gamma)V_t} = \mu_V(t) dt + \sigma_V(t) dB_t \quad (19)$$

where

$$\mu_V(t) = -\frac{\beta}{1-\rho} \left(\frac{c_t^{1-\rho} - ((1-\gamma)V_t)^{\frac{1-\rho}{1-\gamma}}}{((1-\gamma)V_t)^{\frac{1-\rho}{1-\gamma}-1}} \right).$$

From the FOC (17) we then get the dynamics of the state price deflator:

$$d\pi_t = f_c(c_t, V_t) dY_t + Y_t df_c(c_t, V_t). \quad (20)$$

Using Ito's lemma this becomes

$$\begin{aligned} d\pi_t &= Y_t f_c(c_t, V_t) f_v(c_t, V_t) dt + Y_t \frac{\partial f_c}{\partial c}(c_t, V_t) dc_t + Y_t \frac{\partial f_c}{\partial v}(c_t, V_t) dV_t \\ &+ Y_t \left(\frac{1}{2} \frac{\partial^2 f_c}{\partial c^2}(c_t, V_t) (dc_t)^2 + \frac{\partial^2 f_c}{\partial c \partial v}(c_t, V_t) (dc_t)(dV_t) + \frac{1}{2} \frac{\partial^2 f_c}{\partial v^2}(c_t, V_t) (dV_t)^2 \right). \end{aligned} \quad (21)$$

Here

$$\begin{aligned} f_c(c, v) &:= \frac{\partial f(c, v)}{\partial c} = \frac{\beta c^{-\rho}}{((1-\gamma)v)^{\frac{1-\rho}{1-\gamma}-1}}, \\ f_v(c, v) &:= \frac{\partial f(c, v)}{\partial v} = \frac{\beta}{1-\rho} \left(c^{1-\rho} ((1-\gamma)v)^{-\frac{1-\rho}{1-\gamma}} (\rho - \gamma) + (\gamma - 1) \right), \\ \frac{\partial f_c(c, v)}{\partial c} &= -\frac{\beta \rho c^{-\rho-1}}{((1-\gamma)v)^{\frac{1-\rho}{1-\gamma}}}, \quad \frac{\partial f_c(c, v)}{\partial v} = \beta(\rho - \gamma) c^{-\rho} ((1-\gamma)v)^{-\frac{1-\rho}{1-\gamma}}, \end{aligned}$$

$$\frac{\partial^2 f_c}{\partial c^2}(c, v) = \frac{\beta \rho (1 + \rho) c^{-\rho-2}}{((1 - \gamma)v)^{\frac{1-\rho}{1-\gamma}-1}}, \quad \frac{\partial^2 f_c}{\partial c \partial v}(c, v) = \frac{\rho \beta (\gamma - \rho) c^{-\rho-1}}{((1 - \gamma))^{\frac{1-\rho}{1-\gamma}}},$$

and

$$\frac{\partial^2 f_c}{\partial v^2}(c, v) = \frac{\beta (\gamma - \rho) (1 - \rho) c^{-\rho}}{((1 - \gamma)v)^{\frac{1-\rho}{1-\gamma}+1}}.$$

Denoting the dynamics of the state price deflator by

$$d\pi_t = \mu_\pi(t) dt + \sigma_\pi(t) dB_t, \quad (22)$$

from (21) and the above expressions we now have that the drift and the diffusion terms of π_t are given by

$$\begin{aligned} \mu_\pi(t) = & Y_t \left(\frac{\beta^2}{1 - \rho} (\rho - \gamma) c_t^{2(1-\rho)-1} ((1 - \gamma)V_t)^{-\frac{2(1-\rho)}{1-\gamma}+1} \right. \\ & - \frac{(1 - \gamma)\beta^2}{1 - \rho} c^{-\rho} ((1 - \gamma)V_t)^{-\frac{1-\rho}{1-\gamma}+1} - \beta \rho c_t^{-\rho} ((1 - \gamma)V_t)^{-\frac{1-\rho}{1-\gamma}+1} \mu_c(t) \\ & - \beta c_t^{-\rho} (\rho - \gamma) ((1 - \gamma)V_t)^{-\frac{1-\rho}{1-\gamma}} f(c_t, V_t) + \frac{1}{2} \beta \rho (1 + \rho) c_t^{-\rho} ((1 - \gamma)V_t)^{-\frac{1-\rho}{1-\gamma}+1} \sigma'_c(t) \sigma_c(t) \\ & - \beta \rho c_t^{-\rho} (\rho - \gamma) ((1 - \gamma)V_t)^{-\frac{1-\rho}{1-\gamma}+1} \sigma_{cV}(t) \\ & \left. - \frac{1}{2} \beta (\rho - \gamma) (1 - \rho) c_t^{-\rho} ((1 - \gamma)V_t)^{-\frac{1-\rho}{1-\gamma}+1} \sigma'_V(t) \sigma_V(t) \right), \quad (23) \end{aligned}$$

and

$$\begin{aligned} \sigma_\pi(t) = & Y_t \beta c_t^{-\rho} \left((-\rho) \sigma_c(t) ((1 - \gamma)V_t)^{-\frac{1-\rho}{1-\gamma}+1} + \right. \\ & \left. (\rho - \gamma) ((1 - \gamma)V_t)^{-\frac{1-\rho}{1-\gamma}} ((1 - \gamma)V_t) \sigma_V(t) \right) \quad (24) \end{aligned}$$

respectively.

The risk premium is in general given by

$$\mu_R(t) - r_t = -\frac{1}{\pi_t} \sigma_{R\pi}(t), \quad (25)$$

where $\sigma_{R\pi}(t)$ is the instantaneous covariance of the increments of R and π . Interpreting π_t as the price of the consumption good at time t , by the first order condition it is a decreasing function of consumption c since $f_{cc} < 0$. So in "good times" when consumption is high, the state price is low and returns are high for a typical security. Accordingly the covariance rate is negative, accounting for the minus sign.

Furthermore when π_t is low (good times) then the typical risk premium is high, since the state price appears in the denominator. For securities that work as an "insurance" of consumption, just the opposite conclusions hold.

Combining the FOC in (17) with the above result (24), the formula for the risk premium in terms of the primitives of the model is accordingly given by

$$\mu_R(t) - r_t = \rho \sigma_{Rc}(t) + (\gamma - \rho) \sigma_{RV}(t). \quad (26)$$

If $\sigma_V(t) = \sigma_M(t)$, where $\sigma_M(t)$ is the volatility of the return of the value-weighted market portfolio at time t , the intertemporal model is a linear combination of the CCAPM and the classical CAPM. It reduces to the latter when $\rho = 0$ and to the former when $\rho = \gamma$.

When $\rho = 0$ the CAPM-term explains all of the risk premium, in which case the utility function u of consumption in the CES-specification of the felicity index f is of the form $u(c) = c$. This corresponds to neutrality with respect to consumption transfers.

When $\rho \neq \gamma$ the latter term in (26) may be positive or negative. The most reasonable situation for the data in Table 1 is when $\gamma > \rho$, corresponding to early resolution of uncertainty. This results in a higher equilibrium risk premium than for the conventional model, as we shall demonstrate.

We return to a discussion about the volatility term $\sigma_V(t)$ later. Before we do that, we give an expression for the equilibrium interest rate r_t in terms of $\sigma_V(t)$.

5 The equilibrium interest rate

The equilibrium interest rate r_t is given by the general formula

$$r_t = -\frac{\mu_\pi(t)}{\pi_t}. \quad (27)$$

The real interest rate at time t can be thought of as the expected exponential rate of decline of the representative agent's marginal utility, which is π_t in equilibrium.

In order to find an expression for r_t in terms of the primitives of the model, we use the formula for $f(c_t, V_t)$ from (12) in the expression for $\mu_\pi(t)$ in (23). We then obtain the following

$$r_t = \beta + \rho \mu_c(t) - \frac{1}{2} \rho (\rho + 1) \sigma_c'(t) \sigma_c(t) + \rho (\rho - \gamma) \sigma_{cV}(t) + \frac{1}{2} (\rho - \gamma) (1 - \rho) \sigma_V'(t) \sigma_V(t). \quad (28)$$

For the standard utility ($\rho = \gamma$) this reduces to the familiar expression in (4). We observe that it is the time substitution interpretation that is the meaningful one in this new setting for terms two and three on the right hand side. For example, for a value of $\rho < 1$ this may give a low value for the interest rate. First and foremost it is the term related to the growth rate of consumption that now contributes to this lower value. The "precautionary savings" term also works in the right direction since it is negative because this representative agent is also "prudent", but is likely to be relatively small in magnitude. Notice that the concept of "prudence" is now linked to the time preference parameter ρ rather than the risk aversion γ . The two last terms are negative provided $\rho < \gamma$ and $\rho \leq 1$. These relationships between the parameters all seem reasonable.

In order to link the volatility term $\sigma_V(t)$ to an observable (or estimable) quantity in the market, we now specify a model for the financial market.

6 The Financial Market

In this section we present a model for the financial economy along the lines of Cox, Ingersoll, and Ross (1985), who use dynamic programming to find equilibrium. This model has been extended to the case of recursive utility by Duffie and Epstein (1992a-b).

The model requires a Markov state process K satisfying the stochastic differential equation

$$\frac{dK_t}{K_t} = \mu_K(K_t, t)dt + \sigma(K_t, t)dB_t \quad (29)$$

where μ_K and σ_K satisfy standard technical conditions. One interpretation is that K is capital in a production model, in which $K\mu_K$ is the production function. The dimension of the vector σ_K is N , which can also be equal to the dimension d of the Brownian motion B when $N > 1$.

In state k at time t , let $\nu(k, t) \in R^N$ denote the vector of expected rates of return of the N given risky securities in excess of the riskless instantaneous return r_t , and let $\sigma(k, t)$ denote the $N \times N$ matrix of diffusion coefficients of the risky asset prices, normalized by the asset prices, so that $\sigma(k, t)\sigma(k, t)'$ is the instantaneous covariance matrix for asset returns. The combined state process is then (K, W) where W is wealth.

The representative consumer's problem is, for each initial level (k, w) of the state variables to solve

$$\sup_{(c, \varphi)} U(c) \quad (30)$$

subject to the intertemporal budget constraint

$$dW_t = (W_t(\varphi'_t \cdot \nu(K_t, t) + r_t) - c_t)dt + W_t\varphi'_t \cdot \sigma(K_t, t)dB_t, \quad (31)$$

Here $\varphi'_t = (\varphi_t^{(1)}, \varphi_t^{(2)}, \dots, \varphi_t^{(N)})$ are the fractions of total wealth W_t held in the risky securities.

Duffie and Epstein (1992b) establish that the preference ordering represented by recursive utility is *time consistent* in the sense of Johnsen and Donaldson (1985). (The utility function considered by Kreps and Porteus (1978) is time consistent by construction, by their Axiom 3.1.) In the present setting it is claimed that state variables reflecting past consumption are unnecessary, and therefore they proceed with the dynamic programming approach.

According to this idea is the first order condition for the problem (30)-(31) given by the generalized Bellman equation:

$$\sup_{(c, \varphi)} \{ \mathcal{D}^{(c, \varphi)} J(w, k, t) + f(c, J(w, k, t)) \} = 0, \quad (32)$$

with boundary condition

$$J(w, k, T) = 0, \quad w > 0, k > 0, \quad (33)$$

where the differential operator $\mathcal{D}^{(c, \varphi)}$ is given by

$$\begin{aligned} \mathcal{D}^{(c, \varphi)} J(w, k, t) &= J_w(w, k, t)(w\varphi \cdot \nu + rw - c) + J_k k\mu_K + J_t(w, x, t) \quad (34) \\ &+ \frac{w^2}{2} \varphi' \cdot (\sigma \cdot \sigma') \cdot \varphi J_{ww}(w, k, t) + \frac{1}{2} J_{kk}(w, k, t) k^2 \sigma'_K \sigma_K \\ &+ J_{wk}(w, k, t) wk \varphi \sigma \sigma_K. \end{aligned}$$

The function $J(w, x, t)$ is the indirect utility function of the representative consumer at time t when the wealth $W_t = w$, and the state $K_t = k$, and represents future expected utility at time t in "state" (w, k) , provided the optimal portfolio choice strategy is being followed from this time on. Thus $J = V$ in optimum.

6.1 The Consumption/Portfolio Choice: A special case

Explicit solutions to problems of this kind are hard to derive, and few are known in the literature. In order to obtain an estimate of the volatility term σ_V , for the moment we simplify the problem as follows: We omit the state variable K and let the instantaneous covariance matrix of the risky securities be a constant matrix. The differential operator $\mathcal{D}^{(c, \varphi)}$ then simplifies to

$$\begin{aligned} \mathcal{D}^{(c, \varphi)} J(w, t) &= J_w(w, t)(w\varphi \cdot \nu + rw - c) + J_t(w, t) \quad (35) \\ &+ \frac{w^2}{2} \varphi' \cdot (\sigma \cdot \sigma') \cdot \varphi J_{ww}(w, t). \end{aligned}$$

It is a consequence of Ito's lemma, since $J = J(W_t, t)$ is a function of wealth, that J is also an Ito process with dynamics given by

$$dJ(W_t, t) = \mu_J(t) + \sigma_J(t)dB_t \quad (36)$$

where the diffusion term is

$$\sigma_J(t) = J_W W_t \varphi'_t \cdot \sigma \quad (37)$$

This means that $\tilde{\sigma}_V(t) = J_w W_t \varphi'_t \cdot \sigma$, where $\tilde{\sigma}_V(t)$ is the diffusion term of V . It remains to find J_w as well as φ_t . In order to do this, we have to solve the generalized HJB-equation.

The first order condition for c is:

$$J_w = f_c$$

implying that

$$c_t = \left(\frac{1}{\beta} ((1 - \gamma)J)^{\frac{1-\rho}{1-\gamma}-1} J_w \right)^{-\frac{1}{\rho}}$$

The first order conditions for φ are:

$$\varphi_t = \left(- \frac{J_w}{J_{ww} w} \right) (\sigma \sigma')^{-1} \nu \quad (38)$$

Attempting a solution of the form $J(w, t) = \frac{1}{\theta} w^\theta k(t)$, for some θ that may depend on ρ and γ in some way, separation of wealth from time works successfully here when $\theta = (1 - \gamma)$. Employing an analogue of *the verification theorem* of optimal control theory (see e.g., Øksendal (2003), Ch 11), established in Proposition 9 of Duffie and Epstein (1992b), shows that we have found *the* solution. Thus $J_w = k(t) w^{-\gamma}$, the optimal consumption c_t is

$$c_t = \left(\frac{1}{\beta} k(t)^{\frac{1-\rho}{(1-\gamma)}} \right)^{-\frac{1}{\rho}} W(t)$$

where the function $k(t)$ satisfies an ordinary, first order differential equation in t . Also⁵

$$\varphi_t = \frac{1}{\gamma} (\sigma \sigma')^{-1} \nu. \quad (39)$$

Market clearing requires that $(\varphi_t)' \sigma = (\delta_t^M)' \sigma = \sigma_M(t)$ in equilibrium, where σ_M is the volatility of the return on the market portfolio, and δ_t^M are the fractions of the different securities, $j = 1, \dots, N$ held in the value-weighted

⁵This is the standard result of optimal portfolio choice, first established by Mossin (1968).

market portfolio. That is, the representative agent can only hold the market portfolio in equilibrium, by definition, and must consume the aggregate consumption.

This implies that $\tilde{\sigma}_V(t) = \sigma_J(t) = W(t)^{-\gamma} k(t) W_t \sigma_M(\omega)$, so that

$$\sigma_V(t) = \frac{\tilde{\sigma}_V(t)}{(1-\gamma)V_t} = \frac{\tilde{\sigma}_V(t)}{(1-\gamma)J(W(t))} = \frac{W(t)^{-\gamma} k(t) W_t \sigma_M(t)}{(1-\gamma) \frac{1}{(1-\gamma)} W_t^{(1-\gamma)} k(t)},$$

taking into account the transformation (19). As a consequence

$$\sigma_V(t) = \sigma_M(t). \quad (40)$$

This is the link to the market based quantity that we conjectured. This result should at least indicate that the connection (40) may be reasonably accurate also in a wider context.

As equation (39) indicates, however, this particular financial market model is not necessarily consistent with equilibrium, for the same reason as for the conventional model (see e.g., Duffie (2001), p 210).

6.2 A two factor model for the equity premium

In this subsection we present an additional argument why $\sigma_V(t) = \sigma_M(t)$ for (almost) all $t \in [0, T]$ in a model which is consistent with equilibrium.

Returning to the model for the financial market in the beginning of Section 6, since we are primarily concerned with the equity premium (and the risk free interest rate) in this article, we can assume without loss of generality than there is only one risky asset - the index - so that $N = 1$ (but $d > 1$).

Using the generalized Bellman approach indicated in the above, the first order condition for optimal portfolio choice gives for the optimal demand of the risky asset

$$W_t \varphi_t = \left(- \frac{J_w(W_t, K_t, t)}{J_{ww}(W_t, K_t, t)} \right) \left(\frac{\mu(K_t, t) - r_t}{\sigma(K_t, t)\sigma(K_t, t)} \right) + \left(- \frac{J_{kw}(W_t, K_t, t)K_t}{J_{ww}(W_t, K_t, t)} \right) \left(\frac{\sigma(K_t, t)\sigma_K(K_t, t)}{\sigma(K_t, t)\sigma(K_t, t)} \right). \quad (41)$$

where $\mu(k, t) - r = \nu(k, t)$. The representative agent is initially endowed with one share of "the firm", in which case the market clearing condition is $\varphi_t = 1$ a.s. for all t , so that $\varphi\sigma = \sigma_M$ and $\mu = \mu_M$. Notice that this gives a different demand than what follows from (38) for obvious reasons. From the

expression (41) the equity premium can be written

$$\begin{aligned} \mu_M(K_t, t) - r_t = & \left(- \frac{J_{ww}(W_t, K_t, t)W_t}{J_w(W_t, K_t, t)} \right) \sigma_M(K_t, t) \sigma_M(K_t, t) \\ & + \left(- \frac{J_{wk}(W_t, K_t, t)K_t}{J_w(W_t, K_t, t)} \right) \sigma_{MK}(K_t, t), \end{aligned} \quad (42)$$

which is a two-factor model. Assuming $c_t = c(W_t, K_t, t)$ for some smooth function $c : R_+^3 \rightarrow R_+$, by Ito's lemma the diffusion function for consumption can be written

$$\tilde{\sigma}_c(W_t, K_t, t) = c_k \sigma_K K_t + c_w \sigma_M W_t, \quad (43)$$

where c_k and c_w signify partial derivatives of c with respect to k and w respectively.

By (35) the first order condition for optimal consumption choice is again $J_w = f_c$. Differentiating with respect to w gives $J_{ww} = f_{cc}c_w + f_{cw}J_w$ and with respect to k gives $J_{wk} = f_{cc}c_k + f_{cv}J_k$. Under homothetic preference J is homogeneous with respect to wealth so that $J(w, k, t) = h(k, t)w^\theta$ for some h and θ . This gives the connection $f_{cc} = (\theta/w - f_{cw})$. Using (43) and noticing that $\tilde{\sigma}_c(W_t, K_t, t) = c_t \sigma_c(t)$ of (18) in Section 4, where $\sigma_c(t)$ is here constrained to only depend on state variables at time t , the last term in (42) becomes a linear combination of $\sigma_M \sigma_M$ and σ_{Mc} . In this situation, when $\theta = \rho$, Epstein and Duffie (1992b) obtain the following expression for the equity premium

$$\mu_M(K_t, t) - r_t = a \sigma_{Mc}(t) + b \sigma_M(K_t, t) \sigma_M(K_t, t) \quad (DE)$$

for some a and b depending on the parameters γ and ρ ($a = \rho(1 - \gamma)/(1 - \rho)$ and $b = (\gamma - \rho)/(1 - \rho)$).

First, since this model of a financial market is consistent with equilibrium, a comparison with (26) for $R = M$ confirms that $\sigma_V(t) = \sigma_M(t)$.

Second, the functional forms of a and b only coincide with the coefficients we have if $\rho = \gamma$, or $\rho = 0$ i.e., for the conventional model or the pure CAPM. Comparing the first order conditions of our approach given in (17) with the corresponding $f_c = J_w$ in dynamic programming, we see that this is not really surprising. Our FOC depends on past consumption as well as past values of utility along the optimal path, while this is not the case for dynamic programming, which is based on Markov processes uncertainty. This suggests that with recursive utility the Markov structure of uncertainty revelation may not be rich enough to capture the fine points of the theory. A change of the state space can often bring a non-Markovian process into a Markov process. This may be of particular interest in the discrete time framework (e.g., Kreps and Porteus (1979)).

Since our approach is the more general of the two, and works for a wide range of concave and increasing utility function including, for example, for both recursive utility and habit formation, it should be clear that this method gives the solution to the problem if there is any discrepancies between the two methods. The dynamic programming approach is valid under more restrictive assumptions than our method.

Dynamic programming sometimes provides the correct solution to the problem with recursive utility, for example when $\rho = 0$ and $\gamma > 0$ which is the CAPM++ model. It also coincides with our solution for the conventional model where $\rho = \gamma$, as it should.

In general, our representative agent is prepared for a situation where uncertainty related payoffs may partially or completely resolve on or before any time t , and the individual may prefer earlier or later resolution of this uncertainty. Our solution, when calibrated to the data in Table 1, is consistent with early resolution as we have seen. There are also late resolution solutions. For example, with respect to the calibrations in the next section, we have a situation where $\beta = .03, \gamma = 2.40, \rho = .20$. This is a solution of two nonlinear equations in two unknowns, namely γ and ρ , keeping the impatience rate $\beta = .03$ fixed. This system of equations also has another solution, which is $\rho = 23.96, \gamma = 23.64$. This one corresponds to *late* resolution of uncertainty. Because the risk aversion as well as the time preference are both rather large, this solution we consider to be less plausible than the one with early resolution. In our application of the model with data from the securities market from 1889 to 1978, we expect the average investor to be anxious to be informed as *early* as possible.

When inserting the values for the late resolution in the dynamic programming version (*DE*), the model provides an equity premium of .07 which is, in this connection, not too far from the observed .062. Typically (*DE*) does not seem to be consistent with early resolution for the data of Table 1. For most of the other results presented in the literature on recursive utility, primarily using discrete time models and applied to the market data that give rise to the equity premium puzzle, the solutions point in the direction of late resolution of uncertainty (e.g., Epstein and Zin (1991), Weil (1989)). In Weil (1989), for example, large values of gamma together with even larger values of ρ seem to better fit the data, but the values of ρ and γ are both very large.

For temporal problems the use of standard methods may easily give results that can be wrong (e.g., Mossin (1969), Kreps (1988)). To be more concrete, recall the basic definition of a stochastic integral:

$$\int_0^t f(s)dB_s = \text{"lim"} \sum_{t_i} f(s_i)(B(t_{i+1}) - B(t_i)),$$

as the limit in some sense (L^2 or probability) of a discrete sum, where $s_i \in [t_i, t_{i+1}]$. Ito's choice is obtained when $s_i = t_i$, and unlike for ordinary Lebesgue-Stieltjes integrals, this choice matters. For example, if s_i is chosen in the middle of the interval, the so-called Stratonovich integral results, which is another object than the standard Ito integral. Next consider dynamic programming, and recall how the Bellman equation is derived. We fix a time point t , run any control between t and $t + \Delta t$ and then switch to the optimal control thereafter. After applying the principle of optimality, one is faced with, among other things, a stochastic integral from t to $t + \Delta t$ of the derivative of the indirect utility function times a volatility. The final form of the Bellman equation now hinges upon the fact that the conditional expectation of this integral is zero for all $\Delta t > 0$, and then one lets Δt approach zero. For this to hold Ito's choice is the essential ingredient, i.e., the uncertainty dB_t has to "stick out into the future" for the procedure to be valid.

The point we are trying to make is the following: The first order condition derived this way depends in a "stiff" way on how uncertainty appears (always the same), and when it is realized (always late). This may not be compatible with the rather rich structure of recursive utility. Using this approach here is as if there is an implicit bias towards late resolution. With Stratonovich's choice, for example, the resulting Bellman equation will be different, after proper adjustments for the drifts. Now, however, the solution is presumably one where the agent is biased to be indifferent to the resolution times of uncertainty, just as for the standard model, again in a rigid manner. If $s_i = t_{i+1}$ is the convention, early resolution of uncertainty is favored, again a different solution. This is not the situation with utility gradients. Here the principle works for any kind of analysis, so long as we properly adjust for the drift term according to which choice is made for the stochastic integral in the subsequent analysis.

In short, it seems as if the continuous time calculus using the Bellman equation favors late resolution of uncertainty for equilibrium with recursive utility, when applied to the data of Table 1. If this 'constraint' is not binding, the two types of analysis give the same answer. We have pointed out two corner solutions where this is the case, when $\rho = 0$, or for the standard model where $\rho = \gamma$. There may be others.

7 The Final Formulation of The Model

Returning to our earlier expression given in (26) for the risk premium of any risky asset having return rate $\mu_R(t)$ and volatility of return $\sigma_R(t)$, and the

equilibrium interest rate given in (28), we can now formulate our main result:

Theorem 1 *In the model specified in sections 3-6, there exists an equilibrium in which risk premiums of risky assets and the real interest rate are given by*

$$\mu_R(t) - r_t = \rho \sigma_{Rc}(t) + (\gamma - \rho) \sigma_{RM}(t). \quad (44)$$

and

$$r_t = \beta + \rho \mu_c(t) - \frac{1}{2} \rho (\rho + 1) \sigma'_c(t) \sigma_c(t) + \rho (\rho - \gamma) \sigma_{cM}(t) + \frac{1}{2} (\rho - \gamma) (1 - \rho) \sigma'_M(t) \sigma_M(t). \quad (45)$$

respectively, where ρ is the time preference, γ the relative risk aversion, and β the impatience rate.

As claimed risk premiums in the resulting model are linear combinations of the consumption-based CAPM and the market-based CAPM at each time $t \in [0, T]$.

Table 2 illustrates additional parameter values to the ones presented in Section 2.2 for the recursive utility model consistent with the data of Table 1. The "Kelly Criterion" means logarithmic utility in the standard model, which here corresponds to $\gamma = 1$ ⁶. This gives a negative value for ρ which is not plausible, since we really require that $\rho \geq 0$. Thus a relative risk aversion of $\gamma = 1$ is too low for our model to be consistent with the Mehra and Prescott-study, which is interesting, but perhaps not surprising.

All the value sets presented in Table 2 represent exact fits to the data of Table 1. The CAPM++ version has acceptable values for risk aversion and the impatience rate, as we have seen before. By CAPM++ in Table 2 is meant the current version in continuous time, with an associated level of interest rate attached, and based on recursive utility. The original equilibrium model developed by Jan Mossin was in a one period (timeless) setting with consumption only on the terminal time point, in which case wealth equals consumption. Since there was no consumption on the initial time point, no intertemporal aspects of consumption transfers arose in the classical model. This naturally corresponds to $u(c) = c$ for the the felicity index regarding consumption transfers, meaning $\rho = 0$ and $\psi = 1/\rho = +\infty$, and corresponding to perfect substitutability of consumption across time.

All the plausible calibration points are in the early resolution part of the (ρ, γ) -plane where $\gamma > \rho$, which is not surprising given the data in Table 1.

⁶Although $\gamma = 1$ is formally not in the permissible range for this parameter, and really require $h(v) = \ln(v)$, still it works fine in the above.

	γ	ρ	EIS	β
Standard Model	20.91	20.91	.0478	-.08
$\beta = 0.010$ fixed	2.83	.68	1.47	.01
$\beta = 0.020$ fixed	2.61	.44	2.27	.02
$\beta = 0.030$ fixed	2.40	.20	4.84	.03
$\beta = 0.035$ fixed	2.30	.09	11.11	.035
$\rho = 1.00$ fixed	3.20	1.00	1.00	-.003
$\rho = .83$ fixed	2.97	.83	1.2	.004
$\rho = .50$ fixed	2.57	.50	2.0	.016
$\rho = 24.0$ fixed	23.67	24	.042	.031
CAPM ++	2.22	0.00	$+\infty$.039
The Kelly Criterion	1.00	-1.34	-.74	.101
$\gamma = 2.25$ fixed	2.25	.029	34.14	.038
$\gamma = 2.50$ fixed	2.50	.309	3.23	.026
$\gamma = 3.05$ fixed	3.05	.924	1.08	.0002

Table 2: Various Specifications Consistent with Table 1 where $\hat{\kappa}_{Mc} = .5$

It is here that our results deviate from earlier research on recursive utility applied to explaining the historical equity premium and the interest rate. However, corresponding to the value $\beta = .031$ for example, as pointed out in Section 6, there is also the solution $\rho = 24$, ($\psi = 0.042$) and $\gamma = 23.67$ which is a *late* resolution equilibrium (since $\rho > \gamma$). Although this risk aversion as well as the time preference are both too large to be considered reasonable, in fact worse than the risk aversion provided by the conventional model (as observed by Weil (1989)), this equilibrium corresponds to a more plausible value of the impatience rate than does the conventional solution (where $\beta = -.08$).

8 Discussion

In this section we briefly discuss some of the new features of our model.

First, it is really reassuring that the risk premium of any risky asset depends on other investment opportunities in the financial market, and not just on this asset's covariance rate with consumption.

The new term in the risk premium is positive when $\gamma > \rho$, which was the result when $\beta = .01$. For the data of Table 1, the estimate of the CAPM-term $\sigma'_M(t)\sigma_M(t)$ is of the order of magnitude .03, while the estimate of the consumption covariance rate with the stock index $\sigma_{Mc}(t)$ is .003, where the estimate of $\kappa_{Mc}(t)$, $\hat{\kappa}_{Mc} = .5$. That the former term has the potential to fill

in the "gap" in the expression the equity premium of the standard model, we have just demonstrated. In the example with β equal to one per cent, the CAPM term explains 96.7% of the risk premium estimated by Mehra and Prescott.

It is equally satisfying that the return rate on government bonds depend more than just the growth rate and the variance rate of aggregate consumption, but also on characteristics of other investment opportunities in the financial market.

The interest rate r comes down relative to the conventional model for three reasons: The term multiplying the growth rate of consumption μ_c is now of the order of .7 instead of the order of 20 for the standard model. The fourth term is negative when $\gamma > \rho$, and so is the last term provided $\rho < 1$ as well. The precautionary savings term is still negative, but less so than in the standard model.

Faced with increasing consumption uncertainty, the prudent consumer will still save and the interest rate will accordingly fall in equilibrium. This effect is smaller the smaller ρ is. When the covariance rate between consumption growths and the return on the market portfolio increases, the recursive utility consumer buys more government bonds and sells stocks provided $\gamma > \rho$. If the opposite is true, the consumer will borrow and buy stocks. When the uncertainty of the return of the market portfolio increases, the recursive utility agent will buy bonds and sell stocks provided $\gamma > \rho$ and $\rho < 1$, or $\gamma < \rho$ and $\rho > 1$, and will otherwise borrow and buy stocks.

Rewriting the expression for the risk premium we obtain the formula

$$\mu_R(t) - r_t = \rho (\sigma_{Rc}(t) - \sigma_{RM}(t)) + \gamma \sigma_{RM}(t). \quad (46)$$

From this version we notice that the risk premium increases when γ increases, ceteris paribus, for risky securities that satisfy $\sigma_{RM}(t) > 0$. In the conventional model the risk premium decreases when ψ increases for such securities. For recursive utility, on the other hand, this is different. When $\sigma_c(t) < \sigma_M(t)$ as the data show, the risk premium *increases* when the EIS-parameter ψ increases, for risky securities that satisfy $(\sigma_{Rc}(t) - \sigma_{RM}(t)) < 0$. For such securities the individual can handle deterministic variations in consumption better when ψ increases, and a larger reward in the securities market must be offered for these securities to make the representative agent indifferent to status quo. For securities that works as an insurance product, the reverse is true: When $(\sigma_{Rc} - \sigma_{RM}(t)) > 0$, an increased ability to handle deterministic variations in consumption in the economy means that the individual will diversify in the presence of uncertainty, and include such securities in his, or her, portfolio. To restore to status quo, the price of such securities

must increase. When $\sigma_c(t) = \sigma_M(t)$ we get the market-based CAPM, which collapses to the conventional model when $\rho = \gamma$

Turning to the expression for the interest rate, we notice that when $\sigma_{cM}(t) > 0$, the interest rate r_t decreases when γ increases, ceteris paribus, provided $\psi \geq 1$. Even if $\psi < 1$ this may still be true. Several other combinations are possible, illustrating the rather rich structure of the model.

This kind of analysis has no place in the conventional model, since there is no direct connection to the securities market in the expression for the equilibrium interest rate in (4), nor is there any direct connection to the securities market for the risk premium in (3).

8.1 The Relation to Climate Change

Our results have immediate implications for the economics of climate change. The Stern Review, (Stern (2007)), deals with precisely these problems. Stern uses the standard model (3) and (4), but ignores the equity premium in (3) altogether, and sets $\gamma = 1$. Furthermore he assume that the impatience rate is close to zero (.1%). The resulting interest rate that he obtains from the standard model is then $r = .014$, which is much lower that what normally follows from the standard model. This value of r can then support a more dramatic mitigation policy than follows from other, similar cost-benefit analyses.

If N. Stern had taken the model for the risk premium seriously, he would have obtained an interest rate of the order larger than 9%, in order to be consistent with a non-negative impatience rate β , and thus with a reasonable representative agent.

A consequence of our analysis with recursive utility is that the social discount rate may be set very low. For the data set in Table 1 our model interest rate r is already down to .8%, the observed one, i.e., a value less than one percent is consistent with the model, which is much lover that Stern's 1.4%. Moreover, projects with a very long horizon are inherently risky, so the discount rate should really be adjusted for risk.

With respect to climate problems, imagine a 'project' that does not pay off in the future if the state of the climate is in the "good" state, but gives a positive payoff if the future state is "bad". Such a project has the effect of substituting consumption across time and across states of nature. With reference to an insurance setting, this project has a negative correlation with aggregate consumption *and utility*, and will therefore result in a negative risk premium in equilibrium.

The standard model would not be of much help in this regard, since aggregate consumption is so smooth that the correlation of such a project's

return with aggregate consumption will not subtract much from the short rate, unless the risk aversion is very large. If so, the short rate produced by the model is larger than 9% for the data of Table 1 provided one insists on $\beta \geq 0$, and the final conclusion would be to reject any projects designed to mitigate the adverse effects of climate change.

For the recursive model, the project's negative correlation with aggregate consumption and utility translates into a negative correlation with aggregate consumption and the the market portfolio. As we have seen, the latter term may very well result in a significant subtraction from the (already very low) interest rate of less than one per cent, since this model is consistent with both large risk premiums at moderate levels of risk aversion, and very low levels of r . Thus our model is really promising for the economics of climate change, but for reasons different from those given in the Stern Review.

8.2 Extensions

There are many important issues to explore, based on the framework of this paper. The life cycle model, for instance, can be better understood once time preference is separated from risk aversion. This gives new insights in the comparisons of defined benefit to defined contribution pension plans. Other applications are plentiful and will be deferred to future research.

9 Conclusions

We have addressed the well-known empirical deficiencies of the conventional asset pricing model in financial and macro economics. The root of the problem was identified to be the equality between risk aversion and time preference. With recursive utility properly defined, these inherently different properties of an individual are disentangled. This explains the new terms of this analysis; for the risk premiums of risky securities, as well as for the equilibrium real interest rate, provided we use utility gradients rather than dynamic programming to find the first order conditions. Dynamic programming, widely used for this sort of problems, tend to produce solutions consistent with late resolution of uncertainty, and then for not very plausible values of the parameters. Late resolution of uncertainty does not seem reasonable for a rational agent operating in the US stock market. Our solution, on the other hand, is not based on dynamic programming and implies that the agent prefers early to late resolution of uncertainty, and for plausible values of the parameters.

With a Kreps-Porteus specification of the felicity index in a recursive

utility approach of the Duffie-Epstein-Skiadas type, we were able to explain both the Equity Premium Puzzle, as well as the Risk-Free Rate Puzzle with good margin. The resulting model is able to fit the observed high equity premium of 6.18% related to the return on the S&P-500 index in the USA for the period of 1889-1978, the low estimated value of .8% for the risk free real interest rate, the low consumption volatility for the same period, and the low covariance between returns on equity and the growth rate of aggregate consumption, for reasonable parameter values of the utility function.

Other important aspects about the single investor assumption are not discussed in this paper, such as limited market participation, heterogeneity, and incomplete markets. These are important topics to be addressed in this topic area.

Nevertheless, our findings are likely to have broad economic implications. Just to illustrate, we rounded off with an application of our results to the economics of climate change, and showed that the conclusions in the Stern Review can be considerably improved on behalf of the climate, or more precisely, to the benefit of future generations, but for reasons different than those put forth in The Review regarding the discount factor.

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