

Essays on Compound Contingent Claims and Financial Guarantees

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Introduction

The topic of this dissertation is the valuation and hedging of so-called *exotic* contingent claims. We use the term exotic in the same way as in Musiela and Rutkowski (1997), i.e., every contingent claim which is not standard European or American is considered exotic. The topic is approached with the by now widely accepted technique termed *arbitrage pricing*. Arbitrage pricing was initiated in the highly celebrated works of Black and Scholes (1973) and Merton (1973). Later extensions, which this dissertation relies heavily on, were made by Harrison and Kreps (1979) and Harrison and Pliska (1981).

The results of Harrison and Kreps (1979) and Harrison and Pliska (1981) roughly state that the market value of a financial asset is the expected deflated cash flow under an equivalent martingale probability measure Q , where the deflator is the money market account that accrues the short-term interest rate. It has become customary to denote this probability measure with the somewhat unfortunate terms *the* equivalent martingale measure Q and *the* risk-neutral measure. This is by now standard terminology, from which we shall not deviate. However, as shown by Geman, El Karoui, and Rochet (1995), there is not only one, but several equivalent martingale measures; each associated with its own deflator. A deflator also goes under the name *numeraire*. Throughout the dissertation we use several equivalent probability measures with their respective numeraires. Each probability measure is carefully chosen to help solve the problem at hand.

In a *general equilibrium model* the market values of stocks and bonds, assets often referred to as *primary traded assets*, are typically endogenously determined by the preferences of the agents in the economy and by underlying technology factors. In a model where arbitrage pricing is used, the market values of these primary traded assets are exogenously given by stochastic processes, and the model is therefore only a *partial* equilibrium model.

Arbitrage pricing has proved to be very fruitful, not only in theory, but also in practical applications. This, not *despite*, but perhaps *because of* the lack of generality compared to the general equilibrium models. Modelling the investors' preferences is a non-trivial exercise that is superfluous when using arbitrage pricing. The insight that was brought to the financial markets

about the possibility to value financial derivatives without any knowledge about investor preferences and risk premiums has sometimes been credited for much of the rapid innovation that has taken place in the financial markets during the last 20-30 years. For there is no doubt that, especially within the fixed income and derivative markets, where several ideas from arbitrage pricing have been implemented, the changes have been dramatically over this time period. To cite Duffie (1996) "On the applied side, markets have experienced an explosion of new valuation techniques, hedging applications, and security innovation, much of this based on the Black-Scholes and related arbitrage models."

The works of Black and Scholes (1973) and Merton (1973) were mainly done under the assumption of constant or deterministic interest rates, although stochastic interest rates were considered by Merton (1973). At the time, this was perhaps not such an unreasonable assumption, since both interest rates and currency exchange rates had been under strict regulation in most parts of the world. However, as financial markets have been deregulated, also interest rate risk has become important. Stochastic models for interest rates are therefore needed. This dissertation relies heavily on the general term structure model by Heath, Jarrow, and Morton (1992) and the extensions made by Amin and Jarrow (1992). This is a fairly general framework in the sense that it allows several different specifications of the "input" in the model, e.g., volatility structures for interest rates and for the return on risky assets. To obtain more explicit results in terms of closed form solutions, we will mainly work within a *Gaussian* Heath, Jarrow, and Morton framework. By no means will it be asserted that this is a realistic model; it is only chosen for its analytical tractability. However, also a non-Gaussian model will be touched upon.

The dissertation is not concerned with the question, though important it may be, "Should economic agents expose themselves to the risk inherent in the claims analysed in this dissertation?" Nor is it concerned with the question, given that the above question was answered by a "Yes", "What is the *optimal* exposure to these claims?" Instead we take the perhaps somewhat arrogant approach and say "If there is a demand for the claims, it must be so because there is a need", and, thus, knowledge of such claims is important.

The above has been an attempt to give the reader some insight about the framework in which this dissertation is written within. The dissertation is divided into six main chapters and three appendices. Each chapter is written as a self-contained paper and can be read in the order favoured by the reader. The appendices contain material that we have not found suitable to include in the main chapters.

Several exotic contingent claims are analysed throughout the dissertation. However, two claims have received more attention than the others, namely a *compound option* and a *multi-period rate of return guarantee*. The

rest of this introduction gives a short overview of some of the literature related to the analysis done in the dissertation. A closer description of the different parts of the dissertation is also given.

A Short Overview of some of the Existing Literature A compound contingent claim is a contingent claim where the underlying asset also is a contingent claim. Geske (1977) was the first to analyse this type of claim. He considered a risky coupon-bearing bond where the bondholders have the possibility to default on the coupon payments. Geske (1979) used the same approach to value an option on a stock, or more precisely, an option on the equity of a leveraged firm. It is well-known that the equity of a leveraged firm can be viewed upon as a call option on the value of the firm; hence, the option on the stock of a leveraged firm is a compound option. The work on compound options was continued in Hodges and Selby (1987). Fischer (1978) and Margrabe (1978) considered the option to exchange one asset for another. Carr (1988) extended this to include the possibility to exchange one asset for an (exchange) option, i.e., a compound exchange option. He found several interesting applications for this kind of option. Further generalisations were made by Geman et al. (1995) who analysed a compound option under stochastic interest rates. Scaillet (1996) presented pricing formulas for compound and exchange options on zero-coupon bonds, coupon bonds, and yields in the framework of affine term structure models.

The analysis of rate of return guarantees, in particular in the form found in various life insurance products, seems to have been initiated by the seminal paper of Brennan and Schwartz (1976). They showed that a *maturity* guarantee is the same as a portfolio of some risky asset and a put option on this asset. This portfolio gives the investor, regardless of how low the return on the asset becomes, a cash amount at the maturity of the option that can never fall below the exercise price of the option; therefore the name maturity guarantee. Brennan and Schwartz (1976), among other things, calculated the market value of the guarantee and derived hedging strategies.

There have been several extensions and modifications of the results and the assumptions made by Brennan and Schwartz (1976). Just mentioning a few, and without going into details, Brennan and Schwartz (1979) investigated the usefulness of the hedging strategies for the guarantee derived by Brennan and Schwartz (1976) when the hedge portfolio can no longer be continuously rebalanced and transaction costs are present. They found the hedging strategies to give a considerable reduction in the risk exposure from the guarantee. A similar analysis was performed by Boyle and Hardy (1997). In addition to approach the analysis of the guarantee by ideas from financial economics, they also used a simulation model from actuarial sciences to value the guarantee. Grosen and Jørgensen (1997) opened for the possibility of early exercise of the guarantee. In financial terms the guarantee is then

of American type and can be analysed as an optimal stopping problem. The added flexibility that comes from the possibility of early exercise is termed *the surrender option*. Stochastic interest rates were introduced by Bacinello and Ortu (1993) and Bacinello and Ortu (1994). Guarantees are typically embedded in life and pension insurance contracts where the premiums are paid periodically. Both Bacinello and Ortu (1994) and Nielsen and Sandmann (1995) analysed guarantees with periodical premiums under stochastic interest rates. It should be mentioned that periodical premiums also were analysed by Brennan and Schwartz (1976).

Many contracts have *annual* guarantees embedded, i.e., a minimum guaranteed rate of return each year, also known as a *multi-period* (rate of return) guarantee. This guarantee seems first to have been analysed by Hipp (1996). Persson and Aase (1997) and Miltersen and Persson (1999) extended the analysis in important ways, in particular by introducing stochastic interest rates, something that complicated the analysis quite a bit. For a comprehensive treatment of both maturity and multi-period guarantees, see Tiong (2000). Tiong (2000) also considered different *levels of participation*. A level of participation γ means that the return on the risky asset, on which, in the case of a maturity guarantee, the put option is written, is a fraction γ of the return on some risky asset, typically a portfolio. The portfolio could for instance be an insurance company's investment portfolio, i.e., the debit side of the balance sheet.

Real-world life and pension insurance contracts often have guarantees that are more involved than both the maturity and the multi-period guarantee, and there may even be several guarantee elements included in a contract. The terms of the contracts will typically differ in different countries so a "unified" analysis of life and pension insurance contracts is not likely to be possible. Some recent studies which tried to build more realistic models of these contracts were Grosen and Jørgensen (2000), Miltersen and Persson (2002), and Hansen and Miltersen (2002). These articles took into consideration that the distribution of the return on the insurer's investment portfolio between the insurer and the insured may be rather involved and is determined by different legislation and practice in the country in which the contract is issued. Different company policies may also influence on how the return is distributed.

Some other recent work on financial guarantees and related issues includes Hansen (2002b), Hansen (2002a), Steffensen (2001), Bacinello (2002), and Nielsen and Sandmann (2002).

Contents of the Dissertation The compound option is a well-known claim that was first analysed in the literature some 25 years ago. About 20 years went by before the multi-period guarantee was given any attention in the literature. We therefore expect the multi-period guarantee to be less

well-known than the compound option.

Both these claims are analysed in the first chapter. The analysis attempts to show that they can both be obtained as a special case of a more general compound contingent claim. In this chapter we also indicate that an established result in the literature may be flawed.

Chapter 2 - 5 are devoted to analysing rate of return guarantees and the use of these guarantees. Rate of return guarantees are typically embedded in life and pension insurance contracts. Not only the pricing of these guarantees is considered, but also hedging issues. The huge amount of risk these guarantees impose on life insurance companies and pension funds make a sound analysis of the guarantees an important issue. Especially since the pricing of rate of return guarantees does not seem to be very well conducted in practice (see Donselaar (1999)). When we also know that there is a close relationship between pricing and hedging, we may suspect that the lack of pricing may also cause a lack in the risk management. We have also seen life insurance companies that have gone into bankruptcy because they were unable to fulfil liabilities imposed by rate of return guarantees (e.g., Nissan Mutual Life).

The last chapter is concerned with a more, in the finance literature, traditional contingent claim, i.e., a compound option. It builds on the observation in chapter 1 that a closed form solution for the market value of a compound option is not easily obtainable under stochastic interest rates. This chapter deviates from the others in that the focus is on numerical methods, whereas the focus in the other chapters is on the derivation of closed form solutions and hedging strategies. The increasingly complexity of both the theoretical models and the claims that are traded in the markets, make the use of numerical methods a necessity in many situations. Also, the steadily increasing speed of computers makes numerical methods more suitable than ever before.

A more detailed description of the chapters follows below.

Chapter 1: ‘Compound Contingent Claims’ The main focus in this chapter is on a *compound contingent claim*. By a compound contingent claim we mean a contingent claim that is written on another contingent claim. The traditional example of such a claim is a call option written on a call option, see e.g., Geske (1979). A multi-period guarantee is an asset that secures that the holder receives the maximum of the return on the underlying asset and some minimum guaranteed rate of return within each period, see e.g., Miltersen and Persson (1999). Below, in the description of chapter 2, we argue that the multi-period rate of return guarantee has “...a sort of compounding effect...” In this chapter we try to “de-mystify” the multi-period guarantee by comparing it to the more familiar compound option. We look at the special case of a two-period guarantee, and we show that

it, just as the compound option, can be treated as a compound contingent claim. To show this, two *general* contingent claims are constructed. The first is a *simple* contingent claim, i.e., a contingent claim that is written on one or several primary traded assets. The second is a compound contingent claim that is written on the simple claim. The first claim is constructed so that it has the necessary generality to capture both a call option and a maturity guarantee as special cases, while the second has the necessary generality to capture a compound option and a two-period guarantee as special cases. These claims put us in a position where we are able to point out similarities and differences between the different claims.

It turns out that also a wide range of other claims can be seen to be special cases of the two general claims. An attempt to indicate for whom of these the market value can be obtained in closed form solution is given. Some of these are recognised as more or less well-known claims previously analysed in the literature, while other seems more obscure and not very practical applicable.

Geman et al. (1995) presented a closed form solution for the market value of a compound option under stochastic interest rates. Though their result may at first seem appealing, we argue, given the exercise set for the compound option, that their result seems flawed and that obtaining a closed form solution is likely to be non-trivial.

In the end of the chapter some special cases are explained.

Chapter 2: ‘Pricing of Multi-period Rate of Return Guarantees’¹

The work in this chapter is mostly inspired by Miltersen and Persson (1999), but also the results in Hipp (1996) and Persson and Aase (1997) have been motivating for this chapter.

Based on the observation that the guarantees embedded in life insurance contracts often are fundamentally different from the maturity guarantees embedded in the unit-linked life insurance contracts analysed by Brennan and Schwartz (1976), Hipp (1996) analysed a *multi-period* rate of return guarantee. Instead of a guarantee on the average return over the whole contract-period, as for the maturity guarantee, the multi-period rate of return guarantee is a guarantee on the average return within each of at least two sub-periods. This leads to a sort of compounding effect for the multi-period guarantee that is not present in the maturity guarantee. Hipp (1996) analysed this kind of guarantee within the framework of Black and Scholes (1973). Because of the independence between the return in the different periods (under deterministic interest rates and under the equivalent martingale measure Q), the market value of the guarantee is reduced to a rather nice expression. Persson and Aase (1997) and Miltersen and Persson (1999)

¹An earlier version of this chapter was presented at the Nordic Symposium on Contingent Claims in Stockholm in 2001.

generalised the analysis of the multi-period rate of return guarantee to also include stochastic interest rates. However, they only considered guarantees lasting for two periods (Persson and Aase (1997) presented an approximation for the market value of the multi-period guarantee). The challenge of imposing stochastic interest rates lies in the fact that the returns in the different periods are no longer independent. This basically follows since we assume that interest rates follow a continuous process and e.g., high interest rates at the end of one period are therefore followed by high interest rates in the beginning of the next period. Because of this extra source of uncertainty, both Persson and Aase (1997) and Miltersen and Persson (1999) found the market value of the guarantee under stochastic interest rates to be expressed by the bivariate normal probability distribution, while Hipp (1996) found it to be expressed by the corresponding univariate distribution.

In this chapter we show how to value a multi-period rate of return guarantee under stochastic interest rates consisting of any number of periods. We find the expression for the market value of the guarantee to be given as a function of the multivariate normal probability distribution. Unfortunately, although we are able to obtain the market value of the guarantee in closed form solution, evaluating the expression for the market value can actually be quite time consuming. As most cumulative probability distributions, also this one has to be approximated by some numerical integration routine. Although estimating one probability is very fast, the structure of the problem is such that for a guarantee with, say, 30 sub-periods, more than one billion probabilities have to be calculated, reducing the practical usefulness of the results. However, we believe that our results in fact do give a lot of useful information about the structure of such guarantees, and then in particular with respect to hedging issues.

Chapter 3: 'Hedging of Multi-period Rate of Return Guarantees'²

This chapter is based on my Master thesis from 1999.³ Also this chapter relies heavily on Miltersen and Persson (1999), Hipp (1996), and Persson and Aase (1997). In contrast to chapter 2, this chapter tries to establish hedging strategies for multi-period rate of return guarantees. The hedging strategies are both derived under the assumption of deterministic and stochastic interest rates. Compared to the hedging strategy for a traditional European option and a maturity guarantee, we find these strategies to be quite different.

For instance, a European call option can be hedged by trading in the underlying asset and a zero-coupon bond. The functions determining the

²Earlier versions of this chapter were presented at the FIBE conference in Bergen in 2000, at the 10th International AFIR conference in Tromsø in 2000, and at the Nordic Symposium on Contingent Claims in Stockholm in 2001.

³The thesis was named 'Hedging av Finansielle Derivater i en Black&Scholes/Amin&Jarrow model'.

number of units of these two assets to include in the hedging strategy are continuous through time. For multi-period guarantees, we find that these functions may be discontinuous, or more precisely, the hedging strategies are determined by different functions in different periods. This may cause a discontinuity in the number of each asset as we go from one period to another. We first show that the hedge portfolio, under stochastic interest rates, can consist of the underlying asset and a whole portfolio of zero-coupon bonds. There are both long and short positions in the bond portfolio. If the guarantee is not binding at the end of a period, the market value of the bond portfolio is zero. The bond portfolio will typically consist of fewer bonds in later periods than in earlier periods. This reflects the more complex structure of multi-period guarantees under stochastic interest rates than under deterministic interest rates where only one zero-coupon bond is needed. We also show, in the special case of a one-factor model for the short-term interest rate, that a portfolio containing the money market account and one zero-coupon bond can replace the bond portfolio. Thus, the first hedging strategies we derive are fairly general in the sense that they in principle also apply to multi-factor term structure models.

The hedging strategies are illustrated with several numerical examples.

Chapter 4: ‘Relative Guarantees’ The guarantees analysed in chapter 2 are often called absolute guarantees since the minimum guaranteed rate of return is denoted as an absolute, or a fixed, number. In some applications the minimum guaranteed rate of return is stochastic, and then typically equal to the return on some reference portfolio. These guarantees are often called *relative* guarantees, and the valuation of such guarantees is the topic of this chapter.

For instance, Argentina, Chile, and Poland have pension plans where relative guarantees are embedded. However, these guarantees are rather complicated and may not be easily valued in closed form.

A wide range of different kinds of relative guarantees is considered in this chapter, hereunder both maturity and multi-period guarantees. Although one may expect a guarantee with a stochastic minimum guaranteed rate of return to be more complicated than the corresponding guarantee with a deterministic minimum guaranteed rate of return, we find this not necessarily to be the case. For a multi-period guarantee, the reason for this is simply that (again under the equivalent martingale measure Q) the *excess* returns (note that the excess returns can be both positive and negative) over the short-term interest rate across the different periods, both for the underlying asset, the reference portfolio, and also *between* the underlying asset and the reference portfolio, are uncorrelated. This simplifies matters considerably. We further analyse less standard and more complicated guarantees where the minimum guaranteed rate of return is a function of the return on the

reference portfolio. The chapter is ended with an attempt to analyse the guarantee embedded in pension contracts in Chile. The descriptions of this guarantee that we have found in the literature do in fact differ quite considerably. Therefore, we analyse three different types of guarantees which all are related to the Chilean guarantee. However, each of these must be seen as a simplification of the real-world guarantee.

In the first of these, the stochastic minimum guaranteed rate of return is given as the minimum of two different functions of the return on the reference portfolio. The results we derive for this guarantee can also be used to extend the analysis of Stulz (1982) and Johnson (1987) on options written on the maximum or the minimum of two (or several) risky assets to a stochastic interest rate framework. The last two are based on the average return on the reference portfolio. The first is a maturity guarantee while the second is a two-period guarantee.

Although the guarantees we consider in this chapter are idealised and simplified compared to real-world guarantees, we hope that some new insight on relative guarantees is obtained by the analysis performed.

Chapter 5: ‘Defined Contribution and Defined Benefit Based Pension Plans’⁴ As of January 2001 a new law, opening for the use of *contribution based pension plans*, was passed in Norway. With this law in mind, this chapter attempts to present a way for the employees to value their participation in a pension plan.

A seemingly common way of arranging such pension plans, is to let the return on the employees’ pension accounts be a given fraction of the return the pension fund obtains on its investment portfolio. As mentioned, this fraction is termed level of participation. To reduce the risk for the employees, rate of return guarantees may be embedded. Assuming such a structure of the pension plan, we propose several pension plans, both with and without guarantees. We show how these pension plans can be valued using the same techniques as in the previous chapters, i.e., by arbitrage pricing. Especially, we find that the use of *forward-start guarantees* makes it possible to incorporate periodical premium and pension payments in a straightforward manner.

An important aspect of both life and pension insurance is *mortality risk*. A by now standard procedure for dealing with mortality risk in the presence of financial risk is to assume that these two risks are independent. This seems like a reasonable assumption. Another assumption that is important in much of the theory in the actuarial sciences is that the insurer (here the pension fund) is risk neutral with respect to mortality risk. The rationale for this

⁴Earlier versions of this chapter were presented at the FIBE conference in Bergen in 2001 and at the International Symposium on Financial Risk Exposure in Life and Pension Insurance in Bergen in 2001.

assumption is that the insurer can diversify mortality risk by issuing many (in theory *infinitely* many) similar policies. Although our main interest is to value pension plans as seen from the employees' point of view, we argue that this may also be a reasonable assumption in our setting.

A criticism that has been raised against defined contribution based pension plans is that they expose the employees to too much risk. We show, using a numerical example based on Monte Carlo simulation, that these contracts indeed expose the employees to a considerable amount of risk. For comparison, we also give a short analysis of *defined benefit based pension plans*. This analysis shows that these two kinds of pension plans, at least when it comes to risk, are totally different. In our simplified models the financial risk is born by the employees in a defined contribution based pension plan, while the employers bear this risk in a defined benefit based pension plan.

The main purpose of this chapter is to show a practical situation where the guarantees in the previous chapters and similar guarantees can be applied. The chapter deviates from the others in that we most of the time assume deterministic interest rates. The main reason for this assumption is based on a wish to focus on the simple structure of the proposed pension plans. This simple structure is likely to be overshadowed by the extensive notation required under stochastic interest rates.

This chapter ends the analysis of rate of return guarantees.

Chapter 6: 'Numerical Evaluation of Compound Options' This chapter is a joint work with Arne-Christian Lund and was written after a joint participation on the course 'Monte Carlo Methods in Financial Engineering' given by Professor Paul Glasserman at the University of Aarhus in the spring of 2001.

In this chapter the focus is on calculating the market value of a compound option under stochastic interest rates. A compound option has previously been valued under deterministic interest rates by Geske (1977) and Geske (1979) and with some extensions made by Hodges and Selby (1987). We limit our analysis to a call option written on a call option that again is written on a stock. Based on the results in chapter 1, there does not seem to exist any known closed form solution for the market value of this claim under stochastic interest rates, and the valuation problem is therefore approached by numerical methods, or more precisely, by Monte Carlo simulation.

In general, simulation within the Heath, Jarrow, and Morton framework requires the whole term structure over the life of the contingent claim to be simulated. In addition, the stochastic differential equations describing the economy will typically have to be discretised. We present a unified and arbitrage-free way to discretise the stochastic differential equations. In the continuous case a restriction on the drift of the forward rates is

imposed to avoid arbitrage opportunities. We derive a discrete time analogy for the drift restriction that also is arbitrage-free. A similar result can be found in Andersen (1997). Armed with these results and variance reduction techniques, the compound option is valued.

The discretisation of the stochastic differential equations will typically lead to bias in the estimates of the market value. It seems like the estimates tend to be too high, but by increasing the number of time steps, this problem is as good as eliminated. One of the variance reduction techniques, the control variate method, in addition to give a very significant reduction in the variance, also has the nice feature that it eliminates the problem with discretisation bias.

Working within a *Gaussian* Heath, Jarrow, and Morton framework, we show that it is not necessary to perform any discretisation of the stochastic differential equations. Using so-called *exact simulation* the compound option can be valued without generating the whole path followed by the underlying asset price and the interest rates; only the terminal payoffs are needed. This increases the computational speed quite considerably and the problem with discretisation bias is eliminated.

Needless to say, the results in this chapter may have applications far wider than for just the compound option analysed here.

Notation As mentioned, each chapter is written as a self-contained paper. This has caused some overlap, especially in the description of the economic model since basically the same economic model is used in each chapter. However, even though some effort has been put hereon, the notation in the different chapters do differ.

Most of the time we let the volatility functions for the forward rates and for the return on some risky asset be vector valued.⁵ As an example, let (some more intuition and explanations for these expressions are given in the main chapters of the dissertation, see e.g., page 15 and 16)

$$\int_v^t \sigma_f(v, u) du = \begin{pmatrix} \int_v^t \sigma_{f_1}(v, u) du \\ \int_v^t \sigma_{f_2}(v, u) du \end{pmatrix}$$

and

$$\sigma_S(v) = \begin{pmatrix} \sigma_{S_1}(v) \\ \sigma_{S_2}(v) \end{pmatrix}$$

⁵For some $N \times 1$ -dimensional vector $\mathbf{x} = (x_1, x_2, \dots, x_k)$, the (Euclidean) norm is given by (see e.g., Rudin (1976) p. 16)

$$|\mathbf{x}| = (\mathbf{x} \cdot \mathbf{x})^{\frac{1}{2}} = \left(\sum_{i=1}^k x_i^2 \right)^{\frac{1}{2}}.$$

be the volatility functions for the instantaneous forward rates and for the return on the risky asset, respectively. It can then be shown that the variance of the return on the risky asset, under the equivalent martingale measure Q , is given by (see e.g., Musiela and Rutkowski (1997) p. 359)

$$\sigma_{\delta_t}^2 = \int_0^t \left| \sigma_S(v) + \int_v^t \sigma_f(v, u) du \right|^2 dv,$$

or, alternatively

$$\begin{aligned} \sigma_{\delta_t}^2 = & \int_0^t \left(\left(\int_v^t \sigma_{f_1}(v, u) du \right)^2 + \sigma_{S_1}^2(v) + \left(\int_v^t \sigma_{f_2}(v, u) du \right)^2 + \sigma_{S_2}^2(v) \right. \\ & \left. + 2 \left(\sigma_{S_1}(v) \int_v^t \sigma_{f_1}(v, u) du + \sigma_{S_2}(v) \int_v^t \sigma_{f_2}(v, u) du \right) \right) dv. \end{aligned}$$

Throughout the dissertation we write this as

$$\begin{aligned} \sigma_{\delta_t}^2 = & \int_0^t \left(\int_v^t \sigma_f(v, u) du \right)^2 dv \\ & + 2 \int_0^t \sigma_S(v) \int_v^t \sigma_f(v, u) du dv + \int_0^t \sigma_S^2(v) dv, \end{aligned}$$

although this is a slightly abuse of notation since we do not distinguish between vectors and scalars.

Software This dissertation is typeset in \LaTeX . Most of the numerical calculations are done using Ox (see Doornik (1999)), but also Fortran 77, Visual Basic, and Excel have been used. And, not to forget, my Citizen scientific calculator SR35 has also been used.

Chapter 1

Compound Contingent Claims

Abstract

This chapter explores similarities and differences between a compound option and a two-period guarantee. A generalised *compound contingent claim* that captures these two claims as special cases is constructed. The underlying asset of the compound contingent claim is a generalised *simple contingent claim*. Similar parities as the put-call parity are derived for both these claims. Also several other claims captured by the two general claims are revealed. We also show that the derivation of a closed form solution for the market value of a compound option under stochastic interest rates is likely to be non-trivial, if possible at all.

Keywords and phrases: Compound option, multi-period guarantee, Heath, Jarrow, and Morton term structure model of interest rates.

1.1 Introduction

Many seemingly different assets may in fact be more similar than they first appear. In this chapter our main goal is to point out similarities between a compound option and a multi-period guarantee. Once the similarities are pointed out, also some of the differences will be displayed.

Compound options were first analysed by Geske (1977) and Geske (1979). A compound option is an option with another option as the underlying asset. We limit our analysis to a call option written on a call option. The underlying option is assumed written on a stock.

A multi-period guarantee is an asset that secures that the holder gets

the maximum of the return on the underlying asset and some minimum guaranteed rate of return within each period. In this chapter we focus, for simplicity, on a two-period guarantee, see e.g., Miltersen and Persson (1999). We assume that the underlying return of the guarantee is the return on the stock in which the call option above is written on. It is straightforward to generalise to a compound option that is written on another compound option and so on. Also, generalising to guarantees lasting for more than two periods is straightforward. However, these generalisations will make the intuition harder to grasp and will not be necessary for our purposes.

To explore the similarities between these two claims, a general compound contingent claim capturing both claims as special cases is constructed. To this end we start by constructing a generalised *simple* contingent claim, i.e., a claim that is written on primary traded assets such as stocks and bonds, not other contingent claims. This asset has the necessary generality to capture both a call option and a maturity guarantee¹ as special cases. To construct the generalised compound contingent claim, we assume that there exists a contingent claim written on the simple contingent claim described above. This asset captures both the compound option and the two-period guarantee as special cases. It puts us in a position where we can easily see similarities between these two claims. It is our hope, since we have not found any connections in the literature between the compound option, which was first analysed in the literature some 25 years ago, and the relatively newly analysed two-period guarantee, that this will shed some new light into these two claims. Our analysis may also give an alternative introduction to the theory of multi-period guarantees for the reader familiar to compound options and vice versa.

Using different specifications for the two claims we construct, we find that the claims also capture several other claims as special cases, not just the call option, the maturity guarantee, the compound option, and the two-period guarantee. Several of these are trivial in the sense that their payoffs do not represent real-world contingent claims and can even be constants. Some of the possible specifications lead to claims where we are not able to derive closed form solutions for the market value. However, based on more or less well-known results relevant for option pricing, we have pointed out for what specifications we have been able to obtain closed form solutions.

An important difference between our framework and that of Geske (1977) and Geske (1979) is that we work under stochastic interest rates. Although this is in principle a trivial extension, it is interesting to notice that a closed form solution for the market value of a compound option as analysed by Geske (1979), i.e., a call option on a standard Black and Scholes call option, is not trivially obtainable, if obtainable at all. This is caused by difficulties concerning the exercise probability for the compound option.

¹A maturity guarantee is effectively the same as a *one*-period guarantee.

From the put-call parity we know that there is a close relationship between a call option and a put option. The put option has a “mirror imaged” payoff structure of what the call option has and vice versa. We therefore denote the put option the *mirror claim* for the call option. By defining the mirror claims for the two generalised claims, we show how to derive parities for these claims. This is an issue also addressed in Haug (2002).

We have also picked five specifications of the generalised compound contingent claim and given them a more thorough analysis.

The chapter is organised as follows: In section 1.2 we give a description of our economic model and some preliminaries. In section 1.3 a short comparison of a call option and a maturity guarantee is given. In section 1.4 we construct a generalised contingent claim. In section 1.5 a short comparison of a compound option and a two-period guarantee is given. In section 1.6 we construct a generalised compound contingent claim that is written on the general contingent claim constructed in section 1.4. In section 1.7 some claims that are special cases of the general compound contingent claim are given a thorough analysis. The chapter is ended in section 1.8 with some concluding remarks.

1.2 The Economic Model and Preliminaries

We assume a continuous trading economy on the time interval $[0, T]$, for some fixed horizon $T > 0$, and with no transaction costs. A filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ is fixed, where Ω is the state space, \mathcal{F} is a σ -algebra, $\mathbb{F} = \{\mathcal{F}_t, 0 \leq t \leq T\}$ is a filtration where $\mathcal{F}_T = \mathcal{F}$ and $\mathcal{F}_0 = \{\emptyset, \Omega\}$, where \emptyset is the empty set, and P is a probability measure. The σ -algebra is generated by a d -dimensional, $d \geq 1$, Brownian motion, W_t . We further assume a complete market, i.e., there exists one unique equivalent martingale measure Q , see e.g., Harrison and Kreps (1979).

Following the model of Heath et al. (1992), the instantaneous continuously compounded forward rate at time s as seen from time t , $t \leq s \leq T$, under the equivalent martingale measure Q , is given by

$$f(t, s) = f(0, s) + \int_0^t \sigma_f(v, s) \int_v^s \sigma_f(v, u) du dv + \int_0^t \sigma_f(v, s) dW_v,$$

where $\sigma_f(t, s)$ is the volatility function for the instantaneous continuously compounded forward rate at time s as seen from time t , satisfying some technical regularity conditions, see Heath et al. (1992). The short-term interest rate is obtained by setting s equal to t , i.e., $r_t = f(t, t)$. The volatility function is assumed deterministic, implying Gaussian interest rates. Under deterministic interest rates we formally set $\sigma_f(v, u) = 0$. We also assume that there is a continuum of bonds that trade in the market.

We let the market value of the non-dividend paying primary traded securities i , S_t^i , be given under the equivalent martingale measure Q by the equation²

$$S_t^i = S_0^i + \int_0^t r_v S_v^i dv + \int_0^t \sigma_{S^i}(v) S_v^i dW_v,$$

where $r_t S_t^i$ satisfies the integrability condition $\int_0^t |r_v S_v^i| dv < \infty$ almost surely for all t . Here $\sigma_{S^i}(t)$ is the volatility function for the return on asset i and satisfies the square integrability condition $E\left[\int_0^t (\sigma_{S^i}(v) S_v^i)^2 dv\right] < \infty$ (for further details on integrability conditions, see e.g., Duffie (1996)). Also this volatility function is assumed to be a deterministic function of time. This class of assets will be referred to as stocks. For simplicity, when only one stock is present, we write $S_t^1 = S_t$.

We also assume that there exists an instantaneously risk-free asset, a money market account, that accrues interest according to the short-term interest rate, yielding a time t market value of

$$M_t = M_0 + \int_0^t r_v M_v dv, \quad M_0 = 1, \quad (1.1)$$

where $r_t M_t$ satisfies the integrability condition $\int_0^t |r_v M_v| dv < \infty$ almost surely for all t . The return on the money market account, under the equivalent martingale measure Q , over the time period from time T_1 to T_2 is given by (see e.g., Miltersen and Persson (1999))

$$\begin{aligned} \beta_{T_2-T_1} &= \int_{T_1}^{T_2} r_v dv = -\ln F(0, T_1, T_2) + \frac{1}{2} \sigma_{\beta_{T_2-T_1}}^2 + c_{T_2-T_1, T_1} \\ &\quad + \int_0^{T_1} \int_{T_1}^{T_2} \sigma_f(v, u) dudW_v + \int_{T_1}^{T_2} \int_v^{T_2} \sigma_f(v, u) dudW_v, \end{aligned}$$

where $F(0, T_1, T_2)$ is the time 0 forward price for delivery at time T_1 of a zero-coupon bond maturing at time T_2 and is given by

$$F(0, T_1, T_2) = \frac{P(0, T_2)}{P(0, T_1)},$$

where $P(0, t)$ is the time zero market value of a zero-coupon bond maturing at time $t \geq 0$. Here $\sigma_{\beta_{T_2-T_1}}^2$ is the variance of the return on the money market account over the time period from time T_1 to T_2 and is given by

$$\sigma_{\beta_{T_2-T_1}}^2 = \int_0^{T_1} \left(\int_{T_1}^{T_2} \sigma_f(v, u) du \right)^2 dv + \int_{T_1}^{T_2} \left(\int_v^{T_2} \sigma_f(v, u) du \right)^2 dv$$

²In this chapter it is sufficient that $i \in \{1, 2, \dots, 6\}$.

and $c_{T_2-T_1, T_1}$ is the covariance between the return on the money market account over the time period from time 0 to T_1 and from time T_1 to T_2 and is given by

$$c_{T_2-T_1, T_1} = \int_0^{T_1} \left(\int_v^{T_1} \sigma_f(v, u) du \right) \left(\int_{T_1}^{T_2} \sigma_f(v, u) du \right) dv.$$

The return on the stock under the equivalent martingale measure \mathbb{Q} over the same time interval is given by

$$\delta_{T_2-T_1} = \int_{T_1}^{T_2} \left(r_v - \frac{1}{2} \sigma_S(v)^2 \right) dv + \int_{T_1}^{T_2} \sigma_S(v) dW_v,$$

with variance

$$\sigma_{\delta_{T_2-T_1}}^2 = \sigma_{\beta_{T_2-T_1}}^2 + 2 \int_{T_1}^{T_2} \sigma_S(v) \int_v^{T_2} \sigma_f(v, u) du dv + \int_{T_1}^{T_2} \sigma_S^2(v) dv. \quad (1.2)$$

1.3 Options and Guarantees

Let us start by considering a standard call option and a maturity guarantee. The terminal time T payoff for the call option is given by $\max(S_T - X, 0)$ for some exercise price $X \in (0, \infty)$, while the terminal payoff for the maturity guarantee is given by $\max(S_T, X)$, or, equivalently, $\max(S_T - X, 0) + X$. As we can see, there is a close relationship between these two claims.

The call option gives the owner the right to receive one unit of the stock by at the same time delivering X units of account, or, since the face value of a zero-coupon bond is equal to one, X units of the face value of a zero-coupon bond. From Merton (1973) we know that the market value of the call option at time $t < T$ is given by

$$\pi_t^c = S_t \Phi(d_1) - P(t, T) X \Phi(d_2), \quad (1.3)$$

where

$$\begin{aligned} d_1 &= \frac{\ln\left(\frac{S_t}{P(t, T) X}\right) + \frac{1}{2} \sigma_{\delta_{T-t}}^2}{\sigma_{\delta_{T-t}}}, \\ d_2 &= d_1 - \sigma_{\delta_{T-t}}, \end{aligned}$$

$\Phi(\cdot)$ is the cumulative normal probability distribution, and $\sigma_{\delta_{T-t}}^2$ follows from (1.2).

First we notice that the option only will be exercised if the condition $S_T > X$ is satisfied. The market value at time t can be interpreted as consisting of two parts; the first, $S_t \Phi(d_1)$, is the time t market value of the

stock multiplied by the probability of receiving the stock at time T . This probability is under the equivalent probability measure where the stock price is used as numeraire. The second, $P(t, T)X\Phi(d_2)$, is the time t market value of delivering X units of the face value of a zero-coupon bond multiplied by the probability (under the equivalent probability measure where the bond price, $P(t, T)$, is used as a numeraire, i.e., the forward probability measure, see e.g., Jamshidian (1989)) that the face value has to be delivered.

Using the symmetry properties of the normal probability distribution, it follows from (1.3) that the time t market value of the maturity guarantee is given by

$$\pi_t^g = S_t\Phi(d_1) + P(t, T)X\Phi(-d_2).$$

From the above we conclude that the main difference between a call option and a maturity guarantee is that the call option gives the holder the choice between receiving one unit of the stock by delivering X units of the face value of a zero-coupon bond or nothing, while the maturity guarantee gives the holder the right to choose between receiving one unit of the stock or X units of the face value of a zero-coupon bond at no cost. Intuitively, we can think of it as being free to “exercise” the maturity guarantee while it is costly to exercise the call option. However, this is paid for up front since the maturity guarantee has a higher initial market value than the call option.

1.4 A Generalised Simple Contingent Claim

Let us now construct a generalised contingent claim that captures the two claims analysed above as special cases. We denote this a *simple* contingent claim. By a simple contingent claim we mean a contingent claim that is only a function of primary traded assets such as stocks and bonds, not other contingent claims.

There are many different ways in which such a simple contingent claim can be constructed. We let the final time T payoff be given by

$$g_T = \max(A_T - B_T, C_T). \tag{1.4}$$

We further let each of A_T , B_T , and C_T be equal to one of the following:

1. zero,
2. a strictly positive constant, or
3. a positive valued random variable.

By a “positive valued random variable” we mean a linear³ function of the market value of a primary traded asset.

Though the claim in (1.4) may seem somewhat ad-hoc, it does in fact do the job of describing a call option and a maturity guarantee. To obtain a call option, let $A_T = S_T$, $B_T = X$, and $C_T = 0$, i.e.,

$$g_T = \max(S_T - X, 0).$$

If instead $B_T = 0$ and $C_T = X$ we have that

$$g_T = \max(S_T, X),$$

and the maturity guarantee is obtained as a special case.

In general, the time 0 market value of the simple claim can be calculated in the following way

$$\begin{aligned} g_0 &= E_Q \left[e^{-\beta T} \max(A_T - B_T, C_T) \right] \\ &= A_0 Q_1(\mathcal{A}) - B_0 Q_2(\mathcal{A}) + C_0 Q_3(\bar{\mathcal{A}}), \end{aligned} \quad (1.5)$$

where $A_0 \equiv E_Q \left[e^{-\beta T} A_T \right]$, $B_0 \equiv E_Q \left[e^{-\beta T} B_T \right]$, and $C_0 \equiv E_Q \left[e^{-\beta T} C_T \right]$.⁴ We define Q_1 , Q_2 , and Q_3 by

$$\begin{aligned} \frac{dQ_1}{dQ} &= \frac{e^{-\beta T} A_T}{E_Q \left[e^{-\beta T} A_T \right]}, \\ \frac{dQ_2}{dQ} &= \frac{e^{-\beta T} B_T}{E_Q \left[e^{-\beta T} B_T \right]}, \end{aligned}$$

and

$$\frac{dQ_3}{dQ} = \frac{e^{-\beta T} C_T}{E_Q \left[e^{-\beta T} C_T \right]}.$$

Here $\mathcal{A} = \{A_T - B_T > C_T\}$ and $\bar{\mathcal{A}}$ is the complement to \mathcal{A} .

For a constant A_T we define $Q_1 = Q_T$, for B_T constant $Q_2 = Q_T$, and finally for C_T constant $Q_3 = Q_T$, where Q_T is the forward probability measure. Similarly, we define $\frac{e^{-\beta T} A_T}{E_Q[e^{-\beta T} A_T]} \equiv 0$ for $A_T = 0$, $\frac{e^{-\beta T} B_T}{E_Q[e^{-\beta T} B_T]} \equiv 0$ for $B_T = 0$, and $\frac{e^{-\beta T} C_T}{E_Q[e^{-\beta T} C_T]} \equiv 0$ for $C_T = 0$. As an example, assume that $B_T = 0$. (1.5) would then be reduced to $A_0 Q_1(\mathcal{A}) + C_0 Q_3(\bar{\mathcal{A}})$.

³A linear function is a function on the form $y = ax$ for some non-zero constant a .

⁴Notice that these definitions are only used for notational simplicity and do not necessarily mean that $e^{-\beta T} A_T$, $e^{-\beta T} B_T$, or $e^{-\beta T} C_T$ are Q -martingales. For instance, if A_T is a constant, say, A , it follows trivially that $A_0 \neq A$.

So far we have considered two possible specifications of the claim in (1.4); a call option and a maturity guarantee. However, also several other claims can be constructed by choosing other specifications. A natural question that then arises is the following: For what specifications of the claim in (1.4) do there exist a closed form solution for the market value?

The usual definition of a closed form solution is that it is a (deterministic) function that takes its arguments from a set of known parameter values and returns a scalar; the market value. This means that there can be no unknown parameters in the pricing formula such as future stock prices or level of interest rates. All the arguments used at time t have to be \mathcal{F}_t -measurable. Even though, in a Gaussian setting, the cumulative normal probability distribution has to be approximated by some numerical integration routine, we follow tradition and also denote an expression for the market value of a claim containing a cumulative normal probability distribution a closed form solution.

In total, it is possible to construct $3^3 = 27$ different combinations for the claim in (1.4), not all of which are equally interesting. In Table 1.1 - 1.3 we have showed the possible specifications. ($A_T = \bar{A}$ means that A_T is a constant and $A_T = \tilde{A}$ that A is a random variable. The same also applies for B_T and C_T , with the obvious change of notation. “*” indicates no obtainable closed form solution.)

The abbreviations in Table 1.1 - 1.3 define what the market value of the different specifications of the general claim are equal to. They are defined as follows:

- a) = 0.
- b) = a constant.
- c) = a positive valued random variable.
- d) = a call option.
- e) = a put option.
- f) = an exchange option.
- g) = a maturity guarantee.
- h) = e) + b).
- i) = d) + b).
- j) = a spread option + b).
- k) = the maximum of two assets.
- l) = a spread option + b) - c).

Table 1.1: Specifications for the simple claim g for $C_T = 0$.

	$A_T = 0$	$A_T = \bar{A}$	$A_T = \tilde{A}$
$B_T = 0$	a) $\max(0, 0)$	b) $\max(\bar{A}, 0)$	c) $\max(\tilde{A}, 0)$
$B_T = \bar{B}$	a) $\max(-\bar{B}, 0)$	a) or b) $\max(\bar{A} - \bar{B}, 0)$	d) $\max(\tilde{A} - \bar{B}, 0)$
$B_T = \tilde{B}$	a) $\max(-\tilde{B}, 0)$	e) $\max(\bar{A} - \tilde{B}, 0)$	f) $\max(\tilde{A} - \tilde{B}, 0)$

Table 1.2: Specifications for the simple claim g for $C_T = \bar{C}$.

	$A_T = 0$	$A_T = \bar{A}$	$A_T = \tilde{A}$
$B_T = 0$	b) $\max(0, \bar{C})$	b) $\max(\bar{A}, \bar{C})$	f) $\max(\tilde{A}, \bar{C})$
$B_T = \bar{B}$	b) $\max(-\bar{B}, \bar{C})$	b) $\max(\bar{A} - \bar{B}, \bar{C})$	i) $\max(\tilde{A} - \bar{B}, \bar{C})$
$B_T = \tilde{B}$	b) $\max(-\tilde{B}, \bar{C})$	b) or h) $\max(\bar{A} - \tilde{B}, \bar{C})$	j)* $\max(\tilde{A} - \tilde{B}, \bar{C})$

Table 1.3: Specifications for the simple claim g for $C_T = \tilde{C}$.

	$A_T = 0$	$A_T = \bar{A}$	$A_T = \tilde{A}$
$B_T = 0$	c) $\max(0, \tilde{C})$	g) $\max(\bar{A}, \tilde{C})$	f) $\max(\tilde{A}, \tilde{C})$
$B_T = \bar{B}$	c) $\max(-\bar{B}, \tilde{C})$	c) or g) $\max(\bar{A} - \bar{B}, \tilde{C})$	m)* $\max(\tilde{A} - \bar{B}, \tilde{C})$
$B_T = \tilde{B}$	c) $\max(-\tilde{B}, \tilde{C})$	l)* $\max(\bar{A} - \tilde{B}, \tilde{C})$	n)* $\max(\tilde{A} - \tilde{B}, \tilde{C})$

m) = a spread option + c).

n) = an exchange option to deliver $B_T + C_T$ to receive $A_T + c$).

If two (or three) of A_T , B_T , and C_T are equal (or are linear functions of the same random variable), the definitions above may not apply because the claim degenerates to another claim. Notice also that the spread option is defined as a call on the spread.

1.4.1 A Parity for the Simple Contingent Claim

Using the put-call parity, the market value of a call option can be expressed in terms of the market value of a put option, the underlying asset, and the present value of the strike price. In this subsection we find a parity for the simple contingent claim given in (1.4).

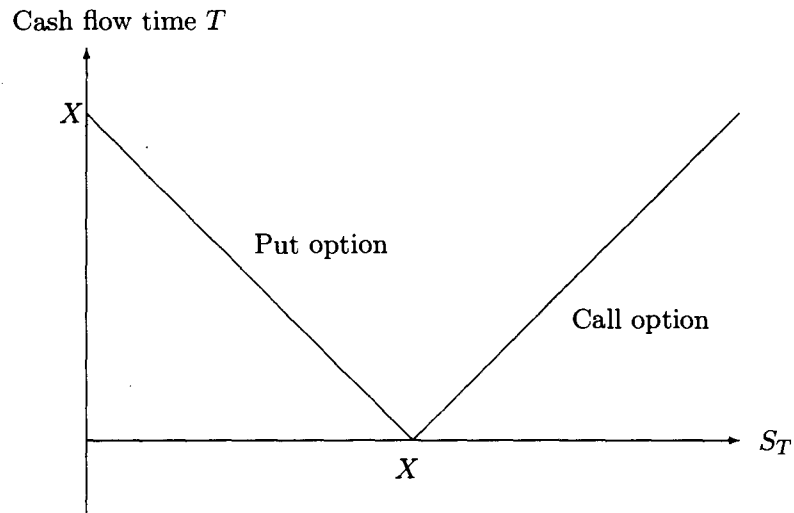


Figure 1.1: The terminal payoff for a call and a put option with exercise price X .

Consider the call and the put option in Figure 1.1 with the market value of the underlying asset on the x -axis and the terminal payoff on the y -axis. When the market value of the underlying asset is greater than X , the payoff of the call option is given by a 45°-line. Otherwise, the market value is given by a horizontal line at $y = 0$. Now, consider placing a vertical two-sided mirror at $x = X$. Looking in the mirror from right to left, we see a 45°-line rising away from us, i.e., the payoff of a put option when the market value of the underlying asset is less than X . On the other hand, looking in the mirror from left to right, we see a horizontal line at $y = 0$ going away from us, i.e., the payoff of a put option when the market value of the underlying asset is greater than X . Because the put option has this “mirror imaged” payoff structure of the call option, we will in the following refer to the put option as the *mirror claim* for the call option and vice versa.

Definition 1.1. For a claim with terminal payoff⁵ $\max(Z_1, Z_2) = (Z_1 - Z_2)^+ + Z_2$, we define the mirror claim as the claim with terminal payoff $-\min(Z_1, Z_2) = -(Z_1 - Z_2)^- - Z_2 = \max(-Z_1, -Z_2)$.

The terminal time T market value of a call option written on a stock with market value S_T is given by $\max(S_T - X, 0)$. From Definition 1.1 we have that the market value of the corresponding put option is given by $-\min(S_T - X, 0)$. Alternatively, the terminal market value of the put option can be found by changing signs (i.e., by multiplying by minus one) *inside* the max-operator in the expression for the terminal market value of the

⁵Let $(Z)^+ = \max(Z, 0)$ and $(Z)^- = \min(Z, 0)$, for some $Z \in \mathbb{R}$.

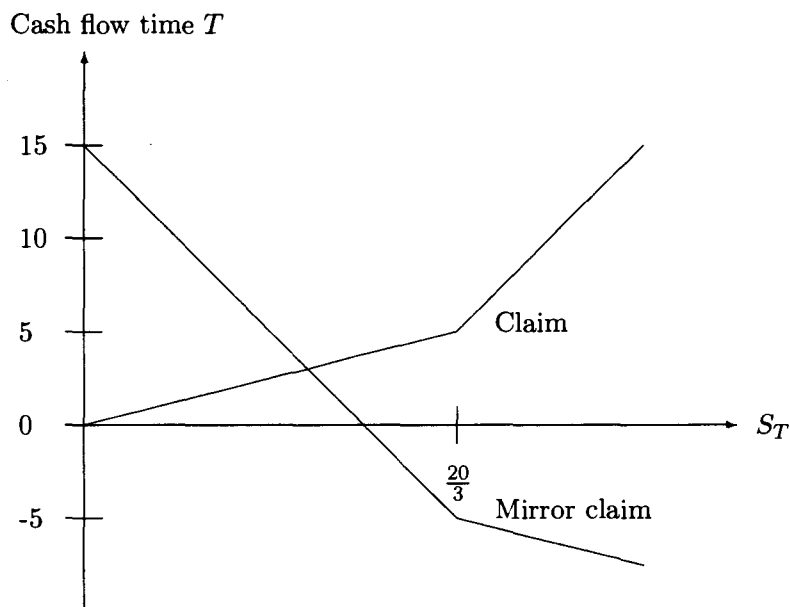


Figure 1.2: Illustration of the terminal cash flow for the claim $\max(3S_T - 15, 0.75S_T)$ and for the mirror claim $\max(15 - 3S_T, -0.75S_T)$.

call option. This gives the more familiar expression for the terminal market value of the put option, i.e., $\max(-1 \cdot S_T - (-1)X, -1 \cdot 0) = \max(X - S_T, 0)$.

Using Definition 1.1 on the simple claim, we find that the terminal market value of the mirror claim is given by $g_T^m = \max(B_T - A_T, -C_T)$. This is illustrated in Figure 1.2 for the simple claim and the mirror claim for $A_T = 3S_T$, $B_T = 15$, and $C_T = 0.75S_T$.

Let g_t and g_t^m be the time t market value of the simple claim and the mirror claim, respectively. Further, define $A_t \equiv E_Q \left[e^{-\int_t^T r_v dv} A_T \right]$, $B_t \equiv E_Q \left[e^{-\int_t^T r_v dv} B_T \right]$, and $C_t \equiv E_Q \left[e^{-\int_t^T r_v dv} C_T \right]$.

Theorem 1.1. *For the simple contingent claim, we have the following parity*

$$g_t = g_t^m + A_t - B_t + C_t.$$

Proof. In the absence of arbitrage, this follows since both the left and the right-hand side of the parity have the same terminal payoff. \square

Another way to justify this interpretation of the mirror claim is the following rewriting (using the terminal market values)

$$\begin{aligned} g_T &= \max(A_T - B_T, C_T) \\ &= \max(A_T - B_T - C_T, 0) + C_T \\ &= \max(B_T - A_T, -C_T) + A_T - B_T + C_T. \end{aligned}$$

1.5 Compound Option and Two-period Guarantee

Let us now consider two somewhat more complicated claims. First we consider a compound option (see e.g., Geske (1979)), i.e., a call option with another call option as the underlying asset. We assume that the compound option can be exercised at time T_1 at a cost of X_1 and that the underlying option is written on a stock and can be exercised at time $T_2 \geq T_1$ at a cost of X_2 . Let π_t^1 be the time $t \leq T_1$ market value of the compound option. We then have that

$$\pi_{T_1}^1(\pi^c) = \max(\pi_{T_1}^c - X_1, 0),$$

where π^c is the underlying call option with time T_1 market value $\pi_{T_1}^c$. Thus, the compound option can be interpreted in the same way as the call option; it gives the holder the right to acquire one unit of the underlying asset by delivering X_1 units of the face value of a zero-coupon bond.

A two-period guarantee secures that the holder receives the maximum of the return on some underlying asset and some minimum guaranteed rate of return in each of the two periods. Assume that the minimum guaranteed rate of return in period i , $i \in \{1, 2\}$, is given by g_i . If the guarantee is written on the return on the stock, the terminal payoff is given by

$$\pi_{T_2}^{mg} = \max\left(\frac{S_{T_1}}{S_0}, e^{g_1}\right) \cdot \max\left(\frac{S_{T_2}}{S_{T_1}}, e^{g_2}\right).$$

The expression $\max\left(\frac{S_{T_2}}{S_{T_1}}, e^{g_2}\right)$ is the same payoff as that of a maturity guarantee over the time period from time T_1 to T_2 and where the initial amount to accrue interest is normalised to one. The time T_1 market value of the two-period guarantee is therefore equal to

$$\pi_{T_1}^{mg}(\pi^g) = \max\left(\frac{S_{T_1}}{S_0}, e^{g_1}\right) \cdot \pi_{T_1}^g,$$

where π^g is the maturity guarantee and $\pi_{T_1}^g$ is the time T_1 market value of the maturity guarantee.

The interpretation of the two-period guarantee is somewhat different than the interpretation of the maturity guarantee. The two-period guarantee gives the holder the opportunity to choose between two different quantities (one of them \mathcal{F}_{T_1} -measurable) of the underlying asset (i.e., the maturity guarantee), whereas the maturity guarantee gave the holder the choice between one unit of the underlying asset and X . This choice can be made at time T_1 at no cost. Comparing this to the compound option, we see that also the holder of the compound option can choose between two different quantities of the underlying asset (i.e., the call option); one or zero units, and if the holder chooses to receive one unit, it comes at a cost.

If we instead think of the maturity guarantee as offering the holder the choice between a stochastic ($\frac{S_T}{S_0}$) and a deterministic (e^g) number of units of account, where one unit of account is equal to 1, the two-period guarantee is almost identical to the maturity guarantee. The main difference is that for the two-period guarantee one unit of account is equal to $\pi_{T_1}^g$.

The above shows that also the two-period guarantee can be interpreted as a compound contingent claim, just as the compound option can. This feature does not seem to have been recognised in the existing literature on multi-period guarantees. In the next section we construct a generalised compound contingent claim that captures these two claims as special cases.

1.6 A Generalised Compound Contingent Claim

We will now, as for the simple contingent claim in section 1.4, construct a generalised *compound contingent claim* that captures the compound option and the two-period guarantee as special cases. By a compound contingent claim we mean a contingent claim that is written on some other contingent claim. In fact, we let the simple contingent claim in section 1.4 be the underlying asset.

Consider now a claim with the following time T_1 market value

$$f_{T_1}(g) = \max(\alpha g_{T_1} - K, \gamma g_{T_1}), \quad (1.6)$$

where each of α , γ , and K is equal to either zero, a strictly positive constant, or a positive valued random variable (i.e., the same possibilities as for A_T , B_T , and C_T in section 1.4). Again, the claim is somewhat ad-hoc; though it has the necessary generality to capture the compound call option and the two-period guarantee as special cases. To show this, let $\alpha = 1$, $K = X_1$, $\gamma = 0$, and $g_{T_1} = \pi_{T_1}^c$. This gives

$$f_{T_1}(g) = \max(\pi_{T_1}^c - X_1, 0),$$

and is equal to the time T_1 market value of a compound call option. If instead $\alpha = \frac{S_{T_1}}{S_0}$, $K = 0$, $\gamma = e^{g_1}$, and $g_{T_1} = \pi_{T_1}^g$, we get

$$f_{T_1}(g) = \max\left(\frac{S_{T_1}}{S_0} \pi_{T_1}^g, e^{g_1} \pi_{T_1}^g\right) = \max\left(\frac{S_{T_1}}{S_0}, e^{g_1}\right) \cdot \pi_{T_1}^g,$$

and this is equal to the time T_1 market value of a two-period guarantee.

Using the results in section 1.4, changing the maturity date for the simple claim from time T to T_2 , and valuing the claim at time T_1 instead of at time 0, the market value can be written as

$$g_{T_1} = A_{T_1} E_{Q_1} \left[1_{\mathcal{A}_2} \middle| \mathcal{F}_{T_1} \right] - B_{T_1} E_{Q_2} \left[1_{\mathcal{A}_2} \middle| \mathcal{F}_{T_1} \right] + C_{T_1} E_{Q_3} \left[1_{\mathcal{A}_2} \middle| \mathcal{F}_{T_1} \right], \quad (1.7)$$

where $\mathcal{A}_2 = \mathcal{A}$ and $\bar{\mathcal{A}}_2$ is the complement to \mathcal{A}_2 . The time 0 market value of the compound contingent claim can be written as

$$f_0(g) = E_Q \left[e^{-\beta T_1} \max(\alpha g_{T_1} - K, \gamma g_{T_1}) \right].$$

Define

$$\alpha A_0 \equiv E_Q \left[e^{-\beta T_1} \alpha A_{T_1} \right],$$

$$\alpha B_0 \equiv E_Q \left[e^{-\beta T_1} \alpha B_{T_1} \right],$$

$$\alpha C_0 \equiv E_Q \left[e^{-\beta T_1} \alpha C_{T_1} \right],$$

$$K_0 \equiv E_Q \left[e^{-\beta T_1} K \right],$$

$$\gamma A_0 \equiv E_Q \left[e^{-\beta T_1} \gamma A_{T_1} \right],$$

$$\gamma B_0 \equiv E_Q \left[e^{-\beta T_1} \gamma B_{T_1} \right],$$

and

$$\gamma C_0 \equiv E_Q \left[e^{-\beta T_1} \gamma C_{T_1} \right].$$

Define further the following Radon-Nikodym derivatives

$$\frac{dQ_4}{dQ} = \frac{e^{-\beta T_1} \alpha A_{T_1}}{E_Q \left[e^{-\beta T_1} \alpha A_{T_1} \right]},$$

$$\frac{dQ_5}{dQ} = \frac{e^{-\beta T_1} \alpha B_{T_1}}{E_Q \left[e^{-\beta T_1} \alpha B_{T_1} \right]},$$

$$\frac{dQ_6}{dQ} = \frac{e^{-\beta T_1} \alpha C_{T_1}}{E_Q \left[e^{-\beta T_1} \alpha C_{T_1} \right]},$$

$$\frac{dQ_7}{dQ} = \frac{e^{-\beta T_1} K}{E_Q \left[e^{-\beta T_1} K \right]},$$

$$\frac{dQ_8}{dQ} = \frac{e^{-\beta T_1} \gamma A_{T_1}}{E_Q \left[e^{-\beta T_1} \gamma A_{T_1} \right]},$$

$$\frac{dQ_9}{dQ} = \frac{e^{-\beta T_1} \gamma B_{T_1}}{E_Q \left[e^{-\beta T_1} \gamma B_{T_1} \right]},$$

and

$$\frac{dQ_{10}}{dQ} = \frac{e^{-\beta T_1} \gamma C_{T_1}}{E_Q \left[e^{-\beta T_1} \gamma C_{T_1} \right]}.$$

In any of the cases where the denominator in the expressions for the Radon-Nikodym derivatives equals zero, we define, as in section 1.4, the Radon-Nikodym derivative to be equal to zero.

Combining the above, the time zero market value of the compound contingent claim can be written as

$$\begin{aligned} f_0(g) = & \alpha A_0 Q_4(\mathcal{A}_1 \cap \mathcal{A}_2) - \alpha B_0 Q_5(\mathcal{A}_1 \cap \mathcal{A}_2) + \alpha C_0 Q_6(\mathcal{A}_1 \cap \bar{\mathcal{A}}_2) \\ & - K_0 Q_7(\mathcal{A}_1) + \gamma A_0 Q_8(\bar{\mathcal{A}}_1 \cap \mathcal{A}_2) - \gamma B_0 Q_9(\bar{\mathcal{A}}_1 \cap \mathcal{A}_2) \\ & + \gamma C_0 Q_{10}(\bar{\mathcal{A}}_1 \cap \bar{\mathcal{A}}_2), \end{aligned} \quad (1.8)$$

where $\mathcal{A}_1 = \{\alpha g_{T_1} - K \geq \gamma g_{T_1}\}$ and $\bar{\mathcal{A}}_1$ is the complement to \mathcal{A}_1 .

To determine the market value of the compound contingent claim we need to be able to determine the exercise probabilities, under the appropriate probability measures, for the claim under consideration. This is the same as saying that we need to determine for what values of the underlying asset(s) the claim will be exercised. For the compound contingent claim this means that we must be able to determine for what value(s) of the underlying asset(s) the following inequality holds with equality

$$\alpha g_{T_1} - K \geq \gamma g_{T_1}. \quad (1.9)$$

We know from the discussion on page 20 that we must be able to determine when (1.9) holds with equality based on the information available at time zero.

As a first example, consider the compound option analysed by Geske (1979), i.e., a call option on a call option under deterministic interest rates. (1.9) then becomes (where d_1 and d_2 are "adjusted" to time T_1)

$$S_{T_1} \Phi(d_1) - P(T_1, T_2) X_2 \Phi(d_2) \geq X_1, \quad (1.10)$$

where the left-hand side of the inequality in (1.10) now is the time T_1 market value of a call option maturing at time $T_2 > T_1$. Since the call option is strictly increasing in the market value of the underlying stock, it follows by the intermediate value property⁶ that there exists a stock price s^* that makes (1.10) hold with equality for all $X_1 \in (0, \infty)$, and the probabilities for the compound option being exercised can then be calculated.

Consider now the setting in this chapter, i.e., stochastic interest rates. Then there is no longer one unique s^* for each X_1 , but several, each as a

⁶See e.g., Rudin (1976) Theorem 4.23.

function of the \mathcal{F}_{T_1} -measurable random variable $P(T_1, T_2)$. This complicates matters quite considerably since there does not seem to exist any trivial relationship between the stock price and the bond price that can be used to determine the exercise probabilities for the compound option. Hence, a closed form solution for the market value of a compound option in a stochastic interest rate framework does not seem to be easily obtainable. Searching the literature, the only work on compound options and stochastic interest rates that we have found is in Geman et al. (1995), but their analysis seems flawed in that they assume that there exists a unique \mathcal{F}_0 -measurable s^* .

If the holder of the compound option instead of delivering X_1 units of the face value of a zero-coupon bond for exercising it at time T_1 could deliver X_1 units of the zero-coupon bond maturing at time T_2 , i.e., $X_1 P(T_1, T_2)$, (1.10) could be simplified to (the only difference is that the maturity date for the bond delivered is changed from time T_1 to T_2)

$$R_{T_1} \Phi(d_1) - X_2 \Phi(d_2) \geq X_1, \quad (1.11)$$

where $R_{T_1} = \frac{S_{T_1}}{P(T_1, T_2)}$ can be interpreted as the market value of the underlying asset of a call option with zero interest rates (see e.g., Carr (1988)). Again using the fact that a call option is strictly increasing in the market value of the underlying asset, it follows that there exists a unique R^* that makes (1.11) hold with equality. Hence, the probabilities for the compound option being exercised can then be calculated.

It seems like if the rewriting above (and similar ones) is possible, it will also be *sufficient* for the derivation of a closed form solution, i.e., the rewriting that makes it possible to calculate the exercise probabilities for the compound option. However, since we have not tried every possible approach, we cannot claim that it is *necessary* to be able to perform such a rewriting for there to exist a closed form solution.

For what specifications of the compound contingent claim do there exist a closed form solution? First, for $g \in \{a, b, c\}$ (see Table 1.1 - 1.3) the claim $f(g)$ is not a compound contingent claim, but at best a contingent claim, and we will therefore not give any attention to these specifications in this section. Since there does not exist a closed form solution for the simple claim when $g \in \{j, l, m, n\}$, we will not be able determine when (1.9) holds with equality, hence, we are not able to find a closed form solution for the market value of the compound contingent claim. It turns out that $g \in \{d, e, f, g, h, i, k\}$ are quite similar.

When the simple claim falls into the categories d), e), and g), the time T_1 market value can be written on the form

$$g_{T_1} = \pm S_{T_1} \Phi(\varphi_1) \pm P(T_1, T_2) X \Phi(\varphi_2),$$

for the categories f) and k)

$$g_{T_1} = \pm \mathcal{S}_{T_1} \Phi(\varphi_1) \pm \mathcal{V}_{T_1} \Phi(\varphi_2),$$

and, finally, for the categories h) and i) as

$$g_{T_1} = \pm \mathcal{S}_{T_1} \Phi(\varphi_1) \pm P(T_1, T_2) X \Phi(\varphi_2) + P(T_1, T_2) \bar{K}.$$

From now on we define \mathcal{S}_{T_1} as the market value of the *first* asset and $P(T_1, T_2)$ and \mathcal{V}_{T_1} as the market value of the *second* asset. Here \bar{K} is a constant. φ_1 and φ_2 will typically not be the same across the different specifications, but it will not be necessary to specify them any closer here. Using the definitions and descriptions below, we have in Table 1.4 - 1.6 showed for what specifications of the compound contingent claim in (1.6) the market value can be obtained in closed form solution.

The abbreviations in Table 1.4 - 1.6 are defined as follows:

- a') = 0.
- b') = a constant number of g .
- c') = a constant number of call options on g .
- d') = exchange K to receive a constant number of g - solvable if K is a function of the second asset.
- e') = a random number of g .
- f') = a call option on a random number of g .
- g') = exchange K to receive a random number of g . Solvable if α is a function of the first asset and K is a function of the second asset.
- h') = $\alpha > \gamma \Rightarrow$ a given number of the Geske (1979)-option + b'), otherwise b').
- i') = $\alpha \leq \gamma \Rightarrow$ b'), $\alpha > \gamma \Rightarrow$ d') + b') if K a function of the second asset, otherwise not solvable.
- j') = the maximum of a random and a constant number of g .
- k') = the maximum of a random number of g subtracted a constant and a constant number of g .
- l') = the maximum of a random number of g subtracted a random variable and a constant number of g .
- m') = the maximum of a constant number of g subtracted a constant and a random number of g .

Table 1.4: Specifications for the claim $f(g)$ for $\gamma = 0$.

	$\alpha = 0$	$\alpha = \bar{\alpha}$	$\alpha = \tilde{\alpha}$
	a')	b')	e')
$K = 0$	$\max(0, 0)$	$\max(\bar{\alpha}g_{T_1}, 0)$	$\max(\tilde{\alpha}g_{T_1}, 0)$
	a')	c')*	f')*
$K = \bar{K}$	$\max(-\bar{K}, 0)$	$\max(\bar{\alpha}g_{T_1} - \bar{K}, 0)$	$\max(\tilde{\alpha}g_{T_1} - \bar{K}, 0)$
	a')	d')	g')
$K = \tilde{K}$	$\max(-\tilde{K}, 0)$	$\max(\bar{\alpha}g_{T_1} - \tilde{K}, 0)$	$\max(\tilde{\alpha}g_{T_1} - \tilde{K}, 0)$

Table 1.5: Specifications for the claim $f(g)$ for $\gamma = \bar{\gamma}$.

	$\alpha = 0$	$\alpha = \bar{\alpha}$	$\alpha = \tilde{\alpha}$
	b')	b')	j')
$K = 0$	$\max(0, \bar{\gamma}g_{T_1})$	$\max(\bar{\alpha}g_{T_1}, \bar{\gamma}g_{T_1})$	$\max(\tilde{\alpha}g_{T_1}, \bar{\gamma}g_{T_1})$
	b')	h')	k')*
$K = \bar{K}$	$\max(-\bar{K}, \bar{\gamma}g_{T_1})$	$\max(\bar{\alpha}g_{T_1} - \bar{K}, \bar{\gamma}g_{T_1})$	$\max(\tilde{\alpha}g_{T_1} - \bar{K}, \bar{\gamma}g_{T_1})$
	b')	i')	l')*
$K = \tilde{K}$	$\max(-\tilde{K}, \bar{\gamma}g_{T_1})$	$\max(\bar{\alpha}g_{T_1} - \tilde{K}, \bar{\gamma}g_{T_1})$	$\max(\tilde{\alpha}g_{T_1} - \tilde{K}, \bar{\gamma}g_{T_1})$

Table 1.6: Specifications for the claim $f(g)$ for $\gamma = \tilde{\gamma}$.

	$\alpha = 0$	$\alpha = \bar{\alpha}$	$\alpha = \tilde{\alpha}$
	e')	j')	o')
$K = 0$	$\max(0, \tilde{\gamma}g_{T_1})$	$\max(\bar{\alpha}g_{T_1}, \tilde{\gamma}g_{T_1})$	$\max(\tilde{\alpha}g_{T_1}, \tilde{\gamma}g_{T_1})$
	e')	m')*	p')*
$K = \bar{K}$	$\max(-\bar{K}, \tilde{\gamma}g_{T_1})$	$\max(\bar{\alpha}g_{T_1} - \bar{K}, \tilde{\gamma}g_{T_1})$	$\max(\tilde{\alpha}g_{T_1} - \bar{K}, \tilde{\gamma}g_{T_1})$
	e')	n')*	q')*
$K = \tilde{K}$	$\max(-\tilde{K}, \tilde{\gamma}g_{T_1})$	$\max(\bar{\alpha}g_{T_1} - \tilde{K}, \tilde{\gamma}g_{T_1})$	$\max(\tilde{\alpha}g_{T_1} - \tilde{K}, \tilde{\gamma}g_{T_1})$

n') = the maximum of a constant number of g subtracted a random variable and a random number of g .

o') = the maximum of two random numbers of g .

p') = the maximum of a random number of g subtracted a constant and a random number of g .

q') = the maximum of a random number of g subtracted a random variable and a random number of g .

As in section 1.4, the above may not apply if two or more of the variables coincide or are linear functions of the market value of the same asset.

1.6.1 A Parity for the Compound Contingent Claim

We will in this subsection derive a parity for the compound contingent claim.

The mirror claim for the compound contingent claim has the following time T_1 market value

$$f_{T_1}^m(g) = \max(K - \alpha g_{T_1}, -\gamma g_{T_1}).$$

We now define the following for $t < T_1$, $K_t \equiv E_Q \left[e^{-\int_t^{T_1} r_v dv} K \right]$, $\alpha g_t \equiv E_Q \left[e^{-\int_t^{T_1} r_v dv} \alpha g_{T_1} \right]$, and $\gamma g_t \equiv E_Q \left[e^{-\int_t^{T_1} r_v dv} \gamma g_{T_1} \right]$.

Theorem 1.2. *For the compound contingent claim, we have the following parity for $t \leq T_1$*

$$f_t(g) = f_t^m(g) + \alpha g_t - K_t + \gamma g_t.$$

Proof. The left and the right-hand side of the parity have the same time T_1 market value, and the result follows therefore in the absence of arbitrage. \square

1.7 Other Claims Captured by (1.6)

In this section we give a closer analysis of some of the claims captured by the general claim in (1.6). The market values are found using the general formula in (1.8). In the proofs we have for simplicity only taken into account the terms in (1.8) that are non-zero.

1.7.1 A Compound Exchange Option

An exchange option seems first to have been analysed by Fischer (1978) and Margrabe (1978). This is a contingent claim that gives the holder the option to exchange a given number of units of one assets in return for one unit of another asset, say, deliver X units of an asset with market value S_T^2 to receive one unit of an asset with market value S_T^1 . Carr (1988) analysed a compound exchange option, i.e., an option to exchange a given number of units of an asset to receive one unit of an exchange option.

Consider the following specification of (1.6): $A_{T_2} = S_{T_2}^1$, $B_{T_2} = X_2 S_{T_2}^2$, $C_{T_2} = 0$, $\alpha = 1$, $K = X_1 S_{T_1}^2$, and $\gamma = 0$. This gives the same payoff as the compound exchange option.

Proposition 1.1. (Carr (1988)) *The time 0 market value of an exchange option on an exchange option is given by*

$$\begin{aligned} f_0(g) = & S_0^1 \Phi(d_3, d_4, \rho) - X_2 S_0^2 \Phi(d_3 - v(T_1), d_4 - v(T_2), \rho) \\ & - X_1 S_0^2 \Phi(d_3 - v(T_1)), \end{aligned}$$

where

$$\begin{aligned} d_3 &= \frac{\ln\left(\frac{R_0}{R^*}\right) + \frac{1}{2}v^2(T_1)}{v(T_1)}, \\ d_4 &= \frac{\ln\left(\frac{S_0^1}{X_2 S_0^2}\right) + \frac{1}{2}v^2(T_2)}{v(T_2)}, \\ \rho &= \frac{v(T_1)}{v(T_2)}, \end{aligned}$$

$$R_0 = \frac{S_0^1}{S_0^2},$$

$$v^2(T_i) = \int_0^{T_i} \left(\sigma_{S^1}^2(v) - 2\sigma_{S^1}(v)\sigma_{S^2}(v) + \sigma_{S^2}^2(v) \right) dv,$$

$\Phi(a, b, p)$ is the cumulative bivariate normal probability distribution evaluated at the points a and b with correlation p , and R^* is the critical ratio of $\frac{S_{T_1}^1}{S_{T_1}^2}$ that makes the time T_1 market value of the underlying exchange option equal to $X_1 S_{T_1}^2$.

Proof. The market value can be found using (1.8). For the compound exchange option it follows that $\alpha A_0 = S_0^1$, $\alpha B_0 = X_2 S_0^2$, and $K_0 = X_1 S_0^2$. The three probability measures Q_4 , Q_5 , and Q_7 are defined by the Radon-Nikodym derivatives

$$\frac{dQ_4}{dQ} = e^{-\frac{1}{2} \int_0^t \sigma_{S^1}^2(v) dv + \int_0^t \sigma_{S^1}(v) dW_v}$$

and

$$\frac{dQ_5}{dQ} = \frac{dQ_7}{dQ} = e^{-\frac{1}{2} \int_0^t \sigma_{S^2}^2(v) dv + \int_0^t \sigma_{S^2}(v) dW_v}.$$

From this we get that

$$f_0(g) = S_0^1 Q_4(\mathcal{A}_1 \cap \mathcal{A}_2) - X_2 S_0^2 Q_5(\mathcal{A}_1 \cap \mathcal{A}_2) - X_1 S_0^2 Q_7(\mathcal{A}_1),$$

where $\mathcal{A}_1 = \{\pi_{T_1}^{eo} > X_1 S_{T_1}^2\}$, $\mathcal{A}_2 = \{S_{T_2} > X_2 S_{T_2}^2\}$, and $\pi_{T_1}^{eo}$ is the time T_1 market value of the underlying exchange option. The result then follows. \square

It is interesting to notice that the result in Proposition 1.1 that is derived under stochastic interest rates is (if $\sigma_{S^i}(v)$ is time independent) identical to the result in Carr (1988) where the result is derived under deterministic interest rates. This is in line with the comment in Carr (1988) that "... there is no presumption that the term structure of interest rates be flat or even known."

Carr (1988) analysed several claims that can be shown to be special cases of his formula and different interpretations of the compound exchange option. All these claims and interpretations are of course also captured by the claim in (1.6).

1.7.2 An Option on a Maturity Guarantee

Another version of a compound contingent claim is the following (this is, to the best of our knowledge, a claim that has not previously been analysed). Assume that one at time T_1 has the right to exchange X_1 units of a zero-coupon bond maturing at time T_2 for one unit of a maturity guarantee maturing at time T_2 . The compound contingent claim in (1.6) and this claim are seen to coincide when using the following specification: $A_{T_2} = S_{T_2}$, $B_{T_2} = 0$, $C_{T_2} = X_2$, $\alpha = 1$, $K = X_1 P(T_1, T_2)$, and $\gamma = 0$.

Proposition 1.2. *The time 0 market value of an option to exchange X_1 units of a zero-coupon bond maturing at time T_2 for one unit of a maturity guarantee maturing at time T_2 is given by*

$$f_0(g) = S_0 \Phi(d_5, d_6, \rho) + X_2 P(0, T_2) \Phi(d_5 - \sigma_{R_{T_1}}, -d_6 + \sigma_{\delta_{T_2}}, -\rho) - X_1 P(0, T_2) \Phi(d_5 - \sigma_{R_{T_1}}),$$

where

$$\begin{aligned} d_5 &= \frac{\ln(\frac{R_0}{R^*}) + \frac{1}{2} \sigma_{R_{T_1}}^2}{\sigma_{R_{T_1}}}, \\ d_6 &= \frac{\ln(\frac{S_0}{X_2 P(0, T_2)}) + \frac{1}{2} \sigma_{\delta_{T_2}}^2}{\sigma_{\delta_{T_2}}}, \\ \sigma_{R_{T_1}}^2 &= \int_0^{T_1} (\int_v^{T_2} \sigma_f(v, u) du)^2 dv + 2 \int_0^{T_1} \sigma_S(v) \int_v^{T_2} \sigma_f(v, u) dudv \\ &\quad + \int_0^{T_1} \sigma_S^2(v) dv, \\ \rho &= \frac{\text{cov}(\ln(R_{T_1}), \delta_{T_2})}{\sigma_{R_{T_1}} \sigma_{\delta_{T_2}}} = \frac{\sigma_{R_{T_1}}}{\sigma_{\delta_{T_2}}}, \end{aligned}$$

$R_0 = \frac{S_0}{P(0, T_2)}$, and R^* is the critical ratio $\frac{S_{T_1}}{P(T_1, T_2)}$ that makes the time T_1 market value of the maturity guarantee equal to $X_1 P(T_1, T_2)$.

Proof. The time 0 market value can be found using (1.8). For the exchange option on the maturity guarantee it follows that $\alpha A_0 = S_0$, $\alpha C_0 = X_2 P(0, T_2)$, and $K_0 = X_1 P(0, T_2)$. The probability measures Q_4 , Q_6 , and Q_7 are defined by the Radon-Nikodym derivatives

$$\frac{dQ_4}{dQ} = e^{-\frac{1}{2} \int_0^t \sigma_S^2(v) dv + \int_0^t \sigma_S(v) dW_v}$$

and

$$\frac{dQ_6}{dQ} = \frac{dQ_7}{dQ} = e^{-\frac{1}{2} \int_0^t (\int_v^{T_2} \sigma_f(v, u) du)^2 dv - \int_0^t \int_v^{T_2} \sigma_f(v, u) dudW_v},$$

respectively. It then follows that

$$f_0(g) = S_0 Q_4(1_{\mathcal{A}_1 \cap \mathcal{A}_2}) + X_2 P(0, T_2) Q_6(1_{\mathcal{A}_1 \cap \bar{\mathcal{A}}_2}) - X_1 P(0, T_2) Q_7(1_{\mathcal{A}_1}),$$

where $\mathcal{A}_1 = \{R_{T_1} > R^*\}$, $\mathcal{A}_2 = \{S_{T_2} > X_2\}$, and $\bar{\mathcal{A}}_2$ is the complement to \mathcal{A}_2 .

Consider now the inequality (where d_1 and d_2 are “adjusted” to time T_1)

$$S_{T_1} \Phi(d_1) + X_2 P(T_1, T_2) \Phi(-d_2) \geq X_1 P(T_1, T_2). \quad (1.12)$$

The left-hand side of (1.12) is the time T_1 market value of the underlying maturity guarantee and the right-hand side is the time T_1 exercise price for the compound contingent claim. Dividing through by $P(T_1, T_2)$, we get

$$R_{T_1} \Phi(d_1) + X_2 \Phi(d_2) \geq X_1. \quad (1.13)$$

That there exists an R^* that makes (1.13) hold with equality follows since the left-hand side of (1.13) can be thought of as the time T_1 market value of a maturity guarantee with $R_{T_1} = \frac{S_{T_1}}{P(T_1, T_2)}$ being the market value of the underlying asset and with zero interest rates. The market value of this claim is strictly increasing in R_{T_1} and there does therefore exist a solution to (1.13), i.e., a parameter R^* .

The result then follows. \square

1.7.3 Instantaneous Compound Contingent Claims

We now analyse a type of contingent claims that we have not found previously to be treated as compound contingent claims. For the assets we have in mind here, the two exercise dates, T_1 and T_2 , coincide and are termed T . These claims do not exactly fit into our general claims. However, replacing the max-operator in the expression for the simple claim by a min-operator, things work out fine.

Consider first a capped call option, i.e., a contingent claim that gives the final time T payoff

$$\begin{aligned} f_T(g) &= \max(\min(S_T, X_2) - X_1, 0) \\ &= \max(-\max(-S_T, -X_2) - X_1, 0), \end{aligned} \quad (1.14)$$

where we assume that $X_2 \geq X_1 > 0$. The expression in (1.14) can be rewritten as

$$\begin{aligned} f_T(g) &= \max(S_T - X_1, \max(S_T - X_2, 0)) - \max(S_T - X_2, 0) \\ &= \max(S_T - X_1, 0) - \max(S_T - X_2, 0), \end{aligned}$$

since $X_2 \geq X_1$. This is the difference between two call options, and from section 1.4 we know that the market value is easily obtainable in closed form solution (corresponds to the case denoted d) in Table 1.1 - 1.3).

This compound contingent claim can be obtained as a special case of (1.6) by using the following specification: $A_T = S_T$, $B_T = 0$, $C_T = X_2$, $\alpha = 1$, $K = X_1$, and $\gamma = 0$.

Another compound contingent claim, though somewhat similar as the one in (1.14), is a call option on the minimum of two assets and has been analysed by Stulz (1982) and Johnson (1987). This claim has the terminal payoff

$$f_T(g) = \max(\min(S_T^1, S_T^2) - X, 0).$$

The specification for the claim in (1.6) that corresponds to a call option on the minimum of two assets is as follows: $A_T = S_T^1$, $B_T = 0$, $C_T = S_T^2$, $\alpha = 1$, $K = X$, and $\gamma = 0$.

1.7.4 A Random Number of Call Options

We end this section by considering a claim that is captured by the general claim in (1.6) but that is not a compound contingent claim. Assume that we at time T_1 will receive a random number of call options, more precisely S_{T_1} units. This could for instance be some sort of a bonus mechanism for the employees. Instead of using more traditional stock options as an incentive, we could strengthen the incentive by also making the number of call options depend on the development in the stock price. This is a sort of a *quanto* option, see e.g., Reiner (1992).

This claim is obtained by the following specification: $A_{T_2} = S_{T_2}$, $B_{T_2} = X$, $C_{T_2} = 0$, $\alpha = S_{T_1}$, $K = 0$, and $\gamma = 0$. What is the value of such a claim?

Proposition 1.3. *The time 0 market value of the claim with time T_2 payoff $S_{T_1} \max(S_{T_2} - X, 0)$ is given by*

$$f_0(g) = \frac{(S_0)^2}{P(0, T_1)} e^{\sigma \delta_{T_1}} \Phi(d_7) - S_0 F(0, T_1, T_2) X e^{-\text{cov}(\delta_{T_2-T_1}, \delta_{T_1})} \Phi(d_8),$$

where

$$d_7 = \frac{\ln\left(\frac{S_0}{XP(0, T_2)}\right) + \frac{1}{2}\sigma_{\delta_{T_2}}^2 + \sigma_{\delta_{T_1}} + \text{cov}(\delta_{T_2-T_1}, \delta_{T_1})}{\sigma_{\delta_{T_2}}},$$

$$d_8 = \frac{\ln\left(\frac{S_0}{XP(0, T_2)}\right) + \frac{1}{2}\sigma_{\delta_{T_2}}^2 - \sigma_{\delta_{T_2-T_1}} - \text{cov}(\delta_{T_2-T_1}, \delta_{T_1})}{\sigma_{\delta_{T_2}}},$$

and

$$\text{cov}(\delta_{T_2-T_1}, \delta_{T_1}) = c_{T_2-T_1, T_1} + \int_0^{T_1} \sigma_S(v) \int_{T_1}^{T_2} \sigma_f(v, u) du dv.$$

Proof. From (1.8) it follows that

$$\alpha A_0 = \frac{(S_0)^2}{P(0, T_1)} e^{\sigma \delta_{T_1}}$$

and

$$\alpha B_0 = S_0 F(0, T_1, T_2) X e^{-\text{cov}(\delta_{T_2-T_1}, \delta_{T_1})}.$$

The exercise set for this claim is given by $\mathcal{A} = \{S_{T_2} > X\}$. Using the Radon-Nikodym derivatives

$$\frac{dQ_4}{dQ} = \frac{S_{T_1} S_{T_2} / M_{T_2}}{E_Q[S_{T_1} S_{T_2} / M_{T_2}]}$$

and

$$\frac{dQ_5}{dQ} = \frac{S_{T_1} / M_{T_2}}{E_Q[S_{T_1} / M_{T_2}]},$$

it follows that the market value can be written as

$$f_0(g) = \frac{(S_0)^2}{P(0, T_1)} e^{\sigma \delta_{T_1}} Q_4(\mathcal{A}) - S_0 F(0, T_1, T_2) X e^{-\text{cov}(\delta_{T_2-T_1}, \delta_{T_1})} Q_5(\mathcal{A}).$$

The result then follows. \square

Another interpretation of this claim can be obtained by replacing $\alpha = S_{T_1}$ with a time T_2 currency exchange rate, say, Y_{T_2} , and then by interpreting the call option as an option on a stock in a foreign economy. By arbitrage arguments, it is easily seen that the time 0 market value of such a claim is equal to $Y_0 g_0$, where g_0 now is the time 0 market value of the call option denoted in the foreign economy's currency.

1.8 Conclusions

We have in this chapter constructed two general contingent claims. The first a simple claim that is written on primary traded assets. Among the claims that were captured by this claim, special attention was given on a call option and a maturity guarantee. The second was a compound contingent claim that was written on the simple claim. First the focus was on the similarities between a compound option and a two-period guarantee. The analysis also showed that the market value of a compound option under stochastic interest rates is not easily obtainable. In addition, also a few of the other claims captured by the general compound contingent claim were given a deeper analysis. Among these, the compound exchange option analysed by Carr (1988) was rediscovered, but this time under stochastic interest rates.

Chapter 2

Pricing of Multi-period Rate of Return Guarantees

Abstract

The basis for this chapter is the pricing of multi-period rate of return guarantees. These guarantees can typically be found in life insurance and pension contracts. We derive closed form solutions, expressed by the cumulative multivariate normal probability distribution, for multi-period rate of return guarantees on both a money market account and a stock. The guarantees of Hipp (1996), Persson and Aase (1997), and Miltersen and Persson (1999) are special cases of our results.

Keywords and phrases: Multi-period rate of return guarantees, Heath, Jarrow, and Morton term structure model of interest rates.

2.1 Introduction

Most financial investments are exposed to the risk of getting a low rate of return. By including a minimum rate of return guarantee in a financial contract, the risk of getting a low rate of return on the investment is eliminated, although the rate of return is still risky.

One of the earliest treatments of guarantees is due to Brennan and Schwartz (1976). They considered maturity guarantees, and they showed, by using the framework and the results of Black and Scholes (1973), that a maturity guarantee is equivalent to holding a European put option and the underlying asset (or, equivalently, a risk-free investment and a European call option). They also included mortality risk and extended the results to

periodic premium payments. This is the same kind of guarantee that can be found in index-linked bonds.

Hipp (1996) recognised that the guarantees included in many life insurance contracts are not maturity guarantees, but annual, or multi-period, guarantees. A multi-period guarantee secures a minimum rate of return in *each* period. This turns out to be a totally different guarantee than the maturity guarantee that can be interpreted as a one-period guarantee. Within the framework of Black and Scholes (1973), Hipp (1996) derived a closed form solution for the market value of a multi-period rate of return guarantee. For deterministic interest rates, the market value of a multi-period guarantee is given by a fairly simple expression. Persson and Aase (1997) investigated a two-period guarantee when interest rates are stochastic. They found that the market value is given as a function of the cumulative bivariate normal probability distribution. This work was continued by Miltersen and Persson (1999) in a Heath, Jarrow, and Morton setting. They found the market value of a two-period rate of return guarantee on both the short-term interest rate and the stock return.¹

This chapter generalises the analysis of Hipp (1996), which is performed under deterministic interest rates, to a setting with stochastic interest rates. The chapter also extends the two-period guarantees analysed by Persson and Aase (1997) and Miltersen and Persson (1999) to guarantees lasting for an arbitrarily number of periods.

Multi-period rate of return guarantees and similar guarantees have received some criticism in the literature for being inefficient in the sense that many investors are likely to prefer other payout structures that can be obtained at the same cost. Dybvig (1988) showed, analysing several trading strategies, including one with a related terminal payoff to the multi-period guarantees analysed in this chapter, that by changing the final payout, the investor can reduce his initial investment quite considerably and still find the final payout to be equally valuable. Also Brennan (1993), though analysing bonus mechanisms and in a somewhat other setting, found life insurance contracts with bonus mechanisms to be inefficient.

Multi-period rate of return guarantees are typically found in life insurance contracts. The narrow focus on the guarantees per se that we have in this chapter does not take into account several important aspects of life insurance contracts. Both legal requirements in different countries and different company policies will determine how returns are distributed between the insurer and the insured. These distribution mechanisms may be fairly involved, and life insurance contracts may therefore be embedded with several option and guarantee elements (see e.g., Grosen and Jørgensen (2000),

¹Reffs (1998) considered an instantaneous rate of return guarantee where the investment, at all times in the contract-period, accrues the maximum of the short-term interest rate and the minimum guaranteed rate of return.

Hansen and Miltersen (2002), and Miltersen and Persson (2002)). Also, mortality factors, periodical premiums, and the surrender option² are important aspects of life insurance contracts.

However, we believe that an isolated analysis of the guarantees is important for several reasons. Based on the findings in Dybvig (1988) and on the fact that numerical examples that are presented in this chapter suggest that multi-period guarantees can be very expensive, it is important to isolate the effect the guarantees have on the price of life insurance contracts. Since the prices of the guarantees are hidden among all the other factors of the life insurance contract, we think our analysis is important since it gives, more or less, an explicit price on one expensive part of the life insurance contract. Another reason, somewhat contradicting the above, is simply that it has been claimed that the pricing among practitioners of guarantees embedded in life insurance contracts often is insufficient. According to Donselaar (1999), as much as 75% of the Dutch life insurers offered minimum rate of return guarantees free of any charge.³ There has been several life insurance companies that have gone into bankruptcy because they were unable to fulfil the liabilities imposed by minimum rate of return guarantees, see e.g., Briys and de Varenne (1997). If the guarantees had been correctly priced, some of these incidents could perhaps have been avoided. The high price we find these guarantees to have should emphasize the importance of pricing them. We think that this demonstrates that the pricing of minimum rate of return guarantees is an important issue.

Since research seems to indicate that these guarantees are inefficient and we find them to be very expensive, an important question is whether they should be embedded in life insurance contracts or not. However, the answer to such a question is outside the scope of this chapter and our focus will only be on determining the market values of the guarantees.

An outline of the chapter goes as follows: In section 2.2 we give a description of the general framework we work within. In section 2.3 we calculate the market value of multi-period rate of return guarantees. In section 2.4 we have implemented the pricing formulas and numerical examples using realistic parameter values are presented. In section 2.5 we end the chapter with some concluding remarks.

2.2 The Economic Model

We work within an extended Heath et al. (1992) model, also called an Amin and Jarrow (1992) model. A description of this model can be found in an

²A surrender option is the right a policy holder has to terminate the policy prior to maturity. This kind of problem can be analysed as an optimal stopping problem, or in financial terms, as an American option. Grosen and Jørgensen (1997) found that the market value of the surrender option can be quite significant.

³It seems unlikely that this is only a Dutch phenomena.

advanced textbook in finance, see e.g., Musiela and Rutkowski (1997).

We assume that trading takes place on a continuous basis on the time interval $[0, T]$, for some fixed horizon $T > 0$. A filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ is fixed, where Ω is the state space, \mathcal{F} is a σ -algebra, $\mathbb{F} = \{\mathcal{F}_t, 0 \leq t \leq T\}$ is a filtration where $\mathcal{F}_T = \mathcal{F}$ and $\mathcal{F}_0 = \{\emptyset, \Omega\}$, where \emptyset is the empty set, and P is a probability measure. The σ -algebra is generated by a d -dimensional, $d \geq 1$, Brownian motion, W_t .⁴

We assume, under the equivalent martingale measure Q , that the instantaneous continuously compounded forward rate at time s , as seen from time t , $f(t, s)$, $0 \leq t \leq s \leq T$, is given by

$$f(t, s) = f(0, s) + \int_0^t \sigma_f(v, s) \int_v^s \sigma_f(v, u) du dv + \int_0^t \sigma_f(v, s) dW_v, \quad (2.1)$$

where sufficient regularity conditions for $\sigma_f(t, s)$, $0 \leq t \leq s \leq T$ are given in Heath et al. (1992).

The short-term interest rate $r_t = f(t, t)$. We will throughout assume that $\sigma_f(t, s)$, is a deterministic function, a fact which implies Gaussian interest rates. When considering deterministic interest rates we formally set $\sigma_f(t, s) = 0$. We also assume that there is a continuum of zero-coupon bonds trading in the market.

We let the market value of a non-dividend paying stock, S_t , be given under the equivalent martingale measure Q by the equation

$$S_t = S_0 + \int_0^t r_v S_v dv + \int_0^t \sigma_S(v) S_v dW_v, \quad (2.2)$$

where $r_t S_t$ satisfies the integrability condition $\int_0^t |r_v S_v| dv < \infty$ almost surely for all $t \leq T$. Here $\sigma_S(t)$ is a volatility function and satisfies the square integrability condition $E\left[\int_0^t (\sigma_S(v) S_v)^2 dv\right] < \infty$ (for further details on integrability conditions, see e.g., Duffie (1996)).

The money market account is an asset where interest accrues according to the short-term interest rate. The market value, M_t , is given by

$$M_t = M_0 + \int_0^t r_v M_v dv, \quad M_0 = 1, \quad (2.3)$$

where $r_t M_t$ satisfies the integrability condition $\int_0^t |r_v M_v| dv < \infty$ almost surely for all $t \leq T$.

From (2.2) and (2.3) we can see that the money market account, under the equivalent martingale measure Q , is just a special case of the stock since the money market account has no diffusion term.

⁴For the case with both stochastic interest rates and a stock, we require, in order to avoid perfect correlation between the stock and the interest rates, that $d > 1$.

In the rest of this chapter we divide the time into periods. Period n will be the time interval between time t_{n-1} and t_n . The initial investment is normalised to one.

Let the return on the money market account in period n , $n \in \{1, 2, \dots, N\}$, under the equivalent martingale measure Q be given by

$$\begin{aligned} \beta_n &= \int_{t_{n-1}}^{t_n} r_v dv = -\ln F(0, t_{n-1}, t_n) + \frac{1}{2} \sigma_{\beta_n}^2 + \sum_{k=1}^n c_{(k-1), n} \\ &\quad + \int_0^{t_{n-1}} \int_{t_{n-1}}^{t_n} \sigma_f(v, u) dudW_v + \int_{t_{n-1}}^{t_n} \int_v^{t_n} \sigma_f(v, u) dudW_v, \end{aligned}$$

where

$$F(0, t_m, t_n) = \frac{P(0, t_n)}{P(0, t_m)}$$

is the time 0 forward price for delivery at time t_m of a zero-coupon bond maturing at time $t_n \geq t_m$ and $P(0, t)$ is the time 0 market value of a zero-coupon bond maturing at time $t \geq 0$ with $P(t, t) = 1$ for all $t \in [0, T]$. Here $\sigma_{\beta_n}^2$ is the variance of the return on the money market account in period n under the equivalent martingale measure Q and is given by

$$\sigma_{\beta_n}^2 = \int_0^{t_{n-1}} \left(\int_{t_{n-1}}^{t_n} \sigma_f(v, u) du \right)^2 dv + \int_{t_{n-1}}^{t_n} \left(\int_v^{t_n} \sigma_f(v, u) du \right)^2 dv$$

and

$$\begin{aligned} c_{m, n} &= \int_0^{t_{m-1}} \left(\int_{t_{m-1}}^{t_m} \sigma_f(v, u) du \right) \left(\int_{t_{n-1}}^{t_n} \sigma_f(v, u) du \right) dv \\ &\quad + \int_{t_{m-1}}^{t_m} \left(\int_v^{t_m} \sigma_f(v, u) du \right) \left(\int_{t_{n-1}}^{t_n} \sigma_f(v, u) du \right) dv \end{aligned}$$

is the covariance between the return on the money market account under the equivalent martingale measure Q in period m and n , $1 \leq m < n$.

The time t_n market value of the money market account can also be written as

$$M_{t_n} = M_{t_{n-1}} e^{\beta_n}.$$

The return on the stock under the equivalent martingale measure Q in period n , $n \in \{1, 2, \dots, N\}$, is given by

$$\delta_n = \int_{t_{n-1}}^{t_n} \left(r_v - \frac{1}{2} \sigma_S(v)^2 \right) dv + \int_{t_{n-1}}^{t_n} \sigma_S(v) dW_v,$$

with variance

$$\sigma_{\delta_n}^2 = \sigma_{\beta_n}^2 + 2 \int_{t_{n-1}}^{t_n} \sigma_S(v) \int_v^{t_n} \sigma_f(v, u) dudv + \int_{t_{n-1}}^{t_n} \sigma_S^2(v) dv.$$

The time t_n market value of the stock can be written as

$$S_{t_n} = S_{t_{n-1}} e^{\delta_n}.$$

The covariance between the return on the stock in period m and the money market account in period n is given by $\bar{c}_{m,n}$ and the covariance between the return on the stock in period m and n is given by $\bar{\bar{c}}_{m,n}$. Using the Itô isometry, we get

$$\bar{c}_{m,n} = c_{m,n} + \int_{t_{m-1}}^{t_m} \sigma_S(v) \int_{t_{n-1}}^{t_n} \sigma_f(v, u) dudv$$

for $n > m$,

$$\bar{c}_{n,n} = \sigma_{\beta_n}^2 + \int_{t_{n-1}}^{t_n} \sigma_S(v) \int_v^{t_n} \sigma_f(v, u) dudv$$

for $m = n$, and

$$\bar{c}_{m,n} = c_{n,m}$$

for $n < m$.

$$\bar{\bar{c}}_{m,n} = c_{n,m} + \int_{t_{m-1}}^{t_m} \sigma_S(v) \int_{t_{n-1}}^{t_n} \sigma_f(v, u) dudv.$$

2.3 Pricing Multi-period Rate of Return Guarantees

Highly inspired by the results of Persson and Aase (1997) and Miltersen and Persson (1999), we follow their approach rather closely when deriving the pricing formulas for the multi-period rate of return guarantees. We find closed form solutions for the initial market value of N -period guarantees, $N \geq 2$, on the return on both the money market account and the stock. The solutions are expressed by the multivariate normal probability distribution. Setting $N = 2$, we obtain the results of Persson and Aase (1997) and Miltersen and Persson (1999) as special cases. For the guarantee on the stock return, the result of Hipp (1996) can be obtained as a special case by setting $\sigma_f(t, s) = 0$. We start by considering the money market account.

2.3.1 Pricing the Guarantee on the Money Market Account

Let N be the total number of periods. The terminal time $t_N < \mathcal{T}$ payoff for the guarantee on the return on the money market account is given by the random variable

$$\pi_{t_N}^\beta = e^{g_1 \vee \beta_1} \cdot e^{g_2 \vee \beta_2} \cdot \dots \cdot e^{g_N \vee \beta_N},$$

where $a \vee b = \max(a, b)$ and g_i is the minimum guaranteed rate of return in period i , $i \in \{1, 2, \dots, N\}$. To find the initial market value of the guarantee we have to find the expected deflated value of $\pi_{t_N}^\beta$ under the equivalent martingale measure. This is given by

$$\pi_0^\beta = E_Q \left[e^{(g_1 - \beta_1) \vee 0} \cdot e^{(g_2 - \beta_2) \vee 0} \cdot \dots \cdot e^{(g_N - \beta_N) \vee 0} \right]. \quad (2.4)$$

An N -period guarantee has two different possibilities in each period; (0) the guarantee is not binding, and (1) the guarantee is binding. For an N -period guarantee, this yields the possibility of in total 2^N different “states” of the world.

To evaluate the expectation in (2.4), we first define some vectors and matrices. Let \mathbf{c}_j , $j \in \{1, 2, \dots, 2^N\}$, be an $N \times 1$ -dimensional vector representing one “state” of the world. The i 'th element of \mathbf{c}_j , $i \in \{1, 2, \dots, N\}$, takes the value 1 when the guarantee is binding in the i 'th period and 0 otherwise. This, of course, yields 2^N unique \mathbf{c}_j 's, each having a unique combination of 0's and 1's.⁵

$\hat{\mathbf{c}}_j$, $j \in \{1, 2, \dots, 2^N\}$, is an $N \times N$ -dimensional symmetric matrix with only non-zero elements on the diagonal. The diagonal of $\hat{\mathbf{c}}_j$ is given by $2\mathbf{c}_j - \mathbf{1}$, where $\mathbf{1}$ is a vector only containing ones, i.e., the i 'th diagonal element of $\hat{\mathbf{c}}_j$ takes the value 1 when the guarantee is binding in the i 'th period and minus one otherwise. The minimum guaranteed rate of return in each period is given by the column vector $\mathbf{g} = (g_1, g_2, \dots, g_N)'$. The expected return on the money market account under the equivalent martingale measure Q is given by Λ , an $N \times 1$ -dimensional vector with i 'th element $\Lambda_i = -F(0, t_{i-1}, t_i) + \frac{1}{2}\sigma_{\beta_i}^2 + \sum_{k=1}^i c_{(k-1),i}$. Σ is the variance-covariance matrix of the multivariate normal distributed random variables $\beta = (\beta_1, \beta_2, \dots, \beta_N)'$. $\hat{\alpha}_j$ is an $N \times 1$ -dimensional vector, whose rational follows from the proof of Proposition 2.1. The i 'th element of $\hat{\alpha}_j$ is given by

$$\hat{\alpha}_{j,i} = \frac{g_i - \Lambda_i + (\Sigma \mathbf{c}_j)_i}{\sigma_{\beta_i}}, \quad (2.5)$$

where $(\Sigma \mathbf{c}_j)_i$ is the i 'th element of the vector $\Sigma \mathbf{c}_j$, and is due to a property for the multivariate normal probability distribution that is given in Lemma 2.1.

⁵To construct all 2^N \mathbf{c}_j 's, consider an $N \times 2^N$ -dimensional matrix with 2^N different columns equal to \mathbf{c}_j . In the first row, let the first 2^{N-1} elements equal 1, and the remaining 2^{N-1} elements equal 0. In row two, let the first 2^{N-2} elements equal 1, the next 2^{N-2} elements equal 0, the next 2^{N-2} elements equal 1, and finally the last 2^{N-2} elements equal 0. Let this continue, so that the elements in row N are equal to 1, 0, 1, ..., 1, and 0. The first column then corresponds to the state where the guarantee is binding in each period, and column 2^N the state where the guarantee is never binding.

Lemma 2.1. For multivariate normal distributed random variables \mathbf{X} with expectation $\boldsymbol{\mu}$, variance-covariance matrix \mathbf{V} , and probability density function $\phi(\mathbf{X}; \boldsymbol{\mu}, \mathbf{V})$, we have that

$$\phi(\mathbf{X}; \boldsymbol{\mu}, \mathbf{V}) \exp(-\mathbf{m}'\mathbf{X}) = \phi(\mathbf{X}; \boldsymbol{\mu} - \mathbf{V}\mathbf{m}, \mathbf{V}) \exp(-\mathbf{m}'\boldsymbol{\mu} + \frac{1}{2}\mathbf{m}'\mathbf{V}\mathbf{m}),$$

where \mathbf{m} can be any column vector with the same dimension as \mathbf{X} .

Proof. For the k -dimensional multivariate normal distributed random variables \mathbf{X} , we have that

$$\begin{aligned} & \phi(\mathbf{X}; \boldsymbol{\mu}, \mathbf{V}) e^{-\mathbf{m}'\mathbf{X}} \\ &= (2\pi)^{-1/2k} |\mathbf{V}|^{-1/2} e^{-1/2(\mathbf{X}-\boldsymbol{\mu})'\mathbf{V}^{-1}(\mathbf{X}-\boldsymbol{\mu}) - \mathbf{m}'\mathbf{X}}. \end{aligned} \quad (2.6)$$

Using the symmetry properties of \mathbf{V} , it follows by straightforward calculations that (2.6) can be rewritten as

$$\begin{aligned} & (2\pi)^{-1/2k} |\mathbf{V}|^{-1/2} e^{-1/2(\mathbf{X}-\boldsymbol{\mu}+\mathbf{V}\mathbf{m})'\mathbf{V}^{-1}(\mathbf{X}-\boldsymbol{\mu}+\mathbf{V}\mathbf{m}) - \mathbf{m}'\boldsymbol{\mu} + 1/2\mathbf{m}'\mathbf{V}\mathbf{m}} \\ &= \phi(\mathbf{X}; \boldsymbol{\mu} - \mathbf{V}\mathbf{m}, \mathbf{V}) \exp(-\mathbf{m}'\boldsymbol{\mu} + \frac{1}{2}\mathbf{m}'\mathbf{V}\mathbf{m}). \end{aligned}$$

□

Finally, $\boldsymbol{\alpha}_j = \hat{\mathbf{c}}_j \hat{\boldsymbol{\alpha}}_j$ is an $N \times 1$ -dimensional vector.

The solution of the expectation in (2.4) is given in Proposition 2.1.

Proposition 2.1. The initial market value of an N -period guarantee on the return on the money market account is given by

$$\pi_0^\beta = \sum_{j=1}^{2^N} e^{\mathbf{c}'_j \mathbf{g} - \mathbf{c}'_j \boldsymbol{\Lambda} + \frac{1}{2} \mathbf{c}'_j \boldsymbol{\Sigma} \mathbf{c}_j} \Phi(\boldsymbol{\alpha}_j, \hat{\mathbf{c}}_j \boldsymbol{\Sigma} \hat{\mathbf{c}}_j),$$

where

$\Phi(\mathbf{a}, \mathbf{V})$ is the cumulative multivariate normal probability distribution evaluated at the points determined by the vector \mathbf{a} and with variance-covariance matrix \mathbf{V} .

Proof. The proof follows the same lines as the proof in the appendix of Persson and Aase (1997). We let the vector \mathbf{c}_j , $j \in \{1, 2, \dots, 2^N\}$, represent a unique state. We let $1_{\mathbf{c}_j}$ be the indicator function for the state \mathbf{c}_j , returning the value 1 when \mathbf{c}_j is true and 0 otherwise. An expectation is a linear operator, and we can therefore split the expectation in (2.4) into the expected deflated payoff in each state, i.e.,

$$\pi_0^\beta = \sum_{j=1}^{2^N} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} 1_{\mathbf{c}_j} \phi(\boldsymbol{\beta}, \boldsymbol{\Lambda}, \boldsymbol{\Sigma}) \exp(\mathbf{c}'_j (\mathbf{g} - \boldsymbol{\beta})) d\beta_N \dots d\beta_2 d\beta_1.$$

For ease of exposition, we rewrite this as follows

$$\begin{aligned}
\pi_0^\beta &= \int_{-\infty}^{g_1} \int_{-\infty}^{g_2} \cdots \int_{-\infty}^{g_N} \phi(\beta, \Lambda, \Sigma) \exp(\mathbf{c}'_1(\mathbf{g} - \beta)) d\beta_N \cdots d\beta_2 d\beta_1 \\
&+ \int_{-\infty}^{g_1} \int_{-\infty}^{g_2} \cdots \int_{g_N}^{\infty} \phi(\beta, \Lambda, \Sigma) \exp(\mathbf{c}'_2(\mathbf{g} - \beta)) d\beta_N \cdots d\beta_2 d\beta_1 \\
&+ \dots + \int_{-\infty}^{g_1} \int_{g_2}^{\infty} \cdots \int_{g_N}^{\infty} \phi(\beta, \Lambda, \Sigma) \exp(\mathbf{c}'_{(N-1)}(\mathbf{g} - \beta)) d\beta_N \cdots d\beta_2 d\beta_1 \\
&+ \int_{g_1}^{\infty} \int_{g_2}^{\infty} \cdots \int_{g_N}^{\infty} \phi(\beta, \Lambda, \Sigma) \exp(\mathbf{c}'_N(\mathbf{g} - \beta)) d\beta_N \cdots d\beta_2 d\beta_1.
\end{aligned}$$

Using the property in Lemma 2.1, this can be rewritten as

$$\begin{aligned}
\pi_0^\beta &= \int_{-\infty}^{g_1} \int_{-\infty}^{g_2} \cdots \int_{-\infty}^{g_N} \phi(\beta, \Lambda - \Sigma \mathbf{c}_1, \Sigma) \cdot \\
&\quad \exp(-\mathbf{c}'_1 \Lambda + \mathbf{c}'_1(\mathbf{g} + \frac{1}{2} \Sigma \mathbf{c}_1)) d\beta_N \cdots d\beta_2 d\beta_1 \\
&+ \int_{-\infty}^{g_1} \int_{-\infty}^{g_2} \cdots \int_{g_N}^{\infty} \phi(\beta, \Lambda - \Sigma \mathbf{c}_2, \Sigma) \cdot \\
&\quad \exp(-\mathbf{c}'_2 \Lambda + \mathbf{c}'_2(\mathbf{g} + \frac{1}{2} \Sigma \mathbf{c}_2)) d\beta_N \cdots d\beta_2 d\beta_1 \\
&+ \dots + \int_{-\infty}^{g_1} \int_{g_2}^{\infty} \cdots \int_{g_N}^{\infty} \phi(\beta, \Lambda - \Sigma \mathbf{c}_{(N-1)}, \Sigma) \cdot \\
&\quad \exp(-\mathbf{c}'_{(N-1)} \Lambda + \mathbf{c}'_{(N-1)}(\mathbf{g} + \frac{1}{2} \Sigma \mathbf{c}_{(N-1)})) d\beta_N \cdots d\beta_2 d\beta_1 \\
&+ \int_{g_1}^{\infty} \int_{g_2}^{\infty} \cdots \int_{g_N}^{\infty} \phi(\beta, \Lambda - \Sigma \mathbf{c}_N, \Sigma) \cdot \\
&\quad \exp(-\mathbf{c}'_N \Lambda + \mathbf{c}'_N(\mathbf{g} + \frac{1}{2} \Sigma \mathbf{c}_N)) d\beta_N \cdots d\beta_2 d\beta_1.
\end{aligned}$$

Next, converting to standard multivariate random variables by using the relation in (2.5), it follows that the limits of the integrals given by \mathbf{g} , are changed to $\hat{\alpha}_j$. Finally, by using standard symmetry properties for the multivariate normal probability distribution, we find that the cumulative multivariate normal probability distribution must be evaluated at the points $\alpha_j = \hat{\mathbf{c}}_j \hat{\alpha}_j$ with variance-covariance matrix $\hat{\mathbf{c}}_j \Sigma \hat{\mathbf{c}}_j$. The desired pricing formula then follows. \square

2.3.2 Pricing the Guarantee on the Stock Return

The terminal time $t_N < T$ payoff for the guarantee on the stock return is given by the random variable

$$\pi_{t_N}^\delta = e^{g_1 \vee \delta_1} \cdot e^{g_2 \vee \delta_2} \cdot \dots \cdot e^{g_N \vee \delta_N}.$$

To find the initial market value of this guarantee, we have to find the expected deflated value of $\pi_{t_N}^\delta$ under the equivalent martingale measure, i.e.,

$$\pi_0^\delta = E_Q \left[e^{(g_1 - \beta_1) \vee (\delta_1 - \beta_1)} \cdot e^{(g_2 - \beta_2) \vee (\delta_2 - \beta_2)} \cdot \dots \cdot e^{(g_N - \beta_N) \vee (\delta_N - \beta_N)} \right]. \quad (2.7)$$

We now introduce some new vectors and matrices. $\bar{\mathbf{c}}_j$, $j \in \{1, 2, \dots, 2^N\}$, is a $2N \times 1$ -dimensional vector only containing -1 's, 0 's, and 1 's. The first N elements are equal to 1 and the remaining N elements are equal to $\mathbf{c}_j - 1$. As in the previous subsection, the i 'th element of $\bar{\mathbf{c}}_j$ is equal to 1 when the guarantee is binding in the i 'th period and 0 otherwise. It then follows that the $N+i$ 'th element of $\bar{\mathbf{c}}_j$, $i \in \{1, 2, \dots, N\}$, is equal to 0 when the guarantee is binding in the i 'th period and -1 otherwise. It is possible to construct 2^N unique $\bar{\mathbf{c}}_j$'s, each corresponding to a state of the world.

The minimum guaranteed rate of return is given by the $N \times 1$ -dimensional vector $\bar{\mathbf{g}} = (g_1, g_2, \dots, g_N)'$. $\bar{\Lambda}$ is a $2N \times 1$ -dimensional vector giving the expectation of $\bar{\beta} = (\beta_1, \beta_2, \dots, \beta_N, \delta_1, \delta_2, \dots, \delta_N)'$ under the equivalent martingale measure. The expectation of the i 'th δ is given by $\bar{\Lambda}_{N+i} = \Lambda_i - \frac{1}{2}\sigma_{\delta_i}^2$. $\bar{\Sigma}$ is the variance-covariance matrix of the multivariate normal distributed random variables $\bar{\beta}$. $\bar{\Sigma}_\delta$ is the variance-covariance matrix of the multivariate normal distributed random variables $\delta = (\delta_1, \delta_2, \dots, \delta_N)'$. $\bar{\alpha}_j$ is a $2N \times 1$ -dimensional vector that gives the points to evaluate the cumulative multivariate normal probability distribution at, and it is given by $\bar{\mathbf{c}}_j \bar{\alpha}_j$, where the i 'th element of $\bar{\alpha}_j$ is given by

$$\bar{\alpha}_{j,i} = \infty$$

for $i \in \{1, 2, \dots, N\}$ and

$$\bar{\alpha}_{j,i} = \frac{g_i - \bar{\Lambda}_i + (\bar{\Sigma} \bar{\mathbf{c}}_j)_i}{\sigma_{\delta_i}}$$

for $i \in \{N+1, N+2, \dots, 2N\}$. $\bar{\alpha}_j^\delta$ is an $N \times 1$ -dimensional vector with i 'th element, $i \in \{1, 2, \dots, N\}$, equal to the $N+i$ 'th element of $\bar{\alpha}_j$.

The solution of the expectation in (2.7) is given in Proposition 2.2.

Proposition 2.2. *The initial market value of an N -period guarantee on the return on the stock is given by*

$$\pi_0^\delta = \sum_{j=1}^{2^N} e^{\mathbf{c}'_j \bar{\mathbf{g}} - \bar{\mathbf{c}}'_j \bar{\Lambda} + \frac{1}{2} \bar{\mathbf{c}}'_j \bar{\Sigma} \bar{\mathbf{c}}_j} \Phi(\bar{\alpha}_j^\delta, \hat{\mathbf{c}}_j \bar{\Sigma}_\delta \hat{\mathbf{c}}_j).$$

Proof. The proof partially follows from the proof for the guarantee on the money market account.

We let $1_{\bar{c}_j}$ be an indicator function returning the value 1 when \bar{c}_j is true and 0 otherwise. Again, using the linearity of the expectation operator, the expectation in (2.7) can be written as

$$\pi_0^\delta = \sum_{j=1}^{2^N} E_Q \left[e^{c'_j \bar{g} - c'_j \bar{\beta}} 1_{\bar{c}_j} \right].$$

Let $E_Q[1_{\bar{c}_j}] = Q(\bar{c}_j)$ be the probability under the equivalent martingale measure Q for the state \bar{c}_j . It follows directly from Lemma 2.1 that we can, for each j , construct a probability measure $Q_{\bar{c}_j}$ equivalent to Q . $Q_{\bar{c}_j}$ is defined by

$$\frac{dQ_{\bar{c}_j}}{dQ} = \frac{e^{-\bar{c}_j \bar{\beta}}}{e^{-\bar{c}_j \bar{\Lambda} + \frac{1}{2} \bar{c}_j \bar{\Sigma} \bar{c}_j}}.$$

It then follows that

$$\pi_0^\delta = \sum_{j=1}^{2^N} e^{c'_j \bar{g} - \bar{c}_j \bar{\Lambda} + \frac{1}{2} \bar{c}_j \bar{\Sigma} \bar{c}_j} Q_{\bar{c}_j}(\bar{c}_j),$$

where the expectation of $\bar{\beta}$ under $Q_{\bar{c}_j}$ is from Lemma 2.1 seen to be given by

$$\bar{\Lambda}_{Q_{\bar{c}_j}} = \bar{\beta} - \bar{\Sigma} \bar{c}_j.$$

$Q_{\bar{c}_j}(\bar{c}_j)$ is determined by the cumulative multivariate normal probability distribution evaluated at the points determined by the vector $\bar{\alpha}_j^\delta$. The i 'th element of $\bar{\alpha}_j^\delta$, $i \in \{N+1, N+2, \dots, 2N\}$, follows after changing to standard multivariate normal random variables under the probability measures $Q_{\bar{c}_j}$ and exploiting symmetry properties for the cumulative multivariate normal probability distribution. We have that

$$\bar{\alpha}_{j,i}^\delta = \frac{g_i - (\bar{\Lambda}_{Q_{\bar{c}_j}})_i}{\sigma_{\delta_i}},$$

where $(\bar{\Lambda}_{Q_{\bar{c}_j}})_i$ is the $N + i$ 'th element of the vector $\bar{\Lambda}_{Q_{\bar{c}_j}}$. Since β has no upper or lower limit as the vector \bar{g} for δ , it is easily seen that the $2N$ -dimensional multivariate normal cumulative probability distribution is reduced to a cumulative N -dimensional multivariate normal probability distribution. Finally, using symmetry properties for the cumulative multivariate normal probability distribution, we can proceed as for the money market account, and it then follows that the distribution must be evaluated at the points determined by the vector $\bar{\alpha}_j^\delta$. The desired pricing formula then follows. □

Example ($N = 2$): Let us consider the same guarantee as in Miltersen and Persson (1999), i.e., $N = 2$, and the stock as the underlying asset. The first guarantee lasts from time 0 to 1, and the second from time 1 to 2. We then have that $\mathbf{c}_1 = (1 \ 1)'$, $\mathbf{c}_2 = (1 \ 0)'$, $\mathbf{c}_3 = (0 \ 1)'$, and $\mathbf{c}_4 = (0 \ 0)'$. We further have that

$$\hat{\mathbf{c}}_1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\hat{\mathbf{c}}_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\hat{\mathbf{c}}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\hat{\mathbf{c}}_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The vector $\bar{\mathbf{\Lambda}}$ under Q is given by

$$\bar{\mathbf{\Lambda}} = \begin{pmatrix} -\ln P(0, 1) + \frac{1}{2}\sigma_{\beta_1}^2 \\ -\ln F(0, 1, 2) + \frac{1}{2}\sigma_{\beta_2}^2 + c_{1,2} \\ -\ln P(0, 1) + \frac{1}{2}\sigma_{\beta_1}^2 - \frac{1}{2}\sigma_{d_1}^2 \\ -\ln F(0, 1, 2) + \frac{1}{2}\sigma_{\beta_2}^2 + c_{1,2} - \frac{1}{2}\sigma_{d_2}^2 \end{pmatrix},$$

where

$$\sigma_{d_i}^2 = \int_{i-1}^i \sigma_S^2(v) dv, \quad i \in \{1, 2\},$$

and $\bar{\mathbf{\Sigma}}$ is given by

$$\bar{\mathbf{\Sigma}} = \begin{pmatrix} \sigma_{\beta_1}^2 & c_{1,2} & \sigma_{\beta_1}^2 + k_1 & c_{1,2} \\ c_{1,2} & \sigma_{\beta_2}^2 & c_{1,2} + k_{1,2} & \sigma_{\beta_2}^2 + k_3 \\ \sigma_{\beta_1}^2 + k_1 & c_{1,2} + k_{1,2} & \sigma_{\delta_1}^2 & c_{1,2} + k_{1,2} \\ c_{1,2} & \sigma_{\beta_2}^2 + k_3 & c_{1,2} + k_{1,2} & \sigma_{\delta_2}^2 \end{pmatrix},$$

where

$$k_1 = \int_0^1 \sigma_S(v) \int_v^1 \sigma_f(v, u) dudv,$$

$$k_{1,2} = \int_0^1 \sigma_S(v) \int_1^2 \sigma_f(v, u) dudv,$$

and

$$k_3 = \int_1^2 \sigma_S(v) \int_v^2 \sigma_f(v, u) dudv.$$

$$\bar{\Sigma}_\delta = \begin{pmatrix} \sigma_{\delta_1}^2 & c_{1,2} + k_{1,2} \\ c_{1,2} + k_{1,2} & \sigma_{\delta_2}^2 \end{pmatrix}.$$

Let $\bar{\rho} = \frac{c_{1,2} + k_{1,2}}{\sigma_{\delta_1} \sigma_{\delta_2}}$. The exponent, $\mathbf{c}'_j \bar{\mathbf{g}} - \bar{\mathbf{c}}'_j \bar{\boldsymbol{\Lambda}} + \frac{1}{2} \bar{\mathbf{c}}'_j \bar{\Sigma} \bar{\mathbf{c}}_j$, becomes

$$\mathbf{c}'_j \bar{\mathbf{g}} - \bar{\mathbf{c}}'_j \bar{\boldsymbol{\Lambda}} + \frac{1}{2} \bar{\mathbf{c}}'_j \bar{\Sigma} \bar{\mathbf{c}}_j = \begin{cases} 0 & \text{for } j = 1, \\ g_2 + \ln F(0, 1, 2) - \sigma_{\delta_1} \sigma_{\delta_2} \bar{\rho} & \text{for } j = 2, \\ g_1 + \ln P(0, 1) & \text{for } j = 3, \\ g_1 + g_2 + \ln P(0, 1) + \ln F(0, 1, 2) \\ = g_1 + g_2 + \ln P(0, 2) & \text{for } j = 4. \end{cases}$$

Only considering $i = 3, 4$, $\bar{\alpha}_{j,i}$ becomes

$$\bar{\alpha}_1 = \begin{pmatrix} \frac{g_1 - \bar{\Lambda}_3 + (\boldsymbol{\Sigma} \mathbf{c}_1)_3}{\sigma_{\delta_1}} \\ \frac{g_2 - \bar{\Lambda}_4 + (\boldsymbol{\Sigma} \mathbf{c}_1)_4}{\sigma_{\delta_2}} \end{pmatrix} = \begin{pmatrix} \frac{g_1 + \ln P(0, 1) - \frac{1}{2} \sigma_{\delta_1}^2}{\sigma_{\delta_1}} \\ \frac{g_2 + \ln F(0, 1, 2) - \frac{1}{2} \sigma_{\delta_2}^2}{\sigma_{\delta_2}} - \sigma_{\delta_1} \bar{\rho} \end{pmatrix},$$

$$\bar{\alpha}_2 = \begin{pmatrix} \frac{g_1 - \bar{\Lambda}_3 + (\boldsymbol{\Sigma} \mathbf{c}_2)_3}{\sigma_{\delta_1}} \\ \frac{g_2 - \bar{\Lambda}_4 + (\boldsymbol{\Sigma} \mathbf{c}_2)_4}{\sigma_{\delta_2}} \end{pmatrix} = \begin{pmatrix} \frac{g_1 + \ln P(0, 1) - \frac{1}{2} \sigma_{\delta_1}^2}{\sigma_{\delta_1}} + \sigma_{\delta_2} \bar{\rho} \\ \frac{g_2 + \ln F(0, 1, 2) + \frac{1}{2} \sigma_{\delta_2}^2}{\sigma_{\delta_2}} - \sigma_{\delta_1} \bar{\rho} \end{pmatrix},$$

$$\bar{\alpha}_3 = \begin{pmatrix} \frac{g_1 - \bar{\Lambda}_3 + (\boldsymbol{\Sigma} \mathbf{c}_3)_3}{\sigma_{\delta_1}} \\ \frac{g_2 - \bar{\Lambda}_4 + (\boldsymbol{\Sigma} \mathbf{c}_3)_4}{\sigma_{\delta_2}} \end{pmatrix} = \begin{pmatrix} \frac{g_1 + \ln P(0, 1) + \frac{1}{2} \sigma_{\delta_1}^2}{\sigma_{\delta_1}} \\ \frac{g_2 + \ln F(0, 1, 2) - \frac{1}{2} \sigma_{\delta_2}^2}{\sigma_{\delta_2}} \end{pmatrix},$$

$$\bar{\alpha}_4 = \begin{pmatrix} \frac{g_1 - \bar{\Lambda}_3 + (\boldsymbol{\Sigma} \mathbf{c}_4)_3}{\sigma_{\delta_1}} \\ \frac{g_2 - \bar{\Lambda}_4 + (\boldsymbol{\Sigma} \mathbf{c}_4)_4}{\sigma_{\delta_2}} \end{pmatrix} = \begin{pmatrix} \frac{g_1 + \ln P(0, 1) + \frac{1}{2} \sigma_{\delta_1}^2}{\sigma_{\delta_1}} + \sigma_{\delta_2} \bar{\rho} \\ \frac{g_2 + \ln F(0, 1, 2) + \frac{1}{2} \sigma_{\delta_2}^2}{\sigma_{\delta_2}} \end{pmatrix}.$$

Inserting these expressions into the formula in Proposition 2.2, the formula in Proposition 5.4 in Miltersen and Persson (1999) is obtained.

2.4 Implementation of the Pricing Formulas

The guarantees considered in this chapter are typically long lasting, and the duration of the majority of the contracts are perhaps in the range from 20 to 40 years. For a guarantee lasting for 30 years there are more than one billion (2^{30}) probabilities that have to be calculated (each given as a 30-dimensional integral over the multivariate normal probability distribution).⁶ Even though we have presented closed form solutions for the market values

⁶Fortran has a routine for calculating multivariate normal probabilities. A Fortran 77 routine can be found at <http://www.sci.wsu.edu/math/faculty/genz/homepage> where a detailed description of the underlying method is given in Genz (1992).

of the multi-period guarantees, evaluating the expressions are likely to be very time consuming.

By specifying the volatility in the Heath, Jarrow, and Morton model as (see e.g., Miltersen and Persson (1999))

$$\sigma_f(v, t) = e^{-\int_v^t \kappa_u du} \sigma_v, \quad (2.8)$$

the model of Vasicek (1977) is obtained. We assume that $\sigma_v = \sigma$ and $\kappa_v = \kappa$ are constants. More precisely, when analysing the money market account we use the specification in (2.8), and when analysing the guarantee on the stock return, we let

$$\sigma_S(t) = \sigma_S \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$\sigma_f(v, u) = \sigma e^{-\kappa(u-v)} \begin{pmatrix} \varphi \\ \sqrt{1-\varphi^2} \end{pmatrix},$$

where φ is a constant.

Using these specifications and inserting into the expressions for the variances and covariances in section 2.2, the following equations follow (note that time $t_n = n$, $t_{n-1} = n - 1$, $t_m = m$, and $t_{m-1} = m - 1$)

$$\begin{aligned} \sigma_{\beta_n}^2 &= \frac{\sigma^2}{2\kappa^3} (2e^{-\kappa} - 2 - e^{-2\kappa n} + 2e^{\kappa(1-2n)} - e^{2\kappa(1-n)} + 2\kappa), \\ c_{m,n} &= \frac{\sigma^2}{2\kappa^3} (-2e^{\kappa(m-n)} - e^{\kappa(-m-n+2)} + 2e^{\kappa(-m-n+1)} \\ &\quad - e^{\kappa(-m-n)} + e^{\kappa(m-n-1)} + e^{\kappa(m-n+1)}), \end{aligned}$$

and

$$\sigma_{\delta_n}^2 = \sigma_{\beta_n}^2 + \frac{2\sigma\sigma_S\varphi}{\kappa^2} (\kappa - 1 + e^{-\kappa}) + \sigma_S^2.$$

$$\bar{c}_{m,n} = c_{m,n} + \frac{\sigma\sigma_S\varphi}{\kappa^2} (e^{-\kappa(n-m-1)} - 2e^{-\kappa(n-m)} + e^{-\kappa(n-m+1)})$$

for $n > m$,

$$\bar{c}_{n,n} = \sigma_{\beta_n}^2 + \frac{\sigma\sigma_S\varphi}{\kappa^2} (\kappa - 1 + e^{-\kappa})$$

for $m = n$, and

$$\bar{c}_{m,n} = c_{n,m},$$

for $n < m$.

We now use the results in Proposition 2.1 and 2.2 to calculate the market values of rate of return guarantees lasting from 2 - 5 periods. We assume the following parameter values (we assume an initial flat term structure of interest rates);

$$S_0 = 1, \quad g = \ln(1.04), \quad \sigma_S = 0.20, \quad r = 0.05,$$

$$\sigma = 0.03, \quad \kappa = 0.10, \quad \varphi = -0.5.$$

The market values are reported in Table 2.1. In addition to calculating the market values of the guarantee on the return on the money market account and on the stock under stochastic interest rates, we have also, for comparison, included the market values of the guarantee on the return on the stock under deterministic interest rates. As we can see, the market values are fairly equal under both stochastic and deterministic interest rates. The market values of the guarantee on the return on the money market account are lower than for the guarantee on the stock return. This is a consequence of the low variance of the return on the money market account.

Table 2.1: Market values of multi-period rate of return guarantees.

	Proposition 2.1	Proposition 2.2	Proposition 2.2 $\sigma_f(t, s) = 0$ (deterministic interest rates)
	π_0^β	π_0^δ	π_0^δ
N=2	1.0105	1.1493	1.1534
N=3	1.0216	1.2341	1.2388
N=4	1.0511	1.3286	1.3304
N=5	1.0643	1.4268	1.4288

As we can see from Table 2.1, these guarantees are rather expensive, especially the guarantee on the stock return. For a guarantee that lasts for five years, an investor has to pay, if he wants his investment to have an annual minimum guaranteed rate of return of 3.92% ($=\ln(1.04)$), an “insurance premium” that amounts to almost 43% of the investment in the stock. This seems very expensive, and most investors would probably not enter into such a contract, but since most of these guarantees are embedded in life insurance contracts, the typical customer may have problems uncovering the high price he is paying for the guarantee (according to Donselaar (1999) many of the customers are *not* paying for the guarantee).

2.5 Conclusions

We have in this chapter derived closed form solutions for the market values of multi-period rate of return guarantees. First we analysed a multi-period guarantee on the short-term interest rate. Secondly we analysed a multi-period guarantee on the return on a stock. We found the expressions for the market values of these two guarantees to be quite similar, and we also found the two-period guarantees analysed by Persson and Aase (1997) and Miltersen and Persson (1999) to be special cases of our results. Also the result of Hipp (1996) is a special case. Finally we gave some remarks on implementation of the pricing formulas, and showed, by using realistic parameter values, numerical examples where the market values were calculated.

Chapter 3

Hedging of Multi-period Rate of Return Guarantees

Abstract

Multi-period guarantees are typically embedded in life insurance contracts. These guarantees expose the life insurers to a considerable amount of risk. In this chapter we show, by deriving self-financing trading strategies, which hedge the market values of multi-period guarantees, how the risk can be managed. We find these strategies to be more complicated than the corresponding strategies for traditional European options. First, for these traditional options, the number of units of the underlying assets to be included in the trading strategies is described by continuous functions. For the multi-period guarantee we find that these functions may be discontinuous and we also find them to differ in the different periods. Second, for traditional options the trading strategies often only consist of one zero-coupon bond, while for the multi-period guarantee we typically find that a whole portfolio of zero-coupon bonds is needed.

Keywords and phrases: Multi-period rate of return guarantees, self-financing hedging strategies, Heath, Jarrow, and Morton term structure model of interest rates.

3.1 Introduction

Life insurance companies were traditionally mainly exposed to mortality risk, a risk they in principle could diversify by issuing a large number of similar and statistically independent policies. However, as more exotic life insurance policies have been offered, such as unit-linked life insurance con-

tracts and policies with bonus mechanisms and minimum rate of return guarantees, life insurance companies have also become exposed to financial risk. The financial risk is non-diversifiable and is likely to affect many, if not most, of the companies outstanding policies in the same direction. Although being non-diversifiable, the financial risk is (at least to some extent) hedgeable. Since the accumulated exposure to financial risk over all the policies issued by a life insurance company can be large, it is important that this risk is hedged so that the company is able to meet its obligations to the policyholders.

Rate of return guarantees are known to impose a lot of financial risk on life insurance companies. For instance, Nissan Mutual Life went into bankruptcy because it was unable to meet the obligations imposed by the minimum rate of return guarantees embedded in the policies it had issued, see e.g., Grosen and Jørgensen (1997) and Briys and de Varenne (1997). At the time when many of the rate of return guarantees were introduced, the minimum guaranteed rate of return was so much lower than the return the insurance companies would normally obtain on their investment portfolios that their financial exposure was basically hedged by their investment portfolios. However, in later years the return has decreased, leading to a need for more advanced hedging strategies.

In this chapter we derive hedging strategies for *multi-period minimum rate of return guarantees*, for short just multi-period guarantees. This is a common type of guarantee to be included in life insurance contracts in several countries. With this guarantee the insured is guaranteed a minimum rate of return in each period, typically in each year, see e.g., Hipp (1996), Persson and Aase (1997), and Miltersen and Persson (1999). The guarantee will typically be on the return on the life insurance companies' investment portfolios.

We derive hedging strategies for multi-period guarantees under both deterministic and stochastic interest rates, and we use both the return on a stock and a money market account as a proxy for the return on the life insurance companies' investment portfolios. The hedging strategies we derive are idealised in the sense that we assume the hedge portfolio can be continuously rebalanced and no transaction costs are present. Also, the guarantees we analyse are rather stylised and do only represent one aspect about life insurance contracts.

The hedge portfolios for traditional European options such as call and put options are known to only consist of long and short positions in the underlying asset and a zero-coupon bond. Also, the number of units of these two assets to include in the hedge portfolios is described by continuous functions of time, or more precisely, by Itô processes. We show that the hedging strategies for the multi-period guarantees are more complicated. First, we find that the functions describing the number of units of the assets to include in the hedge portfolios are discontinuous as we go from one period

to the next. However, we also present a counter example where this is not the case. Second, we find that the hedge portfolios may have to consist of more than two assets under stochastic interest rates. The additional assets, compared to the hedge portfolios for traditional European options, are zero-coupon bonds with different time to maturity. In fact, we find that to hedge a multi-period guarantee one may have to trade, in addition to the underlying asset, a whole portfolio of bonds.

The size of the bond portfolio in terms of how many different zero-coupon bonds that have to be included is determined by two factors:

1. the remaining number of periods until the guarantee matures
2. the number of factors determining the term structure.

Our main results are fairly general and apply to term structures that are determined by any number of factors. We also show that in the special case of a one-factor model that we may apply other hedging strategies. For these strategies the bond portfolio is replaced by one bond and the money market account.

The chapter is organised as follows: In section 3.2 we give a description of the underlying economic model. In section 3.3 we derive hedging strategies for multi-period guarantees, and the strategies are analysed. In subsection 3.3.4 we give several numerical examples showing the hedging strategies for realistic scenarios of the financial market. In section 3.4 we present an alternative approach for hedging the guarantees. Section 3.5 comments on which strategy to use. Section 3.6 concludes. In addition, we have also supplied an appendix with three sections.

3.2 The Economic Model

We work within an extended Heath et al. (1992) model, also called an Amin and Jarrow (1992) model. A description of this model can be found in an advanced textbook in finance, see e.g., Musiela and Rutkowski (1997).

We assume that trading takes place on a continuous basis on the time interval $[0, \mathcal{T}]$, for some fixed horizon $\mathcal{T} > 0$. A filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ is fixed, where Ω is the state space, \mathcal{F} is a σ -algebra, $\mathbb{F} = \{\mathcal{F}_t, 0 \leq t \leq \mathcal{T}\}$ is a filtration where $\mathcal{F}_{\mathcal{T}} = \mathcal{F}$ and $\mathcal{F}_0 = \{\emptyset, \Omega\}$, where \emptyset is the empty set, and P is a probability measure. The σ -algebra is generated by a d -dimensional, $d \geq 1$, Brownian motion, W_t .¹

We assume, under the equivalent martingale measure Q , that the instantaneous continuously compounded forward rate at time s , as seen from time

¹For the case with both stochastic interest rates and a stock, we require, in order to avoid perfect correlation between the stock and the interest rates, that $d > 1$.

$t, f(t, s), 0 \leq t \leq s \leq T$, is given by

$$f(t, s) = f(0, s) + \int_0^t \sigma_f(v, s) \int_v^s \sigma_f(v, u) du dv + \int_0^t \sigma_f(v, s) dW_v, \quad (3.1)$$

where sufficient regularity conditions for $\sigma_f(t, s), 0 \leq t \leq s \leq T$, are given in Heath et al. (1992).

The short-term interest rate $r_t = f(t, t)$. We will throughout assume that $\sigma_f(t, s)$ is a deterministic function, implying Gaussian interest rates. When considering deterministic interest rates we set $\sigma_f(t, s) = 0$. We also assume that there is a continuum of zero-coupon bonds trading in the market and the time t market value of the one maturing at time $T > t$ is given by

$$P(t, T) = P(0, T) + \int_0^t r_v P(v, T) dv - \int_0^t \int_v^T \sigma_f(v, u) du dW_v, \quad (3.2)$$

with $P(T, T) = 1$.

We let the time t market value of a non-dividend paying stock, S_t , be given under the equivalent martingale measure Q by the equation

$$S_t = S_0 + \int_0^t r_v S_v dv + \int_0^t \sigma_S(v) S_v dW_v, \quad (3.3)$$

where $r_t S_t$ satisfies the integrability condition $\int_0^t |r_v S_v| dv < \infty$ almost surely for all t . Here $\sigma_S(t)$ is a time dependent volatility function and satisfies the square integrability condition $E \left[\int_0^t (\sigma_S(v) S_v)^2 dv \right] < \infty$ (for further details on integrability conditions, see e.g., Duffie (1996)).

The money market account is an asset that accrues interest according to the short-term interest rate. The time t market value, M_t , is given by

$$M_t = M_0 + \int_0^t r_v M_v dv, \quad M_0 = 1, \quad (3.4)$$

where $r_t M_t$ satisfies the integrability condition $\int_0^t |r_v M_v| dv < \infty$ almost surely for all t .

From (3.3) and (3.4) we can see that the money market account, under the equivalent martingale measure Q , is a special case of the stock since the money market account has no diffusion term.

In the rest of this chapter we divide the time into periods. Period n will be the time interval between time t_{n-1} and t_n . The initial investment to accrue interest is normalised to one.

3.3 Hedging Multi-period Guarantees

In this section we first give a definition of a *self-financing* trading strategy. We also present the hedging strategy for a maturity guarantee. The

hedging strategy for a multi-period guarantee under deterministic interest rates is derived in subsection 3.3.1, while the corresponding strategies under stochastic interest rates are derived in subsection 3.3.2.

Definition 3.1. *Given some price process Z , a trading strategy γ is said to be self-financing if the following equation holds*

$$\gamma_t Z_t = \gamma_0 Z_0 + \int_0^t \gamma_v dZ_v.$$

Before we derive hedging strategies for multi-period guarantees, we start by showing the hedging strategy for a *maturity guarantee*. A maturity guarantee is a guarantee that only lasts for one period. This guarantee is a useful building block for multi-period guarantees, and it also turns out that the hedging strategy for the maturity guarantee is closely related to the hedging strategies for multi-period guarantees.

Let $X_t \in \{M_t, S_t\}$ and let $\sigma_X^2(t)$ be the variance of the return under the equivalent martingale measure Q over the time interval from time t to T on the asset with market value X_t . The time $t < T$ market value of the maturity guarantee maturing at time T is given by (the Brennan and Schwartz (1976) modification of the seminal results of Black and Scholes (1973) and Merton (1973))

$$\pi_t(X) = \frac{X_t}{X_0} \Phi(d_1) + P(t, T) e^g \Phi(-d_2), \quad (3.5)$$

where

$$\begin{aligned} d_1 &= \frac{\ln(X_t/X_0) - g - \ln P(t, T)}{\sigma_X(t)} + \frac{1}{2} \sigma_X(t), \\ d_2 &= d_1 - \sigma_X(t), \end{aligned}$$

$\Phi(\cdot)$ is the cumulative normal probability distribution, and g is the minimum guaranteed rate of return.

The hedging strategy at time $t < T$ for a maturity guarantee is given by holding $a_t(X)$ units of the underlying asset and $b_t(X)$ units of the bond maturing at the same time as the guarantee. From Brennan and Schwartz (1976) and Persson and Aase (1997) we have that

$$a_t(X) = \frac{1}{X_0} \Phi(d_1)$$

and

$$b_t(X) = e^g \Phi(-d_2).$$

3.3.1 Deterministic Interest Rates

Let us first assume that interest rates are deterministic and that the stock is the underlying asset (indicated by d). The market value of an N -period guarantee at time $t \in [t_{n-1}, t_n)$, for all $n \in \{1, 2, \dots, N\}$, is given by (see Lindset (2001))

$$\pi_t^n(d) = R^n(d)\pi_t(d)\theta(d),$$

where

$$R^n(d) = \prod_{l=1}^{n-1} \max\left(\frac{S_{t_l}}{S_{t_{l-1}}}, e^{g_l}\right)$$

and

$$\theta(d) = \prod_{l=n}^{N-1} \pi_{t_l}(d),$$

where $\pi_{t_l}(d)$ is as in (3.5) with $\frac{X_t}{X_0} = 1$, deterministic interest rates, and lasts for one period and g_n is the minimum guaranteed rate of return in period l . In period n we can interpret $R^n(d)$ as the realised gross return in the previous $n - 1$ periods. Similarly, $\theta(d)$ can be interpreted as the market value (at time t_n) of the guarantees in the remaining $N - n$ periods. For $n = 1$ let $R^n(d) = 1$ and for $n = N$ let $\theta(d) = 1$.

Proposition 3.1. *The following number of units of the stock*

$$a_t^n(d) = R^n(d) \left(\frac{1}{S_{t_{(n-1)}}} \Phi(d_1) \right) \theta(d)$$

and the following number of units of the bond maturing at time t_n

$$b_t^n(d) = R^n(d) \left(e^{g_n} \Phi(-d_2) \right) \theta(d),$$

give a self-financing hedging strategy in the n 'th period.

Proof. Both $R^n(d)$ and $\theta(d)$ are independent of S_t and $P(t, t_n)$. The hedging strategy follows therefore in the same way as the hedging strategy for the maturity guarantee in Brennan and Schwartz (1976). \square

3.3.2 Stochastic Interest Rates: Hedging with Zero-coupon Bonds

For the case with deterministic interest rates, only two assets are needed in each period to hedge the market value of a multi-period guarantee. Under

stochastic interest rates the hedging strategies get somewhat more complicated, and the hedge portfolios have in general to consist of the underlying asset and a whole portfolio of zero-coupon bonds (however, only two assets are needed in the last period). The strategies of this subsection are fairly general. In section 3.4 we show, in the special case of a one-factor model for the short-term interest rate, that the bond portfolio can be replaced by a portfolio containing the money market account and a zero-coupon bond. Thus, if the money market account is the underlying asset, also this hedge portfolio consists of only two assets.

We start by looking at a two-period guarantee on both the return on the money market account and on the stock. The guarantee starts at time t_0 and the first period ends at time t_1 and the second at time t_2 . In order to derive the hedging strategies, we find it convenient to modify the pricing formulas in Miltersen and Persson (1999) so that we can find the market values at any time t in the contract-period, and not just at the initiation of the contract. We do not give any proofs for these modifications, but they should not be too hard to accept.

3.3.2.1 Hedging the Guarantee on the Money Market Account

The time $t \in [t_0, t_1)$ market value of the two-period guarantee on the money market account is given by (β now indicates that we are considering the money market account and the superscript 1 that time t is in the first period)²

$$\begin{aligned} \pi_t^1(\beta) &= \frac{M_t}{M_{t_0}} \Phi(-a_1, -b_1, \rho) + \frac{M_t}{M_{t_0}} F(t, t_1, t_2) e^{g_2 - \rho \sigma_{\beta_1} \sigma_{\beta_2}} \Phi(-a_2, b_2, -\rho) \\ &\quad + P(t, t_1) e^{g_1} \Phi(a_3, -b_3, -\rho) + P(t, t_2) e^{g_1 + g_2} \Phi(a_4, b_4, \rho), \end{aligned}$$

where

$$\begin{aligned} a_1 &= \frac{-\ln(M_t/M_{t_0}) + g_1 + \ln(P(t, t_1)) - \frac{1}{2}\sigma_{\beta_1}^2}{\sigma_{\beta_1}}, \\ b_1 &= \frac{g_2 + \ln(F(t, t_1, t_2)) - \frac{1}{2}\sigma_{\beta_2}^2}{\sigma_{\beta_2}} - \rho \sigma_{\beta_1}, \\ \sigma_{\beta_1}^2 &= \int_t^{t_1} \left(\int_v^{t_1} \sigma_f(v, u) du \right)^2 dv, \\ \sigma_{\beta_2}^2 &= \int_t^{t_1} \left(\int_{t_1}^{t_2} \sigma_f(v, u) du \right)^2 dv + \int_{t_1}^{t_2} \left(\int_v^{t_2} \sigma_f(v, u) du \right)^2 dv, \\ \rho &= \frac{c_{1,2}}{\sigma_{\beta_1} \sigma_{\beta_2}}, \end{aligned}$$

²Notice that $F(t, t_1, t_2)$ is the time t forward price for delivery at time t_1 of a zero-coupon bond maturing at time t_2 and is given by $F(t, t_1, t_2) = \frac{P(t, t_2)}{P(t, t_1)}$.

$$c_{1,2} = \int_t^{t_1} \left(\int_v^{t_1} \sigma_f(v, u) du \right) \left(\int_{t_1}^{t_2} \sigma_f(v, u) du \right) dv,$$

$$a_2 = a_1 + \rho\sigma_{\beta_2}, \quad a_3 = a_1 + \sigma_{\beta_1}, \quad a_4 = a_1 + \rho\sigma_{\beta_2} + \sigma_{\beta_1},$$

$$b_2 = b_1 + \sigma_{\beta_2}, \quad b_3 = b_1 + \rho_t\sigma_{\beta_1}, \quad b_4 = b_1 + \rho_t\sigma_{\beta_1} + \sigma_{\beta_2}.$$

The hedging strategy for the first period is given in Proposition 3.2.

Proposition 3.2. *The following number of units of the money market account*

$$\begin{aligned} a_t^1(\beta) &= \frac{1}{M_{t_0}} \Phi(-a_1, -b_1, \rho) \\ &\quad + \frac{F(t, t_1, t_2)}{M_{t_0}} e^{g_2 - \rho\sigma_{\beta_1}\sigma_{\beta_2}} \Phi(-a_2, b_2, -\rho), \end{aligned}$$

the following number of units of the bond maturing at time t_1

$$\begin{aligned} b_t^1(\beta) &= e^{g_1} \Phi(a_3, -b_3, -\rho) \\ &\quad - \frac{M_t/M_{t_0}}{P(t, t_1)} F(t, t_1, t_2) e^{g_2 - \rho\sigma_{\beta_1}\sigma_{\beta_2}} \Phi(-a_2, b_2, -\rho), \end{aligned}$$

and the following number of units of the bond maturing at time t_2

$$\begin{aligned} y_t^1(\beta) &= e^{(g_1+g_2)} \Phi(a_4, b_4, \rho) \\ &\quad + \frac{M_t/M_{t_0}}{P(t, t_2)} F(t, t_1, t_2) e^{g_2 - \rho\sigma_{\beta_1}\sigma_{\beta_2}} \Phi(-a_2, b_2, -\rho), \end{aligned}$$

give a self-financing hedging strategy in the first period.

Proof. Since the stock is a more general product than the money market account, the proof is just a special case of that given for the guarantee on the stock return in section A.3, and the reader is referred to this proof. \square

Now, let us turn to the second period. The return in the first period has already materialised, and in the second period it can be treated as a constant. It then follows that the market value of the guarantee at time $t \in [t_1, t_2)$ is given by (the superscript 2 indicates that time t is in the second period)

$$\pi_t^2(\beta) = R^2(\beta) \left(\frac{M_t}{M_{t_1}} \Phi(d_1) + P(t, t_2) e^{g_2} \Phi(-d_2) \right),$$

where X_t/X_0 in d_1 and d_2 has to be replaced by M_t/M_{t_1} and $\sigma_X(t)$ by σ_{β_1} where the upper integral limits t_1 are changed to t_2 .

The hedging strategy for the second period is given in Proposition 3.3.

Proposition 3.3. *The following number of units of the money market account*

$$a_t^2(\beta) = R^2(\beta) \frac{1}{M_{t_1}} \Phi(d_1)$$

and the following number of units of the bond maturing at time t_2

$$b_t^2(\beta) = R^2(\beta) e^{g_2} \Phi(-d_2),$$

give a self-financing hedging strategy in the second period.

Proof. Since the stock is a more general product than the money market account, the proof is just a special case of that given for the guarantee on the stock return in section A.2, and the reader is referred to this proof. \square

3.3.2.2 Hedging the Guarantee on the Stock Return

The time $t \in [t_0, t_1)$ market value of the two-period guarantee on the stock return is given by (δ now indicates that we are considering the stock under stochastic interest rates)

$$\begin{aligned} \pi_t^1(\delta) &= \frac{S_t}{S_{t_0}} \Phi(-\bar{a}_1, -\bar{b}_1, \bar{\rho}) + \frac{S_t}{S_{t_0}} F(t, t_1, t_2) e^{g_2 - \bar{\rho} \sigma_{\delta_1} \sigma_{\delta_2}} \Phi(-\bar{a}_2, \bar{b}_2, -\bar{\rho}) \\ &\quad + P(t, t_1) e^{g_1} \Phi(\bar{a}_3, -\bar{b}_3, -\bar{\rho}) + P(t, t_2) e^{g_1 + g_2} \Phi(\bar{a}_4, \bar{b}_4, \bar{\rho}), \end{aligned}$$

where

$$\begin{aligned} \bar{a}_1 &= \frac{-\ln(S_t/S_{t_0}) + g_1 + \ln(P(t, t_1)) - \frac{1}{2}\sigma_{\delta_1}^2}{\sigma_{\delta_1}}, \\ \bar{b}_1 &= \frac{g_2 + \ln(F(t, t_1, t_2)) - \frac{1}{2}\sigma_{\delta_2}^2}{\sigma_{\delta_2}} - \bar{\rho} \sigma_{\delta_1}, \\ \sigma_{\delta_1}^2 &= \sigma_{d_1}^2 + 2k_1 + \sigma_{\beta_1}^2, \\ \sigma_{\delta_2}^2 &= \sigma_{d_2}^2 + 2k_3 + \sigma_{\beta_2}^2, \\ \sigma_{d_i}^2 &= \int_{\max(t, t_{i-1})}^{t_i} \sigma_S^2(v) dv, \quad i \in \{1, 2\}, \\ \bar{\rho} &= \frac{c_{1,2} + k_{1,2}}{\sigma_{\delta_1} \sigma_{\delta_2}}, \\ k_1 &= \int_t^{t_1} \sigma_S(v) \int_v^{t_1} \sigma_f(v, u) dudv, \\ k_{1,2} &= \int_t^{t_1} \sigma_S(v) \int_{t_1}^{t_2} \sigma_f(v, u) dudv, \\ k_3 &= \int_{t_1}^{t_2} \sigma_S(v) \int_{t_1}^{t_2} \sigma_f(v, u) dudv, \end{aligned}$$

$$\bar{a}_2 = \bar{a}_1 + \bar{\rho}\sigma_{\delta_2}, \quad \bar{a}_3 = \bar{a}_1 + \sigma_{\delta_1}, \quad \bar{a}_4 = \bar{a}_1 + \bar{\rho}\sigma_{\delta_2} + \sigma_{\delta_1}$$

$$\bar{b}_2 = \bar{b}_1 + \sigma_{\delta_2}, \quad \bar{b}_3 = \bar{b}_1 + \bar{\rho}\sigma_{\delta_1}, \quad \bar{b}_4 = \bar{b}_1 + \bar{\rho}\sigma_{\delta_1} + \sigma_{\delta_2}.$$

The hedging strategy for the first period is given in Proposition 3.4.

Proposition 3.4. *The following number of units of the stock*

$$\begin{aligned} a_t^1(\delta) &= \frac{1}{S_{t_0}} \Phi(-\bar{a}_1, -\bar{b}_1, \bar{\rho}) \\ &\quad + \frac{F(t, t_1, t_2)}{S_{t_0}} e^{g_2 - \bar{\rho}\sigma_{\delta_1}\sigma_{\delta_2}} \Phi(-\bar{a}_2, \bar{b}_2, -\bar{\rho}), \end{aligned}$$

the following number of units of the bond maturing at time t_1

$$\begin{aligned} b_t^1(\delta) &= e^{g_1} \Phi(\bar{a}_3, -\bar{b}_3, -\bar{\rho}) \\ &\quad - \frac{S_t/S_{t_0}}{P(t, t_1)} F(t, t_1, t_2) e^{g_2 - \bar{\rho}\sigma_{\delta_1}\sigma_{\delta_2}} \Phi(-\bar{a}_2, \bar{b}_2, -\bar{\rho}), \end{aligned}$$

and the following number of units of the bond maturing at time t_2

$$\begin{aligned} y_t^1(\delta) &= e^{(g_1+g_2)} \Phi(\bar{a}_4, \bar{b}_4, \bar{\rho}) \\ &\quad + \frac{S_t/S_{t_0}}{P(t, t_2)} F(t, t_1, t_2) e^{g_2 - \bar{\rho}\sigma_{\delta_1}\sigma_{\delta_2}} \Phi(-\bar{a}_2, \bar{b}_2, -\bar{\rho}), \end{aligned}$$

give a self-financing hedging strategy in the first period.

Proof. See section A.3. □

Let us now turn to the second period (i.e., $t \in [t_1, t_2]$) and see how the market value of the guarantee can be hedged. In the second period, $R^2(\delta)$ is independent of S_t and $P(t, t_2)$, and it then follows that the market value of the guarantee is the same as the market value of a maturity guarantee multiplied by $R^2(\delta)$, i.e.,

$$\pi_t^2(\delta) = R^2(\delta) \left(\frac{S_t}{S_{t_1}} \Phi(d_1) + P(t, t_2) e^{g_2} \Phi(-d_2) \right),$$

where X_t/X_0 in d_1 and d_2 must be replaced by S_t/S_{t_1} and $\sigma_X(t)$ by σ_{δ_1} where the upper integral limits t_1 are changed to t_2 .

The hedging strategy for the second period is given in Proposition 3.5.

Proposition 3.5. *The following number of units of the stock*

$$a_t^2(\delta) = R^2(\delta) \frac{1}{S_{t_1}} \Phi(d_1)$$

and the following number of units of the bond maturing at time t_2

$$b_t^2(\delta) = R^2(\delta) e^{g_2} \Phi(-d_2),$$

give a self-financing hedging strategy in the second period.

Proof. Since $R^2(\delta)$ is \mathcal{F}_{t_1} -measurable, the proof follows by the same lines as the proof for the maturity guarantee in section A.2. □

3.3.2.3 Hedging of an N -period Guarantee

Before we analyse qualitative characteristics of the hedging strategies and illustrate them with numerical examples, we shortly indicate what the *initial* hedging strategy for an N -period guarantee on the money market account looks like (the hedging strategy for the guarantee on the stock return follows in the exact same manner). Unfortunately, the notation required to express the hedging strategy becomes fairly messy, though the “structure” of the strategy is fairly simple.

From chapter 2 we have that the initial market value of an N -period guarantee on the money market account is given by

$$\pi_{t_0}^1(\beta) = \sum_{j=1}^{2^N} e^{c'_j \mathbf{g} - c'_j \Lambda + \frac{1}{2} c'_j \Sigma c_j} \Phi(\alpha_j, \hat{\mathbf{c}}_j \Sigma \hat{\mathbf{c}}_j),$$

where $\mathbf{g} = (g_1, g_2, \dots, g_N)'$ is a vector determining the minimum guaranteed rate of return in period $n \in \{1, 2, \dots, N\}$, Λ is an $N \times 1$ -dimensional vector containing the expectations under the equivalent martingale measure Q of the stochastic variables $\beta = (\beta_1, \beta_2, \dots, \beta_N)$. Σ is the variance-covariance matrix for the stochastic variables β . α_j determines the points to evaluate the cumulative multivariate normal probability distribution, $\Phi(\cdot, \cdot)$, at. c_j is an $N \times 1$ -dimensional vector only containing 0's and 1's and $\hat{\mathbf{c}}_j$ is an $N \times N$ -dimensional symmetric matrix with diagonal equal to $\mathbf{1} - 2c_j$, where $\mathbf{1}$ is an $N \times 1$ -dimensional vector of 1's. For further details, see chapter 2.

The number of the underlying asset to hold in the beginning of the first period is given by

$$a_{t_0}^1(\beta) = \frac{1}{M_{t_0}} \sum_{j=1}^{2^{(N-1)}} e^{\tilde{c}'_j \mathbf{g} - \tilde{c}'_j \Lambda + \frac{1}{2} \tilde{c}'_j \Sigma \tilde{c}_j} \Phi(\alpha_j, \tilde{\mathbf{c}}_j \Sigma \tilde{\mathbf{c}}_j),$$

where $\tilde{\mathbf{c}}_j$ (\tilde{c}_j) is equal to c_j (\hat{c}_j), except that the first element is equal to zero (minus one).

To hedge an N -period guarantee, we can trade N zero-coupon bonds so that we have bonds maturing at the end of each period. The number of units to be invested in the bond maturing at time t_s , $s \in \{1, 2, \dots, N-1\}$, is equal to

$$b_{t_0}^1(\beta) = \frac{1}{P(t_0, t_s)} \sum_{j=1}^{2^{(N-2)}} e^{\hat{c}'_j \mathbf{g} - \hat{c}'_j \Lambda + \frac{1}{2} \hat{c}'_j \Sigma \hat{c}_j} \Phi(\alpha_j, \hat{\mathbf{c}}_j \Sigma \hat{\mathbf{c}}_j) - \frac{1}{P(t_0, t_s)} \sum_{j=1}^{2^{N-2}} e^{\check{c}'_j \mathbf{g} - \check{c}'_j \Lambda + \frac{1}{2} \check{c}'_j \Sigma \check{c}_j} \Phi(\alpha_j, \check{\mathbf{c}}_j \Sigma \check{\mathbf{c}}_j),$$

where \hat{c}_j ($\hat{\mathbf{c}}_j$) is equal to c_j , except that element s ($s-1$) is equal to 1 (0), conditional on element $s-1$ (s) being equal to 0 (1). \check{c}_j ($\check{\mathbf{c}}_j$) is equal to \hat{c}_j ,

except that element s ($s - 1$) is equal to 1 (-1), conditional on element $s - 1$ (s) being equal to -1 (1).

Finally, the number of units of the bond maturing at time t_N is given by

$$y_{t_0}^1(\beta) = \frac{1}{P(t_0, t_N)} \sum_{j=1}^{2^{(N-2)}} e^{\check{c}'_j \mathbf{g} - \check{c}'_j \Lambda + \frac{1}{2} \check{c}'_j \Sigma \check{c}_j} \Phi(\boldsymbol{\alpha}_j, \check{\mathbf{c}} \Sigma \check{\mathbf{c}}),$$

where \check{c}_j ($\check{\mathbf{c}}$) is equal to \mathbf{c}_j ($\hat{\mathbf{c}}_j$), except that the last element is equal to 1 (-1).

By taking the realised return into account, we can proceed with about the same set-up for the hedging strategy in later periods.

3.3.3 Qualitative Analysis of the Hedging Strategies

In this subsection we explore some of the features of the hedging strategies we have found. It is assumed that the guarantees last for two periods, but this can easily be extended to more periods. However, a two-period guarantee brings to light the new and interesting features of the hedging strategies compared to the hedging strategies for European options.

When the guarantee is binding in period n , $n \in \{1, 2\}$, the number of units invested in the underlying asset turns to zero at the end of the period. This can easily be seen by taking the limit of $a_t^n(\eta)$, $\eta \in \{\beta, \delta, d\}$, as $t \rightarrow t_n^-$ and conditioning on $e^{g_n} > \frac{X_t}{X_{t_{n-1}}}$, $\frac{X_t}{X_{t_{n-1}}} \in \{\frac{M_t}{M_{t_{n-1}}}, \frac{S_t}{S_{t_{n-1}}}\}$, i.e.,

$$\lim_{t \rightarrow t_n^-} a_t^n(\eta) \Big|_{e^{g_n} > \frac{X_t}{X_{t_{n-1}}}} = 0. \quad (3.6)$$

This can be explained by the fact that at the end of the period the uncertainty about the return in the period is almost fully resolved, and if the return is going to be equal to the minimum guaranteed rate of return, the underlying asset will no longer be needed. At the beginning of the second period we start all over again, and to be prepared for the uncertainty, we hold a mixture of the underlying asset and the bond. This causes a discontinuity in the number of the underlying asset in the hedging strategy as we go from period one to two. This is also the case when the guarantee is not binding in the first period.

When the guarantee is not binding in period $n = 2$, the number of units invested in the bond maturing at time t_n turns to zero at the end of the period. This follows since

$$\lim_{t \rightarrow t_n^-} b_t^n(\eta) \Big|_{e^{g_n} < \frac{X_t}{X_{t_{n-1}}}} = 0. \quad (3.7)$$

For deterministic interest rates ($\eta = d$), this also holds for $n = 1$.

For the case with stochastic interest rates ($\eta \in \{\beta, \delta\}$) and $n = 1$, three assets are traded. It is interesting to notice that the number of units of the bond maturing at time t_2 is described by a continuous function, also as we go from the first to the second period. This follows since

$$\lim_{t \rightarrow t_1^-} y_t^1(\eta) = b_{t_1}^2(\eta). \quad (3.8)$$

The economic explanation for this is that when interest rates are stochastic we have to, in addition to hedge the market value of the guarantee, also secure that we are able to buy the right amount of the bond at time t_1 .

When the guarantee is not binding in the first period, there will, at the end of the first period, be held a short position in the bond maturing at time t_1 and a long position in the bond maturing at time t_2 , i.e.,

$$\lim_{t \rightarrow t_1^-} b_t^1(\eta) \Big|_{e^{g_1} < \frac{X_t}{X_{t_0}}} = -\frac{X_{t_1}}{X_{t_0}} P(t_1, t_1) e^{g_2} < 0$$

and

$$\lim_{t \rightarrow t_1^-} y_t^1(\eta) \Big|_{e^{g_1} < \frac{X_t}{X_{t_0}}} = \frac{X_{t_1}}{X_{t_0}} e^{g_2} > 0.$$

This means that one has to trade a whole bond portfolio as one approaches time t_1 , both when the guarantee is binding and not. However, it is interesting to notice that the total *amount* invested in the bonds is equal to 0 when the guarantee is not binding at the end of the first period, i.e.,

$$P(t_1, t_2) \frac{M_{t_1}}{M_{t_0}} e^{g_2} - P(t_1, t_1) \frac{M_{t_1}}{M_{t_0}} P(t_1, t_2) e^{g_2} = 0. \quad (3.9)$$

It is also easily seen that when the guarantee is binding in the first period, the amount of money invested at time t , $t \rightarrow t_1^-$, in the bond maturing at time t_1 is equal to the amount invested in the underlying asset at time t_1 , i.e.,

$$\lim_{t \rightarrow t_1^-} P(t, t_1) b_t^1(\eta) \Big|_{e^{g_1} > \frac{X_t}{X_{t_0}}} = X_{t_1} a_{t_1}^2(\eta).$$

3.3.4 Numerical examples

To get some more intuition behind these hedging strategies, we now present some numerical examples where we show the market value of the stock, the money market account, the guarantees, and how many units we must hold of the underlying asset and the bond(s). First we consider the case with deterministic interest rates. We use the following parameter values (we assume an initial flat term structure of interest rates and that the stock price has a yearly drift rate μ);

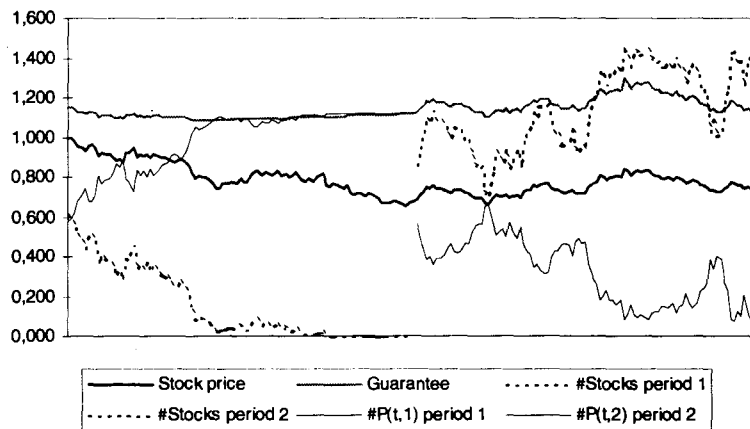


Figure 3.1: Hedging strategy for a two-period guarantee on the stock return under deterministic interest rates. The guarantee is only binding in the first period. The figure shows the market value of the guarantee and the stock and the number of units of the stock and the bond in the hedging strategy.

$$S_0 = 1, \quad g = \ln(1.04), \quad \sigma_S = 0.20, \quad r = 0.05, \quad \mu = 0.12.$$

We let $t_n = n$, $n \in \{0, 1, 2\}$, i.e., each period is of length one.

In Figure 3.1 - 3.4 the time is represented on the x -axis with time 0 to the left, time 1 in the middle of the figure, and time 2 to the right. The market value of the underlying asset and the guarantee and also the number of units of the underlying asset and the bond(s) in the hedging strategy are represented on the y -axis.

The first example is a guarantee on the stock return under deterministic interest rates and is illustrated in Figure 3.1. The guarantee is only binding in the first period, and in accordance with (3.6), we can see that the number of stocks in the hedging strategy is equal to zero at the end of the first period. At the same time the number of units invested in the zero-coupon bond maturing at time 1 is equal to the market value of the guarantee. This follows since $P(1,1) = 1$, and the hedging strategy has indeed the same market value at time 1 as the guarantee. We can also see, in accordance with (3.7), that the number of bonds in the hedging strategy turns to zero at the end of the second period (where the guarantee is not binding). Figure 3.1 also clearly demonstrates the discontinuity in the number of assets in the hedging strategy as we go from period 1 to period 2.

We now extend the example to a stochastic interest rate environment. We use a specification for the volatility of the forward rates that corresponds to the model of Vasicek (1977) (see e.g., Miltersen and Persson (1999)). More

precisely, we let

$$\sigma_S(t) = \sigma_S \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$\sigma_f(v, u) = \sigma e^{-\kappa(u-v)} \begin{pmatrix} \varphi \\ \sqrt{1-\varphi^2} \end{pmatrix},$$

where σ_S , σ , κ , and φ are constants.³

The following additional parameter values are assumed;

$$\sigma = 0.03, \quad \kappa = 0.10, \quad \varphi = -0.5.$$

In Figure 3.2 we have an example of a guarantee on the return on the money market account and where the guarantee is binding in the first period. We can see that the market value of the guarantee and the underlying asset fluctuate less than what was the case in Figure 3.1 where the stock is the underlying asset.

At the end on the first period there is no investment in the money market account and is in accordance with (3.6), while there is no investment in the bond at the end of the second period and is in accordance with (3.7). As showed in (3.8), the number of units invested in the bond maturing at time 2 follows a continuous function when going from period one to two, something we see is not the case for the number of units invested in the money market account (there are of course no investment in the bond maturing at time 1 in the second period). Notice also how the number of units invested in the two bonds in the first period seems to be negatively correlated. The explanation for this is that the second term in the expression for $b_t^1(\beta)$ is subtracted, while it is added for $y_t^1(\beta)$.

We have in Figure 3.3 and 3.4 illustrated the hedging strategies by two examples for the guarantee on the stock return under stochastic interest rates. In Figure 3.3 the guarantee is binding in both periods, and as expected from (3.6), the number of units invested in the stock at the end of each period is equal to zero.

³Under the equivalent martingale measure Q , the short-term interest rate can be expressed by the following stochastic differential equation

$$r_t = r_0 + \int_0^t \kappa(\hat{\theta} - r_v)dv + \int_0^t \sigma dW_v,$$

where $\hat{\theta}$ is the mean reversion level. The parameter κ can then be interpreted as the force of gravitation and σ as the diffusion of r . For details, see e.g., Miltersen and Persson (1999). The parameter φ is used to impose correlation between the process for the stock price and the interest rates (though, under the equivalent martingale measure Q , some correlation already exists since the drift of stock price is the short-term interest rate).

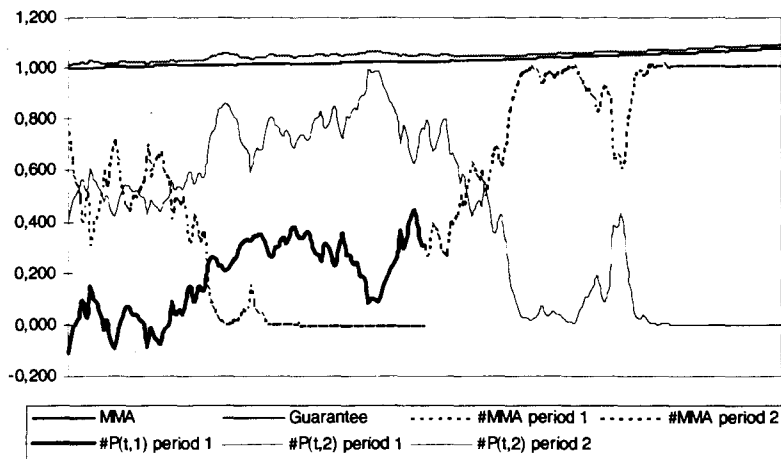


Figure 3.2: Hedging strategy for a two-period guarantee on the return on the money market account under stochastic interest rates. The guarantee is only binding in the first period. The figure shows the market value of the guarantee and the money market account (*MMA*) and the number of units of the money market account and the bonds in the hedging strategy.

In Figure 3.4 the guarantee is only binding in the second period. We can then see, at the end of the first period, that there are a negative number of units of the bond maturing at time 1 in the hedging strategy. Although this cannot be directly seen from the figure, the total amount invested in the two bonds at the end of the first period is equal to zero, cf. (3.9).

3.4 Alternative Hedging Strategies Under Stochastic Interest Rates

Since the model we have used for the interest rates is effectively a one-factor model, the bond maturing at time t_1 can be replicated by a portfolio containing the bond maturing at time t_2 and the money market account. To this end we construct a portfolio with time t market value Π_t that coincides with the market value of the bond maturing at time t_1 for all $t \in [t_0, t_1]$. Let ω_t^2 and ω_t^0 be the amount invested at time t in the bond and the money market account, respectively. The time t market value of the portfolio is then

$$\Pi_t = \omega_t^2 + \omega_t^0.$$

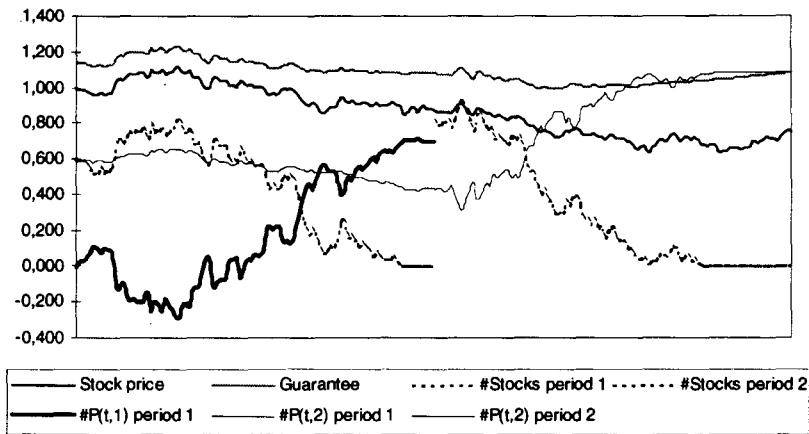


Figure 3.3: Hedging strategy for a two-period guarantee on the stock return under stochastic interest rates. The guarantee is binding in both periods. The figure shows the market value of the guarantee and the stock and the number of units of the stock and the bonds in the hedging strategy.

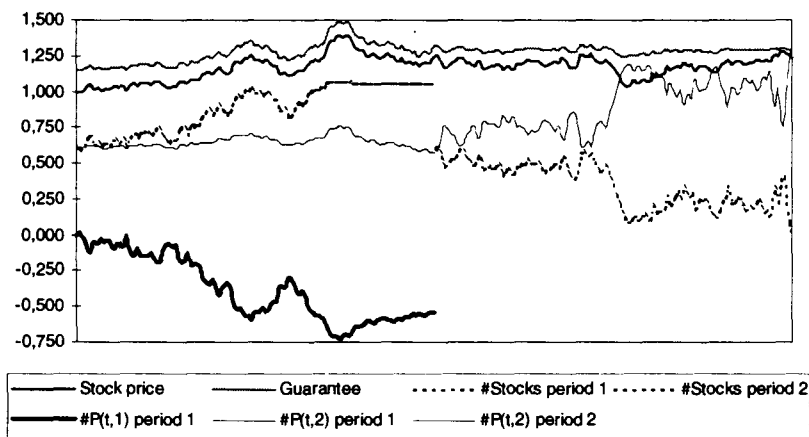


Figure 3.4: Hedging strategy for a two-period guarantee on the stock return under stochastic interest rates. The guarantee is only binding in second period. The figure shows the market value of the guarantee and the stock and the number of units of the stock and the bonds in the hedging strategy.

Lemma 3.1. Let $t \in [t_0, t_1]$. A portfolio with the following amount invested in the money market account

$$\omega_t^0 = P(t, t_1) \left(1 - \frac{\int_t^{t_1} \sigma_f(t, u) du}{\int_t^{t_2} \sigma_f(t, u) du} \right)$$

and the following amount invested in the bond maturing at time t_2

$$\omega_t^2 = P(t, t_1) \frac{\int_t^{t_1} \sigma_f(t, u) du}{\int_t^{t_2} \sigma_f(t, u) du},$$

replicates the market value of the bond maturing at time t_1 .

Proof. From Itô's lemma we have that the market value of the portfolio is given by

$$\Pi_t = \int_{t_0}^t \omega_v^2 (r_v dv - \int_v^{t_2} \sigma_f(v, u) dudW_v) + \int_{t_0}^t \omega_v^0 r_v dv \quad (3.10)$$

and from (3.2) we have that $P(t, t_1)$ is given by

$$P(t, t_1) = \int_{t_0}^t P(v, t_1) (r_v dv - \int_v^{t_2} \sigma_f(v, u) dudW_v). \quad (3.11)$$

We know from the unique decomposition property for Itô processes that if the portfolio in (3.10) is to replicate the bond in (3.11), they must have the same drift and diffusion terms. Thus, for all $t \in [t_0, t_1]$ we have that

$$\begin{aligned} \omega_t^2 \int_t^{t_2} \sigma_f(t, u) du &= P(t, t_1) \int_t^{t_1} \sigma_f(t, u) du \\ \Leftrightarrow \\ \omega_t^2 &= P(t, t_1) \frac{\int_t^{t_1} \sigma_f(t, u) du}{\int_t^{t_2} \sigma_f(t, u) du}. \end{aligned}$$

Further, inserting for ω_t^2 , we find that

$$\begin{aligned} P(t, t_1) \frac{\int_t^{t_1} \sigma_f(t, u) du}{\int_t^{t_2} \sigma_f(t, u) du} r_t dt + \omega_t^0 r_t dt &= P(t, t_1) r_t dt \\ \Leftrightarrow \\ \omega_t^0 &= P(t, t_1) \left(1 - \frac{\int_t^{t_1} \sigma_f(t, u) du}{\int_t^{t_2} \sigma_f(t, u) du} \right). \end{aligned}$$

This completes the proof. \square

3.4.1 The Guarantee on the Money Market Account

Based on the result in Lemma 3.1 we can propose an alternative to the hedging strategy in Proposition 3.2.

Proposition 3.6. *The following number of units of the money market account*

$$\begin{aligned}\hat{a}_t^1(\beta) &= \frac{1}{M_{t_0}} \Phi(-a_1, -b_1, \rho) \\ &\quad + \frac{F(t, t_1, t_2)}{M_{t_0}} e^{g_2 - \rho\sigma\beta_1\sigma\beta_2} \Phi(-a_2, b_2, -\rho) \\ &\quad + b_t^1(\beta)\omega_t^0\end{aligned}$$

and the following number of units of the bond maturing at time t_2

$$\begin{aligned}\hat{y}_t^1(\beta) &= e^{(g_1+g_2)} \Phi(a_4, b_4, \rho) \\ &\quad + \frac{M_t/M_{t_0}}{P(t, t_2)} F(t, t_1, t_2) e^{g_2 - \rho\sigma\beta_1\sigma\beta_2} \Phi(-a_2, b_2, -\rho) \\ &\quad + b_t^1(\beta)\omega_t^2,\end{aligned}$$

give a self-financing hedging strategy in the first period.

Proof. This follows by combining Proposition 3.2 and Lemma 3.1. \square

The hedging strategy in the second period will of course be the same as in Proposition 3.3. The equation

$$\lim_{t \rightarrow t_1} \hat{y}_t^1(\beta) = b_{t_1}^2(\beta),$$

is easily seen to hold since

$$\lim_{t \rightarrow t_1} \omega_t^2 = 0$$

and $b_t^1(\beta)$ is finite for all $t \in [t_0, t_1)$. Hence, also for this hedging strategy the number of units invested in the bond maturing at time t_2 follows a continuous function as we go from the first to the second period. Since the replicating portfolio is self-financing, the number of units invested in the money market account will also follow a continuous function as we go from the first to the second period. This can also be seen from the following equation

$$\lim_{t \rightarrow t_1} \hat{a}_t^1(\beta) = a_{t_1}^2(\beta).$$

3.4.2 The Guarantee on the Stock Return

Following Proposition 3.6, similar hedging strategies can be derived for the guarantee on the stock return.

Proposition 3.7. *The following number of units of the stock $a_t^1(\delta)$, the following number of units of the money market account*

$$\hat{z}_t^1(\delta) = \omega_t^0 b_t^1(\delta),$$

and the following number of units of the bond maturing at time t_2

$$\begin{aligned} \hat{y}_t^1(\delta) &= e^{(g_1+g_2)t} \Phi(\bar{a}_4, \bar{b}_4, \bar{\rho}) \\ &\quad + \frac{S_t/S_{t_0}}{P(t, t_2)} F(t, t_1, t_2) e^{g_2 - \bar{\rho}\sigma_{\delta_1}\sigma_{\delta_2}} \Phi(-\bar{a}_2, \bar{b}_2, -\bar{\rho}) \\ &\quad + b_t^1(\delta) \omega_t^2, \end{aligned}$$

give a self-financing hedging strategy in the first period.

Proof. This follows by combining Proposition 3.4 and Lemma 3.1. \square

As for the hedging strategy in Proposition 3.6, the number of units invested in the bond maturing at time t_2 follows a continuous function. The number of units invested in the stock will, as in Proposition 3.4, have a discontinuity as we go from the first to the second period.

3.5 Which Hedging Strategies to Use?

The answer to this question should probably be based on empirical observations. This is however outside the scope of this chapter. Instead we choose to relate the question to the choice of model for the term structure of interest rates. Let us illustrate by using the following model for the short-term interest rate

$$\mathbf{r}_t = \mathbf{r}_0 + \int_0^t [\boldsymbol{\theta}_v + \boldsymbol{\alpha} \mathbf{r}_v] ds + \int_0^t \boldsymbol{\sigma} d\mathbf{W}_v, \quad (3.12)$$

where $\mathbf{r}_t \in \mathbb{R}^M$, $\boldsymbol{\theta}_t \in \mathbb{R}^M$, $\boldsymbol{\alpha} \in \mathbb{R}^{M \times d}$, and $\boldsymbol{\sigma} \in \mathbb{R}^{M \times d}$.

For instance, assuming that $M = d$ and

$$\begin{aligned} \mathbf{r}_t &= \begin{bmatrix} r_t \\ u_t \end{bmatrix}, \\ \boldsymbol{\theta}_t &= \begin{bmatrix} \theta_t \\ 0 \end{bmatrix}, \\ \boldsymbol{\alpha} &= \begin{bmatrix} -a & 1 \\ 0 & -b \end{bmatrix}, \end{aligned}$$

$$\sigma = \begin{bmatrix} \sigma_{1,1} & 0 \\ \psi\sigma_{2,1} & \sqrt{1-\psi^2}\sigma_{2,1} \end{bmatrix},$$

and

$$\mathbf{W}_t = \begin{bmatrix} W_t^1 \\ W_t^2 \end{bmatrix},$$

the model in (3.12) corresponds to the two-factor model of Hull and White (1994). For this model we have that the bond price is given by

$$P(t, T) = A(t, T)e^{B(t, T)r_t - C(t, T)u_t},$$

for some functions $A(t, T)$, $B(t, T)$, and $C(t, T)$ (see Hull and White (1994) for details). Using Itô's lemma, the bond price can also be written as

$$\begin{aligned} P(t, T) &= P(0, T) + \int_0^t \frac{\partial P}{\partial v} dv + \int_0^t \frac{\partial P}{\partial r_v} dr_v + \frac{1}{2} \int_0^t \frac{\partial^2 P}{\partial r_v^2} (dr_v)^2 \\ &\quad + \int_0^t \frac{\partial P}{\partial u_v} du_v + \frac{1}{2} \int_0^t \frac{\partial^2 P}{\partial u_v^2} (du_v)^2 + \int_0^t \frac{\partial^2 P}{\partial r_v \partial u_v} dr_v du_v, \end{aligned}$$

where P is short-hand notation for $P(v, T)$. Rearranging, we get

$$P(t, T) = P(0, T) + \int_0^t \mu(P)_v dv + \int_0^t \sigma_1(P) dW_v^1 + \int_0^t \sigma_2(P) dW_v^2,$$

where

$$\begin{aligned} \mu(P)_t &= \frac{\partial P}{\partial t} + \frac{\partial P}{\partial r_t}(\theta_t + u_t - ar_t) + \frac{1}{2} \frac{\partial^2 P}{\partial r_t^2} \sigma_{1,1}^2 \\ &\quad - \frac{\partial P}{\partial u_t} bu_t + \frac{1}{2} \frac{\partial^2 P}{\partial u_t^2} \sigma_{2,1}^2 + \frac{\partial^2 P}{\partial r_t \partial u_t} \sigma_{1,1} \psi \sigma_{2,1}, \\ \sigma_1(P) &= \frac{\partial P}{\partial r_t} \sigma_{1,1} + \frac{\partial P}{\partial u_t} \psi \sigma_{2,1}, \end{aligned}$$

and

$$\sigma_2(P) = \frac{\partial P}{\partial u_t} \sqrt{1-\psi^2} \sigma_{2,1}.$$

From the unique decomposition property for Itô processes and the arguments in section 3.4 we see that this bond cannot be replicated by a bond maturing at time $U > T$ and the money market account. We conclude that two bonds maturing after time T , in addition to the money market account, are needed. By induction we reason that for an M -factor model we need M bonds, each maturing at time $U_j > T$, $j \in \{1, 2, \dots, M\}$, and the money market account to replicate the market value of the bond maturing at time T .

For a guarantee lasting for several periods, the hedging strategies in subsection 3.3.2 may involve a rather large bond portfolio. Litterman and Scheinkman (1991) found that two factors capture about 98% of the variance of bond returns, i.e., a portfolio of two bonds and the money market account does a fairly good job of hedging shorter bonds. Extending the hedging strategies in Proposition 3.6 and 3.7 to a two-factor model and more periods may therefore in practice lead to a more cost effective way to hedge the guarantees than the strategies in subsection 3.3.2.

3.6 Conclusions

We have in this chapter derived self-financing hedging strategies for multi-period rate of return guarantees. We showed, both under deterministic and stochastic interest rates, that the hedging strategies typically are given by path-wise continuous Itô processes. However, as we go from one period to the next, there may, though not necessarily, be a discontinuity in the number of units of the assets in the hedging strategy. This is in sharp contrast to the hedging strategies for more traditional European options and maturity guarantees. We also found, under stochastic interest rates that several bonds with different time to maturity may have to be traded in each period. Several numerical examples illustrating the hedging strategies have been presented.

Chapter 4

Relative Guarantees

Abstract

Many real-world financial contracts have some sort of minimum rate of return guarantee included. One class of these guarantees is so-called *relative guarantees*, i.e., guarantees where the minimum guaranteed rate of return is given as a function of the stochastic return on a reference portfolio. These guarantees are the topic of this chapter. We analyse a wide range of different functional specifications for the minimum guaranteed rate of return, hereunder both so-called maturity and multi-period guarantees. Several closed form solutions are presented.

Keywords and phrases: Stochastic minimum guaranteed rate of return, stochastic average minimum guaranteed rate of return, Heath, Jarrow, and Morton term structure model of interest rates.

4.1 Introduction

The rapid innovation in the financial markets during the last 20 to 30 years has led to a wide range of different kinds of investments and savings vehicles. Several of these vehicles have some sort of minimum rate of return guarantee embedded. Examples of such contracts could be guaranteed investment contracts, index-linked bonds, life insurance contracts, and pension plans. The tremendous amount of money under management by life insurance companies and pension funds should justify the analysis of rate of return guarantees.

In the existing literature it seems like the main focus has been on so-called *absolute* guarantees (see e.g., Brennan and Schwartz (1976), Grosen

and Jørgensen (1997), Hipp (1996), Persson and Aase (1997), and Miltersen and Persson (1999)), i.e., guarantees where the minimum guaranteed rate of return is deterministic, typically a constant. These articles can naturally be divided into two categories. The first category contains maturity guarantees, i.e., contracts where the guarantee only can become binding at the end of the contract-period if the average return has been below the minimum guaranteed rate of return. Brennan and Schwartz (1976) and Grosen and Jørgensen (1997) fall into this category, though Grosen and Jørgensen (1997) also allowed for early exercise of the guarantee. The second category is annual, or multi-period, rate of return guarantees and give a minimum guarantee for the rate of return within each period. The work on multi-period guarantees seems to have been initiated by Hipp (1996) and has later been extended by Persson and Aase (1997) and Miltersen and Persson (1999).

Relative guarantees, i.e., guarantees where the minimum guaranteed rate of return is linked to an index, a portfolio, a specific asset, etc. (often called the *reference portfolio*), does not seem to have received the same focus, but some examples are found in Ekern and Persson (1996), Pennacchi (1999), and Romero-Meza (2000). The minimum guaranteed rate of return in these contracts is stochastic, and the guarantee is therefore fundamentally different from the absolute guarantee. Beside Pennacchi (1999) we have not found any work focusing on relative guarantees involving stochastic interest rates, which is the topic of this chapter. Ekern and Persson (1996) analysed unit-linked life insurance contracts with a wide range of different types of guarantees included, hereunder also relative guarantees. Pennacchi (1999) analysed, among other issues, relative rate of return guarantees. His analysis was relatively closely tied up to the guarantees embedded in pension plans in Latin America. He also considered multi-period guarantees, though his interpretation was fundamentally different from our interpretation in that he did not take into account the compounding effect that is present in these guarantees.¹ Romero-Meza (2000) analysed pension plans, and in particular the Chilean pension model that has a relative guarantee included. He approached the valuation of the guarantee element by numerical methods, i.e., by Monte Carlo simulation. However, he treated the pension plan as a maturity guarantee and did not take into account the annual guarantee that, according to Pennacchi (1999), is present in these pension plans.

The absolute guarantee has the feature that it can provide a relatively high rate of return if the market as a whole, or, in particular, the underlying asset of the contract that has the guarantee embedded, has a low rate of return. This can make this guarantee rather expensive. Since many financial assets are positively correlated, a low rate of return on the underlying asset of the contract will often coincide with a low rate of return on the reference

¹Using Monte Carlo simulation, he also allowed for both stochastic interest rates and wage level.

portfolio. This will typically make the relative guarantee cheaper than the absolute guarantee, simply because it gives a poorer protection against lower rate of returns.

This chapter gives several contributions to the literature on relative guarantees, and in particular in the presence of stochastic interest rates. First we show that for some specifications of the guarantee, the market value is independent of the process followed by the interest rates. Since the modelling of interest rates is in practice a complex task, this is an important observation that speaks in favour of marketing relative guarantees instead of absolute guarantees. The chapter also extends the literature on multi-period guarantees by considering multi-period relative guarantees, and it is shown that also these contracts can be constructed so that they are totally independent of interest rates. In real-life situations the minimum guaranteed rate of return is often set lower than the return on the reference portfolio. We consider contracts where the minimum guaranteed rate of return is equal to the return on the reference portfolio subtracted a given amount, contracts where the minimum guaranteed rate of return is equal to a given fraction of the return on the reference portfolio, and contracts where the minimum guaranteed rate of return is a combination of these two. An analysis of the minimum guaranteed rate of return used in several countries (e.g., Argentina, Chile, and Poland) is also given. Finally, the chapter also deals with contracts, both so-called maturity and multi-period, where the stochastic minimum guaranteed rate of return is the average return on the reference portfolio over some given time period. We show that, although some of the terms entering the expressions for the market values of these “average” guarantees are somewhat messy, the structure of these guarantees is fairly simple.

An outline of the chapter goes as follows: In section 4.2 we give a description of the underlying economic model. In section 4.3 we analyse several different specifications of relative guarantees. We end the chapter in section 4.4 with some concluding remarks. We have also supplied an appendix containing, in addition to a proof of Proposition 4.8 and 4.9, several abbreviations used in the same propositions.

4.2 The Economic Model

We assume a continuous trading economy on the time interval $[t_0, T]$, for some fixed horizon $T > t_0$, and with no transaction costs. A filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ is fixed, where Ω is the state space, \mathcal{F} is a σ -algebra, $\mathbb{F} = \{\mathcal{F}_t, t_0 \leq t \leq T\}$ is a filtration where $\mathcal{F}_T = \mathcal{F}$ and $\mathcal{F}_{t_0} = \{\emptyset, \Omega\}$, where \emptyset is the empty set, and P is a probability measure. The σ -algebra is generated by a d -dimensional, $d \geq 1$, Brownian motion, W_t . We further assume a complete market, i.e., there exists one unique equivalent martingale measure

Q , see e.g., Harrison and Kreps (1979) and Harrison and Pliska (1981).

We let the forward interest rates be given by the model of Heath et al. (1992). The instantaneous continuously compounded forward rate at time s as seen from time t , $t_0 \leq t \leq s \leq T$, under the equivalent martingale measure Q , is given by

$$f(t, s) = f(t_0, s) + \int_{t_0}^t \sigma_f(v, s) \int_v^s \sigma_f(v, u) du dv + \int_{t_0}^t \sigma_f(v, s) dW_v,$$

where $\sigma_f(t, s)$ is the volatility function for the instantaneous continuously compounded forward rate at time s as seen from time t , satisfying some technical regularity conditions (for further details, see Heath et al. (1992)). The volatility function is a deterministic function of time, a fact that implies Gaussian interest rates. The short-term interest rate is obtained by setting s equal to t , i.e., $r_t = f(t, t)$.

We assume that there exists a money market account that accrues interest according to the short-term interest rate. This asset is instantaneously risk-free, and the time t market value is given by

$$M_t = M_{t_0} + \int_{t_0}^t r_v M_v dv, \quad M_{t_0} = 1, \quad t_0 \leq t.$$

The return on the money market account from time t_{n-1} to t_n (i.e., in period n) under the equivalent martingale measure Q , β_n , is given by (see e.g., Miltersen and Persson (1999))

$$\begin{aligned} \beta_n &= \int_{t_{n-1}}^{t_n} r_v dv = -\ln F(t_0, t_{n-1}, t_n) + \frac{1}{2} \sigma_{\beta_n}^2 + c_n \\ &\quad + \int_{t_0}^{t_{n-1}} \int_{t_{n-1}}^{t_n} \sigma_f(v, u) du dW_v + \int_{t_{n-1}}^{t_n} \int_v^{t_n} \sigma_f(v, u) du dW_v, \end{aligned}$$

where $F(t_0, t_{n-1}, t_n)$ is the time t_0 forward price for delivery at time t_{n-1} of a zero-coupon bond maturing at time t_n and is given by

$$F(t_0, t_{n-1}, t_n) = \frac{P(t_0, t_n)}{P(t_0, t_{n-1})},$$

where $P(t, T)$ is the market value at time t of a zero-coupon bond maturing at time $T \geq t$. Here $\sigma_{\beta_n}^2$ is the variance of the rate of return on the money market account in period n and is given by

$$\sigma_{\beta_n}^2 = \int_{t_0}^{t_{n-1}} \left(\int_{t_{n-1}}^{t_n} \sigma_f(v, u) du \right)^2 dv + \int_{t_{n-1}}^{t_n} \left(\int_v^{t_n} \sigma_f(v, u) du \right)^2 dv.$$

Further, c_n is the covariance between the return on the money market account over the time intervals from time t_0 to t_{n-1} and time period n . More

generally, the covariance between the return in the two time periods from time t_a to t_b and from time t_c to t_d , $t_0 \leq t_a < t_b \leq t_c < t_d$, is given by

$$\begin{aligned} c_{t_b-t_a, t_d-t_c} &= \int_{t_0}^{t_a} \left(\int_{t_a}^{t_b} \sigma_f(v, u) du \right) \left(\int_{t_c}^{t_d} \sigma_f(v, u) du \right) dv \\ &\quad + \int_{t_a}^{t_b} \left(\int_v^{t_b} \sigma_f(v, u) du \right) \left(\int_{t_c}^{t_d} \sigma_f(v, u) du \right) dv. \end{aligned}$$

We also assume that there exists two non-dividend paying portfolios with time $t \geq t_0$ market value S_t^i , $i \in \{1, 2\}$. The market value of portfolio i under the equivalent martingale measure Q is given by

$$S_t^i = S_{t_0}^i + \int_{t_0}^t r_v S_v^i dv + \int_{t_0}^t \sigma_{S^i}(v) S_v^i dW_v,$$

where $\sigma_{S^i}(t)$ is the volatility function and satisfies $E \left[\int_{t_0}^t (\sigma_{S^i}(v) S_v^i)^2 dv \right] < \infty$.² Also this volatility function is a deterministic function of time, hence, $\ln(S_t^i)$ is Gaussian under the equivalent martingale measure Q .

The return on portfolio i in period n under the equivalent martingale measure Q is given by

$$\delta_n^i = \int_{t_{n-1}}^{t_n} \left(r_v - \frac{1}{2} \sigma_{S^i}(v)^2 \right) dv + \int_{t_{n-1}}^{t_n} \sigma_{S^i}(v) dW_v,$$

with variance (also this under the equivalent martingale measure Q)

$$\sigma_{\delta_n^i}^2 = \sigma_{\beta_n}^2 + 2 \int_{t_{n-1}}^{t_n} \sigma_{S^i}(v) \int_v^{t_n} \sigma_f(v, u) dudv + \int_{t_{n-1}}^{t_n} \sigma_{S^i}^2(v) dv.$$

4.3 Relative Guarantees

4.3.1 The Maximum of Two Assets

We start by considering a contract with time $T > t_0$ payoff $\max(S_T^1, S_T^2)$. If the initial market values of the two portfolios are equal ($S_{t_0}^1 = S_{t_0}^2$), this contract can be thought of as an investment in the first portfolio with a guarantee embedded such that the return on the investment never falls below the return on the second portfolio.

Proposition 4.1. *The time t_0 market value of the guarantee with time T payoff $\max(S_T^1, S_T^2)$ is given by*

$$\pi_{t_0}^{(4.1)} = S_{t_0}^1 \Phi(d_1) + S_{t_0}^2 \Phi(d_2), \quad (4.1)$$

²Notice that in the multidimensional case both $\sigma_f(t, s)$ and $\sigma_{S^i}(t)$ are vectors, but the interpretation should be obvious.

where

$$\begin{aligned} d_1 &= \frac{\ln\left(\frac{S_{t_0}^1}{S_{t_0}^2}\right) + \frac{1}{2}v^2(T)}{v(T)}, \\ d_2 &= \frac{\ln\left(\frac{S_{t_0}^2}{S_{t_0}^1}\right) + \frac{1}{2}v^2(T)}{v(T)}, \\ v^2(T) &= \int_{t_0}^T \left(\sigma_{S^1}^2(v) - 2\sigma_{S^1}(v)\sigma_{S^2}(v) + \sigma_{S^2}^2(v) \right) dv, \end{aligned}$$

and $\Phi(\cdot)$ is the cumulative normal probability distribution.

Proof. The time t_0 market value is given by

$$\begin{aligned} \pi_{t_0}^{(4.1)} &= E_Q \left[e^{-\int_{t_0}^T r_v dv} \max(S_T^1, S_T^2) \right] \\ &= E_Q \left[e^{-\int_{t_0}^T r_v dv} S_T^1 1_A \right] + E_Q \left[e^{-\int_{t_0}^T r_v dv} S_T^2 1_{\bar{A}} \right], \end{aligned}$$

where $A = \{S_T^1 > S_T^2\}$ is the exercise set where the first portfolio is chosen and \bar{A} is the complement to A . Proceeding with a change of probability measure by using the Radon-Nikodym derivative

$$\frac{dQ_{S^i}}{dQ} = e^{-\frac{1}{2} \int_{t_0}^T \sigma_{S^i}(v)^2 dv + \int_{t_0}^T \sigma_{S^i}(v) dW_v}, \quad i \in \{1, 2\},$$

we get

$$\begin{aligned} \pi_{t_0}^{(4.1)} &= S_{t_0}^1 E_{Q_{S^1}} [1_A] + S_{t_0}^2 E_{Q_{S^2}} [1_{\bar{A}}] \\ &= S_{t_0}^1 Q_{S^1}(A) + S_{t_0}^2 Q_{S^2}(\bar{A}) \\ &= S_{t_0}^1 \Phi(d_1) + S_{t_0}^2 \Phi(d_2). \end{aligned}$$

The derivations of d_1 and d_2 are fairly standard and are therefore omitted in both this and the remaining proofs. \square

Interest rates do not enter the formula, and hence, the result is independent of the process followed by the interest rates. See Ekern and Persson (1996) for a derivation under deterministic interest rates.

Let us now consider some special cases that are motivated by an observation of the privately managed pension funds in Chile where relative guarantees have been in use for about 20 years. It has there been observed that the pension portfolios and the reference portfolio are very similar. Translated to our setting, if the return on the first and the second portfolio are equal, the unique decomposition property for Itô processes implies that the volatilities, represented by the vectors σ_{S^1} and σ_{S^2} , are equal, hence, $v^2(T) = 0$. Considering the limiting case as $v^2(T) \rightarrow 0^+$, we have three cases:

1. $S_{t_0}^1 = S_{t_0}^2 \Rightarrow d_1 = d_2 = 0 \Rightarrow \pi_{t_0}^{(4.1)} = S_{t_0}^1 = S_{t_0}^2$,
2. $S_{t_0}^1 > S_{t_0}^2 \Rightarrow d_1 = -d_2 = \infty \Rightarrow \pi_{t_0}^{(4.1)} = S_{t_0}^1$, and
3. $S_{t_0}^1 < S_{t_0}^2 \Rightarrow d_1 = -d_2 = -\infty \Rightarrow \pi_{t_0}^{(4.1)} = S_{t_0}^2$.

In the rest of this chapter we will only consider rate of return guarantees, i.e., we assume that $S_{t_0}^1 = S_{t_0}^2 = 1$. Using this, the expression in (4.1) simplifies to

$$\pi_{t_0}^{(4.1)} = 2\Phi\left(\frac{1}{2}v(T)\right),$$

and since we have that $v(T) > 0$, we can see that $\pi_{t_0}^{(4.1)} > 1$ because $\Phi(0) = \frac{1}{2}$, clearly demonstrating that the market value of the guarantee element in the contract has a positive market value.

4.3.2 A Multi-period Relative Guarantee

In many applications, especially in life insurance, one deals with annual, or periodical, guarantees. Extending the guarantee in Proposition 4.1 to a periodical guarantee, the time t_N payoff is given by

$$\pi_{t_N}^{(4.2)} = \prod_{n=1}^N \max(e^{\delta_n^1}, e^{\delta_n^2}),$$

where $n \in \{1, 2, \dots, N\}$ is the number of the period and is such that period N ends at time t_N . δ_n^1 and δ_n^2 are the return in period n on the first and the second portfolio, respectively.

Proposition 4.2. *The time t_0 market value of the multi-period relative guarantee with time t_N payoff $\pi_{t_N}^{(4.2)} = \prod_{n=1}^N \max(e^{\delta_n^1}, e^{\delta_n^2})$ is given by*

$$\pi_{t_0}^{(4.2)} = \prod_{n=1}^N 2\Phi\left(\frac{1}{2}v_n\right),$$

where

$$v_n^2 = \int_{t_{n-1}}^{t_n} \left(\sigma_{S^1}^2(v) - 2\sigma_{S^1}(v)\sigma_{S^2}(v) + \sigma_{S^2}^2(v) \right) dv.$$

Proof. The time t_0 market value is given by

$$\begin{aligned} \pi_{t_0}^{(4.2)} &= E_Q \left[e^{-\int_{t_0}^{t_N} r_v dv} \prod_{n=1}^N \max(e^{\delta_n^1}, e^{\delta_n^2}) \right] \\ &= E_Q \left[\prod_{n=1}^N e^{-\int_{t_{n-1}}^{t_n} r_v dv} \max(e^{\delta_n^1}, e^{\delta_n^2}) \right] \\ &= E_Q \left[\prod_{n=1}^N \max(e^{\delta_n^1}, e^{\delta_n^2}) \right], \end{aligned} \tag{4.2}$$

where

$$\hat{\delta}_n^i = -\frac{1}{2} \int_{t_{n-1}}^{t_n} \sigma_{S^i}^2(v) dv + \int_{t_{n-1}}^{t_n} \sigma_{S^i}(v) dW_v, \quad i \in \{1, 2\}. \quad (4.3)$$

The second equation follows from the linearity of integrals. From (4.3) it is easily seen that $\max(\hat{\delta}_j^1, \hat{\delta}_j^2)$ and $\max(\hat{\delta}_k^1, \hat{\delta}_k^2)$ are uncorrelated for all $j \neq k$, $j, k \in \{1, 2, \dots, N\}$. (4.2) can then be written as

$$\pi_{t_0}^{(4.2)} = \prod_{n=1}^N E_Q \left[\max(e^{\hat{\delta}_n^1}, e^{\hat{\delta}_n^2}) \right].$$

We proceed by valuing the n 'th guarantee, $\pi(n)$, by a change of probability measure using the Radon-Nikodym derivatives

$$\frac{dQ_{S^i}}{dQ} = e^{-\frac{1}{2} \int_{t_0}^T \sigma_{S^i}^2(v) dv + \int_{t_0}^T \sigma_{S^i}(v) dW_v}, \quad i \in \{1, 2\}.$$

$$\begin{aligned} \pi(n) &= E_Q \left[\max(e^{\hat{\delta}_n^1}, e^{\hat{\delta}_n^2}) \right] \\ &= E_{Q_{S^1}} \left[1_{A_n} \right] + E_{Q_{S^2}} \left[1_{\bar{A}_n} \right] \\ &= Q_{S^1}(A_n) + Q_{S^2}(\bar{A}_n) \\ &= 2\Phi\left(\frac{1}{2}v_n\right), \end{aligned}$$

where $A_n = \{\delta_n^1 > \delta_n^2\}$ and \bar{A}_n is the complement to A_n . This completes the proof. \square

Notice that $v_n > 0$ for all $n \in \{1, 2, \dots, N\}$, so $2\Phi(\frac{1}{2}v_n) > 1$ and $\pi_{t_0}^{(4.2)}$ increases strictly with the number of periods. This clearly demonstrates the importance of pricing these guarantees, since the value can be substantial for contracts lasting for several periods.

If $\sigma_{S^1}(v) = \sigma_{S^2}(v)$ the market value of the contract is equal to the market value of one unit of account accruing the return generated by one of the two portfolios, and is equal to one. Hence, the market value of the guarantee element in the contract is equal to zero.

As for the guarantee in Proposition 4.1 we observe that the market value of this multi-period rate of return guarantee is independent of interest rates and therefore also the process followed by the interest rates.

4.3.3 Additive Reduced Return on Reference Portfolio (1)

Another kind of guarantee gives the time T payoff $\max(e^{\delta_T^1}, e^{\delta_T^2 - \lambda})$, where $\lambda \in \mathbb{R}$. This is a part of the guarantee offered in several countries. In

practice, λ is likely to be positive, and this will then be a cheaper guarantee than the one in Proposition 4.1 since it can never give a higher payoff, but has the possibility of a lower payoff. There can be several rationales for this kind of guarantee. One is simply that this is a cheaper guarantee than the one in Proposition 4.1, and that it may therefore be easier to sell to potential customers.

Proposition 4.3. *The time t_0 market value of the guarantee with time T payoff $\max(e^{\delta_T^1}, e^{\delta_T^2 - \lambda})$ is given by*

$$\pi_{t_0}^{(4.3)} = \Phi(d_3) + e^{-\lambda} \Phi(d_4),$$

where

$$d_3 = \frac{\lambda + \frac{1}{2}v^2(T)}{v(T)}$$

and

$$d_4 = \frac{-\lambda + \frac{1}{2}v^2(T)}{v(T)}.$$

Proof. The proof follows the same lines as the proof of Proposition 4.1. \square

If there is no rate of return guarantee included, this is equivalent to the situation where $\lambda = \infty$, and the time t_0 market value is then equal to one. Including the possibility of getting $e^{\delta_T^2 - \lambda}$ if δ_T^1 is very low must of course have a positive value, hence, $\pi_{t_0}^{(4.3)} > 1$ for a finite λ . For $\lambda = 0$ we obtain the same market value as in Proposition 4.1.

4.3.4 Additive Reduced Return on Reference Portfolio (2)

The guarantee in Proposition 4.3 is easily extended to a multi-period rate of return guarantee with a terminal payoff $\prod_{n=1}^N \max(e^{\delta_n^1}, e^{\delta_n^2 - \lambda_n})$, where $\lambda_n \in \mathbb{R}$.

Proposition 4.4. *The time t_0 market value of a multi-period rate of return guarantee with time T payoff $\prod_{n=1}^N \max(e^{\delta_n^1}, e^{\delta_n^2 - \lambda_n})$ is given by*

$$\pi_{t_0}^{(4.4)} = \prod_{n=1}^N \left(\Phi(d_3^m) + e^{-\lambda_n} \Phi(d_4^m) \right),$$

where

$$d_3^m = \frac{\lambda_n + \frac{1}{2}v_n^2}{v_n}$$

and

$$d_4^m = \frac{-\lambda_n + \frac{1}{2}v_n^2}{v_n}.$$

Proof. The proof follows the same lines as the proof of Proposition 4.2. \square

By the same arguments as in subsection 4.3.3, it follows that $\Phi(d_3^n) + e^{-\lambda_n} \Phi(d_4^n) > 1$ for all $n \in \{1, 2, \dots, N\}$, and also this guarantee is therefore strictly increasing in N .

4.3.5 Multiplicative Reduced Return on Reference Portfolio

Consider now a guarantee with time T payoff $\max(e^{\delta_T^1}, e^{\gamma\delta_T^2})$, $\gamma \in \mathbb{R}$. Also this guarantee can be seen to be a part of the guarantee offered in several countries. In practice, γ is likely to be between zero and one.

The motivation for this guarantee can be that the stochastic guarantee is equal to the average return on several other portfolios. γ would then equal one over the total number of portfolios that δ^2 is made up of. It could also be a positive number less than one, meaning that the stochastic guaranteed rate of return is, e.g., 80% of the return on the reference portfolio. Of course, it can also be a combination of these two interpretations.

Proposition 4.5. *The time t_0 market value of the guarantee with time T payoff $\max(e^{\delta_T^1}, e^{\gamma\delta_T^2})$ is given by*

$$\pi_{t_0}^{(4.5)} = \Phi(d_5) + e^{\kappa_1} \Phi(d_6),$$

where

$$\begin{aligned} \kappa_1 &= (\gamma - 1) \left(-\ln(P(t_0, T)) + \frac{1}{2} \gamma \sigma_{\delta_T^2}^2 \right), \\ d_5 &= \frac{(1 - \gamma) \left(-\ln P(t_0, T) + \frac{1}{2} \sigma_{\delta_T^1}^2 \right) + \gamma \frac{1}{2} v^2(T)}{\sqrt{(1 - \gamma)(\sigma_{\delta_T^1}^2 - \gamma \sigma_{\delta_T^2}^2) + \gamma v^2(T)}}, \end{aligned}$$

and

$$d_6 = -\frac{(1 - \gamma) \left(-\ln P(t_0, T) - \frac{1}{2} \sigma_{\delta_T^1}^2 + \gamma \sigma_{\delta_T^2}^2 \right) - \gamma \frac{1}{2} v^2(T)}{\sqrt{(1 - \gamma)(\sigma_{\delta_T^1}^2 - \gamma \sigma_{\delta_T^2}^2) + \gamma v^2(T)}}.$$

Proof. The time t_0 market value is given by

$$\begin{aligned} \pi_{t_0}^{(4.5)} &= E_Q \left[e^{-\int_{t_0}^T r_v dv} \max(e^{\delta_T^1}, e^{\gamma\delta_T^2}) \right] \\ &= E_Q \left[e^{-\int_{t_0}^T r_v dv} e^{\delta_T^1} 1_A \right] + E_Q \left[e^{-\int_{t_0}^T r_v dv} e^{\gamma\delta_T^2} 1_{\bar{A}} \right], \end{aligned}$$

where $A = \{\delta_T^1 > \gamma\delta_T^2\}$ and \bar{A} is the complement to A . Letting

$$\frac{dQ_{S^1}}{dQ} = e^{-\frac{1}{2} \int_{t_0}^T \sigma_{S^1}^2(v) dv + \int_{t_0}^T \sigma_{S^1}(v) dW_v},$$

it follows by straightforward calculations that

$$\begin{aligned} E_Q \left[e^{-\int_{t_0}^T r_v dv} e^{\delta_T^1} 1_A \right] &= E_{Q_{S^1}} \left[1_A \right] = Q_{S^1}(A) \\ &= \Phi(d_5). \end{aligned}$$

Let

$$\frac{dQ_\gamma}{dQ} = \frac{\xi}{E_Q(\xi)},$$

where

$$\xi = e^{(\gamma-1) \int_{t_0}^T r_v dv - \frac{1}{2} \gamma \int_{t_0}^T \sigma_{S^2}^2(v) dv + \gamma \int_{t_0}^T \sigma_{S^2} dW_v}$$

and is equal to $e^{-\int_{t_0}^T r_v dv + \gamma \delta_T^2}$.

$$E_Q(\xi) = \left(\frac{1}{P(t_0, T)} \right)^{\gamma-1} e^{\gamma(\gamma-1) \frac{1}{2} \sigma_{S^2}^2} = e^{\kappa_1}.$$

It then follows that the second term in the expression for the market value of the guarantee is given by

$$\begin{aligned} E_Q \left[e^{-\int_{t_0}^T r_v dv} e^{\gamma \delta_T^2} 1_{\bar{A}} \right] &= e^{\kappa_1} E_{Q_\gamma} \left[1_{\bar{A}} \right] = e^{\kappa_1} Q_\gamma(\bar{A}) \\ &= e^{\kappa_1} \Phi(d_6). \end{aligned}$$

The time t_0 market value then follows. □

The “extra” term in front of $\Phi(d_6)$, e^{κ_1} , is a consequence of the fact that only a fraction γ of the return on the reference portfolio enters the contract. For $\gamma \in (0, 1)$ the term is positive but less than one and can be interpreted as a “reduction-factor” due to the negative convenience yield³ from the reference portfolio. For $\gamma > 1$ the term will be greater than one. Notice that for $\gamma = 1$ we get, as one would expect, the same time t_0 market value as in Proposition 4.1. A related result under deterministic interest rates can be found in Miltersen and Persson (2002).

The guarantee in Proposition 4.5, and also the ones following in Proposition 4.6 and 4.7, can of course be extended to multi-period guarantees. However, the expressions for the market values of such guarantees are likely to be fairly cumbersome and the guarantees will therefore not be analysed here.

³We can think of convenience yield as a benefit the holder of the asset receives that the holder of a forward contract on the asset does not receive. The dividend a stock is paying is an example of a positive convenience yield, while the cost of carrying one barrel of oil is an example of a negative convenience yield.

4.3.6 Additive and Multiplicative Reduced Return on Reference Portfolio

One could also extend the guarantee to be a combination of the additive and the multiplicative reduced return on the reference portfolio. This yields the time T payoff $\max(e^{\delta_T^1}, e^{\gamma\delta_T^2-\lambda})$, and for a positive λ and $\gamma \in (0, 1)$ this guarantee is of course cheaper than both the guarantee in Proposition 4.3 and 4.5.

Proposition 4.6. *The time t_0 market value of a guarantee with time T payoff $\max(e^{\delta_T^1}, e^{\gamma\delta_T^2-\lambda})$ is given by*

$$\pi_{t_0}^{(4.6)} = \Phi(d_7) + e^{-\lambda+\kappa_1} \Phi(d_8),$$

where

$$d_7 = \frac{(1-\gamma)\left(-\ln P(t_0, T) + \frac{1}{2}\sigma_{\delta_T^1}^2\right) + \gamma\frac{1}{2}v^2(T) + \lambda}{\sqrt{(1-\gamma)(\sigma_{\delta_T^1}^2 - \gamma\sigma_{\delta_T^2}^2) + \gamma v^2(T)}}$$

and

$$d_8 = -\frac{(1-\gamma)\left(-\ln P(t_0, T) - \frac{1}{2}\sigma_{\delta_T^1}^2 + \gamma\sigma_{\delta_T^2}^2\right) - \gamma\frac{1}{2}v^2(T) + \lambda}{\sqrt{(1-\gamma)(\sigma_{\delta_T^1}^2 - \gamma\sigma_{\delta_T^2}^2) + \gamma v^2(T)}}.$$

Proof. The proof follows the same lines as the proof of Proposition 4.5. \square

4.3.7 The “Chilean” Minimum Guarantee

In this and the following two subsections we analyse guarantees that are used in practice, and then in particular in Chile. That we have described the guarantee used in Chile by three different models is both a consequence of the fact that our models must be seen as simplifications of the real-world guarantee and the fact that the descriptions of the guarantee that we have found in the literature are mutual inconsistent. However, we believe that each interpretation of the guarantee contains interesting features in it self.

Consider now a contract where the final payoff at time T is given by

$$\max\left(e^{\delta_T^1}, \min(e^{\delta_T^2-\lambda}, e^{\gamma\delta_T^2})\right),$$

where λ and γ can be interpreted as before. This is the same minimum rate of return guarantee that is embedded in defined contribution based pension plans in Argentina ($\lambda = 0.02$ and $\gamma = 0.7$), Chile ($\lambda = 0.02$ and $\gamma = 0.5$), and Poland ($\lambda = 0.04$ and $\gamma = 0.5$). The first two countries also have a maximum return included in the pension plans, but for simplicity this will not be considered here.

Proposition 4.7. *The time t_0 market value of the guarantee with time T payoff $\max\left(e^{\delta_T^1}, \min(e^{\delta_T^2 - \lambda}, e^{\gamma \delta_T^2})\right)$ is given by*

$$\begin{aligned} \pi_{t_0}^{(4.7)} &= \Phi(a_5, b_5, \bar{\rho}_1) + e^{-\lambda} \Phi(a_6, b_6, -\bar{\rho}_1) + \Phi(a_7, b_7, -\bar{\rho}_2) \\ &\quad + e^{\kappa_1} \Phi(a_8, b_8, \bar{\rho}_2), \end{aligned}$$

where

$$\begin{aligned} a_5 &= \frac{\ln P(t_0, T) - \frac{1}{2}(\sigma_{\delta_T^1}^2 - v^2(T)) + \frac{\lambda}{(1-\gamma)}}{\sigma_{\delta_T^2}}, \\ b_5 &= \frac{\frac{1}{2}v^2(T) + \lambda}{v(T)}, \\ a_6 &= \frac{\ln P(t_0, T) - \frac{1}{2}\sigma_{\delta_T^2}^2 + \frac{\lambda}{(1-\gamma)}}{\sigma_{\delta_T^2}}, \\ b_6 &= -\frac{-\frac{1}{2}v^2(T) + \lambda}{v(T)}, \\ a_7 &= -a_5, \\ b_7 &= \frac{(1-\gamma)\left(-\ln P(t_0, T) + \frac{1}{2}\sigma_{\delta_T^1}^2\right) + \gamma\frac{1}{2}v^2(T)}{\sqrt{(1-\gamma)(\sigma_{\delta_T^1}^2 - \gamma\sigma_{\delta_T^2}^2) + \gamma v^2(T)}}, \\ a_8 &= -\frac{\ln P(t_0, T) - (\gamma - \frac{1}{2})\sigma_{\delta_T^2}^2 + \frac{\lambda}{(1-\gamma)}}{\sigma_{\delta_T^2}}, \\ b_8 &= -\frac{(1-\gamma)\left(-\ln P(t_0, T) - \frac{1}{2}\sigma_{\delta_T^1}^2 + \gamma\sigma_{\delta_T^2}^2\right) - \gamma v^2(T)}{\sqrt{(1-\gamma)(\sigma_{\delta_T^1}^2 - \gamma\sigma_{\delta_T^2}^2) + \gamma v^2(T)}}, \\ \bar{\rho}_1 &= \frac{\frac{1}{2}(1-\gamma)\left(\sigma_{\delta_T^1}^2 - \sigma_{\delta_T^2}^3 - v^2(T)\right)}{(1-\gamma)\sigma_{\delta_T^2} \cdot v(T)}, \end{aligned}$$

and

$$\bar{\rho}_2 = \frac{\frac{1}{2}\left((1-3\gamma+2\gamma^2)\sigma_{\delta_T^2}^2 + (1-\gamma)(\sigma_{\delta_T^1}^2 - v^2(T))\right)}{\sigma_{\delta_T^2} \cdot \sqrt{(1-\gamma)(\sigma_{\delta_T^1}^2 - \gamma\sigma_{\delta_T^2}^2) + \gamma v^2(T)}}.$$

Proof. Let

$$\begin{aligned} A_1 &= \{\delta_T^2 - \lambda > \gamma\delta_T^2\} = \{\delta_T^2 - \gamma\delta_T^2 > \lambda\}, \\ A_2 &= \{\delta_T^1 > \delta_T^2 - \lambda\} = \{\delta_T^1 - \delta_T^2 > -\lambda\}, \end{aligned}$$

and

$$A_3 = \{\delta_T^1 > \gamma\delta_T^2\} = \{\delta_T^1 - \gamma\delta_T^2 > 0\},$$

and let further \bar{A}_1 , \bar{A}_2 , and \bar{A}_3 be the respective complements.

The time t_0 market value of the guarantee can be expressed as

$$\begin{aligned} \pi_{t_0}^{(4.7)} &= E_Q \left[e^{-\int_{t_0}^T r_v dv} \max \left(e^{\delta_T^1}, \min(e^{\delta_T^2 - \lambda}, e^{\gamma\delta_T^2}) \right) \right] \\ &= E_Q \left[e^{-\int_{t_0}^T r_v dv} \left\{ \max \left(e^{\delta_T^1}, e^{\delta_T^2 - \lambda} \right) 1_{A_1} + \max \left(e^{\delta_T^1}, e^{\gamma\delta_T^2} \right) 1_{\bar{A}_1} \right\} \right] \\ &= E_Q \left[e^{-\int_{t_0}^T r_v dv} \left\{ e^{\delta_T^1} 1_{A_1 \cap A_2} + e^{\delta_T^2 - \lambda} 1_{A_1 \cap \bar{A}_2} \right. \right. \\ &\quad \left. \left. + e^{\delta_T^1} 1_{\bar{A}_1 \cap A_3} + e^{\gamma\delta_T^2} 1_{\bar{A}_1 \cap \bar{A}_3} \right\} \right]. \end{aligned} \quad (4.4)$$

Using the following Radon-Nikodym derivatives,

$$\begin{aligned} \frac{dQ_{S^1}}{dQ} &= e^{-\frac{1}{2} \int_{t_0}^T \sigma_{S^1}^2(v) dv + \int_{t_0}^T \sigma_{S^1}(v) dW_v}, \\ \frac{dQ_{S^2}}{dQ} &= e^{-\frac{1}{2} \int_{t_0}^T \sigma_{S^2}^2(v) dv + \int_{t_0}^T \sigma_{S^2}(v) dW_v}, \end{aligned}$$

and

$$\frac{dQ_{S^\gamma}}{dQ} = \frac{e^{\gamma\delta_T^2 - \int_{t_0}^T r_v dv}}{E_Q \left[e^{\gamma\delta_T^2 - \int_{t_0}^T r_v dv} \right]},$$

(4.4) can be written as

$$\begin{aligned} \pi_{t_0}^{(4.7)} &= E_{Q_{S^1}} [1_{A_1 \cap A_2}] + e^{-\lambda} E_{Q_{S^2}} [1_{A_1 \cap \bar{A}_2}] + E_{Q_{S^1}} [1_{\bar{A}_1 \cap A_3}] \\ &\quad + e^{\kappa_1} E_{Q_\gamma} [1_{\bar{A}_1 \cap \bar{A}_3}] \\ &= Q_{S^1}(A_1 \cap A_2) + e^{-\lambda} Q_{S^2}(A_1 \cap \bar{A}_2) + Q_{S^1}(\bar{A}_1 \cap A_3) \\ &\quad + e^{\kappa_1} Q_\gamma(\bar{A}_1 \cap \bar{A}_3). \end{aligned}$$

Recall the definition of A_1 , A_2 , and A_3 , and let

$$\begin{aligned} X_1 &= \delta_T^2 - \gamma\delta_T^2, \\ X_2 &= \delta_T^1 - \delta_T^2, \end{aligned}$$

and

$$X_3 = \delta_T^1 - \gamma\delta_T^2.$$

Based on this, it follows that

$$\bar{\rho}_1 = \frac{E \left[(X_1 - E[X_1])(X_2 - E[X_2]) \right]}{\sqrt{E \left[(X_1 - E[X_1])^2 \right] E \left[(X_2 - E[X_2])^2 \right]}}$$

and

$$\bar{\rho}_2 = \frac{E[(X_1 - E[X_1])(X_3 - E[X_3])]}{\sqrt{E[(X_1 - E[X_1])^2] E[(X_3 - E[X_3])^2]}}$$

Finally, from the above it follows that

$$\begin{aligned} \pi_{t_0}^{(4.7)} &= \Phi(a_5, b_5, \bar{\rho}_1) + e^{-\lambda} \Phi(a_6, b_6, -\bar{\rho}_1) + \Phi(a_7, b_7, -\bar{\rho}_2) \\ &\quad + e^{\kappa_1} \Phi(a_8, b_8, \bar{\rho}_2). \end{aligned}$$

□

The guarantee in this subsection has close resemblance to an option on the maximum or the minimum of two assets, cf. Stulz (1982) and Johnson (1987). Following the approach laid out above, the results of Stulz (1982) and Johnson (1987) can be extended to incorporate stochastic interest rates, see e.g., Lindset (2002).

4.3.8 Average Return on Reference Portfolio (1)

In Chile the stochastic minimum guaranteed rate of return on pension portfolios is given by the average return on the reference portfolio over the last three year period, hence, the return on the underlying asset in, say, year three will be compared with the average return on the reference portfolio in year one, two, and three. Before 1999 the stochastic minimum guaranteed rate of return in one year was the return on the reference portfolio in the same year. However, a problem has been that the pension intermediaries (the private managed pension funds) have been offering very similar portfolios. By calculating the minimum guaranteed rate of return as a three year average, the hope has been to increase the diversity in the offer of pension portfolios.

Assume that you at time t_m , $t_0 \leq t_m \leq t_{N-1}$, would like to value a guarantee on the return from time t_{N-1} to t_N , where the time t_N payoff is given by $\max(e^{\delta_{t_N-t_{N-1}}^1}, e^{\frac{1}{t_N-t_0} \delta_{t_N}^2})$. Here $\delta_{t_N-t_{N-1}}^1$ is the accumulated return on the first portfolio from time t_{N-1} to t_N and $\delta_{t_N}^2$ is the accumulated return on the second portfolio from time t_0 to t_N . See Figure 4.1 for an illustration.

Assume further that the realised return on the reference portfolio from time t_0 to t_m is given by $\delta_{t_m}^2$. Let $R_{t_m}^2 = \frac{\delta_{t_m}^2}{t_m-t_0}$.

Proposition 4.8. *The time t_m market value, $t_0 \leq t_m \leq t_{N-1}$, of the guarantee with time t_N payoff $\max(e^{\delta_{t_N-t_{N-1}}^1}, e^{\frac{1}{t_N-t_0} \delta_{t_N}^2})$ is given by*

$$\pi_{t_m}^{(4.8)} = P(t_m, t_{N-1}) \Phi(d_9) + e^{R_{t_m}^2 + \kappa_2} \Phi(d_{10}),$$

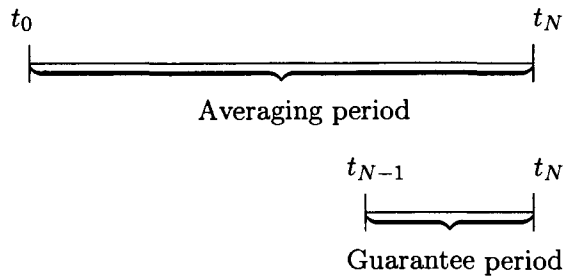


Figure 4.1: Illustration of the averaging period and the guarantee period for the “Average Return on Reference Portfolio (1)”. The return on the first portfolio over the guarantee period, $\delta_{t_N-t_{N-1}}^1$, is compared with the average return on the second (reference) portfolio over the averaging period, $\frac{\delta_{t_N}^2}{t_N-t_0}$.

where

$$\begin{aligned} \kappa_2 &= \left(\frac{1}{\tau} - 1\right) \left(-\ln P(t_m, t_N) + \frac{1}{2\tau} \sigma_{\delta_{t_N-t_m}^2}^2\right), \\ \tau &= t_N - t_m, \\ d_9 &= \frac{-R_{t_m}^2 + \zeta_1}{\hat{\sigma}}, \\ d_{10} &= \frac{R_{t_m}^2 + \zeta_2}{\hat{\sigma}}, \end{aligned}$$

and ζ_1 , ζ_2 , and $\hat{\sigma}$ are defined in section B.1.

Proof. See section B.2. □

Notice that for $t_m = t_{N-1} = t_0$ and $t_N = T$ the result in Proposition 4.8 coincides with the result in Proposition 4.1.

4.3.9 Average Return on Reference Portfolio (2)

Let us now consider a multi-period version of the guarantee in Proposition 4.8. As we will see, this extension complicates matters somewhat, and we therefore restrict our analysis to a guarantee that lasts for two periods. Even though this is likely to be too short for practical applications, our approach shows how these kinds of guarantees can be evaluated. Extending to several periods is in principle straightforward, but a lot of algebra is likely to be required.

The average stochastic minimum guaranteed rate of return will now have some overlapping time for the two periods. We assume that the averaging

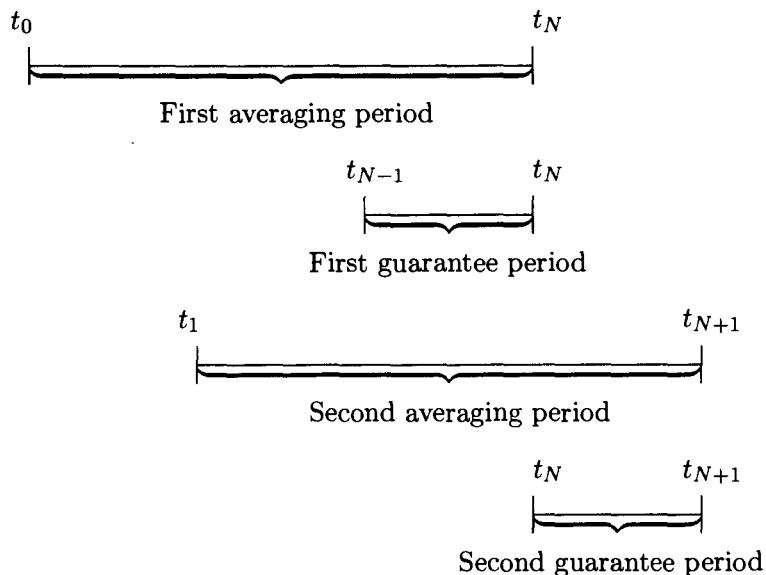


Figure 4.2: Illustration of the two averaging periods and the two guarantee periods for the "Average Return on Reference Portfolio (2)". The return on the first portfolio in the first guarantee period, $\delta_{t_N - t_{N-1}}^1$, is compared with the average return on the second (reference) portfolio over the first averaging period, $\delta_{t_N}^2 / \tau_1$. The return on the first portfolio in the second guarantee period, $\delta_{t_{N+1} - t_N}^1$, is compared with the average return on the second (reference) portfolio in the second averaging period, $\delta_{t_{N+1} - t_1}^2 / \tau_2$.

period for the first guarantee period (from time t_{N-1} to t_N) is the time interval from t_0 to t_N and the averaging period for the second guarantee period (from time t_N to t_{N+1}) is the time interval from t_1 to t_{N+1} . Let $\tau_1 = t_N - t_0$ and $\tau_2 = t_{N+1} - t_1$. An illustration is given in Figure 4.2.

Proposition 4.9. *The time t_0 market value of a guarantee with time t_{N+1} payoff $\max\left(e^{\delta_{t_N - t_{N-1}}^1}, e^{\frac{1}{\tau_1} \delta_{t_N}^2}\right) \cdot \max\left(e^{\delta_{t_{N+1} - t_N}^1}, e^{\frac{1}{\tau_2} \delta_{t_{N+1} - t_1}^2}\right)$ is given by*

$$\begin{aligned} \pi_{t_0}^{(4.9)} &= e^{\kappa_3} \Phi(a_1, b_1, \rho) + e^{\kappa_4} \Phi(a_2, b_2, -\rho) \\ &\quad + e^{\kappa_5} \Phi(a_3, b_3, -\rho) + e^{\kappa_6} \Phi(a_4, b_4, \rho), \end{aligned}$$

where $\kappa_i, i \in \{3, 4, 5, 6\}$, $a_j, b_j, j \in \{1, 2, 3, 4\}$, and ρ are given in section B.3.

Proof. See section B.4. □

Notice the similar structure of the expression for the market value of this two-period guarantee and the two-period guarantees in Persson and Aase (1997) and Miltersen and Persson (1999). The four additive terms is a

consequence of the four different possible combinations of the realised return for the contract. Also notice how this guarantee differs from the multi-period guarantee in Proposition 4.2 where only the cumulative univariate normal probability distribution is used.

4.3.10 Numerical Examples

We end this section with some numerical examples. Assume the following specification of the volatility

$$\sigma_f(v, u) = \sigma e^{-\kappa(u-v)} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \sqrt{1 - \varphi_1^2 - \varphi_2^2} \end{pmatrix},$$

where σ , κ , φ_1 , and φ_2 are constants and φ_1 and φ_2 are such that $\varphi_1^2 + \varphi_2^2 \leq 1$. This choice of volatility corresponds to the model of Vasicek (1977), see e.g., Heath et al. (1992). We further assume that

$$\sigma_{S^1}(v) = \begin{pmatrix} \sigma_{1,1} \\ 0 \\ 0 \end{pmatrix}$$

and

$$\sigma_{S^2}(v) = \begin{pmatrix} \sigma_{2,1} \\ \sigma_{2,2} \\ 0 \end{pmatrix},$$

where $\sigma_{1,1}$, $\sigma_{2,1}$, and $\sigma_{2,2}$ are constants. As our base case we use the following parameter values:

$$\begin{aligned} S_{t_0}^1 &= 1, & S_{t_0}^2 &= 1, & \sigma &= 0.03, & \kappa &= 0.1, & \varphi_1 &= -0.5, \\ \varphi_2 &= -0.25, & \sigma_{1,1} &= 0.2, & \sigma_{2,1} &= 0.1, & \sigma_{2,2} &= 0.15, & \lambda &= 0.1, \\ \gamma &= 0.8, & t_N &= 4. \end{aligned}$$

Based on the choice of volatility structure and parameter values, we have in Table 4.1 calculated the market values of the guarantees in Proposition 4.1 to 4.8.

From Table 4.1 we can see from the case with $\varphi_2 = 0.25$ that changing the correlation between the return on the reference portfolio and the interest rates seems to have little effect on the market values of the guarantees. This

Table 4.1: Market values of the guarantees in Proposition 4.1 - 4.8. Base case parameter values are $S_{t_0}^1 = 1$, $S_{t_0}^2 = 1$, $\sigma = 0.03$, $\kappa = 0.1$, $\varphi_1 = -0.5$, $\varphi_2 = -0.25$, $\sigma_{1,1} = 0.2$, $\sigma_{2,1} = 0.1$, $\sigma_{2,2} = 0.15$, $\lambda = 0.1$, $\gamma = 0.8$, and $t_N = 4$. For the multi-period guarantee $\lambda_n = \frac{\lambda}{t_N}$, $n \in \{1, 2, 3, 4\}$.

	Base case	$\varphi_2 = 0.25$	$\sigma_{2,1} = 0$	$\sigma_{2,1} = -0.2$	$\sigma_{2,2} = 0$
Prop. 4.1	1.14307	1.14307	1.19741	1.33077	1.07966
Prop. 4.2	1.31975	1.31975	1.46131	1.86839	1.16931
Prop. 4.3	1.09382	1.09382	1.14410	1.26947	1.03753
Prop. 4.4	1.25901	1.25901	1.39298	1.77956	1.11794
Prop. 4.5	1.09009	1.08625	1.13623	1.23222	1.05399
Prop. 4.6	1.05471	1.05167	1.09479	1.18334	1.02543
Prop. 4.7	1.08305	1.07951	1.13296	1.23050	1.03452
Prop. 4.8	0.95806	0.96122	0.95550	0.95780	0.95091

	Base case	$\lambda = 0.5$	$\lambda = -0.3$	$\gamma = 0.3$	$\gamma = 1$
Prop. 4.3	1.09382	1.01048	1.39697	1.09382	1.09382
Prop. 4.4	1.25901	1.10129	1.57300	1.25901	1.25901
Prop. 4.5	1.09009	1.08625	1.13623	1.02736	1.14307
Prop. 4.6	1.05471	1.00400	1.29592	1.01415	1.09382
Prop. 4.7	1.08305	1.01048	1.09009	1.01345	1.09382*

* Since the formula is not well defined for $\gamma = 1$, we have used $\gamma = 0.999$.

is in accordance with the knowledge that equity derivatives are not very sensitive to interest rate movements. By setting $\sigma_{2,1} = 0$, i.e., make the return on the two portfolios uncorrelated (except for the correlation through the drift term), the market values increase, except for the guarantee in Proposition 4.8. This guarantee seems to behave somewhat differently from the others and will therefore in the following not be commented on. We see a further increase by setting $\sigma_{1,2} = -0.2$, i.e., introducing negative correlation between the return on the two portfolios. In the same way we can also see a severe reduction in the market values when the correlation is increased through setting $\sigma_{2,2} = 0$. We can also see that decreasing λ increases the market values of the guarantees that are sensitive to λ , while an increase in γ increases the market values of the guarantees that are sensitive to γ . We can also see that when γ is close to 1 that the guarantee in Proposition 4.7 has the same market value as the guarantee in Proposition 4.3. This follows since $\delta_T^2 - \lambda$ will always be less than δ_T^2 for a strictly positive λ and the two guarantees will therefore coincide.

4.4 Conclusions

We have in this chapter analysed and found closed form solutions for the market values of a wide range of different relative rate of return guarantees within a stochastic interest rate framework. Even though the minimum guaranteed rate of return is stochastic in relative guarantees, some of the results we have derived are in fact less complicated than the corresponding results for absolute guarantees. This accounts for the guarantees in Proposition 4.1 and 4.3, and even further so for the guarantees in Proposition 4.2 and 4.4. These multi-period rate of return guarantees have the nice feature, something that is not the case for the absolute multi-period guarantees, that there are no correlation between the returns, after subtracting the risk-free interest rate (under the equivalent martingale measure Q), in the different periods.

We further considered a contract where the minimum guaranteed rate of return was only a fraction of the return on the reference portfolio, and we saw that this complicated matters somewhat. A slightly related contract where the minimum guaranteed rate of return was given as the average return on the reference portfolio over a given time period was also considered. Also a multi-period version of this guarantee was considered. However, these average guarantees led to fairly cumbersome expressions. In addition, the minimum guaranteed rate of return embedded in pension plans in several countries, i.e., Argentina, Chile, and Poland, was also analysed and a closed form solution for the market value was presented. This result also extends the literature on pricing of options on the maximum or the minimum of two assets (and related claims) to a stochastic interest rate framework.

Chapter 5

Defined Contribution and Defined Benefit Based Pension Plans

Abstract

In this chapter we address the problem of valuing (corporate) pension plans, and in particular defined contribution based pension plans. Several pension plans are proposed, both with and without rate of return guarantees. Both maturity and annual guarantees are considered. Emphasis is also given on the risk exposure for the employees' pensions. For comparison, we have also given a short analysis of defined benefit based pension plans.

To tie the analysis closer to real-world problems, we allow for both periodic premium and pension payments and mortality risk is taken into account. Some new results on forward-start guarantees are also derived.

Keywords and phrases: Pension plans, defined contribution based pension plans, defined benefit based pension plans, forward-start guarantees.

5.1 Introduction

The aging of the population in most parts of the western world has led to an increased focus on private pension arrangements, see e.g., *The Economist*, May 11'th, 2002 p. 80. Many employees have pension arrangements through their employers and are members of different corporate pension plans. Traditionally, most of these pension plans have been so-called *defined benefit*

based pension plans. In later years, so-called *defined contribution based pension plans*, where the pensions are a function of (among other things) the return in the financial markets, have increased in popularity. This is likely to be both because of changes in the laws regulating corporate pension plans (for instance, defined contribution based pension plans were in Norway not allowed prior to 2001) and by the fact that the financial markets have historically given a high return on investments.

Pension plans where the employer pays the premiums can be of significant economic value for the employee. For instance, for a person who considers different potential employers, the values of the pension plans the employers are offering should be taken into account when evaluating the offers. The goal of this chapter is to present a way for the employees to value and rank different pension plans. We also propose and analyse several fairly general pension plans.

Many defined contribution based pension plans are not embedded with any sort of guarantees to reduce the financial risk. This risk can be reduced by embedding the pension plans with *nominal* guarantees; i.e., rate of return guarantees. The theory of arbitrage pricing of contingent claims from financial economics has proved to be a useful tool in the valuation of rate of return guarantees, see e.g., Brennan and Schwartz (1976) and Miltersen and Persson (1999). For an excellent treatment of more complicated guarantees, see Tiong (2000).

A crucial assumption underlying arbitrage pricing is that the payoff of the claim that is to be priced can be replicated with some self-financing trading strategy involving the underlying asset(s) of the claim. The price of the claim can by arbitrage arguments be seen to coincide with the initial price of the trading strategy and is termed *the arbitrage price*. Normally, neither the employer nor the employee has the possibility to replicate the (financial) claim that a pension plan represents. However, it does not seem unreasonable to assume that a pension fund has the possibility, at least to some extent, to replicate the (financial) claims that are present in its customers pension plans. In a competitive market, this possibility should therefore make the arbitrage price of the pension plan equal to the price the pension fund can charge for the financial part of a pension plan.

The arbitrage price does not take *mortality risk* into consideration, an important aspect about pension plans. We use the standard assumption about independence between financial and mortality risk. By issuing many similar and statistically independent pension plans, mortality risk can (at least to some extent) be diversified. We therefore also use the assumption about risk neutrality with respect to mortality risk. However, it is not possible for the employees to diversify this risk. Since the employees can neither replicate the arbitrage price of a pension plan nor diversify mortality risk, the *market value*, here defined as the arbitrage price adjusted for mortality risk, will typically not be the only measure that can be used to rank differ-

ent pension plans, but as a price measure it may still be better than other measures not based on economic arguments.

To justify the use of these two assumptions, assume that the employees can sell (or more realistically, borrow against) the future random cash flow from their pension plans and use the funds from the sale to construct portfolios that give them the “optimal” consumption. We can imagine a market for buying these pension plans. If the buyers are sufficiently large in terms of the number of policies bought, they should also be risk neutral with respect to mortality risk. In a competitive market, this and the hedging possibilities for the buyers of the pension plans should make the market value equal to the price the employees can sell their pension plans for. Using these assumptions, the employees can rank different pension plans based on their market values.

This chapter differs from most of the existing literature on pension plans and guarantees, not only in that mortality risk is included, but mainly because we consider both periodic premiums and pension payments. Periodic premiums were also considered by Brennan and Schwartz (1976), but they had another approach based on numerical solution of a partial differential equation. Our approach has the nice feature that it involves closed form solutions based on some extensions of already known pricing results for rate of return guarantees. Using these guarantees and the concepts laid out above, we propose different pension plans and show how they can be valued.

Our main focus is on defined contribution based pension plans. Both pension plans with and without guarantees embedded are analysed. For the sake of comparison, we also give a short and simplified analysis of defined benefit based pension plans. The main purpose of this analysis is to show the difference in financial risk between these two kinds of pension plans. Two of the main differences is *when* the risk is present and *who* gets exposed to the risk. The defined contribution based pension plan, in its simplest form, invests deterministic amounts (the premiums) in the financial market. The premiums accumulate some uncertain return, leading to uncertain pension payments. Again, in its simplest form, a defined benefit based pension plan has deterministic pension payments. To achieve these pension payments when the premiums are invested in an uncertain financial market, the size of the premiums has to change in accordance with the financial market. Hence, the financial risk is borne by the employees in a defined contribution based pension plan, whereas the employers bear this risk in a defined benefit based pension plan.

A criticism that has been raised against defined contribution based pension plans has been that the employees get exposed to too much risk, risk they may not be interested in bearing. We show, by numerical examples with realistic parameter values, that these pension plans indeed are risky, even when rate of return guarantees are included. This is supported by findings in Burtless (2000). He showed by an example, using historical data

from the US, that the annual pension for a “typical” US citizen who retired in 1969 would be nearly 100% of his pre-retirement earnings, while a “typical” US citizen who retired in 1975 only would receive 42%, if they had a defined contribution based pension plan. This illustrates the high risk in these kinds of pension plans and why nominal guarantees are of interest. On the other hand, one of the arguments that has been set in favour for the contribution based pension plans, contra defined benefit based pension plans, is that one should expect a higher rate of return when the premiums are invested in the financial market, leading to a higher expected pension.

The chapter is organised as follows: In section 5.2 we give a description of our economic model and general set-up. In section 5.3 we analyse different pension plans. Defined contribution based pension plans are analysed in subsection 5.3.1 and defined benefit based pension plans are analysed in subsection 5.3.2. Some concluding remarks are given in section 5.4.

5.2 The Economic Model and Preliminaries

5.2.1 Financial Factors

We use the model of Black and Scholes (1973), in the following referred to as a *Black and Scholes economy*. This is a rather simplified, though widely accepted, model of the financial market. We assume there exists a pension fund that has a given investment policy. This policy generates some random return δ_t^T over the time period from time t to T . We further assume that there exists an equivalent martingale measure Q , under which the return is given by the following equation

$$\delta_t^T = \left(r - \frac{1}{2}\sigma_S^2\right)(T - t) + \sigma_S(W_T - W_t),$$

where r is the risk-free interest rate and is assumed constant, σ_S is the instantaneous standard deviation of the return on the pension fund and is also a constant, and W_T is a standard Brownian motion with $W_t = 0$. Sometimes δ will be referred to as the return on the pension fund.

The customers of the pension fund have accounts that are referred to as *pension accounts*. In the rest of this chapter we let the term *level of participation* denote what fraction of the return on the pension *fund* that is the underlying return on the pension *account*. The level of participation is throughout denoted by the parameter γ .

The time t arbitrage price of an investment maturing at time $T > t$ of one unit of account and that accrues the same rate of return as the underlying

return on the pension account, is given by¹

$$\begin{aligned} V_t(e^{\gamma\delta_t^T}) &= E_Q \left[e^{-r(T-t)} e^{\gamma \left((r - \frac{1}{2}\sigma_S^2)(T-t) + \sigma_S(W_T - W_t) \right)} \middle| \mathcal{F}_t \right] \\ &= e^{(\gamma-1)(r + \frac{1}{2}\gamma\sigma_S^2)(T-t)}. \end{aligned} \quad (5.1)$$

More generally, the time t market value of some cash flow X_T at time T is given by

$$V_t(X) = E_Q \left[e^{-r(T-t)} X_T \middle| \mathcal{F}_t \right]. \quad (5.2)$$

5.2.2 Mortality Factors

The remaining lifetime of a person aged x is given by the stochastic variable T_x . The probability for a person aged x to survive t more years we denote by $\hat{P}(T_x > t) = {}_t p_x$. It is assumed that mortality risk is given on another probability space than the uncertainty in the financial market; hence, mortality risk and financial risk are independent by construction.

Given the independence between financial and mortality risk and an assumption about risk neutrality with respect to mortality risk, it can be shown that the market values (as defined on page 96) of the claims in (5.1) and (5.2), if the payment to the investor only will be made if he is alive at time T , are given by

$$\bar{V}_t(e^{\gamma\delta_t^T}) = V_t(e^{\gamma\delta_t^T})_T p_x \quad (5.3)$$

and

$$\bar{V}_t(X) = V_t(X)_T p_x, \quad (5.4)$$

respectively.

5.3 Pension Plans

Let p_i , $i \in \{1, 2, \dots, I\}$, be the size of the premium payment at time t_i if the employee is still alive. Further, let $\hat{p}_i = p_i 1_{\{T_x > t_i\}}$ where $1_{\{T_x > t_i\}}$ is an indicator function returning the value 1 if the employee is alive at time t_i and 0 otherwise. In the same way, let a_j , $j \in \{1, 2, \dots, J\}$, be the size of the pension payment at time T_j and $\hat{a}_j = a_j 1_{\{T_x > T_j\}}$. We define a pension plan as a sequence of payments

$$P = \{-\hat{p}_1, -\hat{p}_2, \dots, -\hat{p}_I, \hat{a}_1, \hat{a}_2, \dots, \hat{a}_J\},$$

¹We let $E_Q[\cdot]$ denote the expectation under the equivalent martingale measure Q and $E_Q[\cdot | \mathcal{F}_t]$ the expectation conditional on \mathcal{F}_t , the information available at time t .

thus, the premiums are paid to obtain pension payments at some later points in time. Further, no premium or pension payments are made after the employee's death. For a lifelong pension we can think of T_J as being infinite.

The terms of the pension plan determine the relationship between the premium payments and the pensions. However, since we are here concerned with how the employee can value the pension plan and the premiums are assumed paid by the employer, only the pension payments are of interest, thus, specifying the premiums will not be necessary. Because of mortality risk, time-lag between the inception of the pension plan and the premium payments, level of participation $\gamma \neq 1$, and rate of return guarantees that may possibly be embedded, the premiums will typically not coincide with the contributions made to the pension account.

For defined contribution based pension plans the *contributions* to the employee's pension account are typically a fraction of his salary², while for defined benefit based pension plans the *pension payments* are typically a fraction of the salary. In real-life situations the salary may fluctuate, but we will implicitly assume that it follows a deterministic function. Since the uncertainty in the financial market is likely to be higher than what is the case for the salary, we believe that the mistake done by this assumption is relatively small.³

5.3.1 Defined Contribution Based Pension Plans

We assume that the employee has a pension account with the pension fund, and at each time of premium payment the account is debited (contributions are being made to the account), and when pension payments are being made, the account is credited. The account will also be debited with the rate of return that the terms of the pension plan prescribe. If the account has any positive balance at the time of death, the balance will be distributed to the pension fund (negative balance will not occur in our pension plans). If the employee's death occurs before time T (the time of retirement), no more premium payments will be made.

A crucial assumption for the analysis performed in this subsection is that any return on the pension fund that exceeds the return debited on the pension account is not in any way distributed back to the employee. How this excess return is distributed between the pension fund and the employer is not of any importance here. However, for the valuation of the pension plan as seen from the stand point of the pension fund and the employer, this is highly important, see e.g., Grosen and Jørgensen (2000), Miltersen and Persson (2002), and Hansen and Miltersen (2002).

²In practice, the contributions may be a more complicated function of the salary.

³Pennacchi (1999) assumed that the salary follows a stochastic process with the same source of uncertainty as a (hypothetical) asset and used this asset in the determination of the market value of the contract.

Maturity Guarantee Since many pension plans have some sort of minimum rate of return guarantee embedded, we will in this and the next paragraph give a short analysis of two of these guarantees.

Let g be the minimum guaranteed rate of return per unit of time (i.e., per year). If the amount $\bar{\lambda}$ is debited on the pension account at time t_0 and with a maturity guarantee embedded and a level of participation γ , the time T payoff is given by $\bar{\lambda} \max(e^{\gamma \delta t_0}, e^{g(T-t_0)})$. If no minimum guaranteed rate of return is included, we can formally set $g = -\infty$, and the payoff then becomes $\bar{\lambda} e^{\gamma \delta T}$. We denote the time t_0 market value of this contract $G(\bar{\lambda}, T - t_0, \bar{\lambda} e^{g(T-t_0)}; \gamma)$ and is given by (see e.g., Tiong (2000))

$$G(\bar{\lambda}, T - t_0, \bar{\lambda} e^{g(T-t_0)}; \gamma) = \bar{\lambda} e^{(\gamma-1)(r+\frac{1}{2}\gamma\sigma_S^2)(T-t_0)} \Phi(d_1) + \bar{\lambda} e^{(g-r)(T-t_0)} \Phi(d_2), \quad (5.5)$$

where

$$d_1 = \frac{\left(-g + \gamma(\gamma-1)(r + (\gamma - \frac{1}{2})\sigma_S^2)\right)(T-t_0)}{\gamma\sigma_S\sqrt{T-t_0}},$$

$$d_2 = \frac{\left(g - \gamma(r - \frac{1}{2}\sigma_S^2)\right)(T-t_0)}{\gamma\sigma_S\sqrt{T-t_0}},$$

and $\Phi(\cdot)$ is the cumulative normal probability distribution.

Definition 5.1. For some function $f(x_1, x_2, \dots, x_n)$, we say that f is homogeneous of degree k if

$$f(tx_1, tx_2, \dots, tx_n) = t^k f(x_1, x_2, \dots, x_n), \quad t > 0.$$

Lemma 5.1. The market value of a maturity guarantee in a Black and Scholes economy is homogeneous of degree one in the number of units it is written on.

Proof. This follows trivially from (5.5) since both d_1 and d_2 are independent of $\bar{\lambda}$. \square

Although the above is trivial and can be thought upon as “price = unit price \times quantity”, it is an important observation in the case where for instance $\bar{\lambda}$ is equal to the time t_0 market value of the underlying asset, something that often is the case for forward-start options and guarantees.

Lemma 5.2. The market value at time $0 \leq t_0$ of the guarantee with time t_0 market value $G(\bar{\lambda}, T - t_0, \bar{\lambda} e^{g(T-t_0)}; \gamma)$ is given by $G(\lambda, T - t_0, \lambda e^{g(T-t_0)}; \gamma)$, where $\lambda = e^{-rt_0} \bar{\lambda}$.

Proof. From Lemma 5.1 we have that $G(\bar{\lambda}, T-t_0, \bar{\lambda}e^{g(T-t_0)}; \gamma)$ can be written as $\bar{\lambda}G(1, T-t_0, e^{g(T-t_0)}; \gamma)$. The time 0 market value is then given by⁴

$$\begin{aligned} E_Q \left[e^{-rt_0} \bar{\lambda} G(1, T-t_0, e^{g(T-t_0)}; \gamma) \right] &= \lambda G(1, T-t_0, e^{g(T-t_0)}; \gamma) \\ &= G(\lambda, T-t_0, \lambda e^{g(T-t_0)}; \gamma). \end{aligned}$$

□

Multi-period guarantee Let us now consider a multi-period (rate of return) guarantee, or annual guarantee. Again, the amount $\bar{\lambda}$ is debited on the pension account at time t_0 , but now with an annual guarantee embedded. The investment lasts for N years (let t_n be the beginning of the $(n+1)$ 'st year and $t_N = T$). The final payoff is given by

$$\bar{\lambda} \prod_{n=0}^{N-1} \max \left(e^{\gamma \delta_{t_n}^{t_{n+1}}}, e^g \right).$$

We denote the time t_0 market value of this contract by $MG(\bar{\lambda}, T-t_0, \bar{\lambda}e^g; \gamma)$. It is easily seen that the market value at time t_0 is given by (see e.g., Tiong (2000))

$$MG(\bar{\lambda}, T-t_0, \bar{\lambda}e^g) = \bar{\lambda} \prod_{n=0}^{N-1} G(1, t_{n+1} - t_n, e^g; \gamma). \quad (5.6)$$

Lemma 5.3. *The time $0 \leq t_0$ market value of the contract with time t_0 market value $MG(\bar{\lambda}, T-t_0, \bar{\lambda}e^g; \gamma)$ is given by $MG(\lambda, T-t_0, e^g \lambda; \gamma)$.*

Proof. This follows trivially since (5.6) is homogenous of degree one in $\bar{\lambda}$. □

The contracts in Lemma 5.2 and Lemma 5.3 give us the market value of some contract with a rate of return guarantee included and that starts to run at some future point in time. We will call these contracts forward-start guarantee and annual forward-start guarantee, respectively. The result in Lemma 5.2 with $\gamma = 1$ can be found in, e.g., Pennacchi (1999), while for $\gamma \neq 1$ and the annual forward-start guarantee in Lemma 5.3 seems to be new results.

⁴Notice that for $t_0 = 0$ we have a standard maturity guarantee as analysed by Tiong (2000) and if we also have that $\gamma = 1$ we have the same guarantee as in Brennan and Schwartz (1976). The result also holds in the case where $\bar{\lambda}$ is a linear function of the market value of the pension fund.

Annuity Contract with No Guarantee The first pension plan we analyse is one with no rate of return guarantee included. At each time t_i , $i \in \{1, 2, \dots, I\}$, of premium payment, an amount $\bar{\lambda}_i$ is debited on the employee's pension account. The balance on the account earns the fraction γ of the return on the pension fund. At the time the employee retires (at time $T \geq t_I$), the balance on the account is converted to an annuity. We assume that the annuity only lasts as long as the employee is alive, or possibly only to some final time horizon $T_J \geq T$. For a lifelong pension, we can think of T_J as being infinite. For a finite T_J , the employee can outlive the pension plan and he may therefore face financial problems if he is still alive after time T_J . Since there is no minimum guaranteed rate of return on the pension account, he also faces the risk of a low rate of return on the pension fund. In particular, his pension is very much exposed to the cumulative return on the pension fund at the time of retirement when the balance on his pension account is converted to an annuity. We let $\lambda_i = e^{-rt_i} \bar{\lambda}_i$.

Proposition 5.1. *If at each time of premium payment, an amount $\bar{\lambda}_i$ is debited on the employee's pension account, the time 0 market value of the pension payments for a man aged x at time 0 is given by*

$$\pi_0 = \sum_{i=1}^I \lambda_i e^{(\gamma-1)(r+\frac{1}{2}\gamma\sigma_S^2)(T-t_i)} {}_T p_x,$$

with an annual pension (the first at time T_1)

$$a = \frac{\sum_{i=1}^I \bar{\lambda}_i e^{\gamma\delta_{t_i}^T}}{\sum_{j=1}^J e^{-r(T_j-T)} ({}_{T_j-T} p_{(x+T)})}.$$

Proof. Since the balance on the employee's pension account accrues a fraction γ of the return on the pension fund, the time T market value of the account is given by $\sum_{i=1}^I \bar{\lambda}_i e^{\gamma\delta_{t_i}^T}$. The employee will only be able to convert the balance on the pension account to an annuity (that at time T has the same market value as the pension account) if he is still alive at time T . Using the results in (5.1) and (5.3) and the definition of λ_i , it follows that the time 0 market value of the pension plan is given by

$$\begin{aligned} \pi_0 &= E_Q \left[e^{-rT} \sum_{i=1}^I \bar{\lambda}_i e^{\gamma\delta_{t_i}^T} \right] {}_T p_x \\ &= \sum_{i=1}^I \lambda_i e^{(\gamma-1)(r+\frac{1}{2}\gamma\sigma_S^2)(T-t_i)} {}_T p_x. \end{aligned}$$

In the spirit of the principle of equivalence⁵, we require that the time T

⁵The principle of equivalence roughly states that the insurance premium should be the present value of the expected cash flow from the insurance contract (see e.g., Persson and Aase (1994)).

market value of the pension payments (i.e., the annuity) must equal the market value of the pension account at time T , i.e., (notice that the man is now aged $x + T$)

$$\sum_{j=1}^J a e^{-r(T_j-T)} {}_{(T_j-T)}p_{(x+T)} = \sum_{i=1}^I \bar{\lambda}_i e^{\gamma \delta_i^T}.$$

It then follows that

$$a = \frac{\sum_{i=1}^I \bar{\lambda}_i e^{\gamma \delta_i^T}}{\sum_{j=1}^J e^{-r(T_j-T)} {}_{(T_j-T)}p_{(x+T)}}.$$

□

Example: We assume that one company hires a man aged 66 and who will retire, if still alive, at age 70. If we let today be time 0, he will retire at time 4. The company is to pay three premiums for the man; at age 67, 68, and 69. He will receive his pensions at the age of 71 and 72, if still alive. No payments are to be made to his heirs. The pension fund is assumed to develop as follows: $\delta_1^2 = 0.25$, $\delta_2^3 = -0.10$, and $\delta_3^4 = 0.06$. With a level of participation $\gamma = 0.75$, this yields a return on the pension account of 18.75%, -7.50%, and 4.50%. We let $r = 0.08$ and $\bar{\lambda}_i = 100$ for all $i \in \{1, 2, 3\}$. Mortality risk is taken into account by using mortality table N1963 (see e.g., Aase (1996)).

The time 0 market value of the pension payments is equal to

$$\begin{aligned} \pi_0 &= \sum_{i=1}^3 e^{-0.08 \cdot i} 100 \cdot e^{(0.75-1) \cdot (0.08 + \frac{1}{2} \cdot 0.75 \cdot 0.2^2) \cdot (4-i)} \cdot {}_4p_{66} \\ &= \underline{214.68}, \end{aligned}$$

where ${}_4p_{66} = 0.8775$. This plan yields an annual pension of

$$\begin{aligned} a &= \frac{100 \cdot e^{0.1875-0.075+0.045} + 100 \cdot e^{-0.075+0.045} + 100 \cdot e^{0.045}}{e^{-0.08} \cdot {}_1p_{70} + e^{-0.08 \cdot 2} \cdot {}_2p_{70}} \\ &= \underline{191.12}, \end{aligned}$$

where ${}_1p_{70} = 0.9597$ and ${}_2p_{70} = 0.9173$.

Annuity Contract with Guarantee Assume that we have the same kind of pension plan as the one above, but now with a minimum rate of return guarantee included. The market value of the pension plan varies with what kind of guarantee that is included. A contract with a maturity guarantee is typically less expensive than a contract with an annual guarantee. With a maturity guarantee included, the employee's pension is not

so sensitive to the accumulated return on the pension fund at the time of retirement as the contract with no guarantee. If the accumulated return on the pension fund is “low” when the employee retires, the guarantee will become binding and the pension payments will be higher than would have been the case if no guarantee was included. The maturity guarantee secures that the average rate of return on the pension plan do not become “low”. However, it is difficult for the employee to make a relatively precise estimate of the size of the pension payments prior to the time of retirement since the accumulated return on the pension fund can fluctuate much. The annual guarantee on the other hand “locks in” the annual return, securing that the average return within each single year is not too low. This guarantee is sometimes given the descriptive name “cliquet”, which is French for rocket. With this guarantee the employee can at the end of each year monitor the realised return on his pension account, reducing the uncertainty in his estimate of the future pension payments. This is probably the reason why the annual guarantee is used in many countries.

The disadvantage of eliminating the probability of a low rate of return on the pension account is that it comes at a cost. In particular the annual guarantee can be quite expensive, and if the employer offers a pension plan where each employee is given some amount to spend on premium payments, the idea of having a rate of return guarantee may not be that appealing. Since the guarantee element comes at a cost, a guarantee will necessarily reduce the amount of the premium that can be debited on the pension account. Even in the case of a low rate of return on the pension fund, one may get a higher pension by choosing the contract with no rate of return guarantee since it allows one to debit a greater amount on the pension account.⁶

Proposition 5.2. *If at each time of premium payment, an amount $\bar{\lambda}_i$ is debited on the pension account and there is a maturity guarantee embedded, then the time 0 market value of the pension payments for a man aged x at time 0 is given by*

$$\pi_0^1 = \sum_{i=1}^I G(\lambda_i, T - t_i, \lambda_i e^{g(T-t_i)}; \gamma) T p_x,$$

⁶Assume that there is only one premium and that it is paid at time 0, and that there is just one pension payment that is made at time T . It is easily seen that the following inequality has to be satisfied for the contract with a maturity guarantee to give a higher pension than the contract with no guarantee

$$\delta_0^T < \frac{gT - \ln G(1, T, e^{gT}; \gamma)}{\gamma}.$$

with an annual pension

$$a^1 = \frac{\sum_{i=1}^I \bar{\lambda}_i \max(e^{\gamma \delta_i^T}, e^{g(T-t_i)})}{\sum_{j=1}^J e^{-r(T_j-T)} (T_j-T) p_{(x+T)}}.$$

Proof. By using the forward-start guarantee in Lemma 5.2 the result follows in the exact same manner as in the proof of Proposition 5.1. \square

Proposition 5.3. *If at each time of premium payment, an amount $\bar{\lambda}_i$ is debited on the pension account and there is an annual guarantee embedded, then the time 0 market value of the pension payments for a man aged x at time 0 is given by*

$$\pi_0^2 = \sum_{i=1}^I MG(\lambda_i, T - t_i, \lambda_i e^g; \gamma)_T p_x,$$

with an annual pension

$$a^2 = \frac{\sum_{i=1}^I \bar{\lambda}_i \prod_{n=i}^{N-1} \max(e^{\gamma \delta_{i_n}^{t_{n+1}}}, e^g)}{\sum_{j=1}^J e^{-r(T_j-T)} (T_j-T) p_{(x+T)}}.$$

Proof. By using the forward-start guarantee in Lemma 5.3 the result follows in the exact same manner as in the proof of Proposition 5.1. \square

Example: Let us now see what would have happened in the above example if the pension plan had a rate of return guarantee included. We assume that the minimum guaranteed rate of return is given by $g \cdot (4 - i)$, $i \in \{1, 2, 3\}$, where $g = 0.04$.

$$\pi_0^1 = \sum_{i=1}^3 G(100 \cdot e^{-0.08 \cdot i}, 4 - i, e^{-0.08 \cdot i} \cdot 100 \cdot e^{0.04 \cdot (4-i)}; 0.75) \cdot 4p_{66} = \underline{228.42},$$

where we have used that

$$\begin{aligned} G(100 \cdot e^{-0.08}, 4 - 1, e^{-0.08} \cdot 100 \cdot e^{0.04 \cdot (4-1)}; 0.75) \cdot 4p_{66} &= 81.38, \\ G(100 \cdot e^{-0.08 \cdot 2}, 4 - 2, e^{-0.08 \cdot 2} \cdot 100 \cdot e^{0.04 \cdot (4-2)}; 0.75) \cdot 4p_{66} &= 76.16, \end{aligned}$$

and

$$G(100 \cdot e^{-0.08 \cdot 3}, 4 - 3, e^{-0.08 \cdot 3} \cdot 100 \cdot e^{0.04 \cdot (4-3)}; 0.75) \cdot 4p_{66} = 70.88.$$

In table 5.1 we have computed the gross return at the time of retirement on the pension account, both with and without a maturity guarantee. As we can see, it is only for the second premium that the guarantee becomes binding.

Table 5.1: The gross return on the pension account at the time of retirement. Maturity guarantee. The "values" are at time T .

	Value of one unit deposited on the pension account	Value of one unit with the minimum guaranteed return	Value of one unit invested in contract with guarantee
First premium	$e^{0.1875-0.075+0.045} = 1.17058$	$e^{0.04 \cdot 3} = 1.12750$	1.17058
Second premium	$e^{-0.075+0.045} = 0.97045$	$e^{0.04 \cdot 2} = 1.08329$	1.08329
Third premium	$e^{0.045} = 1.04603$	$e^{0.04} = 1.04081$	1.04603

Table 5.2: The gross return on the pension account at the time of retirement. Annual guarantee.

	Value at time of retirement of an investment of one unit in the pension contract with an annual rate of return guarantee included
First premium	$\max(e^{0.1875}, e^{0.04}) \cdot \max(e^{-0.075}, e^{0.04}) \cdot \max(e^{0.045}, e^{0.04})$ $= e^{0.1875+0.04+0.045} = 1.31324$
Second premium	$\max(e^{-0.075}, e^{0.04}) \cdot \max(e^{0.045}, e^{0.04}) = e^{0.04+0.045} = 1.08872$
Third premium	$\max(e^{0.045}, e^{0.04}) = e^{0.045} = 1.04603$

Using the numbers in Table 5.1, we find that this plan gives an annual pension of

$$\begin{aligned}
 a^1 &= \frac{100 \cdot e^{0.1875-0.075+0.045} + 100 \cdot e^{0.04 \cdot 2} + 100 \cdot e^{0.045}}{e^{-0.08} \cdot {}_1p_{70} + e^{-0.08 \cdot 2} \cdot {}_2p_{70}} \\
 &= \underline{197.89}.
 \end{aligned}$$

The difference in the annual pension of 6.77 for this plan and the one with no guarantee is fully due to the fact that the guarantee is binding for the second premium, yielding a higher rate of return. I.e.,

$$\frac{100 \cdot (e^{0.04 \cdot 2} - e^{-0.075+0.045})}{e^{-0.08} \cdot {}_1p_{70} + e^{-0.08 \cdot 2} \cdot {}_2p_{70}} = 6.77.$$

If the pension plan had an annual guarantee included, the time 0 market value of the pension payments would be

$$\pi_0^2 = \sum_{i=1}^3 MG(100 \cdot e^{-0.08 \cdot i}, 4 - i, e^{0.04} \cdot 100 \cdot e^{-0.08 \cdot i}; 0.75) \cdot {}_4p_{66} = \underline{237.45}.$$

Using the numbers in Table 5.2, we find that this plan gives an annual pension of

$$a^2 = \frac{100 \cdot e^{0.1875+0.04+0.045} + 100 \cdot e^{0.04+0.045} + 100 \cdot e^{0.045}}{e^{-0.08} \cdot {}_1p_{70} + e^{-0.08 \cdot 2} \cdot {}_2p_{70}} = \underline{206.77}.$$

The increase in the annual pension is now due to the increased return on the first and the second premium.

Alternative Contract with Guarantee In the plans analysed above the employee does not enjoy any “high” return on the pension fund after he has retired. Assume instead a pension plan where the balance on the employee’s pension account accrues a fraction γ of the return on the pension fund until the time of pension payment. There are many ways in which such a plan can be constructed, but we will here only consider one of the possibilities.

We let each premium be divided into J parts, one for each pension payment.⁷ We further equip each part of the premium that is debited on the employee’s pension account with a rate of return guarantee maturing at the time of pension payment, i.e., time T_j , $j \in \{1, 2, \dots, J\}$.

The advantage of this plan, compared to the one where the balance on the pension account is converted to an annuity at the time of retirement, is that the employee now will benefit from any high return on the pension fund also when he is retired. The pension payments are no longer so sensitive to the accumulated return on the pension fund at time T when he retires. This gives a better diversification over time. In addition, he also has a floor for how low the pension can get, though the floor may be lower than the pension the annuity would have given.

We assume that an amount $\bar{\lambda}_i = \sum_{j=1}^J \bar{\lambda}_{i,j}$, $i \in \{1, 2, \dots, I\}$, $j \in \{1, 2, \dots, J\}$, is debited on the pension account at each time t_i . It is further assumed that each $\bar{\lambda}_{i,j}$ is embedded with a guarantee maturing at time T_j . Let $\lambda_{i,j} = e^{-rt_i} \bar{\lambda}_{i,j}$.

Proposition 5.4. *If at each time of premium payment, the J amounts $\bar{\lambda}_{i,j}$, $i \in \{1, 2, \dots, I\}$, $j \in \{1, 2, \dots, J\}$, are debited on the employee’s pension account and there are maturity guarantees embedded, each maturing at each time T_j , then the time 0 market value of the pension payments for a man aged x at time 0 is given by*

$$\pi_0^3 = \sum_{i=1}^I \sum_{j=1}^J G(\lambda_{i,j}, T_j - t_i, \lambda_{i,j} e^{g(T_j - t_i)}; \gamma)_{T_j} p_x,$$

with a pension at time T_j

$$a_j^3 = \sum_{i=1}^I \bar{\lambda}_{i,j} \max \left(e^{\gamma \delta_{i,j}^{T_j}}, e^{g(T_j - t_i)} \right).$$

Proof. By using the forward-start guarantee in Lemma 5.2 the result follows in the exact same manner as in the proof of Proposition 5.1. \square

⁷To avoid too much uncertainty in the return after the employee has retired, we will only consider plans with rate of return guarantees included.

Proposition 5.5. *If at each time of premium payment, the J amounts $\bar{\lambda}_{i,j}$, $i \in \{1, 2, \dots, I\}$, $j \in \{1, 2, \dots, J\}$, are debited on the employee's pension account and there are annual guarantees embedded, each maturing at each time T_j , then the time 0 market value of the pension payments for a man aged x at time 0 is given by*

$$\pi_0^4 = \sum_{i=1}^I \sum_{j=1}^J MG(\lambda_{i,j}, T_j - t_i, \lambda_{i,j} e^g; \gamma)_{T_j} p_x,$$

with a pension at time T_j

$$a_j^4 = \sum_{i=1}^I \bar{\lambda}_{i,j} \prod_{n=i}^{N_j-1} \max(e^{\gamma \delta_{i_n}^{t_n+1}}, e^g),$$

where $t_{N_j} = T_j > t_I$.

Proof. By using the forward-start guarantee in Lemma 5.3 the result follows in the exact same manner as in the proof of Proposition 5.1. \square

Example: Assume, in addition to the assumptions in the previous examples, that $\delta_4^5 = 0.30$ and $\delta_5^6 = -0.15$, i.e., if no guarantee was included, the return on the pension account would have been 22.5% and -11.25%. We let $\bar{\lambda}_{i,5} = \bar{\lambda}_{i,6} = 50$, $i \in \{1, 2, 3\}$.

The time 0 market value of the pension payments with a maturity guarantee included is equal to

$$\begin{aligned} \pi_0^3 &= \sum_{i=1}^3 \sum_{j=1}^2 G(e^{-0.08 \cdot i} \cdot 50, 4 + j - i, e^{-0.08 \cdot i} \cdot 50 \cdot e^{0.04 \cdot (4+j-i)}; 0.75)_{4+j} p_{66} \\ &= \underline{209.93}, \end{aligned}$$

where ${}_5p_{66} = 0.8421$ and ${}_6p_{66} = 0.8049$.

The pension at age 71 and 72 are equal to

$$\begin{aligned} a_{71}^3 &= \max\left(50 \cdot e^{0.1875 - 0.075 + 0.045 + 0.225}, 50 \cdot e^{0.04 \cdot (4+1-1)}\right) \\ &\quad + \max\left(50 \cdot e^{-0.075 + 0.045 + 0.225}, 50 \cdot e^{0.04 \cdot (4+1-2)}\right) \\ &\quad + \max\left(50 \cdot e^{0.045 + 0.225}, 50 \cdot e^{0.04 \cdot (4+1-3)}\right) \\ &= \underline{199.56} \end{aligned}$$

and

$$\begin{aligned}
a_{72}^3 &= \max \left(50 \cdot e^{0.1875-0.075+0.045+0.225-0.1125}, 50 \cdot e^{0.04 \cdot (4+2-1)} \right) \\
&\quad + \max \left(50 \cdot e^{-0.075+0.045+0.225-0.1125}, 50 \cdot e^{0.04 \cdot (4+2-2)} \right) \\
&\quad + \max \left(50 \cdot e^{0.045+0.225-0.1125}, 50 \cdot e^{0.04 \cdot (4+2-3)} \right) \\
&= \underline{182.70},
\end{aligned}$$

respectively. With an annual minimum rate of return guarantee the corresponding numbers are $\pi_0^4 = 231.85$, $a_{71}^4 = 215.90$, and $a_{72}^4 = 224.71$.

Comparison of the Contracts Finally, we end the analysis of defined contribution based pension plans by showing the probability density functions for the pensions received in the different plans. This is done to give a feeling of the risk in each of the pension plans we have considered above, and it also makes it easier to compare the risk in the different plans.

We assume that there are seven premium payments, the first at time 0 and the last at time 6. The employee retires at time 7. The pensions are paid at time 8, 9, and 10. To make a fair comparison between the different plans, the employer will at each time of premium payment pay one unit of account in premium. This premium shall both cover any guarantees included in the plans and the amount to be debited on the employee's pension account.

It is assumed that the fund has a yearly drift rate $\mu = 0.12$, $\sigma_S = 0.20$, and $g = 0.04$. Mortality risk is not included in these calculations and $\gamma = 1$. The density for the pensions at time 8, 9, and 10 are given in Figure 5.1, 5.2, and 5.3, respectively.

The pension plans with the annuity and a maturity guarantee and an annual guarantee embedded are denoted #1 and #2, respectively. The second kind of pension plans with the corresponding guarantees are denoted #3 and #4.

From figure 5.1 - 5.3 we can see, even when rate of return guarantees are included, that the pension is fairly risky in a defined contribution based pension plan, supporting the findings in Burtless (2000).

It should be mentioned that for these pension plans to have a zero time 0 market value, the premiums the employer pays must be e^{r^i} , $i \in \{0, 1, \dots, 6\}$. The difference $e^{r^i} - 1$ can be thought upon as a compensation to the pension fund for "delaying" the premium payments from time 0 to i , though this is not important for the example.

5.3.2 Defined Benefit Based Pensions Plans

For the defined contribution based pension plans we saw that the employee's pension account accrued a fraction γ of the stochastic return on the pension fund, possibly with a minimum rate of return guarantee embedded. This

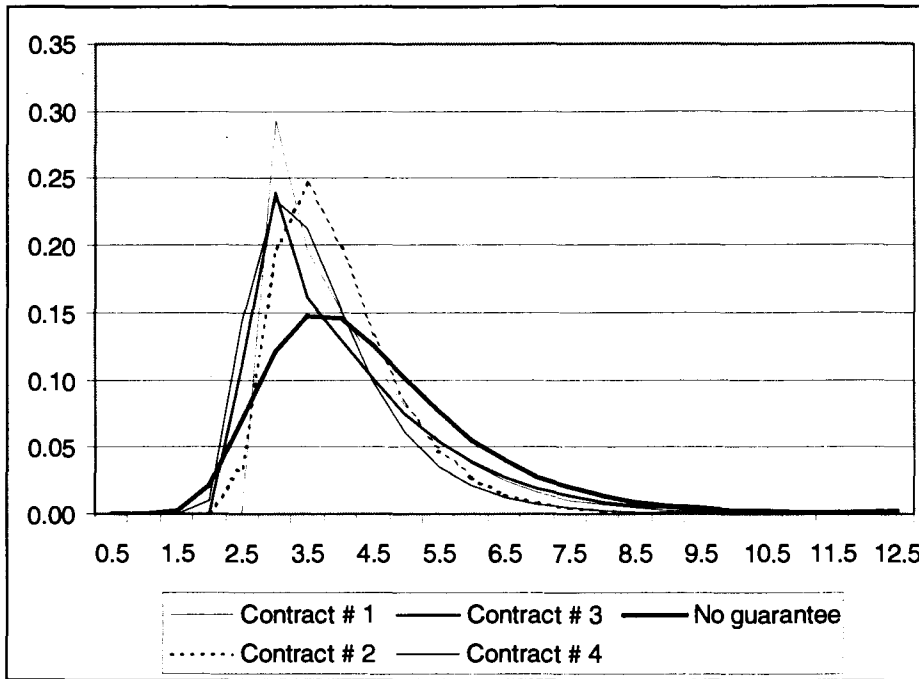


Figure 5.1: Probability density functions for the pensions in the first year. #1 and #2 are contracts with annuity and maturity guarantee and annual guarantee embedded, respectively. #3 and #4 are the second kind of contract with a maturity guarantee and an annual guarantee embedded, respectively. Premiums of one unit of account are paid at time $0, 1, \dots, 6$, with pension payments at time 8, 9, and 10. The employee retires at time 7. Mortality risk is not included. $\mu = 0.12$, $\sigma_S = 0.20$, and $g = 0.04$.

gave some, randomly sized, pension payments when the employee had retired. The size of the contributions to be made by the employer was fixed in the plan. The defined benefit based pension plans are somewhat opposite of the contribution based pension plans. Here it is the size of the pension payments in the plan that is fixed, and it is therefore the premiums that will fluctuate in accordance with the financial market.

Historically, the defined benefit based pension plans have in some countries, e.g., in Norway, been the only available corporate pension plan.

To model the financial uncertainty in a defined benefit based pension plan, we will now assume that there exists a bond market, and the risk-free interest rate will no longer be assumed constant. More precisely, we assume that the instantaneous forward rate at time s , as seen from time $t \leq s$, is given by, under the equivalent martingale measure Q ,

$$f(t, s) = f(0, s) + \int_0^t \sigma_f(v, s) \int_v^s \sigma_f(v, u) du dv + \int_0^t \sigma(v, s) dW_v,$$

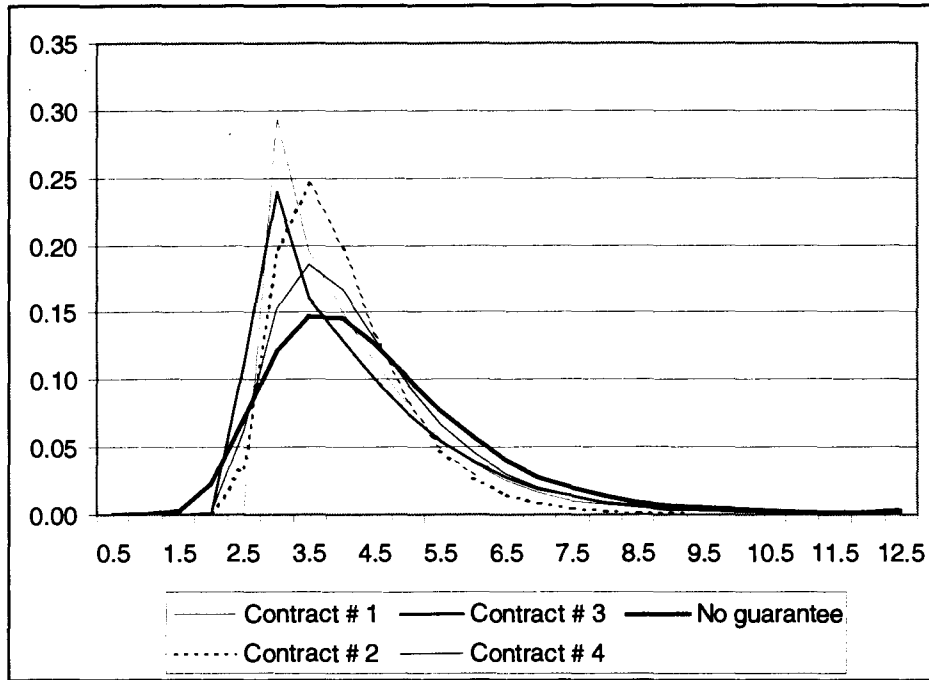


Figure 5.2: Probability density functions for the pensions in the second year. #1 and #2 are contracts with annuity and maturity guarantee and annual guarantee embedded, respectively. #3 and #4 are the second kind of contract with a maturity guarantee and an annual guarantee embedded, respectively. Premiums of one unit of account are paid at time 0, 1, ..., 6, with pension payments at time 8, 9, and 10. The employee retires at time 7. Mortality risk is not included. $\mu = 0.12$, $\sigma_S = 0.20$, and $g = 0.04$.

where $\sigma_f(t, s)$ is a volatility function. The short-term interest rate $r_t = f(t, t)$. This specification of the interest rates is due to Heath et al. (1992).

The analysis performed in this section is highly idealised and its main purpose is only to shed some light into the underlying structure of these kinds of pension plans and to make it easier to see the differences between defined contribution and defined benefit based pension plans.

A Simple Defined Benefit Based Pension Plan Consider a pension plan where the employee receives a pension payment a_j^5 at each time T_j , $j \in \{1, 2, \dots, J\}$, if still alive. This would be a defined benefit of a_j^5 , and would coincide with what pensions the employee typically could get in a defined benefit based pension plan.

Let

$$\alpha_j = a_j^5 e^{-\int_0^{T_j} f(0,v)dv} = a_j^5 P(0, T_j),$$

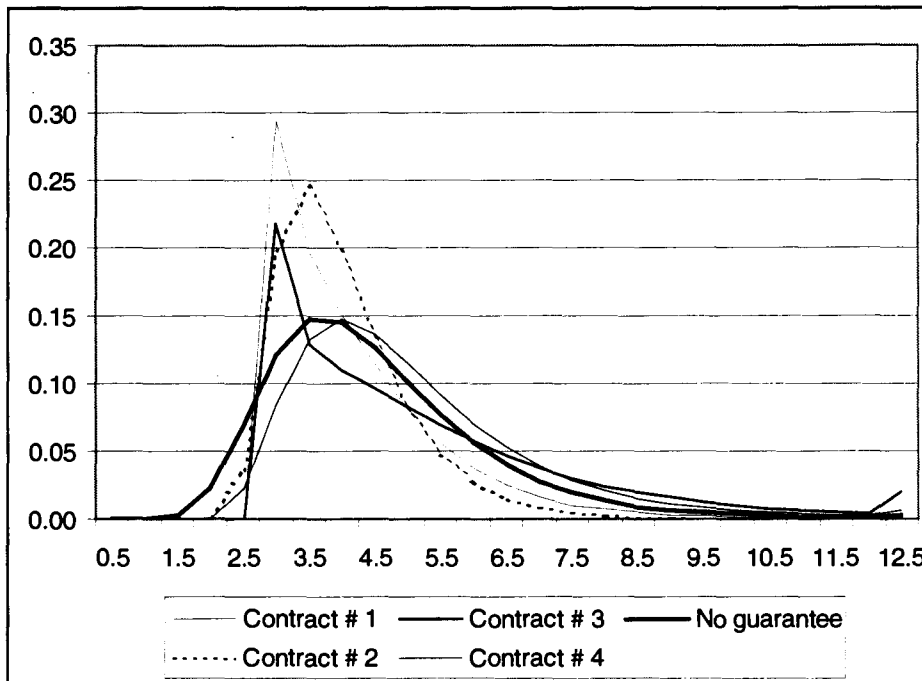


Figure 5.3: Probability density functions for the pensions in the third year. #1 and #2 are contracts with annuity and maturity guarantee and annual guarantee embedded, respectively. #3 and #4 are the second kind of contract with a maturity guarantee and an annual guarantee embedded, respectively. Premiums of one unit of account are paid at time $0, 1, \dots, 6$, with pension payments at time 8, 9, and 10. The employee retires at time 7. Mortality risk is not included. $\mu = 0.12$, $\sigma_S = 0.20$, and $g = 0.04$.

i.e., for $a_j^5 = 1$, α_j is the time 0 market value of a zero-coupon bond maturing at time T_j , $P(0, T_j)$.

Proposition 5.6. *The time 0 market value of the pension payments a_j^5 at time T_j , $j \in \{1, 2, \dots, J\}$, in a defined benefit based pension plan for a man aged x at time 0 is given by*

$$\pi_0^5 = \sum_{j=1}^J \alpha_{jT_j} p_x.$$

Proof. The time 0 market value can be expressed as

$$\begin{aligned}\pi_0^5 &= \sum_{j=1}^J \alpha_j^5 e^{-\int_0^{T_j} f(0,v)dv} T_j p_x \\ &= \sum_{j=1}^J \alpha_j T_j p_x,\end{aligned}$$

and follows from (5.4) and the definition of α_j . \square

Example: Assume that the same employee as in the previous examples instead can get a defined benefit based pension plan. Let the initial term structure of interest be flat and equal to 8.00%, i.e., $f(0, s) = 0.08$ for all $s \in [0, 6]$. The time 0 market value of the pension payments is then given by (for an annual benefit of 100)

$$\pi_0^5 = \sum_{j=5}^6 100 \cdot e^{-\int_0^j 0.08dv} {}_j p_{66} = \underline{106.25}.$$

Protection Against High Premiums Especially for contracts that last for a long time, changes in the level of interest rate can expose the employer to a considerably amount of risk. It can therefore be of interest to reduce some of this risk by using the market for financial derivatives. For instance, assume that the employer wants to secure that a future premium payment will not be too high. This can easily be done by buying call options that are written on a money market account (i.e., an asset that accrues the short-term interest rate) and that matures at the time of premium payment.

For simplicity we assume that there is one premium to be paid at time $t \geq 0$ for a pension to be received at time $T_j > t$. Since we require that the time 0 market value of the premiums must equal the time 0 market value of the pension payments, it is easily seen that the premium at time t is given by

$$p_t = e^{\beta t} P(0, T_j) \frac{T_j p_x}{t p_x},$$

where $e^{\beta t} = e^{\int_0^t r_v dv}$. This follows since the premium only will be paid if the employee is still alive at time t .

If the employer wants the premium to stay below some ceiling X , this is equivalent to saying that he wants the premium \hat{p}_t to satisfy

$$\begin{aligned}\hat{p}_t &= \min(p_t, X) = -\max(-p_t, -X) \\ &= p_t - \max(p_t - X, 0) \\ &= e^{\beta t} P(0, T) \frac{T_j p_x}{t p_x} - P(0, T) \frac{T_j p_x}{t p_x} \max(e^{\beta t} - \hat{X}, 0),\end{aligned}$$

where

$$\hat{X} = \frac{{}_t p_x}{T, p_x P(0, T)} X.$$

This means that if the employer wants the premium not to exceed X , he can achieve this by buying $P(0, T) \frac{{}_T p_x}{{}_t p_x}$ units of a call option maturing at time t and that is written on the money market account and with strike price equal to \hat{X} .

The market value of a call option on the money market account can be found in Persson and Aase (1997) and Miltersen and Persson (1999), and for our problem it is given by

$$c_0 = \Phi(d_5) - \hat{X} P(0, t) \Phi(d_6),$$

where

$$d_5 = \frac{-\ln(\hat{X} P(0, t)) + \frac{1}{2} \sigma_{\beta_t}^2}{\sigma_{\beta_t}},$$

$$d_6 = d_5 - \sigma_{\beta_t},$$

and

$$\sigma_{\beta_t}^2 = \int_0^t \left(\int_v^t \sigma_f(v, u) du \right)^2 dv.$$

Example: It is known that for the model of Vasicek (1977) we have the relationship (see e.g., Miltersen and Persson (1999))

$$\sigma_f(v, t) = e^{-\int_v^t \kappa_u du} \sigma_v.$$

We will assume that $\sigma_v = \sigma$ and $\kappa_u = \kappa$ are constants. By straightforward calculations we find that

$$\sigma_{\beta_t}^2 = \frac{\sigma^2}{2\kappa^3} (2\kappa t - 3 + 4e^{-\kappa t} - e^{-2\kappa t}).$$

We use the same term structure as above. In addition we let $\kappa = 0.10$ and $\sigma = 0.03$. Let us consider a premium to be paid at age 69 for a pension to be received at age 71 for an employee aged 66.

If the employer wants to secure that the average annual interest he is paying for delaying the premium payment from time 0 to 3 does not exceed 10%, this means that $X = 0.824$, yielding $\hat{X} = 1.3495$, where we have used that ${}_3 p_{66} = 0.9109$ and ${}_5 p_{66} = 0.8421$. The market value of one call option is equal to 0.011. This means that for $\alpha = 100$, the employer has to pay, at time 0, 0.68 for the protection against a high premium at time 3. This is about 1.20% of the market value of the pension payment.

5.4 Conclusions

We have in this chapter taken the viewpoint of an employee and showed how he can value defined contribution based pension plans. Both contracts with and without guarantees have been constructed and analysed. Forward-start guarantees have been shown to be a well-suited tool for analysing these pension plans when guarantees are embedded. We have also showed that, even when there is a rate of return guarantee included in the pension plan, that the contribution based pension plan imposes a lot of risk on the employee upon retirement. Finally, we have showed how to evaluate a defined benefit based pension plan by using zero-coupon bonds. In contrast to the defined contribution based pension plan that imposes risk on the employee, we have showed that for a defined benefit based pension plan, the financial risk is basically born by the party paying the premiums, i.e., the employer. We also showed how the employer could use the market for financial derivatives to reduce the risk of high premiums.

Chapter 6

Numerical Evaluation of Compound Options

Abstract

In this chapter we study the pricing of compound options within the model proposed by Amin and Jarrow (1992), i.e., an extension of the model of Heath et al. (1992) to also incorporate risky assets such as stocks. There is, to the best of our knowledge, no known closed form solution for the market value of a compound option under stochastic interest rates, so the pricing issue is approached by Monte Carlo simulation. A unified and arbitrage-free approach for simulation within the Heath, Jarrow, and Morton framework is presented. Using variance reduction techniques, we are able to obtain very efficient estimators and we are also, for practical purposes, able to eliminate known problems with discretisation bias. In addition, we also show that so-called exact simulation can be used within a Gaussian Heath, Jarrow, and Morton framework, leading to very efficient and unbiased estimators.

Keywords and phrases: Compound options, Heath, Jarrow, and Morton term structure model of interest rates, Monte Carlo simulation, stratified sampling, importance sampling, the control variate method.

6.1 Introduction

A compound option is an option that has another option as the underlying asset. Geske (1977) and Geske (1979) were the first to analyse compound options and the focus was primarily on applications in corporate finance.

Other interpretations and applications within the same framework have later been presented; see e.g., Carr (1988) who dealt with an exchange option that was written on another exchange option.

We limit our analysis to a call option written on a call option, which again is written on a stock. The major difference between our framework and that of Geske (1979) is that we allow for stochastic interest rates. This is, to the best of our knowledge, a problem that has not previously been given much attention in the literature. However, an attempt to value the claim in closed form solution was given by Geman et al. (1995). The structure of the compound option is such that a closed form solution for the market value is not easily obtainable under any model with stochastic interest rates. We have therefore analysed the claim by the use of numerical methods, i.e., Monte Carlo simulation. For generality we have adopted the model of Heath et al. (1992) and the extensions made by Amin and Jarrow (1992). This is a framework that includes most of the term structure models of interest rates analysed in the literature as special cases.

In general, simulation within the Heath, Jarrow, and Morton framework requires a discretisation of the stochastic differential equation describing the forward rates. To avoid arbitrage opportunities in the discrete model, a great deal of care is required when choosing the discretisation. We show how the stochastic differential equation must be discretised in order to avoid arbitrage opportunities.¹ We have implemented several variance reduction techniques to reduce the time consumption in the simulations. One of the techniques, the control variate method, also seems to eliminate any problem with discretisation bias.

It is not a general feature of the control variate method that it eliminates the problem with discretisation bias. However, the simplicity of the method makes it very appealing. If we are to obtain a realistic model of the financial market, we often end up with a model that is analytical intractable in the sense that market values cannot be expressed by closed form solutions. By constructing assets within an analytical tractable model that are highly correlated with the assets in an analytical intractable model, the control variate method may often be very effective. We illustrate this at the end of the chapter by using the model of Vasicek (1977) as the analytical tractable model and the model of Cox, Ingersoll, and Ross (1985) as the analytical intractable model. The asset in the analytical tractable model is then used as a control variate in the estimation of the market value of the asset in the analytical intractable model.

As a special case, we show that it is possible to perform the simulation without a discretisation of the stochastic differential equation(s) when assuming a Gaussian Heath, Jarrow, and Morton framework. The advantage

¹A similar result that is better suited for calibration to market data can be found in Andersen (1997).

of this is twofold. First, it makes calculations way faster. For a given time budget, this makes it possible to increase the number of simulations and thereby increasing the precision in the estimate of the market value of the option. Second, the problem with discretisation bias is totally circumvented.

The chapter is organised as follows: In section 6.2 we give a description of our economic model. In section 6.3 a short description of a compound option is given. In section 6.4 we give an overview of Monte Carlo simulation, including some variance reduction techniques. Some of the special features of Monte Carlo simulation within the Heath, Jarrow, and Morton framework are analysed in section 6.5. Section 6.6 analyses so-called exact simulation within a Gaussian Heath, Jarrow, and Morton framework. Numerical results are presented in section 6.7. Section 6.8 concludes. In addition, appendix C contains the derivation of a closed form solution for the market value of a control variate for the compound option. It also shows how exact simulation can be used to estimate the market value of the compound option and the control variate.

6.2 The Economic Model and Preliminaries

We assume a continuous trading economy on the time interval $[0, T]$, for some fixed horizon $T > 0$, and with no transaction costs. A filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ is fixed, where Ω is the state space, \mathcal{F} is a σ -algebra, $\mathbb{F} = \{\mathcal{F}_t, 0 \leq t \leq T\}$ is a filtration where $\mathcal{F}_T = \mathcal{F}$ and $\mathcal{F}_0 = \{\emptyset, \Omega\}$, where \emptyset is the empty set, and P is a probability measure. The σ -algebra is generated by a d -dimensional, $d \geq 1$, Brownian motion, W_t . When W_t is multidimensional, the i 'th element, $i \leq d$, is denoted W_t^i . We further assume a complete market, i.e., there exists one unique equivalent martingale measure Q , see e.g., Harrison and Kreps (1979) and Harrison and Pliska (1981).

Following the model of Heath et al. (1992), the instantaneous continuously compounded forward rate at time s as seen from time t , $t \leq s \leq T$, under the equivalent martingale measure Q , is given by

$$f(t, s) = f(0, s) + \int_0^t \sigma_f(v, s) \int_v^s \sigma_f(v, u) du dv + \int_0^t \sigma_f(v, s) dW_v, \quad (6.1)$$

where $\sigma_f(t, s)$ is the volatility function for the instantaneous continuously compounded forward rate at time s , satisfying some technical regularity conditions, see Heath et al. (1992). This volatility function can be a fairly general function of both time and the forward curve, implying that neither the short-term interest rate, obtained by setting s equal to t , i.e., $r_t = f(t, t)$, nor the forward rates need to be Markov. We also assume that there is a continuum of bonds that trade in the market. Deterministic interest rates correspond formally to $\sigma_f(t, s) = 0$.

We let the market value of a non-dividend paying stock be given under the equivalent martingale measure Q by the equation

$$S_t = S_0 + \int_0^t r_v S_v dv + \int_0^t \sigma_S(v) S_v dW_v,$$

where $r_t S_t$ satisfies the integrability condition $\int_0^t |r_v S_v| dv < \infty$ almost surely for all t . Here $\sigma_S(t)$ is a volatility function and satisfies the square integrability condition $E\left[\int_0^t (\sigma_S(v) S_v)^2 dv\right] < \infty$ (for further details on integrability conditions, see e.g., Duffie (1996)).²

We also assume that there exists an instantaneously risk-free asset that accrues interest according to the short-term interest rate. This asset is denoted a money market account and the time t market value is given by

$$M_t = M_0 + \int_0^t r_v M_v dv, \quad M_0 = 1, \quad (6.2)$$

where $r_t M_t$ satisfies the integrability condition $\int_0^t |r_v M_v| dv < \infty$ almost surely for all t .

To perform numerical evaluation of the compound option, we need a closer specification of the volatility structure. We use two different models.

Model 1 We first study a Gaussian model with the volatility structure

$$\sigma_S(t) = \sigma_S \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$\sigma_f(v, u) = \sigma e^{-\kappa(u-v)} \begin{pmatrix} \varphi \\ \sqrt{1-\varphi^2} \end{pmatrix},$$

where σ_S , σ , κ , and φ are constants. This corresponds to the model of Vasicek (1977), and is Gaussian since the volatilities are only time dependent. Here κ is the force at which the short-term interest rate reverts to some long-term mean level.

The return on the money market account in a Gaussian model, under the equivalent martingale measure Q , is given by (see e.g., Miltersen and Persson (1999))

$$\beta_t = \int_0^t r_v dv = -\ln P(0, t) + \frac{1}{2} \sigma_{\beta_t}^2 + \int_0^t \int_v^t \sigma_f(v, u) dudW_v, \quad (6.3)$$

²Note that in the multidimensional case both $\sigma_f(s, t)$ and $\sigma_S(t)$ are vectors, but the interpretation should be obvious.

where $P(0, t)$ is the time zero market value of a zero-coupon bond maturing at time t and $\sigma_{\beta_t}^2$ is the variance of the return on the money market account and is given by

$$\sigma_{\beta_t}^2 = \int_0^t \left(\int_v^t \sigma_f(v, u) du \right)^2 dv. \quad (6.4)$$

The return on the stock under the equivalent martingale measure Q is given by

$$\delta_t = \int_0^t (r_v - \frac{1}{2} \sigma_S(v)^2) dv + \int_0^t \sigma_S(v) dW_v,$$

with variance

$$\sigma_{\delta_t}^2 = \sigma_{\beta_t}^2 + 2 \int_0^t \sigma_S(v) \int_v^t \sigma_f(v, u) dudv + \int_0^t \sigma_S^2(v) dv. \quad (6.5)$$

Unless otherwise specified, this will be the model that is used throughout the chapter.

Model 2 For the second model, we do the following change in the volatility structure (see e.g., Miltersen and Persson (1999))

$$\sigma_f(v, u) = \frac{4\sigma\gamma^2 e^{\gamma(u-v)}}{\left((\gamma + \kappa + \lambda)(e^{\gamma(u-v)} - 1) + 2\gamma \right)^2} \sqrt{r_v} \begin{pmatrix} \varphi \\ \sqrt{1 - \varphi^2} \end{pmatrix},$$

where

$$\gamma = \sqrt{(\kappa + \lambda)^2 + 2\sigma^2}.$$

In this model the volatility structure depends on the short-term interest rate, and is therefore non-Gaussian.

The Heath et al. (1992) model with the above volatility structure corresponds to a short-term interest rate model of the form

$$r_t = r_0 + \int_0^t \kappa(\theta_v - r_v) dv + \int_0^t \sigma \sqrt{r_v} dW_v,$$

under the equivalent martingale measure Q . This is the term structure model proposed by Cox et al. (1985). As for model 1, κ can also here be interpreted as the force of gravitation. Further, θ_v is associated with the reversion level and σ is a volatility parameter. The parameter λ is associated with the risk premium, $\phi(r_t)$, in the following way (for details, see Heath et al. (1992))

$$\phi(r_t) = -\lambda \sqrt{r_t} \sigma.$$

6.3 The Compound Option

In this chapter we assume that the compound option is written on a standard call option that is written on a stock. From Merton (1973) we know that the time 0 market value of the call option under (Gaussian) stochastic interest rates is given by

$$\pi_0 = S_0\Phi(d_1(T)) - P(0, T)X\Phi(d_1(T) - \sigma_{\delta_T}),$$

where

$$d_1(T) = \frac{\ln\left(\frac{S_0}{XP(0, T)}\right) + \frac{1}{2}\sigma_{\delta_T}^2}{\sigma_{\delta_T}},$$

T is the time to maturity, X is the exercise price, and $\Phi(\cdot)$ is the cumulative normal probability distribution.

Geske (1979) analysed a compound option within the framework of Black and Scholes (1973), i.e., under deterministic interest rates. He found that the time 0 market value is given by the following expression

$$\begin{aligned} \pi_0^1 = & S_0\Phi(\bar{d}_1, \bar{d}_2, \rho) - X_2P(0, T_2)\Phi(\bar{d}_1 - \sigma_S\sqrt{T_1}, \bar{d}_2 - \sigma_S\sqrt{T_2}, \rho) \\ & - X_1P(0, T_1)\Phi(\bar{d}_1 - \sigma_S\sqrt{T_1}), \end{aligned} \quad (6.6)$$

where

$$\begin{aligned} \bar{d}_1 &= \frac{\ln\left(\frac{S_0}{s^*}\right) + \frac{1}{2}\sigma_S^2T_1}{\sigma_S\sqrt{T_1}}, \\ \bar{d}_2 &= \frac{\ln\left(\frac{S_0}{X_2P(0, T_2)}\right) + \frac{1}{2}\sigma_S^2T_2}{\sigma_S\sqrt{T_2}}, \\ \rho &= \sqrt{\frac{T_1}{T_2}}, \end{aligned}$$

and $\Phi(a, b, p)$ is the cumulative standard bivariate normal probability distribution evaluated at the points a and b and with correlation p . The compound option can be exercised at time T_1 at a cost of X_1 , while the call option can be exercised at time $T_2 > T_1$ at a cost of X_2 . Here s^* is the critical value of S_{T_1} that makes the inequality³

$$S_{T_1}\Phi(d_1(T_2 - T_1)) - P(T_1, T_2)X_2\Phi(d_1(T_2 - T_1) - \sigma_{\delta_{T_2 - T_1}}) \geq X_1 \quad (6.7)$$

hold with equality. Thus, s^* is the lowest time T_1 stock price for which the compound option will be exercised. That there exists a unique s^* for

³Now $d_1(T_2 - T_1)$ is of course a function of S_{T_1} and $P(T_1, T_2)$.

all $X_1 \in (0, \infty)$ follows since the market value of the call option is strictly increasing in the stock price.

The only work that we have found that deals with the evaluation of compound options under stochastic interest rates is Geman et al. (1995). They derived an expression for the market value which is equal to equation (6.6).

Of course, also under (Gaussian) stochastic interest rates the inequality in (6.7) has to be satisfied for the compound option to be exercised. However, $P(T_1, T_2)$ is an \mathcal{F}_{T_1} -measurable random variable. Since s^* is a function of $P(T_1, T_2)$, it is also a random variable, not a parameter known at time $t < T_1$. Hence, the result in Geman et al. (1995) is likely to be flawed. To us it seems like a closed form solution for the market value of a compound option is not obtainable under stochastic interest rates.

6.4 Monte Carlo Simulation

Monte Carlo simulation has proved to be a useful tool in the pricing of derivative assets, see e.g., Boyle, Broadie, and Glasserman (1997). The market values of financial derivatives in a complete market are found by calculating expected deflated cash flows under the equivalent martingale measure Q . Monte Carlo simulation can be used to estimate this expectation.

As an illustration we first study the case with a constant interest rate r . Let⁴ $e^{-rT} \hat{\pi}_T(i)$ be a random variable representing the discounted simulated time T payoff of a derivative asset of European type. The variance of $e^{-rT} \hat{\pi}_T(i)$ is defined as σ_π^2 . An estimate of the market value at time 0 using N simulations is given by

$$\hat{\pi}_0 = \frac{1}{N} \sum_{i=1}^N e^{-rT} \hat{\pi}_T(i), \quad (6.8)$$

a random variable with variance

$$\sigma_{\hat{\pi}_0}^2 = \frac{\sigma_\pi^2}{N}.$$

The variance σ_π^2 is normally not known, but can be estimated by (the random variable)

$$\hat{\sigma}_\pi^2 = \frac{N}{N-1} \left(\frac{1}{N} \sum_{i=1}^N (e^{-rT} \hat{\pi}_T(i))^2 - \left(\frac{1}{N} \sum_{i=1}^N e^{-rT} \hat{\pi}_T(i) \right)^2 \right).$$

To get a more efficient estimate of π_0 , we can increase N or use variance reduction techniques to reduce σ_π^2 .

⁴That a quantity is estimated or simulated is emphasised by using a hat.

6.4.1 Variance Reduction Techniques

There is a wide range of different techniques that can be applied to reduce σ_π^2 . For a broad description of several techniques that have proved to be useful in financial applications, see Boyle et al. (1997). We will in this subsection illustrate three techniques; stratified sampling, importance sampling, and control variates. The techniques are illustrated on a standard call option and on a compound call option. We consider a call option that matures in six months ($T = 0.5$). The underlying option of the compound option is assumed to mature in one year, while the compound option has to be exercised in six months ($T_1 = 0.5$ and $T_2 = 1$). We use the results of Black and Scholes (1973) to calculate the time T_1 market value of the underlying option. The initial stock price is set equal to 100. We assume a constant interest rate equal to 8.00%. The exercise price for the compound option (X_1) is set equal to 10. The remaining parameter values, i.e., the volatility and the exercise price for the call option, are changed from example to example.

Using the stock price process in section 6.2, assuming deterministic interest rates and a time independent volatility function, the i 'th simulated time t stock price is given by

$$\hat{S}_t(i) = S_0 e^{(r - \frac{1}{2}\sigma_S^2)t + \sigma_S \sqrt{t}\varepsilon_i},$$

where ε_i is a random variable. For standard Monte Carlo simulation, importance sampling, and the control variate method, we have that $\varepsilon_i \sim \mathcal{N}(0, 1)$. More details about the sampling procedures for the different variance reduction techniques are given in the paragraphs below.

Closed Form Solution (CF) and Monte Carlo Simulation (MC)

The estimates of the market values of the call option and the compound option using the different variance reduction techniques are given in Table 6.1 and 6.2 on page 130 and 131, respectively. For comparison, we have also reported the market values found by the closed form solutions of Black and Scholes (1973) and Geske (1979)⁵ and by standard Monte Carlo simulation. The Monte Carlo simulation is based on 10.000 simulations (the estimates

⁵The only parameter in the formula of Geske (1979) that is not directly observable is the critical time T_1 stock price s^* that makes $\pi_{T_1} = X_1$. s^* can be calculated numerically by Newton's method.

Let $h(s) = \pi_{T_1} - X_1$ for $s = S_{T_1}$. We need to find s^* such that $g(s^*) = 0$. Using Newton's method, this yields, for the $n + 1$ 'st iteration, $n \in \{0, 1, \dots, N - 1\}$,

$$\begin{aligned} s_{n+1} &= s_n - \frac{h(s_n)}{h'(s_n)} \\ &= \frac{X_1 + X_2 P(T_1, T_2) \Phi(d_1 - \sigma_S \sqrt{T_2 - T_1})}{\Phi(d_1)}, \end{aligned}$$

of the standard errors are reported in parenthesis below the estimates of the market values).

Stratified Sampling (SS) Consider dividing the interval $[0, 1]$ into N uniform and disjoint subintervals. These subintervals are known as *strata*. Denote the i 'th stratum, $i \in \{1, 2, \dots, N\}$, by U_i . Then, by sampling a uniformly distributed random variable \tilde{u} , we can obtain a uniformly distributed random variable \tilde{u}_i in stratum U_i by doing the following transformation

$$\tilde{u}_i = \frac{i + \tilde{u} - 1}{N}.$$

By using an inverse transform for the cumulative normal probability distribution, \tilde{u}_i can be used to obtain a random variable ε_i in the i 'th stratum of the normal probability distribution, i.e.,

$$\varepsilon_i = \Phi^{-1}(\tilde{u}_i),$$

where $\Phi^{-1}(\cdot)$ is the inverse cumulative normal probability distribution and has to be approximated numerically. We have used the approximation of Hasting (1955).⁶ By construction, the probability that a normal distributed random variable will lie in the i 'th stratum is equal to $\frac{1}{N}$ for all $i \in \{1, 2, \dots, N\}$.

The stratification that minimises the variance, for a fixed number of random variables, is the stratification that has the same probability in each stratum and where only one random variable is sampled from each stratum, see e.g., Fishman (1996).

When we have a vector with the ε_i 's, the market value can be estimated by the formula in (6.8). The estimates of the market values of the call option and the compound option are given in Table 6.1 and 6.2, respectively. For the estimates of the market values we have used 3.000 strata with one sample from each stratum, and for the estimates of the standard errors we have used 1.000 samples from each stratum. We found that sampling one random variable by stratified sampling is approximately three times as time consuming as sampling one by standard Monte Carlo, and we have therefore only used 3.000 simulations for this approach, compared to 10.000 for the others.

This seems to be a fairly efficient method in terms of low standard errors. Notice three things about the standard errors; they increase with the volatility, they are constant in the exercise price of the call option, and they are also about the same for both the call option and the compound option.

where

$$d_1 = \frac{\ln\left(\frac{s_N}{P(T_1, T_2)X_2}\right) + \frac{1}{2}\sigma_S^2(T_2 - T_1)}{\sigma_S\sqrt{T_2 - T_1}}$$

and N is the smallest integer such that $|h(s_N)| < \epsilon$, where ϵ is small and we let $s^* = s_N$.

⁶This approximation has an error less than 0.45×10^{-3} .

Importance Sampling (IS) Importance sampling takes into account that there may exist another probability measure than Q , under which we can obtain a more efficient estimator for the market value of the derivative asset. As usual, the change of probability measure is done by the use of a Radon-Nikodym derivative, also called a likelihood function. Both Boyle et al. (1997) and Andersen (1995) have used this technique to financial applications with success. For a more general discussion of importance sampling, see e.g., Fishman (1996).

Boyle et al. (1997) showed that by defining a likelihood function L and by replacing the drift term r under Q (assuming a constant interest rate) by μ under some equivalent probability measure Q^μ , the following equation holds

$$E_Q \left[\max(S_T - X, 0) \right] = E_{Q^\mu} \left[\max(S_T - X, 0) L \right],$$

where⁷

$$L = \left(\frac{S_T}{S_0} \right)^{(r-\mu)/\sigma^2} \cdot \exp \left(\frac{(\mu^2 - r^2)T}{2\sigma^2} + \frac{(r - \mu)T}{2} \right).$$

The idea is to find the probability measure Q^μ that minimises the variance of $\max(S_T - X, 0)$. Finding the optimal μ is not trivial, but we can often find a μ that reduces the variance compared to the variance under Q .

The choice of L made by Boyle et al. (1997) is best suited for the case with a stock and deterministic interest rates. We will here pursue a similar approach where we define a Radon-Nikodym derivative L , under a probability measure Q^γ , as⁸

$$L_t = e^{-\frac{1}{2} \int_0^t \gamma_v^2 dv - \int_0^t \gamma_v dW_v^{Q^\gamma}},$$

thus,

$$\hat{\pi}_0 = \frac{1}{N} \sum_{i=1}^N e^{-\hat{\beta}_T} \hat{\pi}_T(i) \hat{L}_T.$$

Andersen (1995) describes this approach as a “reversed” change of probability measure. For instance, we have under the probability measure Q^γ that

$$\beta_t = -\ln P(0, t) + \frac{1}{2} \sigma_{\beta_t}^2 + \int_0^t \gamma_v \int_v^t \sigma_f(v, u) du dv + \int_0^t \int_v^t \sigma_f(v, u) du dW_v^{Q^\gamma}$$

⁷Notice that there is a typo in Boyle et al. (1997) p. 1284. L is there defined as

$$L = \left(\frac{S_T}{S_0} \right)^{(r-\mu)/\sigma^2} \cdot \exp \left(\frac{(\mu^2 - r^2)T}{2\sigma^2} \right).$$

⁸If we want to increase the drift of, say, a stock, the sign in front of the dW -term must be negative and positive otherwise (given that $\int_0^t \gamma_v dv > 0$).

and

$$\delta_t = \beta_t - \frac{1}{2}\sigma_S^2(v)dv + \int_0^t \gamma_v \sigma_S(v)dv + \int_0^t \sigma_S(v)dW_v^{Q^\gamma},$$

where $W_v^{Q^\gamma}$ is a standard Brownian motion under Q^γ . Using this (and assuming some technical regularity conditions), $e^{-\beta t}S_tL_t$ is a martingale under the probability measure Q^γ since $e^{-\beta_0} = L_0 = 1$, hence

$$E_{Q^\gamma} \left[e^{-\beta t} S_t L_t \right] = S_0.$$

The technique is illustrated for the two above claims in Table 6.1 and 6.2. We have assumed that $\gamma_t = \gamma$ is a constant. For each set of parameter values we have used the $\gamma \in [0, 5]$ that minimises the standard error.⁹ Also, the γ 's are reported in square brackets below the estimates of the standard errors (10.000 simulations).

For our examples importance sampling consequently outperforms standard Monte Carlo simulation in terms of standard errors. However, keep in mind that the γ 's are unknown and have to be estimated, something that does not speak in favour of the method.

The Control Variate Method (CV) The general idea behind the control variate method is to use the correlation between the random variable that we wish to estimate the expectation of and some other random variable for which the expectation is known. Let X and Y be two random variables where $E(X)$ is known and $E(Y)$ is not. Let

$$\hat{Y}(b) = \hat{Y} - b(\hat{X} - E(X)). \quad (6.9)$$

$\hat{Y}(b)$ is then an unbiased estimator for $E(Y)$ with variance

$$\sigma_{\hat{Y}(b)}^2 = \sigma_{\hat{Y}}^2 - 2b\sigma_{XY} + b^2\sigma_X^2,$$

where $\sigma_{\hat{Y}}^2$ is the variance of \hat{Y} , σ_X^2 the variance of \hat{X} , and σ_{XY} the covariance between \hat{Y} and \hat{X} . The optimal, or variance minimising, b is given by

$$b^* = \frac{\sigma_{XY}}{\sigma_X^2}.$$

It is easily seen that $\sigma_{\hat{Y}(b)}^2$ is less than $\sigma_{\hat{Y}}^2$ as long as $b \in [0, 2b^*]$ for $\sigma_{XY} \geq 0$ and $b \in [2b^*, 0]$ for $\sigma_{XY} \leq 0$. The ratio, also called the speed-up factor,

$$\frac{\sigma_{\hat{Y}}^2}{\sigma_{\hat{Y}(b^*)}^2} = \frac{1}{1 - \rho_{XY}^2},$$

⁹Our numerical results indicate that the optimal γ lies in this interval.

where

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y},$$

shows by what proportion the variance is reduced by using the optimal b compared to standard Monte Carlo simulation.

Having, say, I control variates, and letting \tilde{X} be an I -dimensional column vector with j 'th element $\tilde{X}_j = \hat{X}_j - E(\hat{X}_j)$, it follows that

$$\hat{Y}(b) = \hat{Y} - b\tilde{X},$$

where b is now an I -dimensional row vector. In practice, b has to be estimated by \hat{b} and can be found by multiple regression with a zero intercept.

The control variate method has earlier been used by e.g., Kemna and Vorst (1990) in the pricing of arithmetic average Asian options by using the geometric average Asian option as a control variate and by Carverhill and Pang (1995) in the pricing of bond options.¹⁰

We have used the underlying stock as a control variate for the call option, i.e., the i 'th simulation of the discounted option payoff, $i \in \{1, 2, \dots, N\}$, where N is the total number of simulations, is given by

$$\hat{\pi}_0(i) = e^{-rT} \max(\hat{S}_T(i) - X, 0) - \hat{b}^*(e^{-rT} \hat{S}_T(i) - S_0),$$

where $\hat{S}_T(i)$ is the i 'th simulated time T stock price. Thus

$$\hat{\pi}_0 = \frac{1}{N} \sum_i^N \hat{\pi}_0(i).$$

We have used two control variates for the compound option; a standard European call option and the underlying stock, i.e., the i 'th simulated discounted compound option payoff is given by

$$\begin{aligned} \hat{\pi}_0^1(i) &= e^{-rT_1} \max(\hat{\pi}_{T_1}(i) - X_1, 0) \\ &\quad - \hat{b}_1^*(e^{-rT_1} \hat{\pi}_{T_1}(i) - \pi_0) - \hat{b}_2^*(e^{-rT_1} \hat{S}_{T_1}(i) - S_0), \end{aligned}$$

where \hat{b}_1^* and \hat{b}_2^* are estimates of the variance minimising weights and are found by multiple regression.

The results are reported in Table 6.1 and 6.2 with estimates of the optimal weights given in curly brackets below the estimates of the standard errors (10.000 simulations).

From the illustration of the control variate method on the standard European call option using the stock price as a control variate, we know that

¹⁰Carverhill and Pang (1995) termed this approach *martingale variance reduction variates*. Note however that in the presence of discretisation bias, discounted asset prices will typically *not* be martingales.

the call option and the stock are highly correlated, causing a possible problem with multicollinearity when using both the call option and the stock as control variates for the compound option, cf. the assumptions behind regression analysis. However, as we can see from the estimates of the standard errors, the weights do in fact reduce the variance quite significantly compared to standard Monte Carlo simulation. Notice that the estimates of the standard errors for the compound option are relatively insensitive to the level of the volatility.

6.5 Simulation within the Heath, Jarrow, and Morton Framework

6.5.1 Simulation of the Whole Term Structure

To use Monte Carlo simulation to estimate the market value of a derivative asset that matures at, say, time T , we need to find a discrete approximation of the integral $\beta_T = \int_0^T r_v dv$. This can be approximated as follows

$$\beta_T \approx \sum_{i=0}^{M-1} f(t_i, t_i)(t_{i+1} - t_i), \quad (6.10)$$

where $0 = t_0 < t_1 < \dots < t_M = T$. Here $f(t_i, t_i)$ is interpreted as the short-term interest rate over the time interval from time t_i to t_{i+1} . More generally, we let $f(t_i, t_j)$, $i \leq j$, be the forward rate from time t_j to t_{j+1} prevailing at time t_i . The initial term structure is given by $f(t_0, t_i)$ for all $i \in \{0, 1, \dots, M-1\}$.

In general, the approximation in (6.10) requires, within the Heath, Jarrow, and Morton framework, the *whole* term structure up to time T to be simulated. To see this, assume a one-factor model and the following discretisation of the forward rates

$$f(t_i, t_j) = f(t_{i-1}, t_j) + \mu(t_{i-1}, t_j)(t_i - t_{i-1}) + \sigma_f(t_{i-1}, t_j)\sqrt{t_i - t_{i-1}}\varepsilon_i,$$

for some drift function $\mu(\cdot, \cdot)$ and where $\varepsilon_i \sim \mathcal{N}(0, 1)$. This approximation may be seen as an Euler scheme of (6.1). For $i = 1$ we have that

$$\begin{aligned} f(t_1, t_1) &= f(t_0, t_1) + \mu(t_0, t_1)(t_1 - t_0) + \sigma_f(t_0, t_1)\sqrt{t_1 - t_0}\varepsilon_1, \\ f(t_1, t_2) &= f(t_0, t_2) + \mu(t_0, t_2)(t_1 - t_0) + \sigma_f(t_0, t_2)\sqrt{t_1 - t_0}\varepsilon_1, \\ &\vdots \\ f(t_1, t_{M-1}) &= f(t_0, t_{M-1}) + \mu(t_0, t_{M-1})(t_1 - t_0) \\ &\quad + \sigma_f(t_0, t_{M-1})\sqrt{t_1 - t_0}\varepsilon_1. \end{aligned}$$

Table 6.1: Market values of a standard call option using closed form solution (CF), standard Monte Carlo simulation (MC), stratified sampling (SS), importance sampling (IS) (the γ 's are reported in square brackets), and the control variate method (CV) with the underlying stock as the control variate (the \hat{b}^* 's are reported in curly brackets). Standard errors are given in parentheses. The parameter values are $S_0 = 100$, $r = 0.08$, and $T = 0.5$. For stratified sampling 3000 strata are used with 1 sample from each stratum (the standard error is based on an estimate of the variance of 1000 samples from each stratum). For the others, 10,000 simulations are used.

		CF	MC	SS	IS	CV
$X = 110$	$\sigma_S = 0.1$	0.90124	0.95745 (0.02460)	0.90312 (0.00076)	0.89088 (0.00628) [2.36]	0.93966 (0.01836) {0.23641}
	$\sigma_S = 0.2$	3.39103	3.55128 (0.07112)	3.39527 (0.00197)	3.35480 (0.02058) [2.03]	3.48610 (0.04292) {0.39834}
	$\sigma_S = 0.3$	6.13620	6.40542 (0.12228)	6.14421 (0.00380)	6.07307 (0.03533) [2.01]	6.27820 (0.06611) {0.47736}
$X = 100$	$\sigma_S = 0.1$	5.15632	5.23034 (0.05540)	5.15799 (0.00076)	5.13692 (0.02017) [1.31]	5.17530 (0.01826) {0.73158}
	$\sigma_S = 0.2$	7.70641	7.87982 (0.10265)	7.71002 (0.00197)	7.66371 (0.03475) [1.54]	7.77200 (0.04003) {0.65911}
	$\sigma_S = 0.3$	10.3881	10.6736 (0.15346)	10.3952 (0.00381)	10.3238 (0.04891) [1.66]	10.5000 (0.06100) {0.65093}
$X = 90$	$\sigma_S = 0.1$	13.5770	13.6512 (0.06982)	13.5787 (0.00076)	13.5595 (0.02128) [0.69]	13.5780 (0.00411) {0.97125}
	$\sigma_S = 0.2$	14.5661	14.7163 (0.12761)	14.5708 (0.00197)	14.5397 (0.04376) [1.11]	14.5740 (0.02523) {0.87133}
	$\sigma_S = 0.3$	16.4094	16.6733 (0.18136)	16.4180 (0.00381)	16.3584 (0.05990) [1.34]	16.4580 (0.04705) {0.80905}

Table 6.2: Market values of a call option on a call option using closed form solution (CF), standard Monte Carlo simulation (MC), stratified sampling (SS), importance sampling (IS) (the γ 's are reported in square brackets), and the control variate method (CV) with the underlying option and the underlying stock as control variates (the \hat{b}^* 's are reported in curly brackets, the first ones are for the call option). Standard errors are given in parentheses. The parameter values are $S_0 = 100$, $r = 0.08$, $X_1 = 10$, $T_1 = 0.5$, and $T_2 = 1$. For the stratified sampling 3000 strata are used with 1 sample from each stratum (the standard errors are based on estimates of the variances of 1000 samples from each stratum). For the others, 10,000 simulations are used.

		CF	MC	SS	IS	CV
	$\sigma_S = 0.1$	0.23303	0.24998 (0.01250)	0.23488 (0.00076)	0.23136 (0.00197) [3.11]	0.24101 (0.00591) {0.50237} {-0.03620}
$X_2 = 110$	$\sigma_S = 0.2$	2.26747	2.30487 (0.05704)	2.27207 (0.00196)	2.25290 (0.01406) [2.27]	2.27749 (0.00745) {0.85089} {-0.04433}
	$\sigma_S = 0.3$	5.35205	5.42900 (0.10946)	5.36017 (0.00380)	5.33203 (0.02808) [2.05]	5.36623 (0.00592) {0.88091} {-0.00059}
	$\sigma_S = 0.1$	2.02371	2.04540 (0.03711)	2.02510 (0.00076)	2.01246 (0.01142) [1.93]	2.00833 (0.01241) {0.94756} {-0.26998}
$X_2 = 100$	$\sigma_S = 0.2$	5.16747	5.23292 (0.08459)	5.17111 (0.00195)	5.15146 (0.02538) [1.78]	5.15495 (0.01120) {0.98250} {-0.13428}
	$\sigma_S = 0.3$	8.59616	8.71115 (0.13620)	8.60374 (0.00380)	8.57661 (0.03855) [1.74]	8.59317 (0.00822) {0.94166} {-0.04795}
	$\sigma_S = 0.1$	7.82306	7.86636 (0.06351)	7.82491 (0.00076)	7.81015 (0.02289) [1.03]	7.81874 (0.00966) {2.27960} {-1.36100}
$X_2 = 90$	$\sigma_S = 0.2$	10.1817	10.2797 (0.11222)	10.1861 (0.00197)	10.1640 (0.03701) [1.34]	10.1896 (0.01278) {1.23120} {-0.32964}
	$\sigma_S = 0.3$	13.20760	13.35370 (0.16305)	13.21590 (0.00380)	13.18430 (0.04897) [1.46]	13.19820 (0.01097) {1.06500} {-0.14843}

For each i there is one forward rate less that has to be simulated. For $i = 2$ this yields

$$\begin{aligned} f(t_2, t_2) &= f(t_1, t_2) + \mu(t_1, t_2)(t_2 - t_1) + \sigma_f(t_1, t_2)\sqrt{t_2 - t_1}\varepsilon_2, \\ &\vdots \\ f(t_2, t_{M-1}) &= f(t_1, t_{M-1}) + \mu(t_1, t_{M-1})(t_2 - t_1) \\ &\quad + \sigma_f(t_1, t_{M-1})\sqrt{t_2 - t_1}\varepsilon_2, \end{aligned}$$

and so on. The sum in (6.10) can now be calculated.

6.5.2 Arbitrage-free Drift Term under Euler Discretisation

The arbitrage-free drift term of the forward rates in the continuous case was given in (6.1). A natural question to ask now is what is the arbitrage-free drift term for the discrete forward rates.

We define a discrete zero-coupon bond

$$\bar{P}(t_i, t_j) = \exp\left(-\sum_{l=i}^{j-1} f(t_i, t_l)(t_{l+1} - t_l)\right) \quad (6.11)$$

on a discrete time grid $0 = t_0 < t_1 \dots < t_i < t_j \dots < t_M = T$. Again, $f(t_i, t_l)$ must be interpreted as the forward rate over the period t_l to t_{l+1} as seen from time $t_i \leq t_l$. This is therefore a discrete approximation of the instantaneous forward rate. We want to formulate an arbitrage-free discrete scheme for the forward rates. For simplicity we study a one-factor model where the forward curve is modelled with an Euler scheme, cf. subsection 6.5.1.

Under the martingale measure Q , the discrete bond price satisfies¹¹

$$\bar{P}(t_i, t_j) = \bar{P}(t_i, t_{i+1})E_Q\left[\bar{P}(t_{i+1}, t_j)\middle|\mathcal{F}_{t_i}\right]. \quad (6.12)$$

Combining (6.11) and (6.12), we find that

$$E_Q\left[\bar{P}(t_{i+1}, t_j)\middle|\mathcal{F}_{t_i}\right] = \frac{\bar{P}(t_i, t_j)}{\bar{P}(t_i, t_{i+1})} = \exp\left(-\sum_{l=i+1}^{j-1} f(t_i, t_l)(t_{l+1} - t_l)\right). \quad (6.13)$$

First we study the expression

$$\bar{P}(t_{i+1}, t_j) = \exp\left(-\sum_{l=i+1}^{j-1} f(t_{i+1}, t_l)(t_{l+1} - t_l)\right).$$

¹¹In the continuous case the bond price is only used as a deflator under the forward probability measure. However, in the discrete case $\bar{P}(t_i, t_{i+1})$ and the inverse of the money market account, i.e., $e^{-\int_{t_i}^{t_{i+1}} f(t_i, t) dt}$, coincide and this bond price can also be used as a deflator under the probability measure Q . In fact, Q and the forward probability measure for time t_{i+1} coincide in the discrete case.

Assume that $f(t_{i+1}, t_l)$ (under Q) is given from the discrete scheme

$$f(t_{i+1}, t_l) = f(t_i, t_l) + \mu(t_i, t_l)[t_{i+1} - t_i] + \sigma_f(t_i, t_l)\sqrt{(t_{i+1} - t_i)}\varepsilon_{i+1} \quad (6.14)$$

for $l \in \{i+1, i+2, \dots, M\}$ and for $i \in \{0, 1, \dots, M-1\}$. We take the volatility structure σ_f as given, and try to find a μ such that the discrete model is arbitrage-free. This discretisation gives that

$$\bar{P}(t_{i+1}, t_j) = \exp(m - \sqrt{v}\varepsilon),$$

where

$$m = - \sum_{l=i+1}^{j-1} [\{f(t_i, t_l) + \mu(t_i, t_l)(t_{i+1} - t_i)\} (t_{l+1} - t_l)],$$

$$v = \left(\sum_{l=i+1}^{j-1} \sigma_f(t_i, t_l)(t_{l+1} - t_l) \right)^2 (t_{i+1} - t_i),$$

and

$$\varepsilon \sim \mathcal{N}(0, 1).$$

This implies that

$$E_Q[\bar{P}(t_{i+1}, t_j) | \mathcal{F}_{t_i}] = E_Q[\exp(m - \sqrt{v}\varepsilon) | \mathcal{F}_{t_i}] = \exp(m + \frac{v}{2}).$$

This combined with (6.13) implies that μ must satisfy

$$\sum_{l=i+1}^{j-1} \mu(t_i, t_l)(t_{l+1} - t_l) = \frac{1}{2} \left(\sum_{l=i+1}^{j-1} \sigma_f(t_i, t_l)(t_{l+1} - t_l) \right)^2.$$

If we let $j-1 = i+1$ this gives that

$$\mu(t_i, t_{i+1}) = \frac{1}{2} \sigma^2(t_i, t_{i+1})(t_{i+2} - t_{i+1}),$$

and $j-1 = i+2$ implies that

$$\begin{aligned} \mu(t_i, t_{i+2})(t_{i+3} - t_{i+2}) &= \frac{1}{2} \left(\sum_{l=i+1}^{i+2} \sigma_f(t_i, t_l)(t_{l+1} - t_l) \right)^2 \\ &\quad - \mu(t_i, t_{i+1})(t_{i+2} - t_{i+1}) \\ &= \frac{1}{2} \left(\sum_{l=i+1}^{i+2} \sigma_f(t_i, t_l)(t_{l+1} - t_l) \right)^2 \\ &\quad - \frac{1}{2} \sigma^2(t_i, t_{i+1})(t_{i+2} - t_{i+1})^2. \end{aligned}$$

Continuing recursively we get

$$\begin{aligned} \mu(t_i, t_j)(t_{j+1} - t_j) &= \frac{1}{2} \left(\sum_{l=i+1}^j \sigma_f(t_i, t_l)(t_{l+1} - t_l) \right)^2 \\ &\quad - \frac{1}{2} \left(\sum_{l=i+1}^{j-1} \sigma_f(t_i, t_l)(t_{l+1} - t_l) \right)^2. \end{aligned}$$

For an approach that is better suited when calibrating to market data, see Andersen (1997).

6.6 Exact Simulation of Model 1

We know from (6.1) that

$$f(t, s) = f(0, s) + M(t, s) + N(t, s),$$

where

$$M(t, s) = \int_0^t \sigma_f(v, s) \int_v^s \sigma_f(v, u) du dv$$

and

$$N(t, s) = \int_0^t \sigma_f(v, s) dW_v.$$

For model 1 we have that

$$M(t, s) = \frac{\sigma^2}{\kappa^2} \left[e^{-\kappa(s-t)} + \frac{1}{2} e^{-2\kappa s} - e^{-\kappa s} - \frac{1}{2} e^{-2\kappa(s-t)} \right].$$

Further, $N(t, s)$ is Gaussian with zero expectation and variance

$$\int_0^t \sigma_f^2(v, s) dv = \frac{\sigma^2}{2\kappa} \left[e^{-2\kappa(s-t)} - e^{-2\kappa s} \right].$$

Using this, we can use exact simulation, i.e., only simulate the terminal values, not the entire path followed by the stock price and the interest rates.

6.6.1 Simulation of $\int r_s ds$

Since $r_s = f(s, s)$ we can write

$$\int_0^T r_s ds = \int_0^T f(0, s) ds + \int_0^T M(s, s) ds + \int_0^T N(s, s) ds.$$

We find that

$$\int_0^T M(s, s) ds = \frac{\sigma^2}{4\kappa^3} [2\kappa T - e^{-2\kappa T} - 3 + 4e^{-\kappa T}]. \quad (6.15)$$

Further,

$$\int_0^T N(s, s) ds = \int_0^T \int_0^s \sigma_f(v, s) dW_v ds = \int_0^T \int_v^T \sigma_f(v, s) ds dW_v.$$

This can be written as

$$\int_0^T N(s, s) ds = \frac{\sigma}{\kappa} \left[\varphi \int_0^T g(v) dW_v^1 + \sqrt{1-\varphi^2} \int_0^T g(v) dW_v^2 \right],$$

where $g(v) = 1 - e^{-\kappa(T-v)}$. The variance of $\int_0^T N(s, s) ds$ is easily seen to be given by (see e.g., Miltersen and Persson (1999))

$$\sigma_{\beta_T}^2 = \frac{\sigma^2}{2\kappa^3} [2\kappa T - e^{-2\kappa T} - 3 + 4e^{-\kappa T}],$$

and based on (6.3) this could have been directly derived from (6.15).

We now illustrate how this may be utilised to price contingent claims under stochastic interest rates of this type.

6.6.2 A Call Option and Exact Simulation

Suppose we want to price a call option on a stock. Let X be the exercise price and T the time of maturity. The time zero market value is then given by

$$\begin{aligned} \pi_0 &= E_Q[e^{-\int_0^T r_v dv} (S_T - X)^+] \\ &= E_Q[e^{-\int_0^T r_v dv} (S_0 e^{(\int_0^T r_v dv - \frac{1}{2}\sigma_S^2 T + \int_0^T \sigma_S dW_s^1)} - X)^+] \\ &= E_Q \left[e^{\left(\ln P(0, T) - \frac{1}{2}\sigma_{\beta_T}^2 - \frac{\sigma}{\kappa} [\varphi \int_0^T g(v) dW_v^1 + \sqrt{1-\varphi^2} \int_0^T g(v) dW_v^2] \right)} \right. \\ &\quad \left. (S_0 e^{(-\ln P(0, T) + \frac{1}{2}\sigma_{\beta_T}^2 - \frac{1}{2}\sigma_S^2 T + \int_0^T (\frac{\sigma}{\kappa} \varphi g(v) + \sigma_S) dW_v^1)} \right. \\ &\quad \left. e^{(\frac{\sigma}{\kappa} \sqrt{1-\varphi^2} \int_0^T g(v) dW_v^2)} - X)^+ \right]. \end{aligned}$$

The covariance between the random variables

$$Z_1 = \frac{\sigma\varphi}{\kappa} \int_0^T g(v) dW_v^1$$

and

$$Z_3 = \int_0^T \left(\frac{\sigma\varphi}{\kappa} g(v) + \sigma_S \right) dW_v^1$$

is given by

$$\begin{aligned}\text{cov}(Z_1, Z_3) &= \int_0^T \frac{\sigma\varphi}{\kappa} g(v) \left[\frac{\sigma\varphi}{\kappa} g(v) + \sigma_S \right] dv \\ &= \varphi^2 \sigma_{\beta_T}^2 + \frac{\sigma_S \varphi \sigma}{\kappa^2} [\kappa T - 1 + e^{-\kappa T}].\end{aligned}$$

Further we find that

$$\sigma_1^2 \equiv \text{var}(Z_1) = \varphi^2 \sigma_{\beta_T}^2$$

and

$$\sigma_3^2 \equiv \text{var}(Z_3) = \sigma_S^2 T + 2 \frac{\sigma_S \varphi \sigma}{\kappa^2} [\kappa T - 1 + e^{-\kappa T}] + \varphi^2 \sigma_{\beta_T}^2.$$

The correlation is

$$\rho = \frac{\text{cov}(Z_1, Z_3)}{\sigma_1 \sigma_3}.$$

The pricing problem can now be solved using exact simulation, i.e.,

$$\pi_0 = E_Q \left[e^{\ln P(0,T) - \frac{1}{2} \sigma_{\beta_T}^2 - \bar{Z}_1 - \bar{Z}_2} (S_0 e^{-\ln P(0,T) + \frac{1}{2} \sigma_{\beta_T}^2 - \frac{1}{2} \sigma_S^2 T + \bar{Z}_2 + \bar{Z}_3} - X)^+ \right]$$

where

$$\begin{aligned}\bar{Z}_1 &= |\varphi| \sigma_{\beta_T} Y_1, \\ \bar{Z}_2 &= \sqrt{1 - \varphi^2} \sigma_{\beta_T} Y_2, \\ \bar{Z}_3 &= \sigma_3 (\rho Y_1 + \sqrt{1 - \rho^2} Y_3),\end{aligned}$$

and

$$Y_i \sim \mathcal{N}(0, 1), \quad i \in \{1, 2, 3\}.$$

Using the methods of this section we are also able to use exact simulation to find the market value of the compound option under stochastic interest rates. The details of this simulation are given in section C.2 on a more analytical tractable claim that is closely related to the compound option.

6.7 Numerical Results

In this section we present numerical results for the estimates of the market values of the compound option under stochastic interest rates. For model 1, i.e., the term structure of Vasicek (1977), both exact simulation and the more general discrete method of subsection 6.5.2 are used. For exact simulation both standard Monte Carlo simulation and the control variate method

are used. For the second approach we have in addition used importance sampling. For model 2 (the term structure of Cox et al. (1985)) we have only used the control variate method. From the examples in section 6.4, we saw that stratified sampling was the superior method. However, we neither found this method to be very practical when the whole price path is needed nor when the Brownian motion is multidimensional, and the method will therefore not be used.

Since the most important dimension with respect to uncertainty in the model stems from the movements in the stock price, we let the Radon-Nikodym derivative L , when using importance sampling, only correlate with the Brownian motion that is common for the stock and the interest rates. Thus, L can be thought of as being driven by a one-dimensional Brownian motion.

We saw in section 6.4 when pricing the compound option using the control variate method that this was a fairly effective method. However, the effectiveness of the method is of course highly dependent on the correlation between the asset and the control variate being high. We have therefore constructed a modified compound option. This asset (the control variate) has the same underlying asset as the compound option. The only difference is that the exercise price is $P(T_1, T_2)X_1$ instead of X_1 . Thus, to exercise the control variate, the holder has to deliver X_1 units of the zero-coupon bond maturing at time T_2 . This asset can be evaluated in closed form (in a Gaussian setting) and is highly correlated with the compound option (see section C.1 for details). Based on the standard errors in Table 6.5, we found the speed-up factor to range from 195 to 8179(!), implying a correlation between 0.9974 to 0.9999. It should be mentioned that the correlation between these two assets can be increased even further by not using the same value of X_1 for the two assets. We can then choose an X_1 for the control variate that makes the time T_1 exercise price closer to the exercise price for the compound option.

For model 1 we use the formula in section 6.3 to calculate the time T_1 market value of the underlying option.

Throughout the chapter we use the following parameter values: $S_0 = 100$, $X_1 = 10$, $f(0, t) = 0.08$ for all $t \in [0, T]$, $\kappa = 0.1$, $\varphi = -0.5$, and $\lambda = -0.2$.¹²

The market values of the control variate using the closed form solution of section C.1 are presented in Table 6.3. Estimates of the market values of the control variate using standard Monte Carlo simulation for both exact simulation and for different number of time discretisations are also presented. As expected, there does not seem to be any evidence or indications of discreti-

¹²The idea behind this choice of λ is simply to assure that (at least for the initial term structure) the Feller condition (see Heath et al. (1992)) is satisfied. However, using $\lambda = 0$ we found no significant changes in the estimates of the market values, but the Feller condition is not satisfied for the initial term structure.

Table 6.3: Market values of the control variate using closed form solution (CF), exact simulation (ES), and standard Monte Carlo simulation for different number of time discretisations, Δt , of the time interval $[0, T_2]$. Estimates of the standard errors are reported in parentheses below the estimates of the market values (1.000.000 simulations). Model 1.

		Standard Monte Carlo simulation				
		CF	ES	$\Delta t = 4$	$\Delta t = 10$	$\Delta t = 20$
$X_2 = 110$	$\sigma_S = 0.1$	0.17334	0.17201 (0.00096)	0.18495 (0.00101)	0.17758 (0.00098)	0.17454 (0.00097)
	$\sigma_S = 0.2$	2.13238	2.12809 (0.00522)	2.17345 (0.00532)	2.14926 (0.00527)	2.13919 (0.00525)
	$\sigma_S = 0.3$	5.21947	5.21525 (0.01034)	5.27530 (0.01045)	5.24087 (0.01040)	5.22815 (0.01038)
$X_2 = 100$	$\sigma_S = 0.1$	1.92893	1.92870 (0.00340)	1.96913 (0.00347)	1.94453 (0.00343)	1.93676 (0.00342)
	$\sigma_S = 0.2$	5.06871	5.06868 (0.00803)	5.12429 (0.00812)	5.08980 (0.00807)	5.07904 (0.00805)
	$\sigma_S = 0.3$	8.50934	8.50918 (0.01306)	8.57303 (0.01317)	8.53351 (0.01312)	8.52069 (0.01301)
$X_2 = 90$	$\sigma_S = 0.1$	8.02641	8.03160 (0.00600)	8.05542 (0.00602)	8.03916 (0.00598)	8.03489 (0.00596)
	$\sigma_S = 0.2$	10.2268	10.2333 (0.01080)	10.2810 (0.01089)	10.2492 (0.01084)	10.2397 (0.01082)
	$\sigma_S = 0.3$	13.2128	13.2184 (0.01578)	13.2773 (0.01589)	13.2391 (0.01584)	13.2272 (0.01581)

sation bias for exact simulation. For 4 time discretisations there are clear evidence of discretisation bias. The results also seem to support the hypothesis that there is bias for 10 time discretisations as well. The estimates seem somewhat high for 20 time discretisations too, but in case of any bias, it is not statistically significant. The estimates are based 1.000.000 simulations. We can notice that the standard errors are approximately the same across the four estimates.

In a further search for bias, we have in Table 6.4 reported estimates of the market values of the compound option using importance sampling and the control variate method for 10 and 20 time discretisations and 100.000 simulations. Exact simulation with the control variate method and 1.000.000 simulations is used as a benchmark.

The compound option and the control variate are two very similar assets. This indicates that the prices have correlation close to one and also variances of the same magnitude. Hence,

$$b^* = \frac{\sigma_{\pi^C, \pi^1}}{\sigma_{\pi^C}^2} \approx \frac{\sigma_{\pi^C, \pi^1}}{\sigma_{\pi^C} \sigma_{\pi^1}} \approx 1,$$

where σ_{π^C, π^1} is the covariance between $\hat{\pi}_0^C$ and $\hat{\pi}_0^1$ (π_0^C is the time 0 market

value of the control variate). Further, $\sigma_{\pi_c}^2$ and $\sigma_{\pi_1}^2$ are the respective variances. By choosing $b = 1$, we found very large speed-up factors (cf. page 137). This confirms that b^* is close to one.

There is some evidence of bias for the estimates based on importance sampling, especially for 10 time discretisations.¹³ There does not seem to be any bias for the control variate method. This can be explained by the fact that the compound option and the control variate are two very similar assets, and any bias is likely to affect the two in the same direction. Because of the way in which the estimator is constructed, the discretisation bias in the simulated discounted payoff for the compound option is cancelled out by the bias in the simulated discounted payoff for the control variate, hence, any bias in the estimator for the market value of the compound option is insignificant.

In Table 6.5 we have reported estimates of the market values of the compound option found using standard Monte Carlo simulation, importance sampling, and the control variate method. The estimates are based on only 10.000 simulations and 10 time discretisations. Only using 10.000 simulations is fast enough for practical applications (just a few seconds). The standard errors of both the estimates found using importance sampling and the control variate method are fairly low, though the estimates based on the control variate method are by far the most efficient in terms of low standard errors.

Finally, we have in Table 6.6 reported estimates of the market values of the compound option using model 2. Both the market value of the underlying call option and the compound option are estimated using the control variate method. We have used a standard call option based on model 1 as a control variate in the estimation of the time T_1 market value of the call option underlying the compound option. The same approach is used in the estimation of the market value of the compound option, but then by using the control variate from section C.1. These assets are highly correlated, and, for the sake of simplicity, b is in both cases set equal to one. The variance is reduced as long as $b \in [0, 2b^*]$ (see Fishman (1996) p. 278). The size of the standard errors seems to confirm that b is in this interval.

The estimate of the market value of the call option, that has to be estimated for each simulation of the discounted payoff for the compound option and is used as a parameter value, is found using 1.000 simulations. The market value of the compound option is then estimated by doing 100 simulations. Using few time discretisations, estimating the market values takes only a few seconds and is fast enough for practical applications. However, only doing 100 simulations results in relatively high standard errors. The

¹³Notice that for $\Delta t = 20$ the variance minimising γ 's are found by doing (only) 100 simulations. For instance, increasing the number of simulations to 1.000 we found γ to change from 3.72 to 3.18 for $X_2 = 110$ and $\sigma_S = 0.1$, though the variance does not seem to be too dependent on the choice of γ .

Table 6.4: Estimates of the market values of the compound option under stochastic interest rates (model 1) using importance sampling (IS) and the control variate method (CV) for different number of time discretisations, Δt , of the time interval $[0, T_2]$. Estimates of the standard errors are reported in parentheses below the estimates of the market values, the γ 's are reported in square brackets below the estimates of the standard errors, and the \hat{b} 's are reported in curly brackets (100.000 simulations). Exact simulation with the control variate method (ESCV) is reported as a benchmark (1.000.000 simulations).

		ESCV	IS		CV	
			$\Delta t = 10$	$\Delta t = 20$	$\Delta t = 10$	$\Delta t = 20$
$X_2 = 110$	$\sigma_S = 0.1$	0.15582 (0.00001)	0.15985 (0.00052) [3.08]	0.15846 (0.00059) [3.72]	0.15560 (0.00024) {1.00}	0.15570 (0.00024) {1.00}
	$\sigma_S = 0.2$	2.04481 (0.00015)	2.06016 (0.00421) [2.20]	2.05215 (0.00443) [2.59]	2.04460 (0.00048) {1.00}	2.04429 (0.00048) {1.00}
	$\sigma_S = 0.3$	5.08585 (0.00017)	5.10426 (0.00863) [2.00]	5.09196 (0.00886) [2.22]	5.08609 (0.00055) {1.00}	5.08589 (0.00055) {1.00}
	$\sigma_S = 0.1$	1.79074 (0.00018)	1.80739 (0.00360) [1.85]	1.79989 (0.00378) [2.12]	1.79137 (0.00056) {1.00}	1.79072 (0.00056) {1.00}
	$\sigma_S = 0.2$	4.89696 (0.00018)	4.91629 (0.00786) [1.75]	4.90492 (0.00801) [1.91]	4.89777 (0.00058) {1.00}	4.89704 (0.00058) {1.00}
	$\sigma_S = 0.3$	8.31815 (0.00018)	8.33754 (0.01199) [1.73]	8.32435 (0.01200) [1.77]	8.31850 (0.00058) {1.00}	8.31830 (0.00058) {1.00}
$X_2 = 100$	$\sigma_S = 0.1$	7.69918 (0.00014)	7.70722 (0.00735) [0.98]	7.70307 (0.00740) [1.00]	7.69989 (0.00046) {1.00}	7.69943 (0.00046) {1.00}
	$\sigma_S = 0.2$	9.95724 (0.00017)	9.97336 (0.01157) [1.33]	9.96442 (0.01158) [1.35]	9.95782 (0.00055) {1.00}	9.95762 (0.00055) {1.00}
	$\sigma_S = 0.3$	12.9591 (0.00018)	12.9770 (0.01527) [1.45]	12.9647 (0.01528) [1.48]	12.9593 (0.00056) {1.00}	12.9595 (0.00056) {1.00}
	$\sigma_S = 0.1$	7.69918 (0.00014)	7.70722 (0.00735) [0.98]	7.70307 (0.00740) [1.00]	7.69989 (0.00046) {1.00}	7.69943 (0.00046) {1.00}
	$\sigma_S = 0.2$	9.95724 (0.00017)	9.97336 (0.01157) [1.33]	9.96442 (0.01158) [1.35]	9.95782 (0.00055) {1.00}	9.95762 (0.00055) {1.00}
	$\sigma_S = 0.3$	12.9591 (0.00018)	12.9770 (0.01527) [1.45]	12.9647 (0.01528) [1.48]	12.9593 (0.00056) {1.00}	12.9595 (0.00056) {1.00}
$X_2 = 90$	$\sigma_S = 0.1$	7.69918 (0.00014)	7.70722 (0.00735) [0.98]	7.70307 (0.00740) [1.00]	7.69989 (0.00046) {1.00}	7.69943 (0.00046) {1.00}
	$\sigma_S = 0.2$	9.95724 (0.00017)	9.97336 (0.01157) [1.33]	9.96442 (0.01158) [1.35]	9.95782 (0.00055) {1.00}	9.95762 (0.00055) {1.00}
	$\sigma_S = 0.3$	12.9591 (0.00018)	12.9770 (0.01527) [1.45]	12.9647 (0.01528) [1.48]	12.9593 (0.00056) {1.00}	12.9595 (0.00056) {1.00}
	$\sigma_S = 0.1$	7.69918 (0.00014)	7.70722 (0.00735) [0.98]	7.70307 (0.00740) [1.00]	7.69989 (0.00046) {1.00}	7.69943 (0.00046) {1.00}
	$\sigma_S = 0.2$	9.95724 (0.00017)	9.97336 (0.01157) [1.33]	9.96442 (0.01158) [1.35]	9.95782 (0.00055) {1.00}	9.95762 (0.00055) {1.00}
	$\sigma_S = 0.3$	12.9591 (0.00018)	12.9770 (0.01527) [1.45]	12.9647 (0.01528) [1.48]	12.9593 (0.00056) {1.00}	12.9595 (0.00056) {1.00}

Table 6.5: Estimates of the market values of the compound option under stochastic interest rates (model 1) using standard Monte Carlo simulation (MC), importance sampling (IS), and the control variate method (CV). Estimates of the standard errors are reported in parentheses below the estimates of the market values, the γ 's are reported in square brackets below the estimates of the standard errors, and the \hat{b} 's are reported in curly brackets (10 time discretisations, 10.000 simulations). Exact simulation with the control variate method (ESCV) is reported as a benchmark (1.000.000 simulations).

		ESCV	MC	IS	CV
$X_2 = 110$	$\sigma_S = 0.1$	0.15582 (0.00001)	0.17921 (0.01040)	0.16398 (0.00167) [3.08]	0.15552 (0.00075) {1.00}
	$\sigma_S = 0.2$	2.04481 (0.00015)	2.09556 (0.05389)	2.07996 (0.01335) [2.20]	2.04470 (0.00151) {1.00}
	$\sigma_S = 0.3$	5.08585 (0.00017)	5.18360 (0.10659)	5.13849 (0.02723) [2.00]	5.08313 (0.00174) {1.00}
$X_2 = 100$	$\sigma_S = 0.1$	1.79074 (0.00018)	1.83029 (0.03390)	1.82384 (0.01140) [1.85]	1.78758 (0.00179) {1.00}
	$\sigma_S = 0.2$	4.89696 (0.00018)	4.98367 (0.08132)	4.94785 (0.02479) [1.75]	4.89365 (0.00183) {1.00}
	$\sigma_S = 0.3$	8.31815 (0.00018)	8.45570 (0.13332)	8.38234 (0.03770) [1.73]	8.31443 (0.00185) {1.00}
$X_2 = 90$	$\sigma_S = 0.1$	7.69918 (0.00014)	7.80066 (0.05965)	7.74471 (0.02200) [0.98]	7.69650 (0.00144) {1.00}
	$\sigma_S = 0.2$	9.95724 (0.00017)	10.1085 (0.10882)	10.0230 (0.03623) [1.33]	9.95340 (0.00173) {1.00}
	$\sigma_S = 0.3$	12.9591 (0.00018)	13.1558 (0.16007)	13.0388 (0.04782) [1.45]	12.9551 (0.00177) {1.00}

Table 6.6: Estimates of the market values of the compound option under stochastic interest rates (model 2) using the control variate method for different number of time discretisations, Δt , of the time interval $[0, T_2]$. A standard call option in model 1 is used as a control variate in the estimation of the market value of the underlying call option (1.000 simulations). The control variate of section C.1 is used as a control variate in the estimation of the market value of the compound option (100 simulations). Estimates of the standard errors are reported in parentheses below the estimates of the market values. $b = 1$.

		$\Delta t = 4$	$\Delta t = 10$	$\Delta t = 20$
$X_2 = 110$	$\sigma_S = 0.1$	0.24403 (0.03094)	0.25087 (0.03717)	0.21103 (0.02413)
	$\sigma_S = 0.2$	2.34732 (0.05440)	2.32260 (0.05081)	2.29933 (0.04010)
	$\sigma_S = 0.3$	5.50743 (0.05644)	5.48185 (0.05132)	5.48741 (0.04989)
$X_2 = 100$	$\sigma_S = 0.1$	2.21802 (0.06424)	2.17076 (0.05721)	2.17252 (0.05561)
	$\sigma_S = 0.2$	5.42402 (0.06153)	5.40027 (0.05908)	5.37470 (0.05501)
	$\sigma_S = 0.3$	8.86248 (0.05711)	8.85306 (0.05683)	8.84663 (0.05118)
$X_2 = 90$	$\sigma_S = 0.1$	8.42344 (0.07390)	8.37126 (0.07860)	8.41119 (0.06434)
	$\sigma_S = 0.2$	10.6477 (0.06213)	10.6396 (0.06419)	10.5990 (0.05522)
	$\sigma_S = 0.3$	13.6285 (0.05666)	13.6240 (0.05678)	13.5834 (0.05013)

size of the standard errors also makes it impossible to detect any discretisation bias.

6.8 Conclusions

We have in this chapter addressed the problem of valuing compound options by Monte Carlo simulation within a Heath, Jarrow, and Morton framework. A general approach for an arbitrage-free discretisation of the forward rates has been applied. Known problems with discretisation bias have been detected for some of the valuation approaches, but methods where this problem seemed to be eliminated have also been used. At the same time, we have also been able to find estimators that are efficient in terms of a low standard error and that require a low computational time. In addition, we have also presented a very fast and unbiased approach for simulation within a Gaussian Heath, Jarrow, and Morton framework. This approach does not require any discretisation of the stochastic differential equations describing

the economy and we can therefore use exact simulation.

The results regarding simulation within the Heath, Jarrow, and Morton framework have of course a much wider application than just for the pricing of compound options.

A problem with numerical evaluation of compound options that has not been considered in this chapter is that the estimator for the true market value of the compound option is biased, even when using exact sampling. As before, let $\hat{\pi}_{T_1}$ be the estimator for π_{T_1} . We know that $(\hat{\pi}_{T_1} - X_1)^+$ is a convex function. Suppose that $E[\hat{\pi}_{T_1} | \mathcal{F}_{T_1}] = \pi_{T_1}$. We then have that

$$\begin{aligned} E[(\hat{\pi}_{T_1} - X_1)^+] &= E\left[E[(\hat{\pi}_{T_1} - X_1)^+ | \mathcal{F}_{T_1}]\right] \geq \\ &E\left[(E[\hat{\pi}_{T_1} | \mathcal{F}_{T_1}] - X_1)^+\right] = E[(\pi_{T_1} - X_1)^+], \end{aligned}$$

i.e., $E[(\hat{\pi}_{T_1} - X_1)^+] \geq E[(\pi_{T_1} - X_1)^+]$. The result follows from Jensen's inequality. However, we have not been able to conclude that this bias has been significant in the estimates that have been presented in this chapter.

Based on the promising results in section 6.4, an interesting topic for future research is to explore the effectiveness of stratified sampling in a setting with stochastic interest rates. Another interesting topic may be to combine importance sampling with exact simulation. This can be of great importance if we do not have a highly correlated control variate. Another natural extension is to explore the effectiveness of using quasi Monte Carlo, or low discrepancy sequences, in the pricing of (compound) options under stochastic interest rates.

Appendix A

Proposition 3.4 and 3.5

A.1 Useful Relationships for Section A.2 and A.3

In this section we supply some relations that turn out to be useful for the calculations done in section A.2 and A.3.

First we can notice that under the equivalent martingale measure Q , the bond price can be expressed in the following way

$$P(t, T) = P(0, T) + \int_0^t r_v P(v, T) dv + \int_0^t \sigma_p(t) P(v, T) dW_v,$$

where

$$\sigma_p(t) = - \int_t^T \sigma_f(t, u) du.$$

In the following we let $\sigma_p(t) = - \int_t^{t_1} \sigma_f(t, u) du$ and $\sigma_Y(t) = - \int_t^{t_2} \sigma_f(t, u) du$.

$$\begin{aligned} \frac{\partial \sigma_{\beta_1}^2}{\partial t} &= - \left(\int_t^{t_1} \sigma_f(t, u) du \right)^2 = - \sigma_p^2(t), \\ \frac{\partial \sigma_{\beta_2}^2}{\partial t} &= - \left(\int_{t_1}^{t_2} \sigma_f(t, u) du \right)^2 = - (-\sigma_Y(t) + \sigma_p(t))^2, \\ \frac{\partial c_{1,2}}{\partial t} &= - \left(\int_t^{t_1} \sigma_f(t, u) du \right) \left(\int_{t_1}^{t_2} \sigma_f(t, u) du \right) \\ &= - \sigma_p(t) (\sigma_Y(t) - \sigma_p(t)), \\ \frac{\partial k_1}{\partial t} &= - \sigma_S(t) \int_t^{t_1} \sigma_f(t, u) du = \sigma_S(t) \sigma_p(t), \\ \frac{\partial k_3}{\partial t} &= - \sigma_S(t) \int_{t_1}^{t_2} \sigma_f(t, u) du = \sigma_S(t) (\sigma_Y(t) - \sigma_p(t)), \end{aligned}$$

$$\begin{aligned}
\frac{\partial \bar{\rho} \sigma_{\delta_1} \sigma_{\delta_2}}{\partial t} &= \frac{\partial (c_{1,2} + k_2)}{\partial t} = (\sigma_S(t) - \sigma_P(t))(\sigma_Y(t) - \sigma_P(t)), \\
\frac{\partial \sigma_{\delta_1}^2}{\partial t} &= -\sigma_S^2(t) - \sigma_S(t) \int_t^{t_1} \sigma_f(t, u) du - \left(\int_t^{t_1} \sigma_f(t, u) du \right)^2 \\
&= -(\sigma_S(t) - \sigma_P(t))^2, \\
\frac{\partial \sigma_{\delta_2}^2}{\partial t} &= -(\sigma_Y(t) - \sigma_P(t))^2, \\
\frac{\partial \Phi(x)}{\partial x} &= \phi(x), \\
\phi(d_2) &= \phi(d_1) \frac{S_t}{e^{g_1} P(t, t_1)}, \\
e^{-\bar{a}_3^2/2} &= e^{-\bar{a}_1^2/2} \frac{S_t}{e^g P(t, t_1)}, \\
e^{-\bar{a}_4^2/2} &= e^{-\bar{a}_2^2/2} e^{-g_2 - \rho \sigma_{\delta_1} \sigma_{\delta_2}} \frac{S_t}{P(t, t_1)}, \\
e^{-\bar{b}_2^2/2} &= e^{-\bar{b}_1^2/2} \frac{1}{F(t, t_1, t_2)} e^{-g_2 + \rho \sigma_{\delta_1} \sigma_{\delta_2}}, \\
e^{-\bar{b}_4^2/2} &= e^{-\bar{b}_3^2/2} \frac{1}{e^{g_2} F(t, t_1, t_2)}.
\end{aligned}$$

For two variables, x and y , and two continuously differentiable functions, w and z , the cumulative bivariate normal probability distribution is given by

$$\Phi(w; z; \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^w \int_{-\infty}^z e^{-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}} dx dy.$$

From this we have that

$$\begin{aligned}
\frac{\partial \Phi(w; z; \rho)}{\partial w} &= \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^z e^{-\frac{(x^2 - 2\rho xw + w^2)}{2(1-\rho^2)}} dx \\
&= \frac{e^{-w^2/2}}{\sqrt{2\pi(1-\rho^2)}} \Phi\left(\frac{z - \rho w}{\sqrt{1-\rho^2}}\right).
\end{aligned}$$

A.2 Proof of Proposition 3.5

In this section we prove that the hedging strategy for the maturity guarantee on the stock return in the Amin and Jarrow economy is self-financing and that it hedges the market value of the guarantee.

Both $a_t(\delta) = a(t, S_t, P(t, t_1))$ ($= \frac{a_t^2(\delta)}{R^2(\delta)}$) and $b_t(\delta) = b(t, S_t, P(t, t_1))$ ($= \frac{b_t^2(\delta)}{R^2(\delta)}$) are twice continuously differentiable on $[0, \infty) \times \mathbb{R} \times \mathbb{R}$. Since S_t and $P(t, t_1)$ are Itô processes, it follows that $a_t(\delta)$ and $b_t(\delta)$ also are Itô processes (see e.g., Øksendal (1995)). Define μ_a , σ_a and μ_b , σ_b as the drift term and

the diffusion term of a_t and b_t , respectively. Applying Itô's lemma to the hedging strategy now yields (without loss of generality we have set $t_0 = 0$, and for simplicity we have written $a_t(\delta)$, $b_t(\delta)$, and $\pi_t(\delta)$ as a_t , b_t , and π_t , respectively)

$$\begin{aligned} a_t S_t + b_t P(t, t_1) &= a_0 S_0 + b_0 P(0, t_1) + \int_0^t a_v dS_v + \int_0^t b_v dP(v, t_1) \\ &+ \int_0^t S_v da_v + \int_0^t P(v, t_1) db_v + \int_0^t da_v dS_v + \int_0^t db_v dP(v, t_1). \end{aligned} \quad (\text{A.1})$$

Condition A.1. *For the hedging strategy for the maturity guarantee to be self-financing, the sum of the last line in (A.1) has to equal zero (see e.g., Duffie (1988)).*

By Itô's lemma the market value of the guarantee can be expressed as

$$\begin{aligned} \pi_t &= \pi_0 + \int_0^t \left[\frac{\partial \pi_v}{\partial v} + \frac{\partial \pi_v}{\partial S_v} r_v S_v + \frac{1}{2} \frac{\partial^2 \pi_v}{\partial S_v^2} S_v^2 \sigma_S^2(v) \right. \\ &+ \frac{\partial \pi_v}{\partial P(v, t_1)} r_v P(v, t_1) + \frac{1}{2} \frac{\partial^2 \pi_v}{\partial P(v, t_1)^2} P(v, t_1)^2 \sigma_P^2(v) \\ &+ \left. \frac{\partial^2 \pi_v}{\partial S_v \partial P(v, t_1)} S_v P(v, t_1) \sigma_S(v) \sigma_P(v) \right] dv \\ &+ \int_0^t \left[\frac{\partial \pi_v}{\partial S_v} S_v \sigma_S(v) + \frac{\partial \pi_v}{\partial P(v, t_1)} P(v, t_1) \sigma_P(v) \right] dW_v. \end{aligned} \quad (\text{A.2})$$

From Condition A.1 we get the following expression for the market value of the hedging strategy

$$\begin{aligned} \theta_t &= a_0 S_0 + b_0 P(0, t_1) + \int_0^t \left(a_v r_v S_v + b_v r_v P(v, t_1) \right) dv \\ &+ \int_0^t \left(a_v S_v \sigma_S(v) + b_v P(v, t_1) \sigma_P(v) \right) dW_v. \end{aligned} \quad (\text{A.3})$$

Condition A.2. *By the unique decomposition property for Itô processes, we know that (A.2) and (A.3) must have the same drift and diffusion term if the hedging strategy is to hedge the market value of the guarantee, i.e., (A.2) - (A.3) = 0, for all $t \in [0, 1]$ almost surely.*

We can now state a proof for that the hedging strategy is self-financing and hedges the market value of the guarantee.

Proof. Calculate the derivatives and check that Condition A.1 and A.2 are satisfied. \square

A.3 Proof of Proposition 3.4

In this section we prove that the hedging strategy in Proposition 3.4 is self-financing and that it hedges the market value of the guarantee.

Both $a_t^1(\delta) = a^1(t, S_t, P(t, t_1), P(t, t_2))$, $b_t^1(\delta) = b^1(t, S_t, P(t, t_1), P(t, t_2))$ and $y_t^1(\delta) = y^1(t, S_t, P(t, t_1), P(t, t_2))$ are twice continuously differentiable on $[0, \infty) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, and are thereby Itô processes.

An application of Itô's lemma gives the following expression for the market value of the hedging strategy (without loss of generality we have set $t_0 = 0$, and for simplicity we have written $a_t^1(\delta)$, $b_t^1(\delta)$, $y_t^1(\delta)$, and $\pi_t^1(\delta)$ as a_t^1 , b_t^1 , y_t^1 , and π_t^1 , respectively)

$$\begin{aligned}
& a_t^1 S_t + b_t^1 P(t, t_1) + y_t^1 P(t, t_2) \\
&= a_0^1 S_0 + b_0^1 P(0, t_1) + y_0^1 P(0, t_2) \\
&+ \int_0^t a_v^1 dS_v + \int_0^t b_v^1 dP(v, t_1) + \int_0^t y_v^1 dP(v, t_2) \quad (\text{A.4}) \\
&+ \int_0^t S_v da_v^1 + \int_0^t P(v, t_1) db_v^1 + \int_0^t P(v, t_2) dy_v^1 \\
&+ \int_0^t da_v^1 dS_v + \int_0^t db_v^1 dP(v, t_1) + \int_0^t dy_v^1 dP(v, t_2).
\end{aligned}$$

Condition A.3. For the hedging strategy in Proposition 3.4 to be self-financing, the sum of the last two lines in (A.4) has to equal zero.

An application of Itô's lemma yields the following expression for the market value of the guarantee

$$\begin{aligned}
\pi_t^1 &= \pi_0^1 + \int_0^t \left[\frac{\partial \pi_0^1}{\partial v} dv + \frac{\partial \pi_0^1}{\partial S_v} + \frac{1}{2} \frac{\partial^2 \pi_v}{\partial S_v^2} (dS_v)^2 + \frac{\partial \pi_v}{\partial P(v, t_1)} dP(v, t_1) \right. \\
&+ \frac{1}{2} \frac{\partial^2 \pi_v^1}{\partial P(v, t_1)^2} (dP(v, t_1))^2 + \frac{\partial \pi_v^1}{\partial P(v, t_2)} dP(v, t_2) \\
&+ \frac{1}{2} \frac{\partial^2 \pi_v^1}{\partial P(v, t_2)^2} (dP(v, t_2))^2 + \frac{\partial^2 \pi_v^1}{\partial S_v \partial P(v, t_1)} dS_v dP(v, t_1) \quad (\text{A.5}) \\
&+ \frac{\partial^2 \pi_v^1}{\partial S_v \partial P(v, t_2)} dS_v dP(v, t_2) \\
&\left. + \frac{\partial^2 \pi_v^1}{\partial P(v, t_1) \partial P(v, t_2)} dP(v, t_1) dP(v, t_2) \right].
\end{aligned}$$

From Condition A.3 we get the following expression for the market value of the hedging strategy

$$\begin{aligned} \theta_t &= a_0^1 S_0 + b_0^1 P(0, t_1) + y_0^1 P(0, t_2) \\ &\quad + \int_0^t \left[a_v^1 dS_v + b_v^1 dP(v, t_1) + y_v^1 dP(v, t_2) \right]. \end{aligned} \quad (\text{A.6})$$

Condition A.4. *By the unique decomposition property for Itô processes, we know that (A.5) and (A.6) must have the same drift and diffusion term if the hedging strategy is to hedge the market value of the guarantee, i.e., (A.5) – (A.6) = 0, for all $t \in [0, 1]$ almost surely.*

We can now prove that the hedging strategy in Proposition 3.4 is self-financing and that it hedges the market value of the guarantee.

Proof. Calculate the derivatives and check that Condition A.3 and A.4 are satisfied. \square

Appendix B

Proposition 4.8 and 4.9

B.1 Abbreviations in Proposition 4.8

$$\begin{aligned}
 \zeta_1 = & -\ln F(t_m, t_{N-1}, t_N) + \frac{1}{2}\sigma_{\delta_{t_N-t_{N-1}}}^2 \\
 & -\frac{1}{\tau}\left\{-\ln P(t_m, t_N) + \frac{1}{2}\sigma_{\beta_{t_N}}^2 - \frac{1}{2}\int_{t_m}^{t_N}\sigma_{S^2}^2(v)dv\right. \\
 & -\int_{t_m}^{t_{N-1}}\int_v^{t_{N-1}}\sigma_f(v, u)du\int_v^{t_N}\sigma_f(v, u)dudv \\
 & +\int_{t_{N-1}}^{t_N}\sigma_{S^1}(v)\sigma_{S^2}(v)dv + \int_{t_{N-1}}^{t_N}\sigma_{S^1}(v)\int_v^{t_N}\sigma_f(v, u)dudv \\
 & \left. -\int_{t_m}^{t_{N-1}}\sigma_{S^2}(v)\int_v^{t_{N-1}}\sigma_f(v, u)dudv\right\},
 \end{aligned}$$

$$\begin{aligned}
 \zeta_2 = & \frac{1}{\tau}\left\{-\ln P(t_m, t_N) + \left(\frac{1}{\tau} - \frac{1}{2}\right)\sigma_{\delta_{t_N}}^2\right\} \\
 & -\left(-\ln F(t_m, t_{N-1}, t_N) + \frac{1}{2}\sigma_{\beta_{t_N-t_{N-1}}}^2 + c_{t_{N-1}-t_m, t_N-t_{N-1}}\right. \\
 & +\left(\frac{1}{\tau} - 1\right)\left\{\int_{t_m}^{t_{N-1}}\int_{t_{N-1}}^{t_N}\sigma_f(v, u)du\int_v^{t_N}\sigma_f(v, u)dudv\right. \\
 & +\left.\int_{t_{N-1}}^{t_N}\left(\int_v^{t_N}\sigma_f(v, u)du\right)^2dv + \int_{t_{N-1}}^{t_N}\sigma_{S^1}(v)\int_v^{t_N}\sigma_f(v, u)dudv\right\} \\
 & +\frac{1}{\tau}\left\{\int_{t_m}^{t_{N-1}}\sigma_{S^2}(v)\int_{t_{N-1}}^{t_N}\sigma_f(v, u)dudv + \int_{t_{N-1}}^{t_N}\sigma_{S^2}(v)\int_v^{t_N}\sigma_f(v, u)dudv\right. \\
 & \left. +\int_{t_{N-1}}^{t_N}\sigma_{S^1}(v)\sigma_{S^2}(v)dv\right\} - \frac{1}{2}\int_{t_{N-1}}^{t_N}\sigma_{S^1}^2(v)dv,
 \end{aligned}$$

and

$$\begin{aligned}
\hat{\sigma}^2 &= \sigma_{\delta_{t_N-t_{N-1}}}^2 + \frac{1}{\tau^2} \sigma_{\delta_{t_N}^2}^2 \\
&+ \frac{2}{\tau} \left\{ - \int_{t_m}^{t_{N-1}} \sigma_{S^2}(v) \int_{t_{N-1}}^{t_N} \sigma_f(v, u) dudv \right. \\
&- \int_{t_m}^{t_{N-1}} \int_{t_{N-1}}^{t_N} \sigma_f(v, u) du \int_v^{t_N} \sigma_f(v, u) dudv \\
&- \int_{t_{N-1}}^{t_N} \left(\int_v^{t_N} \sigma_f(v, u) du \right)^2 dv - \int_{t_{N-1}}^{t_N} \sigma_{S^1}(v) \int_v^{t_N} \sigma_f(v, u) dudv \\
&\left. - \int_{t_{N-1}}^{t_N} \sigma_{S^2}(v) \int_v^{t_N} \sigma_f(v, u) dudv - \int_{t_{N-1}}^{t_N} \sigma_{S^1}(v) \sigma_{S^2}(v) dv \right\}.
\end{aligned}$$

B.2 Proof of Proposition 4.8

The time t_m market value of the guarantee is given by

$$\begin{aligned}
\pi_{t_m}^{(4.8)} &= E_Q \left[e^{-\int_{t_m}^{t_N} r_v dv} \max(e^{\delta_{t_N-t_{N-1}}^1}, e^{\frac{1}{\tau} \delta_{t_N}^2}) \right] \\
&= E_Q \left[e^{-\int_{t_m}^{t_N} r_v dv} e^{\delta_{t_N-t_{N-1}}^1} 1_A \right] + E_Q \left[e^{-\int_{t_m}^{t_N} r_v dv} e^{\frac{1}{\tau} \delta_{t_N}^2} 1_{\bar{A}} \right],
\end{aligned}$$

where $A = \{\delta_{t_N-t_{N-1}} > \frac{1}{\tau} \delta_{t_N}\}$ and \bar{A} is the complement to A . To evaluate the first expectation, we use the Radon-Nikodym derivative

$$\begin{aligned}
\frac{dQ_1}{dQ} &= e^{-\frac{1}{2}(\sigma_{\beta_{t_{N-1}}} + \int_{t_{N-1}}^{t_N} \sigma_{S^1}^2(v) dv)} \\
&\quad e^{-\int_{t_m}^{t_{N-1}} \int_v^{t_{N-1}} \sigma_f(v, u) dudW_v + \int_{t_{N-1}}^{t_N} \sigma_{S^1}(v) dW_v}.
\end{aligned}$$

It then follows (after some algebra) that

$$E_Q \left[e^{-\int_0^{t_N} r_v dv} e^{\delta_{t_N-t_{N-1}}^1} 1_A \right] = P(t_m, t_{N-1}) \Phi(d_9).$$

To evaluate the second expectation, we use the Radon-Nikodym derivative

$$\begin{aligned}
\frac{dQ_2}{dQ} &= \frac{e^{\bar{\delta}_{t_N}^2}}{E_Q(e^{\bar{\delta}_{t_N}^2})} \\
&= e^{-\frac{1}{2}(\frac{1}{\tau}-1)^2 \sigma_{\beta_{t_N}}^2 - \frac{1}{2\tau^2} \int_{t_m}^{t_N} \sigma_{S^2}^2(v) dv - \frac{1}{\tau}(\frac{1}{\tau}-1) \int_{t_m}^{t_N} \sigma_{S^2}(v) \int_v^{t_N} \sigma_f(v, u) dudv} \\
&\quad e^{(\frac{1}{\tau}-1) \int_{t_m}^{t_N} \int_v^{t_N} \sigma_f(v, u) dudW_v + \frac{1}{\tau} \int_{t_m}^{t_N} \sigma_{S^2}(v) dW_v},
\end{aligned}$$

where $\bar{\delta}_{t_N}^2 = \frac{1}{\tau} \delta_{t_N}^2 - \int_{t_m}^{t_N} r_v dv$. Some algebra then leads to the solution of the second expectation

$$E_Q \left[e^{-\int_{t_m}^{t_N} r_v dv} e^{\frac{1}{\tau} \delta_{t_N}^2} 1_{\bar{A}} \right] = e^{\kappa^2} \Phi(d_{10}),$$

and this completes the proof.

B.3 Abbreviations in Proposition 4.9

In this section we write out the remaining expressions of Proposition 4.9.

$$\kappa_3 = \ln P(t_0, t_{N-1}),$$

$$\begin{aligned} \kappa_4 = & \ln P(t_0, t_{N-1}) + \ln F(t_0, TN, t_{N+1}) - c_{t_{N-1}-t_0, t_N-t_{N-1}} \\ & + \int_{t_0}^{t_{N-1}} \int_v^{t_{N-1}} \sigma_f(v, u) du \int_{t_N}^{t_{N+1}} \sigma_f(v, u) dudv \\ & - \int_{t_{N-1}}^{t_N} \sigma_{S^1}(v) \int_{t_N}^{t_{N+1}} \sigma_f(v, u) dudv \\ & + \frac{1}{\tau_2} \left(- \int_{t_0}^{t_1} \int_{t_1}^{t_{N+1}} \sigma_f(v, u) du \int_v^{t_{N-1}} dudv \right. \\ & - \int_{t_1}^{t_{N-1}} \int_v^{t_{N-1}} \sigma_f(v, u) du \int_v^{t_{N+1}} \sigma_f(v, u) dudv \\ & - \int_{t_1}^{t_{N-1}} \sigma_{S^2}(v) \int_v^{t_{N-1}} \sigma_f(v, u) dudv - \int_{t_1}^{t_N} \sigma_{S^2}(v) \int_{t_N}^{t_{N+1}} \sigma_f(v, u) dudv \\ & - \int_{t_0}^{t_1} \int_{t_N}^{t_{N+1}} \sigma_f(v, u) du \int_{t_1}^{t_{N+1}} \sigma_f(v, u) dudv \\ & - \int_{t_1}^{t_N} \int_{t_N}^{t_{N+1}} \sigma_f(v, u) du \int_v^{t_{N+1}} \sigma_f(v, u) dudv \\ & - \int_{t_1}^{t_N} \sigma_{S^2}(v) \int_{t_N}^{t_{N+1}} \sigma_f(v, u) dudv - \int_{t_N}^{t_{N+1}} \left(\int_v^{t_{N+1}} \sigma_f(v, u) du \right)^2 dv \\ & - \int_{t_N}^{t_{N+1}} \sigma_{S^2}(v) \int_v^{t_{N+1}} \sigma_f(v, u) dudv \\ & + \int_{t_{N-1}}^{t_N} \sigma_{S^1}(v) \int_v^{t_{N+1}} \sigma_f(v, u) dudv + \int_{t_{N-1}}^{t_N} \sigma_{S^1}(v) \sigma_{S^2}(v) dv \\ & - \ln F(t_0, t_1, t_{N+1}) + \frac{1}{2} \sigma_{\beta, t_{N+1}-t_1}^2 + c_{t_1-t_0, t_{N+1}-t_1} \\ & \left. + \frac{1}{2\tau_2^2} \sigma_{\delta, t_{N+1}-t_1}^2 \right), \end{aligned}$$

$$\kappa_5 = \left(\frac{1}{\tau_1} - 1 \right) \left(- \ln P(t_0, t_N) + \frac{1}{2\tau_1} \sigma_{\delta, t_N}^2 \right),$$

$$\begin{aligned} \kappa_6 = & \ln P(t_0, t_{N+1}) + \frac{1}{\tau_1} \left(- \ln P(t_0, t_N) + \frac{1}{2} \sigma_{\beta, t_N}^2 - \frac{1}{2} \int_{t_0}^{t_N} \sigma_{S^2}^2(v) dv \right) \\ & + \frac{1}{\tau_2} \left(- \ln F(t_0, t_1, t_{N+1}) + \frac{1}{2} \sigma_{\beta, t_{N+1}-t_1}^2 + c_{t_1-t_0, t_{N+1}-t_1} \right. \\ & - \frac{1}{2} \int_{t_1}^{t_{N+1}} \sigma_{S^2}^2(v) dv \left. + \frac{1}{2\tau_1^2} \left(\int_{t_0}^{t_N} \left(\int_v^{t_N} \sigma_f(v, u) du \right)^2 dv \right. \right. \\ & + \int_{t_0}^{t_N} \sigma_{S^2}^2(v) dv + \int_{t_0}^{t_1} \left(\int_{t_1}^{t_{N+1}} \sigma_f(v, u) du \right)^2 dv \\ & \left. \left. + \int_{t_1}^{t_{N+1}} \left(\int_v^{t_{N+1}} \sigma_f(v, u) du \right)^2 dv + \int_{t_1}^{t_{N+1}} \sigma_{S^2}^2(v) dv \right) \right) \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{\tau_1} \left(\int_{t_0}^{t^N} \int_v^{t^{N+1}} \sigma_f(v, u) du \int_v^{t^N} \sigma_f(v, u) dudv \right. \\
& + \left. \int_{t_0}^{t^N} \sigma_{S^2}(v) \int_v^{t^{N+1}} \sigma_f(v, u) dudv \right) \\
& -\frac{1}{\tau_2} \left(\int_{t_0}^{t^1} \int_v^{t^{N+1}} \sigma_f(v, u) du \int_{t_1}^{t^{N+1}} \sigma_f(v, u) dudv \right. \\
& + \left. \int_{t_1}^{t^{N+1}} \left(\int_v^{t^{N+1}} \sigma_f(v, u) du \right)^2 dv + \int_{t_1}^{t^{N+1}} \sigma_{S^2}(v) \int_v^{t^{N+1}} \sigma_f(v, u) dudv \right) \\
& + \frac{1}{\tau_1^2} \int_{t_0}^{t^N} \sigma_{S^2}(v) \int_v^{t^N} \sigma_f(v, u) dudv \\
& + \frac{1}{\tau_1 \tau_2} \left(\int_{t_0}^{t^1} \int_v^{t^N} \sigma_f(v, u) du \int_{t_1}^{t^{N+1}} \sigma_f(v, u) dudv \right. \\
& + \left. \int_{t_1}^{t^N} \int_v^{t^N} \sigma_f(v, u) du \int_v^{t^{N+1}} \sigma_f(v, u) dudv + \int_{t_1}^{t^N} \sigma_{S^2}(v) \int_v^{t^N} \sigma_f(v, u) dudv \right. \\
& + \left. \int_{t_0}^{t^1} \sigma_{S^2}(v) \int_{t_1}^{t^{N+1}} \sigma_f(v, u) dudv + \int_{t_1}^{t^N} \sigma_{S^2}(v) \int_v^{t^{N+1}} \sigma_f(v, u) dudv \right. \\
& \left. + \int_{t_1}^{t^N} \sigma_{S^2}^2(v) dv \right) + \frac{1}{\tau_2^2} \int_{t_1}^{t^{N+1}} \sigma_{S^2}(v) \int_v^{t^{N+1}} \sigma_f(v, u) dudv,
\end{aligned}$$

$$\begin{aligned}
a_1 &= \frac{\zeta_1^a}{\sigma_a}, & b_1 &= \frac{\zeta_1^b}{\sigma_b}, & a_2 &= \frac{\zeta_2^a}{\sigma_a}, & b_2 &= \frac{\zeta_2^b}{\sigma_b}, & a_3 &= \frac{\zeta_3^a}{\sigma_a}, \\
b_3 &= \frac{\zeta_3^b}{\sigma_b}, & a_4 &= \frac{\zeta_4^a}{\sigma_a}, & b_4 &= \frac{\zeta_4^b}{\sigma_b}, & \rho &= \frac{c}{\sigma_a \sigma_b},
\end{aligned}$$

$$\begin{aligned}
\sigma_a^2 &= \int_{t_0}^{t^{N-1}} \left(\int_{t_{N-1}}^{t^N} \sigma_f(v, u) du \right)^2 dv + \int_{t_{N-1}}^{t^N} \left(\int_v^{t^N} \sigma_f(v, u) du \right)^2 dv + \int_{t_{N-1}}^{t^N} \sigma_{S^1}^2(v) dv \\
& + \frac{1}{\tau_1^2} \left(\int_{t_0}^{t^N} \left(\int_v^{t^N} \sigma_f(v, u) du \right)^2 dv + \int_{t_0}^{t^N} \sigma_{S^2}^2(v) dv \right) \\
& - \frac{2}{\tau_1} \left(\int_{t_0}^{t^{N-1}} \int_{t_{N-1}}^{t^N} \sigma_f(v, u) du \int_v^{t^N} \sigma_f(v, u) dudv \right. \\
& + \left. \int_{t_0}^{t^{N-1}} \sigma_{S^2}(v) \int_{t_{N-1}}^{t^N} \sigma_f(v, u) dudv + \int_{t_{N-1}}^{t^N} \left(\int_v^{t^N} \sigma_f(v, u) du \right)^2 dv \right. \\
& + \left. \int_{t_{N-1}}^{t^N} \sigma_{S^2}(v) \int_v^{t^N} \sigma_f(v, u) dudv + \int_{t_{N-1}}^{t^N} \sigma_{S^1}(v) \int_v^{t^N} \sigma_f(v, u) dudv \right. \\
& + \left. \int_{t_{N-1}}^{t^N} \sigma_{S^1}(v) \sigma_{S^2}(v) dv \right) \\
& + 2 \int_{t_{N-1}}^{t^N} \sigma_{S^1}(v) \int_v^{t^N} \sigma_f(v, u) dudv + \frac{2}{\tau_1^2} \int_{t_0}^{t^N} \sigma_{S^2}(v) \int_v^{t^N} \sigma_f(v, u) dudv,
\end{aligned}$$

$$\begin{aligned}
\sigma_b^2 &= \int_{t_0}^{t^N} \left(\int_{t^N}^{t^{N+1}} \sigma_f(v, u) du \right)^2 dv + \int_{t^N}^{t^{N+1}} \left(\int_v^{t^{N+1}} \sigma_f(v, u) du \right)^2 dv \\
& + \int_{t^N}^{t^{N+1}} \sigma_{S^1}^2(v) dv + \frac{1}{\tau_2^2} \left(\int_{t_0}^{t^1} \left(\int_{t_1}^{t^{N+1}} \sigma_f(v, u) du \right)^2 dv \right.
\end{aligned}$$

$$\begin{aligned}
& + \int_{t_1}^{t_{N+1}} \left(\int_v^{t_{N+1}} \sigma_f(v, u) du \right)^2 dv + \int_{t_1}^{t_{N+1}} \sigma_{S^2}^2(v) dv \\
& - \frac{2}{\tau_2} \left(\int_{t_0}^{t_1} \int_{t_N}^{t_{N+1}} \sigma_f(v, u) du \int_{t_1}^{t_{N+1}} \sigma_f(v, u) dudv \right. \\
& + \int_{t_1}^{t_N} \int_{t_N}^{t_{N+1}} \sigma_f(v, u) du \int_v^{t_{N+1}} \sigma_f(v, u) dudv \\
& + \left. \int_{t_1}^{t_N} \sigma_{S^2}(v) \int_{t_N}^{t_{N+1}} \sigma_f(v, u) dudv \right) + 2 \int_{t_N}^{t_{N+1}} \sigma_{S^1}(v) \int_v^{t_{N+1}} \sigma_f(v, u) dudv \\
& - \frac{2}{\tau_2} \left(\int_{t_N}^{t_{N+1}} \left(\int_v^{t_{N+1}} \sigma_f(v, u) du \right)^2 dv + \int_{t_N}^{t_{N+1}} \sigma_{S^2}(v) \int_v^{t_{N+1}} \sigma_f(v, u) dudv \right. \\
& + \int_{t_N}^{t_{N+1}} \sigma_{S^1}(v) \int_v^{t_{N+1}} \sigma_f(v, u) dudv + \left. \int_{t_N}^{t_{N+1}} \sigma_{S^1}(v) \sigma_{S^2}(v) \right) \\
& + \frac{2}{\tau_2^2} \int_{t_1}^{t_{N+1}} \sigma_{S^2}(v) \int_v^{t_{N+1}} \sigma_f(v, u) dudv,
\end{aligned}$$

$$\begin{aligned}
c = & \int_{t_0}^{t_{N-1}} \int_{t_{N-1}}^{t_N} \sigma_f(v, u) du \int_{t_N}^{t_{N+1}} \sigma_f(v, u) dudv \\
& + \int_{t_{N-1}}^{t_N} \int_{t_v}^{t_N} \sigma_f(v, u) du \int_{t_N}^{t_{N+1}} \sigma_f(v, u) dudv \\
& + \int_{t_{N-1}}^{t_N} \sigma_{S^1}(v) \int_{t_N}^{t_{N+1}} \sigma_f(v, u) dudv \\
& - \frac{1}{\tau_1} \left(\int_{t_0}^{t_N} \int_v^{t_N} \sigma_f(v, u) du \int_{t_N}^{t_{N+1}} \sigma_f(v, u) dudv \right. \\
& + \left. \int_{t_0}^{t_N} \sigma_{S^2}(v) \int_{t_N}^{t_{N+1}} \sigma_f(v, u) dudv \right. \\
& - \frac{1}{\tau_2} \left(\int_{t_0}^{t_1} \int_{t_{N-1}}^{t_N} \sigma_f(v, u) du \int_{t_1}^{t_{N+1}} \sigma_f(v, u) dudv \right. \\
& + \int_{t_1}^{t_{N-1}} \int_{t_{N-1}}^{t_N} \sigma_f(v, u) du \int_v^{t_{N+1}} \sigma_f(v, u) dudv \\
& + \int_{t_1}^{t_{N-1}} \int_{t_{N-1}}^{t_N} \sigma_f(v, u) du \int_v^{t_{N+1}} \sigma_f(v, u) dudv \\
& + \int_{t_1}^{t_{N-1}} \sigma_{S^2}(v) \int_{t_{N-1}}^{t_N} \sigma_f(v, u) dudv + \int_{t_{N-1}}^{t_N} \sigma_{S^2}(v) \int_v^{t_N} \sigma_f(v, u) dudv \\
& + \int_{t_{N-1}}^{t_N} \int_v^{t_N} \sigma_f(v, u) du \int_v^{t_{N+1}} \sigma_f(v, u) dudv \\
& + \left. \int_{t_{N-1}}^{t_N} \sigma_{S^1}(v) \int_v^{t_{N+1}} \sigma_f(v, u) dudv + \int_{t_{N-1}}^{t_N} \sigma_{S^1}(v) \sigma_{S^2}(v) \right) \\
& + \frac{1}{\tau_1 \tau_2} \left(\int_{t_0}^{t_1} \int_v^{t_N} \sigma_f(v, u) du \int_{t_1}^{t_{N+1}} \sigma_f(v, u) dudv \right. \\
& + \int_{t_1}^{t_N} \int_v^{t_N} \sigma_f(v, u) du \int_v^{t_{N+1}} \sigma_f(v, u) dudv \\
& + \left. \int_{t_1}^{t_N} \sigma_{S^2}(v) \int_v^{t_N} \sigma_f(v, u) dudv + \int_{t_0}^{t_1} \sigma_{S^2}(v) \int_{t_1}^{t_{N+1}} \sigma_f(v, u) dudv \right)
\end{aligned}$$

$$+ \int_{t_1}^{t_N} \sigma_{S^2}(v) \int_v^{t_{N+1}} \sigma_f(v, u) du dv + \int_{t_1}^{t_N} \sigma_{S^1}(v) \sigma_{S^2}(v) dv),$$

$$\begin{aligned} \zeta_1^a &= -\ln F(t_0, t_{N-1}, t_N) + \frac{1}{2} \sigma_{\delta_{t_N - t_{N-1}}}^2 \\ &\quad - \frac{1}{\tau_1} \left\{ -\ln P(t_0, t_N) + \frac{1}{2} \sigma_{\beta_{t_N}}^2 - \frac{1}{2} \int_{t_0}^{t_N} \sigma_{S^2}^2(v) dv \right. \\ &\quad - \int_{t_0}^{t_{N-1}} \int_v^{t_{N-1}} \sigma_f(v, u) du \int_v^{t_N} \sigma_f(v, u) du dv \\ &\quad - \int_{t_0}^{t_{N-1}} \sigma_{S^2}(v) \int_v^{t_{N-1}} \sigma_f(v, u) du dv + \int_{t_{N-1}}^{t_N} \sigma_{S^1}(v) \sigma_{S^2}(v) dv \\ &\quad \left. + \int_{t_{N-1}}^{t_N} \sigma_{S^1}(v) \sigma_{S^2}(v) dv \right\}, \end{aligned}$$

$$\begin{aligned} \zeta_1^b &= -\ln F(t_0, t_N, t_{N+1}) + \frac{1}{2} \sigma_{\delta_{t_{N+1} - t_N}}^2 + c_{t_N - t_0, t_{N+1} - t_N} \\ &\quad + \int_{t_0}^{t_{N-1}} \int_{t_N}^{t_{N+1}} \sigma_f(v, u) du \int_v^{t_{N-1}} \sigma_f(v, u) du dv \\ &\quad - \frac{1}{\tau_2} \left\{ -\ln F(t_0, t_1, t_{N+1}) + \frac{1}{2} \sigma_{\beta_{t_{N+1} - t_1}}^2 + c_{t_1 - t_0, t_{N+1} - t_1} \right. \\ &\quad - \frac{1}{2} \int_{t_1}^{t_{N+1}} \sigma_{S^2}^2(v) dv - \int_{t_0}^{t_1} \int_{t_1}^{t_{N+1}} \sigma_f(v, u) du \int_v^{t_{N-1}} \sigma_f(v, u) du dv \\ &\quad - \int_{t_1}^{t_{N-1}} \int_v^{t_{N+1}} \sigma_f(v, u) du \int_v^{t_{N-1}} \sigma_f(v, u) du dv \\ &\quad + \int_{t_{N-1}}^{t_{N+1}} \sigma_{S^1}(v) \int_v^{t_{N+1}} \sigma_f(v, u) du dv \\ &\quad \left. - \int_{t_1}^{t_{N-1}} \sigma_{S^2}(v) \int_v^{t_{N-1}} \sigma_f(v, u) du dv + \int_{t_{N-1}}^{t_{N+1}} \sigma_{S^1}(v) \sigma_{S^2}(v) dv \right\}, \end{aligned}$$

$$\begin{aligned} \zeta_2^a &= -\ln F(t_0, t_{N-1}, T - N) + \frac{1}{2} \sigma_{\delta_{t_N - t_{N-1}}}^2 + c_{t_{N-1} - t_0, t_N - t_{N-1}} \\ &\quad - \int_{t_0}^{t_{N-1}} \int_{t_{N-1}}^{t_N} \sigma_f(v, u) du \int_v^{t_{N+1}} \sigma_f(v, u) du dv \\ &\quad - \frac{1}{\tau_2} \left(\int_{t_0}^{t_1} \int_{t_{N-1}}^{t_N} \sigma_f(v, u) du \int_{t_1}^{t_{N+1}} \sigma_f(v, u) du dv \right. \\ &\quad + \int_{t_1}^{t_{N-1}} \int_{t_{N-1}}^{t_N} \sigma_f(v, u) du \int_v^{t_{N+1}} \sigma_f(v, u) du dv \\ &\quad + \int_{t_1}^{t_{N-1}} \sigma_{S^2}(v) \int_{t_{N-1}}^{t_N} \sigma_f(v, u) du dv + \int_{t_{N-1}}^{t_N} \sigma_{S^2}(v) \int_v^{t_N} \sigma_f(v, u) du dv \\ &\quad \left. + \int_{t_{N-1}}^{t_N} \sigma_{S^1}(v) \int_v^{t_N} \sigma_f(v, u) du dv \right) \\ &\quad + \left(1 + \frac{1}{\tau_2}\right) \left(\int_{t_{N-1}}^{t_N} \int_v^{t_N} \sigma_f(v, u) du \int_v^{t_{N+1}} \sigma_f(v, u) du dv \right. \\ &\quad \left. + \int_{t_{N-1}}^{t_N} \sigma_{S^1}(v) \int_v^{t_{N+1}} \sigma_f(v, u) du dv \right) \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{\tau_1} \left\{ -\ln P(t_0, t_N) + \frac{1}{2} \sigma_{\beta, t_N}^2 - \frac{1}{2} \int_{t_0}^{t_N} \sigma_{S^2}^2(v) dv \right. \\
& - \int_{t_0}^{t_N} \int_v^{t_N} \sigma_f(v, u) du \int_v^{t_{N+1}} \sigma_f(v, u) dudv \\
& + \int_{t_0}^{t_{N-1}} \int_v^{t_N} \sigma_f(v, u) du \int_{t_{N-1}}^{t_N} \sigma_f(v, u) dudv \\
& + \int_{t_{N-1}}^{t_N} \left(\int_v^{t_N} \sigma_f(v, u) du \right)^2 dv + \int_{t_{N-1}}^{t_N} \sigma_{S^1}(v) \int_v^{t_N} \sigma_f(v, u) dudv \\
& - \int_{t_0}^{t_N} \sigma_{S^2}(v) \int_v^{t_{N+1}} \sigma_f(v, u) dudv + \int_{t_0}^{t_{N-1}} \sigma_{S^2}(v) \int_{t_{N-1}}^{t_N} \sigma_f(v, u) dudv \\
& + \int_{t_{N-1}}^{t_N} \sigma_{S^2}(v) \int_v^{t_N} \sigma_f(v, u) dudv + \int_{t_{N-1}}^{t_N} \sigma_{S^1}(v) \sigma_{S^2}(v) dv \\
& + \frac{1}{\tau_2} \left\{ \int_{t_0}^{t_1} \int_v^{t_N} \sigma_f(v, u) du \int_{t_1}^{t_{N+1}} \sigma_f(v, u) dudv \right. \\
& + \int_{t_1}^{t_N} \int_v^{t_N} \sigma_f(v, u) du \int_v^{t_{N+1}} \sigma_f(v, u) dudv \\
& + \int_{t_1}^{t_N} \sigma_{S^2}(v) \int_v^{t_{N+1}} \sigma_f(v, u) dudv + \int_{t_0}^{t_1} \sigma_{S^2}(v) \int_{t_1}^{t_{N+1}} \sigma_f(v, u) dudv \\
& \left. + \int_{t_1}^{t_N} \sigma_{S^2}(v) \int_v^{t_{N+1}} \sigma_f(v, u) dudv + \int_{t_1}^{t_N} \sigma_{S^2}^2(v) dv \right\},
\end{aligned}$$

$$\begin{aligned}
\zeta_2^b & = -\ln F(t_0, t_N, t_{N+1}) + \frac{1}{2} \sigma_{\beta, t_{N+1}-t_N} + c_{t_N-t_0, t_{N+1}-t_N} - \frac{1}{2} \int_{t_N}^{t_{N+1}} \sigma_{S^1}^2(v) dv \\
& - \int_{t_0}^{t_N} \int_{t_N}^{t_{N+1}} \sigma_f(v, u) du \int_v^{t_{N+1}} \sigma_f(v, u) dudv \\
& + \int_{t_0}^{t_{N-1}} \int_{t_N}^{t_{N+1}} \sigma_f(v, u) du \int_{t_{N-1}}^{t_N} \sigma_f(v, u) dudv \\
& + \int_{t_{N-1}}^{t_N} \int_{t_N}^{t_{N+1}} \sigma_f(v, u) du \int_v^{t_N} \sigma_f(v, u) dudv \\
& - \int_{t_N}^{t_{N+1}} \left(\int_v^{t_{N+1}} \sigma_f(v, u) du \right)^2 dv + \int_{t_{N-1}}^{t_N} \sigma_{S^1}(v) \int_{t_N}^{t_{N+1}} \sigma_f(v, u) dudv \\
& - \int_{t_N}^{t_{N+1}} \sigma_{S^1}(v) \int_v^{t_{N+1}} \sigma_f(v, u) dudv \\
& + \frac{1}{\tau_2} \left\{ \int_{t_0}^{t_1} \int_{t_N}^{t_{N+1}} \sigma_f(v, u) du \int_{t_1}^{t_{N+1}} \sigma_f(v, u) dudv \right. \\
& + \int_{t_1}^{t_N} \int_{t_N}^{t_{N+1}} \sigma_f(v, u) du \int_v^{t_{N+1}} \sigma_f(v, u) dudv \\
& + \int_{t_N}^{t_{N+1}} \left(\int_v^{t_{N+1}} \sigma_f(v, u) du \right)^2 dv + \int_{t_1}^{t_N} \sigma_{S^2}(v) \int_{t_N}^{t_{N+1}} \sigma_f(v, u) dudv \\
& + \int_{t_N}^{t_{N+1}} \sigma_{S^2}(v) \int_v^{t_{N+1}} \sigma_f(v, u) dudv + \int_{t_N}^{t_{N+1}} \sigma_{S^1}(v) \int_v^{t_{N+1}} \sigma_f(v, u) dudv \\
& \left. + \int_{t_N}^{t_{N+1}} \sigma_{S^1}(v) \sigma_{S^2}(v) dv \right\}
\end{aligned}$$

$$\begin{aligned}
& - \left(-\ln(t_0, t_1, t_{N+1}) + \frac{1}{2}\sigma_{\beta_{t_1-t_0, t_{N+1}-t_1}} - \frac{1}{2}\int_{t_1}^{t_{N+1}} \sigma_{S^2}(v)dv \right. \\
& - \int_{t_0}^{t_1} \int_v^{t_{N+1}} \sigma_f(v, u)du \int_v^{t_{N+1}} \sigma_f(v, u)dudv \\
& + \int_{t_0}^{t_1} \int_v^{t_{N+1}} \sigma_f(v, u)du \int_{t_{N-1}}^{t_N} \sigma_f(v, u)dudv \\
& + \int_{t_1}^{t_{N-1}} \int_v^{t_{N+1}} \sigma_f(v, u)du \int_{t_{N-1}}^{t_N} \sigma_f(v, u)dudv \\
& + \int_{t_{N-1}}^{t_N} \int_v^{t_{N+1}} \sigma_f(v, u)du \int_v^{t_N} \sigma_f(v, u)dudv \\
& - \int_{t_1}^{t_{N+1}} \left(\int_v^{t_{N+1}} \sigma_f(v, u)du \right)^2 dv + \int_{t_{N-1}}^{t_N} \sigma_{S^1}(v) \int_v^{t_N} \sigma_f(v, u)dudv \\
& - \int_{t_1}^{t_{N+1}} \sigma_{S^2}(v) \int_v^{t_{N+1}} \sigma_f(v, u)dudv + \int_{t_1}^{t_{N-1}} \sigma_{S^2}(v) \int_{t_{N-1}}^{t_N} \sigma_f(v, u)dudv \\
& + \int_{t_{N-1}}^{t_N} \sigma_{S^2}(v) \int_v^{t_N} \sigma_f(v, u)dudv + \int_{t_{N-1}}^{t_N} \sigma_{S^1}(v) \sigma_{S^2}(v)dv \\
& + \frac{1}{\tau_2} \left\{ \int_{t_0}^{t_1} \left(\int_{t_1}^{t_{N+1}} \sigma_f(v, u)du \right)^2 dv + \int_{t_1}^{t_{N+1}} \left(\int_v^{t_{N+1}} \sigma_f(v, u)du \right)^2 dv \right. \\
& \left. + 2 \int_{t_1}^{t_{N+1}} \sigma_{S^2}(v) \int_v^{t_{N+1}} \sigma_f(v, u)dudv + \int_{t_1}^{t_{N+1}} \sigma_{S^1}(v) \sigma_{S^2}(v)dv \right\},
\end{aligned}$$

$$\begin{aligned}
\zeta_3^a &= \frac{1}{\tau_1} \left(-\ln P(t_0, t_N) + \left(\frac{1}{\tau_1} - \frac{1}{2}\sigma_{\delta_{t_N}^2} \right) \right. \\
& \quad - \left\{ -\ln F(t_0, t_{N-1}, t_N) - \frac{1}{2}\sigma_{\delta_{t_{N-t_{N-1}}}^2} \right. \\
& \quad + \frac{1}{\tau_1} \left(\int_{t_0}^{t_{N-1}} \sigma_{S^1}(v) \int_{t_{N-1}}^{t_N} \sigma_f(v, u)dudv \right. \\
& \quad \left. \left. + \int_{t_0}^{t_{N-1}} \sigma_{S^1}(v) \int_{t_{N-1}}^{t_N} \sigma_f(v, u)dudv + \int_{t_{N-1}}^{t_N} \sigma_{S^1}(v) \sigma_{S^2}(v)dv \right) \right\},
\end{aligned}$$

$$\begin{aligned}
\zeta_3^b &= -\ln F(t_0, t_N, t_{N+1}) + \frac{1}{2}\sigma_{\delta_{t_{N+1}-t_N}^2} + \frac{1}{\tau_1} \int_{t_0}^{t_N} \sigma_{S^1}(v) \int_{t_N}^{t_{N+1}} \sigma_f(v, u)dudv \\
& - \frac{1}{\tau_2} \left\{ -\ln F(t_0, t_1, t_{N+1}) + \frac{1}{2}\sigma_{\beta_{t_{N+1}-t_1}} + c_{t_1-t_0, t_{N+1}-t_1} \right. \\
& - \frac{1}{2} \int_{t_1}^{t_{N+1}} \sigma_{S^2}^2(v)dv + \int_{t_N}^{t_{N+1}} \sigma_{S^1}(v) \int_v^{t_{N+1}} \sigma_f(v, u)dudv \\
& + \int_{t_N}^{t_{N+1}} \sigma_{S^1}(v) \sigma_{S^2}(v)dv + \frac{1}{\tau_1} \left(\int_{t_0}^{t_1} \sigma_{S^2}(v) \int_{t_1}^{t_{N+1}} \sigma_f(v, u)dudv \right. \\
& + \int_{t_1}^{t_N} \sigma_{S^2}(v) \int_v^{t_{N+1}} \sigma_f(v, u)dudv + \int_{t_1}^{t_N} \sigma_{S^2}(v)dv \\
& + \left(\frac{1}{\tau_1} - 1 \right) \left(\int_{t_1}^{t_N} \sigma_{S^2}(v) \int_v^{t_N} \sigma_f(v, u)dudv \right. \\
& \left. \left. + \int_{t_0}^{t_1} \int_{t_1}^{t_{N+1}} \sigma_f(v, u)du \int_v^{t_N} \sigma_f(v, u)dudv \right) \right\}
\end{aligned}$$

$$+ \int_{t_1}^{t_N} \int_v^{t_{N+1}} \sigma_f(v, u) du \int_v^{t_N} \sigma_f(v, u) dudv \Big\},$$

$$\begin{aligned} \zeta_4^a = & \frac{1}{\tau_1} \left(-\ln P(t_0, t_N) + \frac{1}{2} \sigma_{\beta_{t_N}}^2 - \frac{1}{2} \int_{t_0}^{t_N} \sigma_{S^2}^2(v) dv \right. \\ & - \int_{t_0}^{t_N} \int_v^{t_N} \sigma_f(v, u) du \int_v^{t_{N+1}} \sigma_f(v, u) dudv \\ & - \int_{t_0}^{t_N} \sigma_{S^2}(v) \int_v^{t_{N+1}} \sigma_f(v, u) dudv + \int_{t_1}^{t_N} \sigma_{S^2}(v) \int_v^{t_N} \sigma_f(v, u) dudv \\ & + \frac{1}{\tau_1} \left(\int_{t_0}^{t_N} \left(\int_v^{t_N} \sigma_f(v, u) du \right)^2 dv + 2 \int_{t_0}^{t_N} \sigma_{S^2}(v) \int_v^{t_N} \sigma_f(v, u) dudv \right. \\ & + \int_{t_0}^{t_N} \sigma_{S^2}^2(v) dv \Big) + \frac{1}{\tau_2} \left(\int_{t_0}^{t_1} \int_v^{t_N} \sigma_f(v, u) du \int_{t_1}^{t_{N+1}} \sigma_f(v, u) dudv \right. \\ & + \int_{t_0}^{t_1} \sigma_{S^2}(v) \int_{t_1}^{t_{N+1}} \sigma_f(v, u) dudv + \int_{t_1}^{t_N} \sigma_{S^2}(v) \int_v^{t_{N+1}} \sigma_f(v, u) dudv \\ & + \int_{t_1}^{t_N} \int_v^{t_N} \sigma_f(v, u) du \int_v^{t_{N+1}} \sigma_f(v, u) dudv + \int_{t_1}^{t_{N+1}} \sigma_{S^2}^2(v) \Big) \\ & - \left\{ -\ln F(t_0, t_{N-1}, t_N) + \frac{1}{2} \sigma_{\beta_{t_N - t_{N-1}}}^2 + c_{t_{N-1} - t_0, t_N - t_{N-1}} \right. \\ & - \frac{1}{2} \int_{t_{N-1}}^{t_N} \sigma_{S^1}^2(v) dv - \int_{t_0}^{t_{N-1}} \int_{t_{N-1}}^{t_N} \sigma_f(v, u) du \int_v^{t_{N+1}} \sigma_f(v, u) dudv \\ & - \int_{t_{N-1}}^{t_N} \int_v^{t_N} \sigma_f(v, u) du \int_v^{t_{N+1}} \sigma_f(v, u) dudv \\ & - \int_{t_{N-1}}^{t_N} \sigma_{S^1}(v) \int_v^{t_{N+1}} \sigma_f(v, u) dudv + \frac{1}{\tau_1} \left(\int_{t_0}^{t_{N-1}} \sigma_{S^2}(v) \int_{t_{N-1}}^{t_N} \sigma_f(v, u) dudv \right. \\ & + \int_{t_{N-1}}^{t_N} \left(\int_v^{t_N} \sigma_f(v, u) du \right)^2 dv + \int_{t_{N-1}}^{t_N} \sigma_{S^2}(v) \int_v^{t_N} \sigma_f(v, u) dudv \\ & + \int_{t_{N-1}}^{t_N} \sigma_{S^1}(v) \int_v^{t_N} \sigma_f(v, u) dudv + \int_{t_{N-1}}^{t_N} \sigma_{S^1}(v) \sigma_{S^2}(v) dv \Big) \\ & + \int_{t_0}^{t_{N-1}} \int_{t_{N-1}}^{t_N} \sigma_f(v, u) du \int_v^{t_N} \sigma_f(v, u) dudv \Big) \\ & + \frac{1}{\tau_2} \left(\int_{t_0}^{t_1} \int_{t_{N-1}}^{t_N} \sigma_f(v, u) du \int_{t_1}^{t_{N+1}} \sigma_f(v, u) dudv \right. \\ & + \int_{t_1}^{t_{N-1}} \int_{t_{N-1}}^{t_N} \sigma_f(v, u) du \int_v^{t_{N+1}} \sigma_f(v, u) dudv \\ & + \int_{t_1}^{t_{N-1}} \sigma_{S^2}(v) \int_{t_{N-1}}^{t_N} \sigma_f(v, u) dudv \\ & + \int_{t_{N-1}}^{t_N} \int_v^{t_N} \sigma_f(v, u) du \int_v^{t_{N+1}} \sigma_f(v, u) dudv \\ & + \int_{t_{N-1}}^{t_N} \sigma_{S^2}(v) \int_v^{t_N} \sigma_f(v, u) dudv + \int_{t_{N-1}}^{t_N} \sigma_{S^1}(v) \int_v^{t_{N+1}} \sigma_f(v, u) dudv \\ & \left. + \int_{t_{N-1}}^{t_N} \sigma_{S^1}(v) \sigma_{S^2}(v) dv \right\}, \end{aligned}$$

$$\begin{aligned}
\zeta_4^b = & \frac{1}{\tau_2} \left(-\ln F(t_0, t_1, t_{N+1}) + \frac{1}{2} \sigma_{\beta, t_{N+1}-t_1}^2 + c_{t_1-t_0, t_{N+1}-t_1} \right. \\
& - \frac{1}{2} \int_{t_1}^{t_{N+1}} \sigma_{S^2}^2(v) dv - \int_{t_0}^{t_1} \int_{t_1}^{t_{N+1}} \sigma_f(v, u) du \int_v^{t_{N+1}} \sigma_f(v, u) dudv \\
& - \int_{t_1}^{t_{N+1}} \left(\int_v^{t_{N+1}} \sigma_f(v, u) du \right)^2 dv - \int_{t_1}^{t_{N+1}} \sigma_{S^2} \int_v^{t_{N+1}} \sigma_f(v, u) dudv \\
& + \frac{1}{\tau_1} \left\{ \int_{t_0}^{t_1} \int_{t_1}^{t_{N+1}} \sigma_f(v, u) du \int_v^{t_N} \sigma_f(v, u) dudv \right. \\
& + \int_{t_1}^{t_N} \int_v^{t_N} \sigma_f(v, u) du \int_v^{t_{N+1}} \sigma_f(v, u) dudv \\
& + \int_{t_0}^{t_1} \sigma_{S^2}(v) \int_{t_1}^{t_{N+1}} \sigma_f(v, u) dudv + \int_{t_1}^{t_N} \sigma_{S^2}(v) \int_v^{t_{N+1}} \sigma_f(v, u) dudv \\
& \left. + \int_{t_1}^{t_N} \sigma_{S^2}(v) \int_v^{t_N} \sigma_f(v, u) dudv + \int_{t_1}^{t_N} \sigma_{S^2}^2(v) dv \right\} \\
& + \frac{1}{\tau_2} \left\{ \int_{t_0}^{t_1} \left(\int_{t_1}^{t_{N+1}} \sigma_f(v, u) du \right)^2 dv + \int_{t_1}^{t_{N+1}} \left(\int_v^{t_{N+1}} \sigma_f(v, u) du \right)^2 dv \right. \\
& + 2 \int_{t_1}^{t_{N+1}} \sigma_{S^2}(v) \int_v^{t_{N+1}} \sigma_f(v, u) dudv + \int_{t_1}^{t_{N+1}} \sigma_{S^2}^2(v) dv \left. \right\} \\
& - \left(-\ln F(t_0, t_N, t_{N+1}) + \frac{1}{2} \sigma_{\beta, t_{N+1}-t_N}^2 + c_{t_N-t_0, t_{N+1}-t_N} \right. \\
& - \frac{1}{2} \int_{t_N}^{t_{N+1}} \sigma_{S^1}^2(v) dv - \int_{t_0}^{t_N} \int_{t_N}^{t_{N+1}} \sigma_f(v, u) du \int_v^{t_{N+1}} \sigma_f(v, u) dudv \\
& - \int_{t_N}^{t_{N+1}} \left(\int_v^{t_{N+1}} \sigma_f(v, u) du \right)^2 dv - \int_{t_N}^{t_{N+1}} \sigma_{S^1}(v) \int_v^{t_{N+1}} \sigma_f(v, u) dudv \\
& + \int_{t_0}^{t_1} \int_{t_N}^{t_{N+1}} \sigma_f(v, u) du \int_{t_1}^{t_{N+1}} \sigma_f(v, u) dudv \\
& + \frac{1}{\tau_1} \left\{ \int_{t_0}^{t_N} \int_{t_N}^{t_{N+1}} \sigma_f(v, u) du \int_v^{t_N} \sigma_f(v, u) dudv \right. \\
& \left. + \int_{t_0}^{t_N} \sigma_{S^2}(v) \int_{t_N}^{t_{N+1}} \sigma_f(v, u) dudv \right\} \\
& + \frac{1}{\tau_2} \left\{ \int_{t_1}^{t_N} \int_{t_N}^{t_{N+1}} \sigma_f(v, u) du \int_v^{t_{N+1}} \sigma_f(v, u) dudv \right. \\
& + \int_{t_N}^{t_{N+1}} \left(\int_v^{t_{N+1}} \sigma_f(v, u) du \right)^2 dv \\
& + \int_{t_1}^{t_N} \sigma_{S^2}(v) \int_{t_N}^{t_{N+1}} \sigma_f(v, u) dudv + \int_{t_N}^{t_{N+1}} \sigma_{S^2}(v) \int_v^{t_{N+1}} \sigma_f(v, u) dudv \\
& \left. + \int_{t_N}^{t_{N+1}} \sigma_{S^1}(v) \int_v^{t_{N+1}} \sigma_f(v, u) dudv + \int_{t_N}^{t_{N+1}} \sigma_{S^1}(v) \sigma_{S^2}(v) dv \right\}.
\end{aligned}$$

B.4 Proof of Proposition 4.9

We have that

$$\pi_{t_0}^{(4.9)} = E_Q \left[e^{-\int_{t_0}^{t_{N+1}} r_v dv} \max \left(e^{\delta_{t_N-t_{N-1}}^1}, e^{\frac{1}{\tau_1} \delta_{t_N}^2} \right) \cdot \max \left(e^{\delta_{t_{N+1}-t_N}^1}, e^{\frac{1}{\tau_2} \delta_{t_{N+1}-t_1}^2} \right) \right].$$

This can also be written as

$$\begin{aligned} \pi_{t_0} &= E_Q \left[e^{-\int_{t_0}^{t_{N+1}} r_v dv} e^{\delta_{t_N-t_{N-1}}^1 + \delta_{t_{N+1}-t_N}^1} 1_{A_1} \right] \\ &+ E_Q \left[e^{-\int_{t_0}^{t_{N+1}} r_v dv} e^{\delta_{t_N-t_{N-1}}^1 + \frac{1}{\tau_2} \delta_{t_{N+1}-t_1}^2} 1_{A_2} \right] \\ &+ E_Q \left[e^{-\int_{t_0}^{t_{N+1}} r_v dv} e^{\frac{1}{\tau_1} \delta_{t_N}^2 + \delta_{t_{N+1}-t_N}^1} 1_{A_3} \right] \\ &+ E_Q \left[e^{-\int_{t_0}^{t_{N+1}} r_v dv} e^{\frac{1}{\tau_1} \delta_{t_N}^2 + \frac{1}{\tau_2} \delta_{t_{N+1}-t_1}^2} 1_{A_4} \right], \end{aligned}$$

where

$$\begin{aligned} A_1 &= \{(\delta_{t_N-t_{N-1}}^1 > \frac{1}{\tau_1} \delta_{t_N}^2) \cap (\delta_{t_{N+1}-t_N}^1 > \frac{1}{\tau_2} \delta_{t_{N+1}-t_1}^2)\}, \\ A_2 &= \{(\delta_{t_N-t_{N-1}}^1 > \frac{1}{\tau_1} \delta_{t_N}^2) \cap (\delta_{t_{N+1}-t_N}^1 < \frac{1}{\tau_2} \delta_{t_{N+1}-t_1}^2)\}, \\ A_3 &= \{(\delta_{t_N-t_{N-1}}^1 < \frac{1}{\tau_1} \delta_{t_N}^2) \cap (\delta_{t_{N+1}-t_N}^1 > \frac{1}{\tau_2} \delta_{t_{N+1}-t_1}^2)\}, \end{aligned}$$

and, finally,

$$A_4 = \{(\delta_{t_N-t_{N-1}}^1 < \frac{1}{\tau_1} \delta_{t_N}^2) \cap (\delta_{t_{N+1}-t_N}^1 > \frac{1}{\tau_2} \delta_{t_{N+1}-t_1}^2)\}.$$

We show how to evaluate the third expectation (the solution is denoted $\pi_{t_0}^3$). The other three expectations can be evaluated in the exact same manner and are therefore not presented here.

Using the Radon-Nikodym derivative

$$\frac{dQ_3}{dQ} = \frac{\xi}{E_Q[\xi]},$$

where

$$\xi = e^{-\int_{t_0}^{t_{N+1}} r_v dv + \frac{1}{\tau_1} \delta_{t_N}^2 + \delta_{t_{N+1}-t_N}^1},$$

some algebra shows that

$$\pi_{t_0}^3 = e^{\kappa^5} E_{Q_3} [1_{A_3}] = e^{\kappa^5} Q_3(A_3^a \cap A_3^b),$$

where

$$\kappa_5 = \left(\frac{1}{\tau_1} - 1\right) \left(-\ln P(t_0, t_N) + \frac{1}{2\tau_1} \sigma_{\delta_{t_N}^2}^2\right).$$

Here $Q_3(\cdot, \cdot)$ denotes the joint probability, under the probability measure Q_3 , for the events $A_3^a = \{\frac{1}{\tau_1} \delta_{t_N}^2 > \delta_{t_N - t_{N-1}}^1\}$ and $A_3^b = \{\delta_{t_{N+1} - t_N}^1 > \frac{1}{\tau_2} \delta_{t_{N+1} - t_N}^2\}$, i.e., the event A_3 . Some more algebra then leads to the solution of the third expectation, i.e.,

$$\pi_{t_0}^3 = e^{\kappa_5} \Phi(a_3, b_3, \rho),$$

where

$$a_3 = \frac{\zeta_3^a}{\sigma_a},$$

$$b_3 = \frac{\zeta_3^b}{\sigma_b},$$

and

$$\rho = \frac{c}{\sigma_a \sigma_b}.$$

As already mentioned, the other three expectations can be found in the exact same manner, but notice that the structures of the problems cause the solutions of the second and the fourth expectation to require some more algebra than the solutions of the first and the third expectation.

Appendix C

A Control Variate for the Compound Option

C.1 Closed Form Solution for the Control Variate

We want to construct a control variate that is identical to the compound call option except for one difference; the exercise price for the control variate is given by $P(T_1, T_2)X_1$ instead of just X_1 as for the compound option.

Define three probability measures, Q_{T_1} , Q_{T_2} , and Q_S equivalent to Q . The probability measures are defined by the Radon-Nikodym derivatives

$$\begin{aligned}\frac{dQ_{T_1}}{dQ} &= \frac{P(t, T_1)/M_t}{P(0, T_1)/M_0}, \\ \frac{dQ_{T_2}}{dQ} &= \frac{P(t, T_2)/M_t}{P(0, T_2)/M_0},\end{aligned}$$

and

$$\frac{dQ_S}{dQ} = \frac{S_t/M_t}{S_0/M_0},$$

respectively.

The time 0 market value of the control variate can now be calculated in the following way (let $A_1 = \{\pi_{T_1} > X_1 P(T_1, T_2)\}$ and $A_2 = \{S_{T_2} > X_2\}$, i.e., A_1 is the exercise set for the control variate and A_2 is the exercise set for the underlying option)

$$\begin{aligned}\pi_0^C &= P(0, T_1)E_{Q_{T_1}} \left[\max \left(P(T_1, T_2)E_{Q_{T_2}} \left[\max(S_{T_2} - X_2, 0) \middle| \mathcal{F}_{T_1} \right] \right. \right. \\ &\quad \left. \left. - X_1 P(T_1, T_2), 0 \right) \right]\end{aligned}$$

$$\begin{aligned}
&= P(0, T_1)E_{Q_{T_1}} \left[\max \left(P(T_1, T_2)E_{Q_{T_2}} \left[(S_{T_2}1_{A_2} - X_21_{A_2}) \middle| \mathcal{F}_{T_1} \right] \right. \right. \\
&\quad \left. \left. - X_1P(T_1, T_2), 0 \right) \right] \\
&= P(0, T_1)E_{Q_{T_1}} \left[\max \left(S_{T_1}E_{Q_S} \left[1_{A_2} \middle| \mathcal{F}_{T_1} \right] \right. \right. \\
&\quad \left. \left. - X_2P(T_1, T_2)E_{Q_{T_2}} \left[1_{A_2} \middle| \mathcal{F}_{T_1} \right] - X_1P(T_1, T_2), 0 \right) \right] \\
&= P(0, T_1)E_{Q_{T_1}} \left[S_{T_1}Q_S(A_2)1_{A_1} - X_2P(T_1, T_2)Q_{T_2}(A_2)1_{A_1} \right. \\
&\quad \left. - X_1P(T_1, T_2)1_{A_1} \right] \\
&= S_0Q_S(A_1 \cap A_2) - X_2P(0, T_2)Q_{T_2}(A_1, A_2) \\
&\quad - X_1P(0, T_2)Q_{T_2}(A_1) \\
&= S_0\Phi(a_1, b_1, \rho) - X_2P(0, T_2)\Phi(a_2, b_2, \rho) - X_1P(0, T_2)\Phi(a_2),
\end{aligned}$$

where

$$\begin{aligned}
a_1 &= \frac{\ln\left(\frac{R_0}{R^*}\right) + \frac{1}{2}\sigma_{R_{T_1}}^2}{\sigma_{R_{T_1}}}, \\
a_2 &= a_1 - \sigma_{R_{T_1}}, \\
b_1 &= \frac{\ln\left(\frac{S_0}{X_2P(0, T_2)}\right) + \frac{1}{2}\sigma_{\delta_{T_2}}^2}{\sigma_{\delta_{T_2}}}, \\
b_2 &= b_1 - \sigma_{\delta_{T_2}}, \\
\sigma_{R_{T_1}}^2 &= \int_0^{T_1} \sigma_S^2(v)dv + 2 \int_0^{T_1} \sigma_S(v) \int_v^{T_2} \sigma_f(v, u)dudv, \\
&\quad + \int_0^{T_1} \left(\int_v^{T_2} \sigma_f(v, u)du \right)^2 dv, \\
\rho &= \frac{\sigma_{R_{T_1}}}{\sigma_{\delta_{T_2}}}, \\
R_0 &= \frac{S_0}{P(0, T_2)},
\end{aligned}$$

and R^* is the critical value that makes

$$R^*\Phi(a_1) - X_2\Phi(a_2) = X_1. \quad (\text{C.1})$$

That such an R^* exists for all $X_1 \in (0, \infty)$ follows since the right-hand side of (C.1) can be interpreted as a call option with R_t being the time t market value of the underlying asset, and we know that the call option is monotonically increasing in the market value of the underlying asset.

Using model 1, it follows that

$$\begin{aligned}
\sigma_{R_{T_1}}^2 &= \sigma_S^2 T_1 + \frac{2\sigma_S\sigma\varphi}{\kappa^2} \left(\kappa T_1 + e^{-\kappa T_2} - e^{-\kappa(T_2-T_1)} \right) \\
&\quad + \frac{\sigma^2}{2\kappa^3} \left(e^{-2\kappa(T_2-T_1)} - 4e^{-\kappa(T_2-T_1)} + 2\kappa T_1 - e^{-2\kappa T_2} + 4e^{-\kappa T_2} \right)
\end{aligned}$$

and

$$\begin{aligned}\sigma_{\delta T_2}^2 &= \sigma_S^2 T_2 + \frac{2\sigma_S\sigma\varphi}{\kappa^2} (\kappa T_2 + e^{-\kappa T_2} - 1) \\ &\quad + \frac{\sigma^2}{2\kappa^3} (2\kappa T_2 - 3 + 4e^{-\kappa T_2} - e^{-2\kappa T_2}).\end{aligned}$$

This formula can be implemented in about the same way as the Geske (1979) - option.

C.2 Exact Simulation of the Control Variate

In this section we present a scheme that gives us the possibility to use exact simulation for model 1 to estimate the market value of the control variate in section C.1. The scheme for the compound option is similar.

We want to price the claim

$$\pi_0^C = E_Q \left[e^{-\int_0^{T_1} r_s ds} (\pi_{T_1} - X_1 P(T_1, T_2))^+ \right],$$

where

$$\pi_{T_1} = E_Q \left[e^{-\int_{T_1}^{T_2} r_s ds} (S_{T_2} - X_2)^+ \middle| \mathcal{F}_{T_1} \right]$$

and

$$P(T_1, T_2) = e^{-\int_{T_1}^{T_2} f(T_1, s) ds} = E_Q \left[e^{-\int_{T_1}^{T_2} r_s ds} \middle| \mathcal{F}_{T_1} \right].$$

Under our functional assumptions the price π_{T_1} is known in closed form solution (cf. section 6.3) as a function of $P(T_1, T_2)$ and the time T_1 stock price S_{T_1} . We call this function $\Pi(S, P)$.

We need to calculate

$$\int_{T_1}^{T_2} r_s ds = \int_{T_1}^{T_2} f(0, s) ds + \int_{T_1}^{T_2} M(s, s) ds + \int_{T_1}^{T_2} N(s, s) ds.$$

We find that

$$\int_{T_1}^{T_2} M(s, s) ds = \frac{\sigma^2}{4\kappa^3} [2\kappa(T_2 - T_1) - e^{-2\kappa T_2} + e^{-2\kappa T_1} + 4(e^{-\kappa T_2} - e^{-\kappa T_1})]$$

and

$$\begin{aligned}\int_{T_1}^{T_2} N(s, s) ds &= \int_{T_1}^{T_2} \int_0^s \sigma_f(v, s) dW_v ds \\ &= \int_{T_1}^{T_2} \int_v^{T_2} \sigma_f(v, s) ds dW_v + \int_0^{T_1} \int_{T_1}^{T_2} \sigma_f(v, s) ds dW_v \\ &= \int_{T_1}^{T_2} \frac{\sigma}{\kappa} \left(\frac{\varphi}{\sqrt{1-\varphi^2}} \right) (1 - e^{-\kappa(T_2-v)}) dW_v \\ &\quad + \int_0^{T_1} \frac{\sigma}{\kappa} \left(\frac{\varphi}{\sqrt{1-\varphi^2}} \right) (e^{-\kappa(T_1-v)} - e^{-\kappa(T_2-v)}) dW_v \\ &\equiv I_1 + I_2.\end{aligned}$$

Here $I_1 \sim \mathcal{N}(0, \sigma_{T_1 T_2})$ and $I_2 \sim \mathcal{N}(0, \sigma_{0 T_1})$ where

$$\sigma_{T_1 T_2}^2 = \frac{\sigma^2}{2\kappa^3} \left[2\kappa(T_2 - T_1) - e^{-2\kappa(T_2 - T_1)} + 4e^{-\kappa(T_2 - T_1)} - 3 \right]$$

and

$$\begin{aligned} \sigma_{0 T_1}^2 &= \frac{\sigma^2}{2\kappa^3} \left[1 - e^{-2\kappa T_1} - 2e^{-\kappa(T_2 - T_1)} \right. \\ &\quad \left. + 2e^{-\kappa(T_1 + T_2)} + e^{-2\kappa(T_2 - T_1)} - e^{-2\kappa T_2} \right]. \end{aligned}$$

Now, the random variable I_1 is independent of the other variables in the problem. The variable I_2 however, correlates with the variables $e^{-\int_0^{T_1} r_s ds}$ and S_{T_1} . Using exact simulation, it is crucial to get this correlation correct. To this end we define

$$\begin{aligned} I_2 &= \int_0^{T_1} \frac{\sigma}{\kappa} \varphi (e^{-\kappa(T_1 - v)} - e^{-\kappa(T_2 - v)}) dW_v^1 \\ &\quad + \int_0^{T_1} \frac{\sigma}{\kappa} \sqrt{1 - \varphi^2} (e^{-\kappa(T_1 - v)} - e^{-\kappa(T_2 - v)}) dW_v^2 \equiv I_2^a + I_2^b \end{aligned}$$

and

$$\begin{aligned} I_3 &= \int_0^{T_1} \frac{\sigma}{\kappa} \left(\frac{\varphi}{\sqrt{1 - \varphi^2}} \right) g_{T_1}(v) dW_v \\ &= \frac{\sigma}{\kappa} \left[\varphi \int_0^{T_1} g(v) dW_v^1 + \sqrt{1 - \varphi^2} \int_0^{T_1} g(v) dW_v^2 \right] \equiv I_3^a + I_3^b. \end{aligned}$$

We find that

$$\begin{aligned} \text{var}(I_2^a) &= \sigma_{I_2^a}^2 = \varphi^2 \sigma_{0 T_1}^2, \\ \text{var}(I_2^b) &= \sigma_{I_2^b}^2 = (1 - \varphi^2) \sigma_{0 T_1}^2, \\ \text{var}(I_3^a) &= \sigma_{I_3^a}^2 = \varphi^2 \sigma_{\beta T_1}^2, \end{aligned}$$

and

$$\text{var}(I_3^b) = \sigma_{I_3^b}^2 = (1 - \varphi^2) \sigma_{\beta T_1}^2.$$

We further find that

$$\begin{aligned} \text{cov}(I_2^a, I_3^a) &= \varphi^2 \psi, \\ \text{cov}(I_2^b, I_3^b) &= (1 - \varphi^2) \psi, \end{aligned}$$

and

$$\begin{aligned} \psi &= \frac{\sigma^2}{2\kappa^3} \left[1 - e^{-\kappa(T_2 - T_1)} - 2e^{-\kappa T_1} \right. \\ &\quad \left. + 2e^{-\kappa T_2} + e^{-2\kappa T_1} - e^{-\kappa(T_1 + T_2)} \right]. \end{aligned}$$

This gives that

$$\rho^* = \rho(I_2^a, I_3^a) = \frac{\varphi^2 \psi}{\sqrt{\varphi^2 \sigma_{0T_1}^2 \varphi^2 \sigma_{\beta T_1}^2}} = \frac{\psi}{\sigma_{0T_1} \sigma_{\beta T_1}} = \rho(I_2^b, I_3^b).$$

We must also calculate the covariance between I_2^a and Z_3 . We find that

$$\text{cov}(I_2^a, Z_3) = \varphi^2 \psi + \frac{\sigma_S \varphi \sigma}{\kappa^2} \left[1 - e^{-\kappa(T_2 - T_1)} - e^{-\kappa T_1} + e^{-\kappa T_2} \right]$$

and

$$\hat{\rho} = \rho(I_2^a, Z_3) = \frac{\text{cov}(I_2^a, Z_3)}{\sigma_{0T_1} \sigma_3}.$$

For the exact simulation we now sample the following time T_1 random variables

$$\begin{aligned} \bar{Z}_1 &= |\varphi| \sigma_{\beta T_1} Y_1, \\ \bar{Z}_2 &= \sqrt{1 - \varphi^2} \sigma_{\beta T_1} Y_2, \\ \bar{Z}_3 &= \sigma_3 (\rho Y_1 + \sqrt{1 - \rho^2} Y_3), \\ \bar{Z}_4 &= \sigma_{0T_1} |\varphi| (\rho^* Y_1 + \alpha Y_3 + \sqrt{1 - (\rho^*)^2 - \alpha^2} Y_4), \end{aligned}$$

and

$$\bar{Z}_5 = \sigma_{0T_1} \sqrt{(1 - \varphi^2)} (\rho^* Y_2 + \sqrt{1 - (\rho^*)^2} Y_5),$$

where

$$\alpha = \hat{\rho} - \frac{\rho \rho^*}{\sqrt{1 - \rho^2}},$$

σ_3 and ρ are as defined in subsection 6.6.2, and $Y_i \sim \mathcal{N}(0, 1)$, $i \in \{1, 2, \dots, 5\}$.

Using exact simulation, the market value is now given by

$$\pi_0^C = E_Q \left[e^{\ln P(0, T_1) - \frac{1}{2} \sigma_{\beta T_1}^2 - \bar{Z}_1 - \bar{Z}_3} (\pi_{T_1} - X_1 P(T_1, T_2))^+ \right],$$

where

$$\begin{aligned} \pi_{T_1} &= \Pi(S_{T_1}, P(T_1, T_2)), \\ S_{T_1} &= S_0 e^{-\ln P(0, T_1) + \frac{1}{2} \sigma_{\beta T_1}^2 - \frac{1}{2} \sigma_S^2 T_1 + \bar{Z}_3 + \bar{Z}_2}, \end{aligned}$$

and $(F(0, T_1, T_2))$ is the time 0 forward price for delivery at time T_1 of a zero-coupon bond maturing at time T_2)

$$\begin{aligned} P(T_1, T_2) &= E_Q \left[e^{-\int_{T_1}^{T_2} r_v dv} \middle| \mathcal{F}_{T_1} \right] \\ &= E_Q \left[e^{\ln F(0, T_1, T_2) - \int_{T_1}^{T_2} M(v, v) ds - \bar{Z}_4 - \bar{Z}_5 - \sigma_{T_1 T_2} Y_6} \right], \end{aligned}$$

where also $Y_6 \sim \mathcal{N}(0, 1)$. Since $\bar{Z}_1, \bar{Z}_2, \dots, \bar{Z}_5$ are \mathcal{F}_{T_1} measurable, we get

$$P(T_1, T_2) = e^{\ln F(0, T_1, T_2) - \int_{T_1}^{T_2} M(s, s) ds - \bar{Z}_4 - \bar{Z}_5 + \frac{1}{2} \sigma_{T_1 T_2}^2}.$$

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