Corporate Finance:
Capital Structure and Hybrid Capital

Aksel Mjøs

Bergen, October 2007

Dissertation submitted to the Department of Finance and Management Accounting at the Norwegian School of Economics and Business Administration, in partial fulfilment of the requirements for the degree of dr. oecon.

To *Kari, Jon Carlos and Jonathan*

Contents

List of Figures

List of Tables

Acknowledgements

This thesis marks the completion of the most demanding phase in my professional career to date. Demanding in terms of complexity, time and effort, but also in the uncertainty and delayed rewards from own work. These 6 years at The Norwegian School of Economics and Business Administration (NHH) have represented an unexpectedly large investment in knowledge and understanding whilst the significant returns in terms of external acceptance of my contributions are yet to materialize. As such, it has also been my most risky professional position, far from the dynamics of business life. However, in terms of overall quality of life, the academic life, even as a doctoral student, has many attractions. My main personal and professional reward has been a fundamentally different analytical understanding of economic issues, both financial and in general. This achievement in combination with my other knowledge hopefully contribute to a constructive understanding of both the 'investments in knowledge' of academic research and the (necessary) 'delivery of results' in business. The efforts, the achievements and the competitive pressure are certainly no less in academics compared to the most demanding parts of international business.

In completing this doctoral program, I have had the great benefit of good advice and cooperation. I am very grateful to Professor B. Espen Eckbo who has been my main supervisor and patient but persistent coach throughout the period. I also had the pleasure of visiting Tuck School of Business, Dartmouth College, as his guest for a short period in 2004. Professor Thore Johnsen was my sponsor when I first approached NHH and deserves many thanks both for accepting and supporting me, and for being in my advisory committee. Professor Hans K. Hvide has contributed significantly as member of the advisory committee and in particular when working on our joint paper "Start-up Financing: Outside

Equity". The three papers related to hybrid capital are the result of an, in my view, very stimulating combination of my market experience and the theoretical competence of Professor Svein-Arne Persson. We have cooperated on these papers for more than 3 years with ever increasing modelling challenges and great fun, and I very much appreciate this partnership. Thanks also to my 'cell-mate' Trond Døskeland for advice and elaborated discussions in most areas. I also thank Gorm Grønnevet for valuable support with the panel regressions and in wrapping up the thesis, Jarle Møen and Ragnhild Balsvik for friendly and supportive guidance in the jungle of econometrics, and Per Østberg for being a supportive and challenging neighbour. I would also like to thank many good colleagues at the Department of Finance and Management Science, elsewhere at NHH, and at The Institute for Research in Economics and Administration for their joint support of an old apprentice.

I am grateful for the financial support for learning, research and necessary travelling during these years. I was a research scholar by NHH until 1st July 2006 and have since been employed as a researcher at The Institute for Research in Economics and Administration, affiliated with NHH. I have also in my current position been allowed to devote much of my time to complete this thesis. Finally, a part of the thesis was funded by The Norwegian Research Council (Finansmarkedsfondet) through the project "An empirical study of the financing choices of Norwegian companies", project no. 178923, gratefully received.

The most important support in this project has been my wife Kari, who has always been both an understanding and a challenging partner for a sometimes frustrated man and student. I am deeply grateful to you. I also thank our dear boys, Jon Carlos and Jonathan, for acceptance and understanding and recalibrated perspectives even if their father sometimes was unusually difficult to live with.

I am a different person now than I was 6 years ago, I have gained a new professional platform and look forward to the continuation.

Aksel Mjøs
Bergen/Hoshovde, October 2007

1

Introduction

1.1 Capital structure and hybrid capital

In understanding Corporate finance we continue to face a map with many white and challenging spots. The specific question - *How does a company choose to finance itself?* - illustrates the breadth of the field. What is a company? Which sensible criteria define a firm? Who make the choices? What are the incentives, information, and rational, as well as less rational, drivers of these choices? Under what conditions and limitations are the choices made - internally and externally? Finally, what constitutes financing, and motivates its various shapes and forms? As economists, the standard approach consists of optimizing an objective function given a set of constraints. The complexity of a corporate finance setting with usually confused objectives and simultaneity of events makes this optimization a particularly demanding task.

The awareness of many of the issues of corporate finance is evident also amongst early economists, even Adam Smith warned about the misdeeds of directors managing other people's money and even against the limited liability corporation as such. In the specific field of capital structure research, Miller and Modigliani (1958) marked the start of more formal research after proposing their fundamental 'irrelevance argument' where they claim that under strict assumptions, in particular of efficient markets, no taxes or bankruptcy costs, how the cashflows from a firm's assets are split between various financiers cannot alter any combined values. Subsequent research effectively introduces deviations from these assumptions in an on-going attempt to better understand observed company behavior. As to the current state of theoretical research in the field, Tirole (2006) on page 132 states: "the [financial contracting] theories ... are often criticized for their lack of robustness; it is also pointed out that they do not account for the diversity of capital structures that characterizes modern operations". He (in Chapter 2) sums up the key themes when studying capital structure as being, tax benefits and bankruptcy loss from leverage, information and incentives, (current) capital, liquidity, collateral[1] and external monitoring.

I include a brief overview of the main theories on capital structure in the first chapter of the thesis. Trade-off theories generally assumes that all firms are moving

[1]Collateral is usually measured as fixed assets in proportion to total assets and is an indicator of available assets that may be used as security for debt.

- at different speeds - towards an optimal composition of their alternative sources of financing. Pecking order predicts that firms will prefer the least informationally sensitive forms of financing, typically internal funds and secured debt, over alternatives where the required return has to compensate for the informational asymmetry and any incentive-related issues between insiders and outsider, the standard example is equity. Within behavioral finance, market timing theories predict that leverage ratios are the result of repeated attempts by management to opportunistically issue equity at overvalued share prices. In addition, theories motivated by inefficiencies caused by adverse selection (of implemented projects) and moral hazard (of ex-post behavior) between outside investors and inside management may be related both to the optimality of financing as well as to marginal financing choices.

The current state of capital structure research includes an extensive empirical literature and significant insights have been gained. Frank and Goyal (2008) provide a useful overview, and sums up the cross-sectional 'stylized facts of leverage'. These are based on US empirical findings, but has also broader support. These are: the stationarity of the leverage ratios, the correlation between firm and industry median leverage, the positive relationship between leverage and collateral, log total assets and expected inflation, the negative relation to market-to-book ratio, profits and dividend payment.

My own motivation for studying capital structure relates not only to the described challenges in the field, but also to many years of involvement in financing firms. As an issuer's representative in charge of capital raising and investor relations, an implicit objective was to lower the level of informational asymmetries towards outside investors - although never explicitly formulated in that manner. As an investment banker, any transaction also had to be financed optimally and the expected impact on the shareprice, i.e. implicit shareholder value optimization, was always a fundamental criteria. In this environment we also faced many challenging incentive issues although seldom explicitly addressed. In addition to providing a fundamental interest in the important field of how firms are financed, I believe the experience represents valuable observations for a researcher.

This thesis consists of 2 empirical and 3 theoretical chapters on capital structure and financing alternatives in a corporate finance perspective. The first paper describes the capital structure of the population of Norwegian companies. In addition to document the current state, recent developments and variations of capital structure, I also challenge the assumption that the reported balance sheet represents the complete picture of a firm's indebtedness. The last part of the paper runs a full dynamically specified panel regression of leverage. The findings in terms of leverage dynamics are very much in line with mainstream US research on listed firms (see e.g. Lemmon, Roberts, and Zender (2007). The second paper, joint with Professor Hans K. Hvide, study a selection of more than 4,000 Norwegian start-ups to better understand why and with what effects external shareholders get involved in start-up companies. The incorporation, founder and firm data are all observed at the start-up date without problems of path-dependency. This allows for a unique test of capital structure theories with limited endogeneity issues. Our main findings are that in the 40 % of sample firms with outside equity, outside equity seems primarily be just an additional 'injection of cash' not motivated by other effects than facilitating the start-up and raising additional debt. The third, fourth and fifth papers, joint with Professor Svein-Arne Persson, are all theoretical papers related to a specific financing instrument, hybrid capital, and model the valuation and issuer bankruptcy impact from having such securities outstanding. Hybrid capital is a perpetual continuously coupon paying debt instrument with a level of risk exposure that makes it accepted as risk capital for financial institutions. The combination of perpetual maturity, junior position, embedded, finitely lived issuer's call option and reduced contingent rights in default makes the modelling not trivial. Our models are developed in a corporate finance perspective although not including optimization of capital structure. The third paper develops the fundamental valuation formula in a barrier option framework and the forth paper expands the model to include realistic specifications of hybrid capital and capital structure. The fifth paper includes methodological support for the two other papers. This paper develops a methodology for valuing infinite and finite claims with different cashflows for various fixed levels of financial distress. The differences between the subjects of these papers are less than they may appear although both capital structure theory, applied option pricing theory and alternative econometric approaches are involved. Strebulaev (2007) is a recent recognized example of a paper that builds his analysis of capital structure tests on asset-based calibrated structural model in the same tradition as our papers on hybrid capital.

To sum up, my thesis starts out with an overall empirical description of actual capital structures, continues with a focused test of the main theories in understanding the effect of outside equity, and ends with theoretical models of the value of a particular form of debt securities in a corporate finance setting. I will present the papers in more detail in the following.

1.1.1 Norwegian Companies' Capital Structure - an overview.

This paper is the first comprehensive documentation of the capital structure of the population of Norwegian private and public companies. By the use of a well specified dataset with 138,990 firms and 852,862 firm-years for the years 1992-2005, I manage both to analyze overall capital structure as well as relevant and still large sub-samples. The subsamples reflect size and listed/unlisted firms, as well as groups expected to differ in terms of financial constraints, e.g., dividend payers or firms that have received formal notifications from their auditors. I also include a limited analysis of capital structure including capitalized rental agreements and find significantly increased adjusted leverage ratios. Leverage ratios are analyzed in a dynamic panel regression using the System GMM-method proposed by Blundell and Bond (1998). The results confirm known US findings regarding 'partial adjustment' of leverage and shows in addition interesting variations by subgroups. The analysis also highlights the sensitivity of the results to the choice of regression method. The main contributions of the paper are to document a national population of firms, show drivers of changes in leverage, illustrate the effects of including capitalized rents in overall leverage and exemplify the importance of panel regression methodology.

1.1.2 Start-up Financing: Outside Equity

This paper is joint work with Professor Hans K. Hvide of University of Aberdeen, Business School and NHH. We investigate the extent to which start-ups use outside equity, and interpret our results in relation to financial contracting theory. We do so by studying the start-up and founder characteristics that are associated with the use of outside equity financing, using a unique dataset from Norway. Our findings suggest that adverse selection are less of a concern for start-ups than ex-post opportunistic behavior (risk shifting) by the entrepreneur as in Myers (1977) and Ravid & Spiegler (1997). One implication of this finding is that outside equity and debt are complements rather than substitutes, and that an extra unit of equity financing has a multiplicative effect on total financing through releasing additional debt financing. We do not find convincing evidence that the use of outside equity has detrimental effects on entrepreneurial effort, nor that a possible shortage of available outside equity leads to investor monopolization and excessive investor returns. Thus we provide evidence that outside equity provides an important avenue for entrepreneurs to escape liquidity constraints.

1.1.3 Callable Risky Perpetual Debt: Options, Pricing and Bankruptcy Implications

This and the subsequent two papers are joint work with Professor Svein-Arne Persson at NHH. Issuances in the USD 260 Bn global market of *perpetual* risky debt are often motivated by capital requirements for financial institutions. However, observed market practices indicate that actual maturity equals first possible call date. We develop a valuation model for callable risky perpetual debt including an initial protection period before the debt may be called. The total market value of debt including the call option is expressed as a portfolio of perpetual debt and barrier options with a time dependent barrier. We analyze how an issuer's optimal bankruptcy decision is affected by the existence of the call option using closed-form approximations. In accordance with intuition, our model quantifies the increased coupon and the decreased initial bankruptcy level caused by the embedded option. Examples indicate that our closed form model produces reasonably precise coupon rates compared to more exact numerical solutions. The credit-spread produced by our model is in a realistic order of magnitude compared to market data.

1.1.4 Bundled Financial Claims - A Model of Hybrid Capital

A large class of infinite horizon financial instruments which incorporates elements of both debt and equity, may collectively denoted "hybrid capital". The Bank for International Settlements (BIS) has devised the fundamental requirements for how hybrid capital may qualify as a part of core ("Tier 1") regulatory capital for banks. We present valuation models for hybrid capital in the set-up of Black and

Cox (1976) and Leland (1994) and derive new valuation formulas incorporating these special features. In particular, we take into account the issuer's right to omit hybrid coupon payments and to call the hybrid capital at par value starting from a given date. In doing so, we build on formulas developed in the previous paper. We show that hybrid capital actually carry risk and clarify interesting links between their valuation and overall corporate capital structure as guidance both for market participants and regulators alike.

1.1.5 Default of Multiple Degrees: The Risk of Lost Debt Coupons.

Motivated by the risk of stopped debt coupon payments from a leveraged company in financial distress we define a multi-level annuity contract which pays an annuity at a rate depending on the value of an underlying asset. The range of possible values of this asset is divided into a finite number of regions. The annuity rate is constant within each region, but may differ between the regions. The annuity payments end at a finite time horizon or upon an earlier bankruptcy, i.e., if the asset value process hits an absorbing boundary. Such annuities occur naturally in models of debt with credit risk in financial economics. Suspension of debt service under the US Chapter 11 provisions is one well-known example. We present closed-form formulas for the market value of multi-level annuities contracts when the market value of the underlying asset is assumed to follow a geometric Brownian motion.

1.2 References

Black, Fisher, and J. Cox, 1976, Valuing corporate securities: Some effects of bond indenture provisions, *Journal of Finance* 31, 351–367.

Blundell, Richard, and Stephen Bond, 1998, Initial conditions and moment restrictions in dynamic panel data models, *Journal of Econometrics* 87, 115–143.

Frank, Murray Z., and Vidhan K. Goyal, 2008, *Trade-off and Pecking Order Theories of Debt*vol. 2 of *Handbook of Corporate Finance: Empirical Corporate Finance* . chap. 12 (Handbooks in Finance Series, Elsevier/North-Holland).

Leland, Hayne, 1994, Corporate debt value, bond covenants, and optimal capital structure, *Journal of Finance* 49, 1213–1252.

Lemmon, Michael L., Michael R. Roberts, and Jaime F. Zender, 2007, Back to the beginning: Persistence and the cross-section of corporate capital structure., *Journal of Finance* Forthcoming.

Miller, Merton H., and Franco Modigliani, 1958, The cost of capital, corporation finance and the theory of investment, *American Economic Review* 48, 655–669.

Strebulaev, Ilya A., 2007, Do tests of capital structure theory mean what they say?, *The Journal of Finance* 62, 1747–1787.

Tirole, Jean, 2006, *The Theory of Corporate Finance* (Princeton University Press: Princeton and Oxford).

2

Norwegian Companies' Capital Structure - an overview

Thanks[1]

Abstract

This paper is the first comprehensive documentation of the capital structure of the population of Norwegian private and public companies. By the use of a well specified dataset with 138,990 firms and 852,862 firm-years for the years 1992-2005, I manage both to analyze overall capital structure as well as relevant and still large sub-samples. The subsamples reflect size and listed/unlisted firms, as well as groups expected to differ in terms of financial constraints, e.g., dividend payers or firms that have received formal notifications from their auditors. I also include a limited analysis of capital structure including capitalized rental agreements and find significantly increased adjusted leverage ratios. Leverage ratios are analyzed in a dynamic panel regression using the System GMM-method proposed by Blundell and Bond (1998). The results confirm known US findings regarding 'partial adjustment' of leverage and shows in addition interesting variations by subgroups. The analysis also highlights the sensitivity of the results to the choice of regression method. The main contributions of the paper are to document a national population of firms, show drivers of changes in leverage, illustrate the effects of including capitalized rents in overall leverage and exemplify the importance of panel regression methodology.

[1] I want to thank Trond Døskeland, B.Espen Eckbo, Gorm Grønnevet, Hans K. Hvide, Thore Johnsen and Jarle Møen. Personal responsibility for all mistakes accepted.

2.1 Introduction and motivation

This paper presents a complete description of the capital structure of all Norwegian private- and public-companies for the period 1992-2005[2]. I construct a set of accounting based capital structure measures and study industries and other relevant sub-samples both descriptively and by panel regressions. The use of subsamples represent an important widening of the study by allowing differences in dynamic relationships not possible to include in a regression of undifferentiated samples. The study of a large sample of predominantly private firms represents a relevant comparison to conventional datasets of large, listed firms studied in current literature. The balance sheets of private companies lack market values of assets and financial claims and may also be incomplete in representing the overall position. Whilst I do not attempt to estimate market values, I in a subsample do adjust the scope of the balance sheets by including capitalized rented/leased assets and calculate adjusted leverage-measures. This approach shows that median interest bearing debt as a percentage of total assets increases by 20 %.

Capital structure becomes interesting when a company deviates from the classical benchmark case - one owner that runs her business financed fully by her own capital. The introduction of outside financing, as debt with fixed return and contingent control or equity with residual return and current control, makes capital structure choices important. Outside capital may serve different purposes in financing growth beyond the entrepreneur's own limits, redistribute risk more efficiently, help liquidate the entrepreneur's investment or be more operationally-related motivated. We know that outside capital also may cause losses or increased risks due to misaligned incentives or asymmetric information between the entrepreneur and new financiers (Jensen (1986)). Examples of these issues include incentive effects for the owner/manager herself, different risk, return and tax-effects between various financiers, changed distress- and bankruptcy probabilities and -costs and various effects on the operations of the company. Agrawal and Knoeber (1996) test the effect of seven mechanisms used to control agency problems between managers and shareholders and conclude that since none of them, except outside directors, have significant effects on firm performance, a given firm's use must be

[2]All companies excluding subsidiaries where parent company financing policy is expected to prevail, se later discussion.

endogenous and assumed economically rational. They define firm performance by Tobin's Q. The causal relationships between financing, related problems and measures to mitigate them, and thus the empirical analysis of capital structure, are particularly challenging to define; why do we observe a particular capital structure and what have been the driving - or limiting - factors?

The immediate difference between a study of private (unlisted) - compared to public companies is the lack of a market in shares issued by private companies. A public listing creates a running valuation of the equity capital of the company and also disciplines the insiders' with respect to financial reporting and otherwise. In a private company, the owner/manager has, limited if any focus on daily valuations and no specific incentives for disclosures beyond the minimum[3].

My analysis of capital structure takes the conventional approach of studying incorporated limited liability companies or consolidated groups as the relevant entities, making rational financing choices that are subsequently reported annually in their accounts. This research approach is increasingly challenged with the growth of new, dynamic, networking business structures (Grossman and Hart (1986)). Partnerships, licensing arrangements, outsourcing, complex return- and risk-sharing contracts with partners and employees combined with increasingly globalized operational- and legal-structures undermine the validity of a traditional study of corporate financing. Such structures are typically situation-specific and seldom disclosed in a way that allows for outside analysis. One attempt to accommodate to this development is to capitalize the values of all sorts of expected cashflows as assets or liabilities for the firm. By capitalizing of rented assets I include a first step towards a more complete understanding of how companies use different financing means to get assets under control.

I initially present some theoretical foundations for capital structure research as well as selected relevant precedent empirical studies. The remainder of paper is structured around five main parts; section 4 presents data and develop and motivate the composition of a comprehensive data-set for the population of Norwegian companies. The descriptive part in section 5 includes a general overview of the leverage all Norwegian companies' using accounting items and ratios, and also by selected categories, including industries. I find fairly large differences leverage

[3]The exception being information prepared for outside financiers.

between subsamples. In section 6 studies a subsample where the leverage is adjusted for capitalized rented assets and shows large adjustments to both leverage and profitability. Section 7 analyze leverage and potential drivers empirically both by univariate- and dynamic panel-regressions. Conclusions are in found Section 8. Variable definition, data quality discussions, some tables and references are found in the appendices.

2.2 Theories on capital structure

Modigliani and Miller (1958) is a natural starting point for an analytical understanding of capital structure. All research that have followed study various forms of deviations from their idealized setting with fully symmetrical information, no taxes, no bankruptcy costs, exogenous cashflows, and efficient markets in all assets. Myers (1984) defines the two main competing theoretical directions; 'Trade-off theory' and 'Pecking order theory'. Harris and Raviv (1991) and Frank and Goyal (2008) provide comprehensive overviews.

2.2.1 Trade-off theory.

Trade-off theory assumes that firms optimally balance the costs of debt, e.g., distress risk and bankruptcy frictions with the benefits, typically tax savings, but also management discipline and optimal scale. Firms are expected to move towards a target leverage and do their marginal financing accordingly, although time their transactions due to costs of adjustments. In a dynamic setting, firms see their targets develop over time and consider today's adjustments through new financing or payouts also in light expected future optimal leverage. Recent papers on dynamic capital structure include Fischer, Heinkel, and Zechner (1989), Goldstein, Ju, and Leland (2001) and Hennessy and Whited (2005). Most of this literature assume a fixed firm scale and that product markets and cash flows as exogenous, disregarding any interaction between leverage and operations. An example of the opposite is Maksimovic (1988) who shows that a firm's debt capacity is a function of industry and firm characteristics, e.g. elasticity of demand and discount rate.

Agency issues are relevant in any financing discussion and the seminal paper is Jensen and Meckling (1976). They develop a theory based on agency costs in structures with separation between ownership and management and as well as outside lenders. The theory predicts that a manager who owns less than 100% of the shares of a firm will have incentives to consume private benefits at the expense of the other owners ('moral hazard'). Shareholders in a leveraged limited liability company will also have incentives to take on riskier projects than without debt since they will receive all positive outcomes, but only a limited share of the downside ('adverse selection'). The authors present these costs as problems that will impact the optimal capital structure for a firm. Agency problems are one of the reasons why the pecking order theory predicts that outside capital is more expensive.

2.2.2 Pecking order theory.

Pecking order theory predicts that due to the information asymmetry between a firm and outside investors regarding the actual value of both current operations and future prospects, outside capital will always be relatively costly compared to internal funds, and equity more so than debt. Outside investors will, as described by Akerlof (1970), require a compensation for their expected informational disadvantage.

Myers and Majluf (1984) argue that information asymmetry will lead to a mispricing of a firm's equity in the marketplace. Aware of the resulting dilution of current shareholders' actual values, firms may not raise new equity even for projects with positive net present values, often denoted 'The under-investment problem'. They predict that firms will choose to finance new investments in ways which minimize this problem and thus avoid new equity issues.

Myers (1984) extends this theory into a 'pecking order' theory of financing. This theory predicts that the existence of asymmetric information will lead a firm to firstly use retained earnings and funds from current owners, then risk-free debt and finally risky debt before eventually raising new equity from outside investors.

A development in this area, Halov and Heider (2004), takes a more sophisticated approach to the issue of asymmetric information by separating uncertainty from risk. The paper is primarily empirical, but novel in that they find that firms prefer to issue equity when risk matters relatively more and debt otherwise. Their argument is that by issuing equity rather than debt, risky firms avoid the adverse selection costs of debt. Berger and Udell (2006) applies a related approach to the issue of credit availability for small- and medium sized companies, discussing how the opaqueness of firms impacts the relevant lending assessment technology.

Pecking order theory predicts marginal financing flows, but has no views as to overall optimal capital structure. It is hard not to assume that extremely high or low leverage will have to impact marginal financing, but this is outside the pecking order theory.

2.2.3 Other attempts to explain capital structure

In a more behavioral attempt to explain capital structure, many, e.g., Baker and Wurgler (2002) strongly argue in favor of 'market timing'. The idea is that firms tend to issue new equity when they perceive the market value of their shares to be high, relative to past book- and market-values. They find that such timing is the best explanation of changes in capital structures and that current leverage is the cumulative effect of historical attempts to time the issuance on new capital. This line of research is in contrast to classical assumptions regarding market efficiency and investor rationality.

2.3 Related empirical studies of capital structure

There are relatively few broad empirical studies of the capital structure of private companies. Most empirical papers analyze either public companies, survey-data on small- and medium companies or start-ups. I am not aware of any papers attempting to describe the whole population of firms in a country, although some papers conduct a between-country test of capital structure theories. Rajan and Zingales (1995) is a classical paper comparing public companies between G7-countries and Booth, Aivazian, Demirguc-Kunt, and Maksimovic (2001) studies public companies in 10 developing countries.

In the field of small businesses or start-ups, Berger and Udell (1998) include an overview of the capital structure of small businesses in the US, with average debt ratio of .50, significantly lower than my findings. Huyghebaert and de Gucht (2007) studies a range of financing features of 244 Belgian start-ups finding an average total debt ratio of .76 (median .82), very close to my overall findings.

Frydenberg (2004) analyzes Norwegian private manufacturing companies' capital structure using data for the period 1990 - 2000. He finds a total debt-ratio of .67, a short-term debt ratio of .44 and a long-term debt ratio of .23. This is based on slightly different definitions of debt, but are still in line with my findings as shown in Tables 2.17 and 2.21 in Appendix C. He finds that fixed assets, size, growth, taxes, return on assets and industry category are the key determinants of capital structure. Hol (2004) examines default-probabilities and debt-maturity choices by studying Norwegian private companies' capital structures using a sample for the period 1995-2000.[4] Carlsen and Nilsen (1993) represent an earlier study on Norwegian listed companies for the years 1984 - 1986. Failing to find strong support for any of the main theories, Carlsen and Nilsen (1993) conclude on certain specific relations between leverage and company features. They in particular find that leverage is increasing in size and long-term assets and decreasing in profitability, a commonly found result.[4]

The economic stability and statistical persistence of capital structure arguments in favor of modelling the changes in capital structure in a panel data set by using a partial adjustment model which includes last years leverage as an explanatory variable as in, e.g., Fama and French (2002), Chang and Dasgupta (2006) and Flannery and Rangan (2006). Lemmon, Roberts, and Zender (2007) takes this further in demonstration that initial (IPO)-leverage actually have a lasting impact on firm leverage, not only last year's realization. This set up implicitly assume a trade-off explanation of leverage, since the lagged leverage ratio plays no role in a pecking-order explanation of financing behavior.

[4]The thesis lacks a summary which would have made comparison of results possible.

2.4 The data

2.4.1 The data source

I use accounting data for all Norwegian limited liability[5] private and public companies for the years 1992 - 2005[6]. The data-set is a combination of single company accounts and consolidated accounts as presented in Table 2.1. The data is made available by Dun&Bradstreet. Companies owning subsidiaries (ownership ≥ 50%) have to file both company accounts and consolidated accounts. This results in two partially overlapping panel-data sets. I make the assumption that the capital structure of a consolidated group is defined by the parent company alone and imposed upon the subsidiaries[7]. I thus merge the two data-sets and exclude single company filings for parent companies, using only their consolidated accounts. I also exclude subsidiary companies as they are consolidated into a Norwegian or a foreign group. The inclusion of all consolidated group accounts will also necessarily include foreign subsidiaries of Norwegian companies, obviously stretching the assumption of parent company financing policy outside its national domicile. The indication of subsidiary status is only given for the year 2005 and my procedure is thus decreasingly accurate for earlier years.[8] The sample remains sufficiently representative and thus appropriate for the purpose of studying capital structure of a national population.

Common with most capital structure research, this paper focuses on the capital structure of privately held limited liability non-financial companies, since financial institutions have fundamentally different financing structures.[9] I have

[5]I include only limited liability companies and exclude legal forms which are not expected to make shareholder-value maximizing financing decisions, e.g., mutual companies and foundations. Sole traders are also excluded.

[6]Norwegian legislation requires companies to have a financial reporting year equal to the calender year and to file their financial accounts with the central company registry "Brønnøysundsregisteret" by end June the following year.

[7]E.g., Rajan and Zingales (1995) study only consolidated accounts.

[8]The annual numbers in Table 2.1 do not add together precisely since a large share of the separate parent company accounts seem to be missing. Parent companies may also themselves be subsidiaries causing hierarchical consolidations and in part explaining the effects.

[9]I have excluded the following industries, according to the classification of NACE Rev. 1.1(NACE: Nomenclature statistique des activiés économiques dans la Communauté Européenne, EU's industry-classification system): 65 Financial intermediation, except insurance and pension funding, 66 Insurance and pension funding, except compulsory social security, 75 Public ad-

also excluded observations (firm years) with turnover or total assets below NOK 1 mill (app. 172,000 USD, September '07).

Table 2.1: **Composition of the data-set.** The table shows the composition of the data-set applied in the analysis. The two first columns show the number of reporting companies and consolidated groups per year. The third column shows the number of consolidated subsidiaries. The fourth column shows the sum of companies and consolidated groups less parent companies of consolidated groups, and subsidiaries. The final column shows the applied dataset after a symmetric, per-year, 5 % 'Return on assets'-winsorization as described in Appendix B. The data-set excludes companies with turnover and total assets below NOK 1 mill and financial and public sector related industries, company forms or ownership.

(Firm years)	Single Companies	Consol. Groups	Subsi-diaries	Combined population set	Applied winsorized data-set
1992	50,123	4,904	10,940	40,815	39,989
1993	54,576	6,057	11,800	44,791	43,880
1994	59,848	6,626	12,919	49,107	48,112
1995	65,149	7,221	14,203	53,273	52,199
1996	70,975	7,855	15,699	57,806	56,612
1997	77,693	8,756	17,635	62,862	61,573
1998	83,634	9,209	19,826	66,755	65,389
1999	87,475	2,902	21,666	66,872	65,484
2000	92,280	2,787	23,760	69,557	68,123
2001	95,712	2,788	25,932	70,811	69,375
2002	96,077	2,734	27,647	69,470	68,054
2003	99,106	2,582	29,629	70,407	68,971
2004	103,043	2,490	31,966	71,990	70,526
2005	110,710	2,444	35,348	76,154	74,581
Totals	1,146,401	69,355	298,970	870,670	852,868
Averages	88,185	5,335	22,998	66,975	65,605

Dun&Bradstreet provides a range of company-specific additional variables like ownership categories, industry-codes, CEO-salary, reported accounting mistakes and any auditor remarks.

ministration and defence; compulsory social security, 91 Activities of business, employers and professional organizations and 95 Activities of households as employers of domestic staff.

2.4.2 The combined data-set - key levels and ratios.

Appendix B discusses quality issues related to private company accounting data in general and also analyzes the effects of alternative economical and statistical standardization of the data-set. Appendix C includes the main accounting items, performance- and leverage-ratios of the companies in the data-set. The accounting items are presented as ratios relative to total asset, calculated by entity and then aggregated. To support clarity, the data is presented for the individual years 1993[10], 1999 and 2005 as well as aggregated across the whole sample. The tables provided are[11]:

- Table 2.17: Balance sheet items.
- Table 2.18: Balance sheet ratios.
- Table 2.19: Income statement items.
- Table 2.20: Performance ratios.
- Table 2.21: Leverage ratios.
- Table 2.22: Univariate table with absolute values of the items analyzed.

I will in the following discuss briefly some aspects of the observed capital structure related data.

Time variations.

The assets reported in Table 2.17 shows increased holdings of cash, (financial) investments and real estate, where the latter two have rather skewed distributions. This may be linked to increased profitability, close owner/manager-links and/or prevalence of agency issues. The financing is increasingly long-term and formal, with reduced use of trade credit and short term interest bearing debt, and increased use of long term interest bearing debt and equity. In Table 2.18 net working capital, invested capital (interest bearing debt + equity) and bank debt show a growing

[10] 1993 is preferable to 1992 since several flow variables are calculated by using balance sheet differences between years and are thus missing for 1992.

[11] All variables and ratios are defined in Appendix A.

trend. The working capital grows since dividend provisions (particularly in 2005) and trade debt falls. In parallel, with improved profitability, both new debt and equity show relatively large increases. The income statement variables in Table 2.19 reflect an overall trend of improved profitability, both relative to turnover and capital. Average (relative) EBIT goes from .09 to .10 and net profit from .03 to .07. Firm cashflow has also almost doubled, except for the drop in the year 2005. The planned increase in dividend taxation caused a large drop in declared dividends as shown in Table 2.2 resulting in a large net reduction in cashflows. Dividends shows a growing trend with some large shifts due to years of taxation changes as discussed below. The capital intensity also increases, shown by the asset turnover-ratio moving from 2.6 (2.1) to 2.0 (1.5) (means, medians).

Table 2.20 converts the increased profitability into increased returns, in particular ROAE[12], whilst ROAA[13] is more stable. The operating margin shows a large dispersion between years, but with a slightly growing median. Table 2.21 compares well with Table 2.17 and shows a trend towards more equity and interest bearing debt and less financing by trade credit, particularly in the early years. This apparently negative relation between profitability and leverage over time supports the pecking order theory. Increased profitability and reduced interest rates seem to result in improved gearing and interest cover.

Dividend taxation changes

Norway has since the 1992-tax reform, with some exceptions, had a fairly neutral taxation of debt vs. equity financing for companies and investor combined[14]. Taxes still have an impact on capital structure. Sophisticated listed companies adapt to an international investor base, and individual Norwegian investors' have been taxed on capital gains beyond their individual share of the company's accumulated retained profits during their holding period. Dividends in Norway are declared annually and normally paid out within 6 months after the end of the financial year. Norwegian-based individual investors have received dividends tax-free throughout

[12] Return on Average Equity, see Appendix A.

[13] Return on Average Assets, see Appendix A.

[14] The modifications being that firstly, the tax neutrality is directly relevant to domestic investors only. Before this tax-reform debt had large tax advantages which would be expected to have lasting effects on debt-levels.

the sample-period except for the period from 5th September 2000 until end-2001. During these 16 months the effective investor dividend tax-rate was 11 %, paid one year in arrear. After 1st January 2006 dividends were again subject to investor taxation, effectively dividends declared for the accounting year 2005. Corporate tax-rates were stable at 28 % during the period. Directly affected individual investors held[15] 7.7 (7.0) % of the Oslo Stock Exchange as of end 2000 (2001, correcting for some large partial privatizations of state-held companies). Foreign investors held 37 % of the Oslo Stock Exchange as of 31.12.2005.

Table 2.2: **Dividend taxation changes, dividends and new financing.** The table shows mean payout ratios, dividends relative to total assets, new interest bearing debt, new equity and combined investor cashflows for the years 1998-2005. New financing is measured relative to beginning-of-year total assets. Investor cashflow is the sum of last year's declared dividend and current year's actual net changes in new equity, both relative to total assets. Dividends received are taxed in the last 4 months of 2000, all of 2001 and from 2006 onwards.

Years, means	**1998**	**1999**	**2000**	**2001**	**2002**	**2003**	**2004**	**2005**
Payout rate	.402	.346	.213	.610	.595	.825	.922	.035
Dividends/ Total assets	.039	.053	.032	.055	.076	.071	.094	.008
New financing:								
-Int.bear.debt	.094	.080	.130	.059	.104	.050	.060	.136
-Equity	.022	.035	.046	.032	.042	.023	.023	.101
Investor cashflow	.023	.034	.044	.047	.064	.065	.058	.080

Table 2.2 shows a marked positive trend in relative dividend-payouts, interrupted by large shifts around the years of tax-changes, even if they only have a substantial effect on relatively few shareholders. The payout rate shows a drop in 2000[16] followed by a large increase in 2001 whilst dividends relative to total assets shows a large increase in 1999 followed by a marked drop in 2000 and a more normal level from 2001. An even more dramatic increase in both rates is seen in 2004 followed by almost no dividends declared for the year 2005. These changes indicate that companies time their dividends in response to announced or expected tax changes. It is difficult to assess the changes in net new financing between the marketwise optimistic year 2000 and the problematic year 2001. All

[15] The share only includes shares held directly by Norwegian individuals, not those they may hold through mutual funds or other savings vehicles.

[16] This discussion relates to the year for which dividends are declared, as they are received by the investor and taxed the following year.

new financing fall, but new debt more than new equity which is contrary to the expected partial effects of tax on the relative attractiveness of debt vs. equity. The relative investor cashflow across the whole period (total of dividends, new issues and share repurchases) shows a stable growth trend which indicates that on average, companies repurchased shares to stabilize their transfers to shareholders. We also see very large annual standard deviations across firms, particularly in the years around the tax changes. These statistics necessarily also reflect overall market developments beyond the tax changes.

Impact of new accounting legislation

With effect from 1 January 1999, the accounting regime for Norwegian companies was subject to a major revision. Except for the payable tax-rate, all selected numbers and ratios seem to be continuous at the same level across the change. I have not explored this subject in more detail at this stage.

2.4.3 Supplementary data on rental-charges.

I include data on annual leasing- and rental-expenses for a limited subsample for the years 1993-2002. These expenses are split between charges for rented real estate and for equipment (cars, machinery etc.), but we lack any contractual details. The data comes from the Norwegian capital database provided by Statistics Norway and documented in Raknerud, Rønningen, and Skjerpen (2004). The database only covers selected industries, primarily manufacturing. Table 2.3 provides an overview. The capital database covers 69 % of the companies in the same 2-digit industry sectors found in the main database and the table includes a comparison of the relevant industry groups. In 2002, a total of 3,883 (93 % of the sample) companies reported rental expenses for either real estate or equipment[17]. The rental expenses have a meaningful magnitude compared to interest expenses.

Assuming that assets rented or leased are not owned by the company's own shareholders, these additional data-items allow for a broader definition of leverage beyond reported debt. In principle, this is achieved by capitalizing the rented assets and adding this value to assets and to interest bearing debt. Rented assets replicate

[17] All Norwegian companies have a unique 9-digit identification number which allows for exact linking of data from different sources, in this case the capital database and the accounting data-set.

regular debt to a large degree in that their use is subject to regular cash-payment, see Myers, Dill, and Bautista (1976) for an early discussion. However, non-payment of rental charges leads to the loss of a specific rented asset whilst non-payment of debt may lead to liquidation of the whole company. The probability of loosing a rented asset is larger than for a debt-financed one, but likely consequences are smaller. Whilst the balance sheet inclusion of rented equipment is easily motivated, owning or renting real estate are two more fundamentally different positions. An owner will experience market value changes on real estate whilst the tenant has no such exposure. When capitalizing real estate rents, I thus only define a 'loan', to avoid imposing a real estate investment decision on a tenant. In this analysis I apply a modified version of the 'Constructive Capitalization' - method introduced by Imhoff Jr., Lipe, and Wright (1991).

The motivation for including rental charges in a study of capital structure is to achieve a more economically correct understanding of the companies' access to alternative sources of financing.[18] In the analysis where rental charges are included, I limit the study to companies covered by the capital data-base, supplemented with single-company[19] accounting data for the relevant years. The limited size of the sample makes it relevant for a targeted analysis, but the overall panel regressions will be done using the main data set.

2.5 Main descriptive findings

This section presents and discusses the capital structure of Norwegian limited liability companies overall, by different measures and categories.

2.5.1 Definitions of debt and leverage

Relevant debt

The broadest definition of 'debt' is as 'non-equity'-financing. A company usually has both short-term liabilities like taxes due and trade credit[20] which do not claim

[18] One might in principle wish to capitalize *any* longer term obligation or contractually received cashflows, but this is for obvious reasons not currently possible.

[19] The capital database only includes single companies and no consolidated groups.

[20] Trade creditors may safely be expected to charge indirectly for their lending, e.g., through discounts for immediate payments. It is, however impossible to calculate these borrowing costs

Table 2.3: **Overview of the capital database.** The table presents selected univariate statistics for the companies included in the capital database 1993-2002. The columns marked Δ show the relative differences between the mean/median of these companies and the sample of single entity companies in the same 2-digit industry-segments in the accounting database. Interest-expenses and -cover are estimated by using annually winsorized credit margins plus annual NIBOR 3 mnth. interest rates. Variable descriptions are found in Appendix A.

(NOK 1000, rates)	n	**Mean**	*Δ(%)*	**Median**	*Δ (%)*	**St.dev.**
Total revenues	51,289	27,808	*6.4*	5,212	*21.7*	201,001
Total assets	51,289	42,380	*32.2*	2,797	*7.8*	1,343,835
Interest bearing debt	51,289	19,317	*60.2*	571	*9.2*	843,881
Equity	51,289	14,110	*11.7*	540	*4.9*	379,733
Interest expenses, est.	40,278	2,135	*45.5*	108	*5.9*	70,205
Property rentals	51,289	340	*n.a.*	67	*n.a.*	1,414
Equipment rentals	51,289	149	*n.a.*	7	*n.a.*	1,061
Total debt/assets	51,274	.88	*(32.3)*	.78	-	7
In.bear.debt/assets	51,266	.35	*(47.8)*	.25	*4.2*	4
Int.bear.debt/EBITDA	51,236	1.50	*(106.3)*	.80	*27.0*	147
Interest cover, est.	40,278	40.00	*(43.7)*	2.20	*10.0*	2,028

explicit interest-payments. Table 2.17 shows that across the sample, liabilities other than equity, interest bearing debt and trade credit constitute on average 30 % of total assets.

I focus on interest-bearing debt (IBD) as the relevant non-equity financing from a financial capital structure perspective. IBD can be expected to have all the regular contractual features of the theoretical concept of 'debt' (Tirole (2006), p. 80) in terms of cashflow, liquidation rank, contingent control rights and possibly security, and will have been raised through active financing negotiations between the company and its' lenders. Debt contracts vary by maturity, call-features, seniority, convertibility, fixed or floating interest rate, security, covenants and possibly other features which indicates that even IBD is a fairly generic term.

Leverage measures

Table 2.21 provides an overview of selected leverage ratios which are defined in Appendix A.2. The total debt ratio (mean 80 % / median 81 %) shows the

---

precisely by outsiders and trade credit is thus not included in the narrow term "Interest bearing debt".

overall importance of non-equity capital for the financing of companies. The other leverage ratios use interest bearing debt as nominator, and total assets, invested capital and EBITDA[21] as denominators. The IBD/total asset ratio (mean 34 % / median 23 %) shows the relative importance of interest bearing debt as part of the overall financing of the company. The IBD/invested capital ratio (mean 67 % / median 49 %) is the narrowest measure. It tells the company's choice between these two fundamental sources of capital. The two other leverage measures constructed are 'flow' measures, Interest bearing debt/EBITDA (mean -9.2 / median .5) and interest cover (mean 53.0 / median 2.2). These measures indicate 'debt capacity' in that they relate the debt to the company's cashflows and profits and thus ability to service debt. In comparison, the balance sheet measures are indicators of the security behind the debt in case of default and liquidation.

The last ratios included in Table 2.21 covers IBD-maturity and trade credit use. On average 79 % (median 100 %) of interest bearing debt is long-term, although the short-term may be underestimated in the data due to lack of specification. Trade credit represents on average 44 % (median 34 %) of the sum of IBD and trade credit. This shows the large importance of trade credit as a financing source, but the lack of specific terms causes me to exclude it from the definition of relevant debt.

It is particularly challenging for the study of leverage ratios that in spite of the standardization as described in Appendix B, 14 % of the firm-year observations have negative equity capital causing analytically 'impossible' ratios. As for now, these observations are not excluded from the data-set.

2.5.2 Capital structure by company categories.

Table 2.4 introduces the subsamples in the analysis. These are defined by dividend payment, scale, auditor's formal remarks and ownership categories as separately reported in Table 2.5. These subsamples will also be separately studied in the panel regressions in section 7.

[21] Earnings before interest, tax, depreciations and amortizations.

Dividend payers

Dividend-payment is shown in the data to be a strong indicator of what may collectively be called 'financial health'. Dividend-payers are typically not financially constrained and use much less debt by any measure, they obtain far more outside new financing both as debt and equity, and deliver a total return twice that of the non-payers, and has a lower return volatility. Although the group-means are significantly different, stable dividend payment may be more of a signal than reflecting current performance.

Table 2.4: **Leverage by dividends, scale and auditors' remarks.** The table presents debt-ratios, financing sources and risk and return ratios split by dividend payment last year, listing and auditor's remarks. Auditor's remarks are those formally noted in the auditor's statement to the annual accounts for at least one of the two years before. All ratios are medians of the variables measured relative to total assets by firm-year. Variable descriptions are found in Appendix A. ***,**,* indicates significantly different means at 1 %, 2 % and 10 %-levels, respectively (T-test).

	Dividend payers?		**Scale**		**Auditor remarks?**	
	Yes	**No**	**Smallest**	**Largest**	**Yes**	**No**
Split(n)	*32.4%*	*67.6%*	*20%*	*20%*	*22.8%*	*77.2%*
Debt ratios:						
IBD/TA	.08***	.32	.20***	.28	.36***	.19
IBD/Inv.cap	.20***	.64	.44***	.51	.77***	.41
IBD/EBITDA	.16**	.86	.00	1.30	.90**	.40
Bank debt	.00***	.10	.00***	.08	.15***	.00
New financing:						
-CF firm	.06	.04	.05**	.04	.05**	.05
-IBD	.00***	.00	.00***	.00	.00***	.00
-LT-debt	.00***	.00	.00***	.00	.00***	.00
-Equity	.00**	.04	.00	.00	.06***	.00
Risk & return:						
ROAA	.15***	.07	.07***	.10	.08***	.10
ROAA, st.dev.	.09***	.09	.09***	.07	.11***	.08

Small and large companies

I scale the companies by the sum of log total asset by sample median and log turnover by sample median, and study firms in the 1st and 5th quintiles of this scaling distribution. Skewed distributions causes all mean debt ratios, except gearing, to be larger for small firms whilst the medians are lower. Lower profitability an more limited access to outside funds explains why small firms get less new funds from cashflow and debt, compared to the largest firms. Our univariate comparison indicates that the expected informational disadvantages of small firms impact only debt financing negatively, not equity.

Companies with historical auditor remarks.

The term 'Auditor remarks' refers to formally noted qualifications by the appointed auditors as to significantly negative accounting or reporting issues. Assuming sequential causality, I classify a company as having auditor remarks if such are noted in at least one of the two years preceding the reporting year. 22.8 % (194,685 firm-years) of the sample falls in this category.[22] The auditors are entitled to full access to all information underlying companies' accounts and negative remarks will only come after hard negotiations with the company and can therefore be seen as a reliable, negative signal. The prediction is that such remarks will impact capital structure in the years to follow. Table 2.4 shows that leverage is significantly larger, most likely caused by historically lower cashflows and returns. Mean bank debt ratio(not reported) is 6 %-points higher, suggesting that the refinancing-role of banks after distress dominate the credit-effects of these negative remarks, low returns and higher volatility. Companies with historic auditor remarks both deliver high cashflows and raise relatively more outside financing. This may indicate a dominance of refinancing after a crisis, and gives more support to the trade-off than the pecking order theory. There is probably also a survivor bias in these statistics.

Ownership and capital structure.

Table 2.5 splits the sample by main ownership categories. Since subsidiaries are excluded, 60 % of the companies are owned by individuals. Listed and foreign owned companies have the largest (median) overall reliance on equity. Companies owned by individuals use the least IBD as part of their overall financing, whilst those held by other firms have the highest use of bank debt. In terms of debt

[22] The remarks are of different gravity, but I do not distinguish them here.

capacity, the typical listed company has the largest gearing and those with unspecified owners the lowest, relative to EBITDA. Companies owned by individuals deliver the largest cashflows and have almost no annual changes in interest bearing debt, raise no new equity and deliver the highest return. Listed companies, as commented above, have generally better access to outside financing probably due to lower information asymmetry. Companies held by foreigners have generally the largest variations and the lowest return of all, both results may be caused by corporate structures and transactions outside the financial accounts.

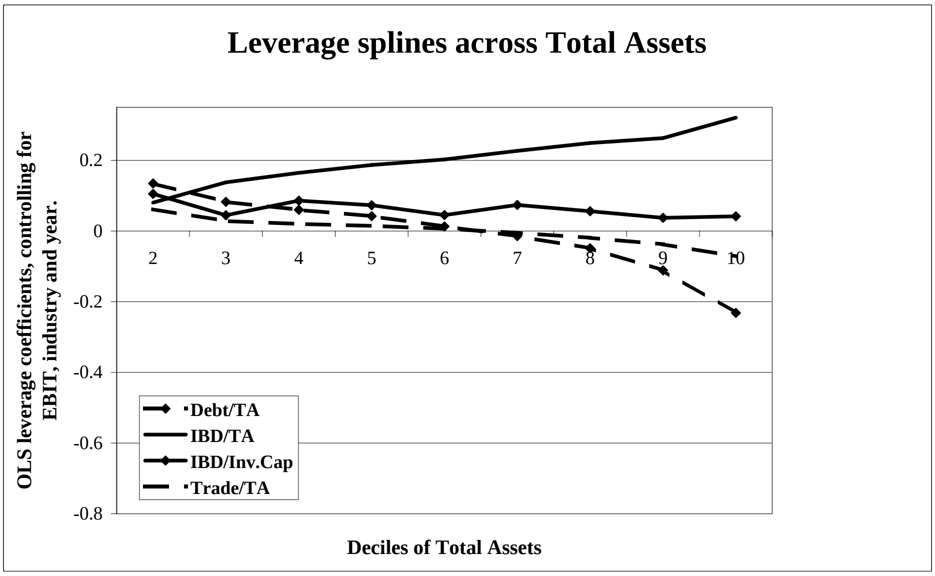
Capital structure by size and profitability.

Trade-off theories predict that leverage will increase in company size due to reduced risk and information asymmetry. Pecking order theory predicts that leverage will fall in profitability since cashflows will be retained or used to repay debt. I have conducted two simplistic illustrations of these two predictions in Figure 2.1. The lines show the changes[23] in selected leverage ratios (Debt/Total assets, IBD/TA, IBD/Inv.cap, Trade debt/TA) by deciles of total assets and EBIT, respectively. Both explanatory variables were lagged one year. The figures are created with equally scaled y-axes for improved comparability. The size graph shows that the relative use of interest bearing debt increases by company size, but the other ratios fall, although all changes are moderate. This graph supports the trade-off theory's predicted increased use of interest bearing debt and also indicates that other debt may be relatively stable. Somewhat contrary, the profitability graph supports the pecking order prediction in that the IBD/TA ratio falls as profits grow. The graph indicates that profits both are retained and used to repay debt, illustrated by the large reduction in the IBD/Invested Capital ratio by profit. The very limited sensitivity of trade debt to the changes both in total assets and in profits supports the separation between Trade debt and Interest bearing debt in this paper.

[23]These graphs are constructed by regressing the selected leverage ratios on dummies for each size-/profit-decile as well as annual industry-leverage levels and year-dummies. In addition, I included lagged EBIT in the total asset-decile-regressions and lagged log total assets in the EBIT-decile-regressions to provide additional controls. The regressions were done with no intercept, using Huber/White-sandwich standard errors and clustered on firms.

Table 2.5: **Leverage by ownership categories.** The table presents debt-ratios, sources of new financing and risk and return ratios split by reported main ownership categories. All variables are relative to total assets by firm per year. Abbreviations: Me: Mean, Md: Median, and Sd: Standard deviation. Variable descriptions are found in Appendix A.

	Split (%)		Debt/ Total assets	IBD/ Total assets	IBD/ Inv. Cap	IBD/ EBITDA	Bank debt	Cash-flow (firm)	*Outside financing:* IBD	Equity	LT-debt	ROA	ROA St. dev.
Listed	.2	Me	.57	.29	.40	(0.44)	.17	.04	-0.05	.10	-0.05	.05	.11
		Md	*.59*	*.25*	*.38*	*1.56*	*.04*	*-0.01*	*.00*	*.04*	*.00*	.08	*.06*
		Sd	.24	.24	.51	126.82	.22	1.51	1.29	.40	1.29	.19	.12
Company owned	9.4	Me	.74	.37	.56	(35.60)	.25	.12	-0.03	-0.01	-0.02	.11	.10
		Md	*.77*	*.30*	*.53*	*1.11*	*.09*	*.04*	*.00*	*.03*	*.00*	.09	*.07*
		Sd	.55	.54	3.25	2,158.12	.39	6.44	2.11	4.97	1.65	.19	.10
Owned by indiv-iduals	59.6	Me	.78	.30	.59	(2.72)	.20	.08	-0.01	.04	.00	.14	.12
		Md	*.80*	*.20*	*.44*	*.52*	*.03*	*.06*	*.00*	*.01*	*.00*	*.11*	*.09*
		Sd	.40	.39	5.22	189.02	.30	1.30	.76	1.48	.67	.19	.10
Foreign owned	.6	Me	.81	.37	.64	(64.70)	.11	.38	-0.18	-0.54	-0.11	.09	.11
		Md	*.76*	*.23*	*.44*	*.61*	-	*.02*	*.00*	*.03*	*.00*	*.08*	*.07*
		Sd	.86	.83	2.65	1,950.86	.70	14.33	8.27	12.79	4.74	.20	.10
Other	30.1	Me	.88	.39	.87	(11.30)	.23	.09	-0.04	.08	-0.03	.07	.12
		Md	*.84*	*.27*	*.58*	*.25*	-	*.04*	*.00*	*.04*	*.00*	*.06*	*.09*
		Sd	1.80	1.67	8.26	4,633.17	.41	3.17	2.42	2.96	2.22	.22	.11
All		Me	.80	.34	.67	(8.77)	.21	.09	-0.02	.04	-0.01	.12	.12
		Md	*.81*	*.23*	*.49*	*.50*	*.03*	*.05*	*.00*	*.02*	*.00*	*.09*	*.09*
		Sd	1.05	.98	6.12	2,632.25	.34	3.00	1.69	2.87	1.44	.20	.10



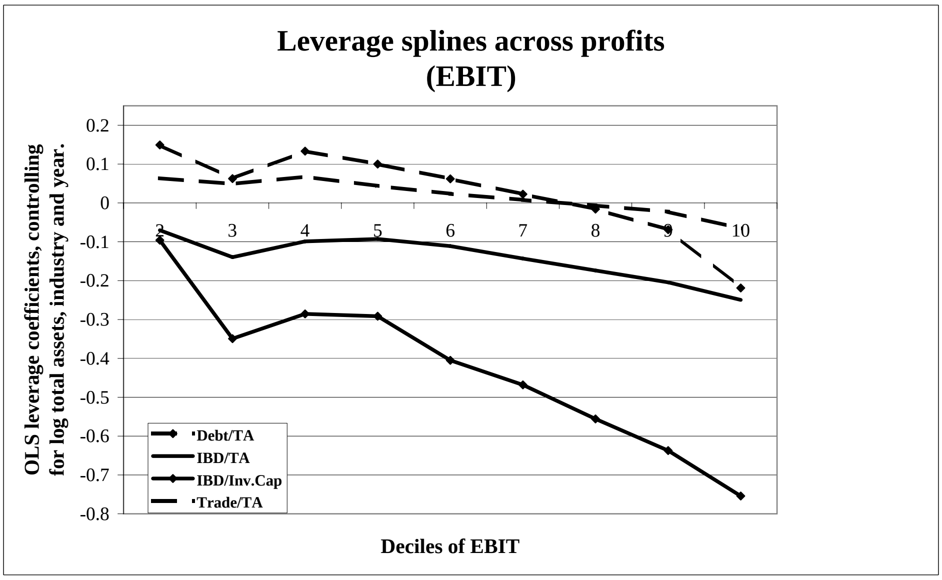


Figure 2.1: The figures shows how four balance sheet leverage ratios vary by grouping the sample into deciles by total assets and profits (EBIT), lagged one year. The lines are drawn between the coefficients for decile dummies in clustered OLS-regressions with no intercepts, but including industry and year dummies, as well as lagged log total assets and EBIT, respectively. All coefficients are significant at 10 % or lower levels, except for IBD/Inv.cap. in the total asset deciles 9 & 10.

2.5.3 Leverage variations across industries

Industry-groups represents a common, secondary grouping approach to capital structure analysis. Industries vary across many dimensions, Frank and Goyal (2004) find that the median industry leverage-variable is the factor with the largest explanatory power on company leverage. This is also confirmed in the panel data regressions to follow in section 7. Table 2.6 in the text and Table 2.23 in Appendix C, give an overview of how industry-groups vary with respect to business profile and capital structure measures. The industry-grouping follow the aggregation of industry codes used by Statistics Norway and are defined as follows:

1. Farming and fishing, including fish-farming which is particularly important.
2. Petroleum related, reflecting the reliance on the oil sector.
3. Manufacturing industry and mining.
4. Construction and energy-production.
5. Retail and wholesale trade, tourism.
6. International shipping and pipelines.
7. Other transport and communications including telecom.
8. Services, R&D, real estate and other rental providers.
9. Culture, media etc.
10. Information technology

Many capital structure studies are limited to manufacturing industries (e.g., Frydenberg (2004)). The scale of my dataset allows for describing and comparing the capital structure of all industry groups to highlight relevant differences.

Industry operating comparisons.

Table 2.23 in Appendix C shows that Oil, Manufacturing industry and Trade & tourism are of comparable size and together cover more that 2/3 of sample revenues, whilst Services is ranked as number 4 with 12 %. The Service-sector,

Table 2.6: **Leverage measures by industry-groups.** The table presents key financing leverage measure by main industry groups using data from 1999 until 2005 to be able to include details on bank debt. All ratios are calculated in relation to individual company total assets and then aggregated or averaged. Medians are denote 'M' and standard deviations 'S'. IBD equals 'Interest bearing debt'. Variable and ratio descriptions are found in Appendix A.

		Sources of new funds:			*Leverage ratios:*							
Industry group		**Cash-flow (Firm)**	**Int. bear. debt**	**Equity**	**Total debt/ Assets**	**IBD/ Assets**	**IBD/ Inv. Cap.**	**IBD/ EBITDA**	**Trade debt/ Ass.**	**Int. cover (est.)**	**Debt Mat.**	**Bank debt**
Agri/fish	M	.04	.00	.00	.79	.37	.58	.1	.02	1.4	1.00	.93
	S	*.97*	*.84*	*2.10*	*.53*	*.51*	*19.00*	*642.0*	*.15*	*11367.0*	*.40*	*.42*
Oil	M	.04	.00	.00	.61	.21	.34	.0	.01	2.4	.97	.00
	S	*51.00*	*53.00*	*48.00*	*.63*	*.61*	*.89*	*13486.0*	*.13*	*2022.0*	*.43*	*.39*
Mfg.ind.	M	.04	.00	.00	.79	.23	.48	.5	.10	2.2	.98	1.00
	S	*4.00*	*4.90*	*.71*	*.47*	*.42*	*5.50*	*1192.0*	*.18*	*1360.0*	*.34*	*.41*
Constr./ energy	M	.04	.00	.00	.83	.12	.36	.2	.14	4.3	1.00	1.00
	S	*.40*	*4.00*	*3.00*	*.32*	*.28*	*4.80*	*10124.0*	*.19*	*1374.0*	*.33*	*.42*
Trade & tourism	M	.05	.00	.00	.84	.20	.49	.2	.16	2.2	1.00	.96
	S	*2.50*	*5.20*	*.37*	*.80*	*.76*	*7.40*	*234.0*	*.25*	*1657.0*	*.90*	*.43*
Shipping	M	.06	.00	.00	.69	.36	.48	.0	.00	1.8	1.00	.80
	S	*1.90*	*2.10*	*3.40*	*1.40*	*1.20*	*8.80*	*4089.0*	*.12*	*2310.0*	*.31*	*.45*
Transport &telecom	M	.06	.00	.00	.83	.27	.56	.7	.08	2.1	1.00	1.00
	S	*.61*	*.72*	*.87*	*1.70*	*1.60*	*2.20*	*2077.0*	*.22*	*1433.0*	*.30*	*.42*
Services	M	.06	.00	.00	.80	.31	.53	.6	.00	2.1	1.00	.94
	S	*3.60*	*5.30*	*5.80*	*.46*	*.47*	*2.80*	*845.0*	*.13*	*9235.0*	*.36*	*.43*
Culture	M	.06	.00	.00	.83	.15	.39	.0	.05	2.1	1.00	.94
	S	*1.60*	*1.40*	*3.00*	*.99*	*.88*	*6.70*	*144.0*	*.26*	*1162.0*	*.28*	*.45*
IT	M	*.03*	*.00*	*.00*	*.77*	*.00*	*.00*	*.0*	*.05*	*2.5*	*1.00*	*.51*
	S	*.56*	*5.30*	*4.90*	*.54*	*.46*	*3.40*	*782.0*	*.19*	*18324.0*	*.44*	*.47*
All	M	.05	.00	.00	.82	.22	.48	.3	.06	2.2	1.00	.97
	S	*3.50*	*5.30*	*4.20*	*.70*	*.67*	*6.20*	*3438.0*	*.21*	*6468.0*	*.58*	*.43*

including real estate companies, uses the largest share (25 %) of sample assets. This structure is also evident in the asset turnover-numbers where Trade & tourism shows 22, Services .5 and Oil .8 compared to an overall average of 6.9. Oil, Shipping and Services have the largest entities with the highest asset-intensity. In terms of performance, these sectors achieve the largest operating margins. After deducting depreciations and debt service, IT and Construction which have very high asset turnover and the lowest asset tangibility, achieve the highest ROAA and ROAE-numbers, although their operating margins are only average.

Industry leverage comparisons.

In the following, I will comment on the leverage position by industry-groups, with reference to Table 2.6. The perspective is on typical companies rather than a more macroeconomic sector view, thus the focus on medians rather than mean ratios. To allow for details on bank debt, I have only used data from the year 1999 onwards.

Agriculture and fishing This small and fragmented sector has the highest use of interest bearing debt with an IBDR ratio of 0.37 and an ICL ratio of .58. Given this high leverage, it is not surprising to also see the second highest ratio of new equity raising (0.04).

Oil The oil sector is large, concentrated and profitable to the extent that annual average cashflow represents 511 % of total assets (not reported). This cashflow is subject to tax-rates of $\geq$ 80%. The profitability explains low net changes in debt and equity and the third largest interest cover. The sector also has the highest equity-ratio (39 %)and seems financially sophisticated with a bank debt ratio of .00 indicating use of market or internal[24] debt.

Manufacturing industries The manufacturing sector is close to the averages across the included variables, the exception being trade financing and the large use of new equity.

[24] Many firms in this sector are subsidiaries of international parent companies which may also be a source of debt financing.

Construction & energy. The construction and energy sector consists both of regular construction companies and large hydroelectric power companies. The sector has the second lowest equity- and IBDR-ratios of .17 and .12 respectively. The largest financing item is trade credit with a ratio of .14. The low amount of IBD also gives the sector the highest interest cover (ICOV).

Trade & tourism This sector has the lowest equity ratio, but the financing profile is otherwise comparable to Construction & energy although the use of both trade credit and IBD is somewhat higher at .20 and .16.

Shipping Shipping has traditionally been an important industry in Norway, although globalized legal and operational structures may well lead to an underrepresentation of companies reported here. The sector has the largest firm cashflow (CF_{fi}) of .07 and no average net outside financing. The financing consists of equity at .31 and IBD at .36. Debt capacity is stretched with ICOV of only 1.8, the second weakest after Agri/Fish. Asset-appreciation beyond book-values have traditionally been an important source of profit in shipping and may partly explain the high book leverage.

Transport & telecom Transport & telecom has overall the highest leverage (Debt/Total assets: .84) and the third highest IBD-ratio. This leverage in combination with average profitability gives a low interest cover of 2.5 and a GE-ratio above average at 1.2

Services Services is a mix of pure service companies with primarily human capital and asset-intense real estate companies. The sector has above average use of IBD (IBDR: 0.31) probably due to good security. New funds come primarily from cashflows.

Culture The culture sector has both a high cashflow (0.07) and is also one of very few sectors with positive net new equity. The other ratios are close to the overall averages.

Information Technology Information Technology (IT) has the largest revenue growth and return ratios as shown in Table 2.23. Consistent with the funding needs of this growth, net firm cashflow is the lowest amongst the industries. IT is typically expected to have lower leverage due to large information asymmetries, in particular with regards to development of new products and services. That the median IBD-related ratios are 0 and the overall equity ratio (0.23) is second only to Oil, are thus consistent. The use of trade credit is also below the average.

These results show how leverage and factors related to leverage vary significantly between industries supporting the use of industry factors when analyzing capital structure. However, I find almost no industry variation in the debt maturity variable.

2.6 Leverage with capitalized rental expenses.

Analysis of 'on-balance-sheet capital structure' for private companies has some important qualifications, primarily due to lack of market valuations and capitalization of additional significant contractual receipts and obligations. In an attempt to compensate for the latter, I estimate a more complete leverage measure by redefining rented assets as effectively 'owned' assets which are debt-financed.[25] Research involving capitalization of leases was introduced by Nelson (1963). Imhoff Jr., Lipe, and Wright (1991) developed a methodology denoted 'constructive capitalization' for operational leases. Fülbier, Lirio Silva, and Pferdehirt (2008) present the empirical literature in the field and apply the approach on German listed companies. I will use a simplified version of this methodology in the following and explore the impact on assets, debt, measures of leverage and specification of profits. I do the analysis on the smaller capital database, see Table 2.3, combined with the single company data-set from Dun&Bradstreet shown in the first column of Table 2.1.

[25] This approach assumes that no assets kept under financial leases are already on the balance sheet, which is a fair assumption under Norwegian GAAP during this period.

2.6.1 Capitalizing rented equipment.

The estimated value of rented equipment, EV, is calculated using a standard annuity-formula:

$$EV = FS_E * R_E * \frac{((1 + r_n)^T - 1)}{r_n * (1 + r_n)^T} \qquad (2.6.1)$$

where R_E is the annual rental payment, nominally fixed, FS_E is the fixed share of the payment that covers the financing costs, r_n is the nominal discount-rate and T is the remaining duration of the contract in years. Table 2.7 shows the sensitivity of the value multiplier for rented equipment to remaining contract duration and assumed financial share of the rental payment. The discount rate is 11 % in this illustration, equal to the median borrowing interest rate of the companies the data-set. The interest rates are estimated with errors from insufficiently specified accounting numbers, but by winsorizing the credit margin symmetrically by 15 % annually and adding the adjusted margin to the nominal annual average NIBOR 3 mnth. interest rates, the distribution becomes reasonable. The interest rates are for simplicity not differentiated between real estate- and equipment-rentals. I implicitly assume that the rental charges can be analyzed as one, representative contract. I also assume that a company's equipment rental contracts have a nominally fixed annual payment, last for 3 more years and that 60 % of the annual rental charge is effectively interest payment for the implicit financing.[26] The remainder of the charge covers servicing costs and instalments. This approach results in a typical capitalization value multiplier on the annual rental charges of 1.47. In the subsequent analysis, capitalizations use the estimated annual borrowing interest rates for each company. Compared with the methodology in Imhoff Jr., Lipe, and Wright (1991), I leave aside any periodic valuation differences due to straight-line depreciation of the asset and the delayed implicit reduction in the liability from a fixed annual annuity contract. The adjustments thus do not impact the book equity of the companies. Analytically, the objective of broadening the the scope of the balance sheet does not suffer from these simplifications.

[26]Leasing contracts often have a 35 % downpayment at the start - I disregard this as a on-balance-sheet asset.

Table 2.7: **Capitalized rental charges - sensitivities.** The table presents valuation multipliers for annual equipment rental charges. The multipliers vary by contract length and the share of the rental charge that effectively covers the interest element of the implicit financing. All values are calculated as present values of annuities using a nominal discount rate of 11 %, equal to the median of the data-set studied.

Rental contract duration in years:				
		2	3	4
Financial	80%	1.37	1.95	2.48
rental	70%	1.20	1.71	2.17
share	60%	1.03	**1.47**	1.86
	50%	.86	1.22	1.55
	40%	.69	.98	1.24

2.6.2 Capitalizing real estate rentals.

Real estate rentals differ by the long expected life and the value appreciation over time. As introduced earlier, I leave out the equity element by assuming that the company either owns the property but has written a call option which cancels all upside potential, or has lent to the landlord an amount equal to the capitalized rents and hold this loan as an asset. These positions are financially equal. Real estate rents are assumed fixed in real value[27] and should be capitalized by a real discount rate, calculated by adjusting the annual company-specific interest rates by the annual changes in the CPI-levels. 80 % of the rents are assumed to be financial and the remainder operational costs. The value of rented real estate, denoted REV, is thus calculated as:

$$REV = FS_{RE} * \frac{R_{RE}}{r_r} \tag{2.6.2}$$

where r_r denotes a real discount rate and the other variables retain their earlier interpretations.

2.6.3 The impact of capitalized rents.

The values of EV and REV are added to the adjusted operating and real estate asset classes respectively, and the combined liability to adjusted interest bearing

[27] Norwegian real estate rental contracts typically have an annual indexing clause linked to the official Consumer Price Index(CPI)-changes.

debt. The adjusted $EBITDA$[28]-profit following these adjustments is calculated as:

$$EBITDA_{adj.} = EBITDA - FS_E * R_E - FS_{RE} * R_{RE} \quad (2.6.3)$$

and the adjustment is added to interest costs keeping net profit unchanged.

The Norwegian interest rate- and inflation-changes have fluctuated a lot during this period, creating volatile discount rates. The 'firm-year'-approach thus creates large swings in capitalized values between years, similar to an annual market-value adjustment of other assets.

Table 2.8: **Capitalized rental charges.** The table shows the effects of capitalizing real estate and equipment rental charges. The top part shows means, percentiles and standard deviations for original accounting items, capitalized values and combined estimates including the capitalized rentals. The lower part of the table shows the relative shares of capitalized rented assets and financial rental charges of the adjusted numbers for real estate, operating long term assets, total assets and EBITDA in percentages in decimal form calculated for the years 1993 - 2002 only, due to data availability. Variable and ratio descriptions are found in Appendix A.

	Mean	**p25**	**Median**	**p75**	**St.dev.**
Real estate, bookv.	2,603	0	0	784	30,485
Rented property, cap.v.	5,209	0	220	1,670	44,790
Total real estate, est.	7,812	0	902	3,578	57,110
Operational assets, bookv.	12,348	189	713	2,649	360,138
Rented op.assets, cap.v.	198	0	0	59	1,593
Total op.assets, est.	12,547	219	787	2,817	360,663
Total assets, bookv.	42,380	1,207	2,797	7,877	1,343,835
Tot. Assets incl. cap.v.	47,787	1,566	3,860	10,781	1,351,418
Interest bearing debt	19,317	45	571	2,414	843,881
Interest bearing debt, incl. cap.r.	24,724	254	1,487	5,125	851,237
EBITDA	2,117	85	353	1,055	31,643
EBITDA, rentals adjusted	2,478	157	479	1,323	32,122
Interest costs	2,293	38	114	351	73,281
Interest costs incl. fin.rent.	2,132	73	196	550	64,728
Capitalized rental charges as share of adjusted items:					
-Real estate	.62	.07	.93	1.00	.43
-Operational assets	.08	.00	.00	.06	.18
-Total assets(liabilities)	.20	.00	.08	.35	.24
-EBITDA	.21	.00	.13	.34	.44
-Interest expenses	.44	.09	.45	.74	.33

[28] I choose to focus on EBITDA to avoid any effects of disregarding depreciations and taxes.

Table 2.9: **Key ratios adjusted for capitalized rental charges.** The table shows the effects on key leverage ratios from capitalizing real estate and equipment rental charges. Variable and ratio descriptions are found in appendix A.

	Mean			**Median**		
	Book values	**Rental adj.**	Δ	**Book values**	**Rental adj.**	Δ
Total debt/Assets	.88	.85	-0.03	.78	.84	.06
Int.bearing debt/Tot.assets	.88	.85	-0.03	.25	.44	.20
Int.bearing debt/Inv. Capital	.71	.61	-0.09	.50	.70	.20
Int.bearing debt/EBITDA	1.51	6.60	5.09	.80	2.18	1.38
Int.bearing debt/Equity	1.50	3.95	2.45	.45	1.27	.82
Interest cover, est.	44.23	18.47	-25.76	2.32	1.73	-0.58
EBITDA/Total Assets	.1	.17	.07	.14	.18	.04

Table 2.8 shows how the capitalized asset/debt adjustment impacts asset-, debt-, EBITDA- and interest expenses levels for the sample. Real estate has the largest share of rentals where rented assets are on average 57 % (median 72 %) of the adjusted values. The distribution of this share is polarized with high densities at 0 and 100 % rented share. Rented operating assets make up only 7 % (median 0 %) of this adjusted asset-class[29]. These differences are due to both the size of the annual rental charges and the capitalization methods. The rented assets' share of total assets is 18 % (median 7 %) showing the significant importance of rented assets as an alternative financing source for Norwegian companies. The adjustments to EBITDA represents 30 % (median 38 %) of the adjusted number[30]. Financial rents represents 42 % of total adjusted interest expenses, but the polarized distribution and inclusion of more observations causes a reduced mean. Table 2.9 reports the original debt-ratios compared with those after the rental adjustments. As shown, the increases are large and consistent with increased leverage, except for the reduced mean of IBD/Invested Capital which is probably caused by distributional reasons. The overall picture illustrates the importance of including rented assets for a more complete capital structure analysis.

Table 2.10 splits the impact of rented assets by the industry groups analyzed in the previous section. Manufacturing industries represent 86 % of the capital database and the representativeness of the ratios for other industries necessarily suffers from small samples. It is clear that companies in the Services-sector rent less real estate and equipment than the average. This is intuitively correct since Services includes the real estate companies letting out properties themselves. The numbers also indicate that Agriculture/fisheries rent much less than the average and Transport & telecom and IT rent most real estate. Construction & energy, Oil and Trade & tourism are the sectors renting most operating equipment. The relative effect on EBITDA is highest for manufacturing industries.

This analysis of the effect of capitalized rents differ from existing literature in the magnitude of the data-set (50, 000 companies across 9 years), the dominance of private companies and the inclusion of real estate. The overall conclusion is that

[29] These numbers are necessarily dependent on the fairly conservative assumptions chosen, se Table 2.7.

[30] The EBITDA-ratio has been winsorized by 2 % symmetrically due to some large outliers since the adjusted EBITDA-level came close to 0.

Table 2.10: **Use of rented assets by industry group** The table shows the effects of capitalizing real estate and equipment rental charges by main industry groups. The table shows the relative shares of capitalized rented assets and financial rental charges of the adjusted numbers for real estate, operating long term assets, total assets and EBITDA in percentages in decimal form calculated for the years 1993 - 2002. The EBITDA-ratio is winsorized by 2 % symmetrically. Variable and ratio descriptions are found in Appendix A.

Industry group	n	Real estate Mean	Real estate Med.	Oper.assets Mean	Oper.assets Med.	Total assets Mean	Total assets Med.	EBITDA Mean	EBITDA Med.
Agri/fish	90	.34	.10	.01	.00	.10	.01	.09	.01
Oil	54	.70	.99	.07	.00	.21	.05	.10	.04
Mfg.ind.	33,222	.64	.98	.08	.00	.20	.09	.22	.14
Constr./ energy	330	.72	1.00	.11	.00	.17	.06	.20	.10
Trade & tourism	1,327	.69	1.00	.09	-	.18	.07	.20	.12
Transport &telecom	28	.84	.98	.03	-	.15	.04	.18	.03
Services	3,031	.38	.22	.05	.00	.15	.04	.15	.07
Culture	75	.79	1.00	.06	-	.17	-	.24	.08
IT	180	.89	1.00	.12	.00	.21	.10	.19	.12
All	38,337	.62	.93	.08	.00	.20	.08	.21	.13

an inclusion of capitalized rents adds important understanding to an analysis of capital structure.

2.7 Company characteristics explaining leverage.

An important descriptive finding is that firms are heterogeneous and show large differences in chosen capital structure. This heterogeneity is of course more apparent in this sample, since the whole population of firms is included, than in conventional samples which are limited to publicly listed firms. The detailed selection of variables including indicators like auditors' remarks makes additional categorizings possible.

In the present section, I will apply regression methodology to better understand what explain observed capital structure. In doing so, I also test the robustness of earlier studies on this data set, particularly in the 'partial adjustment' field (see e.g., Flannery and Rangan (2006)). To be able to utilize the scale of the data set, as well as significant subgroups, and also get results that are comparable to earlier studies, I limit the analysis to the more conventional measures found in the literature. In the first subsection I select and analyze single explanatory variables of leverage and turn in the next subsection to a full scale multivariate dynamic panel regression. To anticipate this analysis, I find that the study of subgroups to a significant degree may cancel out the heterogeneity of the total sample. When estimating annual adjustment rates, we also get relevant differences between the groups.

2.7.1 Univariate drivers of leverage.

There are no definite selection of sufficiently exogenous explanatory variables for a regression of capital structure. Frank and Goyal (2008) discuss theories of leverage and what factors may reliably explain it.[31] They group variables into growth, firm size, asset tangibility, profitability, industry leverage, dividends and expected inflation. Their selection is broadly supported by papers like Fama and French (2002), Lemmon, Roberts, and Zender (2007), Frydenberg (2004) and Petersen (2005), although variables covering volatility, accounting based growth indicators and firm age also are common. Generally following those found in the literature, I have selected 14 company characterizing variables based on expected economical relevance. Due to the dominance of private companies, any market-value related factors are necessarily left out:

- *Scale:* Total assets and total revenues, logged annual observations.
- *Profile:* Firm age in years, asset tangibility measured as long term assets over total assets and payable tax-rate measured relative to pre-tax profit.
- *Cashflow:* Cashflow to equity and to firm, average cashflow to firm for the years preceding the observation.
- *Performance:* Dividends (including intra-group contributions), one-year growth in net profit, return on average total assets, pre-tax (ROAA), current year and average across the firm-years preceding the observation.
- *Risk:* Standard deviation of ROAA and cashflow to firm for the years preceding the observation.

[31] The discussion is based on their empirical findings in Frank and Goyal (2004).

Average and volatility variables are measured across at least 3 years up to and including the current year. Univariate statistics for these variables are given in Table 2.11. The loss in number of observations from the original sample reported in Table 2.1 is primarily caused by the inclusion of mean and standard deviation measures which are not available for the first years of the panel.

Table 2.11: **Company characteristics.** 551,120 observations, the same subsample as applied in Table 2.12.

(NOK 1000, rates)	**Mean**	**p25**	**p50**	**p75**	**St.dev.**
Total Assets	3,502	1,339	2,783	7,005	2,677
Revenues	4,094	1,622	3,815	10,133	3,316
Age	13.59	5.00	9.00	16.00	14.09
Tangibility	.35	.06	.24	.59	.32
Tax-rate, payable	.18	-	.19	.32	.18
Cashflow, eq.	(35)	(192)	60	405	203,032
CF, firm	1,193	(103)	125	544	210,826
CF firm, avg.	250	(57)	72	300	82,066
Dividends	437	-	-	185	14,207
Profit growth, 1 y.	.02	- .06	.00	.07	.33
ROAA, pre-tax	.12	.03	.10	.20	.19
ROAA, average	.13	.06	.11	.18	.14
ROAA, st.dev.	.11	.05	.09	.15	.10
CF firm, st.dev.	6,403	166	419	1,213	160,061

I analyze the linkages between the selected variables and leverage for TDR, IBDR and ICL. Table 2.24 in Appendix C shows the Spearman correlation matrix between the leverage ratios and the variables. Table 2.12 shows the standardized (beta) coefficients of OLS-regressions of these leverage ratios on each of the characterizing variables[32], adding year-dummies and annual industry group leverage medians. I report significance level, F-values and R^2 for each variable/model. This approach allows for a partial analysis of how each variable explains leverage whilst controlling for industry- and calender year-effects.

Since the total debt ratio and the equity-ratio necessarily must sum to unity, the coefficients and significance levels may also be interpreted as explanations of the equity ratio. All 'positioning'-variables and those related to profits (log total assets,

[32]Both in the univariate and multivariate regression analysis to follow, I use accounting variables relative to total assets and all variables lagged one year, except for averages and volatility variables

Table 2.12: **Leverage and company characteristics.** The table presents standardized (beta-) coefficients, F-values and R^2 for models with the different leverage measures Interest bearing debt/Total assets, Int.bearing debt/Invested Capital and Int.bearing debt/Invested capital as dependent variables. The independent variables are each variable independently, measured relative to total assets if not already being a ratio and lagged one year, calender year dummies and industry group average leverage ratios. Regression with robust standard errors clustered by firms over the years. Significance of coefficients at 1 %, 5 % and 10 % levels are conventionally marked with ***,**,* respectively. Variable univariate statistics are given in Table 2.11 and variable descriptions are found in Appendix A. 551,120 observations.

	Debt/TA			**IBD/TA**			**IBD/Inv.Cap.**		
	Coeff.	F	R^2	Coeff.	F	R^2	Coeff.	F	R^2
Scale:									
ln Total assets	-.095***	857	.01	.049***	958	.01	-.017***	22	.00
ln Tot.Revenues	.109***	623	.02	-.136***	1,460	.03	.012***	9	.00
Profile:									
Firm age	-.089***	502	.01	-.036***	727	.01	-.016***	44	.00
Asset tangibility	.047***	317	.01	.277***	5,198	.08	.020***	34	.00
Tax rate, paybl.	-.149***	1,724	.03	-.191***	3,503	.05	-.053***	171	.00
Cashflows:									
-CF equity	-.005**	267	.01	-.009**	703	.01	-.003**	5	.00
-CF firm	-.000	267	.01	-.004**	704	.01	-.001*	5	.00
-CF firm, avg.	.012	266	.01	.002	703	.01	.001	4	.00
Performance:									
Dividends	-.077***	597	.01	-.160***	1,248	.04	-.037***	211	.00
Revenue gwth,1y.	-.001*	266	.01	-.000	700	.01	-.000*	5	.00
ROAA, pre-tax	-.128***	479	.02	-.160***	1,436	.04	-.050***	45	.00
ROAA, avg.	-.161***	822	.03	-.202***	2,018	.05	-.066***	54	.00
Risk:									
ROAA, st.dev.	.106***	502	.02	-.042***	719	.01	.026***	17	.00
CF firm, st.d.	.025**	265	.01	.019**	706	.01	.005*	5	.00

age, payable tax rate, cashflows, dividends, ROAA) are significantly negatively related to the total debt ratio, supporting the view that profits are used to reduce debt-levels over time. Of these, tax rate and average ROAA have both the largest coefficients and most explanatory powers. The standard deviation of ROAA and of firm cashflows carry positive coefficients which indicate that previous volatility either has depleted equity or created a need for new debt of some kind. An alternative explanation may be that with increased risk, equityholders' would want to offload more risk onto lenders through increased leverage. Total revenues has a positive coefficient, the explanation may be that increased revenues creates more trade debt, although the growth itself is weakly negative. Asset tangibility reflects the security available for lenders and is unsurprisingly positively related to leverage across all ratios.

The IBD-ratio gets the same signs, magnitude and significance as the total debt ratio on almost all the variables. The exception, in addition to total revenues, is the large, negative and strongly significant coefficient for the volatility of ROAA. This indicates that past volatility is viewed negatively by lenders and reduces the availability of formal, interest bearing debt for the company. Similarly, the positive coefficient on total assets supports an argument that increased size reduces informational asymmetry making outside financing more efficient. The negative coefficient for total revenues may be a proxy for profits as discussed above since any increases in trade debt necessarily are not included in the IBD-ratio.

The final ratio, ICL, is effectively the company's mix between two alternative forms of capital employed[33]. Most variable coefficients are significant and some models get sensible F-statistics, although none provide any meaningful explanations in terms of R^2. Most coefficients have the same signs and significance as for IBDR and the magnitude also reflects that the dependent variable have larger absolute values.

2.7.2 Multivariate panel regression analysis of leverage.

I have so far analyzed the capital structure of Norwegian companies descriptively and by univariate regressions which provide an overview and primarily partial explanation of capital structure. Given the low R^2-values from the univariate regressions in the previous section, it is clearly a demanding task to define an appropriate regression strategy with this heterogeneous sample. I have chosen to proceed with a regression of the book-leverage ratio in order to analyze both private and public firms. Since the data-set lacks any market values reflecting the cost-of-capital and also have limited information regarding outside financing events, I find it in any case less suitable for a test of the pecking order theory.

The book leverage ratio I study is 'Interest bearing debt relative to total assets' (IBDR). It is well known that this measure is highly persistent. E.g. Lemmon, Roberts, and Zender (2007) shows that in their sample of all non-financial firms

[33] Capital employed and invested capital are synonyms.

in the CRSP/Compustat database for the years 1965 - 2003, an ADL-regression[34] on book leverage, the coefficient on last year's ratio in explaining the current level is .87 with an R^2 of .72. Similarly, my sample including SMEs and primarily private firms shows a coefficient of .85 with an R^2 of .69. Decomposed, the IBDR's standard deviation of .986 is separated into .964 between the 138,990 firms and .467 within the on average 6.14 yearly observations by firm. Industry median leverage shows the same pattern of within group persistence with an overall standard deviation of .075, a between statistic of .078 and a within statistic of .014. By testing this autoregressive feature of IBDR using simple pooled-OLS, I get residuals with significantly positive coefficients between current year and across the whole sample (up to 10 years).

Methodological considerations.

Given the persistence in the leverage ratio, I need to include last year's IBDR as explanatory variable in what is denoted a 'dynamic panel regression'. By including the lagged IBDR-measure, I regress a partial adjustment- model following recent papers like Lemmon, Roberts, and Zender (2007), Chang and Dasgupta (2006) and Flannery and Rangan (2006). Partial adjustment refers to the degree of adjustment a company does from last year's IBDR towards the assumed optimal leverage ratio. The optimal IBDR is predicted annually by a set of current[35] explanatory variables. The generally applied regression model is

$$Y_{it} = \alpha Y_{i,t-1} + \beta X_{i,t-1} + \nu_i + \varepsilon_{it} \tag{2.7.1}$$

where

$$i = 1, \ldots, N; t = 1, \ldots, T,$$

and Y_{it} is interest bearing debt ratio and X_{it} is a matrix of explanatory variables for the i firms across the t years, ν_i is a firm specific fixed effect and the residuals are assumed to be $\varepsilon_{it} \sim IID(0, \sigma_\varepsilon^2)$. The lagged left-hand variable introduces mutual interdependencies between the current level of Y and last period's residuals and vice versa creating a downward bias in fixed effect panel coefficients as shown by Nickell (1981). The dynamic effect on Pooled OLS-coefficients is an upward bias

[34]No additional control variables included.

[35]The actual variables applied are usually lagged one year to reduce the endogeneity problem.

and the estimators are inconsistent for fixed T, whilst the fixed effect estimators regain consistency for large T.

Arellano and Bond (1991) developed a first difference General Methods of Moments (GMM)-model for dynamic panels utilizing as instruments all available lagged first difference of the dependent variable and the first differences of the other explanatory variables. This first differences regression method accepts both sequentially exogenous explanatory variables, and first order autocorrelation in the explained variable, and utilizes as all previous realizations of the variables as instruments. It is regarded as particularly suitable for panels with a large number of cross-sectional observations across relatively few time periods. These models will typically have more instruments than endogenous variables and thus allows for testing whether the model is overidentified or has unobserved variables using Sargan/Hansen test (Hansen (1982)). Arellano and Bover (1995) and Blundell and Bond (1998) demonstrate that first difference GMM performs poorly if the autoregressive coefficient, α, is large and approaches unity and the data has large variance in the fixed effect relative to the idiosyncratic variance. My dataset most likely falls into this category. Blundell and Bond (1998) develop a 'system GMM'-model which estimates the regression in levels by utilizing both lagged levels and first differences as instruments and as such obtain more instruments for a given panel. In my application of this approach in Table 2.13 I also compare the results of this model with more conventional pooled- and panel fixed effect- instrumental regressions, all in levels.

Choice of explanatory variables

As my research focus is to better understand the capital structure choices of firms' managers, I use only lagged variables and no year-dummies to replicate the likely information set of the decision maker for each observation. I do include lagged median industry IBDR to control for common industry effects. The use of lagged variables either directly or as instruments also limits the endogeneity of current-year observations. Starting out with the previously discussed set of 14 likely explanatory variables as analyzed in Table 2.12, I did a stepwise OLS-regression procedure using Wald-tests to suggest variables for multivariate analysis of leverage. The regressions applied Rogers' standard errors (clustered on firms) which are also robust for heteroscedasticity by the Huber/White-sandwich estimator, following Petersen (2005). With the help of this procedure and the univariate analysis of relevance, I selected the reported variables.

Regression results

Table 2.13 reports the results from alternative regressions of IBDR in levels. Column one reports a pooled instrument(IV) regression where 2 lags of IBDR are included as exogenous explanatory variables and the current values of the remaining variables are instrumented by 3 lagged levels. Column two reports the same specification as panel IV regression with a fixed firm effect. Column three reports the system GMM regression following Blundell and Bond (1998) with the same lag structure as in the previous regressions. System GMM also instruments the lagged values of IBDR and differs in this respect different from the two previous specifications. Due to the size of the dataset and computational demanding method, I had to apply the system GMM-regression on smaller samples. To avoid any selection bias, I randomly selected 150 samples (with replacement) of 1 % each from the data-set. Each individual regression sample included approximately 7,000 firms and 47,000 firm-years. The table reports average coefficients and p-values from the system GMM-regression as well as calculated long term steady state coefficients.[36] The IV regressions are run on the whole sample, although the inclusion of lagged variables has caused a reduced regression sample.

The comparison of the results shows that as predicted, the pooled IV- overstates and fixed effect IV-regression understates the coefficient on the lagged IBDR variable. The coefficients on the other variables show no systematic relationship, neither in the short- or long-run. When comparing these three similarly specified regressions, the most striking difference, in addition to the magnitude of the coefficient on the lagged IBDR-variable, is that the other β-coefficients change both sign, significance and magnitude. Most of the variables get the smallest and least significant coefficients in the System GMM-regression and the largest in the pooled IV regression. These results are perhaps not surprising since both pooled IV and fixed effect IV are inconsistent in a dynamic panel specification.

[36] With reference to equation (2.7.1), the long term coefficients of the explanatory variables are calculated as $\frac{\alpha}{(1-\beta)}$, implicity assuming that the firm is in a steady state.

Table 2.13: **Panel regressions of leverage.** The table presents current and long-term coefficients and p-values for cross-sectional and fixed effect instrument-regressions, and a system GMM regression of leverage (Interest Bearing Debt Ratio, IBDR) in a dynamic specification. The system GMM regression result is the average from 150 randomly selected (with replacement) 1 % samples from the dataset. Significance of coefficients at 1%, 5 % and 10 % levels are conventionally marked with ***, ** and *, respectively. ρ is the autoregressive coefficient of ε in a regression with no constant term. Variable descriptions are found in Appendix A.

IBDR	IV	IV LT	XTIV	XTIV LT	GMM	GMM LT
$IBDR_{t-1}$	.821***		.427***		0.648***	
	(0.009)		(0.003)		(0.000)	
$IBDR_{t-2}$	.054***		-0.009***		.048*	
	(0.007)		(0.002)		(0.056)	
ln Total Assets	.002***	.008	.036***	.062	-0.006	-.020
	(0.000)		(0.002)		(0.386)	
ln Total Assets(t-1)					0.023	
					(0.362)	
ln Turnover, rel.	.024***	.192	-0.198***	-.340	0.129	.095
	(0.005)		(0.013)		(0.318)	
ln Turnover, rel.(t-1)					-0.100	
					(0.361)	
Tangibility	.038***	.304	.045***	.077	0.130**	.322
	(0.004)		(0.006)		(0.046)	
Tangibility(t-1)					-0.032	
					(0.288)	
Taxrate, payabl.	-0.038***	-.304	-0.237***	-.407	-0.049	-.184
	(0.006)		(0.014)		(0.372)	
Taxrate, payabl.(t-1)					-0.007	
					(0.480)	
Dividend, rel.	.186***	1.488	-0.499***	-.857	-0.031	.095
	(0.035)		(0.035)		(0.421)	
Dividend, rel.(t-1)					0.060	
					(0.247)	
ROAA	-0.193***	-1.544	.057***	.098	.033	.000
	(0.029)		(0.014)		(0.357)	
ROAA(t-1)					-0.033	
					(0.357)	
ROAA, avg.	.023*	.184	-0.007	.012	-0.677	-.628
	(0.010)		(0.019)		(0.130)	
ROAA,avg(t-1)					0.486	
					(0.136)	
Industry IBDR(t-1)	-0.043	.264	-0.057	.040	-0.235*	-.980
	(0.031)		(0.030)		(0.058)	
Industry IBDR(t-2)	.076*		.080*		0.063	
	(0.031)		(0.034)		(0.421)	
NIBOR	-0.057	3.272	-0.371***	.679	-0.200*	.480
	(0.031)		(0.033)		(0.079)	
NIBOR(t-1)	.379***		.286***		0.372**	
	(0.045)		(0.041)		(0.028)	
NIBOR(t-1)	.087		.480***		-0.026	
	(0.055)		(0.048)		(0.425)	
Constant	-0.035***		.093***		-0.050	
	(0.006)		(0.021)		0.436	
No. of observations:	237,465		237,465		47,000	
R^2	0.7817		0.6469		*Repeated*	
$\rho(\varepsilon_t, \varepsilon_{t-1})$	-0.057***		-0.057***		*samples*	

The unbiased system GMM-regression is useful for estimating the speed of adjustment, but the limited selection of significant explanatory variables reduces the applicability weaker. When comparing the results with Lemmon, Roberts, and Zender (2007), Table 6, there are similarities in terms of switching signs on coefficients between methods, although more of their coefficients are significant. As assumed for the system GMM method, there is first-order autoregression in the residuals, and except for a few samples, no significant autoregression of higher order. However, the Sargan test of overidentification rejects the null-hypothesis of no overidentification in 55 % of the sample regressions. Given that the samples are large, referring to Davidson and G.MacKinnon (2004) page 368, misspecification or unobserved variables may also cause significant rejections of this test. The far fewer rejections of the test in the study of more homogeneous subsamples below indicate that unobserved variables present in the large sample may be the actual cause of the high rejection rates.

2.7.3 Panel data analysis by subsamples

Table 2.14[37] reports selected results of the panel data regressions for the whole sample, the smallest and the largest size-quintiles, firms paying dividends at least 2/3 of the reported years and firms having auditor comments in their accounts in more than 1/4 of the years. These definitions of groups reflect firstly the view that as a consistent dividend payer, a firm have to deliver a stable dividend stream. Secondly, since auditor remarks are assumed to inflict lasting reputational damage, I assume that remarks in 1/4 of the years is sufficientas a classification.

In these regressions, selected variables get meaningful coefficients that allows for comparisons between subsamples. We will return to a discussion of the differences in the speed of adjustment below and leave it for now. The smallest and most distinct group, listed companies, also has the most significant coefficients and broadly confirm conventional results in that leverage is growing in size, tangibility and industry leverage, and falling in profitability and interest rate levels. Another fairly distinct and probably financially distressed group, are the firms with auditor comments. The main difference from the listed firms is that dividends get a large and significant coefficient. Assuming that these firms are financially constrained

[37] Since only significant coefficients are reported in this table, a complete table is found in 2.25.

Table 2.14: **Regressions of leverage by subsamples.** The table presents the significant coefficients from a System GMM regression of leverage (Interest Bearing Debt Ratio, IBDR) in a dynamic specification. The system GMM regression result is the average from repeated sampling from the dataset. Only significant variables are reported. A complete table including all coefficients and p-values is found in Table 2.25 in the Appendix. Variable descriptions are found in Appendix A.

	All firms	Scale: Large	Small	Dividend payers? Yes	No	Auditor remarks? Yes	No	Listed
$IBDR_{t-1}$	.648***	.613***	.682***	.740***	.637***	.625***	.697***	.664***
$IBDR_{t-2}$	.048*	.048*	.077**	.054***	.048**	.047*	.053***	
Tot.Assets				.081**				
Tot.Assets(t-1)				-.076**				.020***
Turnover								-.110***
Turnover(t-1)								-.071**
Tangibility	.130**	.154**		.055*	.161**	.183*	.126**	.133***
Tangibility(t-1)								-.127***
Taxrate								-.040***
Taxrate(t-1)								.064***
Dividend						.241***		
Dividend(t-1)				.155***			.681***	
ROAA				-.140**				
ROAA, avg.						-1.183*		-.180**
ROAA, a.(t-1)							.125**	
Ind.IBDR(t-1)	-.235*	-.321***		-.177**	-.234*		-.244**	.090**
Ind.IBDR(t-1)								.063*
NIBOR	-.200*	-.232*		-.201***		-.241	.595***	
NIBOR(t-1)	.372**	.551***			.403*		.393**	-.455***
NIBOR(t-2)								-.208***
N(firms, avg.)	7,000	3,900	3,200	4,300	5,400	3,700	5,500	166
N(firm-yrs., avg.)	47,000	28,500	17,000	34,000	35,000	22,400	35,000	1,026
Samples	150	75	75	75	75	75	75	All
Sample size (%)	.01	.03	.03	.03	.0075	.03	.0075	All
Sargan***(%)	.55	.76	.01	1.00	.47	.09	.60	.00

and have limited access to debt, this indicates that when there is cashflow available to pay dividends, this is also a useful indicator of access to debt. Comparing the coefficients with the coefficients in Table 2.12, all variables except relative dividends retain their signs although the magnitude may not be compared due to differences in methodology and specification. As to the methodological choices, the regressions by subgroups show that the most distinct groups, small companies, listed and those with auditor remarks, somehow have less problem with unobserved variables and/or overidentification whilst the broader groups, e.g., large firms and dividend payers are still very heterogenous and the Sargan-test rejects the null as discussed above.

Flannery and Rangan (2006) test firm's annual adjustment towards the optimal leverage by defining $(1 - \alpha)$ as the adjustment speed from last year's leverage ratio. Our coefficient on lagged IBDR as reported in Table 2.15 represents a target leverage adjustment speed of 35 %, very close to the 34 % found by Flannery and Rangan (2006) in a fixed effect regression on US Compustat data for the years 1965 - 2003. Their long time dimension may be part of the explanation why there is no apparent bias between their result and my system GMM results. It is also somewhat surprising that the fundamental differences between the selection of firms do not seem to matter in this aspect. We see large variations across the subsamples, between large firms that close 39 % of the gap per year and the the dividend-payers that close only 26 % and thus have the most persistent leverage ratio. The latter may indicate no effective firm-related financing constraints and that dividend-paying firms are less concerned about optimizing their capital structure. Towards the other end of the range, firms with auditor comments may be expected to be more financially constrained and as it seems, be more inclined to optimize their capital structure given today's situation and less satisfied with keeping last year's leverage ratio. The inclusion of 'Speed of adjustment' calculated using fixed effect IV regression highlights how vulnerable such analyzes are to choice of method, as theoretically predicted. As reported in Table 2.15, neither the magnitude of the coefficients nor the internal ranking between the subsamples show similarities between the methods.

Table 2.15: **Speed of adjustment - FE vs. System GMM.** The table presents the estimated speed of adjustment coefficients from a System GMM regression of leverage (Interest Bearing Debt Ratio, IBDR) in a dynamic specification compared with a fixed effect instrumented panel regression. The unreported explanatory variables are log total assets, log Total return, payable taxrate, dividends relative to total assets, return on average total assets (current and average historic), industry median IBDR and annual average NIBOR 3 mnth. interest rate. The system GMM regression result is the average from repeated samplings from the dataset. Variable univariate statistics are given in Table 2.11 and variable descriptions are found in Appendix A.

Method:	**Fixed effect**		**System GMM**	
	Speed of Adj.	Rank	Speed of Adj.	Rank
All	*.573*		*.352*	
Largest quintile	.593	(1)	.387	(1)
Smallest quintile	.573	(4)	.318	(5)
Dividend payers.	.556	(5)	.260	(7)
Non-dividend payers	.576	(3)	.363	(3)
Auditor-comments	.549	(6)	.375	(2)
No auditor comm.	.581	(2)	.303	(6)
Listed	.521	(7)	.336	(4)

2.7.4 Summary on the regressions

The panel regression results have illustrated the challenging task of building a useful regression model for testing capital structure-theories. The well motivated 'System GMM'-method produce both statistically weak and economically less useful coefficients. Compared with other papers that apply this method, the short time-span of the data-set may be part of the explanation. On the other hand, the model seems to produce reasonable estimates for the leverage 'speed of adjustment' between years which allows for comparisons with related research. The comparison in in Table 2.15 is in itself a useful example of the fixed effect bias for shorter panels. Another methodological issue is the variations in the rejection rates of the Sargan test by subsamples as reported in Table 2.14. Only distinct subsamples of either listed firms, small firms or firms with auditor remarks seem to be sufficiently homogenous to eliminate the effects of a potentially common unobserved variable. The comparable sizes of the regression samples implies that these effects are not related to number of observations as such. Future research may further explore alternative sample definitions including industry groups.

2.8 Conclusion

This paper is a comprehensive documentation of the capital structure of a population dataset of private and public Norwegian companies. I conduct a complete descriptive analysis of leverage by alternative ratios, over time, by company categories and finally by regression methods. I show that although Norwegian companies' capital structure has been fairly stable over the period 1992-2005, there are several generic discoveries.

Firstly, the very high degree of persistency in firm-leverage in this data set of primarily private companies gives an estimated speed of adjustment very close to the levels found in far longer dated samples including US publicly listed firms. Using the breadth of the dataset, clear differences in this speed of adjustment are found between more or less distinct subgroups. The latter refers both to the general commonality between firms in a subgroup and similarities in terms of financing possibilities.

Secondly, I document large operational and financial differences between company categories and industry groups through the descriptive analysis This compares well with the US findings on public companies in Frank and Goyal (2004) regarding the importance of industry leverage.

Thirdly, the paper includes an analysis of the capital structure of a selected subsample adjusted by capitalized rents. This analysis finds significantly increased leverage ratios for large parts of the sample, although varying by sectors, and is a first attempt to recognize the importance of off-balance-sheet commitments in the complete study of capital structure. The small sample made it less relevant to base the panel regression analysis on these firms.

Fourthly, the Norwegian taxation of companies and investors allows for a test of use of debt without tax benefits. The results show large debt ratios (Total debt ratio, median: 81 % / Interest bearing debt ratio, median: 25 %). Selected periods with taxation of dividends at investor level show large and predictable swings in dividend payouts, but no changes in the sources of new financing. Univariate data actually indicate reduced use of debt in years with dividend taxation. This subject is in itself an interesting area for further research.

Overall, the dominanting impression from the analysis is the persistence in leverage ratios across the whole sample, although with differences by subgroups.

2.9 References

Agrawal, A., and C. R. Knoeber, 1996, Firm performane and mechanisms to control agency problems between managers and shareholders., *Journal of Financial and Quantitative Analysis* 31, 377–397.

Akerlof, George A., 1970, The market for 'lemons': Quality uncertainty and the market, *Quarterly Journal of Economics* 84, 488–500.

Arellano, M., and S. R. Bond, 1991, Some tests of specifications for panel data: Monte carlo evidence and an application to employment equations, *Review of Economic Studies* 58, 277–297.

Arellano, Manuel, and Olympia Bover, 1995, Another look at the instrumental variable estimation of error-components models, *Journal of Econometrics* 68, 29–51.

Baker, Malcolm, and Jeffrey Wurgler, 2002, Market timing and capital structure, *The Journal of Finance* 57, 1–32.

Berger, Allen N., and Gregory F. Udell, 1998, The economics of small business finance: The roles of private equity and debt markets in the financial growth cycle., *Journal of Banking and Finance* 22, 613–673.

———, 2006, A more complete conceptual framework for sme finance., *Journal of Banking and Finance* 30, 2945–2966.

Blundell, Richard, and Stephen Bond, 1998, Initial conditions and moment restrictions in dynamic panel data models, *Journal of Econometrics* 87, 115–143.

Booth, Laurence, Varouj Aivazian, Asli Demirguc-Kunt, and Vojislav Maksimovic, 2001, Capital structures in developing countries, *Journal of Finance* 56, 87–130.

Carlsen, Fredrik, and Øivind Anti Nilsen, 1993, En empirisk analyse av norske bedrifters gjeldsandel, *Beta. Tidskrift for bedriftsøkonomi* pp. 1–13.

Chang, Xin, and Sudipto Dasgupta, 2006, Target behavior and financing: How conclusive is the evidence?, Working paper.

Coppens, Laurent, and Erik Peek, 2004, An analysis of earnings management by european private firms, Working paper.

Davidson, Russell, and James G.MacKinnon, 2004, *Econometric Theory and Methods* (Oxford University Press).

Fama, Eugene F., and Kenneth R. French, 2002, Testing trade-off and pecking order predictions about dividends and debt, *The Review of Financial Studies* 15, 1–33.

Fischer, E., R. Heinkel, and J. Zechner, 1989, Dynamic capital structure choice: theory and tests., *Journal of Finance* 44, 19–40.

Flannery, Mark J., and Kasturi P. Rangan, 2006, Partial adjustment toward target capital structures, *Journal of Financial Economics* 79, 469–506.

Frank, Murray Z., and Vidhan K. Goyal, 2004, Capital structure decisions: Which factors are reliably important?, Contemporary Corporate Finance Issues III Tuck School of Business at Dartmouth July 9-10, 2004.

———, 2008, *Trade-off and Pecking Order Theories of Debt*vol. 2 of *Handbook of Corporate Finance: Empirical Corporate Finance* . chap. 12 (Handbooks in Finance Series, Elsevier/North-Holland).

Frydenberg, Stein, 2004, Determinants of corporate capital structure of norwegian manufacturing firms, Doctoral thesis Sør Trøndelag University College.

Fülbier, Rolf U., Jorge Lirio Silva, and Marc Henrik Pferdehirt, 2008, Impact of lease capitalization the on financial ratios of listed german companies, *Schmalenbach Business Review* 60.

Goldstein, Robert, Nengjiu Ju, and Hayne Leland, 2001, An ebit-based model of dynamic capital structure, *The Journal of Business* 74, 483–512.

Grossman, S.J., and O. D. Hart, 1986, The cost and benefit of ownership: A theory of vertical and lateral integration., *Journal of Political Economy* pp. 691–719.

Halov, Nikolay, and Florian Heider, 2004, Capital structure, risk and asymmetric information, Discussion paper, NYU Stern School of Business and European Central Bank.

Hansen, Lars Peter, 1982, Large sample properties of generalised method of moment estimators, *Econometrica* 50, 1029–1054.

Harris, Milton, and Arthur Raviv, 1991, The theory of capital structure, *The Journal of Finance* 46, 279–355.

Hennessy, C.A., and T.A. Whited, 2005, Debt dynamics, *Journal of Finance* 60, 1129–1165.

Hol, Suan, 2004, Essays on capital structure, debt maturity, trade credit, and default probability, Ph.D. thesis Norwegian University of Science and Technology.

Huyghebaert, Nancy, and Linda M. Van de Gucht, 2007, The determinants of financial structure, *European Financial Management* 13, 101–133.

Imhoff Jr., Eugene A., Robert C. Lipe, and David W. Wright, 1991, Operating leases: Impact of constructive capitalization., *Accounting Horizons* pp. 51–63.

Jensen, Michael, 1986, Agency costs of free cash flow, corporate finance and takeovers, *American Economic Review* 76, 323–329.

———, and William H. Meckling, 1976, Theory of the firm: Managerial behavior, agency costs and ownership structure, *The Journal Of Financial Economics* 3, 305–360.

Lemmon, Michael L., Michael R. Roberts, and Jaime F. Zender, 2007, Back to the beginning: Persistence and the cross-section of corporate capital structure., *Journal of Finance* Forthcoming.

Maksimovic, Vojislav, 1988, Capital structure in repeated oligopolies, *Rand Journal of Economics* 19, 389–407.

Modigliani, Franco, and Merton H. Miller, 1958, The cost of capital, corporation finance and the theory of investment, *American Economic Review* 48, 655–669.

Myers, S., D. Dill, and A. Bautista, 1976, Valuation of financial lease contracts, *Journal of Finance* 31, 799–819.

Myers, Stewart C., 1984, The capital structure puzzle, *The Journal of Finance* 39, 575–592.

———, and Nicholas S. Majluf, 1984, Corporate financing and investment decisions when firms have information that investors do not have, *Journal of Financial Economics* 13, 187–221.

Nelson, A. Tom, 1963, Capitalizing leases: The effect on financial ratios., *Journal of Accountancy* 41, 49–58.

Nickell, Stephen, 1981, Biases in dynamic models with fixed effects, *Econometrica* 49, 1417–1426.

Petersen, Mitchell A., 2005, Estimating standard errors in finance panel data sets: Comparing approaches, Working Paper 11280 NBER.

Rajan, Raghuram G., and Luigi Zingales, 1995, What do we know about capital structure? some evidence from international data, *The Journal of Finance* I, 1421–1460.

Raknerud, Arvid, Dag Rønningen, and Terje Skjerpen, 2004, Documentation of the capital database, Document 2004/16 Statistics Norway Research Department.

Tirole, Jean, 2006, *The Theory of Corporate Finance* (Princeton University Press: Princeton and Oxford).

2.A Specification of variables.

2.A.1 Accounting and firm variables.

The accounting variables are included as annual observations for 1992-2005. Company fixed variables are given only as of 2005. Selected variables are described below:

- *Investments* consists of securities and other financial investments held as short-term assets and are not netted against debt.
- *Intangible assets* is the sum of goodwill, immaterial assets and, before 1999, activated costs.
- *Interest bearing debt* is the sum of long-term debt, excluding provisions, deferred tax liability or similar items, plus short term specific interest bearing debt. Long-term debt is a reliable number from the accounts. Short term debt includes bank-debt, overdrafts and short-term market debt, but lacks a complete specification of other interest bearing short-term debt. "Short term interest-bearing debt" is probably underestimated.
- *Equity* consists of total shareholders funds plus minorities in the case of consolidated accounts.
- *Invested capital* is the sum of interest bearing debt and equity. Due to no information regarding operationally required levels of cash nor whether investments are financial or strategic, these items have not been deducted to get to invested capital.
- *Net working capital* consists of total short term assets less accounts payable, taxes and duties due, dividends and group contributions payable.
- *New interest bearing debt* is a calculated item being the change in debt over the year plus any instalments paid during the year. Payments on short-term debt within the year is not fully specified in this approach.
- *New equity* is the net change in equity excluding profits and dividends, recorded in cases where the nominal share capital has changed between the years.

- *Interest cost* is calculated as total financing costs less any write-downs. This is a crude measure and the spread after deducting the NIBOR-rates have been symmetrically winsorized by 15 % to standardize the most extreme outliers.

- *NIBOR* The 3 month NOK Norwegian Interbank Offered Rate, annual average as reported by Norges Bank, the Central Bank of Norway.

2.A.2 Definition of calculated variables and ratios

The subscript t denotes the current year.

Leverage related ratios

- Total debt ratio(TDR) is defined by:

$$TDR = \frac{Debt}{TA}$$

  where $Debt$: All non-equity liabilities of any kind on the balance sheet and TA: Total Assets.

- Interest bearing debt ratio($IBDR$) is defined by:

$$IBDR = \frac{IBD}{TA}$$

  where IBD: Interest Bearing Debt.

- Invested Capital Leverage (ICL) is defined by:

$$ICL = \frac{IBD}{IBD + EQ + MINT}$$

  where EQ: Common shareholders' funds and $MINT$: Capital belonging to minority shareholders in consolidated groups. The combined denominator equals Invested Capital (IC).

- Gearing (GE) is defined by:

$$GE = \frac{IBD}{EBITDA}$$

  where $EBITDA$: Earnings before (net)interest costs, taxes, depreciations and amortizations.

- Interest cover ($ICOV$) is defined by:

$$ICOV = \frac{EBIT + FI}{IC}$$

where $EBIT$: Earnings before (net) interest and tax, FI: Financial income and IC: Estimated interest costs using average IBD and winsorized nominal interest rates.

- Debt maturity is defined by

$$GMAT = \frac{LIBD}{IBD}$$

where $LIBD$: Long-term interest bearing debt, maturity 1 year.

Performance variables and ratios.

- Asset turnover is calculated as:

$$ATO = \frac{TR}{((OPA_t + OPA_{t-1})/2)}$$

where TR is total revenues and OPA is long term operating assets.

- Asset tangibility is calculated as:

$$TANG = \frac{OPA}{TA}.$$

- Cashflow to firm is calculated by:

$$CF_{fi} = OP + DA - TAXO \pm NINV \pm NPROV + FININAT.$$

where OP is operating profit, DA depreciations and amortizations, $TAXO$ is payable tax attributable to operating profit, $NINV$ is net new investments, $NPROV$ is net new provisions and $FININAT$ is financial income after tax.

- Cashflow to equity is calculated by:

$$CF_{eq} = CF_{fi} - ICAT - DPAY$$

where $ICAT$ is interest costs after tax and $DPAY$ any other payment to debtholders including instalments. These items are mostly calculated as differences between years since explicit cashflow statements are missing.

- Operating margin is defined by:

$$OPM = \frac{OP}{TR}$$

- Return on average assets (pre-tax) is defined by:

$$ROAA = \frac{EBIT + FI}{((TA_t + TA_{t-1})/2)}$$

- Return on average equity (pre-tax) is defined by:

$$ROAE = \frac{NP}{((EQ_t + EQ_{t-1})/2)}$$

  where NP is net profit after tax.

- Payoutrate is defined by

$$PR = \frac{(DIV + IGC)}{NP}$$

  where DIV is dividends and IGC is intragroup contributions.

All standard deviations are estimated over no less than the 3 years up to and including the current reporting year.

2.B Data-set issues and adjustments

Reported accounting data are produced by companies at fixed intervals rather than being the result of a market clearing process like, e.g., the time-series of share-prices. Any accounting data-set therefore includes a number of potentially important sources of mistakes, in addition to the effect of only yearly frequencies. The accuracy of accounts is subject to a range of managerial considerations which outsiders cannot verify. These include e.g., depreciation-periods/rates/methods, asset valuations or write-offs, immaterial assets and timing of revenues and costs. In particular, publicly listed companies may conduct 'earnings management' in an attempt to improve perceived performance. Coppens and Peek (2004) finds differences in earnings management between public and private firms. The current data-set consists predominantly of private companies (99.8 % share by number, 65 % by assets) exposed to the the additional risk of:

- Mistakes, either in the accounts or, more likely, in the entering of items into the database.
- Negative equity and/or debt caused by off-balance sheet contractual arrangements, large differences between market- and book- values of assets or any undisclosed transitory effects between reporting years.
- Tax-motivated depreciation and timing practices disregarding the impact on financial reporting.
- General disregard of the principles of true and fair accounting for profit and capital elements. This may either be due to irrelevance of outsiders or that relevant outsiders receive reliable information through other channels.

The overriding purpose in this paper of describing Norwegian capital structure is to present the companies' typically chosen financing structure by extracting empirical understanding from leverage data and control-variables. Observations which with reasonable certainty can be deemed extreme or directly inconsistent will not add value to the analysis. Table 2.16 shows the effects of alternative ways to limit the data-set and their effects on the number of observations and on the first moments of total return and leverage. The starting point is the combined population data-set from column 4 in Table 2.1. I analyze the effects from the alternative limitations on two central ratios, *return on assets (pre-tax, end-of-year assets)* and *total debt-to-total asset ratio*. The first set of limitations are economical in that 'unreasonable' observations are dropped. Positive equity, interest-bearing debt-ratio ($IBD/Totalassets$) below 1 and return on assets (ROA) ≥-100 % are all sensible overall exclusion criteria. Observations for companies outside these criteria must either have explanations outside the financial accounts or simply be mistakes. As an example, real estate-companies with low book asset values compared to market values may have negative book equity but still be viewed as solvent, also by informed outsiders, but may violate the suggested limitations. Table 2.16 indicates the outlying observations in the original data-set shown by the large drops in standard deviations even with moderate reductions in the number of firm-years.

The alternative approach is to winsorize the data-set by excluding the tails of the distributions. I illustrate this by excluding the top and bottom 1, 2 and 5 %

of the ROA-observations in each calender year. The ROA- and the debt-ratios have obvious outliers, e.g., a 2% cut in both tails of the ROA-distribution gives a reduction in the standard deviation of ROA by 64.8 % and of the debt-ratio of 43.8 %. This method is purely statistical and involve no economical reasoning or subjective judgement, but highlights the impact of outliers when using the data.

Table 2.16 shows that the two alternative approaches to limiting the data-set works in different ways. The economical limitations increase the means and medians and reduce the variation in the data-set by excluding loss-making and over-levered firm-year observations. The statistical criteria decrease the means, necessarily leave the medians unchanged, and provide a large drop in the variation, since the method cuts the data-set symmetrically.

Based on this, I have chosen to limit the data-set by a 1 % winsorization of the ROA-distribution, each year. The outliers have been dropped and not replaced by the 1 % and 99 % percentiles. This 10 % cut in the number of firm-years leads to a 71.6 % reduction in the ROA -ratio standard deviation and a 47.6 % reduction in the $Debt/(TotalAssets)$-ratio standard deviation. The mean ROA -ratios are unchanged and the mean(median) $Debt/(TotalAssets)$-ratio decrease by 2.3(0.5) %-points.

I have chosen to leave out observations which are excluded by the respective criteria on the grounds that if one variable is an outlier, all data-elements in the whole observation may be misguiding. This applies to single firm-year observations only and not to all observations for a given firm.

Table 2.16: **Alternative adjustments to the accounting data-set.** The table shows the effect on return and leverage statistics of different economical and statistical limitations applied to the original combined population set of accounting data presented in Table 2.1. The economical limitations are: Non-negative equity, the sum of interest-bearing debt(IBD) being smaller than total assets, and finally a pre-tax total return on end-of-year assets ≥ -100% (ROA=(EBIT+financial income)/end-of-year total assets). The statistical limitations include winsorizing the two tails of the ROA-distribution by 1, 2 and 5%, respectively, each calender year. The table shows mean, median and standard deviation for ROA and overall debt-to-total assets-ratio as well as the relative reduction in number of observations resulting from the different limitations. The annual number of companies after the 5 % ROA limitation is shown in Table 2.1, fifth column.

	***ROA* (pre-tax, end-of-year)**			
Alternative limiting criteria:	**n & Δ**	**Mean**	**Median**	**St.dev**
Unadjusted	870,575	.137	.088	27.383
Non-negative equity	-15.2%	.191	.102	26.996
Interest bearing debt < total assets	-3.9%	.154	.091	25.630
ROA - 100%	-0.9%	.199	.090	26.246
All combined:	-15.3%	.198	.102	27.014
Winsorizing - ROA:				
1%	-2.0%	.102	.088	.203
2%	-4.0%	.104	.088	.175
5%	-10.0%	.105	.088	.134

	Debt/(Debt+Equity)			
Alternative limiting criteria:	**n & Δ**	**Mean**	**Median**	**St.dev**
Unadjusted	870,440	.983	.811	28.638
Non-negative equity	-15.2%	.698	.764	.236
Interest bearing debt < total assets	-3.9%	.805	.800	7.851
ROA> - 100%	-0.9%	.874	.809	14.047
All combined:	-15.3%	.698	.765	.235
Winsorizing - ROA:				
1%	-2.0%	.805	.809	1.059
2%	-4.0%	.796	.808	1.043
5%	-10.0%	.781	.804	1.001

2.C Tables

Table 2.17: **Key relative balance sheet values** The table first shows the annual cross-sectional mean, median and standard deviation of selected balance sheet items calculated relative to company total assets for the years 1993, 1999 and 2005, and then means, percentiles, standard deviations and kurtosis for the whole sample at company level. Operational- and intangible assets are only reported as from 1999 due to data-availability. I LT = Long-term and ST = Short-term. Interest bearing debt is defined and calculated as described in Appendix A.

		Annual values			**Whole sample**					
Balance sheet items:		**1993**	**1999**	**2005**	**Mean**	**p25**	**p50**	**p75**	**St.dev.**	**Kurt.**
Assets:										
ST assets,	Mean	.43	.38	.36	.39	.09	.36	.65	.31	2
excl. cash	*Median*	*.41*	*.35*	*.31*						
& investments	St.dev	.31	.31	.31						
Cash	Mean	.19	.19	.21	.20	.03	.11	.30	.22	5
	Median	*.10*	*.10*	*.12*						
	St.dev	.21	.22	.23						
Investments	Mean	.01	.03	.03	.02	-	-	-	.10	46
	Median	-	-	-						
	St.dev	.08	.11	.12						
LT assets	Mean	n.a.	.08	.09	.05	-	-	.00	.16	22
(excl. op.ass.	*Median*	*n.a.*	-	-						
from '99)	St.dev	n.a.	.19	.22						
LT Oper.assets	Mean	.38	.32	.31	.32	.04	.19	.57	.33	2
	Median	*.29*	*.20*	*.17*						
	St.dev	.32	.32	.33						
Real estate	Mean	.17	.17	.19	.17	-	-	.22	.31	4
	Median	-	-	-						
	St.dev	.29	.31	.33						
Intang. assets	Mean	n.a.	.02	.03	.04	-	-	.01	.17	72
	Median	*n.a.*	-	-						
	St.dev	n.a.	.09	.09						
Financing:										
ST debt	Mean	.29	.30	.30	.30	.10	.24	.44	.34	37,508
excl. Trade-	*Median*	*.24*	*.23*	*.24*						
& int.bearing	St.dev	.28	.32	.29						
Trade credit	Mean	.18	.15	.13	.16	.01	.08	.22	.22	332
	Median	*.10*	*.07*	*.05*						
	St.dev	.10	.21	.19						
Int. bear. debt:										
-Short-term	Mean	.08	.06	.04	.06	-	-	.03	.39	109,873
	Median	-	-	-						
	St.dev	.76	.16	.12						
-Long-term	Mean	.29	.26	.30	.28	-	.14	.46	.76	41,477
	Median	*.14*	*.12*	*.15*						
	St.dev	.8	.35	.51						
-Total	Mean	.36	.32	.33	.34	-	.23	.55	.99	57,304
	Median	*.25*	*.22*	*.22*						
	St.dev	1.40	.38	.51						
Provisions	Mean	.00	.02	.01	.01	-	-	-	.04	7,835
	Median	-	-	-						
	St.dev	.01	.05	.04						
Equity	Mean	.16	.22	.23	.20	.08	.19	.38	1.10	54,192
	Median	*.19*	*.21*	*.22*						
	St.dev	1.40	.45	.53						

Table 2.18: **Balance sheet ratios** The table first shows the annual cross-sectional mean, median and standard deviation of selected balance sheet ratios for the years 1993, 1999 and 2005, and then means, percentiles, standard deviations and kurtosis for the whole sample at company level. Bank debt is only reported as from 1999 due to data-availability. The ratios are defined and calculated as described in Appendix A.

		Annual values			Whole sample					
Balance sheet ratios:		**1993**	**1999**	**2005**	**Mean**	**p25**	**p50**	**p75**	**St.dev.**	**Kurt.**
Net working capital	Mean	.32	.30	.35	.29	.08	.27	.50	.31	102
	Median	*.31*	*.27*	*.34*						
	St.dev.	.31	.31	.30						
Invested capital	Mean	.53	.54	.57	.53	.32	.56	.79	.41	18,868
	Median	*.55*	*.57*	*.59*						
	St.dev.	.35	.39	.36						
Bank debt	Mean	n.a.	.20	.21	.21	-	.03	.36	.34	2,622
	Median	*n.a.*	*.01*	*.02*						
	St.dev.	n.a.	.29	.35						
Net annual increases:										
-Interest bearing debt	Mean	.04	.08	.14	.08	- .05	-	.04	4.20	115,328
	Median	-	-	-						
	St.dev.	2.40	5.40	8.20						
-LT debt	Mean	.03	.07	.12	.07	- .04	-	.00	3.90	139,384
	Median	-	-	-						
	St.dev.	2.30	5.40	8.10						
-Equity	Mean	.01	.04	.10	.03	-	-	-	3.30	202,277
	Median	-	-	-						
	St.dev.	.72	4.70	8.60						

Table 2.19: **Key relative income statement items** The table first shows the annual cross-sectional mean, median and standard deviation of selected income statement items calculated relative to company total assets for the years 1993, 1999 and 2005, and then means, percentiles, standard deviations and kurtosis for the whole sample at company level. Tax rate is calculated relative to profit before tax and dividends are calculated relative to net profit. EBIT = Earnings before interest and tax.

		Annual values			Whole sample					
Income statement items:		**1993**	**1999**	**2005**	**Mean**	**p25**	**p50**	**p75**	**St.dev.**	**Kurt**
Total revenues	Mean	2.60	*2.20*	2.00	2.20	.36	1.80	3.00	5.30	58,471
	Median	*2.10*	*1.70*	*1.50*						
	St.dev.	3.70	3.90	*2.50*						
EBIT	Mean	.09	*.08*	.10	.08	- .00	.07	.17	.20	12
	Median	*.08*	*.07*	*.07*						
	St.dev.	.16	.20	*.20*						
Net profit, after tax	Mean	.03	*.04*	.07	.04	- .01	.04	.12	.29	59,617
	Median	*.03*	*.04*	*.05*						
	St.dev.	.28	.25	*.20*						
Cashflow to firm, pre-tax	Mean	.04	*.07*	.02	.17	-	.11	.31	.18	2
	Median	*.02*	*.04*	*- .00*						
	St.dev.	4.30	.90	*3.50*						
Tax rate/(pre-tax profit), payable	Mean	.18	*.16*	.15	.17	-	.11	.31		0
	Median	*.10*	*.12*	*.08*						
	St.dev.	.19	.17	*.16*						
Dividends incl. group c./ (net profit)	Mean	.46	*.35*	.04	.44	-	-	.52	31.00	88,719
	Median	-	-	-						
	St.dev.	25.00	26.00	*6.30*						
Dividends incl. group c./ (Total assets)	Mean	.02	.05	.01	.04	-	-	.04	.13	28,289
	Median	-	-	-						
	St.dev.	.09	.12	.05						

Table 2.20: **Key performance ratios** The table first shows the annual cross-sectional mean, median and standard deviation of selected performance measures calculated for the years 1993, 1999 and 2005, and then means, percentiles, standard deviations and kurtosis for the whole sample at company level. The ratios are defined and calculated as described in Appendix A.

		Annual values			Whole sample					
Performance ratios:		**1993**	**1999**	**2005**	**Mean**	**p25**	**p50**	**p75**	**St.dev.**	**Kurt**
Return on average assets, pre-tax	Mean	.11	.11	.13	.12	.02	.09	.20	.20	8
	Median	*.10*	*.09*	*.09*						
	St.dev.	*.16*	.21	.22						
Return on average equity, post-tax	Mean	.16	.31	.24	.35	-	.21	.64	21.00	67,175
	Median	*.15*	*.19*	*.29*						
	St.dev.	*23.00*	12.00	17.00						
Operating margin	Mean	- .16	- .45	- .13	- .32	.00	.05	.15	67.00	316,436
	Median	*.04*	*.05*	*.07*						
	St.dev.	*10.00*	62.00	9.70						
Revenue growth, annual	Mean	1.30	.72	1.90	2.60	- .08	.04	.20	945.00	331,444
	Median	*.02*	*.03*	*.04*						
	St.dev.	*97.00*	29.00	153.00						
Profit growth, annual	Mean	.05	.01	.05	.05	- .06	.00	.07	9.20	495,740
	Median	*.01*	*.00*	*.00*						
	St.dev.	*2.90*	5.10	2.10						

Table 2.21: **Key leverage ratios** The table first shows the annual cross-sectional mean, median and standard deviation of selected leverage ratios calculated for the years 1993, 1999 and 2005, and then means, percentiles, standard deviations and kurtosis for the whole sample at company level. The ratios are defined and calculated as described in Appendix A.

		Annual values			Whole sample					
Leverage ratios :		**1993**	**1999**	**2005**	**Mean**	**p25**	**p50**	**p75**	**St.dev.**	**Kurt.**
Total debt/	Mean	*.84*	*.78*	*.77*	.80	.62	.81	.92	1.10	54,179
Total assets	*Median*	*.81*	*.79*	*.78*						
	St.dev.	1.4	.46	.53						
Int. bearing debt/	Mean	*.36*	*.32*	.33	.34	.00	.23	.55	.99	57,301
Total assets	*Median*	*.25*	*.22*	*.22*						
	St.dev.	1.4	.38	.51						
Int. bearing debt/	Mean	*.78*	*.64*	.63	.67	.00	.49	.83	6.10	35,799
Invested capital	*Median*	*.53*	*.46*	*.46*						
	St.dev.	6.3	5.7	9.40						
Int.bearing debt/	Mean	*-7.7*	*-7.5*	-14.00	-9.20	.00	.50	3.60	2,644	617,399
EBITDA	*Median*	*.97*	*.39*	*.28*						
	St.dev.	950	352	1029.00						
Interest cover,	Mean	*18*	*62*	77.00	53.00	.51	2.20	7.80	5,038	263,782
winsorized credit	*Median*	*1.7*	*2.1*	*3.40*						
spread	St.dev.	322	1,700	3,074						
Debt maturity, (LT/	Mean	.75	.79	.85	.79	.70	1.00	1.00	.56	80,895
Total interest	*Median*	*1.00*	*1.00*	*1.00*						
bearing debt)	St.dev.	.46	1	.30						
Trade credit/	Mean	.44	.44	.44	.44	.05	.34	.96	.41	553
(Trade credit	*Median*	*.34*	*.33*	*.30*						
+Int. bearing debt)	St.dev.	.39	.43	.41						

Table 2.22: **Summary statistics combined population data-set 1992-2005** Long-term operating assets, intangible assets and bank debt are only present as from 1999.

(NOK 1000)	N	Mean	p25	Median	p75	St.dev.
Income statement:						
Total revenues	853,204	33,428	1,214	2,984	8,323	1,053,267
Operating profit	853,204	2,681	(6)	163	566	192,109
Net profit	853,204	1,388	(25)	91	388	77,007
Cashflow to firm	705,394	786	(130)	102	501	210,255
Dividends	853,204	371	-	-	100	12,578
Assets:						
Current assets	853,204	13,276	493	1,200	3,183	350,586
Cashflow to firm	853,204	3,129	66	258	882	72,740
Investments	853,204	1,256	-	-	-	55,696
LT assets	853,204	22,219	137	711	2,813	742,634
Operational assets (LT)	485,340	19,349	61	425	2,107	737,543
Real estate	853,204	8,563	-	-	979	354,234
Intangible assets	485,340	5,193	-	-	28	212,510
Total assets	853,204	35,495	1,183	2,481	6,445	1,065,244
Financing:						
Short-term debt	853,204	10,039	410	960	2,441	303,453
Trade credit	853,204	3,067	15	156	626	132,611
-Interest bearing debt:						
-Short-term	853,204	1,450	-	-	70	28,069
-Long-term	853,204	11,380	-	260	1,650	256,705
-Total	853,204	12,893	-	450	2,086	268,458
Provisions	853,204	1,579	-	-	-	206,797
Equity	853,204	11,398	112	423	1,536	358,698
Balance sheet based measures:						
Net working capital	853,204	8,047	143	520	1,674	229,369
Invested capital	853,204	24,680	407	1,289	4,039	611,071
Bank debt	485,340	6,898	-	49	1,116	102,371
New outside financing:						
-Interest bearing debt	853,204	1,300	(79)	-	66	87,438
-Long term debt	853,204	1,096	(64)	-	-	87,173
-Equity	853,204	228	-	-	-	43,831

Table 2.23: **Industry characteristics.** The table presents selected accounting items and ratios calculated by main industry groups for the period 1992-2005. The table also includes each group's relative share of assets and revenues across 2004 and 2005. Medians are denoted 'M' and standard deviations 'S' to save space. Variable descriptions are found in Appendix A.

Industry group		Share of overall Rev.	Share of overall Ass.	Total rev.(avg.) (MNOK)	Asset turn-over	Oper-ating marg.	ROAA	ROAE	Annual growth: Rev.	Annual growth: Prof.	Asset tang-ibility
Agri/fish	M	1.5	2.5	2,755	1.50	.06	.07	.12	.05	.00	.34
	S	-	-	86,948	391.00	9.70	.19	52.00	85.00	.50	.34
Oil	M	21.0	20.0	8,537	.80	.08	.07	.12	.03	.00	.40
	S	-	-	21,000,000	253.00	70.00	.24	5.70	42.00	16.00	.37
Mfg.ind.	M	26.0	21.0	4,789	6.90	.04	.09	.18	.04	.00	.27
	S	-	-	862,818	377.00	11.00	.20	19.00	15.00	7.10	.25
Constr./ energy	M	7.9	7.5	3,794	12.00	.05	.12	.33	.08	.01	.19
	S	-	-	326,181	724.00	5.00	.20	17.00	16.00	2.90	.25
Trade & tourism	M	21.0	9.1	4,473	22.00	.03	.09	.23	.04	.00	.12
	S	-	-	256,485	941.00	13.00	.20	19.00	33.00	3.40	.24
Shipping	M	4.2	9.9	3,150	.62	.08	.07	.09	.03	.00	.55
	S	-	-	518,966	2777.00	40.00	.19	16.00	6313.00	73.00	.38
Transport & telecom	M	2.4	1.7	3,533	3.30	.04	.09	.21	.06	.00	.45
	S	-	-	256,855	296.00	8.40	.20	11.00	31.00	.82	.32
Services	M	12.0	25.0	1,316	.53	.17	.09	.18	.03	.00	.47
	S	-	-	373,882	339.00	116.00	.20	14.00	1171.00	3.00	.37
Culture	M	.6	.5	2,209	6.50	.04	.10	.24	.05	.00	.35
	S	-	-	62,855	370.00	8.10	.25	8.70	15.00	1.00	.32
Information technology	M	3.0	3.2	2,851	13.00	.06	.12	.31	.08	.01	.10
	S	-	-	998,539	343.00	10.00	.34	74.00	24.00	6.10	.24
All	M			2,984	6.90	.05	.09	.21	.04	.00	.23
	S			1,053,267	709.00	67.00	.20	21.00	945.00	9.20	.33

Table 2.24: **Spearman correlations between leverage ratios and selected factors.** The table presents the Spearman correlations between the main leverage measures and selected accounting based company variables. All correlations are significant. Variable and ratio descriptions are found in Appendix A. 230,431 observations.

Debt /TA	IBD /TA	IBD/ Inv.cap	TA	TR	Age	Tang	Tax	CF, eq.	CF, fi.	CF, fi.av	Div	Pr. gwth.	ROA	ROAA avg.	ROAA std.d.
.54															
.70	.94														
(0.11)	.19	.10													
.14	(0.30)	(0.15)	(0.43)												
(0.23)	(0.08)	(0.12)	.22	(0.14)											
.09	.49	.39	.24	(0.49)	.02										
(0.24)	(0.33)	(0.35)	(0.01)	.10	.04	(0.15)									
(0.15)	(0.22)	(0.24)	(0.08)	.06	.07	(0.08)	.21								
(0.07)	(0.13)	(0.14)	(0.08)	.04	.06	(0.03)	.12	.98							
(0.14)	(0.25)	(0.26)	(0.10)	(0.01)	.08	(0.06)	.15	.48	.48						
(0.22)	(0.37)	(0.38)	(0.01)	.05	.11	(0.20)	.46	.43	.36	.33					
.06	.01	.03	.06	.11	(0.17)	(0.00)	.05	(0.06)	(0.06)	(0.11)	.04				
(0.17)	(0.28)	(0.30)	(0.03)	.12	(0.01)	(0.18)	.47	.39	.35	.33	.57	.24			
(0.25)	(0.37)	(0.39)	(0.04)	.11	.01	(0.20)	.47	.30	.26	.42	.55	.07	.71		
.04	(0.21)	(0.13)	(0.27)	.27	(0.04)	(0.29)	(0.06)	.04	.04	.05	.10	(0.00)	.13	.23	
.11	.05	.07	(0.01)	(0.04)	.00	(0.02)	(0.11)	(0.00)	.01	.00	(0.01)	(0.01)	(0.01)	.01	.34

Table 2.25: **System GMM regressions by subsamples.** The table presents the coefficients and p-values for System GMM-regressions on the whole dataset and on selected subsamples.Variable and ratio descriptions are found in Appendix A.

		Whole sample		Largest Q		Smallest Q		Dividend payer		Div. non-payer		Auditor's Comm.		No auditor's comm.		Listed	
	Lags	Coeff.	P-value	Coeff.	P-value	Coeff.	P-value	Coeff.	P-value	Coeff.	P-value	Coeff.	P-value	Coeff.	P-value	Coeff.	P-value
IBDR	L1	.648***	(.000)	.613***	(.000)	.682***	(.000)	.740***	(.000)	.637***	(.000)	.625***	(.000)	.697***	(.000)	.664***	(.000)
	L2.	.048*	(.056)	.048*	(.003)	.077**	(.031)	.054***	(.000)	.048**	(.047)	.047*	(.066)	.053***	(.005)	.005	(.411)
ln Total Assets		-.006	(.386)	.040	(.121)	-.017	(.212)	.081**	(.010)	-.010	(.366)	-.067	(.221)	.028	(.264)	.004	(.415)
	L1	.023	(.362)	-.025	(.236)	.050	(.242)	-.076**	(.014)	.026	(.357)	.068	(.190)	-.014	(.318)	.020***	(.000)
ln Turnover		.129	(.318)	.166	(.317)	.147	(.326)	-.071	(.394)	.110	(.395)	.188	(.260)	.007	(.361)	-.110***	(.009)
	L1	-.100	(.361)	-.224	(.217)	-.036	(.405)	.032	(.494)	-.097	(.442)	-.181	(.341)	-.010	(.363)	-.071**	(.019)
Tangibility		.130**	(.046)	.154**	(.022)	.102	(.158)	.055*	(.084)	.161**	(.038)	.183*	(.078)	.126**	(.011)	.133***	(.000)
	L1	-.032	(.288)	-.035	(.207)	-.024	(.333)	-.017	(.415)	-.052	(.297)	-.031	(.376)	-.051	(.264)	-.127***	(.000)
Taxrate, payable		-.049	(.372)	-.079	(.262)	-.107	(.282)	-.029	(.401)	-.057	(.364)	-.049	(.433)	-.037	(.364)	.040***	(.000)
	L1	-.007	(.480)	-.002	(.426)	.012	(.453)	.047	(.112)	.004	(.494)	.028	(.502)	.006	(.423)	.064***	(.000)
Dividend, rel.		-.031	(.421)	.020	(.442)	-.051	(.470)	.023	(.317)	-.070	(.304)	-.098	(.335)	.003	(.461)	.241***	(.000)
	L1	.060	(.247)	.021	(.218)	.137	(.168)	.155***	(.000)	.054	(.315)	.138	(.274)	.071	(.162)	.681***	(.000)
ROAA		.033	(.310)	-.090	(.318)	-.069	(.296)	-.140**	(.038)	.054	(.363)	.068	(.317)	-.064	(.289)	.014	(.506)
	L1	-.033	(.357)	.037	(.296)	-.107	(.262)	.025	(.410)	-.051	(.358)	-.097	(.255)	.013	(.326)	.008	(.162)
ROAA, avg.		-.677	(.130)	-.508	(.198)	-.421	(.298)	-.288	(.195)	-.775	(.116)	-1.183*	(.078)	-.417	(.218)	-.180**	(.018)
	L1	.486	(.136)	.446	(.158)	.275	(.291)	.254	(.149)	.540	(.128)	.734	(.110)	.336	(.189)	.125**	(.040)
Industry leverage	L1	-0.235*	(.058)	-.321***	(.004)	-.144	(.315)	-.177**	(.039)	-.234*	(.093)	-.172	(.379)	-.244**	(.017)	.090**	(.020)
	L2.	.063	(.421)	.137	(.196)	.046	(.486)	.035	(.473)	.102	(.330)	.106	(.493)	.082	(.318)	.063*	(.090)
NIBOR rate		-.200*	(.079)	-232*	(.052)	-.076	(.434)	-.201***	(.005)	-.155	(.237)	.007	(.543)	-.241**	(.021)	.595***	(.000)
	L1	.372**	(.028)	.551***	(.001)	.174	(.351)	.136	(.189)	.403*	(.055)	.299	(.262)	.393**	(.012)	-.455***	(.443)
	L2.	-.026	(.425)	-.131	(.152)	.038	(.521)	.022	(.478)	-.056	(.324)	-.065	(.440)	-.052	(.331)	-.208***	(.001)
Constant		-.050	(.436)	.030	(.444)	-.233	(.248)	.051	(.358)	-.022	(.484)	.118	(.405)	-.025	(.396)	-.097***	(.000)

3

Start-up Financing: Outside Equity

coauthored with Hans K. Hvide [1]

Abstract

We investigate the extent to which start-ups use outside equity, and interpret our results in relation to financial contracting theory. We do so by studying the start-up and founder characteristics that are associated with the use of outside equity financing, using a unique dataset from Norway. Our findings suggest that adverse selection are less of a concern for start-ups than ex-post opportunistic behavior (risk shifting) by the entrepreneur as in Myers (1977) and Ravid & Spiegler (1997). One implication of this finding is that outside equity and debt are complements rather than substitutes, and that an extra unit of equity financing has a multiplicative effect on total financing through releasing additional debt financing. We do not find convincing evidence that the use of outside equity has detrimental effects on entrepreneurial effort, nor that a possible shortage of available outside equity leads to investor monopolization and excessive investor returns. Thus we provide evidence that outside equity provides an important avenue for entrepreneurs to escape liquidity constraints.

[1]Thanks to Thore Johnsen, Eirik G. Kristiansen, Kjell Nyborg, and seminar participants at Aberdeen and NHH for comments and suggestions. Thanks also to Robert Holm at Brønnøysundregisteret, and to Lena Mari Johnsen for excellent research assistance. Hvide has received financial support from the ESRC. Mjøs has received financial support from the Norwegian Research Council (Finansmarkedsfondet).

3.1 Introduction

While a large body of research studies the role of debt in the outside financing of new and small companies (e.g., Bond & Townsend, 1984, Petersen & Rajan, 1994, Cassar, 2004, Paulson et al., 2006), much less is known about the role of outside equity financing. Outside equity financing is apparently quite commonly used by start-ups; aggregate evidence on small firms in the US suggests that equity provided by other than the principal founder amounts to about 13 percent of total funding (Berger & Udell, 1998). This figure is comparable to the amount of lending from commercial banks (19 percent) and trade credit (16 percent), both sources of funding that have been studied extensively. The purpose of this paper is to analyze the extent to which, and why, start-ups use outside equity. In addition, we attempt to analyze the effectiveness of outside equity financing.

We view understanding the role of outside equity as important for two reasons. First, a large theoretical literature has investigated how the outcome of negotiations between entrepreneurs and investors can be understood as responses to informational asymmetries. Little is known, however, about which informational problems are empirically more important for the financing choices of young firms. Through an analysis of outside equity we hope to inform theory about this question.[2] Second, many government policies, such as subsidies and tax breaks, aim to alleviate liquidity constraints of entrepreneurs. As argued by Paulson et al. (2006), to be able to better design policies in this area, we need an improved understanding not only of whether young firms are credit constrained, but also why.

We assemble a unique dataset from Norway that combines initial ownership stakes for a representative sample of start-ups with data on subsequent profitability and on the sociodemographic characteristics of the owners. Defining an external investor as a shareholder who owns at least 10 percent and is neither the largest owner nor employed by the start-up during the first two years of operations, we find that 40 percent of firms in our sample use outside equity.[3] Outside equity is

[2] The need to inform theory on the use of outside equity is illustrated by the following quote from Tirole (2006, p. 132): "the [financial contracting] theories ... are often criticized for their lack of robustness; it is also pointed out that they do not account for the diversity of capital structures that characterize modern operations, and that even small firms sometimes admit outside equity".

[3] This definition of outside investors does not exclude family members. In the current version of the paper we are only able to exclude female spouses, but hope to be able to exclude other close

also a substantial source of financing in absolute terms; at the end of the first year of operations, outside equity comprises on average about 40 percent of outside financing for the whole sample, and 68 percent for firms that use outside equity. These figures are fairly constant across robustness checks, such as constraining attention to companies where the largest owner has a majority stake in the start-up.

Why do start-ups use outside equity? Theory tends to emphasize that start-ups are informationally opaque and prone to problems of asymmetric information between founders and outside investors. One can think of all security design papers as attempts to understand financial contracting under a variety of assumptions about the informational problem. At one end of the spectrum are papers that assume that both parties know the distribution function generating the firm's cash flows, but the investor cannot costlessly observe the realized profit (Townsend, 1979), or cannot force the entrepreneur to stay on in the firm (Hart & Moore, 1994). Papers in this tradition tend to predict that debt is the optimal external financing source.[4] Somewhat related are papers that view profits as observable but the underlying effort of the entrepreneur as non-observable. Papers in this tradition, such as Aghion & Bolton (1992), Aghion & Bolton (1997), and Holmstrom & Tirole (1997) emphasize the distortion in effort incentives arising from outside equity financing and also tends to predict the use of debt as the sole source of outside finance.

Theories that can explain the use of outside equity contain some notion of the entrepreneur having special knowledge of the underlying cash flow distribution. These theories come in two variations. In Myers & Majluf (1984) it is assumed that the return distribution is fixed and known only by the insiders in advance. This informational asymmetry leads the entrepreneur to take up debt if he has favorable private information about the firm's prospects, and to raise outside equity if he has unfavorable private information. According to this pecking-order theory, where equity and debt are substitute financing means, one would expect firms that use outside equity to subsequently perform worse, controlling for observable

family members in later versions.

[4]One exception is Myers (2000) who argues that outside equity can be optimal in a setup resembling Hart & Moore (1994). Myers' setup is not able to generate an optimal financing mix. Hvide & Leite (2006) derive an optimal mixed capital structure in a variation of the costly state verification model of Townsend (1979). Although Hvide & Leite's paper provides new insights on entrepreneur repayment and investor monitoring behavior they need to assume that outside equity bears a lower monitoring cost than debt to justify its existence.

characteristics of the start-up such as founder human capital and wealth.

In the second variation of models with asymmetric information about project returns, entrepreneurs, after financing has taken place, can manipulate the distribution of cash flow through the choice of projects (Myers, 1977, Ravid & Spiegel, 1997). If the amount of equity is small, debt financing would give entrepreneurs incentives to take on projects with a highly dispersed return profile, in order to exploit the upside. This "risk shifting" would make financing with unsecured debt not feasible. If additional equity is provided, however, these risk-shifting incentives can be mitigated and investors willing to provide debt financing.[5] According to this theory, equity is a complement to debt rather than a substitute as under pecking-order, and an extra unit of equity financing has a multiplicative effect on total financing, since it releases additional debt financing.

The pecking-order theory and the risk shifting theory have the similar implication that the founder will use external funding only after internal resources have been exhausted. The theories differ in at least two respects. First, while the pecking-order theory predicts that firms with higher leverage should subsequently perform better, the risk shifting theory is mute in this respect, since there is no necessary connection between the possibility to shift risk and overall returns. Second, the risk-shifting theory implies that firms with less wealthy entrepreneurs (and thus less access to inside equity) are more likely to use outside equity, while the pecking-order theory is mute on this relationship. We would thus, if the risk shifting theory has more validity, expect outside equity to primarily be used by small start-ups with relatively poor entrepreneurs.

Our analysis of which theory best can explain why start-ups use outside equity is only preliminary but yields some interesting results. First, we find that the start-ups that use outside equity are predominantly smaller and have less wealthy founders than those that do not. This gives support to the risk shifting theory. Second, we do not find that a more extensive use of outside equity is associated with worse future profitability. This finding is in contrast to pecking-order theory. Overall, our findings give support to the risk shifting theory and little support to

[5]Rather than requiring more equity, another way for creditors to mitigate the ex-post opportunism problem of these models is for lenders to monitor, as in e.g., Holmstrom & Tirole (1997). Given that the start-up is sufficiently small, getting and staying informed will not be worth the fixed cost.

the pecking-order theory.

Under the risk shifting theory, equity serves the dual role of providing funding per se, and to release debt financing. Our estimates of the magnitude the "equity multiplier" are highly uncertain but suggests that it is significant. For example, evaluated at the means, an increase in initial equity by NOK 50K leads to an increase in debt by about NOK 40K. For firms that use outside equity, the estimated equity multiplier is larger.

Does the use of outside equity have any negative effects? Theory emphasizes that outside ownership dilutes insiders' stake and therefore undermines insiders' incentives to exert effort. To evaluate whether there indeed is such a negative effect of using outside equity, we investigate whether start-ups that use outside equity subsequently perform worse. Our analysis of this question, which follows much the same lines as the test of the pecking-order theory, somewhat surprisingly suggests that this effect is weak or non-existing.

Who provides outside equity? Our findings suggest that outside equity is provided by a relatively small circle of individuals in the social vicinity of the founder. For example, 31 percent of the investors are former co-workers of the founder. A natural question is whether a limited circle of potential investors gives rise to investor monopolization and a shortage of funds available. We investigate this question by analyzing whether start-ups that use outside equity have a higher investor returns than start-ups that do not use outside equity. Our results show that investor returns, as measured by dividend returns, are not significantly higher for start-ups that use outside equity than for start-ups that do not use outside equity. This finding does not suggest that a shortage of outside equity is a major problem. Thus we provide evidence that outside equity provides an important avenue for entrepreneurs to escape liquidity constraints.

The paper is organized as follows. In the next subsection, we review some prior empirical work. In Section 2, we present the data and provide summary statistics. In Section 3, we analyze why start-ups use outside equity. In Section 4, we evaluate the effectiveness of outside equity financing.

3.1.1 Related empirical literature

To our knowledge, this paper is the first that relates financing choices of start-ups to both subsequent profitability and to founder sociodemographic characteristics.

Berger et al. (1998) study the financing choices of small businesses in the U.S. based on data from the National Survey of Small Business Finance. Their main question is whether firms go through a financing cycle, where they are initially financed by informal sources of finance, and then gradually make use of more structured forms of external finance. They find empirical support for this hypothesis, as do Fluck et al. (1997). In contrast, our aim is to understand initial financing. We use far more detailed data on both start-up and founder characteristics than accessible to both Berger et al. (1998) and to Fluck et al. (1997).

Using structural methods, Evans & Jovanovic (1989) find that virtually all the entrepreneurs in their sample are liquidity constrained. Gertler & Gilchrist (1994) contains evidence on the importance of credit constraints for small firms in the U.S., a finding mirrored by Aghion et al. (2007) who use recent firm-level data from 16 countries. Petersen & Rajan (1994) study how relationship lending from banks to small companies can alleviate credit constraints, but do not analyze the role of equity. We contribute to this literature by analyzing the role of outside equity in easening credit contraints.

A large literature tests the pecking-order hypothesis in the context of mature, publicly listed firms (see e.g., Leary & Roberts, 2005, and Frank et al., 2006, for reviews). While this literature has the advantage of having access to cost of capital measures based on market values, a persistent problem is to account for path dependencies and adjustment costs. In this respect, our setting for testing the pecking-order hypothesis is better suited since our sample consists of new companies.

3.2 Data and Summary Statistics

3.2.1 Data

We construct a dataset consisting of 4,271 incorporated limited liability firms started up by individuals between 1994 and 2002 in Norway. The dataset contains incorporation and accounting information on the start-ups in addition to sociodemographic information about the founders based on tax records and other public registries. The dataset is compiled from three different sources:

1. *Accounting information from Dun & Bradstreet's database of accounting figures based on the annual financial statements reported by the companies.*[6] This data includes variables such as sales, assets, and profits for the years 1992 - 2005, as well as 5-digit industry codes.

2. *Data on individuals from 1986 to 2002 prepared by Statistics Norway.* These records include the anonymized personal identification number and yearly sociodemographic variables such as gender, age, education in years, taxable wealth, and income.[7]

3. *Founding documents submitted by new firms to the government agency 'Brønnøysundregisteret'.* These data includes the total capitalization of the start-up, the personal identification number of all founders with more than 10 percent ownership share, and each founder's ownership share.

For each start-up identified in 1) we compile a list of founders identified through 2) and match their associated sociodemographic information from 3). Due to alterations in the reporting requirement in 1997 we were able to match around 80% of the founders in companies founded after 1997 and around 20% before. We are then left with about 8,600 unique founding individuals, of which 1,768 are defined as external investors.[8]

We will understand as the "principal founder" the owner with the largest ownership share. To ensure that we are not counting spin-offs from existing companies as start-ups, we eliminate all new companies that have a company as the principal founder. We also eliminate all start-ups where 1/3 of the ownership or more cannot be identified, either because of missing data or because incorporations

[6] Dun & Bradstreet is Bureau van Dijk's Norwegian subsidiary.

[7] Earnings and wealth figures are public information in Norway. This transparency is generally believed to make tax evasion more difficult and hence our data more reliable.

[8] Because of the tax benefits of incorporation status, many of the start-ups are expected to be continuations of sole proprietorships. Since moving into incorporation status likely indicates some expansion in activity that is a feature we would like to capture, we do not view this as a major problem.

own more than 1/3.[9] In the remaining sample, there exists a largest owner for 67 percent of the start-ups. In 24 percent of the start-ups, two founders are tied for the largest ownership stake and in 5 percent of the start-ups, three founders are tied. In such situations, we define as principal founder the individual defined as internal. In cases where the principal founder is still undetermined, we apply a random selection procedure.

Whilst the principal founder is defined as an insider irrespective of the size of her holding, outside shareholders are those not defined as principal founders and who are not employed by the company for the first two years. Moreover, female founders other than principal founders are excluded as a crude, but conservative, mechanism to avoid defining female spouses as external investors.

Setting up an incorporated company carries tax benefits relative to being self-employed such as more beneficial expensing of items like home office, company car, and computer equipment. Incorporation also implies limited personal risk exposure and certain employment benefits compared to being self-employed. Since the financial barrier to start up an incorporated company in Norway is quite low (one needed to raise NOK 50K in equity until 1998 and NOK 100K thereafter) for any founder with a business above a minimum level, incorporation status will be more tax efficient than becoming or remaining self-employed. Incorporated companies are required to have an authorized auditor certifying the annual reports including the accounting statements, which makes the reliability of the data higher than for the self-employed persons.

An adverse consequence of the low barriers to starting up an incorporated company and its favorable tax treatment, is that many start-ups in Norway are tax-shelters or have minimal activity. This is particularly common within real estate. We deal with this problem in two ways. First, we over-sample start-ups from the manufacturing and IT industries. We believe that tax shelters are less likely to occur in these industries. Moreover, over-sampling these industries creates high variation in the sample since, relative to manufacturing, the IT industry has a high proportion of service-oriented firms with a low capital intensity. We selected all start-ups within the high tech sectors NACE 23-35 and 72 from 1994-1998, and all start-ups within manufacturing and IT, NACE 15-37 and 72 from 1999-2002. We

[9] Our results are robust to including these companies in the analysis.

added a random 25 percent sample of other non-financial private sector start-ups from 1999-2002. We expanded the sample after 1998 because the cost of collecting data for the more recent period is lower. Second, to further reduce the share of "empty shells" we only include companies that have at least NOK 500K in sales and at least two persons employed during one of the first two years of operation.[10]

We note several advantageous features of our data compared to data used in earlier work on start-ups. First, we have access to a long panel with verified yearly measures of founder previous income and wealth, and of start-up profitability. This enables us to investigate the relation between founder characteristics such as wealth and the financial structure. It also allows us to relate financial structure and subsequent profitability, controling for founder characteristics. Second, that the sample is representative means that a whole range of selectivity problems usually associated with datasets on start-ups are not present. Due to the representativeness of the sample, the vast majority of the start-ups are too small or have a too limited growth potential to be of interest for highly structured finance such as venture capital financing. It is also reasonable that for most of the companies, intensive monitoring by banks or other investors would not pay off.

3.2.2 Summary statistics

The main founder variables are age, years of education, gender, previous income and wealth, all evaluated at the start-up date. Age, years of education, and previous income are measures of the founder human capital. Age, gender and education are also likely to be correlated with risk preferences. Wealth captures founder liquidity. The value of property investments and investments in unlisted stocks have an artificially low tax value. Debt, on the other hand is deductible at nominal value (financing property and unlisted stocks by debt, therefore, is a common way to avoid the taxation of wealth). For this reason, we use gross wealth rather than net wealth as our liquidity measure.[11] For previous income and wealth, we reduce

[10]To avoid sampling empty companies is important as the incorporation documents have to be hand-collected by research assistants at a non-negligible cost per unit. However, our selection criteria are deliberately set low in order not to exclude companies of interest.

[11]In Norway, individuals are liable to pay wealth and income tax every year throughout their lives. In contrast, the U.S. tax system requires wealth reporting only in connection with estate tax, which is imposed only on the very rich at the time of death (Campbell, 2006). Our wealth variable

measurement error by constructing the variables as averages across the five years preceding the start-up year.[12]

The main start-up variables can be divided into financing and operating variables. The main financing variables are equity at the incorporation date, and debt, as measured by the sum of interest bearing debt and trade credit. Our main measure of leverage ratio is debt relative to total outside financing. The main operating variables are turnover and EBITDA. We measure profitability by yearly EBITDA margin, that is EBITDA relative to average assets. Our main performance measure will be the asset-weighted average of second and third year EBITDA margin (e.g., Lemmon et. al. 2007).

Annual EBITDA-margin is winsorized symmetrically at the 5 percent level. Variables in absolute levels are reported in NOK 1,000 unless stated otherwise. USD 1 equals about NOK 6 as of Summer 2007. All accounting variables are indexed to 2005-levels using Statistics Norway's CPI. Our final sample, which has non-missing values for all main variables, consists of 3,167 firms.[13]

Table 1 presents descriptive statistics for principal founders and the start-ups, at the end of the first operating year. Founders tend to be experienced, on average 39 years old, and are relatively wealthy. Start-ups are small, on average they have NOK 2.2M in assets at the end of the first year, with the median being considerably lower, and 4.6 employees. 40 percent of the start-ups use outside equity, and the average ownership share held by an outside investor is 16 percent.

3.3 Outside equity

3.3.1 Prevalence of outside equity

Table 2 describes the financing mix of all companies in the sample versus those that have at least one external shareholder.

therefore has much better coverage than the measures used in previous research on U.S. data. The only other work we are aware of with a comparable richness in income and wealth measures is Calvet et al. (2006) using data from Sweden. Hvide & Møen (2007) analyzes the effect of founder liquidity on entrepreneurial performance using the same dataset as in the present paper.

[12] To construct the previous income average, we omit years where the founder is self-employed.

[13] We require that the start-ups survive for three years. Of the 4,271 initial start-ups, 480 do not survive three years of operations, and for 624 principal founders we are unable to calculate a previous income measure.

Table 3.1: Summary statistics, first year

3-year EBITDA-margin is computed as an asset-weighted average of the margin in the second and third year of operations. Turnover and asset growth are defined as growth between the end of the first year and end of the third year of operations. Debt ratio is the sum of interest bearing debt and trade credit relative to total assets. External debt ratio is interest bearing debt and trade credit relative to the sum of such debt and external equity (outside financing). Turnover and cashflow are annualised by using number of months from incorporation until first reporting year-end to make first year operating periods comparable.

	Median	Mean	St. Dev
Principal founder:			
Wealth	393	1,036	(3,972)
Previous income	339	369	(245)
Age	38.0	39.3	(9.7)
Education	12.0	12.0	(3.5)
Female	0	.11	(.31)
Start-up:			
Start-up year	2000	1999	(2.1)
Turnover	1,977	5,613	(18,428)
Cashflow	45	67	(1,826)
Total assets	806	2,164	(7,764)
Equity	109	515	(4,781)
Growth in turnover	.36	.88	(2.47)
Growth in assets	.15	.44	(1.29)
EBITDA-margin, 3 year	.14	.12	(.29)
Debt ratio	.30	.40	(1.06)
External debt ratio	1.00	.81	(.34)
Employees	2.5	4.6	(8.0)
Internal founders	1.0	1.5	(.8)
External founders	0	.5	(.8)
Outside ownership?	0	.40	(.49)
Outside ownership share	0	.16	(.22)
N = 3,167			

Table 3.2: Financing mix by external ownership

The table reports the relative shares of each financing claim as part of total invested capital. Invested capital is defined as the sum of the claims included in the table. Interest bearing debt is debt of all maturities which is explicitly categorized as interest bearing.

		All		*With external investors*	
First report, percentages		Median	Mean (St. Dev)	Median	Mean (St. Dev)
Equity	Total	.40	.49	.50	.54
			(.36)		(.36)
	Inside	.32	.40	.26	.32
			(.32)		(.23)
	Outside	0	.09	.19	.22
			(.15)		(.17)
Debt	Total	.60	.51	.50	.46
			(.36)		(.36)
	Interest bearing	0	.24	0	.23
			(.32)		(.32)
	Trade	.17	.27	.14	.24
			(.27)		(.26)
	N	3,167		1,268	

The invested capital is on average 49 percent equity and 51 percent debt. This is similar to evidence from the U.S. reported by Berger et al. (1998), Table 1, and to findings of Van Auken & Carter (1989).

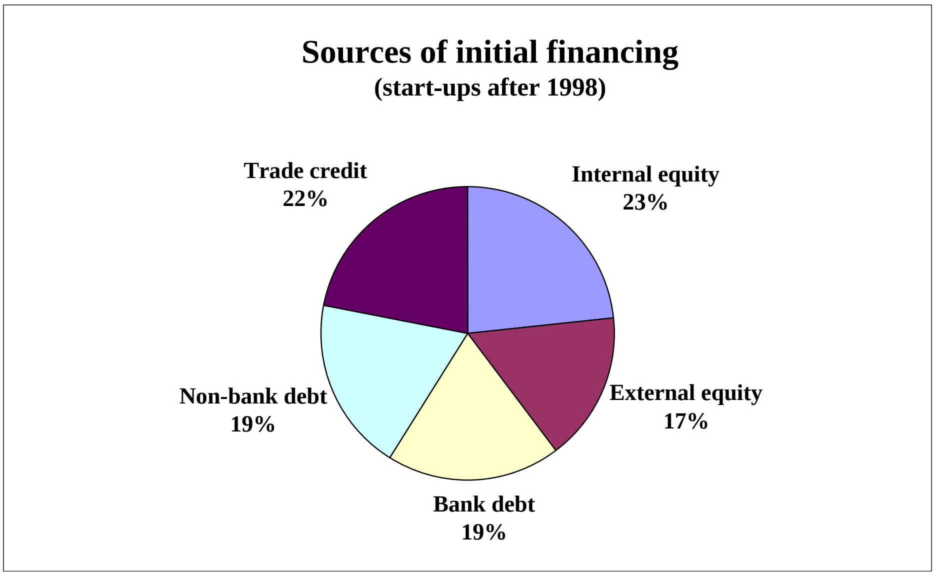
For companies founded after 1998 our data allows us to split debt into bank and non-bank interest bearing debt. Figure 1 illustrates the financing mix for the companies started up after 1998 that use outside equity.[14] Outside equity constitutes on average 17 percent of start-up financing (42 percent of total equity capital), a similar share to bank debt and trade credit.

3.3.2 Firm and entrepreneur characteristics

The following table breaks up summary statistics on firms with and without outside equity.

Table 3 shows that principal founders of start-ups that use outside equity are

[14] 2/3 of the start-ups are founded in 1999 or later.



very similar to those that do not use outside equity in terms of age, education and previous income. The most striking difference is that principal founders that use outside equity are considerably less wealthy than principal founders that do not use outside equity. The use of outside equity can seemingly not fully compensate for the lack of wealth; at the end of first year, the firms using outside equity are significantly smaller than firms not using outside equity, as measured by assets or by turnover. Also, start-ups using outside equity grow faster. These findings suggest that using outside equity is an indication of being credit constrained.[15]

Table 4 runs a probit regressions on whether a company uses outside equity or not. To control for founder liquidity prior to start-up we include a measure of founder wealth. To control for observable aspects of founder human capital, we include founder age, sex, education and previous wage.

[15] We have also analyzed how the maturity of debt varies with ownership structure and start-up size. Maturity increases in the use of equity and increases in start-up size. We are not aware of theories that link the use of outside equity with the maturity of loans and have therefore not pursued these findings further.

Table 3.3: Summary statistics, first year

See the notes to table 1.

External investors?	*Yes*			*No*		
	Median	Mean	St. Dev	Median	Mean	St. Dev
Principal founder:						
Wealth	367	732	(1,873)	408	1,239	(4,886)
Previous income	346	375	(226)	335	366	(256)
Age	37.0	38.7	(9.7)	38.0	39.7	(9.6)
Education	12.0	12.2	(3.4)	12.0	11.9	(3.6)
Female	.00	.10	(.30)	.00	.12	(.32)
Start-up:						
Turnover	1,644	4,232	(19,314)	2,262	6,527	(17,763)
Cashflow	43.8	95.8	(1,641)	45.6	47.2	(1,939)
Total assets	690	1,758	(8,647)	920	2,435	(7,104)
Equity	109	486	(5,551)	109	535	(4,189)
Growth in turnover	.38	.95	(2.43)	.34	.83	(2.49)
Growth in assets	.16	.55	(1.62)	.14	.37	(1.01)
EBITDA-margin, 3-year	.13	.12	(.32)	.14	.13	(.28)
Debt ratio	.24	.42	(1.62)	.32	.39	(.36)
External debt ratio	.61	.50	(.38)	1.00	1.00	(.00)
Employees	2.0	3.6	(7.0)	3.0	5.2	(8.6)
Internal founders	1.0	1.3	(.7)	1.0	1.6	(.9)
External founders	1	1.3	(.8)	0	0	0
Outside ownership share	.40	.41	(.15)	.00	.00	.00
N		1,268			1,899	

The regressions in Table 4 confirm the impression from the univariate analysis that less wealthy founders are more likely to use outside equity. We note that more educated founders, and founders with higher previous income, are more likely to use outside equity. This could be because a higher human capital is associated with better access to potential sources of outside equity.

3.3.3 Outside investor characteristics

Who are the outside investors? One reason to be interested in this question is to see whether outside investors are likely to provide other scarce resources than financing, such as business experience or reputational capital. The following table

Table 3.4: Determinants of external ownership

The dependent variable is a dummy which takes the value of 1 if the firm has at least one external shareholder, and 0 otherwise. Two digit industry dummies and dummies for the year of the start-up are included in (2) and (3), but not reported. The regression method is probit. Robust standard errors in parenthesis. Coefficients marked ***, ** and * are significant at 1 %, 5 % and 10 % levels, respectively.

	(1)	(2)	(3)
Founder:			
ln Wealth	-.045***	-.046***	-.044***
	(-2.78)	(-2.78)	(-2.61)
ln Previous income	.076***	.067***	.067***
	(3.59)	(3.06)	(3.09)
Age	-.045***	-.046***	-.047***
	(- 2.64)	(- 2.65)	(- 2.70)
Age, sq.	.000**	.001**	.001**
	(2.43)	(2.49)	(2.54)
Education	.014**	.011	.010
	(2.13)	(1.58)	(1.55)
Female	-.077	-.073	-.077
	(-1.02)	(-.92)	(-.97)
Start-up:			
ln Equity			-.033
			(-1.31)
Constant	.397	.541	.689
	(1.22)	(.56)	(.71)
R^2	.0089	.0317	.0321
N	3,167	3,157	3,157
Correctly classified	.6012	.6120	.6174

compares the sociodemographic characteristics of the principal founder versus the largest outside investor for start-ups that use outside equity.[16]

Table 3.5: Principal founders and largest investors

	Principal Founders		*Largest investor*	
	Median	Mean (St.dev.)	Median	Mean (St.dev.)
Wealth	367	732	498	1,817
		(1,873)		(2,000)
Previous income	346	375	360	420
		(227)		(395)
Age	37.0	38.7	40.0	41.2
		(9.7)		(11.5)
Education	12.0	12.2	12.0	12.2
		(3.4)		(3.7)
Female	0	.10	0	0
		(.30)		0
Industry experience:				
- 2-digit NACE		.24		.18
- 5-digit NACE		.11		.07
N	1,268		1,055	

Table 5 shows that the principal founder and the largest outside investor are similar in terms of age and education, and outside investors have slightly higher previous income. Principal founders are on average far less wealthy than the largest outside investor. In about 2/3 of the start-ups, the principal founder is less wealthy than the largest outside investor. In terms of industry experience, 18 percent of the investors have at some point in time worked within the same 2-digit- and 7 percent within the same 5 digit industry-code as the newly founded company. In comparison, the principal founders have 24 percent and 11 percent industry experience. 11 percent of the outside investors are self-employed in the start-up year.[17] Overall, it seems that the outside investors are chosen more for their wealth than for their competence.

[16] We include only the largest external investor as he is the investor with the largest exposure and thus likely impact on the start-up.

[17] We do not have information on industry codes for self-employed investors. Given the small percentage of outside investors that are self-employed, this seems unlikely to have any effect.

Where do entrepreneurs and investors meet? 31 percent of all external investors are previous co-workers of the principal founder, having worked with the same employer at some point in time before the new company was incorporated. This is consistent with the typical external investor being a person in the social vicinity of the founder.

3.3.4 Is there a pecking-order?

The main prediction of the pecking-order theory is that firms using outside equity should perform worse than firms only using debt as outside financing. To test this hypothesis, we study whether the use of outside equity is associated with lower profitability.

We use two main econometric specifications. In the first specification, the left hand side variable is a dummy variable that takes the value of zero if the start-up uses outside equity, and the value one if not. In the second specification, the left hand side variable is the external debt ratio. To control for founder liquidity prior to start-up we include a measure of founder wealth. To control for observable aspects of founder human capital, we include founder age, sex, education and previous wage. We report regression results both including and excluding a control for the level of assets.

In the probit specification of (1) we find a negative but nonsignificant relationship between leverage ratio and profitability. In (2) - (4), where start-ups using only inside equity are excluded, we get a significantly negative relationship between profitability and the leverage ratios, directly opposite to the positive relationship predicted by the pecking order theory.

External leverage is growing in the total assets of the firm, as is commonly seen in capital structure research. The lack of any significant coefficients on wealth when controlling for total assets indicates that whilst wealth is important in explaining whether a founder seeks outside equity, it plays little role in the subsequent scale of external debt versus external equity.

Previous wage has a negative effect on external leverage ratio. This is somewhat puzzling, since founders with high human capital should be more likely to have access to debt financing. One possibility is that founders with high human capital are more likely to have private information about the prospects of their

Table 3.6: Determinants of financing ratios

The dependent variable in (1) is a dummy variable which equals 1 if all external funding is in the form of debt, and 0 if not. The regression method is probit. The dependent variable in (2) is the external debt ratio. The dependent variable in (3) is the same as in (2), except that we exclude trade credit. The dependent variable in (4) is debt ratio. The regression method in (2)-(4) is OLS. Two digit industry dummies and dummies for the year of the start-up are included, but not reported. Robust standard errors in parenthesis. Coefficients marked ***, ** and * are significant at 1 %, 5 % and 10 % levels, respectively.

	(1) All debt financing?		(2) D/ (D + Ex)		(3) (D - trade cr)./ (D + Ex - trade cr.)		(4) D/ Total Assets	
Founder:								
ln Wealth	.052***	.017	.013***	-.004	.022***	-.006	.000	-.000
	(3.00)	(.95)	(2.85)	(-.80)	(3.47)	(-.86)	(.04)	(-.01)
ln Prev income	-.071***	-.079***	-.016***	-.019***	-.026***	-.029***	-.005	-.005
	(-3.18)	(-3.47)	(-3.68)	(-4.42)	(-3.76)	(-4.62)	(-.39)	(-.37)
Age	.045**	.046**	.009**	.009**	.013**	.012**	.018**	.018**
	(2.51)	(2.58)	(2.05)	(2.10)	(2.09)	(2.17)	(2.45)	(2.41)
Age, sq	-.001**	-.001**	-.000**	-.000**	-.000**	-.000**	-.000**	-.000**
	(-2.42)	(-2.55)	(-2.24)	(-2.39)	(-2.09)	(-2.27)	(-2.37)	(-2.39)
Education	-.014**	-.013*	-.004*	-.003*	-.010***	-.009***	-.003	-.003
	(-1.97)	(-1.78)	(-1.96)	(-1.69)	(-3.27)	(-3.20)	(-1.17)	(-1.14)
Female	.047	.075	.010	.021	.031	.039	-.072***	-.072**
	(.56)	(.89)	(.52)	(1.10)	(.99)	(1.39)	(-2.85)	(-2.52)
Start-up:								
EBITDA-	-.020	-.071	-.088***	-.107***	-.071**	-.110***	-.273***	-.274***
margin	(-.24)	(-.85)	(-3.46)	(-4.29)	(-2.15)	(-3.43)	(-3.70)	(-3.67)
ln Total Assets		.198***		.084***		.136***		-.002
		(8.88)		(14.18)		(18.19)		(.04)
Constant	-.689	-2.102**	.521***	.063	-.095	-1.016***	.538**	.527***
	(-.74)	(-2.17)	(5.11)	(.62)	(-.68)	(-7.57)	(2.56)	(3.47)
N	2,922	2,922	2,772	2,772	2,172	2,172	2,927	2,927
R^2	.0397	.0600	.1529	.2237	.2195	.3262	.0255	.0255

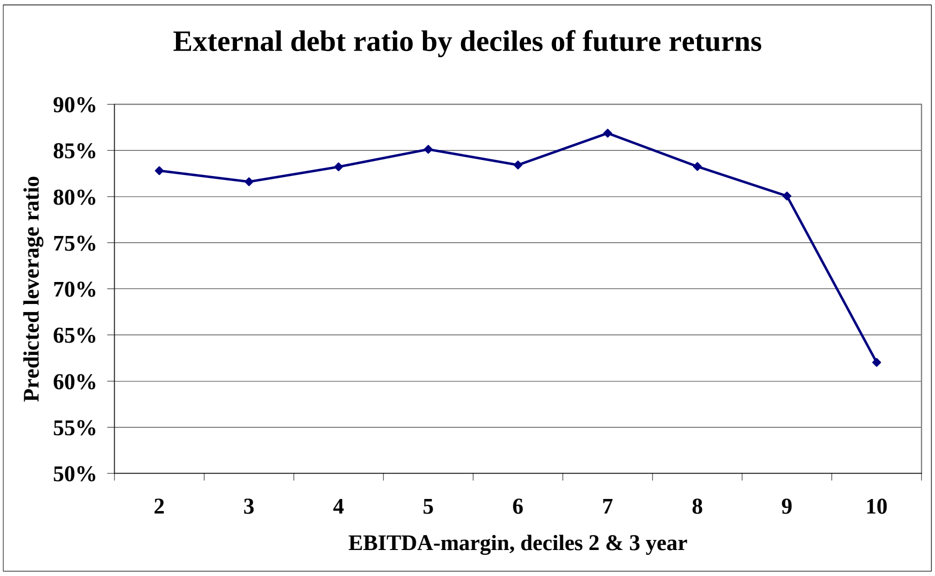
start-up, and use equity as a substitute for debt in a manner consistent with the pecking-order theory. To evaluate this possibility further, we split the sample into two groups; start-ups with principal founders that have low previous wage (less than NOK 500K) and high previous wage (more than NOK 500K). We then estimated the relationship between profitability and leverage for the founders with a high previous wage. Neither this regression gave support to the pecking-order hypothesis.

One explanation for the lack of a positive relation between profitability and leverage could be that firms that are more highly leveraged are slower at becoming profitable. To assess this possibility, we investigated the relation between profitability and initial leverage in yearly regressions. We found that the relation between profitability and initial leverage is significantly negative in the second year of operations, and not significantly different from zero in either the first, third or fourth year of operations.

To test for the robustness of our results, we performed a variety of exercises. First we constructed a debt ratio measure excluding trade credit. This regression, reported in column (3), gives almost identical results to column (2). Second, column (4) reports the regression of the conventional debt ratio. Neither here does the pecking-order theory get any support. Third, we used alternative measures of start-up profitability, such as return on equity. The results were the same. Fourth, we investigated whether the prevalence of outside investors impacts survival probability. We found no such relationship. Fifth, the results are robust for exclusion of first year equity and total assets, as well as replacing total assets by turnover. We conclude that there is little or no support for a pecking-order in the data.

Figure 2 used the estimated coefficients of column (2) to depict predicted leverage ratio for varying deciles of EBITDA-margins, holding the other variables at mean values.

We see that the relation between profitability and leverage is initially flat or slightly increasing and then quite strongly negative in the upper quartile. A possible reason for the drop in the upper quartile is reverse causality: companies that are very profitable in the first year tend to pay down their debt and rely on their cash flow for financing. To investigate this possibility, we looked at a subsample of firms that are started up in October or later in the calendar year. These companies are unlikely to generate enough cash to pay down substantial amounts of debt during



their first calendar year. The predicted relation between profitability and leverage ratio is almost identical to for the full sample.

3.3.5 Discussion

Let us first briefly summarize our findings so far. Outside equity is a commonly used source of financing: 40 percent of the start-ups in our sample use outside equity financing, and for these companies outside equity is of equal magnitude to the use of bank debt and trade credit. The start-ups that use outside equity are more likely to have poorer founders, be smaller, and to have less inside equity. The relation between low founder wealth and the use of outside equity is well in line with the risk shifting theory. The empirical relevance of the risk shifting theory is further corroborated by the main rival, the pecking-order theory, receiving little or no support in the data.

In the next section, we try to assess the effectiveness of outside equity as a financing means. Let us first briefly discuss two alternative explanations for why the start-ups use outside equity. First, the trade-off theory posits that the optimal amount of outside equity balances gains from decreased financial distress costs against reduced tax benefits of debt. Since debt under the Norwegian tax system does not have any advantages over equity financing, this theory cannot explain why firms in our sample are typically heavily leveraged. Second, the more risk averse founders could use outside equity in order to share risk. The age, gender and education variables provide a crude control for risk preferences. Also, if risk preferences explain the use of outside equity, we would expect founders that use outside equity to invest a smaller fraction of their wealth in the start-up. We find that principal founders' median investment in the start-up represents 16 % and 19 % of their gross wealth, for founders that use and do not use outside equity, respectively. Thus it does not seem that risk preferences have a powerful influence on the propensity to use outside equity.

3.4 The effectiveness of outside equity

In this section we try to assess the effectiveness of outside equity in capitalizing the start-ups. The risk shifting theory suggests that equity infusion from outsiders can generate more debt financing for credit constrained start-ups through an "equity multiplier". This section first tries to assess the magnitude of the equity multiplier. We next assess two mechanisms that could limit the effectiveness of outside equity; moral hazard and investor monopolization due to scarce supply of outside investors.

3.4.1 The equity multiplier

Table 7 analyzes the determinants of the amount of debt that a company has raised at the end of first year of operations. To control for founder liquidity prior to start-up we include a measure of founder wealth. To control for observable aspects of founder human capital, we include founder age, sex, education and previous wage.

In the first column, we use the full sample to investigate the relation between total equity and debt level. We see from the first column that both founder wealth

Table 3.7: Determinants of debt

The dependent variable is the natural logarithm of debt plus trade credit. Regression (1) include all start-ups, regressions (2) and (3) include start-ups with external owners. Two digit industry dummies and dummies for the year of the start-up are included, but not reported. The regression method is OLS. Robust standard errors in parenthesis. Coefficients marked ***, ** and * are significant at 1 %, 5 % and 10 % levels, respectively.

	(1)	(2)	(3)
Founder:			
ln Wealth	.121***	.074	.072
	(4.65)	(1.58)	(1.52)
ln Previous income	-.038	.004	-.001
	(-1.36)	(.07)	(-.02)
Age	.044*	.071*	.072*
	(1.75)	(1.71)	(1.73)
Age, sq.	-.001	-.001	-.001
	(-1.44)	(-1.49)	(-1.52)
Education	-.021*	-.024	-.023
	(-1.77)	(-1.32)	(-1.23)
Female	-.221**	.026	.024
	(-1.95)	(.14)	(.12)
Start-up:			
ln Equity	.535***	.705***	
	(14.60)	(11.10)	
ln OUTSIDE			.169**
Equity			(1.98)
ln INSIDE			.528***
Equity			(5.89)
Constant	1.279**	3.55	.830
	(2.21)	(.39)	(.91)
N	2,702	1,038	1,038
R^2	.2533	.3412	.3372

and equity get positive and significant coefficients in explaining debt level.[18] Based on these coefficients, NOK 50K of additional equity at incorporation gives an increase in debt of NOK 40K, calculated at mean levels for debt and equity. In the main regression, reported in column (2), we confine attention to the start-ups that use outside equity. Here we see that the elasticity of debt with respect to equity is around .7. Since not all the start-ups need be credit-constrained, the estimate of the equity-multiplier gives a conservative picture of the maximum debt increase possible after an equity infusion (Evans & Jovanovic, 1989, contains a very similar point).

To examine the role of outside equity further, in column (3), we break up equity into inside and outside equity. Here we see that inside equity seems to have a stronger role in explaining debt level than outside equity.[19] To further analyze the relation between outside equity and debt capacity, Table 8, repeats the regression analysis from Table 7 with inside and outside equity separated and split by quartiles of principal founder wealth.

The relation between outside equity and debt level is of considerable magnitude for the first two wealth quartiles (although only weakly significant), while there is no relation between outside equity and debt for the upper two quartiles. This result confirms the impression that outside equity is of particular importance to less wealthy founders. The estimated coefficients suggest that, evaluated at the means, NOK 50K increased outside equity gives about NOK 25K additional debt financing. Assuming that all firms that use outside equity is credit constrained, this implies that the equity multiplier is around 1.5.

Of the other variables, age carries a large and significant coefficient for the wealthiest quartile, although the squared term is negative, penalizing the oldest. A surprising result is that none of the variables can explain any of the debt-raising for the smallest quartile. Industry dummies explain large parts of the volume of debt, as expected.

[18] A question we plan to pursue is how debt capacity varies with investor human capital (positive since better reputation) and with external share (negative by Holmstrom-Tirole).

[19] Preliminary regressions of debt on investor characteristics indicate that these may provide additional explanations as to the level of debt. We plan to pursue this further in later versions.

Table 3.8: Sample with external investors: Debt determinants by founder wealth quartiles

The dependent variable is the natural logarithm of debt plus trade credit. Regression 1 include all start-ups, regressions 2 and 3 include start-ups with external owners. Two digit industry dummies and dummies for the year of the start-up are included, but not reported. The regression method is OLS. Robust standard errors in parenthesis. Coefficients marked ***, ** and * are significant at 1 %, 5 % and 10 % levels, respectively.

	(1)	(2)	(3)	(4)
Founder:				
ln Wealth	-.075	-.074	.892	.357*
	(-.95)	(-.11)	(1.19)	(1.66)
ln Previous income	.022	.034	-.084	-.028
	(.16)	(.21)	(-.56)	(-.29)
Age	.066	.095	-.097	.258**
	(.56)	(.84)	(-1.07)	(2.47)
Age, sq.	-.001	-.001	.001	-.003**
	(-.33)	(-.57)	(.80)	(-2.54)
Education	.007	-.063	-.043	.057
	(-.20)	(-1.45)	(-1.25)	(1.22)
Female	.291	-.354	.342	-.078
	(.96)	(-.83)	(.56)	(-.16)
Start-up:				
ln OUTSIDE Equity	.330	.373*	.079	.001
	(1.63)	(1.85)	(.37)	(.00)
ln INSIDE Equity	.198	.441**	.485**	.616***
	(.92)	(2.27)	(2.40)	(3.26)
Constant	.505	1.474	-.341	-7.168**
	(.25)	(.38)	(-.06)	(-2.33)
N	259	260	250	269
R^2	.3623	.3765	.4672	.4293

3.4.2 Moral hazard

Theory emphasizes that using outside equity could have a detrimental effect on insiders' incentives to exert effort. To evaluate whether there indeed is such a negative effect of outside equity, we investigate whether companies that use outside equity have lower profitability. As performance variable we use EBITDA-margin. To control for founder liquidity prior to start-up we include a measure of founder wealth. To control for observable aspects of founder human capital, we include founder age, sex, education and previous wage. To control for start-up type, we include level of assets and initial equity, in addition to year and industry controls.

In column (1) we regress profitability on a dummy that equals one if the start-up uses outside equity, controlling for other characteristics of the start-up. We do not find any detrimental effect on performance from using outside equity as a financing source. In column (2) we regress profitability on external ownership share and again find no effect of outside equity. The robustness of the regression has been confirmed by median and robust regressions as well as replacing the return-measure by weighted ROAA for the same period with unchanged results.

3.4.3 Investor returns

Start-ups that use outside equity are more likely to be liquidity constrained, and it is natural to ask whether there is enough outside equity available in the market. For example, if outside equity financing primarily is taken from the entrepreneur's existing network, a notion that our evidence supports, then this network may be too small to provide sufficient equity financing.

Although this hypothesis is difficult to test directly, we can perform an indirect test. If the entrepreneur's network of potential outside investors is small, then it is likely that these investors can exert monopoly power and require a high rate of return to provide financing. With this backdrop, we investigate investor returns. Our empirical strategy is simple. We compare the cash returns of companies with and without external investors, respectively, and view a higher return for companies with external investors as evidence of investor monopolization.

We should note that there are several qualifications to our analysis. First, our data lacks information as to any differences in the share premiums, i.e., the premium paid on top of nominal value for the shares, between internal and external

Table 3.9: Determinants of return

The dependent variable is the asset-weighted average EBITDA-margin over year 2 and 3. Two digit industry dummies and dummies for the year of the start-up are included, but not reported. The regression method is OLS. Robust standard errors in parenthesis. Coefficients marked ***, ** and * are significant at 1 %, 5 % and 10 % levels, respectively.

	(1)	(2)
External owners ?	-.003	
	(-.26)	
External ownership (%)		-.022
		(-.98)
Founder:		
ln Wealth	.009**	.009**
	(2.18)	(2.15)
ln Previous income	.013***	.013***
	(3.24)	(3.27)
Age	-.002	-.002
	(-.54)	(-.56)
Age, sq.	.000	.000
	(.23)	(.25)
Education	.002	.002
	(1.31)	(1.33)
Female	.025	.025
	(1.61)	(1.59)
Start-up:		
ln Equity	-.108***	-.108***
	(-15.04)	(-14.96)
ln Total Assets	.053***	.052***
	(9.79)	(9.66)
Constant	.236***	.246***
	(2.73)	(2.84)
N	3,167	3,167
R^2	.1289	.1291

founding shareholders. We cannot rule out that external investors might have paid a premium which would impact their subsequent investment returns. The more fundamental issue when analyzing returns is that we obviously do not have any market values for the shares in these private companies and any measures of investor returns are thus only based on cash dividends and book equity.[20]

Table 10 compares the investor cash return for companies with and without outside equity. The cash return is calculated annually as declared dividends divided by average invested equity capital for the passed year.[21] The observations are weighted by average invested equity for each cell. The interpretation of the yearly cash return is thus the expected returns for a randomly invested krone of equity capital. We also report averages on whether start-ups pay a dividend or not, and on the payout ratio. Yearly cash return and payout ratio annual measures are winsorized annually by 5 percent symmetrically. To ensure comparability, all returns as well as the subsequent regression analyses are conducted on a subsample where we are able to calculate 4 year cash return.

In Table 10, we find very small differences between the two subsamples across the cash return measures.[22] Firms with external shareholders seem to receive a dividend more seldom, which implies that dividends, if paid, are larger. The payout ratio for start-ups with outside equity is smaller than for firms that do not use outside equity, which again suggests that using outside equity is an indication of being liquidity constrained.[23]

[20] Since the survival rate of the companies in the sample is quite high, we would expect the cash returns to understate the actual returns, so that our estimates of investor returns are conservative.

[21] We do not calculate cash return for the first reported year to avoid any adjustments for differences in operating periods up until first report. Invested equity is the accumulated sum of intial and subsequent equity-issues by the firm. The 2 percent of firm-years with negative average invested equity are left out.

[22] Moskowitz & Vissing-Jorgensen (2002) analyze the returns to private equity using a variety of data sources and find that the average yearly returns are around 8 percent. Given that the firms in the current sample are likely to carry higher risk, the overall risk-adjusted returns seem of similar magnitude.

[23] A natural question is how investors enforce repayments. Potentially there is opportunity for the internal founders to divert funds, by for example raising their wages and lower dividend payments. Theory suggests that such diversion problems should be met by outside investors requiring a debt contract rather than an equity contract (e.g., Townsend, 1979). The fact that outside equity is used to a considerable extent, and that investor returns reported in Table 10 are relatively high, suggests that the diversion problem is not of great importance to the external investors of the start-ups we observe. It could still be that the lack of enforcement mechanisms

Table 3.10: Shareholders' return – descriptive statistics

External investors?	*Yes*			*No*		
	Median	Mean	St. Dev	Median	Mean	St. Dev
Dividend?						
2nd year	.00	.09	(.28)	.00	.30	(.46)
3rd year	.00	.12	(.32)	.00	.20	(.40)
4th year	.00	.10	(.30)	.00	.10	(.30)
Payout ratio						
2nd year	.00	.07	(23)	.00	.11	(.26)
3rd year	.00	.09	(.29)	.00	.18	(.41)
4th year	.00	.07	(.25)	.00	.20	(.46)
Cash return						
2nd year	.00	.14	(.70)	.00	.15	(.66)
3rd year	.00	.18	(.76)	.00	.16	(.70)
4th year	.00	.11	(.64)	.00	.11	(.62)
4 year cash return						
	.00	.26	(1.08)	.00	.29	(1.59)
N		1,128			1,702	

In Table 11, we regress shareholder cash return on founder and start-up characteristics.

Column (1) shows that after controlling for founder and firm characteristics, start-ups using outside equity are somewhat less likely to pay dividends. Our estimates in column (3), however, only weakly suggest that the investor returns in companies that use outside equity is lower than in companies that do not use outside equity. It therefore does not seem monopolization on the supply side of the outside equity market poses a grave problem to the start-ups in the sample.

could stop would-be entrepreneurs from obtaining outside equity.

Table 3.11: Determinants of shareholders' return

The dependent variable in column (1) is a dummy variable that equals 1 if the start-up pays dividends in that year, and 0 if not. The regression method is probit. The dependent variable in column (2) is payout as a percentage of net profit. The dependent variable in column (3) is annual cash return, measured as dividend by average invested equity capital. The dependent variable in column (4) is the first 4 years weighted average cash return. The regression method in column (2) - (4) is OLS. In all regressions, two digit industry and start-up year dummies are included, but not reported. Robust standard errors in parenthesis. Coefficients marked ***, ** and * are significant at 1 %, 5 % and 10 % levels, respectively.

	(1) Dividend-dummy	(2) Payout ratio	(3) Cash return	(4) 4 year cash return
External ownership, %	-.149*	-.066*	-.035	-.156
	(-1.69)	(-1.68)	(-.48)	(-1.54)
ln Wealth	.048***	.010**	.015	.029
	(3.06)	(2.14)	(1.51)	(1.33)
ln Previous income	.091***	.018**	.011	.070***
	(4.61)	(2.32)	(1.06)	(3.04)
Age	-.004	.007	.014	.011
	(-.27)	(.78)	(1.31)	(.55)
Age, sq.	.000	-.000	-.000	-.000
	(.41)	(-.66)	(-1.43)	(-.76)
Female	.031	.101	.053	.136
	(.46)	(1.11)	(1.41)	(1.34)
Education	.007	.002	-.004	.011*
	(1.20)	(1.23)	(-1.06)	(1.74)
ln Total assets	.302***	.067***	.000	.405***
	(15.47)	(13.90)	(.04)	(6.83)
ln Equity, inc.	-.283***	-.030	-.043***	-.520***
	(-10.85)	(-.67)	(-5.15)	(-9.94)
Firm age: 2nd year	.119***	-.011	.043*	
	(4.08)	(-.20)	(1.75)	
Firm age: 3rd year	.087**	.004	(dropped)	
	(2.31)	(.07)		
Firm age: 4th year	.036	-.011	-.085***	
	(.76)	(-.19)	(-2.87)	
Constant	-.772**	-.201	-.258	-2.074
	(-2.24)	(-.44)	(-1.10)	(-3.22)
N	11,131	11,204	8,469	2.830
R^2	.1050	.0148	.0664	.1381

3.5 Conclusion

We have investigated the extent to which start-ups use outside equity, and interpreted our results in relation to financial contracting theory. We have also tried to evaluate how well outside equity performs as a source of financing. We have done so by studying the start-up and founder characteristics that are associated with the use of outside equity financing, using a unique dataset from Norway.

Our findings suggest that adverse selection are less of a concern for start-ups than ex-post opportunistic behavior (risk shifting) by the entrepreneur as in Myers (1977) and Ravid & Spiegler (1997). Actual financing looks similar to what one would expect from the risk-shifting theory of Myers (1977).

One implication of this finding is that outside equity and debt are complements rather than substitutes, and equity serves the dual role of providing funding per se, and to release debt financing. Our estimates suggest that the "equity multiplier" for outside equity is at least 1.5.

According to theory, financing by outside equity could have the negative effect of reducing insiders' incentives to exert effort. To evaluate whether there indeed is such a negative effort effect, we investigated whether companies that use outside equity show signs of having weaker profitability. Although our findings here are preliminary, we do not find convincing evidence that the use of outside equity has detrimental effects on start-up performance, nor that a possible shortage of available outside equity leads to investor monopolization and excessive investor returns. Thus we provide evidence that outside equity provides an important avenue for entrepreneurs to escape liquidity constraints.

We should caution that although adverse selection in its traditional form do not seem to be main problems for the start-ups that we observe, we cannot rule out that these mechanisms could stop potential entrepreneurs from obtaining financing to start up a business. To make headway on this question, one needs a more structural approach, which seems like a promising avenue for future research.

3.6 References

Aghion, P. and P. Bolton ,1992, An Incomplete Contracts Approach to Financial Contracting. *Review of Economic Studies*, 59, 473-94.

Aghion, P., T. Fally and S. Scarpetta ,2007, Credit Constraints as a Barrier to Entry and Post-Entry Growth of Firms. Mimeo, Harvard University, Economics Department.

Aghion and Bolton ,1997, A Theory of Trickle-Down Growth and Development. *Review of Economic Studies*, 59, 151-72.

Ando, F. ,1985, Access to Capital by Subcategories of Small Business. Report prepared for the U.S. Small Business Administration. Fort Washington, Pa.: JACA Corp.

Berger, A. N. and G. F. Udell ,1998, The Economics of Small Business Finance: The Roles of Private Equity and Debt Markets in the Financial Growth Cycle. *Journal of Banking and Finance*, 22, 613-73.

Bond, P. and R. Townsend ,1984, Formal and Informal Financing in a Chicago Ethnic Neighborhood. *Economic Perspectives*, 3-25. Federal Reserve Bank of Chicago.

Calvet, L., J. Y. Campbell and P. Sodini ,2006, Down or Out: Assessing the Welfare Costs of Household Investment Mistakes. NBER working paper No. 12030. Forthcoming, *Journal of Political Economy*.

Campbell, J. Y. ,2006, Household Finance. *Journal of Finance*, 61, 1553-1604.

Cassar, G. ,2004, The Financing of Business Start-ups. *Journal of Business Venturing*, 19, 261-83.

Evans, D. S. and B. Jovanovic ,1989, An Estimated Model of Entrepreneurial Choice under Liquidity Constraints. *Journal of Political Economy*, 97, 8-27.

Fluck, Z., D. Holtz-Eakin and H. S. Rosen ,1997, Where Does the Money Come From? The Financing of Small Entrepreneurial Enterprises. Working paper, New York University.

Frank, M. and V. K. Goyal ,2006, Trade-off and Pecking Order Theories of Debt. Forthcoming in Handbook of Corporate Finance, Elsevier/ North-Holland Press. Edited by B. Espen Eckbo. Available at
http://mba.tuck.dartmouth.edu/pages/faculty/espen.eckbo/NewBooks.html.

Gertler, M. and S. Gilchrist ,1994, Monetary Policy, Business Cycles, and the

Behavior of Small Manufacturing Firms. *Quarterly Journal of Economics*, 109, 309-40.

Hart, O., and J. Moore. ,1994, A Theory of Debt Based on the Inalienability of Human Capital. *Quarterly Journal of Economics*, 109, 841-79.

Holmstrom, B. and J. Tirole ,1997, Financial Intermediation, Loanable Funds, and the Real Sector. *Quarterly Journal of Economics*, 62, 663-91.

Hvide, H. K. and T. Leite ,2004, Capital Structure under Imperfect Enforcement. CEPR working paper 4757.

Hvide, H. K. and J. Møen ,2007, Liquidity Constraints and Entrepreneurial Performance. CEPR working paper. Available at www.ssrn.com.

Kaplan, S. and P. Stromberg ,2003, Financial Contracting Theory Meets the Real World. *Review of Financial Studies*, 70, 281-315.

Lemmon, M. L., M. R. Roberts and J. F. Zender ,2007, Back to the beginning: Persistence and the Cross-Section of Corporate Capital Structure. *Journal of Finance* ,Forthcoming 2007)

Moskowitz, T. and A. Vissing-Jorgensen ,2002, The Returns to Entrepreneurial Investment: A Private Equity Premium Puzzle? *American Economic Review*, 92, 745-78.

Myers, S. C. ,1977, Determinants of Corporate Borrowing. *Journal of Financial Economics*, 5, 147-75.

Myers, S. and N. Majluf ,1984, Corporate Financing and Investment Decisions when Firms Have Information that Investors do not Have. *Journal of Financial Economics*, 13, 187-221.

Myers, S. C. ,2000, Outside Equity. *Journal of Finance*, 55, 1005-37.

Ng, C. K., J. K. Smith and R. L. Smith ,1999, Evidence on the Determinants of Credit Terms Used in Interfirm Trade. *Journal of Finance*, 54, 1109-29.

Paulson, A., R. M. Townsend and A. Karaivanov ,2006, Distinguishing Limited Liability from Moral Hazard in a Model of Entrepreneurship. *Journal of Political Economy*, 114, 100-144.

Petersen, M. A. and R. G. Rajan ,1994, The Benefits of Lending Relationships: Evidence from Small Business Data. *Journal of Finance*, 49, 3-37.

Ravid, S. A. and M. Spiegel ,1997, Optimal Financial Contracts for a Start-Up with Unlimited Operating Discretion. *Journal of Financial and Quantitative Analysis*, 32, 269-86.

Stiglitz, J. and A. Weiss ,1981, Credit Rationing in Markets with Imperfect Information. *American Economic Review*, 71, 393-410.

Tirole, J. ,2006, The Theory of Corporate Finance. Princeton University Press, Princeton and Oxford.

Townsend, R. M. ,1979, Optimal Contracts and Competitive Markets with Costly State Verification. *Journal of Economic Theory*, 21, 265-293.

Van Auken, H. E. and R. B. Carter ,1989, Acquisition of Capital by Small Businesses. *Journal of Small Business Management*, April issue, 1-9.

4

Callable Risky Perpetual Debt: Options, Pricing and Bankruptcy Implications

coauthored with Svein-Arne Persson [1]

Abstract

Issuances in the USD 260 Bn global market of *perpetual* risky debt are often motivated by capital requirements for financial institutions. However, observed market practices indicate that actual maturity equals first possible call date. We develop a valuation model for callable risky perpetual debt including an initial protection period before the debt may be called. The total market value of debt including the call option is expressed as a portfolio of perpetual debt and barrier options with a time dependent barrier. We analyze how an issuer's optimal bankruptcy decision is affected by the existence of the call option using closed-form approximations. In accordance with intuition, our model quantifies the increased coupon and the decreased initial bankruptcy level caused by the embedded option. Examples indicate that our closed form model produces reasonably precise coupon rates compared to more exact numerical solutions. The credit-spread produced by our model is in a realistic order of magnitude compared to market data.

[1]We want to thank Petter Bjerksund, Beate Breuer, B.Espen Eckbo, Trond Døskeland, Michael Genser, Hans K. Hvide, Thore Johnsen, Elyès Jouini, Hayne Leland, Kristian R. Miltersen, Tommy Stamland, Gunnar Stensland, and Josef Zechner. Earlier versions of this paper have been presented at FIBE 2005, NHH Skinance 2005 (Hemsedal), Essec Business School Paris 2006, European Financial Management Conference Madrid 2006, 10th Conference of Swiss Society for Financial Market Research Zürich 2007, and at internal seminars at the Norwegian School of Economics and Business Administration.

4.1 Introduction

Perpetual debt securities seldom turn out to be particularly long-lived - in spite of their *ex ante* infinite horizon. The contractual horizon gives the securities a, using regulatory language, *permanence*, which is crucial when banks and other financial institutions are allowed to include them as regulatory required risk capital. However, the contracting parties, the issuing institution and the investors in the securities, typically appreciate financing flexibility and may thus prefer a more tractable finite horizon. These apparently conflicting objectives are resolved by embedding such perpetual securities, almost without exceptions, with an issuer's call-option, facilitating a finite realized horizon.

We develop a closed form valuation model for perpetual debt securities including this option. Our model allows for calibration of coupon rate and optimal bankruptcy asset level. We also show, analytically and through examples, how the finitely lived option embedded in the perpetual security impacts coupon rates and bankruptcy for alternative assumptions in the period before the option expires. These securities are junior and unsecured and thus far more exposed to the issuer's credit quality than regular senior bonds. Market practise indicates that issuers' typically pay a credit margin on top of a market reference interest rate, and are thus not directly exposed to the nominal interest rate levels. We reflect this in our model by including risk through the volatility of the cashflow(EBIT)-process and not using a stochastic interest rate process. This is in line with the corporate finance perspectives of precedents like Black and Cox (1976) and Leland (1994), but contrary to related research in the asset pricing field like, e.g., Acharya and Carpenter (2002).

The riskiness of deeply subordinated perpetual debt securities typically favors issuers that are heavily regulated, highly rated, well capitalized and have previously issued securities in the capital markets. Consequently, the issuers of such securities are typically banks, but also include insurance companies and utilities. Favorable changes to US tax treatment of coupons/dividends, RBC-factors[2] for US insurers and the rating agencies' views have caused a strong growth in new issuance in

[2]RBC: Risk Based Capital requirements is used by the National Association of Insurance Commissioners (NAIC) to regulate the capital adequacy of US insurers. Within this system, investment assets are assigned capital requirements according to their riskiness.

2004/2005. Table 4.1 provides an overview of estimated outstanding large issues of perpetual debt securities as of end-2005 from Pomper and Varma (2005).

Table 4.1: Global outstanding large perpetual debt capital securities end-2005.

Amounts and relative shares are calculated from Lehman Brothers' index of debt- and debt-like risk capital securities. Percentages relate to share of each market. Source: Lehman Brothers.

Market	Outstanding amount (USD Bn)	Tier I (%)	Upper Tier II (%)	Non-financials' share(%)
U.S. dollar	82.8	75	17	8
Euro	105.6	43	16	41
Sterling	71.4	29	57	13
Total	259.8	50	28	23

Table 4.1 shows that the total issuance is evenly spread across the main markets, but that there are large regional variations in mix of securities. The largest category, denoted 'Tier I', refers to securities that qualify as the highest quality risk capital for issuing financial institutions[3]. 'Upper Tier II'-capital is also perpetual but less risky and within the subordinated debt-category for financial institutions. The latter dominates the GBP Sterling market whilst utilities and other non-financial issuers have a large share of the Euro-market. The globally accepted principles for capital adequacy and classification of risk capital were specified by Bank for International Settlements (BIS) in 1988[4]. National variations are caused by regulatory, tax and capital market differences.

Mapondera and Bossert (2005) include 50 large European banks in their research universe and show that amongst those, the volume of new perpetual securities equals 28% of the volume of newly issued senior market debt for the years 2000 - 2005. This category is split between Tier I capital, representing 20%-points and Upper Tier II capital covering the remaining 8%-points. The report also lists all individual new issues of new perpetual securities by these banks during 2004/2005 and all of them have a delayed issuer's call option, typically exercisable 10 years from date of issue. These calls are contractually *American* in that they may be

[3] In this context, financial institutions primarily represent banks and insurance companies.
[4] See Committee on Banking Supervision (1988)

exercised at any coupon-paying dates after year 10. The coupon rate is also typically stepped-up by 75-150 bp at the first call date. In this paper we, however, model the call as a *European*, finitely lived option expiring at the first possible exercise date. In any case, the obtained European option values are lower bounds for American options. Practice indicates that all issuers exercise them at the first possible date see, e.g., Ineke, Guillard, and Mareels (2003) who state: 'To our knowledge, there has only ever been one instance of an issuer not calling a bond and allowing it to step-up and this was actually done unintentionally'.

We follow the approach by Black and Cox (1976) and Leland (1994), including full information and efficient market assumptions. In line with Goldstein, Ju, and Leland (2001) we assume that the issuing company's assets produce a stream of cashflows that follows a geometric Brownian motion. For a given capital structure, including an infinite horizon debt contract, there exists a constant asset value level where it is optimal for the company to go bankrupt. After introducing a finitely lived option on the debt, this bankruptcy level is no longer independent of time to expiration of the option. The bankruptcy level after expiration of the option equals the constant Black and Cox (1976)-level.

One could alternatively consider the situation where third parties trade options on publicly traded debt. Naturally, the existence of such options would neither influence the pricing of the bonds at issue or in the marketplace nor the issuing company's optimal choice of bankruptcy level. In this paper, however, we consider a corporate setting where the issuer's call option is an integrated part of the bond(debt) contract. That is, the option is written by the debtholders in favor of the equityholders. We refer to such a call option as an *embedded* option. The existence of this option will influence both the issue-at-par coupon on the debt and the issuer's bankruptcy considerations before the option's expiration date. Intuition suggests that the coupon is increased to compensate for the embedded option, whereas the optimal bankruptcy level is decreased due to the option value - both compared to the case without an option.

We show in Section 3 that the market value of infinite horizon debt is not lognormally distributed and this fact impacts the valuation of options on such instruments. The standard Black and Scholes (1973) and Merton (1973) option pricing formulas are thus not directly applicable. The time 0 market value of perpetual debt according to Black and Cox (1976) can be interpreted as a riskfree value of an infinite stream of coupons, from which the market value of the debtholder's net loss in case of bankruptcy is subtracted. The market value of the debtholder's net loss is equivalent to the market value of the equityholders' default-option in a limited liability company. The market value of this net loss has the required lognormal properties and can be used as a modified underlying asset replacing the market value of infinite horizon debt itself. By this reformulation the standard Black-Scholes-Merton formulas can be applied using modified time 0 market value, exercise price and volatility.

We develop pricing formulas for both plain vanilla European options and down-and-out barrier European options on infinite horizon continuous coupon paying debt. Down-and-out barrier options are relevant since the debt options may only be exercised at the future time T if the issuing company has not gone bankrupt earlier. The asset-level which defines optimal bankruptcy before the option expires is thus the barrier used in the barrier option formulas.

For analytical tractability we assume that the time dependent barrier is an increasing exponential function. This is a straightforward way to model a time dependent barrier and a natural first attempt, but still an arbitrary choice. To investigate the significance of time dependency, we test the effect of alternative bankruptcy barrier assumptions. Our examples show that the effect of time dependency on the coupon-rate is limited, but increasing in cashflow volatility and time to expiration of the option.

4.1.1 Economic interpretation and insights from our analysis

In our application of the barrier option formulas on debt with embedded options, we want in particular to analyze the impact on debt payoff and optimal bankruptcy decisions. In this paper we denote the market value of total company assets at time t by A_t and the market value of its' debt at time t by D_t.

The payoff to debtholders at expiration of the embedded option

The payoff to debtholders when the option expires is illustrated in Figure 4.1[5].

[5] This and the next graphical presentation use the same base case parameters as in Table 4.2 in Section 5 of the paper: Time 0 EBIT (earnings before interest and tax) $\delta_0 = 3$, asset level $A_0 = 100$, par value of debt $D = 70$, expiration date of option $T = 10$ years, volatility of EBIT and assets

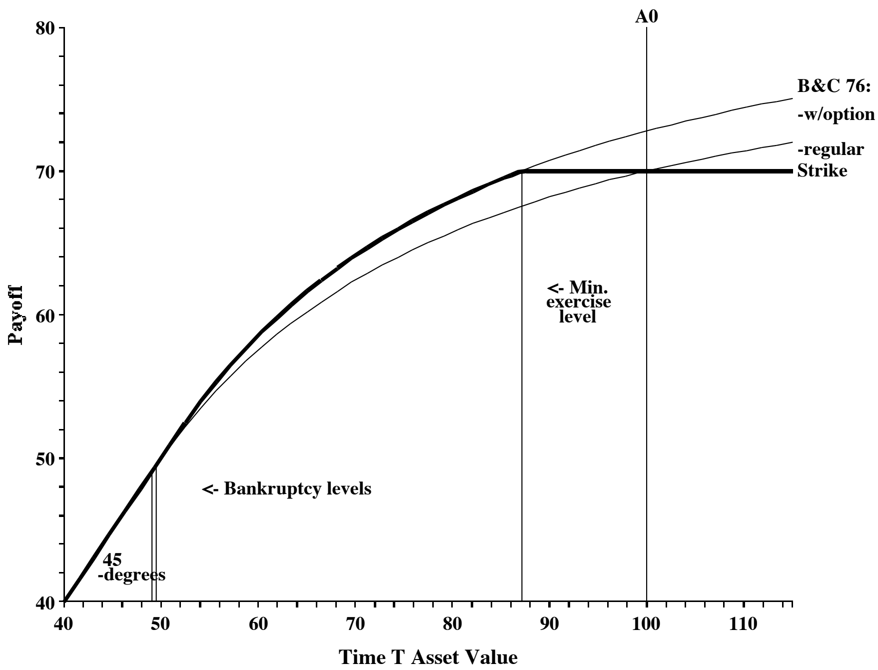


Figure 4.1: Payoff from perpetual debt with and without embedded option at time T as a function of asset level A_T. See Table 4.2 for parameter values.

The payoff to debtholders is shown as a function of asset value, A_T, for debt with and without embedded option, assuming that the absolute priority rule is followed. The leftmost part of the graph shows that in bankruptcy, debtholders receive all assets as payoff, indicated by the 45-degree line. Beyond the bankruptcy asset level, the thicker/upper line indicates the payoff to debt with embedded option whilst the thinner/lower line represents payoff to regular perpetual debt. The optimal bankruptcy asset levels for these structures are different due to the different coupons. At time T the option does not impact optimal bankruptcy level anymore and it is only the higher coupon that causes a higher optimal long-term bankruptcy asset level.

The more interesting issue is for which levels of A_T the option is rationally

σ = 0.20, constant riskfree interest rate r = 5% and drift of the EBIT(cashflow) process, μ = 2%. These parameters yield issue-at-par coupon rates of 5.718 % for perpetual debt without option and 5.998 % for the equivalent with embedded option, solved analytically.

exercised. *Ceteris paribus*, perpetual debt with a higher coupon will be more valuable than debt with a lower coupon. In our model, all uncertainty stems from the EBIT-process. By not exercising the option at time T, the issuer is left with regular perpetual debt with a higher coupon than time T issued identically risky debt. The explanation is that the coupons were fixed at time 0 and that a part of the historical coupon was a compensation to debtholders for the embedded, but at time T expired, option. The issuer is therefore willing to exercise the option at lower levels of A_T relative to the time 0 value of A_0 to avoid paying this relatively high coupon in the future. In the example in Figure 4.1, where the exercise level of the option is par value of the debt, (70), the indifference asset level is appr. 87, compared to the time 0 asset level[6] of 100. At this indifference level, the issue-at-par optimal coupon for newly issued straight debt will exactly equal the original coupon for debt with option. This is valid irrespective of any refinancing considerations which are in any case outside our model[7].

The 'smiling' bankruptcy-level

Our analysis combines the infinite debt contract with an embedded finite option. The classical infinite setting from Black and Cox (1976) leads to a constant bankruptcy level. The market value of a finite option depends on its' time to expiration. After introducing a finite embedded option, the optimal bankruptcy level therefore becomes time-dependent. Intuition tells us that the existence of an option with positive value will lower the optimal bankruptcy asset level. The value of such options is also in itself dependent on the bankruptcy risk of the issuer. To model options with inherent bankruptcy risk, we use barrier options. We have illustrated this in Figure 4.2, using the same parameters as above. This figure shows how the combined market value of all option-elements taken from expression (4.4.5) in Section 4 varies varies by elapsed time. The graph shows the

[6] Our analysis provides the calibrated coupon level for debt with embedded option to ensure issue-at-par. The indifference level of A_T is found by using this coupon in the valuation expression for regular perpetual debt (6.3.10) setting $D(A_T)$ equal to the exercise level (par) and solve for A_T.

[7] Mauer (1993) also claims that the value of a call-option is the value of the opportunity to repurchase a non-callable bond with the same coupon and principal. This approach is intuitive at the time when the option expires, but in a case without any exercise premium implicit in the option, such a comparison is impossible at time of issuance simply because the coupon will itself incorporate the option-premium.

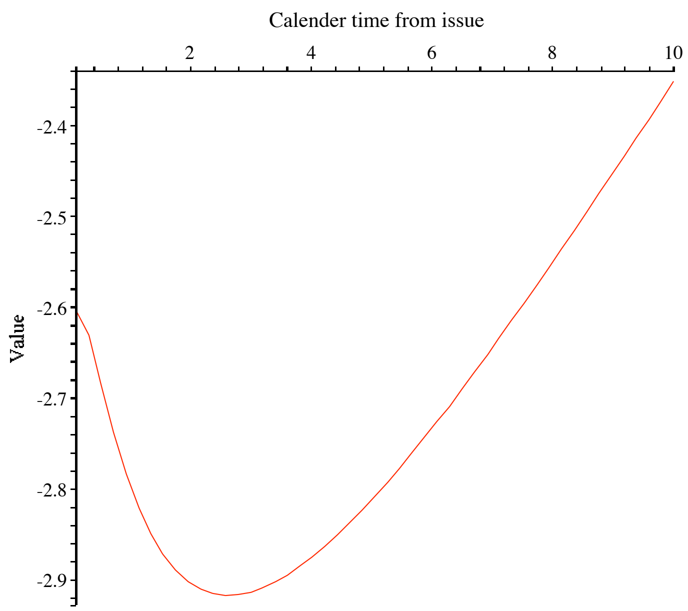


Figure 4.2: Value of the debtholders' short barrier call option position as a function of calender time when call option expires at time $T = 10$. See Table 4.2 for parameter values.

value to debtholders and is therefore negative. The explanation of the 'smile'-shape is that the market value of barrier options do not vary monotonically with time like regular options due to the inherent changes in bankruptcy risk. King (2007) study empirical the implicit value of embedded options in various U.S. agency bonds, their levels and relationship to maturity and interest rates. The issuers are less risky than banks, but it is still interesting to see that embedded calls in bonds issued by Federal Home Loan Mortgage Corporation on average represents 3.9 % of the bond values in the call protection period of up to 10 years.

4.1.2 Literature overview

The related literature may broadly be separated into research on debt-based derivatives on one hand and on perpetual debt on the other hand.

Central to the classic literature on valuing bonds with embedded derivatives are papers like Ingersoll (1977) and Brennan and Schwartz (1977). Our paper differs from these primarily with regards to the analysis of long deferral of the embedded option, its impact on bankruptcy risk, the revised formulation of the underlying security in options on bonds and the use of barrier option methodology to reflect default risk before expiry of the security. Kish and Livingston (1992) test for determinants of calls included in corporate bond contracts. Their findings are that the interest rate level, agency costs and bond maturity significantly affect whether a bond comes with an embedded call option. Sarkar (2001) is the closest precedent to our paper in his focus on callable perpetual bonds modelled in the tradition of Leland (1994). The main difference is that the calls are assumed to be *American* and immediately exercisable, i.e., without a protection period, and a main part of the paper thus deals with the optimal exercise timing of the call. The paper does neither include analytical valuation of the options nor optimal coupon- or bankruptcy levels. Bank (2004) values call options on debt in a similar manner, but without calibrating coupons nor taking into account the fact that debt-values are not log-normally distributed. The paper also lacks a clear distinction between infinite securities and finite options.

Jarrow and Turnbull (1995) model various derivatives on fixed maturity debt securities, but do not include any analysis of the impact on endogenous bankruptcy decisions. Acharya and Carpenter (2002) develop valuation formulas for callable defaultable bonds with stochastic interest rates and asset values. Through decomposing the bonds into a riskfree bond less two options, they explore how the call option impacts optimal default in line with our results. They analyze fixed maturity bonds and the hedging aspects of callable bonds through the options' impact on bond duration, but without developing exact valuation formulas for the specific bonds. Toft and Prucyk (1997) develop modified equity option expressions based on Leland (1994) for leveraged equity and various capital structure and bankruptcy assumptions. The infinite horizon property of equity makes it comparable to our work although the specific issues related to embedded options on debt are not covered directly. Rubinstein (1983) is related to our approach with the use of a modified asset process, labelled a 'displaced diffusion process', to modify the standard Black-Scholes approach. Johnson and Stulz (1987) defined the concept of 'vulnerable options' i.e. options where the counterparty may default on the contract. Hull and White (1995) categorize risky derivative contracts into classes by default risk of counterparty and credit-risk of underlying asset. Embedded options which are the focus of our paper are thus in a class of vulnerable options with credit-risky underlying assets in that the two risks may not be separated. Whilst these authors focus primarily on the risk at option maturity, the literature on barrier options, in particular Bjork (2004), effectively include the risk of bankruptcy before the option matures.

In the perpetual debt pricing tradition, starting with Black and Cox (1976), this paper is related to the paper by Emanuel (1983) which develops a valuation of perpetual preferred stock, based on the option-methodology of Black-Scholes. Preferred stock can be viewed as perpetual debt for analytical purposes. Emanuel's analysis allows unpaid dividends to accumulate as arrearage due to the junior position of the instrument, which is relevant for financial institutions, but beyond the scope of our paper. He does not cover options on preferred stock as such. Sarkar and Hong (2004) extend Sarkar (2001) and analyze the impact from callability on the duration of perpetual bonds and find that a call reduces the optimal bankruptcy level and thus extends the duration of a bond. Their reduced optimal bankruptcy level matches our intuition and results.

4.1.3 Outline of the paper

Our main contribution is a complete valuation expression for callable perpetual continuous coupon paying debt. To this end we develop option and barrier option formulas on perpetual debt contracts, handling both the lack of lognormal distribution of the market values of debt and the finite option expiration embedded in an infinite security. We thus expand the results of Black and Cox (1976) to integrate an issuer's call option into the valuation. The level and time-dependency of optimal bankruptcy for a given capital structure are consequently also changed.

Our analysis forms a basis for improved understanding of the pricing of such securities and their impact on the optimal bankruptcy level of the issuing company. Numerical examples indicate that our closed form model produces correct coupon-rates compared to more exact numerical solutions. Compared to market data the spread produced by our model is in a realistic order of magnitude.

The structure of the paper is as follows: In Section 2 we present the model and the basic results. In Section 3 the option formulas are developed and Section 4 contains the complete expressions for perpetual debt with embedded options.

Section 5 compares the base case analytical solutions with a numerically solved binomial tree, Section 6 tests the assumptions regarding the time dependency in the bankruptcy barrier, Section 7 covers the numerical sensitivities, and Section 8 concludes the paper. Supporting technical derivations and results are enclosed in an appendix.

4.2 The model and basic results

We consider the standard Black-Scholes-Merton economy and impose the usual perfect market assumptions:

- All assets are infinitely separable and continuously tradeable.
- No taxes, transaction cost, bankruptcy costs, agency costs or short-sale restrictions.
- There exists a known constant riskless rate of return r.

4.2.1 The EBIT-based market value process

We study a limited liability company with financial assets and a capital structure consisting of two claims, infinite horizon continuously coupon paying debt and common equity. In line with Goldstein, Ju, and Leland (2001), we assume that the assets generate an EBIT (earnings before interest and tax) cashflow denoted δ_t given by the stochastic differential equation

$$d\delta_t = \mu\delta_t dt + \sigma\delta_t dW_t, \tag{4.2.1}$$

where μ and σ are constants representing the drift and volatility parameters respectively, and δ_0 is the fixed initial cashflow level. Here W_t is a standard Brownian motion under a fixed equivalent martingale measure. The total time t market value $\hat{A}_t$ of the assumed perpetual EBIT stream from the assets equals

$$\begin{aligned} \hat{A}_t &= E_t^Q\left[\int_t^\infty e^{-r(s-t)}\delta_s ds\right] \\ &= \frac{\delta_t}{r-\mu} \end{aligned} \tag{4.2.2}$$

The market value of this EBIT stream is the solution to the stochastic differential equation process

$$\begin{aligned} d\hat{A}_t &= (r\hat{A}_t - \delta_t)dt + \sigma\hat{A}_t dW_t \\ &= \mu\hat{A}_t dt + \sigma\hat{A}_t dW_t. \end{aligned} \tag{4.2.3}$$

The quantity $\hat{A}$ is elsewhere in the literature referred to as *the unlevered value of the firm's assets*.

In this setting there is a level of $\hat{A}_t$ where it is optimal for the company to stop paying debt coupons and declare bankruptcy. In the classic case this level is independent of time, i.e., constant.

4.2.2 The standard Black and Cox (1976) results

The time 0 market value of infinite horizon debt with continuous constant coupon payment is

$$D(A) = \frac{cD}{r} - (\frac{cD}{r} - \bar{A})(\frac{A}{\bar{A}})^{-\beta}, \tag{4.2.4}$$

where c is the constant coupon rate, D is the par value of the debt-claim and cD is the continuous coupon payment rate. The ratio $(\frac{A}{\bar{A}})^{-\beta}$ can be interpreted as the *current market value* of one monetary unit paid upon bankruptcy, i.e., when the process A_t hits the bankruptcy level $\bar{A}$. Here

$$\beta = \frac{\mu - \frac{1}{2}\sigma^2 + \sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2\sigma^2 r}}{\sigma^2}. \tag{4.2.5}$$

Expression (6.3.10) for the market value of debt carries a nice intuition. Observe that $\frac{cD}{r}$ is the current market value of infinite horizon default-free debt. Upon bankruptcy the debtholder looses infinite coupon payments which at the time of bankruptcy have market value $\frac{cD}{r}$. On the other hand the debtholder receives the remaining assets with a value equal to $\bar{A}$. We can therefore interpret $(\frac{cD}{r} - \bar{A})$ as the debtholder's *net loss* upon bankruptcy. The time 0 market value of this net loss, $(\frac{cD}{r} - \bar{A})(\frac{A}{\bar{A}})^{-\beta}$, therefore represents the reduction of the time 0 total market value of debt due to default risk. In our model, this is the only source of risk for the debt.

The value of equity as the residual claim on the assets is in this setting determined by

$$E(A) = A - D(A) = A - \frac{cD}{r} + (\frac{cD}{r} - \bar{A})(\frac{A}{\bar{A}})^{-\beta} \tag{4.2.6}$$

In the classic case of no embedded options, i.e., with a constant bankruptcy level, Black and Cox (1976) determine the *optimal* bankruptcy level for a given capital structure (E, D) from the perspective of the equityholders (found by differentiating expression (5.3.8) with respect to $\bar{A}$) as

$$\bar{A} = \frac{\beta}{\beta + 1} \frac{cD}{r}. \tag{4.2.7}$$

4.2.3 The modified process.

Our initial exercise is to price an embedded, finitely lived option on infinite horizon debt. Due to the finite horizon of this option, the optimal bankruptcy level of the issuer depends on remaining time to expiration. In order to capture this aspect we introduce a time-dependent bankruptcy asset level B_t,

$$B_t = Be^{\gamma t},$$

for a given time 0 level B and a constant γ. The time of bankruptcy is given by the stopping time τ defined as

$$\tau = \inf\{t \geq 0, \hat{A}_t = B_t\}$$

where $\hat{A}_t$ is given in expression (5.3.3).

By modifying the asset process this stopping time can equivalently be expressed as

$$\tau = \inf\{t \geq 0, A_t = B\},$$

where A_t is

$$dA_t = (\mu - \gamma)A_t dt + \sigma A_t dW_t, \tag{4.2.8}$$

Compared to equation (5.3.3), the modified process has a a negative drift adjustment of γ. Although γ determines the curvature of the bankruptcy level, it can formally be interpreted as a constant dividend yield on A_t. Again formally, this transformation allows us to analyze the simpler setting of a constant bankruptcy level B, although no economic fundamentals have been changed. In Section 5, we numerically compare our analytical approximation with the actual optimal barrier deriver numerically from a binomial tree.

4.3 Option formulas for finite options on infinite debt claims

We develop formulas for European options and barrier-options applying the standard approach from financial economics. We denote by T the exercise-date of these *European*-type options.

As shown below, the market value of the underlying asset, an infinite horizon debt contract, is not lognormally distributed. We solve this problem by reinterpreting the underlying asset.

We maintain the assumption of an exponential bankruptcy barrier before time T, $B_t = Be^{\gamma t}$. In section 2 we showed that interpreting γ as a dividend yield in the asset price process, see expression (4.2.8), allows us to work with a constant bankruptcy barrier B.

In this section we introduce a general bankruptcy level M after the expiration of the embedded option at time T. At this point we do not consider whether M is optimal or not, we only assume that it is above the time 0 bankruptcy level, $B \leq M$. We have argued in the introduction that this is a reasonable assumption.

We develop the option pricing formulas sequentially: We first derive formulas for *European* plain vanilla options on perpetual debt, disregarding any default- or credit-risk. Secondly, we follow Johnson and Stulz (1987) and Jarrow and Turnbull (1995), and acknowledge the bankruptcy risk of the issuer of the option and the underlying security at time T by introducing a certain minimum asset value, equal to the time T bankruptcy level. Finally, we also include bankruptcy risk before time T by requiring a minimum asset value B before time T, using a barrier option approach following Bjork (2004). In our subsequent application of these formulas we study embedded options on corporate debt where the counterparty- and credit-risk in the underlying security are inseparable.

4.3.1 The generalized debt dynamics

We assume that the underlying asset of our option formulas is given by

$$D_t = \frac{cD}{r} - JF_t, \tag{4.3.1}$$

where

$$F_t = (\frac{e^{\lambda t} A_t}{M})^{-\kappa}.$$

where A_t is given in expression (4.2.8). Before we explain this expression, observe that the parameter choices $J = \frac{cD}{r} - \bar{A}$, $\lambda = \gamma$, $M = \bar{A}$, and $\kappa = \beta$, yield the time t value of standard Black and Cox (1976) debt as in expression (6.3.10).

This comparison with standard Black and Cox debt explains why we need the parameter λ. The time T payoff of any option depends on the *actual* market value of debt, a function of $\hat{A}_T$ in expression (4.2.3). To incorporate the time dependent bankruptcy level, as explained above, we work with a modified asset value process $\{A_t, t \geq 0\}$ in expression (4.2.8). Observe that $\hat{A}_T = e^{\gamma T} A_T$, thus the parameter λ allows us to express the time T *actual* option payoff in terms of the *modified* asset value process.

The constant M represents a fixed (but not necessarily optimal) longterm bankruptcy level. Also F_t is interpreted as the time t market value of one unit of account payable the first time the process $e^{\lambda t} A_t$ hits the level M from above. The constant J represents the debtholders' net loss in case of bankruptcy and depends typically on other parameters of the model, see the example above. Our option pricing formulas may readily be used for other debt contracts with different net losses (e.g., as a result of different seniority) by applying alternative specifications of J. Finally, κ is the positive solution to the quadratic equation

$$\frac{1}{2}\sigma^2\kappa(\kappa+1) - \kappa(\mu - \gamma + \lambda) - r = 0 \tag{4.3.2}$$

given by

$$\kappa = \frac{\mu - \gamma + \lambda - \frac{1}{2}\sigma^2 + \sqrt{(\mu - \gamma + \lambda - \frac{1}{2}\sigma^2)^2 + 2\sigma^2 r}}{\sigma^2}. \tag{4.3.3}$$

We study finitely lived options embedded in perpetual debt contracts. The general time T payoff of such a *call* option with exercise price K is

$$(\frac{cD}{r} - JF_T - K)^{+} 1\{e^{\lambda T} A_T > M\} 1\{\tau > T\}).$$

The first factor represents a plain vanilla payoff $(D_T - K)^+$ disregarding any default risk. The first indicator function cancels the time T payoff when $e^{\lambda T} A_T$ is less than the time T bankruptcy level M. The second indicator function cancels the time T payoff upon earlier default, i.e., if $\inf_{0,T} A_t \leq B$.

4.3.2 Properties of F_t and D_t

An application of Itô's lemma on F_t using expressions (4.2.8) and (4.3.2) yields

$$dF_t = rF_t dt - \kappa\sigma F_t dW_t, \tag{4.3.4}$$

which we recognize as a geometric Brownian motion. It has drift parameter r and volatility parameter $-\kappa\sigma$. Furthermore, F_t is a function of A_t, and can therefore also be interpreted as a tradable asset. Observe that the parameters γ and λ determine the difference between the 'true' $\hat{A}_t$ and the modified process A_t. These parameters enter the F_t process only through the volatility parameter κ. Furthermore, note that κ depends only on the difference between these two parameters, see expression (4.3.3).

Applying Itô's lemma on expression (4.3.1) shows that

$$\frac{dD_t}{D_t} = (r - \frac{cD}{D_t})dt + \sigma\kappa(\frac{cD}{rD_t} - 1)dW_t \tag{4.3.5}$$

which is not a geometric Brownian motion (the right-hand side depends on D_t), and is thus not lognormally distributed. Options on D_t can therefore not be valued directly using standard option pricing formulas.

The use of F_T as underlying asset allows us to use the standard option approach on debt-options and is, as such, fundamental to our results.

4.3.3 European call and put options

First we consider the 'plain vanilla' version of standard European put and call options. These call and put options have time T payoffs

$$(D_T - K)^+ = (\frac{cD}{r} - JF_T - K)^+ = J(X - F_T)^+,$$

and

$$(K - D_T)^+ = (K - \frac{cD}{r} + JF_T)^+ = J(F_T - X)^+$$

respectively, where the *modified exercise price* is

$$X = \frac{\frac{cD}{r} - K}{J}.$$

We have shown[8] that one call option on debt with exercise price K is equivalent to J put options on F_T with a modified exercise price X. Similarly, one put option on debt with exercise price K is equivalent to J call options on F_T with a modified exercise price X.

From the above expression for the payoffs of the plain vanilla call and put options, we see that the following value of A_T

$$\dot{A} = \theta e^{-\lambda T} M \tag{4.3.6}$$

where

$$\theta = \left(\frac{J}{\frac{cD}{r} - K} \right)^{\frac{1}{\kappa}}$$

produces payoffs of zero for both the plain vanilla put and call options. Note that K represents the exercise price relative to D_T, and $\dot{A}$ similarly can be interpreted as the exercise price relative to A_T, see Figure (4.3). The factor $e^{-\lambda T}$ scales M down to the adjusted A_t-process.

In general our valuation formulas depend on

- four asset process parameters $(\mu, \gamma, \sigma, \delta_0)$,
- five debt parameters (c, D, J, M, λ),
- three option parameters (K, T, B),

in addition to r, in total 13 parameters.

For notational simplicity we write the expressions as functions of A and K only.

Proposition 1. *The time zero market prices of European plain vanilla put and call options on infinite horizon continuous coupon paying debt claims as described above are*

$$\begin{aligned} P_0^D(A, K) &= J C_0^F(F_0, X) \\ &= J(\frac{A}{M})^{-\kappa} N(d_1) - (\frac{cD}{r} - K) e^{-rT} N(d_2) \end{aligned} \tag{4.3.7}$$

[8] A similar call/put-relationship was also pointed to by Sarkar (2001)(page 510).

and

$$C_0^D(A,K) = JP_0^F(F_0,X) = (\frac{cD}{r} - K)e^{-rT}N(-d_2) - J(\frac{A}{M})^{-\kappa}N(-d_1), \tag{4.3.8}$$

where

$$d_1 = \frac{\ln(\frac{M}{A}) - \frac{1}{\kappa}(\ln(\frac{cD}{r} - K) - \ln J) - (\mu - \gamma + \lambda - \frac{1}{2}\sigma^2 - \sigma^2\kappa)T}{\sigma\sqrt{T}},$$

$$d_2 = d_1 - \sigma\kappa\sqrt{T}$$

and $A = \frac{\delta_0}{r-\mu}$.

Proof. We have shown how one call [put] option on D_T equivalently can be seen as J put [call] options on F_T with a modified exercise price. Under no-arbitrage assumptions these options must have (pairwise) the same market value at any point in time before expiration. Options on F_T can immediately be calculated by the Black-Scholes-Merton formulas, by using $F_0 = (\frac{A}{M})^{-\kappa}$ as the time 0 market value of the underlying asset, $|-\kappa\sigma| = \kappa\sigma$ as the volatility parameter[9] and X as the exercise price. □

We remarked earlier that κ depends on the difference between γ and λ. The above option formulas also depend on this difference through d_1 (in addition to through κ).

Compared to the payoffs from regular options, the payoffs at maturity T from options and barrier-options on perpetual debt are non-linear, not piecewise linear, functions of A_T. The payoffs at maturity T for plain vanilla options are illustrated in Figure 4.3.

These option pricing formulas do not take into account that the issuer of the underlying security may be bankrupt at time T, i.e., if A_T is below $e^{-\lambda T}M$ or A_t has hit B before time T. The formulas are still useful building-blocks in the following formulas which include both mentioned types of default risk.

[9] Option prices on assets with negative volatility, as F_t, are, in this setting, calculated by inserting the absolute value of the volatility parameter into the option pricing formula, see e.g., Aase (2004).

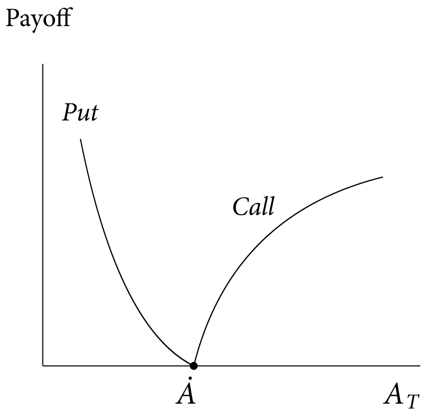


Figure 4.3: The payoff at maturity for plain vanilla put and call options on infinite coupon paying debt as functions of the time T market value of the firm A_T.

4.3.4 European put and call options with time T default risk

Denote the time T cash flow of a European *call* option on D_T with exercise price K and expiration at time T by $C_T^D(A_T, K)$. The option only has positive payoff if the issuer of the underlying security is *not* bankrupt at time T, i.e., if $A_T > e^{-\lambda T}M$. Similarly to the plain vanilla case, the time T call option cashflow is

$$\begin{aligned} C_T^D(A_T, K) &= (D_T - K)^+ 1\{A_T > e^{-\lambda T}M\} \\ &= J(X - F_T)^+ 1\{A_T > e^{-\lambda T}M\} = JP_T^F(F_T, X). \end{aligned} \tag{4.3.9}$$

The time T cash flow of a European *put* option on D_T with exercise price K and expiration at time T is

$$\begin{aligned} P_T^D(A_T, K) &= (K - D_T)^+ 1\{A_T > e^{-\lambda T}M\} = \\ &= J(F_T - X)^+ 1\{A_T > e^{-\lambda T}M\} = JC_T^F(F_T, X). \end{aligned} \tag{4.3.10}$$

To develop option pricing formulas which reflect that the issuing company may be bankrupt at time T, it is useful to distinguish between the cases where $\theta > 1$ and $\theta < 1$, ref. equation (4.3.6).

Proposition 2. *In the case when $\theta > 1$ the time zero market prices of European put and call options on infinite horizon continuous coupon paying debt claims, with positive payoff only when $A_T > e^{-\lambda T}M$, are*

$$P_0(A,K)^\theta = P_0^D(A,K) - \Big(J(\frac{A}{M})^{-\kappa}N(f_1) - (\frac{cD}{r} - K)e^{-rT}N(f_2)\Big), \qquad (4.3.11)$$

where

$$f_1 = \frac{\ln(\frac{M}{A}) - (\mu - \gamma + \lambda - \frac{1}{2}\sigma^2 - \sigma^2\kappa)T}{\sigma\sqrt{T}},$$

$$f_2 = f_1 - \sigma\kappa\sqrt{T},$$

and $P_0^D(A,K)$ *is given in expression* (4.3.7), *and*

$$C_0(A,K)^\theta = C_0^D(A,K) = (\frac{cD}{r} - K)e^{-rT}N(-d_2) - J(\frac{A}{\bar{A}})^{-\beta}N(-d_1), \qquad (4.3.12)$$

where $C_0^D(A,K)$ *is given in expression* (4.3.8).

Proof. In the case of the put option we must calculate the time 0 market value of the 'chopped' claim with the pay-off

$$(K - D_T)^+ 1\{A_T > e^{-\lambda T}M\}.$$

First observe that

$$(K - D_T)^+ 1\{A_T > e^{-\lambda T}M\} =$$
$$(K - D_T)^+ - (K_1 - D_T)^+ - (K - K_1)1\{A_T \le e^{-\lambda T}M\},$$

i.e., as a difference between two plain vanilla put options from which a constant is subtracted for values of A_T less than $e^{-\lambda T}M$. See Figure (4.4). Here K_1 is a modified exercise price calculated as follows: The second put option must have zero payoff for values of $A_T > e^{-\lambda T}M$, and we therefore choose the exercise price, denoted by K_1, so that $\dot{A} = e^{-\lambda T}M$. From expression (4.3.6) this is

$$K_1 = \frac{cD}{r} - J.$$

The constant $K - K_1$ represents the net difference in the payoff of a long position in the first and a short position in the second option for values of A_T less than $e^{-\lambda T}M$. The above identity is then verified.

The market value of the above claim is easily calculated and the result given by the formula $P_0(A,K)^\theta$ above.

The call formula has a strictly positive payoff only for values of $A_T > \dot{A}$. In this case $\theta > 1$, so $\dot{A} > e^{-\lambda T}M$, thus the inclusion of time T default risk has no effect on the payoff, see Figure (4.5). □

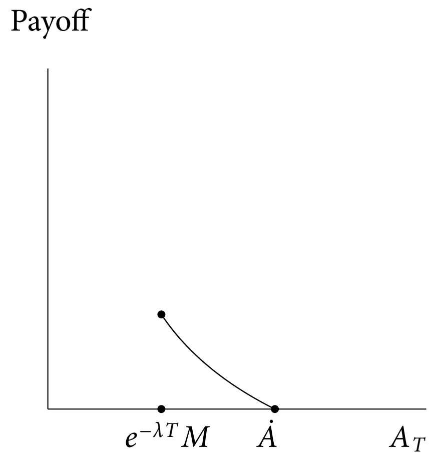


Figure 4.4: The payoff at maturity for a vulnerable *put* option on infinite coupon paying debt when $e^{-\lambda T}M < \dot{A}$ ($\theta > 1$), as a function of the firm value A_T.

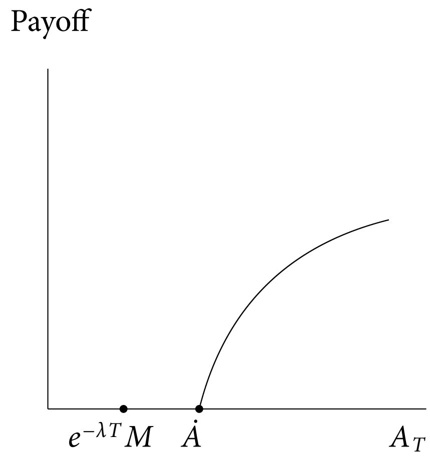


Figure 4.5: The payoff at maturity for a vulnerable *call* option on infinite coupon paying debt when $e^{-\lambda T}M < \dot{A}$ ($\theta > 1$), as a function of the firm value A_T.

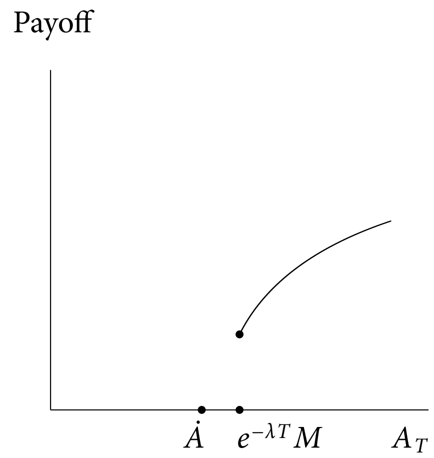


Figure 4.6: The payoff at maturity for a vulnerable *call* option on infinite coupon paying debt when $e^{-\lambda T}M > \dot{A}$ ($\theta < 1$), as a function of the firm value A_T.

Proposition 3. *In the case when $\theta < 1$ the time zero market prices of European put and call options with positive payoff only when $A_T > Me^{-\lambda T}$ on infinite horizon continuous coupon paying debt claims are*

$$P_0(A, K)_\theta = 0, \tag{4.3.13}$$

$$C_0(A, K)_\theta = (\frac{cD}{r} - K)e^{-rT}N(-f_2) - J(\frac{A}{M})^{-\kappa}N(-f_1). \tag{4.3.14}$$

Proof. In the case when $\theta < 1$, $\dot{A} < e^{-\lambda T}M$, so the chopped put option does not have positive payoff for any values of A_T.

The time T payoff of the chopped call option is

$$(D_T - K)^+ 1\{A_T > e^{-\lambda T}M\}.$$

This can be written as

$$(D_T - K_1)^+ + (K_1 - K)1\{A_T > e^{-\lambda T}M\},$$

where K_1 is given in the proof of Proposition 2. See also Figure (4.6).

The market value of the above claim is easily calculated and is given by the formula $C_0(A, K)_\theta$ above. □

4.3.5 Down-and-out barrier put and call options

The previous section includes the possibility of default at time T. In this section we also include the possibility of an earlier default. We assume that the issuing company defaults if the market value process A_t drops below the constant B before time T.

We treat the following cases separately. Recall that $B < e^{-\lambda T}M$ by our previous assumption.

- Case 1: $\theta > 1$, $B < e^{-\lambda T}M < \dot{A}$.
- Case 2: $\theta < 1$, $B < \dot{A} < e^{-\lambda T}M$ or $\dot{A} < B < e^{-\lambda T}M$.

The time T cashflows of down-and-out barrier put and call options on infinite debt-claims with barrier B for the asset-process A_t and exercise price K are

$$P_T^{do}(A_T, K) = (K - \frac{cD}{r} + J(\frac{e^{\lambda T}A_T}{M})^{-\kappa})^{+}1\{m_T^A > B\}1\{A_T > e^{-\lambda T}M\}, \quad (4.3.15)$$

and

$$C_T^{do}(A_T, K) = (\frac{cD}{r} - J(\frac{e^{\lambda T}A_T}{M})^{-\kappa} - K)^{+}1\{m_T^A > B\}1\{A_T > e^{-\lambda T}M\}, \quad (4.3.16)$$

where $1\{\cdot\}$ represents the usual indicator function and the minimum function $m_T^A = \min\{A_t; 0 \leq t \leq T\}$.

The payoff at maturity from barrier options is not only dependent on the asset level A_T as plain vanilla options, but also on the two relevant bankruptcy barriers, B for $t < T$ and $e^{-\lambda T}M$ for $t = T$.

4.3.6 Case 1: Down-and-out barrier options when $\theta > 1$.

Proposition 4. *The time zero market values of the down-and-out barrier put and call options on infinite horizon continuous coupon paying debt claims, with $B < e^{-\lambda T}M < \dot{A}(\theta > 1)$ and exercise price K are, respectively*

$$P_1^{do}(A, K) = P_0(A, K)^{\theta} - (\frac{B}{A})^{(\frac{2(\mu-\gamma)}{\sigma^2}-1)}P_0(\frac{B^2}{A}, K)^{\theta}, \quad (4.3.17)$$

and

$$C_1^{do}(A, K) = C_0^D(A, K) - (\frac{B}{A})^{(\frac{2(\mu-\gamma)}{\sigma^2}-1)}C_0^D(\frac{B^2}{A}, K). \quad (4.3.18)$$

Proof. The results follow immediately from Theorem 18.8 in Bjork (2004). □

4.3.7 Case 2: Down-and-out barrier call option when $\theta < 1$

Proposition 5. *The time zero market values of down-and-out barrier call options on infinite horizon continuous coupon paying debt claims, with $B < \dot{A} < e^{-\lambda T} M$ and exercise price K is*

$$C_2^{do}(A,K) = C_0(A,K)_\theta - \left(\frac{B}{A}\right)^{\left(\frac{2(\mu-\gamma)}{\sigma^2}-1\right)} C_0\left(\frac{B^2}{A},K\right)_\theta. \tag{4.3.19}$$

Proof. The formula follows immediately from Theorem 18.8 in Bjork (2004). □

The down-and-out barrier put option in this case has a market value identical to zero also when including bankruptcy risk before time T.

The option formulas in the this section are also applicable in situations where third parties trade options on corporate perpetual debt. In such situations the existence of an option contract will neither influence the pricing of the debt nor the issuing company's own optimal choice of bankruptcy level. The option pricing formulas above can thus be applied by third parties using $B = e^{-\lambda T} M = e^{-\lambda T} \bar{A}$ and $\gamma = \lambda = 0$. Recall that $\bar{A}$ represents the constant optimal bankruptcy level in the case of infinite horizon debt claims with no embedded call option.

4.4 Value of perpetual debt including embedded option

In this section we analyze the case with an issuer's European call option as an integrated part of the debt contract, i.e., the option is written by the debtholders in favor of the equityholders. Thus, the existence of the option will influence both the issue-at-par coupon and the issuer's bankruptcy level before the option's expiration date. Intuition suggests that the coupon is increased to compensate for the added option, whereas the optimal bankruptcy level is decreased - both compared to the case without an option. As long as the option has a positive market value equityholders will rationally keep the company going for lower asset levels than without an option.

We analyze a company with a simple capital structure, equity and one class of Black and Cox (1976)-debt with net loss J in case of bankruptcy.

In this section we assume that no further options are present after time T. We are at that time then back to the classic Black and Cox (1976) setting and the bankruptcy level from time T onwards is given by $\bar{A}$ in expression (4.2.7).

Let D_T^c denote the time T payoff of perpetual debt including an embedded option to repay debt at par value D, given no prior bankruptcy. Also denote the time zero value of cashflows before time T, i.e., coupon and potential bankruptcy payments, by $L_0(A)$. The time zero value of debt including the embedded option $D_0^c(A)$ equals the time zero value of the time T cashflow D_T^c plus $L_0(A)$, i.e.,

$$D_0^c(A) = V_0(D_T^c) + L_0(A), \tag{4.4.1}$$

where $V_0(\cdot)$ represents the market value operator.

In this section we make no assumptions regarding if, or how, the hybrid capital is refinanced at time T if the embedded option is exercised. We assume that $M = \bar{A}$, the constant, optimal, longterm bankruptcy level.

4.4.1 The time T payoff of debt with embedded option

We assume that $D > \bar{A}$, i.e., that the debt is risky in case of liquidation. The time T payoff of perpetual debt including an embedded option is

$$D_T^c = \begin{cases} 0 & \text{for } \tau < T, \\ e^{\lambda T} A_T & \text{for } \tau > T \text{ and } A_T < e^{-\lambda T}\bar{A}, \\ D_T & \text{for } \tau > T \text{ and } e^{-\lambda T}\bar{A} < A_T < \dot{A}, \\ D & \text{for } \tau > T \text{ and } A_T > \dot{A}, \end{cases}$$

where D_T is given by expression (4.3.1) and τ is the time of bankruptcy as defined in Section 2. The time T payoff D_T^c is depicted in Figure (4.1). This expression can be rewritten under the process A_t as

$$D_T^c = \begin{cases} 0 & \text{for } \tau < T, \\ e^{\lambda T} A_T & \text{for } \tau > T \text{ and } A_T < e^{-\lambda T}\bar{A}, \\ D_T - \max(D_T - D, 0) & \text{for } \tau > T \text{ and } A_T > e^{-\lambda T}\bar{A}, \end{cases} \tag{4.4.2}$$

This shows how D_T^c equals D_T minus the payoff from a *call*-option on the debt with exercise price *par*, in the case where $\tau > T$ and $A_T > e^{-\lambda T}\bar{A}$.

4.4.2 Calculation of $V_0(D_T^c)$

From expression (4.4.2) and standard financial pricing theory, see, e.g., Duffie (2001), the time 0 market value $V_0(D_T^c)$ can be written as

$$\begin{aligned} V_0(D_T^c) = & E^Q[A_T e^{(\gamma-r)T} 1\{\tau > T\} 1\{A_T < e^{-\lambda T}\bar{A}\}] \\ & + E^Q[(D_T - 0)^+ e^{-rT} 1\{\tau > T\} 1\{A_T > e^{-\lambda T}\bar{A}\}] \\ & - E^Q[e^{-rT}(D_T - D)^+ 1\{\tau > T\} 1\{A_T > e^{-\lambda T}\bar{A}\}]. \end{aligned} \tag{4.4.3}$$

Below we calculate the three terms on the right hand side separately. We denote the first term by V_k and calculate

$$\begin{aligned} V_k(A) &= E^Q[e^{(\gamma-r)T} A_T 1\{\tau > T\} 1\{A_T < e^{-\lambda T}\bar{A}\}] \\ &= A e^{(\mu-r)T} \bar{Q}(1\{\tau > T\} 1\{A_T < e^{-\lambda T}\bar{A}\}) \\ &= A e^{(\mu-r)T} \left(N(g_1) - N(g_2) + (\frac{A}{B})^{-\frac{2(\mu-\gamma)}{\sigma^2}-1} [N(-g_3) - N(-g_4)], \right) \end{aligned}$$

where

$$g_1 = \frac{\ln(\frac{A}{B}) + (\mu - \gamma + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}},$$

$$g_2 = \frac{\ln(\frac{A}{\max(\bar{A},B)}) + (\mu - \gamma + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}},$$

$$g_3 = \frac{\ln(\frac{A}{B}) + \ln(\frac{\max(\bar{A},B)}{B}) - (\mu - \gamma + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}},$$

$$g_4 = \frac{\ln(\frac{A}{B}) - (\mu - \gamma + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}.$$

We recognize the second and third terms of expression (4.4.3) as down-and-out barrier call options with common parameters $\lambda = \gamma$, $M = \bar{A}$ and $J = \frac{cD}{r} - \bar{A}$.

The second term is the time zero market value of a down-and-out barrier call option as analyzed in Section 3 with $K = 0$ and, consequently, $\theta < 1$. The time zero market value of this option is calculated in Proposition 5 as $C_2^{do}(A, 0)$, equation (4.3.19).

Finally, the third term is the time zero market value of a down-and-out barrier call option with $K = D$ and, consequently, $\theta > 1$. The time zero market value of this option is calculated in Proposition 4 as $C_1^{do}(A, D)$, equation (4.3.18).

To summarize our calculations so far, we have that

$$V_0(D_T^c) = V_k(A) + C_2^{do}(A, 0) - C_1^{do}(A, D).$$

4.4.3 Calculation of $L_0(A)$

We now turn to the calculations of the time zero value of cashflows before time T,

$$\begin{aligned}
L_0(A) &= E^Q\Big[\int_0^{\tau\wedge T} cDe^{-rs}ds\Big] + E^Q[Be^{-r\tau}1_{\{\tau\le T\}}], \qquad (4.4.4)\\
&= \frac{cD}{r} - \frac{cD}{r}e^{-rT}Q(\tau > T) - (\frac{cD}{r} - B)E^Q[e^{-r\tau}1\{\tau \le T\}],\\
&= \frac{cD}{r} - \frac{cD}{r}e^{-rT}Q(\tau > T) - (\frac{cD}{r} - B)E^Q[e^{-r\tau}(1 - 1\{\tau > T\})],\\
&= \frac{cD}{r} - (\frac{cD}{r} - B)(\frac{A}{B})^{-\kappa} - \frac{cD}{r}e^{-rT}Q(\tau > T) + (\frac{cD}{r} - B)E^Q[e^{-r\tau}1\{\tau > T\})],\\
&= \frac{cD}{r} - (\frac{cD}{r} - B)(\frac{A}{B})^{-\kappa} - \frac{cD}{r}e^{-rT}Q(\tau > T) + (\frac{cD}{r} - B)E^Q[e^{-rT}(\frac{A_T}{B})^{-\kappa}1\{\tau > T\})],\\
&= \frac{cD}{r} - (\frac{cD}{r} - B)(\frac{A}{B})^{-\kappa} - E^Q[e^{-rT}(\frac{cD}{r} - (\frac{cD}{r} - B)(\frac{A_T}{B})^{-\kappa})1\{\tau > T\})],\\
&= \frac{cD}{r} - (\frac{cD}{r} - B)(\frac{A}{B})^{-\kappa} - C_2^{do}(A, 0).
\end{aligned}$$

This represents the time 0 market value of coupon payments and any compensation in case of bankruptcy before time T. $Q(\tau > T)$ is a standard result and included in Mjøs and Persson (2007). The bankruptcy compensation to debtholders during this period is the time-dependent bankruptcy level(barrier) $B_t = Be^{\gamma t}$. The down-and-out barrier call option for $\theta < 1$ is given in equation (4.3.19), using $M = B$, $\lambda = 0$ and $J = \frac{cD}{r} - B$.

4.4.4 The time 0 market value of perpetual debt with embedded option

Proposition 6. *The time 0 value of infinite horizon continuous coupon-paying debt claims including an embedded option to repay debt at par value D at time T is*

$$D_0^c(A) = V_k(A) + C_2^{do}(A, 0) \qquad (4.4.5)$$

$$-C_1^{do}(A,D)+\frac{cD}{r}-(\frac{cD}{r}-B)(\frac{A}{B})^{-\kappa}-C_2^{do}(A,0).$$

Our expression for $D_0^c(A)$ can be interpreted as follows: The first term represents the time 0 value of bankruptcy payoff if $\tau \geq T$ i.e. the firm has survived until time T, but $e^{\lambda T}A_T < \bar{A}$ and bankruptcy occurs at time T[10]. The second term represents the time 0 market value of a call option on debt at time T with exercise price 0. The third term represents the short, embedded, call option on debt exercisable at time T with a strike equal to par value D. This possibility to refinance in case of improved available terms at time T is exactly the purpose of the embedded option included in the time 0 debt contract. The last two terms represent the time 0 market value of all cashflows before time T, modelled as the difference between immediately starting perpetual debt and a forward starting perpetual debt expressed as a barrier call option with exercise price 0 at time T. This combined expression allows for calibrating both the "issue-at-par" coupon rate reflecting the embedded option and a time-dependent endogenously calibrated issuer bankruptcy level before the option expires.

4.5 Base case parameter calibrations

In this section we calculate calibrated coupon rates and bankruptcy levels for realistic parameter values. In particular, we test whether our closed form produces correct coupon rates compared to more exact numerical solutions. The difference between the coupon rates is a benchmark for the precision of our closed form approach.

In order to calibrate the bankruptcy barrier parameters B and γ, we implement a binomial tree following the binomial lattice methodology from Broadie and Kaya (2007). They assume that debt coupons are paid from a firm's cashflow and that any shortfalls are covered by new equity either from existing or new shareholders as long as the market value of equity is positive. This approach also provides an exact coupon-rate, denoted by c_n, which is independent of any analytically assumed shape of the bankruptcy barrier.

We generally assume that the bankruptcy level is continuous at time T due to the correlation between asset levels and option payoff. The payoff-profile in

[10] If the bankruptcy barrier is assumed to be continuous at time T, this terms equals 0.

Figure 4.1 shows that the levels of A_T for which the option is in-the-money are well above any optimal bankruptcy levels and will therefore have no impact on optimal bankruptcy. Equivalently, distressed firms with asset values approaching the bankruptcy level will not expect any option payoffs.

The derived value of B, the initial bankruptcy level, is then used in the closed form solution to calculate the coupon-rate, here denoted by c_c. We do this by adjusting the coupon-rate c_c in equation (4.4.5) to achieve $D_0^c(A) = D$, i.e., that the market value of debt with embedded option is par.

In our binomial approach we apply the base case parameters in Table 4.2 and run 100,000 steps per year for 10 years. The chosen level of asset volatility is taken from Leland (1994), whereas the level of riskfree interest rate is common in similar illustrations. The base case time to expiration of the option resembles the actual option maturities in most publicly listed perpetual bonds issued by financial institutions.

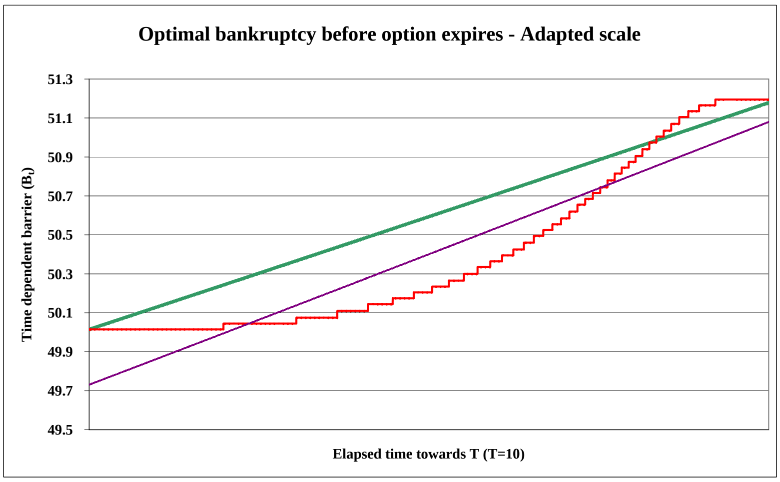
Table 4.2: Base case parameters, all rates are annualized.

δ_0	3	Initial EBIT
μ	2 %	Drift of EBIT
D	70	Face value of debt
T	10	Expiration date of option
σ	0.2	Volatility of EBIT
r	5 %	Riskfree interest rate
A_0	100	Total asset value at time 0

Consistent with our assumed analytical form of the bankruptcy barrier $B_t = Be^{\gamma t}$ we calculate $\gamma = \frac{1}{T}\ln(\frac{B_T}{B})$, where B is calculated by the binomial approach and B_T is equal to the long-term bankruptcy level, $\bar{A}$. Observe that by this formulation γ only depends on the time 0 and time T values of the bankruptcy barrier and not on intermediary values.

We also test the functional form $B_t = Be^{\gamma t}$ by using Ordinary Least Squares (OLS) to estimate γ. The estimation is based on the complete sequence of numerically calculated values of B_t and regress $\ln(B_t)$ on $\ln(B) + \gamma t$.

Figure 4.7 shows the development of B_t as a function of elapsed time to expiration from the binomial approach, the analytical approximation and the OLS-



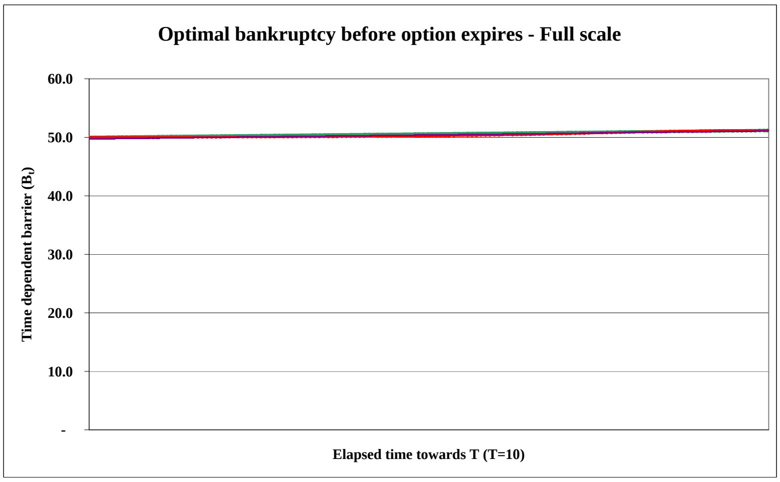


Figure 4.7: The numerically calculated optimal, and the analytically solved bankruptcy asset level B_t as a function of elapsed time until expiration of the call option embedded in perpetual debt. The lower line represents an OLS-regression of the numerically calculated B_t-values. See Table 4.2 for parameter values.

regression. The latter is shown by the lower line. We include two illustrations of the barriers to emphasize the limited impact of the time-dependencies compared to the long-term bankruptcy level. The appropriate choice of bankruptcy assumptions is discussed in the next section.

Table 4.3: Calibrated values of the coupon-rates and long-term bankruptcy levels using the base case parameters.

Alternative solutions:	B	γ	c_c/c_n	$\bar{A}$
Analytical				
- regular(B&C'76)	*Not applicable*		5.718 %	49.04
- with option, B from model	4.91	0.250123	6.840 %	58.68
- with option, B from tree	50.15	0.0023026	5.998 %	51.437
Binomial				
- with option	50.15	0.0023026	5.969 %	51.18

Table 4.3 compares the calibrated coupon-rates and bankruptcy barriers for alternative approaches. The simplification in our analytical model of assuming a fixed level of γ yields a very low starting-level B and high level of γ, compared to the binomial solution. We have left for future research to develop an analytical solution that better captures the time-variations of this barrier, and choose to apply the starting level of B from the tree as an input-variable in our calculations. Given this approach, our structural model generates results that are very close to the binomial solution and gives us confidence in analyzing sensitivities on this basis.

The results support our intuition that an embedded option increases the closed form coupon c_c (from 5.718 % to 5.998 %) even when in a model where the only source of risk is the loss in case of bankruptcy. This increase also changes the long-term bankruptcy-levels $\bar{A}$. As an overall assessment, we find that the coupon-rates c_c and c_n are reasonably close (the difference is less than 5 basispoints). The OLS-approach yields $\gamma = 0.00268$, $B = 49.73$, and $R^2 = 87.5\%$. The high value of R^2 supports the assumed linearity of $\ln(B_t)$. The estimated $\gamma's$ from the two approaches are close (The ratio is 0.86.), and as expected, the OLS-approach underestimates the starting point B.

Observe in Figure 4.7 that both the numerically calculated and the modelled bankruptcy levels are below the constant long-run level $\bar{A}$. Additional analysis shows that for large T, the effect of the option disappears and B approaches $\bar{A}$.

4.6 Relevance of the time-dependent bankruptcy barrier

An important assumption in our analysis is the exponential shape of the time-dependent bankruptcy barrier before the expiration of the embedded option. In this section, we compare the numerically calculated coupon-rate c_n to the closed-form coupon-rate c_c for variations in maturity and volatility to illustrate the effects of alternative assumptions regarding time-dependency. We compare our base case bankruptcy barrier to two alternatives profiles, a constant barrier and a constant barrier with two levels, illustrated in Figure 4.8. The numerical results produce the correct bankruptcy barrier before time T whilst our analytical model necessarily will represent an approximation.

We initially in Table 4.4 show the sensitivity of the time-dependency growth parameter γ for changes in volatility and option maturity. The values of γ are calculated from our binomial solutions as a benchmark for the choice of analytical assumptions. We find that γ is positively related to the value of our barrier-options, increasing in volatility σ and decreasing in option maturity T. As γ increases with volatility, time dependency becomes increasingly important in these situations.

Table 4.4: Numerical values of the time-dependency parameter γ using alternative parameters for maturity(T) and annual EBIT(δ)-volatility (σ). 10,000 steps per year.

γ - **sensitivities**	*Volatility*(σ)		
Maturity(T)	0.10	0.20	0.30
5 years	0.000134	0.004716	0.011182
10 years	0.000219	0.002303	0.003786
20 years	0.000080	0.000652	0.000912

In Table 4.5 we show the coupon-rate differences, $c_c - c_n$. The differences are reported in basispoints for maturities 5, 10 and 20 years, respectively, for different

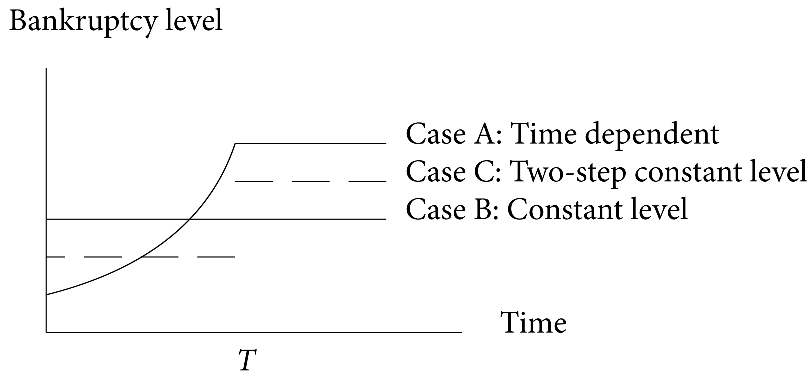


Figure 4.8: Alternative bankruptcy barrier profiles with different degrees of time dependency for $t < T$.

values of σ and T and by alternative bankruptcy barrier assumptions. This test illustrates the relevance and precision of our assumed time-dependent bankruptcy barrier by comparing the following alternatives:

- *Case A:* The base case time dependent approach with assumed exponential barrier, $B \neq \bar{A}$ and $\gamma \neq 0$. B is found by solving a binomial tree, as explained in Section 5.

- *Case B:* A constant barrier for all t, $B = \bar{A}$ and $\gamma = 0$. This alternative disregards any impact the finite option may have on the optimal bankruptcy level.

- *Case C:* A two-step constant barrier with different levels before and after time T, $B \neq \bar{A}$ and $\gamma = 0$. This approach recognizes that the bankruptcy level may be lower before time T because of the option, but disregards any additional time dependencies.

The table shows that the differences are relatively insensitive to the choice of barrier. When analyzing the longest maturities, 10 and 20 years, both alternatives with constant barriers (Cases B and C) produce comparably good results. Assuming a 5 year horizon, Case B with one, constant barrier performs best. We

Table 4.5: Differences in basispoints between the closed form calibrated coupon rates c_c and the numerical solution c_n using alternative parameters for annual EBIT(δ)-volatility (σ) and alternative approaches for bankruptcy barriers denoted Cases A, B and C and described in the text. c_n is calculated using 10,000 steps per year.

Δ Coupon-rates, $(c_c - c_n)$, bp			
Volatility (σ)	0.10	0.20	0.30
Maturity: 5 years			
Case A	-0.1	2.8	21.3
Case B	-0.1	-0.2	-0.4
Case C	N.A.	27.9	27.0
Maturity: 10 years			
Case A	-0.1	2.9	8.9
Case B	-0.1	-0.2	-0.9
Case C	N.A.	-0.2	-0.9
Maturity: 20 years			
Case A	0.1	1.0	2.7
Case B	0.0	-0.2	-0.3
Case C	0.0	-0.1	0.0

are not able to analytically estimate the coupon-rates for Case C with the lowest volatility and maturities below 20 years. The overall result is that in applications where one may test different combinations of σ and T, Case B with one, constant barrier is the preferable choice. This alternative also has the additional benefit of not requiring B as input-variable, as well as computational simplicity. Our base case (Case A), performs equally well except for the shortest maturity and highest volatility. For long maturity (20 years) all barrier alternatives produce reasonably correct coupon rates.

Our conclusion is somewhat counterintuitive, but is that the effects of time-dependency are negligible for realistic parameter values. Our model includes simplifying assumptions which limits the general validity of this conclusion. However, the fact that we assume 70 % debt financing to magnify the effects strengthen the conclusion.

In our discussions of sensitivities we apply the assumption of one, constant bankruptcy barrier, Case B.

4.7 Sensitivities and market reference

Table 4.6 shows the sensitivity of the analytically calculated coupon-rate c_c for alternative combinations of EBIT-volatility and time until the option expires. Here c_c is strongly increasing in volatility, in accordance with the classical result of increasing option values with volatility. Contrary to the standard effect of maturity on plain vanilla option values, the barrier option specifications in our setting decreases value by longer maturities reflecting the risk of bankruptcy. See Figure 4.2 in Section 1 for an illustration.

Table 4.6: Analytically calculated values of the coupon rate c_n for callable perpetual debt with embedded option expiring after T years using alternative parameters for maturity(T) and annual EBIT(δ)-volatility (σ). Assumed constant bankruptcy level.

Coupon-rates (%)	*Volatility*(σ)		
Maturity(T)	0.10	0.20	0.30
5 years	5.103	6.056	7.730
10 years	5.095	5.967	7.457
20 years	5.080	5.849	7.185

Table 4.7 shows the sensitivity of the long-term optimal bankruptcy level $\bar{A}$ following from the coupon-rates in Table 4.6. The main difference in the sensitivities is that $\bar{A}$ decreases both for increases in volatility and in option maturity. Increased volatility will, at least when viewed in isolation, always increase the value of equity and thus reduce optimal $\bar{A}$. Increased maturity reduces optimal coupon-rate as shown in Table 4.6 and $\bar{A}$ then follows by equation (4.2.7).

As a market reference, Figure 4.9 shows the yield spreads of Iboxx-indices[11] of UK Tier 1 perpetual debt securities (hybrid capital) including embedded option compared to UK banks' senior debt, both relative to UK government bonds(Gilts) reported weekly for the period December 2004 - September 2007. The hybrid

[11]Index yields are sourced from Iboxx via Datastream

Table 4.7: Analytical values of the longterm bankruptcy asset level $\bar{A}$ for callable perpetual debt with embedded option after T years using c_n and alternative parameters for maturity(T) and annual EBIT(δ)-volatility (σ). Assumed constant bankruptcy level

Bankruptcy level $\bar{A}$	*Volatility*(σ)		
Maturity(T)	0.10	0.20	0.30
5 years	59.53	51.92	48.51
10 years	59.44	51.17	46.80
20 years	59.27	50.16	45.09

capital yield-spread produced by our stylized model is of the same magnitude as the observed yield-spreads in the latter part of the period, except for the most recent months. The higher riskiness of the hybrid security is exemplified both by the standard deviation of this annualized spread being 26 basispoints compared to 12 basispoints for senior debt, and the relatively larger increases towards the end of the period. The latter relates to the credit-crisis in Summer 2007. Both reflect that Tier 1 securities are more risk-exposed than senior bonds, as expected.

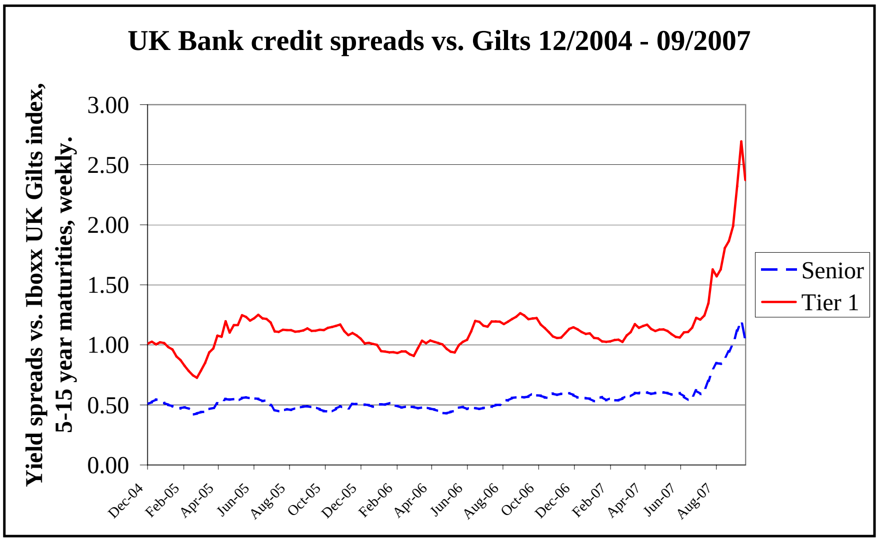


Figure 4.9: The graph shows the redemption yield spread of the Iboxx-indices for UK banks' senior debt and UK banks' Tier I capital after deducting the yield on the UK Gilts index for maturities 5-15 years. Weekly observations for the period December 2004 until September 2007. Source: Iboxx(via Datastream).

4.8 Concluding remarks and further research

We show how a European embedded option in perpetual debt impacts both the value of debt and the issuer's rational economic behavior with regards to bankruptcy. Specifically, the embedded option impacts the bankruptcy decision, level of debt coupons, and the optimal exercise of the option. We derive closed form solutions based on an approximation of the optimal bankruptcy level before the option expires. We show that for realistic parameter values an assumed constant bankruptcy level produces the most robust correct coupon-rates for alternative volatilities and option maturities. Perhaps surprisingly, our model, with its heroic assumptions, produces coupon spreads that appear to be in a realistic order of magnitude compared to observed market spreads.

The equityholders pay for the embedded option through a higher fixed coupon on the perpetual debt, compared to regular perpetual debt. The equityholders choice of optimal bankruptcy-level is impacted by the debt with embedded option in two ways; an increased coupon and the existence of a potentially valuable option. The increased coupon raises the optimal long-term bankruptcy-level, whilst the embedded option lowers it.

The market values of perpetual debt with and without option are different after expiration in the situation when the option has not been exercised. A higher coupon in the first case reflects the historical cost of the expired option and is a major motivation for the exercise of such options. This higher coupon rate causes exercises also in significantly worse future states compared to the situation at time of issue. It is common in the marketplace to contractually agree that coupons are even 'stepped-up' post-expiry of the option to further incentivice exercise.

For our analytical purpose, we have developed some European option and barrier option pricing formulas on perpetual debt. The barrier option formulas are applied to reflect default risk before the option expires. These formulas are quite general and may be used for valuing both embedded and third-party options.

Our model can be extended along a number of dimensions such as introducing frictions (taxes, bankruptcy costs), different liquidation priorities (hybrid/preferred stock), and American type options.

4.9 References

Aase, Knut K., 2004, Negative volatility and the survival of the western financial markets, *WILMOTT magazine* 64-67 Reprinted in Haug, E.: Derivatives: Models on Models, pp. 238-244, John Wiley & Sons, Ltd. 2007.

Acharya, Viral A., and Jennifer N. Carpenter, 2002, Corporate bond valuation and hedging with stochastic interest rates and endogeneous bankruptcy, *Review of Financial Studies* 15, 1355–1383.

Bank, Matthias, 2004, On the pricing of options on corporate debt, the term structure of default spreads, and the pricing of risky forward loans, Working paper University of Innsbruck, Department of Finance.

Bjork, Tomas, 2004, *Arbitrage Theory in Continous Time. Second edition* (Oxford University Press: London).

Black, Fisher, and J. Cox, 1976, Valuing corporate securities: Some effects of bond indenture provisions, *Journal of Finance* 31, 351–367.

Black, Fisher, and Myron Scholes, 1973, The pricing of options and corporate liabilities, *Journal of Political Economy* 81, 637–654.

Brennan, Michael J., and Eduard S. Schwartz, 1977, Convertible bonds: Valuation and optimal strategies for call and conversion, *Journal of Finance* 32, 1699–1715.

Broadie, Mark, and Özgür Kaya, 2007, A binomial lattice method for pricing corporate debt and modeling chapter 11 proceedings, *Journal of Financial and Quantitative Analysis* Forthcoming.

Committee on Banking Supervision, Basle, 1988, International convergence of capital measurement and capital standards, Report Bank for International Settlements (BIS).

Duffie, David, 2001, *Dynamic Asset Pricing Theory. Third Edition* (Princeton University Press: Princeton, NJ, USA).

Emanuel, David, 1983, A theoretical model for valuing preferred stock, *Journal of Finance* 38, 1133–1155.

Goldstein, Robert, Nengjiu Ju, and Hayne Leland, 2001, An ebit-based model of dynamic capital structure, *The Journal of Business* 74, 483–512.

Hull, J., and A. White, 1995, *Derivative Credit Risk - Advances in Measurement and Management* (Risk: London) Introduction, Ch. 5.

Ineke, Jackie, Olivia Guillard, and Carlo Mareels, 2003, Bank capital a-z, Fixed income research Morgan Stanley.

Ingersoll, Jonathan E., 1977, A contingent-claims valuation of convertible securities, *Journal of Financial Economics* 4, 289–321.

Jarrow, Robert A., and Stuart M Turnbull, 1995, Pricing derivatives on financial securities subject to credit risk, *Journal of Finance* 50, 53–85.

Johnson, Herb, and René Stulz, 1987, The pricing of options with default risk, *Journal of Finance* 42, 267–280.

King, Tao-Hsien Dolly, 2007, Are embedded calls valuable? evidence from agency bonds., *Journal of Banking and Finance* 31, 57–79.

Kish, Richard J., and Miles Livingston, 1992, Determinants of the call option on corporate bonds, *Journal of Banking and Finance* 16, 687–703.

Lando, David, 2004, *Credit Risk Modeling: Theory and Applications* (Princeton University Press: Princeton, NJ, USA).

Leland, Hayne, 1994, Corporate debt value, bond covenants, and optimal capital structure, *Journal of Finance* 49, 1213–1252.

Mapondera, Leslie, and Alexandra Bossert, 2005, European banks: 2006 new issue forecasts, Financial institutions Credit Suisse First Boston.

Mauer, David C., 1993, Optimal bond call policies under transaction costs, *Journal of Financial Research* XVI, 23–37.

Merton, Robert C., 1973, Theory of rational option pricing, *Bell Journal of Economics and Management Science* 4, 141–183.

Mjøs, Aksel, and Svein-Arne Persson, 2007, Default of multiple degrees: The risk of lost debt coupons, Unpublished working paper Norwegian School of Economics and Business Administration.

Pomper, Marc, and Shantanu Varma, 2005, Hybrid capital advisor, Fixed income u.s. credit strategy Lehman Brothers.

Rubinstein, Mark, 1983, Displaced diffusion option pricing., *Journal of Finance* 38, 213–217.

Sarkar, Sudipto, 2001, Probability of call and likelihood of the call feature in a corporate bond, *Journal of Banking and Finance* 25, 505–533.

———, and Gwangheon Hong, 2004, Effective duration of callable corporate bonds: Theory and evidence, *Journal of Banking and Finance* 28, 499–521.

Toft, Klaus Bjerre, and Brian Prucyk, 1997, Options on leveraged equity: Theory and empirical tests., *Journal of Finance* 52, 1151–1180.

4.A Present value of 1 payable at first hitting time before a finite horizon.

In this appendix we collect some technical results. Consider the Itô process

$$X_t^z = zt + W_t, \tag{4.A.1}$$

where z is a constant, and the stopping time

$$\tau = \inf\{t \geq 0, X_t = b\}$$

where b is a constant. Define another constant

$$w = \sqrt{z^2 + 2r},$$

where r represent the constant riskfree interest rate. We are concerned about the present value of one currency unit payable at the first hitting time of a lower boundary if this occurs before the horizon T and define

$$V_0 = E^Q[e^{-r\tau}1\{\tau \leq T\}].$$

where $E^Q[\cdot]$ denotes the expectation under the equivalent martingale measure. E.g., Lando (2004) shows that

$$V_0 = e^{b(z-w)}Q^w(\tau \leq T),$$

where

$$Q^w(\tau \leq T) = N(\frac{b - wT}{\sqrt{T}}) + e^{2wb}N(\frac{b + wT}{\sqrt{T}}), \tag{4.A.2}$$

represents the cumulative probability distribution of τ as a function of the parameter w. The above result can be rewritten as

$$V_0 = e^{b(z-w)}N(\frac{b - wT}{\sqrt{T}}) + e^{b(z+w)}N(\frac{b + wT}{\sqrt{T}}). \tag{4.A.3}$$

The constants z and b for our problem, see Section 2, which may be plugged into expression (4.A.3), are:

$$z = \frac{1}{\sigma}(\mu - \gamma - \frac{1}{2}\sigma^2) \tag{4.A.4}$$

and

$$b = \frac{1}{\sigma}\ln(\frac{B}{A}). \tag{4.A.5}$$

The revised expression becomes

$$V_0 = (\frac{A}{B})^{\beta+2\frac{\mu-\gamma-\frac{1}{2}\sigma^2}{\sigma^2}}N(n_1) + (\frac{A}{B})^{-\beta}N(n_2), \tag{4.A.6}$$

where

$$n_1 = \frac{(\mu-\gamma-\frac{1}{2}\sigma^2-\sigma^2\beta)T+\ln(\frac{B}{A})}{\sigma\sqrt{T}}$$

and

$$n_2 = \frac{-(\mu-\gamma-\frac{1}{2}\sigma^2-\sigma^2\beta)T+\ln(\frac{B}{A})}{\sigma\sqrt{T}}$$

and β is given in (4.3.3).

5

Bundled Financial Claims - A Model of Hybrid Capital

coauthored with Svein-Arne Persson [1]

Abstract

A large class of infinite horizon financial instruments which incorporates elements of both debt and equity, may collectively denoted "hybrid capital". The Bank for International Settlements (BIS) has devised the fundamental requirements for how hybrid capital may qualify as a part of core ("Tier 1") regulatory capital for banks. We present valuation models for hybrid capital in the set-up of Black and Cox (1976) and Leland (1994) and derive new valuation formulas incorporating these special features. In particular, we take into account the issuer's right to omit hybrid coupon payments and to call the hybrid capital at par value starting from a given date. In doing so, we build on formulas developed in Mjøs and Persson (2005). We show that hybrid capital actually carry risk and clarify interesting links between their valuation and overall corporate capital structure as guidance both for market participants and regulators alike.

[1]The authors thank Petter Bjerksund, J. David Cummins, Neil A. Doherty, Elyès Jouini, Kristian R. Miltersen, Alexander Muermann, and Stephen H. Shore for valuable suggestions and discussions. Earlier versions of this paper have been presented at the *FIBE* conference at the Norwegian School of Economics and Business Administration(NHH), Bergen, January 2004, at the Rosen - Huebner - McCahan Seminar Series at the Insurance and Risk Management department at The Wharton School, January 2004, at European Group of Risk and Insurance Economists (EGRIE) 31st seminar Marseilles, September 2004, University of Bonn, May 2004, at the Norwegian University of Science and Technology, June 2004, at the European Financial Management Association, Vienna, June 2007, and at The Nordic Academy of Management Conference, Bergen, 2007.

5.1 Introduction

Increasingly complex structured securities are a consequence of the sophistication of financial markets. The commercial drive behind this development is - as always - to seek any remaining arbitrage opportunities either in terms of risks or rewards. Complexity is a function of issuer and investor preferences, investment banks' commercial creativity and finally regulatory, legal or tax frameworks. In the case of hybrid capital[2] which encompass elements of both debt and equity, the regulatory and tax-considerations define the most unique characteristics. Typical 'hybrid capital' is a perpetual coupon paying security, senior only to common equity, includes an issuer option to call the security as from 10 years after the date of issue and the right of the issuer to forego coupon-payments without it constituting a default. In sum, the commercial "raison d'être" for hybrid capital is that it is a qualifying form of risk-carrying capital which in most cases have tax-deductible dividends or coupons. In this paper we contribute with better understanding of the valuation of and capital structure impacts from hybrid capital generally, and particularly the typical structures issued by banks. Standard asset pricing models value each separate element included in a complex security and conduct a security-specific "sum-of-the-parts" valuation. This approach disregards the need to any resulting effects on the securities and the issuer caused by a combined set of fixed and conditional claims. Our paper models, values and contributes to a better understanding of hybrid capital and comments on its effect on optimal shareholder bankruptcy behavior for a given capital structure. The complexity makes the valuation a particularly challenging research task, even under strict assumptions regarding to market efficiency and symmetric information. Mjøs and Persson (2005) developed fundamental valuation formulas using a barrier-options-approach, whilst the contribution of this paper is a completely specified version of hybrid capital in a complete capital structure setting. The specification of hybrid capital resembles standard market practice.

[2] *Hybrid capital* is a class of infinite horizon coupon-paying securities for which the value is bounded above similarly as for debt, but which carry almost the same downside risk as equity. These securities have many names but the generic structures are very similar on a global basis due to commonality in the regulation of the issuers, primarily banks and insurers. Preferred stock, Trust Preferred Stock and Capital Securities are terms used on variants of hybrid capital, in particular in the US.

5.1.1 The market for hybrid capital

As of mid 2005, the (global) stock of outstanding hybrid capital was estimated at $376 billion[3] Amongst the issuers were 57 % banks, 8 % insurance companies, 15 % utilities and 12 % industrial companies. This dominance of regulated industries reflects both that the securities are tailor-made to specific regulations and that the infinite horizon maturity is well suited to sectors where regulations and supervision reduce the expected default risk. An international comparison also shows that this dominance is even clearer in markets outside the US.

All banks in the developed part of the world are under strict regulations compared to other sectors, in particular regarding risk exposure, risk management and required capital. The foundation for the global regime for risk capital in banks is the 1998 Basle Accord on Capital Standards, see Committee on Banking Supervision (1988). This document laid out the general principles for calculating the prudent capital requirements for banks and the criteria for what constituted acceptable forms of risk capital, as well as certain deductions from this capital. We describe these regulations in more detail below. In addition to strengthening the global regulatory focus on risk capital in financial institutions, the Accord became a standard which secured a large degree of commonality between capital regulations between jurisdictions. An element of this was a more standardized global structure for hybrid capital. National variations include criteria for tax-deductability and contingent control rights in case of financial distress. This high degree of standardization of hybrid capital issued by banks and to a large degree also insurance companies causes us to focus our examples on hybrid capital for financial institutions. The valuation formulas and related analysis are generic and may usefully be applied also in other settings.

We analyze under the assumption that an issuer's main motivation for issuing hybrid capital is to optimize between regulatory requirements, after-tax cost-of-capital and capital-structure considerations.[4] A bank may issue hybrid capital instead of raising new common equity and thus both potentially save tax and avoid the direct and indirect costs related to seasoned equity offerings. Benston, Irvine, Rosenfeld, and Sinkey (2003) analyze 105 issues of hybrid capital by US

[3]Source: Lehman Brothers "The Capital Securities Market. Composition and Trends".

[4]The full understanding of drivers of issuance is a separate research subject in itself and not covered here.

bank holding companies in the years 1995-1997 with regards to what characterizes issuers vs. non-issuers. They find distinct differences in that issuers are larger, have higher tax-rates, more uninsured funding and lower equity ratios. Their event study also finds significant common stock abnormal returns for issuers who filed after the Federal Reserve's new favorable tax-treatment of hybrid capital[5] was announcement on 21 October 1996. The latter is taken as a confirmation of the positive shareholder value impact from such issues.

5.1.2 Financial institutions' regulatory framework

Banks and insurance companies are subject to extensive regulations. Lacking a consensus view as to the full justification of the regulations, Santos (2000) points to two commonly accepted motivations: Firstly, the risk of a systemic crisis when the banks as liquidity providers experience a "run" from depositors. Secondly, fragmented depositors' limited incentives and ability to properly monitor banks as lenders, and thus the implicit risk of moral hazard in banks. These risks are met by deposit insurance, rules and supervision regulating the creation and operation of banks, and finally specific capital requirements to capture a sufficient part of the remaining risk exposure of both society and depositors. Under the current regulations[6] the assets of a bank are weighted primarily according to credit risk and the minimum amount of capital is 8 % of this weighted sum of assets. This requirement may be met by capital of different priority and risk-exposure. Tier I includes common equity and hybrid capital, Tier II constitutes subordinated debt and is split between a perpetual "upper-level" and a dated "lower-level" category. Tier III is short term subordinated debt which may only cover market price risks.

The basic requirements defined by BIS for hybrid capital to qualify as part of Tier I capital are[7]:

"Hybrid (debt/equity) capital instruments. This heading includes a range of instruments which combine the characteristics of equity capital and of debt. Their

---

[5] Trust Preferred Securities (TPS)

[6] BIS is well under way with a new set of capital requirements popularly labelled "Basel II" which includes a broader range of risks evaluated when setting the capital requirements. Basel II also represents a significant improvement in how individual institutions may calculate their exact capital requirements. These regulations are being implemented gradually during the years to come.

[7] Committee on Banking Supervision (1988) Annex 1, D, (d), page 18-19.

precise specifications differ from country to country, but they should meet the following requirements:

- they are unsecured, subordinated and fully paid-up,
- they are not redeemable at the initiative of the holder without the prior consent of the supervisory authority,
- they are available to participate in losses without the bank being obliged to cease trading (unlike conventional subordinated debt),
- although the capital instrument may carry an obligation to pay interest that cannot permanently be reduced or waived (unlike dividends on ordinary shareholders' equity), it should allow service obligations to be deferred (as with cumulative preference shares) where the profitability of the bank would not be supported.

Cumulative preference shares, having these characteristics would be eligible for inclusion in this category."

National implementations of these principles are necessarily more practically phrased in line with the following:

- Hybrid capital is only senior to equity capital in case of distress and liquidation.
- Hybrid capital have to be fully paid-in, any authorized payments are not included.
- The issuer have the right to call the securities at coupon-payment[8] dates after 10 years from the date of issue. The execution of this option requires explicit approval from the supervisory authority in charge of the bank to secure that a repayment of the hybrid capital does not make the issuer too weak in capital terms.

[8] We choose to denote the amounts paid to service the securities "coupons" even though some versions of hybrid capital are categorized as preference shares which receive dividend payments and not coupons.

- To absorb risk in financially distressed situations, non-payment of coupons should not cause the bank to default.

- Hybrid capital should not exceed 15 % of total Tier I capital.

The main variations between different countries' regulations regards whether any unpaid coupons are permanently omitted or may be accumulated up to a maximum period (e.g., 5 years) but then constitutes a default, the position of the capital relative to equity when a distressed bank is refinances, the maximum step-up of the coupon-rate if the call option is not exercised at the first possible date, the tax-treatment of coupons for both issuer and investor, and the maximum amount of hybrid capital accepted as Tier I capital. Some countries allow direct hybrid issuance where the coupons are fully treated as debt-coupons, these include, e.g., the Scandinavian countries and Spain. In the USA the most common form used by bank holding companies (BHCs) has been Trust Preferred Capital which is issued by a special purpose vehicle which then re-lend the funds to the bank holding company as Upper Tier II subordinated debt. This structure allows for Tier I treatment of the funds in the consolidated bank-group accounts and tax-treatment as for regular debt, but no Tier I effect on the bank holding company separately.

We model hybrid capital as infinite horizon subordinated debt with a finite, embedded call-option, a step-up in the coupon-rate and an exogenously given risk of non-payment of coupons. These combined features characterize most issues of hybrid capital. To be able to analyze the impact from hybrid-capital on the issuer's other outstanding liabilities, we also include a model of perpetual senior debt.

5.1.3 Literature overview

Hybrid capital has been studied from various angles, explaining issuance (e.g., Benston, Irvine, Rosenfeld, and Sinkey (2003) as discussed above), understanding its role in a regulatory perspective (e.g., Santos (2000)) and some event-studies of the effect of issuance on common stock values (e.g., Krishnan and Laux (2005) as commented earlier).

The most direct precedence to our work is the paper by Emanuel (1983) which develops a valuation of preferred stock, equivalent to our term hybrid capital, based on the option-methodology of Black and Scholes. Emanuel applies a geometric Brownian motion for the development of the total value of the firm and develops partial differential equations for the value of equity, debt and preferred stock. The paper gives a comprehensive motivation for management's dividend payment decisions and argues that in this context, preferential dividend rights and voting rights conditional on passed dividends are equivalent. The basic model assumes that as soon as firm value exceeds initial value, current and any accumulated preferred dividends will be paid. The derivative of preferred stock with respect to firm value is subject to current firm value and the size of any arrearage of unpaid preferred dividends, exemplifying the mix of debt and equity features. In comparison, we provide an alternative way of modelling passed hybrid coupons. Common to both approaches is that hybrid coupons are passed when the asset value process hits an exogenously defined boundary which is assumed to be somewhat above the optimal bankruptcy/liquidation level. The choice of the first boundary, the omission level, affects the optimal bankruptcy level in our model. Another contribution compared to Emanuel's paper is the analysis of the issuer's embedded call option commonly found in practice. We develop closed form formulas for this option and also show how the existence of this option affects the optimal bankruptcy level and valuation of other contingent claims on the firm, in particular debt.

Regarding the existence and impact of hybrid capital per se, some specific papers are worth mentioning:

Engel, Erikson, and Maydew (1999) analyze the issuance of trust preferred stock using a sample of 158 issuances for the period 1993-1996. They find that issuers are willing to incur on average USD 10 million in direct and USD 43 million in indirect costs to reduce the debt-to-asset ratio by 12.8 % and also that the Net Present Value of the tax-benefits in replacing traditional preferred stock with trust preferred stock are around 28 %. These tax benefits are partially transferred to investors (implicit taxes) through a higher pre-tax yield on tax-advantaged vs. regular preferred stock.

Krishnan and Laux (2005) study the impact of issuance of trust preferred stock on the return of common stock. They find significant positive abnormal return when the issuer gains specific financial benefits from the issue.

Beatty (2005) analyze whether changes in accounting treatment of hybrid capital impacts banks' propensity to issue hybrid capital when capital regulations are unchanged. The paper shows that market discipline is important in that bank behavior actually changed following the accounting changes.

Engel, Erikson, and Maydew (1999), Benston, Irvine, Rosenfeld, and Sinkey (2003) and Krishnan and Laux (2005) all include extensive descriptions of the market for and structure of hybrid capital, in particular as seen in a US perspective.

5.1.4 Focus of the paper

Lack of contingent control rights when coupon-payments are defaulted, subordination after all debt, perpetual maturity and non-cumulative coupons are all 'equity-like' features of hybrid capital. On the other hand, seniority before common equity, tax-deductible coupons, issuer call-features and no proportional share in any profits are clearly 'debt-like' features. We develop formulas for the valuation of these instruments in a given capital-structure setting, including its impact on optimal shareholder bankruptcy behavior. Interesting topics like agency-problems and issuance dynamics are left for later research.

Our model is based on the Merton (1974)/Black and Cox (1976) models as extended by Leland (1994) and Goldstein, Ju, and Leland (2001) although we do not introduce market frictions like tax or bankruptcy costs. Our model cannot in its current form be used to directly analyze optimal capital structure.

In our model, risk is represented by the volatility of the EBIT-process following the tradition of, e.g., Ingersoll (1977) and this differs from more recent research on callable bonds (see e.g., Acharya and Carpenter (2002)) which use a stochastic interest rate-process. We motivate our choice of process by two different arguments. Firstly, hybrid capital are typically an issuers most risky securities and capture far more of the issuer's credit risk than regular senior bonds. Secondly, market practise indicates that issuers of hybrid capital, either directly or through an interest rate swap, pay a fixed credit margin plus a floating market interest. Issuers are thus hedged against the nominal interest rate level. These factors implies that an EBIT-process which replicates the firm specific credit quality developments replicates well hybrid capital risk.

Our paper is organized as follows: Section 2 presents our specifications of hybrid capital. Section 3 reproduces the standard results of Black and Cox (1976) with one class of debt. Section 4 explains our main results. Section 5 presents numerical examples. Section 6 discusses various specific features of hybrid capital in more detail. Section 7 contains numerical sensitivities and graphical illustrations and section 8 concludes. Some calculations are left for in an appendix.

5.2 Our specification of hybrid capital

We include the following properties of hybrid capital:

1. Hybrid capital is a perpetual, continuously coupon-paying debt instrument that has priority after senior debt, but before equity. In terms of priority it is similar to regular junior or subordinated debt. To incorporate this property hybrid capital is modelled as Black and Cox (1976) junior debt. In cases with several classes of debt with different priorities, hybrid capital will always have the lowest rank.

2. Issuer has the right to call the hybrid capital at par at coupon-dates (usually quarterly) after a fixed period (usually 10 years) from the date of issue. For simplicity we assume that this option is *European* and only exercisable at the first possible date. Our assumed exercise strategy is in correspondence with observed market practice, see Mjøs and Persson (2005), who develop valuation formulas capturing this aspect of hybrid capital.

3. Unpaid coupons represent an irrevocable loss for the investor. They do not trigger bankruptcy, nor are they accumulated as additional debt[9]. To explicitly model the possibility to omit payment of hybrid capital coupons we introduce an exogenous asset threshold level under which coupon payments are not paid.

4. If the call option is not exercised at the first possible date, the annual coupon-rate is increased by a contractually agreed *step-up* (typically 75 - 150 bp).

[9] In the USA and some other jurisdictions, also hybrid capital for financial institutions with either cumulative interest or certain rights to the investors following a defined number (e.g. 20) of consecutive missed coupon payments qualifies as core/Tier 1 capital whilst most European and other supervisory authorities require no cumulation of missed coupons. We choose to focus on the non-cumulative setting. In Norway, issuers may omit coupons provided no dividends are paid on common stock.

It is well known that also regular perpetual junior debt may contain a call option and coupon rate step-up. The defining difference between junior debt and hybrid capital is the issuer's right to omit coupons without causing an event of default. BIS as well as leading rating agencies consider this feature as critical to accept hybrid capital as the highest ranking (Tier I) risk capital and not just as junior(Tier II) debt for financial institutions.

5.3 The valuation model and basic results

We consider the standard Black-Scholes-Merton economy and impose the usual perfect market assumptions:

- All assets are infinitely separable and continuously tradeable.
- No taxes, transaction cost, bankruptcy costs, agency costs or short-sale restrictions.
- There exists a continuously compounded constant riskless rate of return r.

We study a limited liability company with a capital structure consisting of three claims, infinite horizon debt, hybrid capital, and common equity. In line with Goldstein, Ju, and Leland (2001), we assume that the assets generates an EBIT (earnings before interest and tax)-cashflow denoted by δ_t given by the stochastic differential equation

$$d\delta_t = \mu\delta_t dt + \sigma\delta_t dW_t, \tag{5.3.1}$$

where μ and σ are constants representing the drift and volatility respectively, and δ_0 is given. Here W_t is a standard Brownian motion under the equivalent martingale measure. The total time t market value A_t of the assumed perpetual EBIT stream from the assets equals

$$\begin{aligned} A_t &= E_t^Q\left[\int_t^\infty e^{-r(s-t)}\delta_s ds\right] \\ &= \frac{\delta_t}{r-\mu}. \end{aligned} \tag{5.3.2}$$

This quantity is elsewhere in the literature referred to as *the unlevered value of the firm's assets*. This market value follows the process

$$\begin{aligned} dA_t &= (rA_t - \delta_t)dt + \sigma A_t dW_t \\ &= \mu A_t dt + \sigma A_t dW_t \end{aligned} \tag{5.3.3}$$

Observe that the volatility parameter of this market value is identical to the volatility of the cashflow process. We use the notation $A = A_0 = \frac{\delta_0}{r-\mu}$.

A general claim f on the assets under these assumptions satisfies the partial differential equation, see, e.g., Merton (1974),

$$\frac{1}{2}\sigma^2 A_t^2 f_{AA} + \mu A_t f_A - rf + f_t + C(A_t, t) = 0, \tag{5.3.4}$$

where f_t denotes the partial derivative of f with respect to (elapsed or calendar) time, f_A the partial derivative of f with respect to A_t, and f_{AA} the partial derivative of f_A with respect to A_t. Here $C(A_t, t)$ represents the time t continuous net coupon rate received by the owner of the claim f.

5.3.1 A recollection of the Black and Cox (1976) results.

We review the results of Black and Cox (1976) and Leland (1994) for the simple case with one class of regular infinite horizon debt. Observe that we assume an underlying EBIT-process following Goldstein, Ju, and Leland (2001) and that equity-holders receive any cashflows in excess of debt coupons as dividends. This assumption leads to a drift parameter μ different from the risk free rate r in the asset price process.

We denote the face value of the debt by D and assume that contractual debt payments per unit of time are given by cD, where the coupon rate c is assumed constant.

Let $\bar{A}$ denote the lower boundary of A_t where debt payments are stopped and the shareholders transfer the assets of the firm to the debtholders. We refer to $\bar{A}$ as the *bankruptcy level*.

The *time t market value* of one monetary unit paid upon bankruptcy, i.e., when the process A_t hits the boundary $\bar{A}$ is

$$F_t = F_t(A_t, \bar{A}) = (\frac{A_t}{\bar{A}})^{-\beta}, \tag{5.3.5}$$

where[10]

$$\beta = \frac{\mu - \frac{1}{2}\sigma^2 + \sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2\sigma^2 r}}{\sigma^2} > 0. \tag{5.3.6}$$

5.3.2 Valuation of debt and equity

The time 0 market value of infinite horizon debt with continuous constant coupon payment is

$$D(A) = \frac{cD}{r} - J_D F_0(A, \bar{A}), \tag{5.3.7}$$

where

$$J_D = \frac{cD}{r} - \bar{A}.$$

Expression (5.3.7) for the market value of debt carries a nice intuition. $\frac{cD}{r}$ is the time 0 market value of infinite horizon default-free debt receiving a coupon payment rate of cD. In the case of risky debt ($\bar{A} < D$), the debtholder upon bankruptcy looses a stream of infinite coupon payments which at the time of bankruptcy has market value $\frac{cD}{r}$. This loss has a time 0 market value of $F_0 \frac{cD}{r}$. The time 0 market value of debt when $\bar{A} = 0$ is then the difference between the market values of these two coupon streams. In a more realistic setting, $\bar{A} > 0$ and this is also the liquidation payoff to debt in case of bankruptcy. We can therefore interpret $J_D = (\frac{cD}{r} - \bar{A})$ as the debtholder's *net loss* upon bankruptcy. The time 0 market value of this net loss $J_D F_0$ represents the reduction of the time 0 market value of riskfree debt due to default risk. This is the only risk-exposure of lenders in our model, assuming constant riskfree interest rate and liquid, efficient markets.

Since we have not included any market inefficiencies, e.g., taxes, liquidation costs, or strategic debt service opportunities, the time 0 value of equity is the market value of assets less the market value of debt, or

$$E(A) = A - D(A) = A - \frac{cD}{r} + J_D F_0(A, \bar{A}), \tag{5.3.8}$$

using equation (5.3.7).

[10] It can actually be shown that $\beta > \frac{2\mu}{\sigma^2}$.

5.3.3 The optimal bankruptcy level

Given c and D, we assume that the equityholders choose $\bar{A}$ so that the value of equity is maximized. By maximizing expression (5.3.8) with respect to $\bar{A}$ we determine the optimal $\bar{A}^*$ as

$$\bar{A}^* = \frac{\beta}{(\beta+1)}\frac{cD}{r}. \tag{5.3.9}$$

5.4 Main results

We model hybrid capital including the properties described in section 2 and assume a more realistic capital structure including both infinite horizon senior debt, hybrid capital and equity. Our valuation results facilitate, for a given capital structure, the calibration of optimal bankruptcy levels and coupon rates.

We develop our valuation formulas stepwise by first determining payoffs and market values of all claims at time T, the time of expiry of the hybrid capital embedded option. We subsequently determine the time zero value of all claims, partially by applying barrier option formulas with the time T market values as underlying assets.

Our analysis is based on the assumption that the bankruptcy asset level before the expiration of the option is constant, but possibly different from the long term bankruptcy asset level. This assumption ignores any further time dependencies caused by the decreasing time to maturity of the hybrid capital embedded option. A companion paper, Mjøs and Persson (2005), demonstrate that this assumption has negligible influence on the resulting calibrated coupon rates in a simpler set-up for reasonable parameter values. They also find that a common, constant bankruptcy level was most robust. In this paper, we study a more complex security and capital structure and choose to apply two constant bankruptcy levels to maintain analytical flexibility.

We define the stopping time τ as

$$\tau = \inf\{t \geq 0; A_t = B\}. \tag{5.4.1}$$

Here τ can be interpreted as the time of bankruptcy if $\tau \leq T$.

5.4.1 Time T valuation of the claims.

The market values of senior debt, hybrid capital and equity at time T represent the first step in the valuation of the financial claims at time zero. These valuation formulas implicitly assume that the issuing company has not gone bankrupt before or at time T, i.e. $\tau > T$. The value of any payout in case of bankruptcy at time T will be included in the time 0 valuation formulas.

Valuation of senior debt at time T.

Proposition 7. *The time T market value of senior debt with face value D and coupon rate c_S for a given bankruptcy asset level $\bar{A}$ is*

$$D_T^*(A_T) = \frac{c_S D}{r} - J_D^* F_T(A_T, \bar{A}), \tag{5.4.2}$$

where

$$J_D^* = \frac{c_S D}{r} - \min(\bar{A}, D),$$

and F_T is defined in expression (6.3.1).

Proof. This is the standard result for regular Black and Cox (1976) debt from expression (5.3.7) where the bankruptcy payoff term is replaced with $\min(\bar{A}, D)$ to explicitly include the case when senior debt carries no default risk, i.e., $\bar{A} > D$. □

Valuation of hybrid capital at time T.

The payoff to hybrid capital at time T is

$$H_T^* = \begin{cases} H_T - \max(H_T - H, 0) & \text{for } \tau > T \\ 0 & \text{otherwise,} \end{cases} \tag{5.4.3}$$

where H_T is given by expression (5.3.7) for standard Black and Cox (1976) debt. Note, however, the following differences: The face value of debt D is replaced by the face value of hybrid capital H. The coupon rate c is replaced by the stepped up coupon-rate $c_H + k$. Here c_H represents the coupon rate of hybrid capital before time T and k represents the increase *(step-up)* in this rate at time T provided that the option to call the hybrid capital has not been exercised. U represents the exogenous asset level below which hybrid capital coupons are not paid.

The total time T payoff from hybrid capital H_T^* equals infinite horizon debt less the payoff from a *European call*-option on the hybrid capital exercisable at time T with exercise price H, provided the company is not bankrupt at or before time T.

Proposition 8. *The time T market value of hybrid capital with face value H and priority after senior debt with face value D, coupon rate c_H, time T coupon rate step-up k, for a given bankruptcy asset level $\bar{A}$, and given asset threshold value level U below which hybrid capital coupons are not paid, is*

$$H_T^*(A_T) = \frac{(c_H + k)H}{r} - J_H^* F_T(A_T, \bar{A}), \tag{5.4.4}$$

where

$$J_H^* = \frac{(c_H + k)H}{r} \frac{(\beta y^{-\alpha} + \alpha y^{\beta})}{(\alpha + \beta)} - (\bar{A} - D)^+, \tag{5.4.5}$$

$$\alpha = \beta + 1 - \frac{2\mu}{\sigma^2},$$

$$y = \frac{U}{\bar{A}} \geq 1, \tag{5.4.6}$$

and β is given in expression (5.3.6).

Proof. See Appendix A. Expression (5.4.4) follows from equation (5.A.5), replacing c by $c_H + k$ to include the coupon-rate step-up at time T, and G by $(\bar{A} - D)^+$ to adjust the liquidation payoff due to the existence of senior debt. □

As opposed to J_D in the original debt expression (5.3.7), J_H^* represents the potential net loss to holders of hybrid capital both from bankruptcy and unpaid coupons, i.e., when $\bar{A} < A < U$.

The constant y represents the ratio between the coupon omission asset level U and the longterm bankruptcy asset level $\bar{A}$. In the case when $k = 0$ (no coupon rate step-up) and $y = 1$ (no risk of omitted coupons) expression (5.4.4) is simplified to the standard case of junior and senior debt. The term $(\bar{A} - D)^+$ in expression (5.4.5) reflects that hybrid capital has priority after senior debt in case of bankruptcy.

Valuation of equity at time T.

The market value of equity at time T follows from the identity $E^*_T(A_T) = A_T - D^*_T(A_T) - H^*_T(A_T)$ and equations (5.4.4) and (5.4.2)

$$E^*_T(A_T) = A_T - \frac{c_S D + (c_H + k)H}{r} + (J^*_D + J^*_H)F_T(A_T, \bar{A}). \tag{5.4.7}$$

The optimal long term bankruptcy level at time T, denoted by $\bar{A}^*$, is found by differentiating expression (5.4.7) with respect to $\bar{A}$. The solution is

$$\bar{A}^* = \frac{\beta}{\beta + 1}(\frac{c_D D}{r} + \frac{(c_H + k)H}{r} y^{-\alpha}). \tag{5.4.8}$$

In the case where $k = 0$ and $y = 1$, $\bar{A}^*$ is identical to the similar quantity in the standard case with junior and senior debt.

5.4.2 Time 0 valuation of the claims.

We now value the different claims at time 0 and include the effect of the embedded option in hybrid capital. The embedded call option has a fixed maturity, and its market value depends on remaining time to expiry.

We assume a constant, lower, bankruptcy level B before time T reflecting the value of the embedded option. The long term bankruptcy level $\bar{A}^*$ is affected by this assumption through the coupon rates c_H and c_S.

Valuation of senior debt at time 0.

The time 0 value of senior debt is equal to the time 0 market value of receiving D^*_T at time T if the company has not gone bankrupt, plus the time 0 market value of cashflows received before time T, as well as any bankruptcy payment at time T. The last item is the payoff if the firm has survived until time T, but $A_T < \bar{A}$, and bankruptcy occurs.

Proposition 9. *The time zero market value of senior debt is*

$$D^*_0(A) = \frac{c_S D}{r} - J^0_D V_0(B) - J^*_D V_1(\bar{A}) + V_s(A), \tag{5.4.9}$$

where

$$V_0(B) = (\frac{A}{B})^{\alpha} N(d_1) + (\frac{A}{B})^{-\beta} N(d_2),$$

$$ds_1 = \frac{\ln(\frac{B}{A}) + (\mu - \frac{1}{2}\sigma^2 - \sigma^2\beta)T}{\sigma\sqrt{T}}$$

$$ds_2 = \frac{\ln(\frac{B}{A}) - (\mu - \frac{1}{2}\sigma^2 - \sigma^2\beta)T}{\sigma\sqrt{T}}$$

$$V_1(\bar{A}) = (\frac{A}{\bar{A}})^{-\beta}N(-d_2) - (\frac{\bar{A}}{B})^{\beta}(\frac{A}{B})^{\alpha}N(d_1),$$

$$J_D^0 = \frac{c_S D}{r} - \min(B, D),$$

$$V_s(A) = Ae^{-rT}\left(N(gs_1) - N(gs_2) + (\frac{A}{B})^{-\frac{2(\mu)}{\sigma^2}-1}[N(-gs_3) - N(-gs_4)],\right)$$

where

$$gs_1 = \frac{\ln(\frac{A}{B}) + (\mu + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}},$$

$$gs_2 = \frac{\ln(\frac{A}{\max(\bar{A},B)}) + (\mu + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}},$$

$$gs_3 = \frac{\ln(\frac{A}{B}) + \ln(\frac{\max(\bar{A},B)}{B}) - (\mu + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}},$$

$$gs_4 = \frac{\ln(\frac{A}{B}) - (\mu + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}.$$

and β is given in expression (5.3.6) *and J_D^* in Proposition 1.*

Proof. The time 0 market value of debt consists of the market value of time T debt plus the market value of coupon payments and a potential bankruptcy payoff before or at time T, i.e.,

$$D_0^*(A) = E^Q[D_T^* e^{-rT} 1_{\{\tau > T\}}] + E^Q[\int_0^{\tau \wedge T} c_S D e^{-rs} ds] \quad (5.4.10)$$
$$+ E^Q[\min(B, D)e^{-r\tau} 1_{\{\tau \le T\}}] + E^Q[A_T e^{(-r)T} 1_{\{\tau > T\}} 1_{\{A_T < \bar{A}\}}].$$

Denote

$$V_0(B) = E^Q[e^{-r\tau} 1_{\{\tau \le T\}}],$$

where τ is defined in expression (5.4.1). From equation (5.4.2) we can write the first term of expression (5.4.10) as

$$E^Q[D_T^* e^{-rT} 1_{\{\tau > T\}}] = \frac{c_S D}{r} Q(\tau > T)e^{-rT} - J_D^* E^Q[e^{-rT} F_T(A_T, \bar{A}^*) 1_{\{\tau > T\}}].$$

Denote

$$V_1(\bar{A}) = E^Q[e^{-rT}F_T(A_T,\bar{A})1_{\{\tau>T\}}].$$

The integral in the second term of expression (5.4.10) can be written as

$$E^Q[\int_0^{\tau\wedge T} c_S De^{-rs}ds] = \frac{c_S D}{r}(1 - Q(\tau > T)e^{-rT} - V_0(B))$$

The final term equals

$$V_s(A) = E^Q[e^{-rT}A1_{\{A_T<\bar{A}\}}1_{\{\tau>T\}}].$$

The calculations of $V_0(B)$, $V_1(\bar{A})$ and $V_s(A)$ are standard, see e.g., Mjøs and Persson (2005). The result follows by collecting terms. □

The valuation expression for senior debt at time 0 may be decomposed into the value of a perpetual, riskfree coupon-stream, less the loss in case of bankruptcy before or after time T, plus any bankruptcy payout at time T in the case when $B < A_T < \bar{A}$ and $\tau > T$.

Valuation of hybrid capital at time 0.

The time 0 value of hybrid capital is equal to the time 0 market value of receiving H_T^* at time T plus the time 0 market value of cashflows received before time T and any bankruptcy payout at time T. H_T^* is given in expression (5.4.4).

Proposition 10. *The time 0 market value of hybrid capital is*

$$H_0^*(A) = \frac{(c_H + k)H}{r}Q(\tau > T)e^{-rT} - J_H^* V_1(\bar{A}) \qquad (5.4.11)$$
$$- C + L + (B - D)^+ V_0(B) + V_h(A),$$

where

$$C = C_0(A, H) - (\frac{B}{A})^{\beta-\alpha}C_0(\frac{B^2}{A}, H),$$

$$C_0(A, H) = (\frac{(c_H + k)H}{r} - H)e^{-rT}N(-d_2) - J_H^*(\frac{A}{\bar{A}})^{-\beta}N(-d_1),$$

$$dh_1 = \frac{\ln(\frac{\bar{A}}{A}) - \frac{1}{\beta}(\ln(\frac{(c_H+k)H}{r} - H) - \ln(J_H^*)) + (\frac{r}{\beta} + \frac{1}{2}\sigma^2\beta)T}{\sigma\sqrt{T}},$$

$$dh_2 = dh_1 - \sigma\beta\sqrt{T},$$

$$Q(\tau > T) = N(dh_3) - (\frac{A}{B})^{\alpha-\beta} N(-dh_4),$$

$$dh_3 = \frac{\ln(\frac{A}{B}) + (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}},$$

$$dh_4 = \frac{\ln(\frac{A}{B}) - (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}},$$

$$L = \frac{c_H H}{r}(1-\kappa),$$

$$\kappa = \frac{\alpha}{\alpha+\beta}(\frac{A}{U}) - \beta(1 - \hat{Q}(\tau > T, A_T > U) + e^{-rT} Q(A_T > U, \tau > T)$$
$$+ \frac{\beta}{\alpha+\beta}\left((\frac{A}{U})^{\alpha} \bar{Q}(A_T > U, \tau > T) + (\frac{B}{U})^{\alpha} V_0(B)\right),$$

$$V_h(A) = Ae^{-rT}\left(N(gh_1) - N(gh_2) + (\frac{A}{B})^{-\frac{2(\mu)}{\sigma^2}-1}[N(-gh_3) - N(-gh_4)]\right) -$$
$$Ke^{-rT}\left(N(gh_5) - N(gh_6) + (\frac{A}{B})^{-\frac{2(\mu)}{\sigma^2}-1}[N(-gh_7) - N(-gh_8)]),\right)$$

where

$$gh_1 = \frac{\ln(\frac{A}{\max(\bar{A},B)}) + (\mu + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}},$$

$$gh_2 = \frac{\ln(\frac{A}{\bar{A}}) + (\mu + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}},$$

$$gh_3 = \frac{\ln(\frac{A}{B}) + \ln(\frac{\bar{A}}{B}) - (\mu + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}},$$

$$gh_4 = \frac{\ln(\frac{A}{B}) + \ln(\frac{\max(\bar{A},B)}{B}) - (\mu + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}.$$

$$gh_5 = \frac{\ln(\frac{A}{\max(\bar{A},B)}) + (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}},$$

$$gh_6 = \frac{\ln(\frac{A}{\bar{A}}) + (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}},$$

$$gh_7 = \frac{\ln(\frac{A}{B}) + \ln(\frac{\bar{A}}{B}) - (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}},$$

$$gh_8 = \frac{\ln(\frac{A}{B}) + \ln(\frac{\max(\bar{A},B)}{B}) - (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}.$$

α and J_H^* are from Proposition 2, $V_0(B)$ is from Proposition 3, β is given in expression (5.3.6) *and $Q(A_T > U, \tau > T)$, $\hat{Q}(\tau > T, A_T > U)$, and $\bar{Q}(A_T > U, \tau > T)$ are from Mjøs and Persson (2007).*

Proof. The time 0 market value of hybrid capital consists of the market value of hybrid capital at time T including an embedded call option plus the market value of coupon payments and a potential bankruptcy payoff before time T, i.e.,

$$\begin{aligned} H_0^*(A) &= E^Q[H_T^* e^{-rT} 1_{\{\tau>T\}}] - E^Q[(H_T^* - H)^+ e^{-rT} 1_{\{\tau>T\}}] \\ &+E^Q[\int_0^{\tau\wedge T} c_H H 1_{\{A_s>U\}} e^{-rs} ds] + E^Q[(B-D)^+ e^{-r\tau} 1_{\{\tau\leq T\}}] \\ &+E^Q[(A_T - D)^+ e^{-rT} 1_{\{A_T<\bar{A}\}} 1_{\{\tau>T\}}]. \end{aligned} \tag{5.4.12}$$

Observe that equation (5.4.12) differs from equation (5.4.10) in two ways. First, it includes the hybrid capital embedded call option (the second term on the right hand side). Second, the coupon payments before time T only take place if $A_t > U$.

Denote

$$C = E^Q[(H_T^* - H)^+ e^{-rT} 1_{\{\tau>T\}}]$$

and

$$L = E^Q[\int_0^{\tau\wedge T} c_H H 1_{\{A_s>U\}} e^{-rs} ds].$$

We calculate

$$E^Q[H_T^* e^{-rT} 1_{\{\tau>T\}}] =$$

$$\frac{(c_H + k)H}{r} Q(\tau > T)e^{-rT} - J_H^* E^Q[e^{-rT} F_T(A_T, \bar{A}) 1_{\{\tau>T\}}],$$

C and L are calculated in Mjøs and Persson (2005). The result follows by collecting terms using the notation previously introduced. □

Valuation of equity at time 0.

The market value of equity as the residual claim follows from equations (5.4.9) and (5.4.11)

$$E_0^*(A) = A - D_0^*(A, 0) - H_0^*(A)) \tag{5.4.13}$$

5.4.3 Default risk of senior debt and hybrid capital.

To better understand the differences in default risk in senior debt and hybrid capital, we discuss the case when $t > T$ and disregard the embedded call.

In case of liquidation, $A_t = \bar{A}$, and the absolute priorities define the payoffs to the different claimants as

$$\begin{cases} D_L = \min(D, \bar{A}) \\ H_L = \min(H, (A_t - (\bar{A} - D))^+) \\ E_L = (\bar{A} - H_L - D_L)^+, \end{cases}$$

where D_L, H_L and E_L represent the liquidation payoffs to debt, hybrid capital and equity, respectively. Equation (5.3.9) and intuition makes it reasonable to assume that for an optimally leveraged firm, $\bar{A} < D + H$, i.e., equityholders receive no payoffs in case of bankruptcy. The analytically relevant cases in our setting are the following:

- *Case A:* $\bar{A} < D$, no payoff to hybrid capital and senior debt is risky in case of bankruptcy.
- *Case B:* $\bar{A} > D$, positive liquidation payoff to hybrid capital and senior debt is risk-free in case of bankruptcy.

The riskiness of hybrid capital in Case A is closer to equity, whilst hybrid capital in Case B resembles conventional junior debt. Our valuation formulas are generic and applicable in both situations.

Senior debt carries the conditional control rights in case of unpaid coupons, whilst hybrid capital in our specification does not. The issuer will still pay hybrid capital coupons to be permitted to pay dividends on common equity[11]. In our model, bankruptcy occurs if and when equity-holders find it no longer optimal to service the senior debt and hybrid-capital and therefore declare bankruptcy. In case of senior debt, the results are standard. For hybrid capital, we may assume two different asset-levels of distress. The first, previously denoted U, is when the company stop paying coupons. The bankruptcy asset level, $\bar{A}$, represents the state

[11]This is a common requirement to discipline equityholders from immediately exploiting holders of hybrid capital, although such behavior would effectively terminate any market for hybrid capital.

when the equityholders stop paying all coupons and let the company enter into bankruptcy. This level necessarily causes immediate default on all claims on the company. In a capital structure with both senior debt and hybrid capital, we have three levels of coupon payment:

$A > U$:	All coupons are paid.
$U \leq A > \bar{A}$:	Only senior debt coupons are paid.
$A \leq \bar{A}$:	No coupons are paid and the company enters bankruptcy.

We assume that all coupons are treated equally in the first and the latter situations and thus that the bankruptcy asset level $\bar{A}$ is optimized with respect to all coupon payments. However, as equation (5.4.8) shows, the risk of unpaid coupons to hybrid capital impacts the calculation of the optimal bankruptcy level through $y^{-\alpha}$ which reflects the relationship between the hybrid capital distress-level U and the bankruptcy-level $\bar{A}$.

5.5 Base case numerical example.

We illustrate our findings by showing the calibrated coupon-rates for senior debt and hybrid capital for a set of reasonable parameter values, given in Table 5.1. The calibrations meet two criteria. The first optimization find the short term bankruptcy level, B, which minimizes the combined value of senior debt and hybrid capital, the equivalent of maximizing the value of equity. Simultaneously, the coupon-rates of senior debt, c_S, and hybrid capital, c_H, are adjusted to achieve 'issue-at-par'for both claims. The valuation formulas for hybrid capital, equation (5.4.11), and senior debt, equation (5.4.9), as well as the combination, has global minimum values for B for reasonable parameter values which allows for optimizations. We will return later to discussions of sensitivities and alternative assumptions.

The comparison shows that hybrid capital featuring junior position, risk of lost coupons, embedded call option and coupon-rate step-up is a riskier claim demanding a significantly higher coupon rate compared to straight perpetual debt. Equivalently, senior debt becomes less risky if the capital structure includes hybrid capital than the case with only equity, demanding 28 bp lower coupon-rate for the

Table 5.1: Base case parameters

Parameters	Values	Explanations
$\delta_0 c$	3	Initial EBIT
μ	2 %	Drift of EBIT
σ	0.2	Volatility of EBIT
r	5 %	Riskfree interest rate
A_0	100	Total asset value at time 0
D	60	Face value of senior debt
H	15	Face value of hybrid capital
E	25	Face value of equity
T	10	Expiration year of option
k	100	Hybrid coupon rate step-up at time T
y	1.05	The factor on the relevant bankruptcy levels indicating the asset level where hybrid coupons are unpaid.

Table 5.2: Numerical calibrations

	Coupon-	Barrier levels		
Alternative calibrated claims	**rates**	B	$\bar{A}$	U
Senior, (Black&Cox) debt:				
- Par value: 60	5.512 %	n.r.	40.52	n.r.
- Par value: 75	5.849 %	n.r.	53.74	n.r.
Capital structure with senior debt and fully specified hybrid capital:				
- Senior debt ($D = 60$)	5.232 %	55.71	56.30	n.r.
- Hybrid capital ($H = 15$)	9.489 %	55.71	56.30	59.12

same share of the financing. If hybrid capital were replaced by the same amount of senior debt, senior debt becomes riskier and demands a rate-increase of 34 bp, both due to increased share of financing and the loss of hybrid capital in absorbing risk. The risk-absorption by hybrid capital helps explain its high coupon-rate.

5.6 Discussion of certain hybrid capital properties

In this section we discuss the most important features of hybrid capital separately. We focus on issues above and beyond those covered by Mjøs and Persson (2005).

5.6.1 The risk of unpaid coupons

The obligation to pay regular coupons is one of the defining characteristics of debt and equivalently motivates why hybrid capital typically need to have non-cumulative interest payments to qualify as Tier I risk capital. The critical implication is that any missed payment does not cause a default irrespective of whether coupons are permanently lost or just accumulated. An issuer will have to trade off any savings from missed coupon payments against any possible reputational damage potentially impacting future capital market access.[12] We do not explore these considerations further, but choose to treat the equityholders' decision when to stop paying coupons as exogenous in our model.

The asset threshold U, denoted the *omission level*, represents the level where hybrid coupons are dropped.[13]If asset values first drop below U and then again increase above U, only hybrid coupon payments for this period will remain unpaid

.

When $U = \bar{A}$, $y = 1$ and the optimal bankruptcy level equals the optimal bankruptcy level in the case of regular debt, cf. equation (5.3.9). For large values of U and y, $y^{-\alpha}$ approaches zero. In the limit the optimal bankruptcy asset level also goes towards zero since no hybrid coupon payments would ever be paid. For any other value of U, the optimal bankruptcy asset level will also be lower than with no risk of omitted coupons. The introduction of coupon risk via U

[12] Issuers considering dropping coupons are probably is such a critical position that the importance of future regular financing is comparatively small.

[13] In our formulas, we utilize the ratio y on the short term bankruptcy level B to define the omission level before time T.

into the model highlights the important connection between securities valuation and capital structure effects when analyzing hybrid capital. The omission level, U, impacts not only the value of hybrid capital as a claim on the company, but also bankruptcy risk, coupon rates and values of any other claims issued by the company. In the case of financial institutions, this relationship illustrates how hybrid capital with coupon-risk may reduce the bankruptcy risk compared to, e.g., regular subordinated debt and may therefore be a useful part of the risk-carrying capital. We analyze this in Section 7.

The coupon rate for the hybrid capital given $\bar{A}$ and U is calculated from expression (5.4.11) and applying (5.4.6) as

$$c_h = \frac{r}{F_0 y^{\alpha-\beta} - 1}\left(F_0 \frac{\bar{A} y^{-\beta}}{H} - 1\right). \tag{5.6.1}$$

Equation (5.6.1) shows that c_h is falling in y since $(\alpha - \beta) > -\beta$, both exponentials being positive as discussed in Appendix A. The intuition behind this somewhat surprising result is that the reduction in the optimal long term bankruptcy level $\bar{A}$ for an increasing y contributes more to the market value of hybrid capital than the lost value of the omitted coupons.

5.6.2 Embedded issuer's call option and coupon rate step-up

To increase the value of the option and thus the probability that the hybrid capital is called, most hybrid capital securities include a 75-150 basispoints coupon rate-increase as from the first possible expiration date of the option. The step-up reflects a natural difference in views between capital markets and regulators. The former prefers predictable, finite maturities for risky securities. The regulators, on the other hand, expects risk-carrying capital to have long maturities. Regulators solve this disagreement by limiting the size of the step-up to support the permanence of the hybrid capital.

In our valuation-formula (5.4.11) the step-up rate, k, only directly impacts payoffs and values as from time T since no step-ups happen earlier.

A hybrid issuer whose situation has deteriorated at the expiry of the option will not exercise the call, leaving the investor with finite maturity only in the cases where the credit has improved and the investor generally would have preferred to remain invested. It is intuitively easy to see why this call feature has value for the issuer.

The lower the eventual time T market value of the firm's assets, A_T, is, the higher the required coupon-rate of newly issued hybrid capital would be. It is therefore only optimal to let the option expire without exercise when the observed market *issue-at-par*-coupon rate at time T is higher than the existing contract coupon rate, including any agreed step-up. This situation will in our model occur for relatively low values of A_T, i.e., when the company is in a bad state. The existence of the step-up, k, will further reduce this level and make options called more frequently.

5.7 Analysis of alternatives and sensitivities

In this section we apply the common set of base case parameters from Table 5.1 in Section 5. The common set of assumptions provides comparability between the various examples, but the relatively small size of the hybrid capital will naturally cause the variations in the observed coupon rates to be small.

5.7.1 Sensitivities.

Figure 5.1 shows the calibrated coupon-rates for senior debt and hybrid capital assuming alternative levels of annual EBIT-volatility. The graph shows that the hybrid capital coupon rates are - as expected - very sensitive to firm volatility whilst senior debt coupon rates are not.

Table 5.3 shows the sensitivities of the calibrated coupon-rates for perpetual senior debt and hybrid capital for combinations of option maturities and EBIT-volatilities. As expected, the coupon rates for hybrid capital grow in both volatility and maturity, reflecting increased risk exposure. The senior debt coupon-rate is close to the risk free rate with rates which grow in volatility but fall in option maturity.

Figure 5.2 shows how the coupon rates change for different assumed ratios y between the coupon omission level and the bankruptcy level. As expected, the senior debt coupon rate has a very limited sensitivity to increased risk of lost coupons. The more surprising result is that the hybrid coupon rate increases only

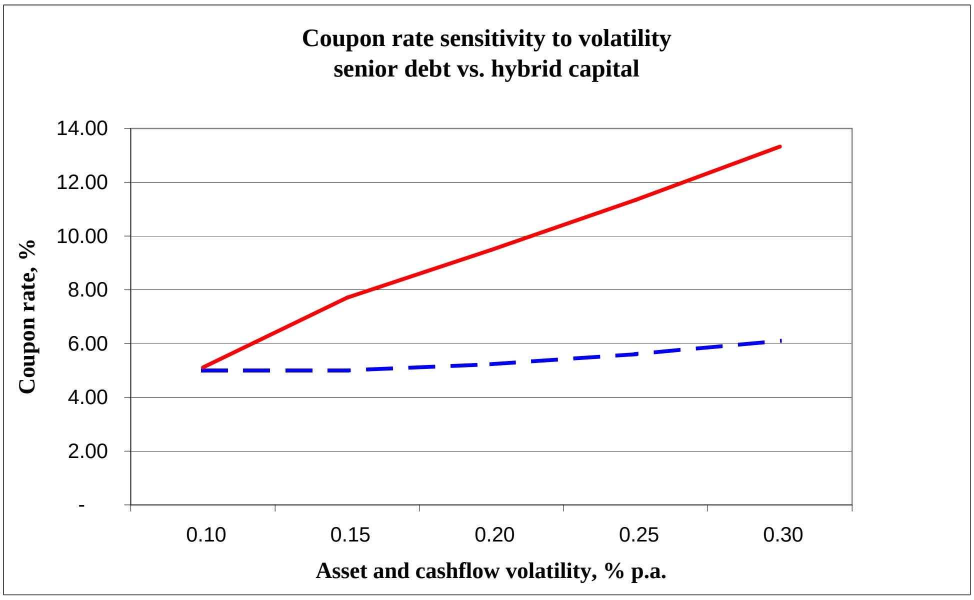


Figure 5.1: The figure shows the analytically calibrated issue-at-par coupon rates for hybrid capital and regular perpetual senior debt for assumed discrete levels of annual cashflow(EBIT)-volatility ranging from 0.10% to 0.30%. See Table 5.1 for parameter values.

for fairly high levels of y. This reflect the results from (5.6.1) that increased risk of lost coupons is balanced by an overall reduction in bankruptcy asset level.

We have also analyzed the coupon rate sensitivities to changes in the hybrid coupon-rate step-up at time T if the option is not exercised. An increase in the step-up from 0 to 250 bp decreases the senior debt coupon by 11 bp whilst the hybrid capital coupon increase by 37 bp.

5.7.2 A bank depositor's exposure.

The regulatory rationale for accepting hybrid capital as part of the risk capital of financial institutions is its effect on the overall riskiness of the institution and exposure of the depositors. In our model, the calibrated coupon-rate of senior

Table 5.3: Calibrated values of the coupon rates for senior perpetual debt and callable perpetual debt with embedded option, coupon rate step-up and exogenously defined risk of omitted hybrid coupons.

The table shows combinations of number of years to expiry of the option (T) and annual EBIT(δ)-volatility (σ). Parameters are given in Table 5.1

Coupon-rates (%)	*Volatility*(σ)					
	0.10		0.20		0.30	
Maturity(T)	Senior	Hybrid	Senior	Hybrid	Senior	Hybrid
5 years	5.000	5.115	5.182	10.179	5.958	15.178
10 years	5.000	5.115	5.232	9.489	6.108	13.328
20 years	5.000	5.138	5.292	8.649	6.188	12.028

debt ('quasi-deposits') may be seen as a reasonable proxy for the risk-exposure of depositors. We do a simplified test of the effects of hybrid capital by comparing the calibrated senior debt coupon rate for three stylized bank cases[14]:

- *Case A:* A traditional bank with 20% equity capital and the remainder as senior deposits.
- *Case B:* A bank that raises 10% hybrid capital as additional risk-capital whilst retaining the equity ratio from Case 1.
- *Case C:* A more aggressive bank that replaces 1/2 of the equity by hybrid capital leaving the sum of risk-carrying capital constant at 20%.

The capital structure compares reasonably well to a simplified financial institution and the results are shown in Table 5.4, assuming an optimized short term bankruptcy level B.

Our illustration shows that the required senior debt coupon rate is reduced if a stylized bank increases its risk-capital by raising hybrid capital in addition to its existing equity capital. We also see that replacing one half of equity by hybrid capital actually leads to a marginal reduction in the coupon rate required by senior claimants ('depositors'). This effect is obviously captured by the hybrid capital which requires a coupon rate increase of 270 bp in compensation for the increased

[14] The actual equity-ratio is somewhat larger than typically found in regular banks.

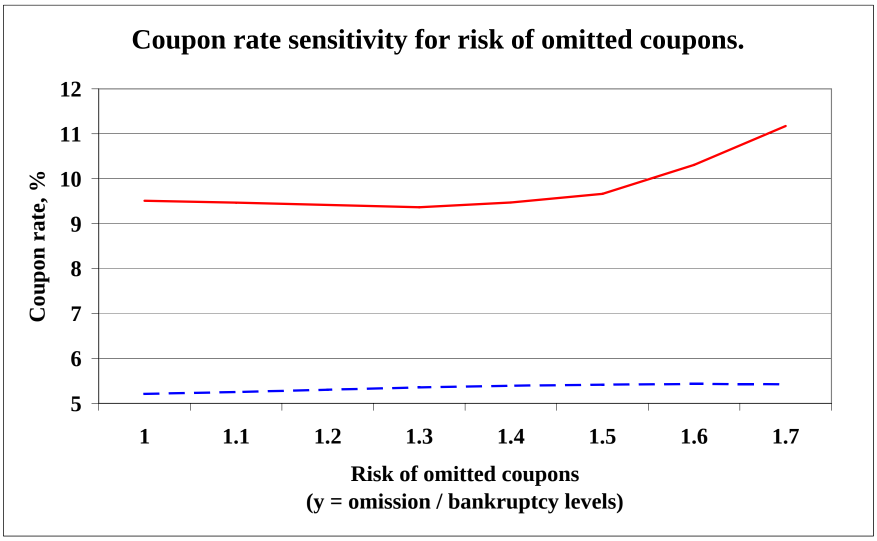


Figure 5.2: The figure shows the analytically calibrated issue-at-par coupon rates for senior debt and hybrid capital for increasing risk of omitted hybrid capital coupons. The risk of lost coupons is modelled as an exogenous asset level above the optimal bankruptcy level. If the asset process is between these two levels, no coupons are paid nor accumulated. The ratio between these levels is denoted y and represents the x-axis in the graph. See Table 5.1 for parameter values.

risk of Case C compared to Case B. This result, although preliminary, is supporting the usefulness of hybrid capital in absorbing risk in the capital structure of financial institutions.

We find in the regressions, except for the lowest volatility, that the calibrated levels of B and $\bar{A}$ are very close. This indicates that a model with one, constant bankruptcy level might be sufficiently precise as was the conclusion in Mjøs and Persson (2005). We have not studied this closer as of now.

Table 5.4: Calibrated values for the coupon rate c_H for hybrid capital and c_d for senior debt for alternative capital structures.

The coupon rates are calculated assuming a short term bankruptcy level B that minimizes the value of senior debt and is reported separately. Remaining parameters are given in Table 5.1.

Bank depositor's exposure	*Cases*		
Capital structure:	A	B	C
Senior debt	80	70	80
Hybrid capital	0	10	10
Equity capital	20	20	10
Senior debt coupon	6.007	5.632	5.850
Hybrid capital coupon	n.r.	6.807	9.510
B - short term bankruptcy	n.r.	57.16	69.77
$\bar{A}$ - long term bankruptcy	58.88	57.27	69.88

5.8 Conclusions and further research

Hybrid capital/preferred stock forms a significant part of the capital of many companies, particularly financial institutions, and is a challenging research area. The valuation of this highly structured instrument also impacts the optimal capital structure of the issuer. We believe that our paper represents a first attempt to develop complete theoretical valuation models for hybrid capital including the common features found in the marketplace, notably issuer call-option, increased coupon-rate at the expiry date of the option and the right to omit coupon payments. The methodology is developed by using barrier call options to allow for bankruptcy risk before the exercise date. The exogenous decision by the company to omit coupons on hybrid capital impacts bankruptcy level and cost-of-capital across all claims issued. This feature, together with the junior rank in liquidation, are critical requirements when regulatory bodies acknowledges hybrid capital as part of the solvency capital for banks and insurance companies.

We have left many areas for future research, in particular the issue regarding when a company will choose to omit coupons. Insight from game theory or informational economics can probably be applied in this respect. The effects of taxes and various bankruptcy practises are also left for future research. There is obviously also a need for research into both the theory and the empirical evidence regarding the issuing decision. The valuation models will need to be empirically tested on actual values and yields both in the primary and secondary market of such securities.

This paper represents the first comprehensive valuation paper on hybrid capital including its many features. We provide closed form solutions for values and insight into what drives values and risk-elements.

5.9 References

Acharya, Viral A., and Jennifer N. Carpenter, 2002, Corporate bond valuation and hedging with stochastic interest rates and endogeneous bankruptcy, *Review of Financial Studies* 15, 1355–1383.

Beatty, Anne, 2005, Do accounting changes affect the economic behavior of financial firms?, Working paper Fisher College of Business, Ohio State University.

Benston, George, Paul Irvine, Jim Rosenfeld, and Joseph F. jr. Sinkey, 2003, Bank capital structure, regulatory capital and securities innovation, *Journal of Money, Credit and Banking* 35, 301–322.

Black, Fisher, and J. Cox, 1976, Valuing corporate securities: Some effects of bond indenture provisions, *Journal of Finance* 31, 351–367.

Committee on Banking Supervision, Basle, 1988, International convergence of capital measurement and capital standards, Report Bank for International Settlements (BIS).

Emanuel, David, 1983, A theoretical model for valuing preferred stock, *Journal of Finance* 38, 1133–1155.

Engel, E., M. Erikson, and E. Maydew, 1999, Debt-equity hybrid securities, *Journal of Accounting Research* 37, 249–274.

Goldstein, Robert, Nengjiu Ju, and Hayne Leland, 2001, An ebit-based model of dynamic capital structure, *The Journal of Business* 74, 483–512.

Ingersoll, Jonathan E., 1977, A contingent-claims valuation of convertible securities, *Journal of Financial Economics* 4, 289–321.

Krishnan, C.N.V., and P.A. Laux, 2005, Misreaction, *Journal of Financial and Quantitative Analysis* 40, 403–434.

Leland, Hayne, 1994, Corporate debt value, bond covenants, and optimal capital structure, *Journal of Finance* 49, 1213–1252.

Merton, Robert C., 1974, On the pricing of corporate debt: The risk structure of interest rates, *Journal of Finance* 29, 449–470.

Mjøs, Aksel, and Svein-Arne Persson, 2005, Callable risky perpetual debt: Options, pricing and bankruptcy implications, Discussion Paper 2005/22 Norwegian School of Economics and Business Administration.

———, 2007, Default of multiple degrees: The risk of lost debt coupons, Unpublished working paper Norwegian School of Economics and Business Administration.

Santos, João A. C., 2000, Bank capital regulation in contemporary banking theory: A review of the literature, BIS Working Papers 90 Bank for International Settlements.

5.A Derivation of the market value for hybrid capital with risk of omitted coupons

In this case the hybrid coupons are only paid when $A_t > U$. The actual coupon-payment is thus dependent on A_t. We solve equation (5.3.4) separately for each region.

The general solution of equation (5.3.4) for the two regions is:

$$H(A) = \begin{cases} \frac{cH}{r} + K_1 A^{\alpha} + K_2 A^{-\beta} & \text{for } A \geq U \\ C_1 A^{\alpha} + C_2 A^{-\beta} & \text{for } \bar{A} \leq A \leq U. \end{cases} \quad (5.A.1)$$

where $\alpha, \beta > 0$, and are the positive solutions to the following equations:

$$\frac{1}{2}\sigma^2 \alpha(\alpha - 1) + \mu\alpha - r = 0$$
$$\frac{1}{2}\sigma^2 \beta(\beta + 1) - \mu\beta - r = 0$$

The solutions may be expressed as $\alpha = \lambda + \xi$ and $\beta = -\lambda + \xi$. Observe that $\alpha + \beta = 2\xi$, where

$$\lambda = \frac{\sigma^2 - 2\mu}{2\sigma^2},$$
$$\xi = \frac{1}{2\sigma^2}\sqrt{(\sigma^2 - 2\mu)^2 + 8\sigma^2 r}$$

The constants C_1 and K_1 in expression (5.A.1) can be found from the boundary conditions

$$\lim_{A \to \infty} H'(A) = 0 \Rightarrow K_1 = 0.$$

For $A = \bar{A}$, i.e. in bankruptcy, the boundary condition is $H(\bar{A}) = G$, where G represents the payoff to hybrid capital in case of bankruptcy. We obtain

$$C_1 \bar{A}^{\alpha} + C_2 \bar{A}^{-\beta} = G$$

$$C_1 = \bar{A}^{-\alpha}[G - C_2 \bar{A}^{-\beta}].$$

By inserting these expressions into the equation (5.A.1) we get

$$H(A) = \begin{cases} \frac{c_H H}{r} + K_2 A^{-\beta} & \text{for } A \geq U, \\ C_2(A^{-\beta} - \frac{A^{\alpha}}{\bar{A}^{\alpha+\beta}}) + G(\frac{A}{\bar{A}})^{\alpha} & \text{for } \bar{A} \leq A \leq U. \end{cases} \quad (5.A.2)$$

The first of the two additional equations necessary to determine K_2 and C_2 are found by assuming $H(A)$ is continuous for $A = U$, yielding

$$\frac{c_H H}{r} + K_2 U^{-\beta} = C_2 U^{-\beta} + (\frac{U}{\bar{A}})^{\alpha}[G - C_2 \bar{A}^{-\beta}] \tag{5.A.3}$$

$$K_2 = C_2(1 - (\frac{U}{\bar{A}})^{\alpha+\beta}) - \frac{c_H H}{r} U^{\beta} + G\frac{U^{\alpha+\beta}}{\bar{A}^{\alpha}}.$$

The second equation is found by the smooth pasting condition, i.e., assuming that the first derivatives of $H(A)$ are continuous at $A = U$,

$$-\beta K_2 U^{-(\beta+1)} = \alpha U^{(\alpha-1)} \bar{A}^{-\alpha}[\bar{A} - C_2 \bar{A}^{-\beta}] - \beta C_2 U^{-(\beta+1)},$$

$$K_2 = C_2(1 + \frac{\alpha}{\beta}(\frac{U}{\bar{A}})^{(\alpha+\beta)}) - G\frac{\alpha}{\beta}\frac{U^{\alpha+\beta}}{\bar{A}^{\alpha}} \tag{5.A.4}$$

The solution to equations (5.A.3) and (5.A.4) is found by equating them and solving for C_2:

$$C_2 = G\bar{A}^{\beta} - \frac{cH}{r}\frac{\beta}{(\alpha+\beta)}\frac{\bar{A}^{\alpha+\beta}}{U^{\alpha}}$$

The expression for C_2 may now be inserted into the expression for K_2:

$$K_2 = G\bar{A}^{\beta} - \frac{cH}{r}\frac{U^{\beta}}{(\alpha+\beta)}[\beta(\frac{\bar{A}}{U})^{\alpha+\beta} + \alpha].$$

and the complete solution based on equation (5.A.1) is

$$H(A) = \begin{cases} \frac{c_H H}{r} - [\frac{c_H H}{r}\frac{1}{(\alpha+\beta)}(\beta(\frac{\bar{A}}{U})^{\alpha} + \alpha(\frac{\bar{A}}{U})^{-\beta}) - G](\frac{A}{\bar{A}})^{-\beta} & \text{for } A \geq U, \\ \frac{c_H H}{r}\frac{\beta}{\alpha+\beta}(\frac{A}{U})^{\alpha} - [\frac{c_H H}{r}\frac{\beta}{\alpha+\beta}(\frac{A}{U})^{\alpha} - G](\frac{A}{\bar{A}})^{-\beta} & \text{for } \bar{A} \leq A \leq U. \end{cases} \tag{5.A.5}$$

This equation equals expression (5.3.7) when $U = \bar{A}$.

6

Default of multiple degrees: The risk of lost debt coupons

coauthored with Svein-Arne Persson [1]

Abstract

Motivated by the risk of stopped debt coupon payments from a leveraged company in financial distress we define a multi-level annuity contract which pays an annuity at a rate depending on the value of an underlying asset. The range of possible values of this asset is divided into a finite number of regions. The annuity rate is constant within each region, but may differ between the regions. The annuity payments end at a finite time horizon or upon an earlier bankruptcy, i.e., if the asset value process hits an absorbing boundary. Such annuities occur naturally in models of debt with credit risk in financial economics. Suspension of debt service under the US Chapter 11 provisions is one well-known example. We present closed-form formulas for the market value of multi-level annuities contracts when the market value of the underlying asset is assumed to follow a geometric Brownian motion.

[1]The authors thank Zheng Huang and Bernt Øksendal for comments and discusssions.

6.1 Introduction

Debt obligations are commonly defaulted on by companies in financial distress. A default may be defined as stopped or reduced coupon payments. The reduction in coupon payments may even be contractual for particularly risky debt. Irrespective of the specific causes of these non-payments, they represent a challenge for the valuation of corporate debt. Chapter 11 in the US bankruptcy code is an important example of regulations that allow a company to default without necessarily being declared bankrupt and liquidated.

Broadie, Chernov, and Sundaresan (2007) determine debt and equity values in a model which distinguishes between default and liquidation, motivated by US legislation. They analyze conflicts of interest between debtholders and shareholders and solve their model numerically using the binomial approach of Broadie and Kaya (2007). An example of *contractual* non-payments is hybrid risk capital for financial institutions which incorporates elements of both equity and debt. One common feature of such claims is the issuer's right to omit coupon payments under certain conditions, see e.g, Mjøs and Persson (2005a).

Motivated by the risk of lost coupon payments we define a multi-level annuity contract with a finite number of asset value levels of 'financial health'. As such, it allows for different, but constant, coupon rates between the different levels. Both coupon rates and financial health levels are assumed to be exogenous. We derive closed form solutions for the market value of the multi-level annuity contract both in the cases of finite and infinite horizons. The choice of the market price process of a company's assets rather than a common financial market factor such as, e.g., a interest rate, as exogenous stochastic process is motivated by the fact that distress is primarily caused by firm specific factors rather than general market factors.

Mathematically we solve a boundary value problem, see e.g., Øksendal (2005, Chapter 9). First, we find the market value of the multi-level annuity contract in the case of an infinite horizon using the standard assumption of *smooth-pasting*, see e.g., Dixit and Pindyck (1994). The multi-level annuity contract can be decomposed into a portfolio of simpler annuities. The market value of the multi-level annuity is calculated as the sum of the market values of these annuities. In the case of a finite horizon T we apply the standard argument that a finite annuity may be considered as an immediately starting infinite annuity from which another infinite annuity starting at time T is subtracted. To take into account the possibility of bankruptcy before the maturity date T, we replace the infinite annuity starting at time T with a call option on this annuity with exercise price equal to zero, exercisable only at time T. Given our model this option is a European down-and-out barrier call. Such options in a similar setting for debt with credit risk have been analyzed in Mjøs and Persson (2005b).

The bankruptcy asset level is modeled as an absorbing barrier In the structural debt modeling framework of Black and Cox (1976) and Leland (1994) . Both Broadie, Chernov, and Sundaresan (2007) and Mjøs and Persson (2005a) apply one additional financial distress (default) level. In this paper we extend this idea to multiple, although exogenous, financial health levels, see Figure 1. In order to interpret these levels as various degrees of financial distress the natural assumption is that the initial asset value is above all these levels. Our approach is general and our formulas are applicable for other assumptions regarding the initial asset level as well.

Our analytical solution may be applied to parts of the valuation problem of Broadie, Chernov, and Sundaresan (2007), although their model contains time-dependencies which severely complicate the use of closed-form solutions. Closed form solutions, as the ones we present, may increase computational speed, provide benchmarks for numerical solutions, and enhance economic understanding of the problems.

This paper is organized as follows: Section 2 contains the set-up. Section 3 treats the case of infinite horizon. Section 4 develops the results for the case of forward starting annuities. In Section 5 results from sections 3 and 4 are combined into results for finite horizon annuities. Conclusions and areas for further research are indicated in Section 6. Some technical results are collected in an appendix.

6.2 Set-up

A filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, Q)$ is given. In particular, Q represents a fixed *equivalent martingale measure*. We furthermore impose the standard frictionless, continuous time market assumptions of financial economics, see e.g., Duffie (2001).

We assume that the underlying asset process is given by a geometric Brownian motion

$$dA_t = \mu A_t dt + \sigma A_t dW_t,$$

where the initial value $A_0 = A$ is a constant. Here μ and σ are constants and W_t represents a standard Brownian motion.

Let T be the finite time horizon, let the constant C be the bankruptcy barrier, and define the stopping time τ as

$$\tau = \inf\{t \geq 0, A_t = C\}.$$

There are n additional constant levels or non-absorbing barriers $B_1, \ldots, B_n$ so that $B_1 > \cdots > B_n > C$. The constant annuity rate is c_1 when $A_t > B_1$, c_i when $B_{i-1} > A_t > B_i$, $i = 2, \ldots, n$, and c_{n+1} when $B_n > A_t > C$. All c_i's are constants. The initial value of the process is above the highest barrier, i.e., $A > B_1$.

Let r be the constant riskfree interest rate. Note that we allow $\mu \neq r$.

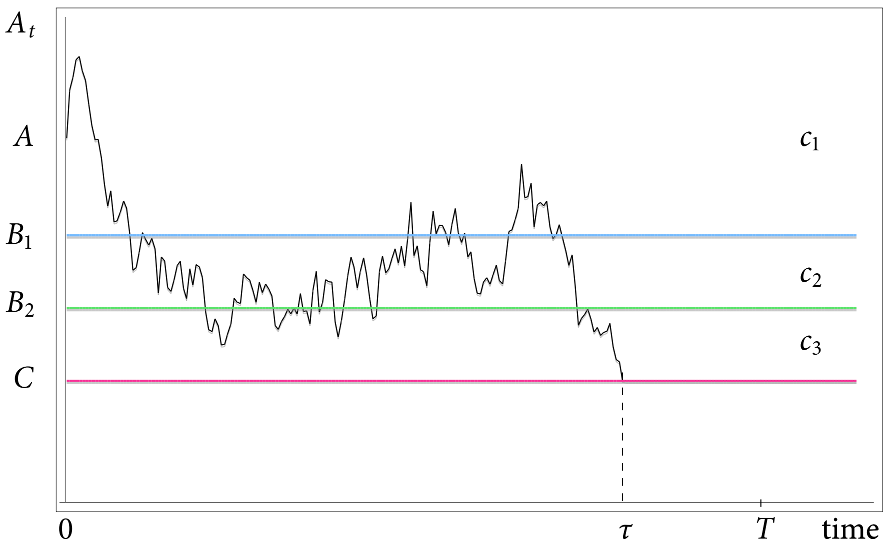


Figure 6.1: An illustration of the case where $n = 2$. The picture contains an example of a path of A_t and indicates in which regions the annuity rates are c_1, c_2, and c_3, respectively. Also, A, B_1, B_2, C, T, and τ are depicted.

We study the problem

$$V_0(A) = \mathrm{E}\left[\int_0^{\tau\wedge T}\sum_{i=1}^{n+1} c_i e^{-rs} 1\{B_{i-1} > A_s > B_i\} ds\right],$$

where $1\{\cdot\}$ denotes the standard indicator function, $E[\cdot]$ denotes the expectation under the equivalent martingale measure. For notational convenience $B_0 = \infty$ and $B_{n+1} = C$.

6.3 The infinite case

In this section we consider infinite horizon claims, assuming that $T = \infty$. Let f be the market value of an arbitrary infinite horizon claim on A_t, and denote the first and second order partial derivatives by $f_A = \frac{\partial f}{\partial A}$ and $f_{AA} = \frac{\partial^2 f}{\partial A^2}$, respectively. Then the partial differential equation

$$\frac{1}{2}\sigma^2 A^2 f_{AA} + \mu A f_A - rf + c(A) = 0 \tag{6.3.1}$$

holds, subject to appropriate boundary conditions. Here $c(A)$ represents the annuity payment rate (to be interpreted as dividends or coupons, depending on the nature of the claim) to the owner of the claim f. The general solution to the homogeneous part, obtained by letting $c(A) = 0$, of equation (6.3.1) is

$$f^*(A) = K_1 A^\alpha + K_2 A^{-\beta}, \tag{6.3.2}$$

where

$$\alpha = \frac{\frac{1}{2}\sigma^2 - \mu + \sqrt{(\frac{1}{2}\sigma^2 - \mu)^2 + 2\sigma^2 r}}{\sigma^2} \qquad (> 1)\,, \tag{6.3.3}$$

$$\beta = \frac{\mu - \frac{1}{2}\sigma^2 + \sqrt{(\frac{1}{2}\sigma^2 - \mu)^2 + 2\sigma^2 r}}{\sigma^2} \qquad \left(> \frac{2\mu}{\sigma^2} > 0\right). \tag{6.3.4}$$

and constants K_1 and K_2 determined by boundary conditions. The general solution to equation (6.3.1) is $f(A) = f^*(A) + f^s(A)$, where $f^s(A)$ is any *special solution* of equation (6.3.1).

First we derive market values of some simpler claims, which subsequently will be used in the valuation of the multi-level annuity. We denote initial market values by capital letters, possibly with subscripts, e.g., U, or $U(A, B)$ to emphasize the dependence on the initial value of the process and on the barrier.

6.3.1 The value of 1 at the initial hit of a barrier

Let U denote the time 0 market price of a claim which pays 1 when $A_t = B$ for the first time.

$$U(A, B) = \begin{cases} U^a = (\frac{A}{B})^{-\beta} & \text{when } A \geq B, \\ U^b = (\frac{A}{B})^{\alpha} & \text{when } A \leq B \end{cases} \tag{6.3.5}$$

The superscripts a and b signify that A_t hits the barrier from *above* or *below*, respectively. These results are standard, but we include a proof for the completeness of the exposition.

Proof. U does not pay any dividend so $c(A) = 0$ in expression (6.3.1). U^a is calculated from equation (6.3.2) using the boundary conditions $\lim_{A\to\infty} U^a = 0 \Rightarrow K_1 = 0$ and $U^a(B) = 1$. U^b is calculated from the boundary conditions $\lim_{A\to 0} U^b = 0 \Rightarrow K_2 = 0$ and $U^b(B) = 1$. □

We remark that $U(\cdot, B))$ is continuous at B, but does not satisfy the *smooth pasting* condition at B.

6.3.2 The value of an above-annuity

Let V_A denote the time 0 market price of an annuity which pays the rate c when $A_t > B$ (*above-annuity*).

$$V_A(A,B) = \begin{cases} V_A^a = \frac{c}{r}\left(1 - \frac{\alpha}{\alpha+\beta}\left(\frac{A}{B}\right)^{-\beta}\right) & \text{when } A \geq B, \\ V_A^b = \frac{c}{r}\frac{\beta}{\alpha+\beta}\left(\frac{A}{B}\right)^{\alpha} & \text{when } A \leq B. \end{cases} \quad (6.3.6)$$

Observe that $V_A^b = 0$ when $B = \infty$.

Proof. V_A pays c only when $A_t > B$, so in expression (6.3.1) $c(A) = c$ when $A_t > B$, and $c = 0$ otherwise. Observe that $f^S(A) = \frac{c}{r}$ solves equation (6.3.1) when $A > B$. The relevant boundary conditions are $\lim_{A\to\infty} V_A^a = \frac{c}{r} \Rightarrow K_1 = 0$ and $\lim_{A\to 0} V_A^b = 0 \Rightarrow K_2 = 0$. To determine K_2 for V_A^a and K_1 for V_A^b we require continuity and smooth pasting at B, i.e., $V_A^a(B) = V_A^b(B)$ and $\frac{\partial}{\partial A} V_A^a(B) = \frac{\partial}{\partial A} V_A^b(B)$. □

6.3.3 The value of a below-annuity

Let V_B denote the time 0 market price of an annuity which pays c when $A_t < B$ (*below-annuity*).

$$V_B(A,B) = \begin{cases} V_B^a = \frac{c}{r}\frac{\alpha}{\alpha+\beta}\left(\frac{A}{B}\right)^{-\beta} & \text{when } A \geq B, \\ V_B^b = \frac{c}{r}\left(1 - \frac{\beta}{\alpha+\beta}\left(\frac{A}{B}\right)^{\alpha}\right) & \text{when } A \leq B. \end{cases} \quad (6.3.7)$$

Observe that $V_B^b = \frac{c}{r}$ when $B = \infty$. Also observe that $V_B^b = \frac{c}{r} - V_A^b$, if $A < B$, an infinite annuity with payments below B equals an infinite annuity from which an annuity with payments only above B is subtracted. Also $V_A^a = \frac{c}{r} - V_B^a$, if $A > B$, an annuity with payments above B equals an infinite annuity from which an annuity with payments only below B is subtracted.

Proof. V_B pays c only when $A_t < B$, so in expression (6.3.1) $c(A) = c$ when $A_t < B$, and $c = 0$ otherwise. Observe that $f^s(A) = \frac{c}{r}$ solves equation (6.3.1) when $A < B$. The relevant boundary conditions are $\lim_{A\to\infty} V_B^a = 0 \Rightarrow K_1 = 0$ and $\lim_{A\to 0} V_B^b = 0 \Rightarrow K_2 = 0$. To determine K_2 for V_B^a and K_1 for V_B^b we also here require continuity and smooth pasting at B, i.e., $V_B^a(B) = V_B^b(B)$ and $\frac{\partial}{\partial A} V_B^a(B) = \frac{\partial}{\partial A} V_B^b(B)$. □

6.3.4 The value of a corridor-annuity

Let $V_C(A, B_i)$ denote the market value of an annuity which pays c when $B_{i+1} < A_t < B_i$ (*corridor-annuity*).

$$V_C(A,B_i) = \begin{cases} V_C^a = \frac{c}{r}\frac{\alpha}{\alpha+\beta}\left(\left(\frac{A}{B_i}\right)^{-\beta} - \left(\frac{A}{B_{i+1}}\right)^{-\beta}\right), & A \geq B_i, \\ V_C^c = \frac{c}{r}\left[1 - \frac{\beta}{\alpha+\beta}\left(\frac{A}{B_i}\right)^{\alpha} - \frac{\alpha}{\alpha+\beta}\left(\frac{A}{B_{i+1}}\right)^{-\beta}\right] & B_{i+1} \leq A \leq B_i, \\ V_C^b = \frac{c}{r}\frac{\beta}{\alpha+\beta}\left(\left(\frac{A}{B_{i+1}}\right)^{\alpha} - \left(\frac{A}{B_i}\right)^{\alpha}\right), & A \leq B_{i+1}. \end{cases} \quad (6.3.8)$$

Here V_C^c denotes the market value when the initial value A is inside the corridor.

Proof. The corridor annuity is equivalent to a below annuity with barrier B_i from which a below annuity with barrier B_{i+1} is subtracted,

$$V_C = V_B(A, B_i) - V_B(A, B_{i+1}).$$

Equivalently, the corridor annuity can be seen as an above annuity with barrier B_{i+1} from which an above annuity with barrier B_i is subtracted,

$$V_C = V_A(A, B_{i+1}) - V_A(A, B_i).$$

□

6.3.5 The values of the above, below, and corridor annuities in the case with bankruptcy risk

Let D_j denote the value of claim V_j where $j \in \{A, B, C\}$, including the lower absorbing barrier C representing bankruptcy.

$$D_j(A, B_i) = V_j(A, B_i) - V_j^b(C, B_i)U^a(A, C) \quad (6.3.9)$$

Proof. Upon bankruptcy, i.e., at time τ, the value of the claim V_j is $V_j^b(C, B_i)$. Because $C < B_i$ for all $i \leq n$, $V_j = V_j^b$. $V_j^b(C, B_i)$ therefore represents the reduction in value of the claim V_j due to bankrupty at the time of bankruptcy. The initial value of this claim is found by discounting by $U = U^a$ because $A > C$. □

Below we calculate the market values of the three annuities considered, including an absorbing bankruptcy barrier C. First the result for the above-annuity

$$D_A(A, B) = \tag{6.3.10}$$

$$\begin{cases} D_A^a = \frac{c}{r}\left[1 - \left(\frac{\alpha}{\alpha+\beta}(\frac{C}{B})^{-\beta} + \frac{\beta}{\alpha+\beta}(\frac{C}{B})^{\alpha}\right)(\frac{A}{C})^{-\beta}\right] & \text{when } A \geq B, \\ D_A^b = \frac{c}{r}\frac{\beta}{\alpha+\beta}\left[(\frac{A}{B})^{\alpha} - (\frac{C}{B})^{\alpha}(\frac{A}{C})^{-\beta}\right] & \text{when } A \leq B. \end{cases}$$

We then calculate the market value of the below annuity as

$$D_B(A, B) = \tag{6.3.11}$$

$$\begin{cases} D_B^a = \frac{c}{r}\left[\frac{\alpha}{\alpha+\beta}(\frac{C}{B})^{-\beta} + \frac{\beta}{\alpha+\beta}(\frac{C}{B})^{\alpha} - 1\right](\frac{A}{C})^{-\beta} & \text{when } A \geq B, \\ D_B^b = \frac{c}{r}\left[1 - \frac{\beta}{\alpha+\beta}(\frac{A}{B})^{\alpha} - (1 - \frac{\beta}{\alpha+\beta}(\frac{C}{B})^{\alpha})(\frac{A}{C})^{-\beta}\right] & \text{when } A \leq B. \end{cases}$$

Finally, the market value of the corridor annuity is

$$D_C(A, B_i) = \tag{6.3.12}$$

$$\begin{cases} D_C^a = & \frac{c}{r(\alpha+\beta)}\left[\alpha((\frac{C}{B_i})^{-\beta} - (\frac{C}{B_{i+1}})^{-\beta}) + \beta((\frac{C}{B_i})^{\alpha} - (\frac{C}{B_{i+1}})^{\alpha})\right](\frac{A}{C})^{-\beta} \\ & \text{for } A \geq B_i, \\ D_C^c = & \frac{c}{r(\alpha+\beta)}\left[\alpha + \beta - \beta(\frac{A}{B_i})^{\alpha} - \alpha(\frac{A}{B_{i+1}})^{-\beta} - \beta((\frac{C}{B_{i+1}})^{\alpha} - (\frac{C}{B_i})^{\alpha})(\frac{A}{C})^{-\beta}\right] \\ & \text{for } B_{i+1} \leq A \leq B_i, \\ D_C^b = & \frac{c}{r}\frac{\beta}{\alpha+\beta}\left[(\frac{A}{B_{i+1}})^{\alpha} - (\frac{A}{B_i})^{\alpha} - ((\frac{C}{B_{i+1}})^{\alpha} - (\frac{C}{B_i})^{\alpha})(\frac{A}{C})^{-\beta}\right], \\ & \text{for } A \leq B_{i+1}. \end{cases}$$

6.3.6 The value of an infinite multi-level annuity with bankruptcy risk

In this section we calculate the time zero market values of two multi-level annuity with infinite horizon. For simplicity we only treat the case where $A > B_1$.

Denote the time zero value of the infinite version of the multi-level annuity in the case of no bankruptcy risk (no absorbing barrier) by $V_0^\infty(A)$.

The cashflows from a multi-level annuity is identical to the cashflow of a particular portfolio of annuities. The annuities are: An above annuity with barrier B_1 and annuity rate c_1, $n-1$ corridor-annuities with coupon c_i and barriers B_{i-1} and B_i, $i = 2, \dots n$, and a below annuity with barrier B_n and annuity rate c_{n+1}.

The initial market value of the infinite multi-level annuity without bankruptcy risk is equal to the sum of the market values of the annuities in the portfolio which replicates the cashflow of the infinite multi-level annuity.

For $A > B_i$ for all i, the market value of an infinite multi-level annuity in the case of no default risk is

$$V_0^\infty(A) = \frac{c_1}{r} - \frac{\alpha}{\alpha+\beta}\sum_{i=1}^{n}\frac{c_i - c_{i+1}}{r}\left(\frac{A}{B_i}\right)^{-\beta}. \tag{6.3.13}$$

Proof. As explained above

$$V_0^\infty(A) = V_A^a(A, B_1) + \sum_{i=1}^{n.1} V_C^a(A, B_i) + V_B^a(A, B_n),$$

where the annuity rates in V_A, $V_C(A, B_{i-1})$ for $i = 2, \dots n$, and V_B are c_1, c_i for $i = 2, \dots n$, and c_{n+1}, respectively. The formula follows by direct calculations. □

The above arguments also hold when there is a constant default barrier C. Denote the time zero value of the infinite version of the multi-level annuity in the case of bankruptcy risk by $D_0^\infty(A)$.

For $A > B_i$ for all i, the market value of an infinite multi-level annuity in the case of default risk is

$$D_0^\infty(A) = \frac{c_1}{r} - \sum_{i=1}^{n+1}\frac{c_i - c_{i+1}}{r(\alpha+\beta)}\left(\alpha\left(\frac{C}{B_i}\right)^{-\beta} + \beta\left(\frac{C}{B_i}\right)^{\alpha}\right)\left(\frac{A}{C}\right)^{-\beta}, \tag{6.3.14}$$

where for notational convenience $c_{n+2} = 0$ and $B_{n+1} = C$.

Proof. Similar to the previous proof

$$D_0^\infty(A) = D_A^a(A, B_1) + \sum_{i=1}^{n-1} D_C^a(A, B_i) + D_B^a(A, B_n),$$

where the annuity rates in D_A, $D_C(A, B_{i-1})$ for $i = 2, \dots n$, and D_B are c_1, c_i for $i = 2, \dots n$, and c_{n+1}, respectively. The formula follows by direct calculations. □

6.4 Forward starting infinite annuities

In this section we calculate the time 0 market values of infinite annuities which start at a future time $T > 0$.

6.4.1 Some preliminary standard results

In the following section we apply the change of measure technique introduced in finance by Geman, El Karoui, and Rochet (1995).

Let Z be any $\mathcal{F}_T$-measurable event. Denote its associated indicator function by 1_Z.

First, the time zero market value of a claim with time T market value $U^a(A_T, B)$, given in expression (6.3.5), receivable at time T only if the event Z occurs is

$$\begin{aligned} V_\beta(B) &= \mathrm{E}[e^{-rT} U^a(A_T, B) 1_Z], \\ &= U^a(A, B)\mathrm{E}[1_Z e^{-\frac{1}{2}\sigma^2\beta^2 T - \sigma\beta W_T}], \\ &= U^a(A, B) Q^\beta(Z), \end{aligned} \tag{6.4.1}$$

where the probability measure Q^β is defined by $\frac{\partial Q^\beta}{\partial Q} = \exp(-\frac{1}{2}\sigma^2\beta^2 T - \sigma\beta T)$, and the dynamics of A_t under Q^β is $dA_t = (\mu - \sigma^2\beta)A_t dt + \sigma A_t dW_t$ (abusing notation by letting W_t also denote a standard Brownian motion under Q^β).

Similarily, the time zero market value of a claim with time T market value $U^b(A_T, B)$, given in expression (6.3.5), receivable at time T only if the event Z occurs is

$$\begin{aligned} V_\alpha(B) &= \mathrm{E}[e^{-rT} U^b(A_T, B) 1_Z], \\ &= U^b(A, B)\mathrm{E}[1_Z e^{-\frac{1}{2}\sigma^2\alpha^2 T + \sigma\alpha W_T}], \\ &= U^b(A, B) Q^\alpha(Z), \end{aligned} \tag{6.4.2}$$

where the probability measure Q^α is defined by $\frac{\partial Q^\alpha}{\partial Q} = \exp(-\frac{1}{2}\sigma^2\alpha^2 T + \sigma\alpha T)$ and the dynamics of A_t under Q^α is $dA_t = (\mu + \sigma^2\alpha)A_t dt + \sigma A_t dW_t$ (repeatedly abusing notation by letting W_t also denote a standard Brownian motion under Q^α).

The time 0 market value of a claim which pays 1 upon bankruptcy (when A_t hits C) if bankruptcy occurs after time T is

$$\begin{aligned} V_\tau(C) &= \mathrm{E}[e^{-rT}U^a(A_T, C)1\{\tau > T\}], \\ &= U^a(A, C)\mathrm{E}[e^{-\frac{1}{2}\sigma^2\beta^2 T - \sigma\beta W_T}1\{\tau > T\}], \\ &= U^a(A, C)Q^\beta(\tau > T), \\ &= (\frac{A}{C})^{-\beta}[N(d_1^\beta) - (\frac{A}{C})^{(\alpha+\beta)}N(-d_2^\beta)], \\ &= (\frac{A}{C})^{-\beta}N(d_1^\beta) - (\frac{A}{C})^{\alpha}N(-d_2^\beta), \end{aligned}$$

where $N(\cdot)$ represents the cumulative standard normal distribution function and d_1^β and d_2^β are defined in Appendix A.

Finally, the time 0 market value of a claim which pays 1 upon bankruptcy (when A_t hits C) if bankruptcy occurs before time T is

$$\begin{aligned} V_{\tau\le T}(C) &= U^a(A, C)Q^\beta(\tau \le T), \\ &= (\frac{A}{C})^{-\beta}N(-d_1^\beta) + (\frac{A}{C})^{\alpha}N(-d_2^\beta). \end{aligned}$$

6.4.2 Forward starting above annuity

Denote the time zero market value of a forward starting above annuity by W_A. Then

$$W_A = \frac{c}{r}\left(e^{-rT}N(d_3) - \frac{\alpha}{\alpha+\beta}(\frac{A}{B})^{-\beta}N(d_3^\beta) + \frac{\beta}{\alpha+\beta}(\frac{A}{B})^{\alpha}N(-d_3^\alpha)\right).$$

Proof.

$$\begin{aligned} W_A &= \mathrm{E}[e^{-rT}V_A(A_T, B)] \\ &= \mathrm{E}\left[e^{-rT}\left(V_A^a(A_T, B)1\{A_T > B\} + V_A^b(A_T, B)1\{A_T < B\}\right)\right]. \\ &= \mathrm{E}\left[e^{-rT}\frac{c}{r}\left((1 - \frac{\alpha}{\alpha+\beta}U^a(A_T, B)))1\{A_T > B\} + \frac{\beta}{\alpha+\beta}U^b(A_T, B))1\{A_T < B\}\right)\right]. \\ &= \frac{c}{r}\left(e^{-rT}Q(A_T > B) - \frac{\alpha}{\alpha+\beta}V_\beta(B) + \frac{\beta}{\alpha+\beta}V_\alpha(B)\right), \end{aligned}$$

where the event Z is specialized to $\{A_T > B\}$ for $V_\beta(B)$ and $\{A_T < B\}$ for $V_\alpha(B)$. Observe that

$$e^{-rT}Q(A_T > B) = e^{-rT}N(d_3),$$

where d_3 is defined in Appendix A. The result follows from the expressions (6.4.1) and (6.4.2). □

6.4.3 Forward starting below annuity

Denote the time zero market value of a forward starting above annuity by W_B. Then

$$W_B = \frac{c}{r}\left(e^{-rT}N(-d_3) + \frac{\alpha}{\alpha+\beta}(\frac{A}{B})^{-\beta}N(d_3^{\beta}) - \frac{\beta}{\alpha+\beta}(\frac{A}{B})^{\alpha}N(-d_3^{\alpha})\right).$$

Proof.

$$\begin{aligned}
W_B &= \mathrm{E}^Q[e^{-rT}V_B(A_T,B)] \\
&= \mathrm{E}^Q\left[e^{-rT}\left(V_B^a(A_T,B)1\{A_T > B\} + V_B^b(A_T,B)1\{A_T < B\}\right)\right]. \\
&= \mathrm{E}^Q\left[e^{-rT}\frac{c}{r}\left(\frac{\alpha}{\alpha+\beta}U^a(A_T,B)1\{A_T > B\} + (1-\frac{\beta}{\alpha+\beta}U^b(A_T,B))1\{A_T < B\}\right)\right]. \\
&= \frac{c}{r}e^{-rT}\left(Q(A_T < B) + \frac{\alpha}{\alpha+\beta}V_\beta(B) - \frac{\beta}{\alpha+\beta}V_\alpha(B)\right),
\end{aligned}$$

using similar definitions of Z as in the previous proof. By symmetry $Q(A_t \le B) = N(-d_3)$ and the result follows from the expressions (6.4.1) and (6.4.2). □

6.4.4 Forward starting corridor annuity

Denote the time zero market value of a forward starting corridor annuity, which pays c when $B_{i+1} < A_t < B_i$ by W_C. In this case we alter the notation of the d arguments of the standard cumulative nprmal distribution function to emphasize the dependence on the barrier, i.e., we write $d(B)$ instead of just d.

Then

$$\begin{aligned}
W_C = \frac{c}{r}e^{-rT}&[N(d_3(B_{i+1})) - N(d_3(B_i))] \\
&- \frac{c}{r}\frac{\alpha}{\alpha+\beta}A^{-\beta}[B_{i+1}^{\beta}N(d_3^{\beta}(B_{i+1})) - B_i^{\beta}N(d_3^{\beta}(B_i))] \\
&+ \frac{c}{r}\frac{\beta}{\alpha+\beta}A^{\alpha}[B_{i+1}^{-\alpha}N(-d_3^{\alpha}(B_{i+1})) - B_i^{-\alpha}N(-d_3^{\alpha}(B_i))].
\end{aligned}$$

Proof. As before,

$$W_C = W_A(B_{i+1}) - W_A(B_i).$$

Direct calculations. □

6.4.5 Forward starting default claims

Denote by X the time zero market value of the forwarding starting claim V delivered at time T upon no prior default. Then

$$X = \mathrm{E}^{\mathrm{Q}}[e^{-rT}V(A_T)1\{\tau > T\}].$$

Forward starting above annuity

Denote the time zero market value of a forward starting above annuity with default risk by X_A. Then

$$\begin{aligned}
X_A &= E[e^{-rT}V_A(A_T)1\{\tau > T\}] \\
&= \mathrm{E}\left[e^{-rT}\left(V_A^a(A_T, B)1\{A_T > B\} + V_A^b(A_T, B)1\{A_T < B\}\right)1\{\tau > T\}\right] \\
&= \frac{c}{r}\left(e^{-rT}\mathrm{Q}(A_T > B, \tau > T) - \frac{\alpha}{\alpha+\beta}(\frac{A}{B})^{-\beta}Q^{\beta}(A_T > B, \tau > T)\right) \\
&+ \frac{c}{r}\frac{\beta}{\alpha+\beta}(\frac{A}{B})^{\alpha}Q^{\alpha}(A_T \le B, \tau > T),
\end{aligned}$$

where the Q, Q^{β}, and Q^{α} probabilities are given in Appendix A.

Forward starting below annuity

$$\begin{aligned}
X_B &= E[e^{-rT}V_B(A_T)1\{\tau > T\}] \\
&= \mathrm{E}\left[e^{-rT}\left(V_B^a(A_T, B)1\{A_T > B\} + V_B^b(A_T, B)1\{A_T < B\}\right)1\{\tau > T\}\right] \\
&= \frac{c}{r}\left(e^{-rT}\mathrm{Q}(A_T \le B, \tau > T) + \frac{\alpha}{\alpha+\beta}(\frac{A}{B})^{-\beta}Q^{\beta}(A_T > B, \tau > T)\right) \\
&- \frac{c}{r}\frac{\beta}{\alpha+\beta}(\frac{A}{B})^{\alpha}Q^{\alpha}(A_T \le B, \tau > T).
\end{aligned}$$

Forward starting defaultable above annuity

Denote the time zero market value of a forward starting above annuity with default risk by X_A. Then

$$\begin{aligned}
X_C &= E\left[e^{-rT} D_A(A_T) 1\{\tau > T\}\right] \\
&= \mathrm{E}\left[e^{-rT}\left(D_A^a(A_T, B) 1\{A_T > B\} + D_A^b(A_T, B) 1\{A_T < B\}\right) 1\{\tau > T\}\right] \\
&= \frac{c}{r}\left(e^{-rT} Q(A_T > B, \tau > T) - \frac{\alpha}{\alpha+\beta}\left(\frac{A}{B}\right)^{-\beta} Q^{\beta}(A_T > B, \tau > T)\right) \\
&+ \frac{c}{r}\frac{\beta}{\alpha+\beta}\left(\left(\frac{A}{B}\right)^{\alpha} Q^{\alpha}(A_T \leq B, \tau > T) - \left(\frac{C}{B}\right)^{\alpha}\left(\frac{A}{C}\right)^{-\beta} Q^{\beta}(\tau > T)\right).
\end{aligned}$$

Forward starting defaultable below annuity

$$\begin{aligned}
X_D &= E\left[e^{-rT} D_B(A_T) 1\{\tau > T\}\right] \\
&= \mathrm{E}\left[e^{-rT}\left(D_B^a(A_T, B) 1\{A_T > B\} + D_B^b(A_T, B) 1\{A_T < B\}\right) 1\{\tau > T\}\right] \\
&= \frac{c}{r}\left(e^{-rT} Q(A_T \leq B, \tau > T) + \frac{\alpha}{\alpha+\beta}\left(\frac{A}{B}\right)^{-\beta} Q^{\beta}(A_T > B, \tau > T)\right) \\
&+ \frac{c}{r}\frac{\beta}{\alpha+\beta}\left(\left(\frac{C}{B}\right)^{\alpha}\frac{A}{C}\right)^{-\beta} Q^{\beta}(\tau > T) - \left(\frac{A}{B}\right)^{\alpha} Q^{\alpha}(A_T \leq B, \tau > T)\Big) \\
&- \frac{c}{r} Q^{\beta}(\tau > T)\left(\frac{A}{C}\right)^{-\beta}.
\end{aligned}$$

These above and below annuities can in the, by now, familiar way be combined into corridor and multi-level annuities.

6.5 The case of a finite T

In this section we give some examples of how the results for the previous annuities, all with infinite horizon, can be combined into results for annuities with finite horizon. Our results are based on the idea that a finite annuity may be considered as an immediately starting infinite annuity from which the time zero value of another infinite annuity starting at the future date T, is subtracted if $\tau > T$, i.e., bankruptcy has not occurred prior to time T.

We assume in this section that $A > B$.

6.5.1 Finite above annuity without default risk

The time zero market price of a default free finite above annuity is calculated as

$$\begin{aligned} V_A^T(A,B) &= V_A(A,B) - W_A(A,B) \\ &= \frac{c}{r} - \frac{c}{r}\left(e^{-rT}N(d_3) + \frac{\alpha}{\alpha+\beta}(\frac{A}{B})^{-\beta}N(-d_3^{\beta}) + \frac{\beta}{\alpha+\beta}(\frac{A}{B})^{\alpha}N(-d_3^{\alpha})\right). \end{aligned}$$

6.5.2 Finite below annuity without default risk

The time zero market price of a default free finite below annuity is calculated as

$$\begin{aligned} V_B^T(A,B) &= V_B(A,B) - W_B(A,B) \\ &= \frac{c}{r}\left(-e^{-rT}N(-d_3) + \frac{\alpha}{\alpha+\beta}(\frac{A}{B})^{-\beta}N(-d_3^{\beta}) + \frac{\beta}{\alpha+\beta}(\frac{A}{B})^{\alpha}N(-d_3^{\alpha})\right). \end{aligned}$$

Observe that the time zero value of a finite annuity which pays both above and below B is $V_A^T(A,B) + V_B^T(A,B) = \frac{c}{r}(1-e^{-rT}$, a familiar result.

6.5.3 Finite corridor annuity without default risk

The time zero market price of a default free finite corridor annuity is calculated as

$$\begin{aligned} V_C^T(A,B_i) &= V_A^T(A,B_{i+1}) - V_A^T(A,B_i) \\ &= \frac{c}{r}\xi, \end{aligned}$$

where

$$\begin{aligned} \xi = e^{-rT}&(N(d_3(B_i) - N(d_3(B_{i+1})) \\ &+ \frac{\alpha}{\alpha+\beta}\left((\frac{A}{B_i})^{-\beta}N(d_3^{\beta}(B_i)) - (\frac{A}{B_{i+1}})^{-\beta}N(d_3^{\beta}(B_{i+1}))\right) \\ &\quad + \frac{\beta}{\alpha+\beta}\left((\frac{A}{B_i})^{\alpha}N(-d_3^{\alpha}(B_i)) - (\frac{A}{B_{i+1}})^{\alpha}N(-d_3^{\alpha}(B_{i+1}))\right). \end{aligned}$$

6.5.4 Finite above annuity *with* default risk

The time zero market price of finite above annuity with default risk is calculated as

$$\begin{aligned} D_A^T(A,B) &= V_A(A,B) - X_A(A,B) - V_A^b(C,B)V_{\tau\le T}(C) \\ &= D_A(A,B) - X_C(A,B) \\ &= \frac{c}{r}\psi, \end{aligned}$$

where

$$\begin{aligned} \psi = 1 - \frac{\alpha}{\alpha+\beta}(\frac{A}{B})^{-\beta}(1 - Q^\beta(A_T > B, \tau > T)) \\ - \frac{\beta}{\alpha+\beta}\Big(Q^\alpha(A_T \le B, \tau > T) + (\frac{C}{B})^\alpha(\frac{A}{C})^{-\beta}Q^\beta(\tau \le T)\Big) \\ - e^{-rT}Q(A_T > B, \tau > T). \end{aligned}$$

6.5.5 Finite below annuity *with* default risk

The time zero market price of finite above annuity with default risk is calculated as

$$\begin{aligned} D_B^T(A,B) &= V_B(A,B) - X_B(A,B) - V_B^b(C,B)V_{\tau\le T}(C) \\ &= D_B(A,B) - X_D(A,B) \\ &= \frac{c}{r}\eta, \end{aligned}$$

where

$$\begin{aligned} \eta = \frac{\alpha}{\alpha+\beta}(\frac{A}{B})^{-\beta}(1 - Q^\beta(A_T \le B, \tau > T)) \\ + \frac{\beta}{\alpha+\beta}\Big(Q^\alpha(A_T \le B, \tau > T)(\frac{A}{B})^\alpha + (\frac{C}{B})^\alpha(\frac{A}{C})^{-\beta}Q^\beta(\tau \le T)\Big) \\ - e^{-rT}Q(A_T \le B, \tau > T) - (\frac{A}{C})^{-\beta}Q^\beta(\tau \le T). \end{aligned}$$

6.6 Conclusions and areas of further research

We present closed form solutions for the market value of multi-level annuities applicable to debt with credit risk. Possible applications include US Chapter 11 regulations, hybrid capital for financial institutions, and strategic debt service. Our results may also be applied to more sophisticated situations with endogenous coupons and financial health levels.

It is also straight forward to generalize our results to the case where all barriers, including the bankruptcy barrier, are time dependent and exponential, i.e., on the form $B_t = Be^{\gamma t}$ for a constant γ, identical for all barriers. We may also envision other mathematical forms of the barriers.

An interesting extension of our results would be to include jumps in the underlying asset value process.

6.7 References

Black, Fisher, and J. Cox, 1976, Valuing corporate securities: Some effects of bond indenture provisions, *Journal of Finance* 31, 351–367.

Broadie, Mark, Mikhail Chernov, and Suresh Sundaresan, 2007, Optimal debt and equity values in the presence of chapter 7 and chapter 11, *Journal of Finance* 62, 1341–1377.

Broadie, Mark, and Özgür Kaya, 2007, A binomial lattice method for pricing corporate debt and modeling chapter 11 proceedings, *Journal of Financial and Quantitative Analysis* Forthcoming.

Dixit, Avinash K., and Robert S. Pindyck, 1994, *Investment under Uncertainty* (Princeton University Press: New Jersey).

Duffie, David, 2001, *Dynamic Asset Pricing Theory. Third Edition* (Princeton University Press: Princeton, NJ, USA).

Geman, Hélyette, Nicole El Karoui, and Jean-Charles Rochet, 1995, Changes of numeraire, changes of probability measure and option pricing, *Journal of Applied Probability* 32, 443–458.

Leland, Hayne, 1994, Corporate debt value, bond covenants, and optimal capital structure, *Journal of Finance* 49, 1213–1252.

Mjøs, Aksel, and Svein-Arne Persson, 2005a, Bundled financial claims: A model of hybrid capital, Discussion paper, Norwegian School of Economics and Business Administration Unpublished.

———, 2005b, Callable risky perpetual debt: Options, pricing and bankruptcy implications, Discussion Paper 2005/22 Norwegian School of Economics and Business Administration.

Øksendal, Bernt, 2005, *Stochastic Differential Equations. Sixth edition* (Springer-Verlag: Berlin Heidelberg).

6.A Some standard results

In this appendix we consider the process below under different probability measures. Consider

$$X_t = \ln(A_t) = \ln(A) + \hat{\mu}t + \sigma W_t,$$

where W_t is defined under a fixed probability measure P, and $\ln(A)$, $\hat{\mu}$, and σ are constants. The process X_t represents the logarithmic version of the process A_t used in the paper. Define the stopping time

$$\tau = \inf\{t : X_t = \ln(C)\}$$

The following results are standard

$$P(\tau > T) = N(\frac{\ln(\frac{A}{C}) + \hat{\mu}T}{\sigma\sqrt{T}}) - (\frac{A}{C})^{\frac{-2\hat{\mu}}{\sigma^2}} N(-\frac{\ln(\frac{A}{C}) - \hat{\mu}T}{\sigma\sqrt{T}}).$$

$$P(X_t > \ln(B), \tau > T) =$$
$$N(\frac{\ln(\frac{A}{B}) + \hat{\mu}T}{\sigma\sqrt{T}}) - (\frac{A}{C})^{\frac{-2\hat{\mu}}{\sigma^2}} N(-\frac{\ln(\frac{A}{C}) + \ln(\frac{B}{C}) - \hat{\mu}T}{\sigma\sqrt{T}}).$$

Observe that $\lim_{B \downarrow C} P(X_t > \ln(B), \tau > T) = P(\tau > T)$. Trivially,

$$P(X_t < \ln(B), \tau > T) = N(\frac{\ln(\frac{A}{C}) + \hat{\mu}T}{\sigma\sqrt{T}}) - N(\frac{\ln(\frac{A}{B}) + \hat{\mu}T}{\sigma\sqrt{T}}) +$$
$$(\frac{A}{C})^{\frac{-2\hat{\mu}}{\sigma^2}} \left(N(-\frac{\ln(\frac{A}{C}) + \ln(\frac{B}{C}) - \hat{\mu}T}{\sigma\sqrt{T}}) - N(-\frac{\ln(\frac{A}{C}) - \hat{\mu}T}{\sigma\sqrt{T}}) \right).$$

6.A.1 Probability measure Q

Under the probability measure Q

$$\hat{\mu} = \mu - \frac{1}{2}\sigma^2.$$

Then

$$Q(\tau > T) = N(d_1) - (\frac{A}{C})^{\alpha-\beta} N(-d_2),$$

where

$$d_1 = \frac{\ln(\frac{A}{C}) + (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}},$$

and

$$d_2 = \frac{\ln(\frac{A}{C}) - (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}.$$

Also,

$$Q(X_t > \ln(B), \tau > T) = N(d_3) - (\frac{A}{C})^{\alpha-\beta} N(-d4),$$

where

$$d_3 = \frac{\ln(\frac{A}{B}) + (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}},$$

$$d_4 = \frac{\ln(\frac{A}{C}) + \ln(\frac{B}{C}) - (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}},$$

and

$$Q(X_t < \ln(B), \tau > T) = N(d_1) - N(d_3) + (\frac{A}{C})^{\alpha-\beta}(N(-d_4) - N(-d_2)).$$

6.A.2 Probability measure Q^α

Under the probability measure Q^α

$$\hat{\mu} = \mu + \sigma^2\alpha - \frac{1}{2}\sigma^2.$$

Then

$$Q^\alpha(\tau > T) = N(d_1^\alpha) - (\frac{A}{C})^{-(\alpha+\beta)} N(-d_2^\alpha),$$

where

$$d_1^\alpha = \frac{\ln(\frac{A}{C}) + (\mu + \sigma^2\alpha - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}},$$

and

$$d_2^\alpha = \frac{\ln(\frac{A}{C}) - (\mu + \sigma^2\alpha - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}.$$

Also,

$$Q^\alpha(X_t > \ln(B), \tau > T) = N(d_3^\alpha) - (\frac{A}{C})^{-(\alpha+\beta)} N(-d_4^\alpha),$$

where

$$d_3^\alpha = \frac{\ln(\frac{A}{B}) + (\mu + \sigma^2\alpha - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}},$$

$$d_4^\alpha = \frac{\ln(\frac{A}{C}) + \ln(\frac{B}{C}) - (\mu + \sigma^2\alpha - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}},$$

and

$$Q^\alpha(X_t < \ln(B), \tau > T) = N(d_1^\alpha) - N(d_3^\alpha) + (\frac{A}{C})^{-(\alpha+\beta)}(N(-d_4^\alpha) - N(-d_2^\alpha)).$$

6.A.3 Probability measure Q^β

Under the probability measure Q^β

$$\hat{\mu} = \mu - \sigma^2\beta - \frac{1}{2}\sigma^2.$$

Then

$$Q^\beta(\tau > T) = N(d_1^\beta) - (\frac{A}{C})^{(\alpha+\beta)}N(-d_2^\beta),$$

where

$$d_1^\beta = \frac{\ln(\frac{A}{C}) + (\mu - \sigma^2\beta - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}},$$

and

$$d_2^\beta = \frac{\ln(\frac{A}{C}) - (\mu - \sigma^2\beta - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}.$$

Also,

$$Q^\beta(X_t > \ln(B), \tau > T) = N(d_3^\beta) - (\frac{A}{C})^{(\alpha+\beta)}N(-d_4^\beta),$$

where

$$d_3^\beta = \frac{\ln(\frac{A}{B}) + (\mu - \sigma^2\beta - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}},$$

$$d_4^\beta = \frac{\ln(\frac{A}{C}) + \ln(\frac{B}{C}) - (\mu - \sigma^2\beta - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}},$$

and

$$Q^\beta(X_t < \ln(B), \tau > T) = N(d_1^\beta) - N(d_3^\beta) + (\frac{A}{C})^{(\alpha+\beta)}(N(-d_4^\beta) - N(-d_2^\beta)).$$