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A Treatise on Asset Pricing Theory with Additive Nonseparable von Neumann-Morgenstern Utility

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CHAPTER 1

Introduction

This thesis consists of three main chapters that investigate different theoretical aspects of equilibrium asset pricing with additive nonseparable von Neumann-Morgenstern utility. In this chapter I discuss how the next three chapters relate to the existing literature on equilibrium asset pricing models. I do this by first presenting the gist of the traditional model of Lucas (1978) and Breeden (1979), which I will refer to as the *standard model.* I then argue that the standard model is not satisfactory according to both positive and normative criteria. Several alternative models have been suggested, and I continue with an informal presentation of the most central ones. This should identify the problems, methodology, and models that are most closely related to the main work of this thesis.

I conclude this chapter with a short summary of each of the next three chapters, and highlight their major contributions. Although closely related, their contributions are rather different in nature: computational, study of a specialized economy, and generalization of a class of economies including those of the preceding chapters. The time-constrained reader should therefore consult the last section of this chapter to get an indication of which of the remaining chapters are of any relevance.

1. The Standard Model

Consider a pure exchange economy, and assume there is a representative consumer with additive separable von Neumann-Morgenstern utility:

$$U(c) = \mathrm{E}\left\{\int_0^T v(t, c_t)\, dt\right\} \tag{1}$$

The consumer selects a consumption strategy c, an Itô process, that for each time t and state ω yields a consumption level of $c_t(\omega)$. The consumer evaluates a consumption level of $c_t(\omega)$ according to the ***felicity*** function $v(t, c_t(\omega))$.[1] There are

[1] v is also called 'subutility', 'Bernoulli utility', or 'atemporal utility' (See e.g., Merton 1990, Deaton 1992). The utility functional U is called ***additive separable*** since it is additive across time

two securities available for trade, one riskless and one risky,[2] and the risky security follows an Itô process. The risky security's return has a conditional instantaneous expectation of μ_t and a conditional instantaneous variance of σ_t^2.

Assume further that markets are complete and there are no market frictions. In equilibrium, the riskless interest rate r_t, and the risk premium of the risky security $\mu_t - r_t$, satisfy Breeden's (1979) Consumption-based Capital Asset Pricing Model (CCAPM):[3]

$$r_t = -\frac{1}{v_c(t,c_t)}\left[v_{ct}(t,c_t) + v_{cc}(t,c_t)\mu_c(t) + \tfrac{1}{2}v_{ccc}(t,c_t)\sigma_c^2(t)\right], \tag{2}$$

$$\mu_t - r_t = -\frac{v_{cc}(t,c_t)}{v_c(t,c_t)}\sigma_t\sigma_c(t). \tag{3}$$

v_c is the partial derivative of v with respect to its second argument, and v_{ct}, v_{cc}, and v_{ccc} are higher order partial derivatives.

2. The Need for Alternative Models

The standard model fails along several dimensions. Two simple examples will illustrate some of the shortcomings of intertemporal models based on preferences represented by (1).[4]

Consider two intertemporal lotteries based on coin flips in each of ten periods. Lottery HR consists of flipping one coin at time $t = 1$, yields unity for ten periods if head, and zero for ten periods otherwise. Lottery LR consists of a sequence of ten independent coin flips, one in each time period $t = 1, \ldots, 10$. For each coin flip it yields either unity or zero. It is easy to check that preferences represented by (1) assigns the same utility index to both lotteries. In other words, an individual acting according to this utility functional is indifferent between the lotteries, at odds with intuition. One would expect lottery LR to be more attractive to a risk averse individual, since it entails some form of time-diversification relative to lottery HR.

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and states, and it is separable across time. The separability is due to the effect of a perturbation h to c on $v(t, c_t)$. The felicity $v(t, c_t)$ is affected only by the perturbation h_t, and unaffected by $\{h_s\}_{s \neq t}$. This attribute is also called 'time-separable', or 'temporally independent'. Merton (1990) gives examples of multiplicative and nonseparable utility functionals.

[2]This is in fact at no loss of generality since two-fund separation obtains (Merton 1971, Breeden 1979).

[3]See section 4 in chapter 3 for a formal derivation of the formula.

[4]Several other arguments can also be raised against the standard model. For instance Epstein (1992), and Duffie and Epstein (1992) offer more examples.

Consider next how the utility functional (1) treats risk aversion and intertemporal substitution in consumption. Since it is additive in both states and time, and it is separable across time, variations in consumption across states are treated symmetrically to variations in consumption across time. An individual represented by (1) who dislikes lotteries across states will therefore also dislike variations in consumption over time. The former governs equilibrium compensation for bearing risk, e.g., the equity risk premium. The latter governs equilibrium compensation for deferring consumption, e.g., the riskless interest rate. Within the standard model it is therefore not possible to, say, increase the equity risk premium without also increasing the riskless interest rate. A widely cited illustration of this relationship is the *equity premium puzzle* of Mehra and Prescott (1985). To illustrate the equity premium puzzle it is useful to assume in addition that

- the felicity is iso-elastic with constant subjective discount rate $\beta \geq 0$, $v(t,c) = e^{-\beta t}\gamma^{-1}c^{\gamma}$, $\gamma < 1$, and
- per capita consumption follows a geometric Brownian motion $dc_t = \mu_c c_t\, dt + \sigma_c c_t\, dB_t$, with strictly positive constants μ_c and σ_c.

This implies that the consumer has an atemporal Arrow-Pratt coefficient of relative risk aversion of $\kappa = (1-\gamma)$. With these stronger assumptions equations (2) and (3) simplify to

$$r_t = \beta + \kappa\mu_c - \tfrac{1}{2}(\kappa^2 + \kappa)\sigma_c^2 \tag{4}$$

$$\mu_t - r_t = \kappa\sigma_t\sigma_c = \kappa\,\mathrm{Cov}_t(dS_t/S_t, dc_t/c_t), \tag{5}$$

where the risky security is denoted S_t. Reasonable estimates of the parameters of consumption and security returns are $\mu_c = 0.018$, $\sigma_c = 0.035$, $r_t = 0.01$ $\mu_t = 0.08$, and $\sigma_t = 0.16$. By substituting these estimates into equations (4) and (5) it is trivial to solve for the two unknowns β and κ. The solutions are $\kappa = 12.5$ and $\beta = -0.11$. Hence, not only do we get a relatively high κ,[5] but also the solution for the subjective discount rate is at odds with the assumption that it is non-negative. The first aspect of the solution, that κ is relatively high, is the equity premium

[5]Chapter 2 has an illustration of the implication of this magnitude of atemporal Arrow-Pratt relative risk aversion (p. 20). It is also too high relative to empirical estimates of it, which typically are less than 3.

puzzle as posed by Mehra and Prescott (1985). The second aspect of the solution, that β is negative, is the ***riskless rate puzzle*** suggested by Weil (1989). If β is forced to take on a reasonable size then the riskless interest rate becomes too high relative to its historical value, as evident in both equation (2) and (4). As Kocherlakota (1996) points out, these anomalies are inherent in equations (2) and (3).

To reiterate the intuition above in the current setting: When the risk aversion of the consumer increases, represented by an increase in κ, then the security must offer a higher return for the consumer not to reduce his demand for it. Thus, a higher equity risk premium is attained by increasing the consumer's risk aversion. On the other hand, a high κ implies low intertemporal elasticity of substitution, represented by κ^{-1}. By increasing κ the consumer wants to smooth consumption over time. In an economy with growing per capita consumption the consumer will therefore want to borrow against future income, and interest rates must increase to maintain a fixed level of equilibrium demand. The only assumptions necessary to reproduce these puzzles are additive separable von Neumann-Morgenstern utility, complete markets, and no market frictions.

One might hope that the unattractive behavioral implications of additive separable von Neumann-Morgenstern utility really are not that important, and that the equity premium puzzle is an empirical artifact. Below I offer a quick review of suggested empirical remedies, and conclude that they are not satisfactory.[6]

Extending the Sample. Mehra and Prescott (1985) use data spanning the period 1889–1978. By extending the time period to the years 1800–1990, Siegel (1992) finds that estimates of the riskless interest rate are higher than that of Mehra and Prescott, while the real returns on stocks are relatively stable. The equity risk premium for this longer period is estimated to be about 0.045.

Three issues indicate that this is not sufficient support for additive separable utility. First, the equity risk premium is still too large to imply a reasonable atemporal Arrow-Pratt coefficient of relative risk aversion. Second, there is no record of per capita consumption for the time period used by Siegel. The real implications of these estimates are therefore indeterminate. Finally, it is unlikely

[6]For alternative short reviews with references to the relevant literature see Abel (1991), or Siegel and Thaler (1997). More extensive reviews are offered by Kocherlakota (1996), and Cochrane (1997).

that the riskiness and institutional structure of the US securities markets have been constant throughout such a long time period. In particular, it is likely that the riskiness of government bonds and the costs of transactions have declined, both contributing to strengthen the puzzles.

Survivorship Bias. Brown, Goetzmann, and Ross (1995) posits that the equity risk premium is high because investors require compensation for the possibility that the operation of the securities markets cease. They argue that several non-US markets have closed down, and that this risk is not reflected by studying historical data only from the US, since those markets have survived. If true, this will inflate σ_t and one can achieve a given equity risk premium with a lower κ, as evident from equation (5).

There are two major problems with this argument (Siegel and Thaler 1997). First, Mehra and Prescott's (1985) data does include an economic catastrophe, that of the 1929 stock market crash and the 1930's depression. Second, financial crises have historically coincided with hyperinflation. Financial crises reduce equity value, while hyperinflation reduce the real return on bonds. The net effect is a greater historical equity premium in "catastrophic" economies, for instance in Germany and Japan. Hyperinflation clearly does nothing to ameliorate the riskless rate puzzle either.

Aggregation. Mehra and Prescott (1985) assume that per capita consumption is equal to consumption of individual investors. If this is false, and their stochastic properties are different the alleged puzzle might be an artifact due to aggregation of the consumption of individuals who do not own stocks. Mankiw and Zeldes (1991) finds that only a small proportion of Americans actually hold stocks. Further, they find that the consumption of those that do own stocks covaries more with the stock market than that of those that do not own stocks. This reduces the degree of risk aversion necessary to reconcile the standard model with historical data, represented by κ in the above model. Still, using only consumption data from those individuals that do own stocks does not bring κ down to conventional levels.

Conclusion. It seems clear from this short survey of the empirical literature on the equity premium puzzle that the puzzle is resilient to different ways of selecting the historical data used in estimation. The standard model is not capable of explaining other important aspects of the data either (See for instance Cochrane 1997, Campbell and Cochrane 1999). Unless the future will be very different from the past,[7] the standard model is not satisfactory.

3. Alternative Models

This section offers a short survey of classes of models that weaken the assumption on preferences, complete markets, or no market frictions in the standard model. This will hopefully identify how the work of this thesis relates to other approaches to determine the equilibrium value of financial assets. Since this thesis weakens assumptions on preferences I will devote special attention to this approach.

Alternative preferences are presented first, with an informal introduction to some important concepts in dynamic choice theory. This is followed by a discussion of models that explicitly analyze the effects of incomplete markets. The last subsection is devoted to models that allow for market frictions.

Alternative Preferences. Consider the probability space $(\Omega, \mathcal{F}, P)$ with information structure represented by the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$. Let C_t denote cumulative consumption up until time t. Duffie and Epstein (1992, Theorem 1), and Duffie and Skiadas (1994, p. 117) show that a large class of intertemporal utility functionals can be represented as the unique solution U_t to the integral equation

$$U_t = \mathrm{E}\left\{ \int_t^T f_s\left[Z_s(C), U_s(C)\right] ds \,|\mathcal{F}_t \right\}, \tag{6}$$

under weak regularity conditions on $f(\omega, t, \cdot, \cdot) : \mathbb{R}^K \times \mathbb{R} \to \mathbb{R}$ and Z. The salient feature of (6) is that most of the interesting models studied in financial economics can be reproduced within this framework by judicial choice of f and Z.

To get an impression of the generality of (6), restrict attention to a utility functional U with generic argument the rate $c : \Omega \times [0, T] \to \mathbb{R}_+$ adapted to $\mathbb{F}$. Let $G_t = \cup_{F \in \mathcal{F}_t} (\Omega \setminus F)$, the set of unrealized states given the information at time t. For any $\mathcal{F}_t \in \mathbb{F}$ let $c_{t,T}|_{\mathcal{F}_t}$ denote the consumption path $c(\omega, t)$ when t runs through $[t, T]$ and $\omega \in F \in \mathcal{F}_t$. Let $U|_{\mathcal{F}_t}(\cdot) = U(c_{0,t}|_{\mathcal{F}_t}, c_{t,T}|_{G_t}, \cdot) : \Omega \times [t, T] \to \mathbb{R}$ be the

[7] Both Cochrane (1997) and Welch (1998) argue that one cannot expect the equity risk premium of, say, the past 50 years to be maintained the next 50 years.

restriction of U to the consumption path realized up until time t, and with future, stochastic, consumption path $c_{t,T}|_{\mathcal{F}_t}$ as generic argument. As illustrated in figure 1, the relationship between several classes of intertemporal utility functionals can be identified with these constructs.

FIGURE 1. The Relationship Between Utility Classes

Intertemporally Consistent
$U(c) \geq U(\hat{c}) \Rightarrow U_t(c_{t,T}|_{\mathcal{F}_t}) \geq U_t(\hat{c}_{t,T}|_{\mathcal{F}_t})$

$\cup$

Weakly Recursive
$U(c_{0,t}|_{\mathcal{F}_t}, c_{t,T}|_{G_t}, c_{t,T}|_{\mathcal{F}_t}) \geq U(\hat{c}_{0,t}|_{\mathcal{F}_t}, c_{t,T}|_{G_t}, \hat{c}_{t,T}|_{\mathcal{F}_t}) \Rightarrow U(c_{0,t}|_{\mathcal{F}_t}, c^*_{t,T}|_{G_t}, c_{t,T}|_{\mathcal{F}_t}) \geq U(\hat{c}_{0,t}|_{\mathcal{F}_t}, c^*_{t,T}|_{G_t}, \hat{c}_{t,T}|_{\mathcal{F}_t})$

$\supset$ **Additive Nonseparable**
$f_t(z, v) = u(z) - \beta_t v$

$\cup$

Recursive
$U_t(c_{t,T}|_{\mathcal{F}_t}) \geq U_t(\hat{c}_{t,T}|_{\mathcal{F}_t}) \iff U(c_{t,T}|_{\mathcal{F}_t}) \geq U(\hat{c}_{t,T}|_{\mathcal{F}_t})$

$\cup$

Stochastic Differential
$dU_t = \mu_U(t)\, dt + \sigma_U(t)\, dB(t)$

$\cup$

Kreps-Porteus
$f_t(z, v) = \beta_t \frac{(z^\rho - v^\rho)}{\rho v^{\rho - 1}} + \frac{1}{2}\frac{\alpha - 1}{v}\sigma_U^2(t)$, and $Z_t(C) = c_t$

$\cup$

Additive Separable
$f_t(z, v) = u(z) - \beta_t v$, and $Z_t(C) = c_t$

The definitions in the figure are largely based on the survey paper by Epstein (1992), who uses the concept of intertemporally consistent choice suggested by Johnsen and Donaldson (1985). To simplify the notation let $U|_{\mathcal{F}_t} = U_t$. The inclusions in the figure refer to significant subsets of classes of continuous-time utility functionals. There are for instance esoteric cases of Kreps-Porteus utility that are recursive, but which do not allow a decomposition as an Itô process.

Intertemporal consistency is a natural requirement and is necessary to validly apply dynamic programming techniques. Consider two consumption paths $c \succsim \hat{c}$ that are equal except at $F \in \mathcal{F}_t$, $P(F) > 0$. Informally, if the individual prefers c to $\hat{c}$ because they differ on F then the individual must still prefer c to $\hat{c}$ when knowing that $\omega \in F$. The utility functional U is said to be ***weakly recursive*** if at all times the restricted functional $U|_{\mathcal{F}_t}$ is independent of unrealized consumption paths $c_{t,T}|_{G_t}$. A weakly recursive utility functional is ***recursive*** if in addition $U|_{\mathcal{F}_t}$ is independent of the consumption history $\{c_s\}_{0 \leq s < t}$, and it is ordinally equivalent to U. That is, both $U|_{\mathcal{F}_t} : \Omega \setminus G_t \times [0,T] \to \mathbb{R}$ and $U : \Omega \setminus G_t \times [0,T] \to \mathbb{R}$ give rise to the same ordering of consumption paths $c_{t,T}|_{\mathcal{F}_t}$.

The preceding classes of intertemporal utility functionals are quite general. Although no explicit parametrizations can summarize any one of them, the dynamic utility in (6) covers a substantial subset of them. By requiring that information arrives according to a Brownian filtration recursive utility can be represented in continuous time by an Itô process. Duffie and Epstein (1992) calls this *stochastic differential utility.* Consider now a lottery at time t, represented by the distribution of continuation utility U_{t+s}, $s \geq 0$. *Kreps-Porteus utility* obtains if it is required that certainty equivalents (CE) for such lotteries are given by von Neumann-Morgenstern utility, $CE(U_{t+s}) = v^{-1}(\mathrm{E}\{v(U_{t+s})\,|\mathcal{F}_t\})$. In figure 1 this is represented by the term $(\alpha - 1)/U_t$, which penalizes volatility in continuation utility, $\sigma_U(t)$, if $\alpha < 1$. Duffie and Epstein (p. 367) show that $\alpha = \rho$ in the parametrization of Kreps-Porteus utility gives rise to the standard model with *additive separable utility. Additive nonseparable utility* functionals induce history dependent continuation utility, and are therefore only weakly recursive. The similarity to additive separable utility is striking, and is due to both being additive across states. They are both also additive across time in the aggregator $Z_t(C)$, but not in the rate c_t. This class of utility is therefore different from the recursive ones in that they are nonseparable in $\{c_t\}_{0\leq t\leq T}$ across time. Some special cases of recursive utility and additive nonseparable utility seems to perform well in explaining historical data, and in ranking uncertain intertemporal consumption bundles—the subject of the rest of this subsection.

Recursive Utility. When information arrives according to a Brownian filtration Duffie and Epstein (1992) define *stochastic differential utility* to be the solution $U_t = U_0 + \int_0^t \mu_U(t)\,dt + \int_0^t \sigma_U(t)\,dB_t$, $U_T = 0$, to the integral equation

$$(7) \qquad U_t = \mathrm{E}\left\{\int_t^T -\mu_U(s)\,ds\,|\mathcal{F}_t\right\}, \qquad t \in [0,T]$$

$$\mu_U(t) = -a(c_t, U_t) - \tfrac{1}{2}A(U_t)\sigma_U(t)\cdot\sigma_U(t),$$

for a given choice of a and A, given that the certainty equivalent for U locally is of the von Neumann-Morgenstern type. Intuitively, A describes the individual's attitudes towards atemporal lotteries, and it evidently plays no role in describing the individual's attitudes towards deterministic consumption plans, in which case $\sigma_U \equiv 0$. Thus, there is a form of separation between atemporal risk aversion and intertemporal substitution in consumption.[8] Clearly then, by suitable parametrization it is possible to resolve the problem of the inverse relationship between attitudes towards risk and intertemporal substitution.[9] Further, it can be shown that several possible parametrizations of (7) will distinguish between lottery HR and LR presented in section 2 (Duffie and Epstein 1992, p. 366).

The problem with simply separating attitudes towards risk and intertemporal substitution is that it does not resolve the main problem of the equity premium puzzle (Kocherlakota 1996, p. 54). It does make it possible to fit both the historical riskless interest rate and risk premium by separately choosing a and A to induce willingness to substitute across time, and unwillingness to substitute across states. These preferences therefore resolve the riskless rate puzzle. Still, it is necessary to have a high degree of risk aversion to fit the historical risk premium, which is the main thrust of the equity premium puzzle argument.

Additive Nonseparable Utility. This class of utility functionals is obtained from (6) by choosing $f_t(z,v) = u(z) - \beta_t v$. Again, if $Z_t(C) = c_t$ the standard model obtains. History dependent preferences, which therefore do not belong to the recursive utility class but rather to the weakly recursive class (Epstein 1992, pp. 21,28), can be introduced by setting $Z_t(C) = (c_t, z_t)$ with $z_t = \int_0^t g(s, c_s)\, ds$. For instance, preferences are said to exhibit habit formation if $\frac{\partial}{\partial z}u(c,z) \leq 0$. In general, these are not indifferent to time diversification, nor do they have as tight a relationship between intertemporal substitution and risk aversion as the standard model. Compared to recursive utility, it is not possible to isolate the effects on equilibrium allocations or prices from changes in individuals' attitudes to atemporal gambles.

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[8]Notice that while A does not play a role in determining willingness to substitute across time, a does play a major role in determining the individuals attitudes towards risk. The standard model is a special case by choosing $a(c,v) = u(c) - \beta v$ and $A(x) \equiv 0$, evidently mixing attitudes towards risk and intertemporal substitution. While any parameter specific to A only affects attitudes towards risk, no parameter specific to f is ensured only to affect attitudes towards deterministic consumption plans.

[9]One example is Kreps-Porteus utility (Duffie and Epstein 1992, p. 367), where $a(c,v) = \beta(c^\rho - v^\rho)/(\rho v^{\rho-1})$, and $A(x) = (\alpha - 1)/x$. Attitudes towards atemporal gambles is thus governed by α while attitudes towards intertemporal substitution is governed largely by ρ. Still, since the standard model obtains when $\alpha = \rho$, ρ not only governs attitudes towards intertemporal substitution but also attitudes towards atemporal gambles.

Further, as opposed to recursive utility, they are not affected by the timing of the resolution of uncertainty. Even so, they have the capacity to reconcile theoretical models of securities returns with historical data.

Sundaresan (1989) and Constantinides (1990) show that habit formation can explain the first moments of both the historical equity premium and the riskless interest rate. Habit formation therefore potentially resolves both the equity premium puzzle and the riskless rate puzzle.[10] However, the two puzzles are resolved only by creating a new one. The volatility of the riskless interest rate implied by these models is far larger than that observed in historical data.

External habit formation (also known as 'Catching Up with the Joneses') (Abel 1990) is different from habit formation in that the previous standards of living, $\{z_t\}$, are exogenous to the consumer's optimization problem.[11] Not only do models based on these preferences overcome the problem of a volatile riskless interest rate, but they also seem to be able to match other statistical properties of historical data (Campbell and Cochrane 1999). On the other hand, since they retain the additive nonseparable structure they do not allow a separation between substitution and risk aversion to the same extent as recursive utility. In contrast to preferences exhibiting habit formation, preferences exhibiting external habit formation will typically not be able to distinguish between the lotteries HR and LR.[12] Even though external habit formation seemingly explains the dynamics of historical returns well, they suffer from the same problem as models based on recursive utility. A high equity risk premium is attained only with a very high degree of aversion to atemporal gambles.

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[10]Kocherlakota (1996, p. 47) claims that habit formation models calibrated to historical data implies an implausible high risk aversion, as with Kreps-Porteus utility. This is not obvious from his argument, which falsely identifies the parameter γ of $u(c_t - z_t) = \gamma^{-1}(c_t - z_t)^\gamma$ with the Arrow-Pratt measure of relative risk aversion. The latter is not well defined for additive nonseparable utility functionals in terms of the felicity u. The reason for this is that the marginal utility (or rather it's Riesz representation) is not necessarily equal to the marginal felicity.

[11]These preferences do not fit directly into the class of dynamic utility encompassed by (6). An extension of (6) is given in chapter 4, and external habit formation is shown to obtain by letting Z_t depend not only on the consumer's cumulative consumption C, but also on a state variable X representing aggregate consumption. Hence, external habit formation obtains by letting $Z_t(C, X) = (c_t, z_t)$ with $z_t = \int_0^t g(s, X_s)\, ds$ in the consumer's optimization problem. In equilibrium $C \equiv X$.

[12]The two lotteries will typically be independent of aggregate consumption. It is therefore easy to show, using the law of iterated expectations, that the two lotteries will have the same utility index. Consider for instance $v(t, c, x) = \beta^t u(c_t / x_{t-1})$ where c_t is unity with probability p and zero otherwise. For both HR and LR the expected utility is $\sum_{t=0}^{T} \beta^t \left[p\mathrm{E}\left\{u(1/x_{t-1})\right\} + (1-p)u(0)\right]$.

Incomplete Markets. Any equilibrium in the standard model is assured to be Pareto optimal by the First Welfare Theorem. Under mild regularity conditions this implies the existence of a representative consumer, who must necessarily consume the aggregate endowment. *Per capita* consumption will thereby enter the CCAPM in equations (2) and (3). Individual consumption will typically be more volatile than per capita consumption, and equations (2) and (3) must hold for each individual. The equity risk premium implied by the theoretical model can thereby seemingly be reconciled with historical averages. Further, the increased riskiness in consumption will induce prudent consumers to save more, reducing the riskless interest rate.

There are two major problems with the preceding argument. First, observed historical individual consumption is not sufficiently volatile to produce an implied equity premium as large as the historical equity premium. Second, the extra volatility implied by these models is generally negligible without further qualifications: Even in the absence of insurance markets, and in the presence of borrowing constraints, consumers can create dynamic self-insurance against income shocks through trade in the securities markets. They can build up a financial reserve by simple savings, and subsequently liquidate these holdings during periods of low income (Kocherlakota 1996, Cochrane 1997). These conjectures are confirmed in numerical analyses of incomplete economies (for instance by Heaton and Lucas 1996).

Heterogeneous Consumers. One qualification that succeeds in explaining historical equity and bond returns is to assume that individual income shocks are permanent (Constantinides and Duffie 1996). In this case there is no scope for dynamic self-insurance. Constantinides and Duffie show that their economy can replicate the dynamics of any time series of aggregate consumption and securities returns by tailoring individual income dynamics. It remains an empirical question whether or not the dynamics of observed individual income is close enough to that required by their model. Kocherlakota (1996) argues that observed income does not comply with their model, while Cochrane (1997) argues that even if it complies, it implies a high degree of atemporal risk aversion.

Discontinuous Information Structure. Large, unpredictable changes in aggregate consumption is another source of incompleteness (Back 1991, Aase 1993a, Aase

1993b, Aase 1997). In contrast to the Survival Hypothesis of Brown, Goetzmann, and Ross (1995), this does not necessarily prescribe any changes in consumption of catastrophic proportions. Instead the resolution of uncertainty prevents consumers from attaining the allocation of the corresponding complete economy through dynamic trading.[13] When the magnitude of the jumps in aggregate consumption is sufficiently large, these economies can be calibrated to the data of Mehra and Prescott (1985) with viable risk aversion and positive subjective discount rates. It remains an open question how these models fit other aspects of historical returns. It also remains an open empirical question if historical data can be reconciled with the dynamics of aggregate consumption required by these economies to resolve the two puzzles.[14]

Market Frictions. There is a wide range of market frictions in real securities markets. Trading costs and borrowing constraints are probably the most important ones in terms of their impact on allocations and prices. I therefore restrict attention to these.

Trading Costs. If it is more costly to trade stocks than bonds, then the consumer will require extra compensation to hold stocks relative to a frictionless economy that offers compensation only for the extra risk inherent in stocks. The question is how large transactions costs must be to explain the historical equity premium. Kocherlakota (1996, pp. 64–65) offers a simple example, backed up by numerical simulation studies. These establish that only large differences in transactions costs between bonds and stocks can resolve the puzzle. Transactions costs are therefore unlikely to be a major explanation for the puzzles.

Liquidity Constraints. Market clearing forces the riskless interest rate to fall when borrowing constraints are introduced. Since aggregate demand for borrowing is forced to decrease, the interest rate must decrease to make it less attractive to save. As such, borrowing constraints represent a resolution of the riskless rate puzzle. Population-wide constraints do not, however, affect the equity risk premia.

[13] "The corresponding complete economy" is somewhat ambiguous. The economy can be made complete either by removing the jumps in a jump-diffusion model, or by restricting the size of the jumps to be deterministic in a pure jump or jump-diffusion model.
[14] Kocherlakota (1996, p. 52) shows that the puzzles are quite robust to the dynamics of consumption growth.

Any consumer restricted in the credit market must also be restricted in the stock market. Hence, all returns should be affected (Kocherlakota 1996, pp. 63–64).

Constantinides, Donaldson, and Mehra (1998) offer an interesting twist to this story. They weaken the condition that all consumers face the same constraints. Instead they study an overlapping generations model consisting of borrowing constrained 'young' with endowment income, unconstrained 'middle-aged' with high-risk wage income, and unconstrained 'old' without income. They thereby introduce different needs for borrowing and lending in the different segments of the population. Not only do they calibrate the model to historical returns using additive separable utility, but they also explain the low demand for securities in general, and the low demand for equities relative to bonds in particular. This is achieved with a reasonable level of risk aversion. They do not, however, study the effects of different relative sizes of the population segments. In the limit, as the proportion of young goes to zero, the borrowing constraint will obviously have no effect. Although a promising approach, there is still need for more analysis. On one hand, it is necessary to establish if the model is robust to changes in relative population sizes. On the other hand, it is necessary to determine the actual size of the population of borrowing constrained young. If it turns out to be negligible and the effect of relative population size is significant, then the model cannot resolve the puzzles.

One Question, Several Answers? It seems quite clear from the current and the preceding sections that asset pricing models that assume additive separable von Neumann-Morgenstern utility, complete markets, and no market frictions are not entirely satisfactory descriptions of historical market returns—they are not likely to be saved by econometric or empirical considerations. It also seems quite clear that we do not have a satisfactory understanding of the competing models. Several of them seem capable of doing a better job at describing historical returns, but it is not clear which is the better one. Further, none of them seem to resolve the fundamental issue of the equity premium puzzle: to explain the observed equity risk premium in an economy populated by consumers with "reasonable" aversion to atemporal gambles. It is therefore seemingly much empirical and theoretical work left to be done. The motivation for this thesis is to increase our understanding of a small subset of the competing models. I retain the assumptions of complete markets and no market frictions, but allow the utility functional of the representative consumer to be additive nonseparable. Rather than trying to resolve any of the puzzles promoted in the literature, the focus is on results that should be useful in trying to understand the effects on equilibrium prices and returns of modeling choices within these economies. Hopefully, the excursion of these last two sections has made it clear that this is a worthwhile endeavor.

4. Outline of the Thesis

The next three chapters are written as self contained papers, and some repetition of basic assumptions and setup is therefore unavoidable. The notation should in large be consistent throughout the thesis.

As indicated in the previous sections, the main topic of this thesis is characterizations of equilibrium prices and returns in pure exchange representative consumer economies. Common to all of the chapters are

- a continuous information structure, with endowments following square integrable Itô processes,
- the notion of a representative consumer spot-securities market equilibrium introduced by Lucas (1978), and
- restrictions on equilibrium returns based on the work of Merton (1973), Breeden (1979), and Detemple and Zapatero (1991).

With that said, it might be illuminating to state which are the related topics that I do not treat in this thesis; production, optimal portfolio and consumption choice, and aggregation:

I restrict attention throughout to pure exchange economies. Aggregate consumption is therefore necessarily exogenous. Given the aim of the thesis, including *production* will add little beyond more complicated notation. Any outcome of an endogenous Itô production process can be implemented in the pure exchange economies under study in chapters 3 and 4.[15] Still, this highlights another, more

[15]The production economy corresponding to the exchange economy in chapter 2 has been studied extensively in the literature (Sundaresan 1989, Constantinides 1990). Furthermore, it has been treated as a general pure exchange economy, with endowments restricted to general Itô processes (Detemple and Zapatero 1991).

interesting, topic that I do not treat: *optimal portfolio and consumption choice* of the individual consumers. The reason I do not treat this admittedly important problem is twofold. First, it has been treated at a relatively general level by Detemple and Zapatero (1992), and Detemple and Giannikos (1996). Any analysis of optimal portfolio and consumption choice within the economies of this thesis will largely be a mechanical extension of their results. Second, their analyses show that there is really not much to learn about the optimal demand functions in abstract economies, apart from existence results. In order to gain any economic insight it is necessary to make far more restrictive assumptions on preferences and security returns than done in this thesis, as for instance in the analyses by Hindy, Huang, and Zhu (1997a, 1997b), and Cuoco and Liu (1998).

This again highlights a serious weakness of the representative consumer asset pricing literature at its current state: The lack of general *aggregation* results. As long as we do not have a more general theory for aggregating individual portfolio and consumption choice, the representative consumer economies will not represent fully general equilibrium models—as pointed out already by the originator (Lucas 1978). This thesis does not contribute in improving upon this situation.

Habit Formation. In chapter 2, I study special parametric cases of an economy where the consumer derives felicity $v(t, c_t, z_t)$ from the rate of purchases of a commodity c_t, and previous standards of living $z_t = \int_0^t g(s, c_s)\, ds$.[16] Since $v_z(t, c, z) \leq 0$, z is interpreted as the consumer's level of consumption habits. This economy has been extensively studied previously for the special case where $v(t, c, z) = f(t)\gamma^{-1}(c - z)^\gamma$, called *additive* or *linear habit formation*. The economy has also been studied in the most general case with no parametric restriction on v.

Additive habit formation has been widely criticized for inducing "consumption addiction," in that the marginal felicity goes to infinity as c approaches z. The general case, with no parametric restriction on v, is too general to evaluate this criticism. In chapter 2 I derive restrictions on equilibrium returns not only for linear habit formation, but also for *multiplicative* or *nonlinear habit formation*, $v(t, c, z) = f(t)\gamma_c^{-1} c^{\gamma_c} z^{\gamma_z}$. Restrictions on returns are derived for both models using the same restrictions on the exogenous endowment. In both cases the equilibrium

[16] I.e., in equation (6), $Z_t(C) = (c_t, z_t)$, and $f_t(Z_t(C), U_t) = u(c_t, z_t) - \beta_t U_t$ with $u_z \leq 0$.

state price deflator and returns are expressed as infinite sums of moments of c and z. If γ_z is a natural number the upper limit of the sums collapses to $\gamma_z - 1$. It is therefore trivial to conduct numerical analysis and comparative statics of the derived restrictions across the two models of habit formation. This makes it possible to determine to what extent it matters how habit formation is modelled in the consumer's felicity.

Closely Related Work. Sundaresan (1989), and Constantinides (1990) study optimal portfolio and consumption strategies in production economies when the utility functional exhibits additive habit formation, $f(t)\gamma^{-1}(c-z)^{\gamma}$. Constantinides use the procedure of Cox, Ingersoll, Jr., and Ross (1985) to derive restrictions on equilibrium returns. Detemple and Zapatero (1991) derive equilibrium prices and returns in a pure exchange economy with a general felicity $v(t, c, z)$. Chapter 2 is a special case of the economy of Detemple and Zapatero.

Main Contribution. I derive explicit restrictions on equilibrium returns with multiplicative habit formation. It is important to be able to evaluate the effect of this parametrization relative to additive habit formation, especially since Schroder and Skiadas (1999) derive an isomorphism between additive habit formation and the standard model with additive separable utility. They are not able to derive a similar relationship for multiplicative habits.

In addition, the methodology is important also in numerical studies of other, more general economies with additive nonseparable utility functionals. It is for instance trivial to extend the techniques presented in this chapter to the economy of chapter 3. Hence, the techniques developed in chapter 2 enable comparisons of economies with quite different preferences, within a unified framework, using efficient and accurate numerical analysis and comparative statics.

Local Substitution and Distant Complementarity. Chapter 3 utilizes the framework offered by Duffie and Skiadas' (1994) economy, represented here by equation (6). The consumer derives felicity $v(t, y, z)$ from services $y_t = \int_0^t g^y(s, c_s)\, ds$ received from purchases of a commodity c. In addition, the felicity also depends on previous standards of living $z_t = \int_0^t g^z(s, c_s)\, ds$. A possible interpretation is that y represents the stock of a durable commodity, while z represents the level of consumption habits. The marginal felicities are restricted by $v_y(t, y, z) > 0$ and $v_z(t, y, z) \leq 0$.[17] The former restriction introduces local substitution in c, while the latter restriction introduces distant complementarity. Equilibrium prices and returns turn out to consist of two symmetric factors, one relating mainly to the level of services y, and the other relating mainly to the level of habits z.

The equity risk premium increases when habit formation is introduced in an economy with only local substitution, since habit formation effectively increases the consumer's risk aversion. Local substitution has previously been found to decrease risk premia in production economies (Hindy and Huang 1993, Hindy, Huang, and Zhu 1997b). This is also the general tendency in this economy, but a counter example is given. A comparison to Breeden's (1979) CCAPM, given here in equation (3), shows that local substitution can increase risk premia.

Closely Related Work. This chapter mainly uses results developed by Duffie and Skiadas (1994), Detemple and Zapatero (1991), and Hindy, Huang, and Zhu (1997b). Duffie and Skiadas develop a general characterization of state prices. Detemple and Zapatero show how the Itô decomposition of said state prices can be derived. This makes it possible to use the link between state prices and equilibrium returns developed by Duffie and Zame (1989), and Back (1991). All of these results are used to study the exchange economy counterpart of the production economy developed by Hindy, Huang, and Zhu.

Main Contribution. The main contribution of this chapter is twofold. First, it is the first investigation of its kind into equilibrium prices and returns in a continuous-time economy with durability and habit formation.[18] Second, in contrast to any economy with habit formation without local substitution (Sundaresan 1989, Constantinides 1990, Detemple and Zapatero 1991, and chapter 2), the suggested economy ensures positive state prices with quite conventional restrictions

[17] In terms of the dynamic utility of Duffie and Skiadas (1994) in equation (6), $Z_t(C) = (y_t, z_t)$, and $f_t(Z_t(C), U_t) = u(y_t, z_t) - \beta_t U_t$. The present economy is somewhat similar to that of Detemple and Zapatero (1991), treated in chapter 2, with one important difference. Their economy assumes $Z_t(C) = (c_t, z_t)$. Hindy, Huang, and Kreps (1992), and Hindy and Huang (1992) introduce a class of preferences that exhibit ***local substitution***, which they argue is attractive on normative grounds. Intuitively, an individual with local substitution does not want to eat dinner right after lunch. This is not a behavioral implication shared by any felicity that depends on $Z_t(C) = (c_t, \ldots)$, unless it is linear in c_t.

[18] This claim might seem surprising to readers familiar with the title of the paper of Hindy, Huang, and Zhu (1997b). An argument supporting this claim can be glanced from sections 1 and 6 in chapter 3.

on the representative consumer's felicity, and no additional restrictions on the consumer's endowment. I thus offer sufficient conditions to prevent a potentially serious problem that has been shown to occur in certain economies with additive habit formation (Chapman 1998).

Additive Nonseparable Utility. Chapter 4 is in the spirit of Duffie and Skiadas (1994). They study existence of the state price deflator induced by a class of preferences that encompass recursive utility as well as additive nonseparable utility. In addition they allow for a large class of information structures. In comparison, I restrict both. Utility is additive across states but not necessarily additive across time. Further, information is restricted to arrive according to a Brownian filtration. This makes it possible to derive much stronger implications for equilibrium returns than at the level of generality studied by Duffie and Skiadas, while still retaining a large degree of freedom. In addition, I allow for multiple commodities and state variables. This is an extension of their approach, which is necessary to encompass several utility functionals studied in the asset pricing literature (for instance Dunn and Singleton 1986, Abel 1990, Campbell and Cochrane 1999).

Closely Related Work. The intellectual debt to the work by Duffie and Skiadas (1994) is apparent. As in chapter 3, I also borrow heavily from Duffie and Zame (1989), Back (1991), and Detemple and Zapatero (1991) for the ideas of this chapter. As alluded to previously, Duffie and Zame, and Back derive "state price beta models," the most general possible versions of consumption-based capital asset pricing models. Their techniques are combined with those of Detemple and Zapatero to characterize equilibrium returns.

Main Contribution. This chapter brings together a large number of asset pricing models within a coherent framework, as illustrated in table 1 (p. 83). The framework presented in this chapter is hopefully transparent enough to increase our understanding of how extant models relate to each other, and sufficiently general to facilitate the development of new models. In addition, the framework is hopefully explicit enough to ease these developments. In other words, it is my hope that I have chosen a fertile compromise between being general and being sufficiently parsimonious to gain new insights.

CHAPTER 2

Habit Formation

I deduce closed form expressions for the equilibrium state price deflator, riskless interest rate and risk premia in an exchange economy where the representative consumer's preferences exhibit habit formation. The framework presented makes it possible to analyze the implications of modeling choices within economies with habit formation. In particular, solutions are derived for the two dominant modeling choices of additive and multiplicative habits. The explicit characterizations facilitate analyses of viable ranges for the parameters that enter these economies for the two modeling choices, within a common framework.

JEL CLASSIFICATION: C63, D51, G12.

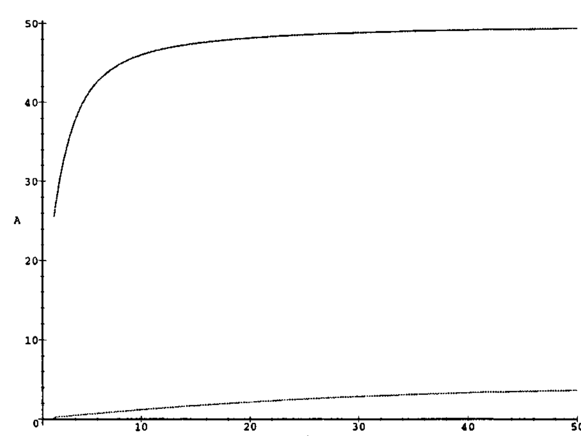
KEY WORDS: Multiplicative (nonlinear) and additive (linear) habit formation; Closed form solutions; Equilibrium asset pricing; CCAPM; Malliavin calculus.

1. Introduction

One of the major contributions of equilibrium theory to financial economics is that it makes it possible to derive implications for equilibrium returns from first principles. Comparing the implications with observations it is possible to evaluate the validity of the basic assumptions being made. A well known and important empirical discrepancy is the "equity premium puzzle" discovered by Mehra and Prescott (1985). It posits that the observed equity premium is too large and the riskless interest rate is too low compared to that predicted by a representative consumer equilibrium model. This anomaly can be reproduced in any representative consumer model with additive separable utility, complete markets, and no market frictions. The puzzle arises because the model fits the observed mean and variance of consumption only with an Arrow-Pratt coefficient of relative risk aversion, $R_r(\cdot)$, between 30–40. Figure 1 shows the certainty equivalents for two binary lotteries when the consumer's fortune changes 50% with a probability of 1/2, or changes

5% with the same probability. The fraction of initial wealth that the consumer is willing to forego to get rid of the lottery gets close to the maximum loss of the most risky lottery when $R_r(\cdot) > 10$, constituting 46% of initial wealth when $R_r(\cdot) = 10$. The ask price is not that unreasonable for the less risky lottery, where it constitutes 1% of initial wealth. Further, even if a high risk aversion is accepted the additive

FIGURE 1. Ask price for a Bernoulli lottery



The graph of the ask price A is given by the equation $u(w - A) = \frac{1}{2}[u(w(1+x)) + u(w(1-x))]$, where $u(w) = (1-b)^{-1}w^{1-b}$, $x \in \{.05, .5\}$, $b \in [1, 50]$, and $w = 100$. The steep and the flat curves represent the large and the small lotteries respectively.

separable model implies a high riskless rate of return. One way to resolve this is to allow the subjective discount rate $\beta < 0$. Hence, the additive separable model fits observations only with unlikely risk aversion and subjective discount rates.[1]

The reason for this bond between the riskless interest rate and the equity premium is that the consumer's risk aversion is tied to the consumer's willingness to substitute consumption over time in the additive separable model. The Arrow-Pratt coefficient of relative risk aversion is a good measure of consumers' attitudes towards atemporal gambles over wealth within this framework. Denoting the consumer's *felicity* (also known as 'Bernoulli utility') from wealth by $v(t, W) = f(t)u(W)$,

[1]Estimates of β in the extant literature have often been negative. Kocherlakota (1990) discusses a possible explanation why this is not a true estimate of the consumers' subjective discount rate, and that the true value is positive.

$R_r(W) = -Wu_{WW}(W)/u_W(W)$. The higher the consumer's $R_r(\cdot)$, the higher is the risk premium necessary to keep demand for the lottery constant. For additive separable utility the instantaneous elasticity of consumption is the limit of the intertemporal elasticity of substitution in consumption:

$$\begin{aligned}\text{IESC}(c; s > t) &\triangleq -\frac{v_c(c_s)/v_c(c_t)}{c_s/c_t}\left[\frac{\partial\{v_c(c_s)/v_c(c_t)\}}{\partial\{c_s/c_t\}}\right]^{-1} \\ &\longrightarrow_{s\to t} -\frac{u_c(c_t)}{u_{cc}(c_t)c_t} \triangleq \text{IES}(c_t).\end{aligned}$$

When IES$(\cdot)$ increases the consumer's willingness to substitute consumption today for consumption tomorrow increases, and the riskless interest rate decreases due to an increase in savings. In an equilibrium for a representative consumer pure exchange economy the optimal consumption plan is to consume the aggregate endowment at each instant in time. For a state price π the wealth of the representative consumer is thus $W_t = \pi_t c_t$, and a gamble over wealth is equivalent to a gamble over consumption. It follows that IES$(\cdot) = R_r(\cdot)^{-1}$. This is the relationship that is believed to cause the coexistence of the equity premium and riskless rate puzzles. For reasonable values of β one can increase the equity premium by increasing the consumer's risk aversion only by also increasing the riskless interest rate.

Several approaches have been suggested to alleviate the implications of the model used by Mehra and Prescott. One is to use preferences that are not time and state additive. Marginal utility will then have terms in addition to the marginal felicity $v_c(\cdot)$, and the inverse relationship between risk aversion and willingness to substitute over time will not necessarily hold. Another approach is to make alternative assumptions on for instance the endowment process in an exchange economy, or the technology for transformation in a production economy. A third approach is to examine the data set used in testing the models. I choose to focus on the first approach.

1.1. Theoretical Work. Durabilities, habit formation, and Kreps-Porteus preferences (Kreps and Porteus 1978) have received the most attention among the approaches that use alternative specifications of the consumers' preferences. The latter separate risk preferences and certainty preferences. It should thus be possible to achieve an increase in the risk premium by using a high degree of risk aversion, while separately calibrating the parameter governing substitution to achieve a low riskless interest rate. Hence, in principle this represents a solution to the puzzles.

Durabilities (that induce local substitution) and habit formation (that induces distant complementarity) both keep the state additive framework of von Neumann-Morgenstern preferences. Hence, the inverse relationship between the consumer's relative risk aversion and elasticity of substitution in consumption is kept (Detemple and Zapatero 1991, p. 1639).[2] Durabilities decrease the risk premium (Hindy and Huang 1993),[3] while habit formation increases it (Sundaresan 1989, Constantinides 1990, Detemple and Zapatero 1991, Hindy, Huang, and Zhu 1997b). Habits make the consumer more averse to changes in consumption. Therefore, one attains a higher risk premium for a given variance in consumption, relative to the additive separable model. Durabilities are interesting in combination with habit formation as empirical research find it to increase the predictive power of a model using habit formation.

1.2. Empirical Findings. Epstein and Zin (1990, 1991), and Bekaert, Hodrick, and Marshall (1997) investigate to what extent first order risk aversion,[4] based on Kreps-Porteus preferences, can explain the size and predictability of risk premia typically observed in securities markets. Epstein and Zin find empirical support for this class of nonexpected utility, but do not take Roll's critique into account. Bekaert, Hodrick, and Marshall take an alternative approach, not subject to Roll's critique. They do not find empirical support for these preferences. Kocherlakota (1990) shows that when increments of the aggregate endowment are *i.i.d.* then the Riesz representation of the utility functional of Epstein and Zin (1989), which is a representation of Kreps-Porteus preferences, coincides with the Riesz representation of an additive separable von Neumann-Morgenstern utility functional. Hence, it is impossible to econometrically distinguish between the two specifications. The result is shown to be robust to the introduction of serial correlation. The issue is

[2]Note that this does not imply that the Arrow-Pratt coefficient of relative risk aversion is the inverse of the instantaneous elasticity of substitution. See also e.g. Campbell and Cochrane (1999) for details.

[3]This is not universally true, but seems to hold for felicities with desirable properties. Chapter 3 offers a counter-example.

[4]Approximating von Neumann-Morgenstern utility by a Taylor series, the risk premium is approximately linear in the variance of small risks. This is the case of ***second order risk aversion***. The utility functional is said to exhibit ***first order risk aversion*** if the utility functional is approximately linear in the standard deviation of small risks.

still open though, as it has been shown that it is in principle possible to distinguish between them when Kocherlakota's hypotheses do not hold (Wang 1993, Ma 1998).

Using quadratic felicity, Heaton (1993) finds support for a combination of local substitution and habit formation. In light of the theoretical results of Hindy and Huang (1993), and Hindy, Huang, and Zhu (1997b) though, Heaton's results can alternatively be interpreted as an indication that more powerful econometric methodology is needed.

The conclusion to be drawn from the theoretical and empirical findings is not necessarily that one specification of preferences is superior to the others. Rather, it is clear that there is need for more theoretical and empirical work in order to better understand the empirical regularities, and to understand the implications from the theoretical models.

The motivation of this chapter is to achieve a better understanding of preferences that depend on habits. Until now no study has analyzed the effects of different specifications of the felicity within a common framework with habit formation. Further, no study has so far derived explicit characterizations of returns without making undesirable assumptions on the structure of the consumer's felicity. I deduce simple analytic expressions for equilibrium returns, that easily lend themselves to comparative static analysis, and that will allow a comparison of additive versus multiplicative habits. This is accomplished while retaining more desirable properties of the felicity function than in previous studies. Another motivation is the mathematical problem itself, of finding an explicit solution to the general model of habit formation derived by Detemple and Zapatero (1991), whose economy is the foundation for the results derived herein.

This chapter is organized as follows. The economic primitives necessary to deduce equilibrium returns on financial securities are presented in section 2. These primitives are first used to derive the state price deflator in section 3. Equilibrium returns are then explicitly characterized in section 4. This is the main result of the chapter, and the proof is conducted in section 5. The last section concludes.[5]

[5]An economic analysis of the models derived in this chapter is conducted in a companion paper (Haug 1998).

2. The Economy

Consider a continuous-time pure exchange economy with a composite commodity as in Duffie and Huang (1985). The fundamental uncertainty is represented by a complete probability space $(\Omega, \mathcal{F}, P)$. $B : \Omega \times [0,T] \rightarrow \mathbb{R}^d$ is a d-dimensional standard Brownian motion defined on the preceding triple. Uncertainty resolves over time according to the augmented filtration $\mathbb{F} \triangleq \{\mathcal{F}_t^B \triangleq \sigma(B_t) \cup \mathcal{F}_0 : t \in [0,T]\}$. By convention all uncertainty is resolved at time T. In the following I write $\mathcal{F}_t$ for any $\mathcal{F}_t^B \in \mathbb{F}$. For simplicity $\mathcal{F}_0$ is equal to the trivial σ-algebra augmented with the P-negligible events in $\mathcal{F}$.

The general setup is a representative consumer economy in the spirit of Lucas (1978). The consumer has preferences defined over the consumption rate c and standard of living z, represented by the von Neumann-Morgenstern utility functional

$$U(c) = \mathrm{E}\left\{\int_0^T v(t, c_t, z_t)\, dt\right\}, \tag{8}$$

where

$$z_t = e^{-\alpha t} z_0 + \delta \int_0^t e^{-\alpha(t-s)} c_s\, ds, \quad z_0, \alpha, \delta \geq 0$$

and

$$v(t, c_t, z_t) = f(t) u(c_t, z_t).$$

So far explicit solutions in continuous time have been found for additive habits of the form

$$v(t, c_t, z_t) = e^{-\beta t} u(c_t - z_t)$$

when u is an exponential or a power function (Sundaresan 1989, Constantinides 1990, respectively). This specification has received wide criticism in the literature as it induces consumption addiction in the case of power felicity, in that $\partial u(x)/\partial x \xrightarrow[x \downarrow 0]{} \infty$ (see e.g. Detemple and Zapatero 1991, Hindy, Huang, and Zhu 1997b). This has severe consequences in an exchange economy, where the consumers cannot protect themselves fully from current consumption falling below the standard of living. To prevent prices from being unbounded it is necessary to restrict endowments such that $e \gg z$.

For the general result of Detemple and Zapatero (1991) it is necessary to restrict attention to felicities satisfying certain mild concavity assumptions. A technical restriction on the felicity is also necessary to ensure a positive state price process.[6]

ASSUMPTION 2.1. *$v \in C^{1,3,2}([0,T] \times \mathbb{R}_+ \times \mathbb{R}_+)$, $v_c > 0$, $v_{cc} < 0$, $v_z < 0$, $v_{cc}v_{zz} - v_{cz}^2 \geq 0$, and v satisfies the Inada conditions wrt. its second argument. In addition, letting β_t be the subjective discount rate,*

$$u_c(e_t, z_t) + \delta \mathrm{E} \left\{ \int_t^T e^{-\int_t^s \beta_u + \alpha \, du} u_z(e_s, z_s) \, ds \, | \mathcal{F}_t \right\} > 0$$

and $e \gg 0$.

ASSUMPTION 2.2. *Aggregate endowments $\{e_t\}$ follow a strictly positive Itô process $de_t = \mu_e(t)\, dt + \sigma_e(t)\, dB_t$, $e(0) = e_0$, with bounded $\mathcal{F}_t$-adapted coefficients.*

The following two assumptions are special cases of the preceding ones. They are sufficient to arrive at explicit characterizations of equilibrium prices and returns.

ASSUMPTION 2.3 (Multiplicative habits). *The felicity of the representative consumer is a generalized power function, homogeneous of degree $\gamma_1 + \gamma_2$:*

$$v(t, c_t, z_t) = e^{-\beta t} \frac{1}{\gamma_1} c_t^{\gamma_1} z_t^{\gamma_2}.$$

Habit formation restricts $\gamma_1 \gamma_2 < 0$. This restriction is not necessary in the subsequent analysis though, and is therefore dropped. When $\gamma_1 \gamma_2 > 0$, $v_z(t, c, z) > 0$, and the consumer experiences an increase in felicity from an increase in z. The classic interpretation of this case is that z represents the flow of services from previous purchases of the commodity c, i.e., the case of a durable commodity deteriorating at the rate α.

[6]Chapman (1998) investigates the restrictions put on the parameters of the model by this assumption. He derives explicit characterizations of the equilibrium risk premia by assuming that the net consumption rate $c_t - z_t$ follows a geometric Brownian motion, and hence exogenously determines the dynamics of the standard of living z_t.

ASSUMPTION 2.4 (Endowments). *Aggregate endowments* $\{e_t\}$ *follow a geometric Brownian motion, or in differential notation* $de_t = \mu_e e_t\, dt + \sigma_e e_t\, dB_t$, $\mu_e \in \mathbb{R}, \sigma_e \in \mathbb{R}^d \setminus \{0\}$.

The variance of the rate of change in the endowment process is $\hat{\sigma}_e^2 \triangleq \sigma_e \sigma_e^\top = \sum_{i=1}^d (\sigma_e^i)^2 \in \mathbb{R}_{++}$. Since $\hat{\sigma}_e \hat{B}_t \overset{d}{=} \sigma_e B_t$ when $\hat{B}$ is independent of B, the subsequent results will not change if I abuse notation by $\sigma_e^2 \triangleq \hat{\sigma}_e^2$ and let $\sigma_e = \sqrt{\hat{\sigma}_e^2}$.

All previous work of habit formation in continuous time has been done using additive habits $u(c-z)$. To enable analysis of this case within the framework of the economy induced by assumption 2.4, state prices and returns are also determined under this form of habits, again with power felicity.

ASSUMPTION 2.5 (Additive habits). *The representative consumer has an isoelastic felicity function of consumption net of habits:*

$$v(t, c_t, z_t) = e^{-\beta t} \frac{1}{\gamma} (c_t - z_t)^\gamma.$$

The consumer is allowed to choose c from some consumption space $C \subseteq L_+^2(P \times \lambda)$ of $\mathcal{F}_t$-adapted processes such that $U : C \to \mathbb{R}$ is well defined. Further, the consumer receives endowments in the form of a process $e \in L_+^2(P \times \lambda)$. The square integrability conditions will allow any positive linear functional on C to have a Riesz representation $\pi \in L^2(P \times \lambda)$. From Hölder's Inequality we know that for any $c \in L_+^p(\mu)$ and $d \in L_+^q(\mu)$, $\int cd\, d\mu \leq \|c\|_p \|d\|_q$ for some measure μ, when q is the Hölder conjugate of p. The integrability condition thus ensures that the consumption and state price deflator have finite second moments, and that the functionals on C take finite values. The functional of interest is of course the price functional given by the representative consumer's marginal rates of substitution in equilibrium.

To attain a consumption stream other than e the consumer is allowed to trade continuously and frictionlessly in $N+1$ long lived securities with price processes $S \in \mathbb{R}_+^{N+1}$. Security j entitles its holder to a cumulative dividend process $D^j, j = 0, \ldots, N$ such that the gains process $G \triangleq D + S$ is an Itô process. When G is an Itô process a version of the Consumption-based Capital Asset Pricing Model (CCAPM) of Breeden (1979) can easily be derived as in Duffie and Zame (1989).

G can be written in differential notation as

$$dG_t = \mu_G(t)\,dt + \sigma_G(t)\,dB_t,\ G_0 = g \in \mathbb{R}^{N+1}_{++},$$

for sufficiently well behaved functions $\mu_G : \mathbb{R}^{N+1} \times [0,T] \to \mathbb{R}^{N+1}$ and $\sigma_G : \mathbb{R}^{N+1} \times [0,T] \to \mathbb{R}^{(N+1)\times d}$. Alternatively, if $\mathbf{I}_S(t)$ is a diagonal matrix with $S^i(t)$ on its i'th diagonal element, $\mu_t = \mathbf{I}_S(t)^{-1}\mu_G(t)$ and $\sigma_t = \mathbf{I}_S(t)^{-1}\sigma_G(t)$, then

$$dG_t = \mathbf{I}_S(t)[\mu_t\,dt + \sigma_t\,dB_t].$$

G^0 is assumed to be of bounded variation. The locally riskless interest rate r_t induced by S^0_t is innocuously assumed bounded and adapted to $\mathbb{F}$. In total, we can describe the economy by

$$\mathcal{E} = \{(\Omega,\mathcal{F},\mathbb{F},P); B; (U,e); D\}.$$

The aim of this article is to use $\mathcal{E}$ to explicitly characterize S under the additional assumptions 2.3, 2.4, and 2.5.

3. The State Price Deflator

A consumption plan c^* and state price deflator π constitutes an Arrow-Debreu *equilibrium* if c^* is optimal for the consumer, c^* is budget feasible in the sense that $\mathrm{E}\left\{\int c^*\pi\right\} \le \mathrm{E}\left\{\int e\pi\right\}$, and spot markets clear; $c_t \le e_t\ \forall\ t \in [0,T]$ P-a.s.

The first order condition for optimality of a representative consumer is sufficient to support an Arrow-Debreu equilibrium as stated above. Several approaches to characterizing the first order condition are available. One approach is to solve the optimal consumption and investment problem separately, using Pontryagin's Maximum Principle. A shorter route is to use the utility gradient approach of Duffie and Skiadas (1994) to find the Riesz representation of the Gâteaux derivative of the consumer's utility functional.

PROPOSITION 2.1 (Duffie and Skiadas). *Consider the economy $\mathcal{E}$ and assume that assumptions 2.1 and 2.2 hold. Then there exists an Arrow-Debreu equilibrium (e,π) only if, modulo a constant*

$$(9)\quad \pi_t = e^{-\int_0^t \beta_u\,du} u_c(e_t, z_t) + \delta e^{-\int_0^t \beta_u\,du} \mathrm{E}\left\{ \int_t^T e^{-\int_t^s (\beta_u+\alpha)\,du} u_z(e_s, z_s)\,ds \,|\mathcal{F}_t \right\}.$$

It is worth noting that Detemple and Zapatero (1991), and Chapman (1998) mistake the quantity $\xi_t e^{-\int_0^t r_u - \beta_u\,du} \triangleq \pi_t e^{\int_0^t \beta_u\,du}$ for the state price deflator. The correct relationship between the state price deflator and the Radon-Nikodym derivative restricted to $\mathcal{F}_t$ must be given by $\pi_t = \xi_t e^{-\int_0^t r_u\,du}$ P-a.s. for equivalence between an Arrow-Debreu equilibrium and the existence of a unique equivalent martingale measure to hold.

Proposition 2.1 easily lends itself to economic interpretations, as is done by Detemple and Zapatero (1991). Still, to be able to conduct computations involving the state price deflator further restrictions are needed. The following result gives a characterization when the representative consumer has power felicity and multiplicative habits.

PROPOSITION 2.2 (Multiplicative habits). *Consider the economy in Proposition 2.1 and assume in addition that assumptions 2.3 and 2.4 hold. Then, modulo a constant, the state price deflator is given by*

$$\pi_t = e^{-\beta t} e_t^{\rho_1} z_t^{\gamma_2} + e^{-\beta t} \frac{\delta \gamma_2}{\gamma_1} e_t^{\gamma_1} S_k^n(\rho_2; t) \frac{k!}{\sigma_e^{2k}} \sum_{j=0}^{k} c_{j,k}^{\nu(\gamma_1)} f_1(T, \varpi_1)$$

where $\nu = \alpha + \mu_e - \frac{1}{2}\sigma_e^2$, $\nu(\gamma) \triangleq \nu + \gamma\sigma_e^2$, $\rho_i = \gamma_i - 1$, *and*

$$\begin{aligned}
\varpi_1 &\triangleq \tfrac{1}{2} j^2 \sigma_e^2 + j\nu(\gamma_1) + \gamma_1[\mu_e - \tfrac{1}{2}\sigma_e^2(1-\gamma_1)] - \alpha\rho_2 - (\alpha+\beta),\\
S_k^n(x;t) &\triangleq \sum_{n=0}^{\infty} \frac{\Gamma(x+1)}{n!\,\Gamma(x+1-n)} \left(z_0 e^{-\alpha t}\right)^{x-n} \sum_{k=0}^{n} \tbinom{n}{k} (\delta e_t)^k \bar{z}_t^{\,n-k},\\
\bar{z}_t &\triangleq z_t - z_0 e^{-\alpha t},\\
f_1(s_1;\phi_1) &\triangleq \frac{\exp\{\phi_1(s_1 - t)\} - 1}{\phi_1}.
\end{aligned}$$

3.1. Technical Lemmas. To prove Proposition 2.2 it is necessary to be able to characterize the n'th moment of the exponential of Brownian motion with drift,

$$\mathrm{E}\left\{ \left(\int_0^t \exp\{\mu s + \sigma B_s\}\,ds \right)^n \right\},$$

when $n \in \mathbb{N}$. The first moment can be computed by applying Fubini's theorem. For higher moments this approach is not viable and the following Lemma will be useful.

LEMMA 2.1 (Yor). *Let $B \in \mathbb{R}$ be a standard Brownian motion, and let $A^{\mu}_{0,t}$ denote $\int_0^t e^{\mu s+\sigma B_s}\,ds$. For any $\sigma \in \mathbb{R}\setminus\{0\}$, $\mu \in \mathbb{R}$ and $n \in \mathbb{Z}_+$ the n'th moment of $A^{\mu}_{0,t}$ is*[7]

$$\mathrm{E}\left\{\left(A^{\mu}_{0,t}\right)^n\right\} = \frac{n!}{\sigma^{2n}}\sum_{j=0}^{n} c^{\mu}_{j,n}\exp\left\{\left(\frac{j^2\sigma^2}{2}+j\mu\right)t\right\}, \tag{10}$$

where

$$c^{\mu}_{j,n} \triangleq \begin{cases} \prod\limits_{\substack{j\neq k\\ k\in\{0,\dots,n\}}} \left[\frac{\mu}{\sigma^2}(j-k)+\frac{1}{2}(j^2-k^2)\right]^{-1} & n\in\mathbb{N},\\ 1 & n=0. \end{cases}$$

PROOF. The Lemma is simply Yor's Corollary 2 restricted to Brownian uncertainty and trivially extended to allow for $n = 0$. □

The following extension of Lemma 2.1 will also prove useful in the proofs of several results.

LEMMA 2.2. *Let $A^{\mu}_{0,t}$ be as in Lemma 2.1. For any $k \in \mathbb{Z}_+$ and any $dP^m/dP \triangleq \exp\{m\sigma B_T - \frac{1}{2}m^2\sigma^2 T\}$, $m \in \mathbb{R}$,*

$$\mathrm{E}^{P^m}\left\{\left(A^{\mu}_{0,s}\right)^k|\mathcal{F}_t\right\} = \sum_{n=0}^{k}\binom{k}{n}\left(A^{\mu}_{0,t}\right)^{k-n}e^{(\mu t+\sigma B_t)n}\mathrm{E}^{P^m}\left\{\left(A^{\nu_m}_{0,s-t}\right)^n\right\}, \qquad s>t,$$

where $\nu_m \triangleq \mu + m\sigma^2$.

PROOF. Let $B^m_t = B_t - m\sigma t$, a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{F}, P^m)$ by Girsanov's Theorem. By the additivity of the Lebesgue integral, Newton's binomial formula, the linearity of the abstract integral, and that B^m_t is $\mathcal{F}_t$ measurable

$$\begin{aligned}\mathrm{E}^{P^m}\left\{\left(A^{\mu}_{0,s}\right)^k|\mathcal{F}_t\right\} &= \mathrm{E}^{P^m}\left\{\left(A^{\nu_m}_{0,s}\right)^k|\mathcal{F}_t\right\}\\ &= \mathrm{E}^{P^m}\left\{\left[A^{\nu_m}_{0,t}+A^{\nu_m}_{t,s}\right]^k|\mathcal{F}_t\right\}\end{aligned}$$

[7]The restriction on σ is due to the definition of $c^{\mu}_{j,n}$. See Yor's Corollary 1 for the more general case of $\sigma \in \mathbb{R}$. Of course, with $\sigma = 0$ the expectation is trivially equal to $\mu^{-n}(e^{\mu T}-1)^n$.

$$= \mathrm{E}^{P^m}\left\{\sum_{n=0}^{k}\binom{k}{n}\left(A_{0,t}^{\nu_m}\right)^{k-n}\left(A_{t,s}^{\nu_m}\right)^n|\mathcal{F}_t\right\}$$

$$(11) \qquad = \sum_{n=0}^{k}\binom{k}{n}\left(A_{0,t}^{\nu_m}\right)^{k-n}\mathrm{E}^{P^m}\left\{\left(A_{t,s}^{\nu_m}\right)^n|\mathcal{F}_t\right\}.$$

It remains to evaluate the P^m-conditional expectation. Using the independence of $B_u^m - B_t^m,\ u \geq t$ from $\mathcal{F}_t$, and the substitution $u = v + t$,

$$\mathrm{E}^{P^m}\left\{\left(A_{t,s}^{\nu_m}\right)^n|\mathcal{F}_t\right\} = \mathrm{E}^{P^m}\left\{\left(\int_t^s e^{\nu_m u+\sigma(B_u^m-B_t^m)}\,du\right)^n\right\}e^{n\sigma B_t^m}$$
$$= \mathrm{E}^{P^m}\left\{\left(\int_0^{s-t} e^{\nu_m(v+t)+\sigma(B_{v+t}^m-B_t^m)}\,dv\right)^n\right\}e^{n\sigma B_t^m}.$$

This P^m-expectation can easily be transformed into Yor's canonical form by constructing a new $(P^m, \mathcal{F}_v)$ standard Brownian motion $B_v^\Delta \triangleq B_{v+t}^m - B_t^m$. Thereby,

$$\mathrm{E}^{P^m}\left\{\left(A_{t,s}^{\nu_m}\right)^n|\mathcal{F}_t\right\} = e^{n(\nu_m t+\sigma B_t^m)}\mathrm{E}^{P^m}\left\{\left(\int_0^{s-t} e^{\nu_m v+\sigma B_v^\Delta}\,dv\right)^n\right\}$$

$$(12) \qquad = e^{n(\nu_m t+\sigma B_t^m)}\mathrm{E}^{P^m}\left\{\left(A_{0,s-t}^{\nu_m}\right)^n\right\}.$$

Substituting (12) into (11) and changing to Brownian motion under P proves the Lemma. □

The following result from the proof of Lemma 2.2 will be useful in later proofs.

COROLLARY 2.1. *With definitions as in Lemma 2.1,*

$$\mathrm{E}\left\{\left(A_{t,s}^{\nu}\right)^n|\mathcal{F}_t\right\} = e^{n(\nu t+\sigma B_t)}\mathrm{E}\left\{\left(A_{0,s-t}^{\nu}\right)^n\right\}.$$

PROOF. The result follows from (12), with P in place of P^m. □

Another useful Corollary is

COROLLARY 2.2. *With definitions as in Lemma 2.1,*

$$\mathrm{E}\left\{\left(A_{t,s}^{\nu}\right)^n A_{t,s}^{\zeta}\right\} = \sum_{k=0}^{n}\binom{n}{k}\exp\{(n-k)[\nu+\tfrac{1}{2}(n+k+2)\sigma^2]t\}$$
$$\times\int_t^s \exp\{[\zeta+k\nu+\tfrac{1}{2}(k+1)^2\sigma^2]u\}\mathrm{E}\left\{\left(A_{0,s-u}^{\nu}\right)^k\right\}$$
$$\times\,\mathrm{E}^{P^{k+1}}\left\{\left(A_{0,u-t}^{[\nu+(k+1)\sigma^2]}\right)^{n-k}\right\}du.$$

Note that the Brownian motion in the P^{k+1}-expectation is the corresponding standard Brownian motion B^{k+1}. Further, both expectations are explicitly known by Lemma 2.1. Although the result uses no new techniques I include its proof for completeness.

PROOF. By Fubini's Theorem and Newton's binomial formula

$$\begin{aligned}\mathrm{E}\left\{\left(A^{\nu}_{t,s}\right)^{n} A^{\zeta}_{t,s}\right\} &= \int_t^s \mathrm{E}\left\{\left(A^{\nu,\sigma}_{t,u} + A^{\nu,\sigma}_{u,s}\right)^n e^{\zeta u+\sigma B_u}\right\} du\\ &= \int_t^s \sum_{k=1}^{n}\binom{n}{k}\mathrm{E}\left\{\left(A^{\nu}_{t,u}\right)^{n-k}\left(A^{\nu}_{u,s}\right)^{k} e^{\zeta u+\sigma B_u}\right\} du,\end{aligned}$$

by the Law of Total Probability (also known as 'Law of Iterated Expectations')

$$\begin{aligned}&= \int_t^s \sum_{k=1}^{n}\binom{n}{k}\mathrm{E}\left\{\left(A^{\nu}_{t,u}\right)^{n-k} e^{\zeta u+\sigma B_u}\mathrm{E}\left\{\left(A^{\nu}_{u,s}\right)^{k}|\mathcal{F}_u\right\}\right\} du\\ &= \sum_{k=1}^{n}\binom{n}{k}\int_t^s \mathrm{E}\left\{\left(A^{\nu}_{0,s-u}\right)^{k}\right\}\\ &\times \mathrm{E}\left\{\left(A^{\nu}_{t,u}\right)^{n-k} e^{(\zeta+k\nu)u+(k+1)\sigma B_u}\right\} du,\end{aligned}$$

using Corollary 2.1. Define a sequence of probability measures by

$$\frac{dP^{k+1}}{dP} = \exp\left\{\int_0^T (k+1)\sigma\, dB_t - \tfrac{1}{2}\int_0^T (k+1)^2\sigma^2\, dt\right\}.$$

Then by Girsanov's Theorem $B^k_t \triangleq B_t - k\sigma t$ is a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{F}, P^k)$ for each k. Hence

$$\begin{aligned}\mathrm{E}\left\{\left(A^{\nu}_{t,s}\right)^{n} A^{\zeta,\sigma}_{t,s}\right\} = \sum_{k=1}^{n}\binom{n}{k}\int_t^s &\exp\{[\zeta + k\nu + \tfrac{1}{2}(k+1)^2\sigma^2]u\}\\ &\times \mathrm{E}\left\{\left(A^{\nu}_{0,s-u}\right)^{k}\right\}\mathrm{E}^{P^{k+1}}\left\{\left(A^{\nu+(k+1)\sigma^2}_{t,u}\right)^{n-k}\right\} du.\end{aligned}$$

Applying the Law of Total Probability and Corollary 2.1 to the last term the result obtains. □

3.2. Proof of Proposition 2.2. With these results in hand the proof of Proposition 2.2 reduces to manipulations of conditional expectations.

PROOF. To complete the proof the right hand side of equation (9) must be computed under the additional assumptions of the proposition. Assume for now that $e_s^{\gamma_1} z_s^{\rho_2} \in L^1(P \times \lambda)$. By Fubini's Theorem

$$\pi_t = e^{-\beta t} e_t^{\rho_1} z_t^{\gamma_2} + \delta e^{-\beta t} \mathrm{E}\left\{ \int_t^T e^{-(\beta+\alpha)(s-t)} \frac{\gamma_2}{\gamma_1} e_s^{\gamma_1} z_s^{\rho_2} \, ds \, | \mathcal{F}_t \right\}$$

$$= e^{-\beta t} e_t^{\rho_1} z_t^{\gamma_2} + e^{-\beta t} \frac{\delta \gamma_2}{\gamma_1} \int_t^T e^{-(\beta+\alpha)(s-t)} \mathrm{E}\left\{ e_s^{\gamma_1} z_s^{\rho_2} \, | \mathcal{F}_t \right\} ds, \tag{13}$$

since $e_t^{\gamma_1}, z_t^{\rho_2} > 0$ and both are $P \times \lambda$ measurable, i.e., the change of integration is valid regardless of finiteness of the λ-integral (see e.g. Royden 1988).

The conditional expectation $\mathrm{E}\{e_s^{\gamma_1} z_s^{\rho_2} \, | \mathcal{F}_t\}$ can be further manipulated to allow an application of Lemma 2.2. Clearly

$$\mathrm{E}\{e_s^{\gamma_1} z_s^{\rho_2} \, | \mathcal{F}_t\} = \mathrm{E}\left\{ e_s^{\gamma_1} \sum_{n=0}^{\infty} \frac{\Gamma(\gamma_2)}{\Gamma(\gamma_2 - n) n!} \left(\frac{z_0}{e^{\alpha s}} \right)^{\rho_2 - n} \left(\delta \int_0^s e^{-\alpha(s-u)} e_u \, du \right)^n | \mathcal{F}_t \right\}$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(\gamma_2)}{\Gamma(\gamma_2 - n) n!} (z_0 e^{-\alpha s})^{\rho_2 - n} \left(\frac{\delta e_0}{e^{\alpha s}} \right)^n \mathrm{E}\left\{ e_s^{\gamma_1} \left(A_{0,s}^{\nu}\right)^n | \mathcal{F}_t \right\} \tag{14}$$

simplifying notation by $\nu \triangleq \alpha + \mu_e - \frac{1}{2}\sigma_e^2$. Since $f(x) = x^{\gamma_1} \in C^{2,1}(\mathbb{R} \times [0,T])$ it follows from Itô's Lemma that $e_t^{\gamma_1}$ is again a stochastic integral. As $\exp\{\frac{1}{2}\int_0^T \gamma_1^2 \sigma_e^2 \, dt\}$ is integrable, $\gamma_1 \sigma_e$ satisfies Novikov's condition. Hence, by Girsanov's theorem there exists a measure $\tilde{P} \sim P$ such that $\tilde{B}_t = B_t - \gamma_1 \sigma_e t$, $t \in [0,T]$, is a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{F}, \tilde{P})$, $T < \infty$, and $d\tilde{P}/dP = \exp\{\int_0^T \gamma_1 \sigma_e \, dB_t - \frac{1}{2}\int_0^T \gamma_1^2 \sigma_e^2 \, dt\} = e_T^{\gamma_1} / \mathrm{E}\{e_T^{\gamma_1} | \mathcal{F}_0\} > 0$ P.a.s. Using the properties of conditional expectation

$$\mathrm{E}\left\{ e_s^{\gamma_1} \left(A_{0,s}^{\nu}\right)^n | \mathcal{F}_t \right\} = \mathrm{E}\{e_s^{\gamma_1} | \mathcal{F}_t\} \mathrm{E}^{\tilde{P}}\left\{ \left(A_{0,s}^{\nu}\right)^n | \mathcal{F}_t \right\}, \quad s > t. \tag{15}$$

The last conditional expectation is well defined as $\left(A_{0,s}^{\nu}\right)^n \in L^1(\tilde{P})$ by Lemma 2.1 and 2.2. The P-expectation of $e_t^{\gamma_1}$ is finite since any real power of a lognormally distributed stochastic variable is again lognormally distributed.

The result now follows by applying Lemma 2.2 to (15) and using the additivity of the Lebesgue integral (innocuously assuming $\rho_2 < \infty$). □

Using the classical assumption on the structure of habits leads to the following characterization of the state price deflator:

PROPOSITION 2.3 (Additive habits). *Consider the economy in Proposition 2.1 and assume in addition that assumptions 2.4 and 2.5 hold. Modulo a constant, the* *state price deflator is given by*

$$\text{(16)}\quad \pi_t^a = e^{-\beta t}(e_t - z_t)^\rho + \delta e^{-\beta t}\sum_{n=0}^{\infty}\frac{\Gamma(\gamma)}{\Gamma(\gamma-n)n!}e_t^{\rho-n}(-1)^n \times S_j^k(n;t)\frac{j!}{\sigma_e^{2j}}\sum_{m=0}^{j} c_{m,j}^{\nu(\rho-n)} f_1(T,\hat{\varpi}_1)$$

where

$$\hat{\varpi}_1 \triangleq \tfrac{1}{2}m\sigma_e^2 + m\nu(\rho-n) + (\rho-n)\left[\mu_e + \tfrac{1}{2}\sigma_e(\kappa-n)\right] - \alpha(n-k+j) - (\alpha+\beta).$$

PROOF. Observing that, by Fubini's Theorem and Newton's binomial formula

$$\mathrm{E}\left\{\int_t^T e^{-(\alpha+\beta)(s-t)}(e_s - z_s)^\rho\,ds\,|\mathcal{F}_t\right\} = \sum_{n=0}^{\infty}\frac{\Gamma(\gamma)}{\Gamma(\gamma-n)n!}(-1)^n\int_t^T e^{-(\alpha+\beta)(s-t)}\mathrm{E}\left\{e_s^{\rho-n}z_s^n\,|\mathcal{F}_t\right\}\,ds,$$

the proof is identical to that of Proposition 2.2 by the substitutions $\gamma_1 \to \rho - n$ and $\gamma_2 \to n$. □

4. Equilibrium Returns

This section begins with deducing Breeden's (1979) CCAPM when the representative consumer has von Neumann-Morgenstern additive separable utility. This provides the basis for an extension to additive nonseparable utility.

Consider the deflated gains process $G_t^\pi = \pi_t S_t + \int_0^t \pi_s\,dD_s + \int_0^t \sigma_D(s)\sigma_\pi(s)^\top\,ds$. Under assumptions 2.1 and 2.2 π is an Itô process. As before, $\mu_t = \mathbf{I}_S(t)^{-1}\mu_G(t)$ and $\sigma_t = \mathbf{I}_S(t)^{-1}\sigma_G(t)$. Hence, by an application of Itô's Lemma,

$$\text{(17)}\quad G_t^{\pi,i} = \pi_0 G_0^i + \int_0^t \mu_\pi(u)S_u^i + \pi_u\mu_{S^i}(u) + \sigma_\pi(u)\sigma_{S^i}(u)^\top + \pi_u\mu_{D^i}(u) + \sigma_{D^i}(u)\sigma_\pi(u)^\top\,du + \int_0^t \sigma_\pi(u)S_u^i + \pi_u\sigma_{S^i}(u) + \pi_u\sigma_{D^i}(u)\,dB_u.$$

Since $\pi \gg 0$ is a state price deflator, deflated gains will be martingales, i.e., $G_t^\pi = \mathrm{E}\{G_s^\pi | \mathcal{F}_t\}, s \geq t$. Thus the drift in (17) must satisfy

$$\mu_\pi(t)S_t^i + \pi_t[\mu_{S^i}(t) + \mu_{D^i}(u)] + \sigma_{S^i}(t)\sigma_\pi(t)^\top + \sigma_{D^i}(t)\sigma_\pi(t)^\top = 0 \qquad P\text{-a.s.}$$

or, if $S^i \gg 0$,

$$\frac{\mu_{G^i}(t)}{S_t^i} + \frac{\mu_\pi(t)}{\pi_t} = -\frac{\sigma_{G^i}(t)\sigma_\pi(t)^\top}{S_t^i\pi_t}. \tag{18}$$

$\mu_\pi(t)/\pi_t$ and $\mu_{G^i}(t)/S_t^i$ can be interpreted as the instantaneous conditional expected rates of "return" on the state price and the security respectively. The term $-\sigma_{G^i}(t)\sigma_\pi(t)^\top/(S_t^i\pi_t)$ can be interpreted as the instantaneous conditional covariance between the rate of change of the state price deflator and the security. Consider now the locally riskless security S^0, in that $\sigma_{G^0}(t) = 0$ a.e. Denoting $\mu_{G^0}(t)/S^0(t)$ by r_t we have

$$r_t = -\frac{\mu_\pi(t)}{\pi_t}, \tag{19}$$

which gives economic meaning to the conditional expected "rate of return on the state price". That π is related to the marginal utility of the representative consumer allows for further interpretations.

4.1. Additive Separable Utility. In the additive separable case it is straightforward to show that $\pi_t = v_c(t, e_t) = \partial v(t,x)/\partial x|_{x=e_t}$, where v is the felicity of the representative consumer. Thus,

$$\begin{aligned}\mu_\pi(t) &= v_{cc}(t, e_t)\mu_e(t) + v_{ct}(t, e_t) + \tfrac{1}{2}v_{ccc}(t, e_t)\sigma_e(t) \cdot \sigma_e(t)\\ \sigma_\pi(t) &= v_{cc}(t, e_t)\sigma_e(t).\end{aligned}$$

Together with equations (18) and (19) these coefficients give rise to the CCAPM of Breeden:

$$\mu_t - r_t\mathbf{1} = R_a(e_t)\sigma_t\sigma_e(t)^\top, \tag{20}$$

where $R_a(\cdot)$ is the Arrow-Pratt coefficient of absolute risk aversion.

4.2. Additive Nonseparable Utility. It is evident from (18) and (20) that to model the equilibrium risk premium it is necessary to find an expression for $\sigma_\pi(t)$, and of course the Riesz representation π itself. Proposition 2.1 illustrates that additional terms to v_{cc} must be computed to characterize σ_π when the felicity is path-dependent. Under technical restrictions on $X \in \mathcal{F}$ there is a process θ in a suitable space s.t.[8]

$$X = \mathrm{E}\{X\} + \int_0^T \theta_t^\top \, dB_t \qquad P\text{-a.s.},$$

In principle it is thus possible, for any Riesz representation of the form[9]

$$\pi_t = v_c(t, e_t, z_t) + X_t, \qquad X_t \in \mathcal{F}_t, \quad dz_t = f(t, e_t, z_t)\, dt,$$

to characterize equilibrium returns by using the approach of Duffie and Zame (1989):

$$r_t = -\frac{\mu_{v_c}(t)}{\pi_t} \tag{21}$$

$$\begin{aligned} \mu_t - r_t \mathbf{1} &= -\frac{1}{\pi_t}\sigma_t[\sigma_{v_c}(t)^\top + \theta_t] \\ &= \hat{R}_a(e_t)\sigma_t\sigma_e(t)^\top - \frac{\sigma_t\theta_t}{v_c(e_t) + X_t}, \end{aligned} \tag{22}$$

where

$$\mu_{v_c}(t) = \mu_{\pi_t} = v_{cc}\mu_e(t) + v_{cz}f(t, e_t, z_t) + v_{ct} + \tfrac{1}{2}v_{ccc}\sigma_e(t)^2$$

$$\sigma_{v_c}(t) = \sigma_\pi(t) - \theta_t^\top = v_{cc}\sigma_e(t).$$

The challenge is to uniquely determine θ. Even though θ does not appear in $\mu_{v_c}(t)$ the effect of θ is ambiguous as it enters r_t through the denominator. Looking at equation (22) it is possible to achieve an increase in the risk premium by choosing a felicity with $\theta < 0$, which also increases r_t. Hence one does not necessarily achieve an increase in the risk premium without also increasing the riskless interest rate—the riskless rate puzzle (a concept suggested by Weil 1989).

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[8] Itô's Representation Theorem requires that $X \in L^2(P)$ and ensures the existence of a $\theta \in L^2(P \times \lambda)$. Several related results have variations of this measurability and integrability hypothesis.

[9] By Leibniz' formula the standard of living in this study has dynamics $dz_t = (\delta e_t - \alpha z_t)\, dt$.

Detemple and Zapatero (1991) characterize θ when the path-dependence is due to habit formation, using the Clark-Ocone Characterization Theorem. From this result $\theta_t^\top = \mathrm{E}\{\mathfrak{D}_t X \,|\mathcal{F}_t\}$, where $\mathfrak{D}_t$ denotes the Malliavin derivative operator. Their general characterization of equilibrium returns is based on the traditional definition of equilibrium. Consider a consumption plan c^*, a trading strategy θ^*, a state price deflator π, and securities prices S. $(\theta^*, c^*; \pi, S)$ constitutes a *spot-security market equilibrium* if (θ^*, c^*) is optimal for the consumer given prices (π, S), and markets clear $c_t^* \le e_t$ and $\theta_t^* = \mathbf{0}\ \forall\, t \in [0,T]$ P-a.s.

THEOREM 2.1 (Detemple and Zapatero). *Consider the economy in Proposition 2.1. If* $S_t = \pi_t^{-1}\mathrm{E}\left\{\pi_T S_T + \int_t^T \pi_s\, dD_s + \int_t^T \sigma_D(s)\sigma_\pi(s)^\top\, ds \,|\mathcal{F}_t\right\}$ *then* $(\mathbf{0}, e; \pi, S)$ *is a spot-security market equilibrium.*[10] *Further, the equilibrium riskless interest rate is characterized by*[11]

$$
(23)\quad r_t = \beta_t - \frac{e^{-\int_0^t \beta_u\, du}}{\pi_t}\Bigg\{ u_{cc}(e_t, z_t)\mu_e(t) + \tfrac{1}{2}u_{ccc}(e_t, z_t)\sigma_e(t)\sigma_e(t)^\top
$$

$$
+ u_{cz}(e_t, z_t)(\delta e_t - \alpha z_t) + \delta\left[\alpha \mathrm{E}\left\{\int_t^T e^{-\int_t^s \beta_u + \alpha\, du} u_z(e_s, z_s)\, ds \,|\mathcal{F}_t\right\} - u_z(e_t, z_t)\right]\Bigg\},
$$

and the risk premia are

$$
(24)\qquad \mu_t - r_t\mathbf{1} = -\frac{\sigma_t \sigma_e(t)^\top}{\pi_t e^{\int_0^t \beta_u\, du}}\Bigg\{ u_{cc}(e_t, z_t) + \delta \mathrm{E}\Bigg[\int_t^T e^{-\int_t^s \alpha + \beta_u\, du}\Bigg(u_{cz}(e_s, z_s)
$$

$$
+ \delta u_{zz}(e_s, z_s)\int_t^s e^{-\alpha(s-u)}\, du\Bigg)\, ds \Bigg|\mathcal{F}_t\Bigg]\Bigg\}
$$

$$
- \frac{\sigma_t}{\pi_t e^{\int_0^t \beta_u\, du}}\delta \mathrm{E}\Bigg\{\int_t^T e^{-\int_t^s \beta_u + \alpha\, du}\Bigg[-u_z(e_s, z_s)\int_t^s \mathfrak{D}_t \beta_u\, du
$$

$$
+ u_{cz}(e_s, z_s)\left(\int_t^s \mathfrak{D}_t \mu_e(u)\, du + \int_t^s \mathfrak{D}_t \sigma_e(u)\, dB_u\right)
$$

$$
+ u_{zz}(e_s, z_s)\delta \int_t^s e^{-\alpha(s-u)}\Bigg(\int_t^u \mathfrak{D}_t \mu_e(v)\, dv
$$

$$
+ \int_t^u \mathfrak{D}_t \sigma_e(v)\, dB_v\Bigg)\, du\Bigg]\, ds \Bigg|\mathcal{F}_t\Bigg\},
$$

[10] The simpler price functional $S_t = \mathrm{E}\left\{\pi_T S_T + \int_t^T \pi_s\, dD(s) \,|\mathcal{F}_t\right\}$ often seen in the literature (see e.g. Detemple and Zapatero (1991), and Duffie and Zame (1989)) holds only when the dividend process D is of bounded variation. It also holds whenever the deflator π is of bounded variation, but this is obviously not the case in any interesting equilibrium model.

[11] The expression for the riskless interest rate differs slightly from that derived by Detemple and Zapatero (1991). The difference is due to their misinterpretation of the state price deflator, alluded to in section 3.

where $\mathfrak{D}_t$ denotes the Malliavin derivative operator evaluated at $t \in [0, T]$.

Even with assumption 2.1 fresh in mind it is impossible to learn much in terms of economic content from this result. The authors point out that "[e]xplicit formulas are necessary to understand the effects of habits on risk premia." (Detemple and Zapatero 1991, p. 1646). It turns out that Theorem 2.1 is subject to significant simplifications under assumptions similar to those of Proposition 2.2. Proposition 2.4 presents expressions for the equilibrium returns that are amenable to numerical analysis and comparative static analysis.

PROPOSITION 2.4 (Multiplicative habits). *Under the additional assumptions of 2.3 and 2.4 the equilibrium riskless interest rate and risk premia are given by*

$$
(25)\quad r_t = \frac{1}{\pi_t e^{\beta t}} \Big\{ \alpha e_t^{\rho_1} z_t^{\gamma_2} + \frac{\delta \gamma_2}{\gamma_1} e_t^{\gamma_1} z_t^{\rho_2} - \rho_1 e_t^{\rho_1} z_t^{\gamma_2} \mu_e - \tfrac{1}{2} \rho_1 \kappa_1 e_t^{\rho_1} z_t^{\gamma_2} \sigma_e^2 - \gamma_2 e_t^{\rho_1} z_t^{\rho_2} (\delta e_t - \alpha z_t) \Big\} - \alpha,
$$

and

$$
\begin{aligned}
\mu_t - r_t \mathbf{1} = {} & \frac{-\sigma_t \sigma_e}{\pi_t e^{\beta t}} \Big\{ \rho_1 e_t^{\rho_1} z_t^{\gamma_2} \\
& + \delta \gamma_2 e_t^{\rho_1} S_k^n(\rho_2; t) \frac{k!}{\sigma_e^{2k}} \sum_{j=0}^{k} c_{j,k}^{\nu(\rho_1)} f_1(T, \vartheta_1) \\
& + \delta^2 \frac{\rho_2 \gamma_2}{\alpha \gamma_1} e_t^{\gamma_1} S_k^n(\kappa_2; t) \frac{k!}{\sigma_e^{2k}} \sum_{j=0}^{k} c_{j,k}^{\nu(\gamma_1)} \left[f_1(T, \varrho_1) - f_1(T, \varpi_1) \right] \Big\} \\
& - \frac{\sigma_t \delta}{\pi_t e^{\beta t}} \Big\{ \gamma_2 \mu_e \sigma_e e_t^{\gamma_1} \frac{\mu_e + \rho_1 \sigma_e^2}{\mu_e} S_k^n(\rho_2, t) \sum_{j=0}^{k} \tbinom{k}{j} \varsigma_{j;j}^{k}(-j-1, -1, \psi_2) \\
& + \gamma_2 \sigma_e^3 e_t^{\gamma_1} S_k^n(\rho_2; t) k \sum_{j=0}^{k-1} \tbinom{k-1}{j} \varsigma_{j+1;j}^{k}(-j, 0, \omega_2) \\
& + \frac{\delta \rho_2 \gamma_2}{\gamma_1} e_t^{\gamma_1 + 1} S_k^n(\kappa_2; t) \sum_{j=0}^{k} \tbinom{k}{j} \sigma_e (\mu_e + (\gamma_1 + j) \sigma_e^2) \frac{j!}{\sigma_e^{2j}} \sum_{i=0}^{j} c_{i,j}^{\nu(\gamma_1)} \\
& \quad \times \sum_{m=0}^{k-j} \tbinom{k-j}{m} \varsigma_{m;j}^{k-j}(0, m, \phi_3) \\
& + \frac{\delta \rho_2 \gamma_2}{\gamma_1} \sigma_e^3 e_t^{\gamma_1 + 1} S_k^n(\kappa_2; t) \sum_{j=0}^{k} \tbinom{k}{j} \frac{j!}{\sigma_e^{2j}} \sum_{i=0}^{j} c_{i,j}^{\nu(\gamma_1)} (k - j) \sum_{m=0}^{k-j-1} \tbinom{k-j-1}{m}
\end{aligned}
$$

$$\times \varsigma_{m+1;j}^{k-j}(1, m+1, \theta_3) \Big\}$$

Common to all terms is

$$S_k^n(x;t) \triangleq \sum_{n=0}^{\infty} \frac{\Gamma(x+1)}{n!\Gamma(x+1-n)} \left(z_0 e^{-\alpha t}\right)^{x-n} \sum_{k=0}^{n} \binom{n}{k} (\delta e_t)^k \bar{z}_t^{n-k},$$

where

$$\bar{z}_t = z_t - z_0 e^{-\alpha t}.$$

Further,

$$\varsigma_{m;j}^{k}(x, y, \phi_i) \triangleq \frac{m!(k-m)!}{\sigma_e^{2k}} \sum_{a=0}^{m} \sum_{b=0}^{k-m} c_{a,m}^{\nu(\gamma_1+j+x)} c_{b,k-m}^{\nu(\gamma_1+j+1+y)} f_i(T; \phi_1, \dots, \phi_i),$$

where

$$f_1(s_1; \phi_1) = \frac{\exp\{\phi_1(s_1 - t)\} - 1}{\phi_1},$$

$$f_i(s_i; \phi_1, \dots, \phi_i) = \int_t^{s_i} \exp\{\phi_i(s_{i-1} - t)\} f_{i-1}(s_{i-1}; \phi_1, \dots, \phi_{i-1})\, ds_{i-1}.$$

The constants are $\nu(\gamma) \triangleq \nu + \gamma\sigma_e^2$, $\rho_i = \gamma_i - 1$, *and* $\kappa_i \triangleq \gamma_i - 2$.

It is interesting to note that only integer moments of the standard of living enter the risk premium (and state price deflator). This is puzzling since the standard of living does not satisfy Carleman's criterion (Yor 1992, p. 522), which is a necessary condition for a distribution to be determined by its moments. It is for instance easy to show that the lognormal density cannot be determined by its moments (Durrett 1996, p. 107). Hence, the consumer's utility depends on the moments of the standard of living, but not the whole distribution.

Before proceeding with the proof one more auxiliary result is needed. See e.g. Øksendal (1996) for background material on the Malliavin calculus.

LEMMA 2.3. *With definitions as in Lemma 2.1*

$$\mathfrak{D}_v \left(A_{0,t}^{\nu}\right)^n = \sigma_e n \left(A_{0,t}^{\nu}\right)^{n-1} A_{v,t}^{\nu}.$$

PROOF. Letting $B_t = \omega(t)$ and differentiating $F(\omega) = \int_0^T e^{\omega(t)}\,dt$ in the direction $\gamma(t) = \int_0^t g(s)\,ds$, we have

$$\begin{aligned}
\mathfrak{D}_\gamma F(\omega) &\triangleq \frac{d}{d\epsilon} F(\omega + \epsilon\gamma)\bigg|_{\epsilon=0} \\
&= \frac{d}{d\epsilon} \int_0^T e^{\omega(t)+\epsilon\gamma(t)}\,dt\bigg|_{\epsilon=0} \\
&= \int_0^T \int_0^t g(s)\,ds\,e^{\omega(t)}\,dt \\
&= \int_0^T \int_s^T e^{\omega(t)}\,dt\,g(s)\,ds.
\end{aligned}$$

It follows that $\mathfrak{D}_t F(\omega) = \int_s^T e^{\omega(t)} 1_{[0,t]}(s)\,dt = \int_t^T e^{\omega(u)}\,du$, where $1_{[0,t]}(\cdot)$ is the indicator function. Thus

$$\begin{aligned}
\mathfrak{D}_v \left(A^\nu_{0,t}\right)^n &= n\left(A^\nu_{0,t}\right)^{n-1} \int_0^t \mathfrak{D}_v \exp\{\nu u + \sigma_e B_u\}\,du \\
&= n\left(A^\nu_{0,t}\right)^{n-1} \int_0^t \exp\{\nu u + \sigma_e B_u\}\sigma_e 1_{[0,u]}(v)\,du \\
&= \sigma_e n\left(A^\nu_{0,t}\right)^{n-1} \int_v^t \exp\{\nu u + \sigma_e B_u\}\,du \\
&= \sigma_e n\left(A^\nu_{0,t}\right)^{n-1} A^\nu_{v,t}.
\end{aligned}$$

It remains to verify that $\left(A^\nu_{0,t}\right)^n \in \mathbb{D}_{1,2}$, i.e., that

$$\left\|\left(A^\nu_{0,t}\right)^n\right\|_{1,2} \triangleq \left\|\left(A^\nu_{0,t}\right)^n\right\|_{L^2(P)} + \left\|\mathfrak{D}_v \left(A^\nu_{0,t}\right)^n\right\|_{L^2(P\times\lambda)} < \infty.$$

It suffices to show that $\mathfrak{D}_v \left(A^\nu_{0,t}\right)^n \in L^2(P\times\lambda)$ (Øksendal 1996), as

$$\left\|\left(A^\nu_{0,t}\right)^n\right\|^2_{L^2(P)} = \mathrm{E}\left\{\left(A^\nu_{0,t}\right)^{2n}\right\} < \infty$$

by Lemma 2.1. As shown above,

$$\begin{aligned}
\left\|\mathfrak{D}_v \left(A^\nu_{0,t}\right)^n\right\|^2_{L^2(P\times\lambda)} &= \int_\Omega \int_{[0,t]} \left[\sigma_e n\left(A^\nu_{0,t}\right)^{n-1} A^\nu_{v,t}\right]^2 \lambda(dv)P(d\omega). \\
&= (\sigma_e n)^2 \sum_{k=0}^{2(n-1)} \int_{[0,t]} \mathrm{E}\left\{\left(A^\nu_{0,v}\right)^{2(n-1)-k} \exp\{(k+2)(\nu t + \sigma_e B_t)\}\right\} \\
&\qquad\qquad \times \mathrm{E}\left\{\left(A^\nu_{0,t-v}\right)^{k+2}\right\} \lambda(dv),
\end{aligned}$$

by using the Fubini Theorem and Corollary 2.1. This expression is finite by Lemma 2.1, after a change of measure. □

5. Proof of Proposition 2.4

The riskless interest rate in (25) follows by substitution of the marginals and addition and subtraction of $u_c(e_t, z_t)$ in (23). To get an explicit characterization of the risk premium it is necessary to compute the conditional expectations in (24).

5.1. First Expectation in (24). Inserting for the marginals and using the linearity of the Lebesgue and abstract integrals,

$$\delta \mathrm{E}\left\{\int_t^T e^{-(\alpha+\beta)(s-t)}\left[u_{cz(e_s,z_s)} + \delta u_{zz}(e_s, z_s)\frac{1}{\alpha}\left(e^{-\alpha(s-t)} - 1\right)\right] ds \Big| \mathcal{F}_t\right\}.$$

$$= \delta\gamma_2 \mathrm{E}\left\{\int_t^T e^{-(\alpha+\beta)(s-t)} e_s^{\rho_1} z_s^{\rho_2}\, ds \,|\mathcal{F}_t\right\}$$

$$+ \frac{\delta^2\gamma_2\rho_2}{\alpha\gamma_1}\mathrm{E}\left\{\int_t^T e^{-(\alpha+\beta)(s-t)} e_s^{\gamma_1} z_s^{\kappa_2}\left(e^{-\alpha(s-t)} - 1\right) |\mathcal{F}_t\right\},$$

and since $e_t^{\gamma_1}, e_t^{\rho_1}, z_t^{\rho_2}, z_t^{\kappa_2} \in L^2(P)$ are strictly positive it is viable to appeal to Fubini's Theorem,

$$(26) \qquad = \delta\gamma_2 \int_t^T e^{-(\alpha+\beta)(s-t)} \mathrm{E}\left\{e_s^{\rho_1} z_s^{\rho_2} \,|\mathcal{F}_t\right\} ds$$

$$+ \frac{\delta^2\gamma_2\rho_2}{\alpha\gamma_1}\int_t^T e^{-(\alpha+\beta)(s-t)}\mathrm{E}\left\{e_s^{\gamma_1} z_s^{\kappa_2} \,|\mathcal{F}_t\right\}\left(e^{-\alpha(s-t)} - 1\right) ds,$$

and the problem is reduced to computing the two conditional expectations in (26). The first conditional expectation is almost equivalent to that of the state price deflator in the proof of Proposition 2.2. I.e., $d\hat{P}/dP \triangleq e_T^{\rho_1}/\mathrm{E}\{e_T^{\rho_1}\}$ induces a P-equivalent probability measure $\hat{P}(G) \triangleq \mathrm{E}\left\{\frac{d\hat{P}}{dP}1_G\right\} \forall G \in \mathcal{F}$. By Girsanov's Theorem $\hat{B}_t \triangleq B_t - \rho_1\sigma_e t$ is a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{F}, \hat{P})$. Hence, by letting

$$\vartheta_1 \triangleq \varpi_1|_{\gamma_1 \to \rho_1}$$

the characterization of this expectation is complete.

The second expectation induces a probability measure $\tilde{P} \sim P$, as in the proof of Proposition 2.2. Hence, by defining ϖ_1 as in Proposition 2.2 and

$$\varrho_1 \triangleq \varpi_1|_{\rho_2 \to \kappa_2} - \alpha$$

the expectation is fully characterized.

5.2. Second Expectation in (24).

First Term. Since $\mathfrak{D}_t(k) = 0 \ \forall k \in \mathbb{R}$

$$\mathrm{E}\left\{\int_t^T e^{-(\alpha+\beta)(s-t)} \int_t^s \mathfrak{D}_t(\beta)\, du\, u_z(e_s, z_s)\, ds\, |\mathcal{F}_t\right\} = 0.$$

Second Term. By the chain rule of Malliavin differentiation

$$\mathfrak{D}_t(ke_u) = k\mathfrak{D}_t(e_u) = k\sigma_e 1_{[0,u]}(t) e_u, \ \forall k \in \mathbb{R}$$

and the problem is to evaluate

$$\mathrm{E}\left\{\int_t^T e^{-(\alpha+\beta)(s-t)} \gamma_2 e_s^{\rho_1} z_s^{\rho_2} \left(\int_t^s \mathfrak{D}_t \mu_e(u)\, du + \int_t^s \mathfrak{D}_t \sigma_e(u)\, dB_u\right) ds\, |\mathcal{F}_t\right\},$$

which by Fubini's Theorem and a change of measure, using Girsanov's Theorem, leads to

$$\begin{aligned}
&\gamma_2\sigma_e \int_t^T e^{-(\alpha+\beta)(s-t)} \mathrm{E}\left\{e_s^{\rho_1}\, |\mathcal{F}_t\right\} \\
&\times \mathrm{E}^{\hat{P}}\left\{z_s^{\rho_2}\left((\mu_e + \rho_1\sigma_e^2)\int_t^s e_u\, du + \sigma_e \int_t^s e_u\, d\hat{B}_u\right) |\mathcal{F}_t\right\}.
\end{aligned}$$

Applying Newton's binomial formula this equals

$$\begin{aligned}
&\gamma_2\sigma_e \int_t^T e^{-(\alpha+\beta)(s-t)} \mathrm{E}\left\{e_s^{\rho_1}\, |\mathcal{F}_t\right\} \sum_{n=0}^{\infty} \frac{\Gamma(\gamma_2)}{n!\Gamma(\gamma_2 - n)} (z_0 e^{-\alpha s})^{\rho_2 - n} \left(\frac{\delta e_0}{e^{\alpha s}}\right)^n \\
&\times \mathrm{E}^{\hat{P}}\left\{\left(A_{0,t}^{\nu(\rho_1)} + A_{t,s}^{\nu(\rho_1)}\right)^n \left((\mu_e + \rho_1\sigma_e^2)\int_t^s e_u\, du + \sigma_e \int_t^s e_u\, d\hat{B}_u\right) |\mathcal{F}_t\right\},
\end{aligned}$$

which by another application of the binomial formula, a suitable change of variables, and the independence of Brownian increments

$$= \gamma_2\sigma_e \int_t^T e^{-(\alpha+\beta)(s-t)} \mathrm{E}\left\{e_s^{\rho_1}\, |\mathcal{F}_t\right\} \sum_{n=0}^{\infty} \frac{\Gamma(\gamma_2)}{n!\Gamma(\gamma_2 - n)} (z_0 e^{-\alpha s})^{\rho_2 - n} \left(\frac{\delta e_0}{e^{\alpha s}}\right)^n$$

$$\times \sum_{k=0}^{n} \binom{n}{k} \left(A_{0,t}^{\nu}\right)^{n-k} \left(\frac{e^{\alpha t}}{e_0}\right)^k e_t^{k+1}$$
$$\times \mathrm{E}^{\hat{P}} \left\{ \left(A_{0,s-t}^{\nu(\rho_1)}\right)^k \left((\mu_e + \rho_1 \sigma_e^2) \int_0^{s-t} \frac{e_u}{e_0}\, du + \sigma_e \int_0^{s-t} \frac{e_u}{e_0}\, d\hat{B}_u \right) \right\},$$

Consider first the Lebesgue integral in the $\hat{P}$-expectation. Let $\hat{s} = s - t$.

$$\mathrm{E}^{\hat{P}} \left\{ \left(A_{0,\hat{s}}^{\nu(\rho_1)}\right)^k \int_0^{\hat{s}} \frac{e_u}{e_0}\, du \right\} = \mathrm{E}^{\hat{P}} \left\{ \left(A_{0,\hat{s}}^{\nu(\rho_1)}\right)^k A_{0,\hat{s}}^{\nu(\rho_1)-\alpha} \right\},$$

which by appealing to Corollary 2.2

$$= \sum_{j=0}^{k} \binom{k}{j} \int_0^{\hat{s}} \exp\{[(j+1)\nu(\rho_1) - \alpha + \tfrac{1}{2}(j+1)^2 \sigma_e^2] u\}$$
$$\times \mathrm{E}^{\hat{P}} \left\{ \left(A_{0,\hat{s}-u}^{\nu(\rho_1)}\right)^j \right\} \mathrm{E}^{\hat{P}^{j+1}} \left\{ \left(A_{0,u}^{\nu(\rho_1+j+1)}\right)^{k-j} \right\} du.$$

Substituting this expression back into the original expression, applying Lemma 2.1, integrating in s and rearranging gives the characterization when

$$\psi_1 \triangleq \tfrac{1}{2} b^2 \sigma_e^2 + b\nu(\rho_1 + j + 1) - \tfrac{1}{2} a^2 \sigma_e^2 - a\nu(\rho_1) + (j+1)\nu(\rho_1) - \alpha + \tfrac{1}{2}(j+1)^2 \sigma_e^2,$$
$$\psi_2 \triangleq \tfrac{1}{2} a^2 \sigma_e^2 + a\nu(\rho_1) + \rho_1[\mu_e - \tfrac{1}{2}\sigma_e^2(1-\rho_1)] - \rho_2 \alpha - (\alpha + \beta).$$

Consider now the Itô integral. Since $\left(A_{0,s-t}^{\nu(\gamma_1-1)}\right)^k \in \mathcal{F}_{s-t}^{\hat{B}}$ the generalized Clark-Ocone Characterization Theorem is applicable (Ocone and Karatzas 1991, Theorem 2.5). Letting $\phi_v \triangleq \mathfrak{D}_v \left(A_{0,s-t}^{\nu(\rho_1)}\right)^k$, we get:

$$\mathrm{E}^{\hat{P}} \left\{ \left(A_{0,\hat{s}}^{\nu(\rho_1)}\right)^k \int_0^{\hat{s}} \frac{e_u}{e_0}\, d\hat{B}_u \right\}$$
$$= \mathrm{E}^{\hat{P}} \Big\{ \Big[\mathrm{E}^{\hat{P}} \left\{ \left(A_{0,\hat{s}}^{\nu(\rho_1)}\right)^k \right\} + \int_0^{\hat{s}} \mathrm{E}^{\hat{P}} \Big\{ \phi_v$$
$$- \left(A_{0,\hat{s}}^{\nu(\rho_1)}\right)^k \int_v^{\hat{s}} \mathfrak{D}_v \rho_1 \sigma_e\, d\hat{B}_u | \mathcal{F}_v \Big\} d\hat{B}_v \Big] \int_0^{\hat{s}} \frac{e_u}{e_0}\, d\hat{B}_u \Big\},$$

using that $\mathrm{E}^{\hat{P}}\left\{\left(A_{0,\hat{s}}^{\nu(\rho_1)}\right)^k\right\} \in \mathcal{F}$. Thus,

$$= \mathrm{E}^{\hat{P}}\left\{\left(A_{0,\hat{s}}^{\nu(\rho_1)}\right)^k\right\}\mathrm{E}^{\hat{P}}\left\{\int_0^{\hat{s}} \frac{e_u}{e_0}\,d\hat{B}_u\right\}$$
$$+\,\mathrm{E}^{\hat{P}}\left\{\int_0^{\hat{s}} \mathrm{E}^{\hat{P}}\left\{\phi_v\,|\mathcal{F}_v\right\}d\hat{B}_v \int_0^{\hat{s}} \frac{e_u}{e_0}\,d\hat{B}_u\right\},$$

which, using that the Itô integral is a $(\hat{P}, \mathcal{F}_t)$-martingale and the Itô isometry

$$= \mathrm{E}^{\hat{P}}\left\{\int_0^{\hat{s}} \mathrm{E}^{\hat{P}}\left\{\phi_v\,|\mathcal{F}_v\right\}\frac{e_v}{e_0}\,dv\right\} = \int_0^{\hat{s}} \mathrm{E}^{\hat{P}}\left\{\phi_v \frac{e_v}{e_0}\right\}dv,$$

by Fubini's Theorem and the Law of Total Probability. Changing measure s.t. $d\check{P}/d\hat{P} = \exp\{\sigma_e \hat{B}_T - \frac{1}{2}\sigma_e^2 T\}$ and $\check{B}_t = \hat{B}_t - \int_0^t \sigma_e\,ds = B_t - \gamma_1 \int_0^t \sigma_e\,ds$ and substituting for ϕ_v in the previous $\hat{P}$ expectation,

$$\begin{aligned}\mathrm{E}^{\hat{P}}\left\{\phi_v e_v/e_0\right\} &= \exp\{(\mu_e + \rho_1\sigma_e^2)v\}\mathrm{E}^{\check{P}}\left\{\phi_v\right\} \\ &= k\sigma_e \exp\{(\mu_e + \rho_1\sigma_e^2)v\}\mathrm{E}^{\check{P}}\left\{\left(A_{0,\hat{s}}^{\nu(\gamma_1)}\right)^{k-1} A_{v,\hat{s}}^{\nu(\gamma_1),\sigma_e}\right\},\end{aligned}$$

and by Newton's binomial formula this is equal to

$$k\sigma_e \exp\{(\mu_e + \rho_1\sigma_e^2)v\}\sum_{j=0}^{k-1}\binom{k-1}{j}$$
$$\times\,\mathrm{E}^{\check{P}}\left\{\left(A_{0,v}^{\nu(\gamma_1)}\right)^{k-1-j}\mathrm{E}^{\check{P}}\left\{\left(A_{v,\hat{s}}^{\nu(\gamma_1)}\right)^{j+1}|\mathcal{F}_v\right\}\right\}.$$

Further, appealing to Lemma 2.2 we get

$$k\sigma_e \exp\{(\mu_e + \rho_1\sigma_e^2)v\}\sum_{j=0}^{k-1}\binom{k-1}{j}$$
$$\times\,\mathrm{E}^{\check{P}}\left\{\left(A_{0,v}^{\nu(\gamma_1)}\right)^{k-j-1}\exp\{(j+1)[\nu(\gamma_1)v + \sigma_e\check{B}_v]\}\right\}$$
$$\times\,\mathrm{E}^{\check{P}}\left\{\left(A_{0,\hat{s}-v}^{\nu(\gamma_1)}\right)^{j+1}\right\}.$$

The first of the $\check{P}$ expectations is given on explicit form by Corollary 2.1 after a change of measure. The last $\check{P}$ expectation is given explicitly in Lemma 2.1.

Substituting back into the original expression and integrating,

$$\delta\gamma_2\sigma_e^3 e_t^{\gamma_1}\sum_{n=0}^{\infty}\frac{\Gamma(\gamma_2)}{n!\Gamma(\gamma_2-n)}(z_0e^{-\alpha t})^{\rho_2-n}\sum_{k=0}^{n}\binom{n}{k}\bar{z}_t^{n-k}k(\delta e_t)^k$$
$$\times\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(j+1)!(k-j-1)!}{\sigma_e^{2k}}\sum_{a=0}^{j+1}\sum_{b=0}^{k-j-1}c_{a,j+1}^{\nu(\gamma_1)}c_{b,k-j-1}^{\nu(\gamma_1+j+1)}$$
$$\times\int_t^T\exp\{\omega_2\hat{s}\}\frac{\exp\{\omega_1\hat{s}\}-1}{\omega_1}\,d\hat{s},$$

where

$$\omega_1\triangleq\tfrac{1}{2}b^2\sigma_e^2+b\nu(\gamma_1+j+1)-\tfrac{1}{2}a^2\sigma_e^2-a\nu(\gamma_1)+\mu_e+\rho_1\sigma_e^2$$
$$+(j+1)[\nu(\gamma_1)+\tfrac{1}{2}(j+1)\sigma_e^2],$$
$$\omega_2\triangleq\psi_2$$

Third Term. Using Fubini's Theorem, changing measure s.t. $\tilde{B}_t=B_t-\gamma_1\sigma_e$ is a standard Brownian motion on $(\Omega,\mathcal{F},\mathbb{F},\tilde{P})$ according to Girsanov's Theorem, Bayes' formula, two applications of Newton's binomial formula, another application of Fubini's Theorem, the chain rule of the Malliavin derivative, a suitable change of variables, and the independence of Brownian increments, it follows that

$$\mathrm{E}\left\{\int_t^T e^{-(\alpha+\beta)(s-t)}\frac{\gamma_2\rho_2}{\gamma_1}e_s^{\gamma_1}z_s^{\kappa_2}\delta\int_t^s e^{-\alpha(s-u)}\left[\int_t^u\mathfrak{D}_t\mu_e(v)\,dv\right.\right.$$
$$\left.\left.+\int_t^u\mathfrak{D}_t\sigma_e(v)\,dB_v\right]du\,ds|\mathcal{F}_t\right\}$$
$$=\frac{\delta\gamma_2\rho_2}{\gamma_1}\int_t^T e^{-(\alpha+\beta)(s-t)}\mathrm{E}\left\{e_s^{\gamma_1}|\mathcal{F}_t\right\}\sum_{n=0}^{\infty}\frac{\Gamma(\rho_2)}{n!\Gamma(\rho_2-n)}(e^{-\alpha s}z_0)^{\kappa_2-n}\left(\frac{\delta e_0}{e^{\alpha s}}\right)^n$$
$$\times\sum_{k=0}^{n}\binom{n}{k}\left(A_{0,t}^{\nu}\right)^{n-k}\left(\frac{e^{\alpha t}}{e_0}\right)^k e_t^{k+1}\int_t^s e^{-\alpha(s-u)}$$
$$\times\mathrm{E}^{\tilde{P}}\left\{\left(A_{0,s-t}^{\nu(\gamma_1)}\right)^k\left[(\mu_e+\gamma_1\sigma_e^2)\sigma_e\int_0^{u-t}\frac{e_v}{e_0}\,dv+\int_0^{u-t}\sigma_e^2\frac{e_v}{e_0}\,d\tilde{B}_v\right\}\right]du\,ds,$$

and applying Newton's binomial formula, using Corollary 2.1, and constructing a sequence of probability measures $\tilde{P}^j$ under which the corresponding $\tilde{B}_t^j = \tilde{B}_t - j\sigma_e t$, $j = 0, 1, \ldots$ are standard Brownian motions according to Girsanov's Theorem, by Bayes' formula, we get

$$\frac{\delta\gamma_2\rho_2}{\gamma_1}\int_t^T e^{-(\alpha+\beta)(s-t)}\mathrm{E}\left\{e_s^{\gamma_1}|\mathcal{F}_t\right\}\sum_{n=0}^{\infty}\frac{\Gamma(\rho_2)}{n!\Gamma(\rho_2-n)}(e^{-\alpha s}z_0)^{\kappa_2-n}\left(\frac{\delta e_0}{e^{\alpha s}}\right)^n$$
$$\times\sum_{k=0}^{n}\binom{n}{k}\left(A_{0,t}^{\nu}\right)^{n-k}\left(\frac{e^{\alpha t}}{e_0}\right)^k e_t^{k+1}\int_t^s e^{-\alpha(s-u)}\sum_{j=0}^{k}\binom{k}{j}\exp\{j(\nu(\gamma_1)+\tfrac{1}{2}j\sigma_e^2)\hat{u}\}$$
$$\times\mathrm{E}^{\tilde{P}}\left\{\left(A_{0,s-u}^{\nu(\gamma_1)}\right)^j\right\}\mathrm{E}^{\tilde{P}^j}\left\{\left(A_{0,\hat{u}}^{\nu(\gamma_1+j)}\right)^{k-j}\right.$$
$$\left.\times\left[(\mu_e+(\gamma_1+j)\sigma_e^2)\sigma_e\int_0^{\hat{u}}\frac{e_v}{e_0}\,dv+\int_0^{\hat{u}}\sigma_e^2\frac{e_v}{e_0}\,d\tilde{B}_v^j\right]\right\}du\,ds$$

It remains to compute the $\tilde{P}^j$ expectation. Consider first the Lebesgue integral.

$$\mathrm{E}^{\tilde{P}^j}\left\{\left(A_{0,\hat{u}}^{\nu(\gamma_1+j)}\right)^{k-j}\int_0^{\hat{u}}\frac{e_v}{e_0}\,dv\right\}=\mathrm{E}^{\tilde{P}^j}\left\{\left(A_{0,\hat{u}}^{\nu(\gamma_1+j)}\right)^{k-j}A_{0,\hat{u}}^{\nu(\gamma_1+j)-\alpha}\right\},$$

using Corollary 2.2 and Lemma 2.1

$$=\sum_{m=0}^{k-j}\binom{k-j}{m}\frac{m!(k-j-m)!}{\sigma_e^{2(k-j)}}\sum_{a=0}^{m}\sum_{b=0}^{k-j-m}c_{a,m}^{\nu(\gamma_1+j)}c_{b,k-j-m}^{\nu(\gamma_1+j+m+1)}$$
$$\times\exp\{[\tfrac{1}{2}a^2\sigma_e^2+a\nu(\gamma_1+j)]\hat{u}\}\frac{\exp\{\zeta_1\hat{u}\}-1}{\zeta_1},$$

where

$$\begin{aligned}\zeta_1 =&(m+1)\nu(\gamma_1+j)-\alpha+\tfrac{1}{2}(m+1)^2\sigma_e^2+\tfrac{1}{2}b^2\sigma_e^2\\&+b\nu(\gamma_1+j+m+1)-\tfrac{1}{2}a^2\sigma_e^2-a\nu(\gamma_1+j).\end{aligned}$$

Define

$$\begin{aligned}\zeta_2 =&\tfrac{1}{2}(a^2-i^2)\sigma_e^2+a\nu(\gamma_1+j)-i\nu(\gamma_1)+j[\nu(\gamma_1)+\tfrac{1}{2}j\sigma_e^2]+\alpha\\ \zeta_3 =&\tfrac{1}{2}i^2\sigma_e^2+i\nu(\gamma_1)-\alpha(\rho_2)-(\alpha+\beta)+\gamma_1(\mu_e+\tfrac{1}{2}\sigma_e^2\rho_1).\end{aligned}$$

Substituting the expression in ζ_1 back into the original expression (disregarding the Itô integral for now) we have by Lemma 2.1

$$\frac{\delta\gamma_2\rho_2}{\gamma_1}e_t^{\gamma_1+1}\sum_{n=0}^{\infty}\frac{\Gamma(\rho_2)}{n!\Gamma(\rho_2-n)}(\frac{z_0}{e^{\alpha t}})^{\kappa_2-n}\sum_{k=0}^{n}\binom{n}{k}\bar{z}_t^{n-k}(\delta e_t)^k\sum_{j=0}^{k}\binom{k}{j}(\mu_e+(\gamma_1+j)\sigma_e^2)\sigma_e$$
$$\times\frac{j!}{\sigma_e^{2j}}\sum_{i=0}^{j}c_{i,j}^{\nu(\gamma_1)}\sum_{m=0}^{k-j}\binom{k-j}{m}\frac{m!(k-j-m)!}{\sigma_e^{2(k-j)}}\sum_{a=0}^{m}\sum_{b=0}^{k-j-m}c_{a,m}^{\nu(\gamma_1+j)}c_{b,k-j-m}^{\nu(\gamma_1+j+m+1)}$$
$$\times\int_t^T\int_t^s\exp\{\zeta_3(s-t)\}\frac{\exp\{(\zeta_1+\zeta_2)\hat{u}\}-\exp\{\zeta_2\hat{u}\}}{\zeta_1}\,d\hat{u}\,ds.$$

Consider now the Itô integral part of the $\tilde{P}^j$ expectation. By the generalized Clark-Ocone Characterization Theorem, letting $\phi_v \triangleq \mathfrak{D}_t\left(A_{0,\hat{u}}^{\nu(\gamma_1+j)}\right)^{k-j}$, the Itô isometry, Fubini's Theorem, and the Law of Total Probability

$$\mathrm{E}^{\tilde{P}^j}\left\{\left(A_{0,\hat{u}}^{\nu(\gamma_1+j)}\right)^{k-j}\int_0^{\hat{u}}\frac{e_v}{e_0}\,d\tilde{B}_v^j\right\}=\int_0^{\hat{u}}\mathrm{E}^{\tilde{P}^j}\left\{\phi_v\frac{e_v}{e_0}\right\}dv.$$

Changing measure s.t. $\tilde{B}_t^{j+1}=\tilde{B}_t+(j+1)\sigma_e t$ is a standard Brownian motion on $(\Omega,\mathcal{F},\mathbb{F},\tilde{P}^{j+1})$ according to Girsanov's Theorem, using Bayes' formula, and using Lemma 2.3, we get

$$\int_0^{\hat{u}}\exp\{[\mu_e+(j+\gamma_1)\sigma_e^2]v\}(k-j)\sigma_e\sum_{m=0}^{k-j-1}\binom{k-j-1}{m}$$
$$\times\mathrm{E}^{\tilde{P}^{j+1}}\left\{\left(A_{0,v}^{\nu(\gamma_1+j+1)}\right)^{k-j-1-m}\exp\{(m+1)[\nu(\gamma_1+j+1)+\sigma_e\tilde{B}_v^{j+1}]\}\right\}$$
$$\times\mathrm{E}^{\tilde{P}^{j+1}}\left\{\left(A_{0,\hat{u}-v}^{\nu(\gamma_1+j+1)}\right)^{m+1}\right\}dv,$$

which by another application of Girsanov's Theorem and Bayes' formula, and using Lemma 2.1 yields

$$\sigma_e(k-j)\sum_{m=0}^{k-j-1}\binom{k-j-1}{m}\frac{(m+1)!(k-j-m-1)}{\sigma_e^{2(k-j)}}\sum_{a=0}^{m+1}\sum_{b=0}^{k-j-m-1}c_{a,m+1}^{\nu(\gamma_1+j+1)}$$
$$\times c_{b,k-j-m-1}^{\nu(\gamma_1+j+m+2)}\exp\{[\tfrac{1}{2}a^2\sigma_e^2+a\nu(\gamma_1+j+1)]\hat{u}\}\frac{\exp\{\theta_1\hat{u}\}-1}{\theta_1},$$

where

$$\theta_1=\mu_e+(j+\gamma_1)\sigma_e^2+(m+1)[\nu(\gamma_1+j+1)+\tfrac{1}{2}(m+1)\sigma_e^2]+\tfrac{1}{2}b^2\sigma_e^2$$

$$+ b\nu(\gamma_1 + j + m + 2) - \tfrac{1}{2}a^2\sigma_e^2 - a\nu(\gamma_1 + j + 1).$$

Define

$$\theta_2 = \zeta_2 + a\sigma_e^2,$$

$$\theta_3 = \zeta_3.$$

Substituting the expression in θ_1 back into the original expression (disregarding the Lebesgue integral) we have by Lemma 2.1

$$\frac{\delta\gamma_2\rho_2}{\gamma_1}\sigma_e^3 e_t^{\gamma_1+1}\sum_{n=0}^{\infty}\frac{\Gamma(\rho_2)}{n!\Gamma(\rho_2-n)}(e^{-\alpha t}z_0)^{\kappa_2-n}\sum_{k=0}^{n}\tbinom{n}{k}\bar{z}_t^{n-k}(\delta e_t)^k\sum_{j=0}^{k}\tbinom{k}{j}$$
$$\times\frac{j!}{\sigma_e^{2j}}\sum_{i=0}^{j}c_{i,j}^{\nu(\gamma_1)}(k-j)\sum_{m=0}^{k-j-1}\tbinom{k-j-1}{m}\frac{(m+1)!(k-j-m-1)!}{\sigma_e^{2(k-j)}}\sum_{a=0}^{m+1}\sum_{b=0}^{k-j-m-1}$$
$$\times c_{a,m+1}^{\nu(\gamma_1+j+1)}c_{b,k-j-m-1}^{\nu(\gamma_1+j+m+2)}\int_t^T\int_t^s e^{\theta_3(s-t)}\frac{\exp\{(\theta_1+\theta_2)\hat{u}\}-\exp\{\theta_2\hat{u}\}}{\theta_1}\,d\hat{u}\,ds.$$

□

PROPOSITION 2.5 (Additive habits). *Consider the economy in Theorem 2.1, and assume in addition that assumptions 2.4 and 2.5 hold. Then the equilibrium riskless interest rate and risk premia are given by*

$$(27)\quad r_t = \frac{1}{\pi_t e^{\beta t}}\Big\{(\alpha-1)(e_t-z_t)^{\rho} - \rho(e_t-z_t)^{\kappa}\left[e_t(\mu_e-\delta)+\alpha z_t\right]$$
$$-\tfrac{1}{2}\rho\kappa(e_t-z_t)^{\kappa-1}e_t^2\sigma_e^2\Big\} - \alpha,$$

and

$$\mu_t - r_t\mathbf{1} = \frac{-\sigma_t\sigma_e}{\pi_t e^{\beta t}}\rho(e_t-z_t)^{\kappa}$$
$$+\sum_{n=0}^{\infty}\frac{\Gamma(\rho)}{\Gamma(\rho-n)n!}(-1)^n e_t^{\kappa-n}S_j^k(n;t)\Big\{$$
$$\frac{-\sigma_t\sigma_e}{\pi_t e^{\beta t}}\delta\rho\frac{j!}{\sigma_e^{2j}}\sum_{m=0}^{j}c_{m,j}^{\nu(\kappa-n)}\left[\frac{\delta}{\alpha}f_1(T,\hat{\varrho}_1) - f_1(T,\hat{\varpi}_1) - f_1(T,\hat{\vartheta}_1)\right]$$
$$-\frac{\sigma_t\delta}{\pi_t e^{\beta t}}\Bigg[-\rho e_t\sigma_e[\mu_e+(\kappa-n)\sigma_e^2]\sum_{m=0}^{j}\binom{j}{m}\varsigma_{m;m}^{j}(-m,0,\hat{\psi}_2)$$

$$
\begin{aligned}
&- \rho e_t \sigma_e^3 \sum_{m=0}^{j} \binom{j-1}{m} \varsigma_{m+1;m}^{j}(-m+1, 1, \hat{\omega}_2) \\
&+ \delta \rho e_t \sigma_e \sum_{m=0}^{j} \binom{j}{m} [\mu_e + (\kappa - n + m)\sigma_e^2] \frac{m!}{\sigma_e^{2m}} \sum_{i=0}^{m} c_{i,m}^{\nu(\kappa-n)} \\
&\times \sum_{l=0}^{j-m} \binom{j-m}{l} \varsigma_{l;m}^{j-m}(0, l, \hat{\zeta}_3) \\
&+ \delta \rho e_t \sigma_e \sum_{m=0}^{j} \binom{j}{m} \frac{m!}{\sigma_e^{2m}} \sum_{i=0}^{m} c_{i,m}^{\nu(\kappa-n)} (j-m) \\
&\times \sum_{l=0}^{j-m-1} \binom{j-m-1}{l} \varsigma_{l+1;m}^{j-m}(1, l+1, \hat{\theta}_3) \Bigg] \Bigg\}.
\end{aligned}
$$

Common to all terms are $S_k^n(x;t)$, $\varsigma_{m;j}^k(x, y, \phi_i)|_{\gamma_1 \to \kappa - n}$, $f_i(s_i; \phi_1, \dots, \phi_i)$, $\bar{z}_t$, and the constants $\nu(\gamma)$, ρ, and κ, as defined in Proposition 2.4. Further, all of the arguments x in $\varsigma_{m;j}^k(\cdot, \cdot, x)$ are as in Proposition 2.4, with the appropriate adjustments.

PROOF. Applying the binomial formula to $(e_s - z_s)^\gamma$ yields the expression $\sum_{n=0}^{\infty} \frac{\Gamma(\rho)}{\Gamma(\rho-n)n!} e_s^{\gamma-n}(-z_s)^n$. Hence, by Fubini's Theorem the computations are similar to the case of multiplicative habits under the infinite sum. I.e., let

$$
\frac{\partial^k}{\partial e^{k-j} \partial z^j} \hat{u}(e_s, z_s) \triangleq e_s^{\hat{\gamma}_1 - (k-j)} z_s^{\hat{\gamma}_2 - j},
$$

where $\hat{\gamma}_1 = \gamma - n$ and $\hat{\gamma}_2 = n$ and change indexing using the substitutions $n \to k \to j \to m \to l$. □

It is clear from the proof that additive habit formation has the structure of a weighted sum of modified felicities exhibiting multiplicative habit formation. The modification consists in summing over felicities with decreasing degrees of Arrow-Pratt risk aversion and increasing degrees of aversion to previous standards of living.

6. Conclusion

The chapter's contribution is twofold. It presents explicit characterizations of the state price deflator and risk premia when the representative consumer's marginal utility depends on past and expected standard of living. This is the first study to derive such expressions in a pure exchange economy with multiplicative habits. The results are based on the general characterization of Detemple and

Zapatero (1991), thus enabling extensive analysis of the effect of habits on equilibrium risk premia. By applying the methodology used in this analysis to other specifications of path-dependent utility functionals (as for power utility with additive habits, derived herein), it will be possible to investigate modeling implications within a common framework. Hence, this study makes available tools to better understand the economic effects of habit formation as well as other path-dependencies in consumption.

As a technical contribution, the main result derived herein illustrates how the Malliavin calculus plays a central role in characterizing the integrand in representations by means of Itô integrals, of stochastic variables adapted to Brownian filtrations. This enables evaluation of relatively complex expectations. Further, the results also show how Yor's (1992) characterization of arithmetic averages of geometric Brownian motion can play an important role in the study of economies with path-dependence based on geometric Brownian motion stochastic processes.

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CHAPTER 3

Local Substitution and Distant Complementarity

I characterize the equilibrium state price deflator, riskless rate, and risk premia in an exchange economy where the representative consumer derives felicity from a durable commodity and previous standards of living. The results extend previous numerical analyses of the corresponding production economy, and complement analyses of economies with a nondurable commodity and habit formation. Introducing durabilities into a pure exchange economy with habit formation effectively resolves the problem of assuring a positive state price deflator with viable parameter values.

JEL CLASSIFICATION: D51, G12.

KEY WORDS: Local substitution, durability; Distant complementarity, habit formation; Additive nonseparable utility; Equilibrium asset pricing; ICAPM; CCAPM.

1. Introduction

It has long been recognized that the traditional capital asset pricing models of Lucas (1978) and Breeden (1979) are not satisfactory descriptions of equity risk premia and the riskless interest rate (Mehra and Prescott 1985). A possible explanation is that they assume additive separable utility functionals. One line of research investigates if more general von Neumann-Morgenstern utility functionals will lead to asset pricing models that match observations better. Allowing for durable commodities and introducing habits in consumption seem to be two promising approaches.

Habit formation introduces complementarity of consumption at distant dates (I will use the terms 'habit formation' and 'distant complementarity' interchangeably). An increase in today's consumption necessitates an increase in future consumption to prevent a decrease in the consumer's future felicity (also known as 'Bernoulli utility'). Sundaresan (1989) and Constantinides (1990) use production economies with habit formation to show respectively that the endogenous consumption paths and returns can be reconciled with observations. Both apply the general characterizations of equilibrium returns developed by Cox, Ingersoll, Jr., and Ross (1985), and thereby impose strong restrictions on security returns. Detemple and Zapatero (1991) show that the effects of habit formation on equity risk premia and the riskless rate have the same sign in a pure exchange economy, but do not calibrate the model to observations. In pure exchange economies only mild restrictions on the security returns are necessary, while the aggregate consumption process is exogenous. It might therefore not be optimal for the consumer to consume all of a large windfall of the commodity. A large increase in consumption today will lead to an increase in the standard of living, and potential deterioration of future felicity. To ensure it is optimal to consume the entire endowment it is necessary to impose an implicit restriction on the parameters of the model. This is achieved by a positivity constraint on the state price deflator.

Hindy and Huang (1993) study local substitution (with interpretation as 'durabilities') in a production economy, again with restrictive assumptions on security returns. They find that the equity premium is lower in their economy than in an economy with additive separable utility. Motivated by empirical findings by Heaton (1993) in support of a combination of durabilities and habits, Hindy, Huang, and Zhu (1997b) do a numerical study of a production economy with both of these path-dependencies. The qualitative aspects of their solution to the optimal consumption problem are puzzling, and far removed from observed aggregate time series of consumption.[1] The analysis is best viewed as an investigation into individual consumers' behavior with utility derived from a dual-attribute durable commodity. The analysis is thus of limited value for asset pricing purposes without a proper theory for aggregating individual investment and consumption choices. Since their analysis cannot be said to be one of the effects of both durability and habits, there

[1]Their parametrization is not in agreement with the traditional interpretation of habit formation. In particular, they study a consumer with felicity $v(t,y,z) = e^{-0.5t}y^{0.2}z^{0.8}$, where y is a "durable" commodity and z represents "habits". Since $v_z(t,y,z) > 0$ their interpretation is at odds with the common use of "habit formation" (Sundaresan 1989, Abel 1990, Constantinides 1990, Ingersoll, Jr. 1992, Detemple and Zapatero 1991). Despite this, they introduce the effect of distant complementarity in consumption by assuming $v_{yz}(t,y,z) > 0$.

is in effect no theoretical work to date that analyzes the implications of durability and habits for equilibrium prices and returns.

In the present chapter I investigate consumers' aggregate behavior in a pure exchange economy when the representative consumer derives felicity from a durable commodity and has preferences that depend on previous standards of living. The first order condition for individual consumer optimality is used to analytically derive the equilibrium state price deflator. Equilibrium returns are then derived, using the approach of Duffie and Zame (1989), and Back (1991). Again the solutions are given in terms of analytic expressions. This is achieved by representing the state price deflator as an Itô process using the the Clark-Ocone Characterization Theorem. The results are derived under mild regularity conditions, for a large class of felicities and endowment processes. As in all pure exchange economies with path-dependent utility, a positivity assumption on the state price deflator is necessary to ensure that the equilibrium conditions are indeed satisfied. I argue that this is a mild condition in an economy with both habits and durabilities, in contrast to an economy with habits and where utility is derived from consumption of a nondurable commodity.

This chapter is organized as follows. The structure and major assumptions of the economy are presented in section 2. Section 3 characterizes the equilibrium state price deflator using the utility gradient principle, and establishes sufficient conditions for it being positive. This result is then used to derive equilibrium returns in section 4. Section 5 compares the equilibrium returns derived in the previous section to those derived under alternative or more restrictive assumptions on the consumer's preferences. The last section concludes and discusses some relevant unresolved questions.

2. The Economy

Consider a representative consumer with life span $[0, T]$, represented by a utility functional U. The consumer's beliefs and information are described by the complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$. A simple interpretation is that Ω represents the set of possible states of nature, while $\mathcal{F}$ is the σ-algebra containing the observable events of Ω. The probability measure P on $\mathcal{F}$ represents the beliefs that the consumer holds about the likelihood of realizations of the observable events. The canonical uncertainty is given by a standard Brownian motion $B : \Omega \times [0,T] \to \mathbb{R}^d$ on $(\Omega, \mathcal{F}, P)$, and $\mathbb{F}$ is the augmentation of the family of σ-algebras $\{\sigma(B_t)\}_{0 \le t \le T}$ generated by the Brownian motion. The consumer derives utility from the service flows from purchases of a composite commodity $c \in L^2_+(P \times \lambda)$, and has available an endowment $e \in L^2_{++}(P \times \lambda)$.[2] To attain a consumption stream other than e the consumer is allowed to trade continuously and frictionlessly in $N+1$ long lived securities with price processes $S \in \mathbb{R}^{N+1}_{++}$. The consumer conducts trade by deciding on a trading strategy $\theta = (\theta^0, \dots, \theta^N)^\top$, a real valued, locally bounded, and $\mathcal{F}_t$-adapted process.

Security j entitles its holder to a cumulative dividend process $D^j, j = 0, \dots, N$ such that the gains process $G \triangleq D + S$ is an Itô process with representation $dG_t = \mu_G(t)\,dt + \sigma_G(t)\,dB_t$. If $\mathbf{I}_S(t)$ is a diagonal matrix with $S^i(t)$ on its i'th diagonal element, $\mu_t = \mathbf{I}_S^{-1}(t)\mu_G(t)$ and $\sigma_t = \mathbf{I}_S^{-1}(t)\sigma_G(t)$, then G can be written in differential notation as

$$dG_t = \mathbf{I}_S(t)[\mu_t\,dt + \sigma_t\,dB_t],$$

for sufficiently well behaved drift vector $\mu_G : \Omega \times [0,T] \to \mathbb{R}^{N+1}$ and dispersion matrix $\sigma_G : \Omega \times [0,T] \to \mathbb{R}^{(N+1)\times d}$.[3] The security S^0 is assumed to be locally riskless, meaning that its gains process is of bounded variation. While G is allowed to be a general Itô process it is necessary to require that endowments are square integrable. This ensures that equilibrium returns are well defined stochastic processes.

ASSUMPTION 3.1 (Endowments). *The aggregate endowment process is an Itô process with decomposition*

$$e(t) = e(0) + \int_0^t \mu_e(s)\,ds + \int_0^t \sigma_e(s)\,dB(s), \quad \forall t \in [0,T],$$

where $\mu_e, \sigma_e \cdot \mathbf{1} \in L^2(P \times \lambda)$.

[2] λ is the Lebesgue measure on $\mathcal{B}([0,T])$, the Borel-measurable subsets of $[0,T]$. $L^p_+(\nu)$ is the positive cone of $L^p(O, \mathcal{O}, \nu)$, while $L^p_{++}(\nu)$ is the interior of $L^p_+(\nu)$.

[3] The setup requires that $|\mu_G + \sigma_G + \operatorname{tr}(\sigma_G\sigma_G^\top)| \in L(\lambda)$ P-a.s. for the corresponding stochastic integrals to be well defined.

The consumer derives utility from consuming a composite, durable commodity c. Previous consumption choices affect how the consumer evaluates current consumption through service flows y from previous purchases of the commodity, and through previous standards of living z. While retaining expected utility I allow for preferences that depend on the history of the economy.

ASSUMPTION 3.2 (Preferences). *The representative consumer's preferences allow an additive nonseparable von Neumann-Morgenstern representation*

$$U(c) = \mathrm{E}\left\{ \int_0^T v(t, y_t, z_t)\, dt \right\},$$

where the felicity $v \in C^{1,2}([0,T] \times \mathbb{R}^2_+)$ *is time separable with impatience rate* $\beta \in L^2_+(P \times \lambda)$ *adapted to* $\mathbb{F}$, *i.e.,* $v(t,y,z) = e^{-\int_0^t \beta_u\, du} u(y,z)$, $u_y > 0$, $u_{yy} < 0$, $u_z \leq 0$ *and* u *is concave in* (y,z). *Further,*

$$y_t = y_0 e^{-\alpha_y t} + \delta_y \int_0^t e^{-\alpha_y (t-s)} c_s\, ds,$$

with z_t *similarly parametrized. The constants* $y_0, z_0, \alpha_y, \alpha_z, \delta_z \in \mathbb{R}_+$, *and* $\delta_y \in \mathbb{R}_{++}$. *Moreover, the marginal felicities satisfy the uniform growth condition*[4]

$$\exists\ k \in \mathbb{R}_+ \ \textit{such that}\ |v_y(t,y,z)| + |v_z(t,y,z)| \leq k(1 + |y| + |z|)\ \forall\ t \in [0,T].$$

The setup does not contain the additive separable von Neumann-Morgenstern model as a special case. Setting y_0 and α_y to zero and δ_y to unity, utility is derived from *cumulative* consumption $C_t \triangleq \int_0^t c_s\, ds$. Letting the consumer's felicity depend on the current consumption rate effectively removes the effect of local substitution. Hardly any commodity gives rise only to instantaneous satisfaction. As such, it can be argued that letting felicity depend on the service flow from past consumption spending is more natural (Hindy, Huang, and Kreps 1992, Hindy and Huang 1992, Hindy and Huang 1993, Hindy, Huang, and Zhu 1997b). The present setup differs from that of Hindy and Huang (1993), and Hindy, Huang, and Zhu (1997b) in that the endowment is exogenous and the consumer makes no production decisions. Hence, this form of local substitution will not induce the consumer to consume in

[4]This growth condition includes the more common formulation of the right hand side, $k(1 + \|(y,z)\|)$. Using Minkowski's Inequality $k(1+\|(y,z)\|) = k(1+\|(y,0)+(0,z)\|) \leq k(1+|y|+|z|)$.

gulps, by optimally choosing singular cumulative consumption paths as in the two aforementioned studies.

Since $u_y > 0$ a consumer who purchases c_s of the commodity at time $s \leq t$ derives felicity from the services y received from it at time t, i.e., $\delta_y e^{-\alpha_y(t-s)} c_s \, ds$. The flow of services deteriorates exponentially at the rate α_y. In addition, as $u_z \leq 0$ the purchasing decision at time t affects utility by affecting the consumer's future standard of living $\{z_s\}_{s>t}$. Again the effect of a purchase c_s on the standard of living deteriorates exponentially, at the rate α_z. In both cases felicity is derived indirectly from the purchase of the commodity, through its effect on the level of services and habits. The consumer starts out with the initial levels of services y_0 and standard of living z_0.

Since endowments are exogenous in a pure exchange economy it is necessary to ensure that the equilibrium state price is strictly positive.

CONDITION 3.1 (Consistency). *The parameters of the endowment process, the utility functional, and the parameters of* (y, z) *are such that* $\forall\, t \in [0, T]$

$$\mathrm{E}\left\{ \int_t^T e^{-\int_t^s \beta_u \, du} \left[\delta_y e^{\alpha_y(t-s)} u_y(y_s, z_s) + \delta_z e^{\alpha_z(t-s)} u_z(y_s, z_s) \right] ds \,|\mathcal{F}_t \right\} > 0,$$

where y *and* z *are the processes of service flows and habits derived from aggregate purchases of the commodity,* e.

It is for instance evident from this assumption why it is necessary that δ_y is strictly positive. Compared to the economies of Detemple and Zapatero (1991), and Detemple and Giannikos (1996) it is clearer in the present setting how to ensure a strictly positive state price deflator.[5] Sufficient conditions for the parameters of the economy to satisfy Condition 3.1 are given in section 3.2.

[5] Detemple and Giannikos (1996) study an economy with two commodities. Their representative consumer derives utility from a nondurable commodity $\hat{c}$, and a dual-attribute durable commodity c with attributes y and z as in the present model. Their model does not nest the present economy, as they do not introduce path-dependence in the numeraire $\hat{c}$. They thereby attain the usual characterization of the state price deflator, and positivity is trivial. Even so, their model puts strong implicit restrictions on the parameters of endowments and felicity to ensure that the demand functions are consistent with market clearing. They also consider equilibrium returns when the durable commodity serves as numeraire. This latter economy is identical to that studied by Detemple and Zapatero (1991), only with a positive marginal felicity with regard to the service flow y from c.

3. Existence and Characterization of Equilibrium

To arrive at a full characterization of a spot-security market equilibrium I follow the procedure of Duffie and Skiadas (1994). They prove a key result linking a representative consumer's preferences to the martingale property of properly deflated gains. First, some structure must be imposed on θ, since the consumer's trading strategy will play some role in deriving the results of this section.

A trading strategy θ is said to finance a consumption stream c using (S, D) if

$$\theta_t^\top S_t = \theta_0^\top S_0 + \int_0^t \theta_s^\top \, dG_s - \int_0^t c_s \, ds, \quad \forall\, t \in [0, T],$$
$$\theta_T^\top S_T = 0.$$

DEFINITION 3.1 (Admissible Trading). *For any strictly positive Itô process Y let G^Y denote deflation of G by Y. The set of admissible trading strategies, Θ, consists of those θ satisfying:*

i) *If G^Y is a $(P, \mathcal{F}_t)$-martingale for some Y, then $\int_0^t \theta_s^\top \, dG^Y$ is a $(P, \mathcal{F}_t)$-martingale $\forall t \in [0, T]$.*

ii) *$\exists\, c \in L^2(P \times \lambda)$ such that θ finances $c - e$.*

The pair (θ, c), $\theta \in \Theta$, is *budget feasible* if θ finances $c - e$ using (S, D), and $\theta_0^\top S_0 \leq e_0 - c_0$ (i.e., prices are in real terms, using the commodity as numeraire). A budget feasible (θ^*, c^*) is *optimal* if for all other budget feasible (θ, c), $U(c^*) \geq U(c)$. Given a budget feasible portfolio-consumption strategy $(\hat{\theta}, \hat{c})$ let the set of *feasible directions* be given by

$$F(\hat{\theta}, \hat{c}) \triangleq \{(\theta, c) \in \Theta \times L^2(P \times \lambda) : \theta \text{ finances } c \text{ using } (S, D),$$
$$(\hat{\theta}, \hat{c}) + \epsilon(\theta, c) \in \Theta \times L_+^2(P \times \lambda)\ \forall\, \epsilon \in (0, \gamma), \gamma > 0\}.$$

Let F_Θ be the projection of F on Θ, and F_C the projection on $L_+^2(P \times \lambda)$. Further, let $\pi(c)$ be the Riesz representation of the utility gradient $\delta U(c; h)$, $h \in F_C(c)$ when it exists.[6] $\pi(c^*)$ will turn out to be the state price deflator in the economy. I.e.,

[6]The utility gradient, when it exists, is defined as $\delta U(c; h) \triangleq \lim_{\epsilon \to 0}[U(c + \epsilon h) - U(c)]/\epsilon = \frac{d}{d\epsilon} U(c + \epsilon h)|_{\epsilon=0}$ for $h \in F_C(c)$. Its Riesz representation $\pi(c)$ is a function of c, but this dependence will be suppressed whenever it is clear at which point it is evaluated.

$\pi(c^*)$-deflated gains will be martingales. To ensure that state price deflated gains are martingales some additional structure must be imposed on the economy.

CONDITION 3.2. *Given any stopping time τ and security n $\exists$ $\{\theta(m)\}_{m\in\mathbb{N}}$ such that*

i) $\pm\theta(m) \in F_\Theta$ $\forall m \in \mathbb{N}$.

ii) $\forall m \in \mathbb{N}$, $\theta^n(m) = 1_{[0,\tau]}(t)$ *and* $\theta^k(m) = 0$ $\forall k \neq 0, n$, *where* $1_{[0,\tau]}(\cdot)$ *is the indicator function.*

iii) $\lim_{m\to\infty} \mathrm{E}\left\{\int_\tau^T \theta_t^\top(m)\, dG_t^{\pi(c^*)}\right\} = 0$.

The condition ensures the existence of two types of sequences of trading strategies. One is to invest all dividends in the locally riskless security up until time τ, while the other is to invest also in another arbitrary security n; condition ii). In both cases condition iii) ensures that the consumer is able to quickly consume all available wealth at time τ before time T. In addition, the negative of these trading strategies must be feasible perturbations of the consumers portfolio-consumption strategy; condition i). It is a technical condition that ensures that the following result, proved for possibly singular cumulative consumption by Duffie and Skiadas (1994), also holds for economies where cumulative consumption is absolutely continuous (see Duffie and Skiadas' section 4 and 5 for an in-depth discussion).

THEOREM 3.1 (Duffie and Skiadas). *If Condition 3.2 is satisfied then (θ^*, c^*) is optimal iff $G^{\pi(c^*)}$ is a $(P, \mathcal{F}_t)$-martingale.*

An easy consequence of Theorem 3.1 is the continuous time version of Lucas' (1978) asset pricing model,

COROLLARY 3.1. *If G^π is a $(P, \mathcal{F}_t)$-martingale then*

$$S_t = \frac{1}{\pi_t}\mathrm{E}\left\{\int_t^T \pi_s\, dD_s + \int_t^T \sigma_D(s)\sigma_\pi^\top(s)\, ds\, |\mathcal{F}_t\right\}. \tag{28}$$

An application of Theorem 3.1 to characterize equilibrium prices and returns requires largely the completion of three steps. Existence of a utility gradient is proved, and its Riesz representation is characterized. It is then proved that the consumer's optimization problem has a solution, and that $c^* = e$. Finally, a certain class of trading strategies is shown to consist of feasible perturbations of a fixed trading strategy. I.e., it is shown that Condition 3.2 is satisfied for $\pi(e)$.

3.1. Existence and Characterization of the Utility Gradient.

THEOREM 3.2 (State Price Deflator). *If Assumptions 3.1 and 3.2 hold then the utility gradient $\delta U(c; h)$, $h \in F_C(c)$, exists, and is a linear bounded functional with Riesz representation*

$$\pi_t = \delta_y \mathrm{E}\left\{\int_t^T e^{-\alpha_y(s-t)} v_y(s, y_s, z_s)\, ds \,|\mathcal{F}_t\right\} + \delta_z \mathrm{E}\left\{\int_t^T e^{-\alpha_z(s-t)} v_z(s, y_s, z_s)\, ds \,|\mathcal{F}_t\right\}.$$

Note that $f^y(t-s, x) \triangleq e^{-\alpha_y(t-s)}x$ trivially satisfies a growth condition similar to that for the marginal felicity. It is therefore easy to generalize to aggregators $y_t = y_0 e^{-\alpha_y t} + \int_0^t f^y(t-s, c_s)\, ds$ where f^y and f^z (similarly defined) satisfy such growth conditions (See chapter 4 for a more extensive generalization of the current economy).

In models where consumers derive felicity from the current consumption rate the price of consumption equals the marginal benefit from the consumption purchase. In this model the consumer only start to derive benefits from current consumption purchases at time $t + dt$. This is in fact more in line with how utility is derived from consumption in the real world. A kid buying a lollipop, however impatient, doesn't begin to derive pleasure from sucking it until an instant dt later. Moreover, the state price equals the conditional *expected* net benefit from current purchases, taking both the past and expected future consumption pattern into consideration, given the present information contained in $\mathcal{F}_t$. Although not allowed in this model, the price of consumption would equal zero whenever the expected marginal benefits from the purchase equal the expected marginal disutility.

PROOF. For simplicity let $\hat{v}(x) \triangleq v(t, e^{-\alpha_y t}y_0 + \int_0^t f^y(t-s, x_s)\, ds, e^{-\alpha_z t}z_0 + \int_0^t f^z(t-s, x_s)\, ds)$. The restrictions on preferences in Assumption 3.2 ensures that the Gâteux differential of U exists (Duffie and Skiadas 1994, Theorem 2). It is therefore valid to interchange differentiation and integration. Thus, the differential

$\delta U(c; h)$ in the direction of $h \in F_C(c)$ is

$$\frac{d}{d\epsilon}U(c+\epsilon h)|_{\epsilon=0} = \mathrm{E}\left\{\int_0^T \frac{d}{d\epsilon}\hat{v}(c+\epsilon h)|_{\epsilon=0}\, dt\right\}$$

$$= \mathrm{E}\left\{\int_0^T v_y(t, y_t, z_t)\int_0^t f^y(t-s, h_s)\, ds + v_z(t, y_t, z_t)\int_0^t f^z(t-s, h_s)\, ds\, dt\right\}.$$

By the Riesz Representation Theorem $\delta U(c; h) = \mathrm{E}\left\{\int_0^T h_t \pi_t\, dt\right\}$ for some $\mathcal{F}_t$-adapted $\pi \in L^2_+$. Interchanging the order of integration, taking conditional expectations, and using that h is $\mathcal{F}_t$-adapted the last expression characterizes π. □

The following proposition establishes the existence of *equilibrium*: There are optimal (c^*, θ^*) such that markets clear, i.e., $c^* = e$ and $\theta^* = \mathbf{0}$.

PROPOSITION 3.1 (Existence of Equilibrium). *Under Assumptions 3.1 and 3.2, if Condition 3.1 hold then a representative consumer equilibrium exists, with state price deflator* $\pi(e)$, $\theta^* = \mathbf{0}$, *and* $c^* = e$.

PROOF. Note that $L^2(P \times \lambda)$ is its own dual. The set of feasible consumption paths is weak* compact by Alaoglu's Theorem, and U is norm-continuous and thus weak*-continuous on $L^2(P \times \lambda)$. U therefore has a maximum on the set of feasible consumption paths (Luenberger 1969, Thm. 5.10.2). Condition 3.1 implies that $\delta U(e; h) < 0,\ h \in F_C(e)$. $c^* = e$ is therefore a global maximum due to Assumption 3.2. □

It remains to validate the condition in Theorem 3.1.

PROPOSITION 3.2. *Condition 3.2 is satisfied for* $\pi = \pi(e)$ *if*

i) $\exists\, \bar{r} < \infty$ *such that* $\operatorname{ess\,sup}|r| \leq \bar{r}$.

ii) $\mathrm{E}\{D_s | \mathcal{F}_t\} \geq D_t\ \forall\ 0 \leq t < s \leq T$.

iii) $\mu_D^n, \sigma_D^{n,k} \in L^2(P \times \lambda)$, $n = 0, \ldots, N$, $k = 1, \ldots, d$.

PROOF. See Appendix A. □

Hypothesis i) ensures a bounded locally riskless interest rate, almost surely, and is standard in the literature. It is an implicit restriction on preferences and endowments. Hypothesis ii) states that the dividend process is a submartingale. I.e., based on information today, one expects cumulative dividends to increase in the future. Both of the hypotheses can be weakened. Nevertheless, they are sufficiently weak to cover all interesting asset pricing applications, and they greatly simplify the proof. Note that securities prices are submartingales whenever the dividend processes are submartingales, for any non-negative state price deflator.

In an economy with additive separable utility an increase in the consumption level will unambiguously decrease the state price deflator. In the current economy this need not hold. A sufficient condition for a non-negative change in consumption to increase the current state price deflator is recorded in the following proposition.

PROPOSITION 3.3. *Assume $\alpha_y = \alpha = \alpha_z$ and that the Gâteaux differential $\delta\pi_t(e;h)$ exists for any $\mathcal{F}_t$-adapted $h \in L^2_+(P \times \lambda)$. If*

$$v_{yz}(s,y_s,z_s) > \left|\min\left(v_{yy}(s,y_s,z_s)\frac{\delta_y}{\delta_z}, v_{zz}(s,y_s,z_s)\frac{\delta_z}{\delta_y}\right)\right|, \quad \forall\, s \in [t,T]$$

then $\pi_t(e) < \pi_t(e+\epsilon h)$ for some $\epsilon > 0$.

PROOF. The Gâteaux differential in the direction of $h \in L^2_+(P \times \lambda)$ is given by

$$\begin{aligned}\delta\pi_t(e;h) = \mathrm{E}\Bigg\{\int_t^T & v_{yy}(s,y_s,z_s)\delta_y^2 e^{-\alpha(s-t)}\int_0^s e^{-\alpha(s-u)}h_u\,du \\ &+ v_{zz}(s,y_s,z_s)\delta_z^2 e^{-\alpha(s-t)}\int_0^s e^{-\alpha(s-u)}h_u\,du \\ &+ 2v_{yz}(s,y_s,z_s)\delta_y\delta_z e^{-\alpha(s-t)}\int_0^s e^{-\alpha(s-u)}h_u\,du\,|\mathcal{F}_t\Bigg\}.\end{aligned}$$

By the main hypothesis of the proposition $\delta\pi_t(e;h) > 0$. □

The main requirement of Proposition 3.3 is that $v_{yz}(\cdot) > 0$. This is a desirable property which is satisfied by for instance Cobb-Douglas felicity. The restriction given in Proposition 3.3 is uniform in $[t,T]$, and places only an implicit restriction on the dynamics of $\{(y_t, z_t)\}$. The next result illustrates that Proposition 3.3 holds under desirable parametric restrictions on the consumer's felicity.

COROLLARY 3.2 (Cobb-Douglas felicity). *Assume $u(y,z) = \frac{1}{\gamma_y}y^{\gamma_y}z^{\gamma_z}$ satisfies the hypotheses of Assumption 3.2. If $\alpha_y = \alpha = \alpha_z$, and either $y_t > \frac{1-\gamma_y}{\gamma_z}z_t$ or $z_t > \frac{1-\gamma_z}{\gamma_y}y_t$ uniformly in t then $\pi_t(e) < \pi_t(e+\epsilon h)$ for some $\epsilon > 0$.*

PROOF. Case 1: $u_{yz}(y,z) > |u_{yy}(y,z)| \geq |u_{zz}(y,z)|$. The strict inequality is satisfied iff $y > \frac{1-\gamma_y}{\gamma_z} z$. The weak inequality is satisfied iff $y \geq \sqrt{\frac{1-\gamma_y}{\gamma_z} \frac{\gamma_y}{1-\gamma_z}} z$. If in addition $\gamma_z > 1 - \gamma_y$ then they will hold jointly. Since u is concave in (y,z) this inequality is satisfied.

Case 2: $u_{yz}(y,z) > |u_{zz}(y,z)| \geq |u_{yy}(y,z)|$. The result obtains by following the same steps as in case 1. □

The concavity of u in (y,z) implies that $\gamma_z > 1 - \gamma_y$. Hence, the hypothesis $y > \frac{1-\gamma_y}{\gamma_z} z$ is weaker than the restriction $y > z$. The last inequality is trivially satisfied if $y_0 \geq z_0$, and $\delta_y > \delta_z$.

3.2. Positivity of the State Price Deflator. An inherent problem with all pure exchange economies is that a positive state price deflator is not ensured without implicit restrictions on the exogenous endowment process and preferences. For models with additive separable utility functionals this is easily resolved by using felicity functions without stationary points on $\mathbb{R}_+$. It is not, however, trivial to establish positivity with more general additive nonseparable utility functionals.[7] Intuitively, if the effect from services and habits on the felicity of the consumer is of the same order, then positivity will be ensured if the pairs (α_y, δ_y) and (α_z, δ_z) are chosen such that habits z do not grow "too fast" relative to services. It is illustrative to consider a simple example.

EXAMPLE 3.1 (Cobb-Douglas felicity). Assume $u(y,z) = \frac{1}{\gamma_y} y^{\gamma_y} z^{\gamma_z}$ with $\gamma_y < 0$ and $\gamma_z \geq 1 - \gamma_y$. The marginal felicities u_y and u_z will be equal in absolute value iff $z = \frac{-\gamma_z}{\gamma_y} y$ or $z = 0$. As a reference point consider the values $\gamma_y = -1$, which implies an atemporal Arrow-Pratt coefficient of relative risk aversion of 2, and $\gamma_z = 2$. In this case equality obtains whenever $z = 2y$. If for instance $z_t > 2y_t$ for all $t \in [0,T]$ it is highly unlikely that Condition 3.1 is satisfied.

The following proposition offers easily verifiable sufficient conditions for Condition 3.1 to hold for the case of Cobb-Douglas felicity.

[7]Chapman (1998) illustrates this for the case of additive habit formation. It is worth noting that while Hindy, Huang, and Zhu's (1997b) economy does not incorporate the traditional notion of habit formation, their parametrization of distant complementarity ensures that the state price deflator is strictly positive. The state price deflator in their economy is identical to the one in the present economy, with $v_z(t,y,z) > 0$.

PROPOSITION 3.4 (Cobb-Douglas felicity). *Assume $u(y,z)$ is as in Example 3.1. If either $z_s = 0 \ \forall\ s \in [t,T]$, or $z_0 \leq y_0$, $\alpha_z \geq \alpha_y$, and $\delta_z \leq \delta_y$ with at least one strict inequality, then $\{\pi_s\}_{t \leq s < T} \gg 0$.*

PROOF. Example 3.1 shows that

$$v_y(t, y_t, z_t) > |v_z(t, y_t, z_t)|$$

iff $z_t = 0$ or $z_t < -\frac{\gamma_z}{\gamma_y} y_t$. Since $-\gamma_z/\gamma_y > 1$ the last inequality is satisfied if the stated hypotheses on the parameters of y and z hold. Under the same hypotheses $e^{-\alpha_y(s-t)} \geq e^{-\alpha_z(s-t)} > 0 \ \forall\ s \in [t,T]$. Hence, $e^{-\alpha_y(s-t)} v_y(s, y_s, z_s) + e^{-\alpha_z(s-t)} v_z(s, y_s, z_s) > 0$. □

4. Equilibrium Returns

I now follow the procedure of Duffie and Zame (1989), and Back (1991) to derive a version of Breeden's (1979) Consumption-based Capital Asset Pricing Model (CCAPM) in the present framework. The idea is to use that π_t is a smooth functional of Itô processes and that gains deflated by π_t are martingales under the canonical measure P.

Consider the deflated gains process $G_t^\pi \triangleq \pi_t S_t + \int_0^t \pi_s \, dD_s + \int_0^t \sigma_D(s) \sigma_\pi^\top(s)\, ds$.[8] As before, $\mu_t = \mathbf{I}_S^{-1}(t)\mu_G(t)$ and $\sigma_t = \mathbf{I}_S^{-1}(t)\sigma_G(t)$. Under assumptions 3.1 and 3.2 π is an Itô process. By an application of Itô's Lemma,

$$(29)\quad G_t^{\pi,i} = \pi_0 G_0^i + \int_0^t \left[\sigma_\pi(u) S_u^i + \pi_u \sigma_{S^i}(u) + \pi_u \sigma_{D^i}(u)\right] dB_u$$
$$+ \int_0^t \left[\mu_\pi(u) S_u^i + \pi_u \mu_{S^i}(u) + \sigma_\pi(u) \cdot \sigma_{S^i}(u) + \pi_u \mu_{D^i}(u) + \sigma_\pi(u) \cdot \sigma_{D^i}(u)\right] du.$$

Since $\pi \gg 0$ is a state price deflator, deflated gains are martingales, i.e., $G_t^\pi = \mathrm{E}\{G_s^\pi \,|\mathcal{F}_t\}, s \geq t$. Thus the drift in (29) must satisfy

$$\mu_\pi(t) S_t^i + \pi_t \mu_{G^i}(t) + \sigma_\pi(t) \cdot \sigma_{G^i}(t) = 0 \qquad P\text{-a.s.}$$

[8]The term $\int_0^t \sigma_D(s)\sigma_\pi^\top(s)\, ds$ must be included to ensure that an investment strategy is self financing regardless of numeraire. Formally, θ is a self financing portfolio strategy with respect to $G = S + D$ if $\theta_t^\top S_t = \theta_0^\top S_0 + \int_0^t \theta_u^\top \, dG_u$. Numeraire invariance holds only if $D_t^\pi \triangleq \int_0^t \pi_u \, dD_u + \int_0^t \sigma_D(u)\sigma_\pi^\top(u)\, du$, for any strictly positive Itô process $\pi \in L^2(P \times \lambda)$. Case in point: Assume θ is self financing wrt. $G \iff d(\theta_t^\top S_t^\pi) = d(\theta_t^\top S_t)\pi_t + \theta_t^\top S_t \, d\pi_t + d(\theta_t^\top S_t)\, d\pi_t = (\theta_t^\top dG_t)\pi_t + \theta_t^\top S_t \, d\pi_t + (\theta_t^\top dG_t)\, d\pi_t = \theta_t^\top[d(\pi_t S_t) + dD_t^\pi] = \theta_t^\top dG_t^\pi$.

or, since $S^i \gg 0$,

$$\frac{\mu_{G^i}(t)}{S_t^i} + \frac{\mu_\pi(t)}{\pi_t} = -\frac{\sigma_\pi(t) \cdot \sigma_{G^i}(t)}{\pi_t S_t^i}. \tag{30}$$

$\mu_\pi(t)/\pi_t$ and $\mu_{G^i}(t)/S_t^i$ can be interpreted as the conditional expected rates of "return" on the state price and the security respectively. The term $-\sigma_\pi(t) \cdot \sigma_{G^i}(t)/(\pi_t S_t^i)$ can be interpreted as the instantaneous conditional covariance rate between the rate of change of the state price deflator and the security. Consider then the locally riskless security S^0, in that $\sigma_{G^0}(t) = 0$ a.s. Denoting $\mu_{G^0}(t)/S^0(t)$ by r_t we have

$$r_t = -\frac{\mu_\pi(t)}{\pi_t}, \tag{31}$$

which gives economic meaning to the conditional expected "rate of decline of the state price", since it has the interpretation as the locally riskless interest rate in the economy. Substituting back into (30) the general form of the consumption-based CAPM obtains,

$$\mu_t - r_t \mathbf{1} = -\frac{1}{\pi_t}\sigma_t \sigma_\pi^\top(t). \tag{32}$$

Since π_t is characterized in the previous section it is possible to derive a more explicit form of the CCAPM under the more restrictive assumptions in this economy. By characterizing the drift and dispersion terms μ_π and σ_π of π, one thus attains the next result.

THEOREM 3.3 (Equilibrium returns). *Under Assumption 3.1 and 3.2, and if Condition 3.1 is satisfied, the equilibrium riskless rate is given by*

$$r_t = \frac{e^{-\int_0^t \beta_u \, du}}{\pi_t} \left[\rho_t^y + \rho_t^z\right], \tag{33}$$

where

$$\rho_t^y \triangleq \delta_y u_y(y_t, z_t) - \alpha_y \delta_y \mathrm{E}\left\{ \int_t^T e^{-\int_t^s \beta_u + \alpha_y \, du} u_y(y_s, z_s) \, ds \,|\mathcal{F}_t \right\},$$

with ρ_t^z similarly defined. The risk premia are given by

$$\mu_t - r_t \mathbf{1} = -\frac{e^{-\int_0^t \beta_u \, du}}{\pi_t} \sigma_t \left[\lambda_t^y + \lambda_t^z\right], \tag{34}$$

where

$$\begin{aligned}
\lambda_t^y \triangleq \delta_y \mathrm{E}\Bigg\{ \int_t^T e^{-\int_t^s \beta_u + \alpha_y \, du} \Bigg[& -u_y(y_s, z_s) \int_t^s \mathfrak{D}_t \beta_u \, du \\
& + u_{yy}(y_s, z_s) \delta_y \int_t^s e^{-\alpha_y (s-u)} \mathfrak{D}_t e(u) \, du \\
& + u_{yz}(y_s, z_s) \delta_z \int_t^s e^{-\alpha_z (s-u)} \mathfrak{D}_t e(u) \, du \Bigg] ds | \mathcal{F}_t \Bigg\},
\end{aligned}$$

and similarly for λ_t^z. Finally,

$$\mathfrak{D}_t e(u) = \int_t^u \mathfrak{D}_t \mu_e(s) \, ds + \int_t^u \mathfrak{D}_t \sigma_e(s) \, dB_s + \sigma_e(t)$$

is the Malliavin derivative of the endowment process.

The Malliavin differential of an Itô process is simply a form of Gâteaux differential, where the perturbation is around the Brownian motion (Nualart 1995, Øksendal 1996).

PROOF. The idea is to apply the Clark-Ocone Characterization Theorem to π_t to find explicit expressions for the integrands in Itô's Representation Theorem (Ocone and Karatzas 1991, Øksendal 1996). Let

$$X_y(T) = \int_0^T e^{-\int_0^s \beta_u + \alpha_y \, du} u_y(y_s, z_s) \, ds.$$

The Malliavin derivative of $X_y(T)$ at u is

$$\begin{aligned}
\mathfrak{D}_u X_y(T) &= \int_0^T e^{-\int_0^s \beta_v + \alpha_y \, dv} \Bigg[-u_y(y_s, z_s) \int_0^s \mathfrak{D}_u \beta_v 1_{[0,v]}(u) \, dv \\
&\qquad + u_{yy}(y_s, z_s) \mathfrak{D}_u y_s + u_{yz}(y_s, z_s) \mathfrak{D}_u z_s \Bigg] 1_{[0,s]}(u) \, ds \\
&= \int_u^T e^{-\int_0^s \beta_v + \alpha_y \, dv} \Bigg[-u_y(y_s, z_s) \int_u^s \mathfrak{D}_u \beta_v \, dv \\
&\qquad + u_{yy}(y_s, z_s) \mathfrak{D}_u y_s + u_{yz}(y_s, z_s) \mathfrak{D}_u z_s \Bigg] ds,
\end{aligned}$$

where

$$\begin{aligned}\mathfrak{D}_u y_s &= \delta_y \int_u^s e^{-\alpha_y(s-v)} \mathfrak{D}_u e_v \, dv \\ &= \delta_y \int_u^s e^{-\alpha_y(s-v)} \left[\int_u^s \mathfrak{D}_u \mu_e(v) \, dv + \int_u^s \mathfrak{D}_u \sigma_e(v) \, dB_v + \sigma_e(u) \right] dv.\end{aligned}$$

Since, by the Clark-Ocone Characterization Theorem,

$$\mathrm{E}\{X_y(T) \,|\mathcal{F}_t\} = \mathrm{E}\{X_y(T)\} + \int_0^t \mathrm{E}\{\mathfrak{D}_u X_y(T) \,|\mathcal{F}_u\} \, dB_u,$$

and

$$\pi_t = \delta_y e^{\int_0^t \alpha_y \, du} \left[\mathrm{E}\{X_y(T) \,|\mathcal{F}_t\} - X_y(t)\right] + \delta_z e^{\int_0^t \alpha_z \, du} \left[\mathrm{E}\{X_z(T) \,|\mathcal{F}_t\} - X_z(t)\right],$$

the state price deflator can be represented by the Itô process

$$\pi_t = 1 + \int_0^t \mu_\pi(s) \, ds + \int_0^t \sigma_\pi(s) \, dB_s,$$

where

$$\begin{aligned}\mu_\pi(t) = \delta_y e^{\alpha_y t} &\left\{ \alpha_y \left(\mathrm{E}\{X_y(T) \,|\mathcal{F}_t\} - X_y(t) \right) - e^{-\int_0^t \beta_s + \alpha_y \, ds} u_y(y_t, z_t) \right\} \\ &+ \delta_z e^{\alpha_z t} \left\{ \alpha_z \left(\mathrm{E}\{X_z(T) \,|\mathcal{F}_t\} - X_z(t) \right) - e^{-\int_0^t \beta_s + \alpha_z \, ds} u_z(y_t, z_t) \right\},\end{aligned}$$

and

$$\sigma_\pi(t) = \delta_y e^{\alpha_y t} \mathrm{E}\{\mathfrak{D}_t X_y(T) \,|\mathcal{F}_t\} + \delta_z e^{\alpha_z t} \mathrm{E}\{\mathfrak{D}_t X_z(T) \,|\mathcal{F}_t\}.$$

The result now follows by substituting for the integrands in the conditional expectations. □

5. Economic Implications

By the Envelope Theorem the consumer's marginal indirect utility for wealth (the value of investing an additional dollar in the consumer's portfolio) is equal to the consumer's marginal utility (the value of one additional unit of the commodity). In the present economy the state price deflator represents the latter. The major insight of Breeden (1979) is that in an economy with additive separable utility the marginal indirect utility for wealth is equal to the marginal felicity of the consumer. Since this is obviously not the case in the present economy, one cannot expect Breeden's (1979) CCAPM to obtain. This observation and sufficient conditions for the CCAPM to obtain are formalized in the first subsection.[9] The second and third subsections deal with comparative statics of the riskless interest rate and risk premia respectively.

5.1. Fund Separation. Rewriting the risk premia (34) in Theorem 3.3 shows that the two-fund separation implied by Breeden's (1979) CCAPM does not hold without further restrictions on aggregate endowments.

COROLLARY 3.3 (Mutual Fund Separation). *The consumer is compensated for covariance risk between the securities σ_t and the aggregate endowment $\sigma_e(t)$, and changes in the production technology $\mathfrak{C}_t e(u) \triangleq \mathfrak{D}_t e(u) - \sigma_e(t)$:*[10]

$$\mu_t - r_t \mathbf{1} = \Lambda_t^{(1)} \sigma_t \sigma_e^\top(t) + \sigma_t \Lambda_t^{(2)\top}, \tag{35}$$

where $\Lambda_t^{(n)} = \Lambda_t^{(n,y)} + \Lambda_t^{(n,z)}$ *for* $n = 1, 2$ *and*

$$\Lambda_t^{(1,y)} \triangleq \frac{e^{-\int_0^t \beta_u\, du}}{\pi_t} \delta_y \mathrm{E}\Bigg[\int_t^T e^{-\int_t^s \beta_u + \alpha_y\, du} \Big\{ u_{yy}(y_s, z_s) \frac{\delta_y}{\alpha_y} \left(e^{-\alpha_y(s-t)} - 1\right) + u_{yz}(y_s, z_s) \frac{\delta_z}{\alpha_z} \left(e^{-\alpha_z(s-t)} - 1\right) \Big\} ds \Big| \mathcal{F}_t \Bigg],$$

$$\Lambda_t^{(2,y)} \triangleq -\frac{e^{-\int_0^t \beta_u\, du}}{\pi_t} \delta_y \mathrm{E}\Bigg[\int_t^T e^{-\int_t^s \beta_u + \alpha_y\, du} \Big\{ -u_y(y_s, z_s) \int_t^s \mathfrak{D}_t \beta_u\, du + u_{yy}(y_s, z_s) \delta_y \int_t^s e^{-\alpha_y(s-u)} \mathfrak{C}_t e(u)\, du + u_{yz}(y_s, z_s) \delta_z \int_t^s e^{-\alpha_z(s-u)} \mathfrak{C}_t e(u)\, du \Big\} ds \Big| \mathcal{F}_t \Bigg].$$

Detemple and Zapatero (1991) find that a similar three fund separation result obtains when utility is derived from a nondurable commodity, and preferences depend on previous standards of living. Clearly then, defining preferences over a

[9]To be more specific, I show that the CCAPM holds in the weaker sense that risk premia are *proportional* to the instantaneous conditional covariance between security returns and aggregate consumption. The coefficient of proportionality will not, however, be equal to the atemporal Arrow-Pratt coefficient of absolute risk aversion.

[10]To arrive at meaningful interpretations of some of the elements of the risk factors it is useful to view the endowment as the output of a production process.

durable rather than a nondurable commodity does not add an extra mutual fund. This is as expected. Letting the felicity depend on services received from the commodity rather than the commodity itself does not add any new source of uncertainty when habit formation is already present. It is therefore clear that the two-fund separation property obtains whenever the marginal utility is not affected by shifts in the coefficients of aggregate endowments (Detemple and Zapatero 1991). This is the case whenever security returns are independent of changes in technology or when the technology is deterministic. The former holds whenever the last term in (35) vanishes, while the latter holds whenever the level of the endowment process is a sufficient statistic for changes in its drift and dispersion coefficients.

COROLLARY 3.4 (Two-fund Separation). *If* $\Lambda_t^{(2)} \cdot \sigma_t = 0 \;\forall\; t \in [0,T]$ *P-a.s. then*[11]

$$\mu_t - r_t \mathbf{1} = \Lambda_t^{(1)} \sigma_t \sigma_e^\top(t).$$

If $de_t = e_t[a(t)\,dt + b(t)\,dB_t]$ *and* $a, \beta \in L^2(\lambda)$, $b \in L^2(\lambda)^d$, *then*

$$\mu_t - r_t \mathbf{1} = [\Lambda_t^{(1)} + \Lambda_t^{(2)} \cdot b_e^{-1}(t)] \sigma_t e_t b^\top(t),$$

where $b_e^{-1}(t) = \frac{1}{e_t}(1/b(t)^{(1)}, \dots, 1/b(t)^{(d)})$.

PROOF. By an application of Itô's Lemma the solution to the SDE for e_t is

$$e_t = e_0 \exp\left(\int_0^t a(s) - \tfrac{1}{2} b(s) \cdot b(s)\,ds + \int_0^t b(s)\,dB_s\right).$$

Thereby, $\mathfrak{D}_t e_u = e_u b(t) 1_{[0,u]}(t)$ and $\mathfrak{C}_t e_u = [e_u - e_t] b(t)$. □

5.2. The Riskless Rate. Intuitively local substitution should increase the riskless interest rate. When a commodity is durable the consumer is less exposed to variations in the output of the commodity. Consequently there is less need to save as a protection against unfavorable shifts in output. Since all securities are in zero net supply the riskless interest rate must increase to induce consumers not to change their net savings. Likewise, habit formation should lead to an increased

[11] A sufficient condition for the orthogonality of $\Lambda_t^{(2)}$ and σ_t is that $\Lambda_t^{(2)}$ is $\mathcal{F}_t'$-adapted, σ_t is $\mathcal{F}_t''$-adapted, and that $\mathcal{F}_t'$ and $\mathcal{F}_t''$ are orthogonal (Detemple and Zapatero 1991, Proposition 6.1). Orthogonality of $\mathcal{F}_t'$ and $\mathcal{F}_t''$ occurs if they are generated by independent Brownian motions.

demand for saving, thereby driving interest rates down. It turns out that it is in no way trivial to formally confirm these conjectures without imposing more structure on the economy. The discussion in this subsection will therefore be mainly informal.

Before discussing the difficulties involved in doing comparative statics it is useful to interpret the somewhat unusual structure of the riskless interest rate by partial considerations.[12] Recall from equation (33) that it is proportional to $\rho_t^y + \rho_t^z$ with ρ_t^y defined by

$$\delta_y u_y(y_t, z_t) - \alpha_y \delta_y \mathrm{E}\left\{ \int_t^T e^{-\int_t^s \beta_u + \alpha_y \, du} u_y(y_s, z_s)\, ds \,|\mathcal{F}_t \right\}, \tag{36}$$

and similarly for ρ_t^z. The riskless rate depends on past, current, and expected future purchases of the commodity, as in models of habit formation without local substitution. The factors ρ^y and ρ^z are the innovations in the marginal felicity of services and the marginal felicity of habits respectively. It is interesting to note that Breeden's result does not generalize to preferences with local substitution, in contrast to models of habit formation without local substitution (Detemple and Zapatero 1991, Theorem 5.1). In the present economy no factors are proportional to the conditionally expected growth in consumption, or the conditional variance of consumption.[13]

An unexpected increase in future marginal felicity of services will reduce the riskless interest rate. This occurs either if future consumption is reduced, i.e., the consumer desires to save more, or if $u_{yz}(y, z) > 0$ and the consumer expects an increase in the future standard of living. The latter effect cannot be expected to be significant though, as an increased future standard of living can occur only if future consumption increases. A future increase in consumption will naturally lead to a reduction in savings, and thereby an increase in the riskless interest rate. If,

[12]The reader should let aside a rigid interpretation of the model at this stage. In particular with respect to statements about partial changes in (y, z), since they are intimately linked. Statements like " ... when y increases ... " should be interpreted for instance as an exogenous shift in the parameters governing services, α_y and δ_y. An interpretation more in line with the mathematics of the economy is to interpret this as $\{y_u\}_{t\leq u\leq T}$ increasing "much more" than $\{z_u\}_{t\leq u\leq T}$ as $\{e_u\}_{t\leq u\leq T}$ increases, for instance due to δ_y being significantly larger than δ_z.

[13]Recall that in the classical CCAPM

$$r_t = \beta_t - \frac{u_{cc}(e_t)}{u_c(e_t)}\mu_e(t) - \frac{1}{2}\frac{u_{ccc}(e_t)}{u_c(e_t)}\sigma_e^2(t),$$

which is typically increasing in the expected growth (since consumers plan to save less), and decreasing in the level of consumption uncertainty (since consumers prefer smooth consumption paths).

on the other hand, $u_{yz}(y,z) < 0$, the increase in future marginal felicity of services can be brought on by a decrease in future standards of living.

An unexpected increase in the future marginal felicity of habits will increase the riskless interest rate. This situation occurs if there is a positive shift in the future standard of living, in which case it is evident from Theorem 3.2 that the cost of current consumption falls. The consumers' propensity to save will decline, and interest rates must increase to clear the debt market. The decrease in future marginal felicity can also be the result of an increase in future services received by the consumer. This requires that $u_{yz}(y,z) < 0$. The case where $u_{yz}(y,z) > 0$ is not important in this case, as argued in the previous paragraph.

The net effect of changes in ρ_t^y and ρ_t^z will depend on the consumer's felicity as well as the parameters of services and habits. The challenge is of course to extend this very partial analysis to a situation that fully incorporates the relationship between ρ_t^y and ρ_t^z.[14] Looking back at equation (33) it is clear that habit formation reduces the riskless interest rate if $\rho_t^z \leq 0$, $\rho_t^y \leq \rho_t^y|_{\delta_z=0}$, and $\pi_t \geq \pi_t^y|_{\delta_z=0}$. It is in no way obvious which conditions that will ensure that these inequalities hold.

The rest of this subsection discusses comparisons of the riskless interest rate across three economies. An economy with local substitution is compared to the traditional economy with additive separable utility. The felicities of the two utility functionals are naturally taken to be identical, except for assumptions on their arguments, (t, e_t) versus (t, y_t, z_t). Subsequently, the effect of introducing habit formation into the economy with local substitution is investigated. It turns out that few unambiguous answers are available at the current level of abstraction. To facilitate these analyses it is convenient to rewrite the riskless rate by introducing a simplifying assumption for the economy with habit formation.

[14]It is tempting to use an indirect approach, using the relationship $\pi_t = e^{-\int_0^t r_s\,ds}\xi_t$, $\xi_t \triangleq \exp\{\int_0^t \psi_u\,dB_u - \frac{1}{2}\int_0^t \psi_u \cdot \psi_u\,du\}$, where ψ is the market price of risk. If π_t^y and π_t^z are supermartingales (for instance with additive separable utility, with increasing and concave felicity, and growing per capita consumption) it can be shown that the price of a zero coupon bond $D_{t,s}$ at time t that matures at time $s > t$ satisfies $D_{t,s} \leq D_{t,s}|_{\delta_z=0}$. Using the relationship $\pi = e^{-\int r}\xi$ one might falsely infer that $r_t|_{\delta_z=0} \leq r_t$, counter to intuition. This does not, however, take into account that the price of risk $\psi \neq \psi|_{\delta_z=0}$. One is therefore left with the exercise of finding conditions that for instance ensure $\pi_t^z \leq 0$ for all possible combinations of (y, z).

PROPOSITION 3.5. *If $\alpha_y = \alpha_z = \alpha$ then*

$$r_t = \frac{1}{\pi_t}\left[\delta_y v_y(t, y_t, z_t) + \delta_z v_z(t, y_t, z_t)\right] - \alpha.$$

PROOF. Combining Theorem 3.2 and Theorem 3.3 the riskless interest rate can be rewritten as

$$\begin{aligned} r_t &= \frac{1}{\pi_t}\left[\delta_y v_y(t, y_t, z_t) + \delta_z v_z(t, y_t, z_t) - \alpha_y \pi_t^y - \alpha_z \pi_t^z\right] \\ &= \frac{1}{\pi_t}\left[\delta_y v_y(t, y_t, z_t) + \delta_z v_z(t, y_t, z_t) - \alpha_y \pi_t + (\alpha_y - \alpha_z)\pi_t^z\right], \end{aligned}$$

and the result follows by setting $\alpha_y = \alpha_z$. □

Local Substitution. The extent to which the comparative statics in Breeden's model generalize to a model with local substitution is addressed by comparing the riskless rate with local substitution

$$r_t^d \triangleq \frac{v_y(t, y_t, z_0)}{\mathrm{E}\left\{\int_t^T e^{-\alpha_y(s-t)} v_y(s, y_s, z_0)\, ds \,|\, \mathcal{F}_t\right\}} - \alpha_y,$$

to one derived from an additive separable utility

$$r_t^a \triangleq \beta_t - \frac{u_{yy}(e_t)}{u_y(e_t)}\mu_e(t) - \tfrac{1}{2}\frac{u_{yyy}(e_t)}{u_y(e_t)}\sigma_e \cdot \sigma_e.$$

A comparison of how they respond to changes in the subjective discount rate involves comparison of four different economies. These are summarized in the following table:

	r_t^a	r_t^d
$\beta_t \uparrow$	$\uparrow$	–
$\{\beta_u\}_{u \neq t} \uparrow$	–	$\uparrow$

An increase in β_t increases r_t^a. Since an increase in β_t only represents a discontinuity in $\{\beta_t\}_{0 \le t \le T}$ it does not affect the Lebesgue integral $\int_0^t \beta_u\, du$, and r_t^d is unaffected. With additive separable utility a uniformly higher subjective discount rate, except at time t, has no effect on the current interest rate. The effect in the economy with local substitution is to increase the riskless interest rate. Not only does the historical subjective discount rate affect the current riskless rate in an economy with local substitution, but also the prospective future subjective discount rate. This is summarized in the following proposition.

PROPOSITION 3.6. *Consider r_t^d as a functional of $\beta \in L_+^2(P \times \lambda)$, denoted by $r^d(t, \beta)$. For any $b \in L_+^2(P \times \lambda)$ adapted to $\mathbb{F}$ and $\epsilon > 0$, $r^d(t, \beta) \leq r^d(t, \beta + \epsilon b)$.*

PROOF. Let $f(t) = \int_0^t b_u \, du$ and $\psi(t) = e^{-\int_0^t \beta_u + \alpha_y \, du} u_y(y_s, z_0)$. Assuming the Gâteaux differential exists it is given by[15]

$$
\begin{aligned}
\delta r^d(t, \beta; b) &= \frac{d}{d\epsilon} r^d(t, \beta + \epsilon b)|_{\epsilon=0} \\
&= -f(t) r^d(t, \beta) - \frac{\mathrm{E}\left\{\int_t^T [-f(s)]\psi(s) \, ds \,|\mathcal{F}_t\right\}}{\mathrm{E}\left\{\int_t^T \psi(s) \, ds \,|\mathcal{F}_t\right\}} r^d(t, \beta).
\end{aligned}
$$

The result follows by observing that the numerator in the fraction satisfies

$$
\begin{aligned}
\mathrm{E}\left\{\int_t^T f(s)\psi(s) \, ds \,|\mathcal{F}_t\right\} &\geq \mathrm{E}\left\{\int_t^T f(t)\psi(s) \, ds \,|\mathcal{F}_t\right\} \\
&= f(t)\mathrm{E}\left\{\int_t^T \psi(s) \, ds \,|\mathcal{F}_t\right\},
\end{aligned}
$$

using that $f(s) \leq f(t)$ whenever $s \leq t$, and that $f(t)$ is adapted to $\mathbb{F}$. □

Consider next a shift in the contemporary and future growth of consumption $\{\mu_e(u)\}_{t \leq u \leq T}$. Since $u_{yy} < 0$, r_t^a will increase. Similarly, since $v_y(t, y_t, z_0)$ is unaffected and $\pi_t^y|_{\alpha_z, \delta_z = 0}$ decreases, r_t^d will also increase.

The consumer with additive separable utility has decreasing absolute risk aversion only if $u_{yyy}(e_t) > 0$. Hence, for most attractive felicity functions r_t^a is decreasing in the conditional instantaneous variance of consumption. Consider now a positive shift in current and future uncertainty $\{\sigma_e(u)\}_{t \leq u \leq T}$. This is a mean preserving spread of $\{e_u\}_{t \leq u \leq T}$. A trivial application of Fubini's Theorem and the martingale property of Itô integrals shows that it also represents a mean preserving spread of y_t, since y_t is a linear functional of $\{e_u\}_{0 \leq u \leq t}$. If $v_{yyy}(t, y) > 0$ then $v_y(t, y)$ is a decreasing and convex function of y. Thereby, $\pi_t^y|_{\alpha_z, \delta_z = 0}$ will increase. Since current marginal felicity $v_y(t, y_t, z_0)$ remains unaffected r_t^d must decrease.

[15]It is not necessary for the current result that b is $P \times \lambda$-square integrable. Still, this restriction is necessary to ensure that risk premia are well defined in the economy with shifted subjective discount rate $\beta + \epsilon b$.

Qualitative changes in r_t^a and r_t^d to changes in key parameters evidently are similar. These results are summarized in the following corollary:

COROLLARY 3.5. *r_t^d is increasing in $\{\beta_u\}_{0\le u\le T}$ and $\{\mu_e(u)\}_{t\le u\le T}$. If in addition $v_{yyy}(t,y) > 0$ then r_t^d is decreasing in $\{\sigma_e(u)\}_{t\le u\le T}$.*

Now to the question of the relative sizes of the two riskless interest rates. It would be useful in the study of this problem if the effects of α_y and δ_y on r_t^d were unambiguous. This is not so, and the computations are relegated to Appendix B. To confirm our intuition that $r_t^a < r_t^d$ it is necessary that

$$\beta_t - \frac{v_{yy}(t,e_t)\mu_e(t) - \frac{1}{2}v_{yyy}(t,e_t)\sigma_e(t)\cdot\sigma_e(t)}{v_y(t,e_t)} < \frac{v_y(t,y_t)}{\mathrm{E}\left\{\int_t^T e^{-\alpha_y(s-t)}v_y(s,y_s)\,ds\,|\mathcal{F}_t\right\}} - \alpha_y.$$

Inspection of these expressions confirms that one cannot hope to find simple sufficient conditions for this inequality to hold.

Habit Formation. The question of the effects of habit formation is addressed by comparing $r_t^d = r_t|_{\delta_z=0}$ to r_t. This analysis will fully utilize the restriction $\alpha_y = \alpha_z$ (and hence $\alpha_z > 0$).

Consider the case where $v_{yz}(t,y,z) \le 0$. Since $v_z(t,y,z) \le 0$ the numerator of r_t is smaller in all states than the numerator of r_t^d. Despite this observation no conclusions can be drawn without more restrictive assumptions on the felicity function and endowments, since the following lemma documents that the net effect on r_t by forcing $\delta_z = 0$ is ambiguous.

LEMMA 3.1. *If $v_{yz}(t,y,z) \le 0$ then $\pi_t \le \pi_t|_{\delta_z=0}$. If $v_{yz}(t,y,z) < 0$ then the implied inequality is strict.*

PROOF. $\pi_t = \pi_t^y + \pi_t^z \le \pi_t^y \le \pi_t^y|_{\delta_z=0} = \pi_t|_{\delta_z=0}$. □

When $v_{yz}(t,y,z) > 0$ even the effects on the numerator and denominator of the riskless interest rate are indeterminate. Again, stronger assumptions must be made about the consumer's felicity and endowments to conclude whether $r_t \lessgtr r_t^d$.

It is also of interest to consider what happens to the local volatility of the riskless interest rate when habit formation is introduced. By an application of Itô's

Lemma and use of the results in Theorem 3.3 the dispersion matrices are

$$\sigma_{r^d}(t) = \frac{\delta_y v_y(t, y_t, z_0 e^{-\alpha t})}{(\pi_t^y|_{\delta_y=0})^2} e^{-\int_0^t \beta_u \, du} \lambda_t^y|_{\delta_y=0}$$

$$\sigma_r(t) = \frac{\delta_y v_y(t, y_t, z_t) + \delta_z v_z(t, y_t, z_t)}{(\pi_t)^2} e^{-\int_0^t \beta_u \, du} (\lambda_t^y + \lambda_t^z),$$

where I have made use of the simplifying assumption in Proposition 3.5. As usual, the conditional instantaneous variances are given by $\sigma_{r^d}(t) \cdot \sigma_{r^d}(t)$ and $\sigma_r(t) \cdot \sigma_r(t)$ respectively. Again, it is evident that the effect of habit formation is ambiguous without stronger restrictions on the economy. It is tempting, however, to conjecture that the riskless interest rate in the present economy typically will be less volatile than one in an economy with habit formation but without local substitution.

5.3. Risk Premia. Consider an economy where the consumer derives felicity from purchases of the commodity and previous standards of living, $v(t, c_t, z_t)$. Detemple and Zapatero (1991) find that habit formation inflates risk premia in an economy without local substitution if $v_{cz}(t, e, z) \leq 0$, and if the atemporal Arrow-Pratt measure of absolute risk aversion is nondecreasing in z. Similar sufficient conditions in the present economy are supplied by the following proposition.

PROPOSITION 3.7 (Effect of Habit Formation). *If $\Lambda_t^{(2)} \geq 0$, $u_{yz}(y, z) \leq 0$, and $u_{yy}(y, z)$ is non-increasing in z then $\mu_t - r_t \mathbf{1}|_{\delta_z=0} \leq \mu_t - r_t \mathbf{1}$. The implied inequality is strict whenever any of the sufficient inequalities are strict.*

PROOF. By Lemma 3.1, $\pi_t \leq \pi_t^y|_{\delta_z=0}$, so habit formation inflates risk premia through a reduction of the state price deflator. From Corollary 3.3 it is sufficient to establish that $\Lambda_t^{(1)} = \Lambda_t^{(1,y)} + \Lambda_t^{(1,z)} \geq \Lambda_t^{(1)}|_{\delta_z=0}$.

Consider first $\Lambda_t^{(1,z)}$. By Assumption 3.2 $u(y, z)$ is concave in (y, z). The implied restrictions on the principal subdeterminants of the Hessian matrix of u require that $u_{zz}(y, z) \leq 0$. Now,

$$\Lambda_t^{(1,z)} \propto \mathrm{E}\left\{ \int_t^T e^{-\int_t^s \beta_u + \alpha_z \, du} \left[u_{zz}(y_s, z_s) \frac{\delta_z}{\alpha_z} \left(e^{-\alpha_z (s-t)} - 1 \right) \right.\right.$$

$$\left.\left. + u_{yz}(y_s, z_s) \frac{\delta_y}{\alpha_z} \left(e^{-\alpha_y (s-t)} - 1 \right) \right] ds \,|\mathcal{F}_t \right\}.$$

By Assumption 3.2 $u_{zz} \le 0$, and by hypothesis $u_{yz} \le 0$. As $\alpha_y > 0$, $e^{\alpha_y(s-t)} - 1 \le 0$, and similarly for α_z. Thereby $\Lambda_t^{(1,z)} \ge 0 = \Lambda_t^{(1,z)}|_{\delta_z=0}$. Similar arguments show that $\Lambda_t^{(1,y)} \ge \Lambda_t^{(1,y)}|_{\delta_z=0}$. □

Let $R_v^a(y) \triangleq -v_{yy}(y,z)/v_y$ denote the atemporal Arrow-Pratt measure of absolute risk aversion.[16] Similarly, let $R_v^r(y) \triangleq yR_v^a(y)$ denote the atemporal Arrow-Pratt measure of relative risk aversion. Notice that the hypotheses on the felicity imply that the consumer is increasingly risk averse with previous standards of living. *Ceteris paribus*, an increase in z_t reduces both $u_y(y_t, z_t)$ and $u_{yy}(y_t, z_t)$.[17] Since the latter is negative both $R_v^a(y;z)$ and $R_v^r(y;z)$ increase.

The conditions in Proposition 3.7 do not represent properties of common felicity functions, as for instance Cobb-Douglas felicity. Note, however, that the conditions are far stronger than necessary for an increase in risk premia to occur. They require that habit formation inflate risk premia at all times and in all states. One can expect this effect to hold under much weaker conditions, but such conditions are not possible to derive formally with the current level of generality. It is possible to compute these effects by applying the methodology developed in chapter 2.

It is also of interest to investigate Hindy and Huang's (1993) finding that local substitution reduces risk premia relative to an economy with additive separable utility. A simple counter example shows their observation need not be valid for a risk averse consumer.

Assume for simplicity $u(y,z) = u(y)$, and let the additive separable utility be defined in terms of rates $\{e_t\}_{0\le t\le T}$, and the one with local substitution be defined in terms of the weighted aggregate rate, $\{y_t\}_{0\le t\le T}$. The question can now be posed in terms of an inequality:

$$\Lambda_t^1 \sigma_t \sigma_e^\top(t) + \sigma_t \Lambda_t^2 \overset{?}{\le} -\frac{u_{yy}(e_t)}{u_y(e_t)} \sigma_t \sigma_e^\top(t), \tag{37}$$

[16]The interpretation of this measure is somewhat problematic as the Envelope Theorem for indirect utility no longer applies. I.e., letting $J(W)$ denote the indirect utility of wealth, it is no longer the case that, say, $J_W(W) = u_y(y,z)$. Still, it is clear that a higher R_v^a coincides with more risk averse preferences.

[17]The qualifier *ceteris paribus* is meaningful in this partial analysis as z_t is changed by changing δ_z, not by increasing $\{e_s\}_{0\le s\le t}$.

where the right hand side is the risk premia in the CCAPM of Breeden (1979). If we assume as in Proposition 3.7 that $\Lambda_t^{(2)} \geq 0$ it is quite clear that this inequality need not hold.

COUNTER EXAMPLE 3.1. If $u(y,z) = u(y)$, β_u is independent of δ_y, $\Lambda_t^{(2)} \geq 0$, and $u_{yyy}(y) \leq 0$ then $\exists\ \underline{b} \in \mathbb{R}_+$ such that inequality (37) does not hold for any $\delta_y > \underline{b}$.

PROOF. In the restricted economy with local substitution

$$\Lambda_t^{(1)} = -\frac{\mathrm{E}\left\{\int_t^T e^{-\int_t^s \beta_u + \alpha_y\, du} u_{yy}(y_s) \frac{\delta_y}{\alpha_y}\left(1 - e^{-\alpha_y(s-t)}\right) ds \,|\mathcal{F}_t\right\}}{\mathrm{E}\left\{\int_t^T e^{-\int_t^s \beta_u + \alpha_y\, du} u_y(y_s)\, ds\,|\mathcal{F}_t\right\}}.$$

The denominator is a deceasing function of δ_y as $u_{yy}(y) < 0$. The effect on the numerator from a small increase in δ_y is

$$\mathrm{E}\left\{\int_t^T e^{-\int_t^s \beta_u + \alpha_y\, du}\left(1 - e^{-\alpha_y(s-t)}\right) \times \left[u_{yyy}(y_s)\frac{\delta_y}{\alpha_y}\int_0^s e^{-\alpha_y(s-u)} e_u\, du + u_{yy}(y_s)\frac{1}{\alpha_y}\right] ds\,|\mathcal{F}_t\right\},$$

which is non-positive if $u_{yyy}(y) \leq 0$. □

The intuition is quite simple. In terms of variance, the volatility of y is proportional to δ_y^2. For small δ_y, $\{y_t\}_{0\leq t\leq T}$ will typically be a less volatile stochastic process than $\{e_t\}_{0\leq t\leq T}$. As δ_y increases this relationship will eventually be reversed. Preferences satisfying the conditions of the example are not very attractive though. $u_{yyy}(y) \leq 0$ imply the consumer is not prudent, and therefore that the atemporal Arrow-Pratt measure of absolute risk aversion is increasing in y. For instance, the iso-elastic felicity chosen by Hindy and Huang implies $u_{yyy}(y) > 0$.

6. Concluding Remarks

Several authors have made impressive contributions to our understanding of more realistic economies than that offered by the seminal model of Lucas (1978). The contributions of the present analysis are best understood in light of some of the previous analyses of related economies.

Hindy, Huang, and Zhu (1997b) use a production economy to endogenize per capita consumption when the consumer has preferences similar to those in the present model. Nevertheless, this is not necessarily a desirable approach if the aim is to analyze dynamics of equilibrium security prices and returns. This is exemplified by the aggregate consumption series that would result from the homogeneous consumer economy based on their model. Aggregate consumption is much smoother than what their model predicts. Furthermore, as alluded to in the introduction, the parametrization of their model does not allow a study of the traditional notion of habit formation. They assume the marginal felicity is positive with regard to both services and "habits". This implies that the consumer will find it optimal to retain an as large as possible, fixed proportion of services relative to habits.[18] Such behavior is not implied by habit formation. Hence, the present chapter is the only theoretical analysis to date of the implications of durabilities and habits for equilibrium prices and returns.

Chapman (1998) illuminates the dangers of specifying the endowment process exogenously in a representative consumer pure exchange economy. It makes possible events where the state price deflator is negative, and hence the consumer abstains from consuming the entire endowment. As the author points out, events such as these can be eliminated by using a production economy to endogenously arrive at the aggregate endowment process. Still, the analysis of Hindy, Huang, and Zhu (1997b) illustrates the difficulties of this approach. Instead of attacking the difficult problem of aggregating such single consumer consumption strategies over heterogeneous consumers, I choose to study the effects of habits and durabilities within a pure exchange economy, thus exposing the model to the objections of unlikely events. The analysis of the present economy demonstrates that durabilities potentially resolves the problem of negative state prices (Proposition 3.4).

It should also be (embarrassingly) clear from the present analysis that there are still many unresolved questions about the behavior of prices and returns in more general economies than that implied by additive separable utility. Within the present economy there are two main directions of attack. One is to make sufficiently

[18]It is easy to see this in a plot of the consumer's felicity $u(y,z) = \gamma_y^{-1} y^{\gamma_y} z^{\gamma_z}$ as a function of services y and "habits" z.

strong assumptions to be able to carry out more extensive comparative statics. Alternatively, a numerical approach can be used to evaluate if the effects of local substitution and distant complementarity are significant. Such an analysis can be carried out at a high level of generality using for instance Monte Carlo simulation. Alternatively one can derive exact formulas under stronger assumptions using the approach developed in chapter 2. Finally, it is worth noting that the current economy trivially extends to the one studied by Hindy, Huang, and Zhu (1997b). They basically study an economy where the consumer derives utility from consuming a dual-attribute commodity. Local substitution and distant complementarity effects are introduced by assuming the marginal felicity in one attribute is increasing in the other attribute. Hence, the present economy easily facilitates an analysis of how their approach to modelling distant complementarity compares to the traditional approach of habit formation.

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APPENDIX A

1. Proof of Proposition 3.2

The following proof makes use of the locally riskless rate, defined in section 4. It is characterized by $r_t = -\mu_\pi(t)/\pi_t$, where $\mu_\pi(t)$ is the drift of the Itô process representation of π_t.

PROOF. The proof consists in checking that the hypotheses of Proposition 4 of Duffie and Skiadas (1994) are satisfied. Their hypotheses are that for any stopping time $\tau \leq T$ there exists a $c \in L^2_+(P \times \lambda)$ such that:

a) $P(|e_t| > \epsilon\ \lambda\text{-a.e.}) = 1$ for some $\epsilon > 0$,

b) $\left(V^n_\tau \bar{M}_{\tau,T}\right) \in L^2(P)$, where V and $\bar{M}$ are defined below, and

c) $c_0 + \int_t^\tau c_s\, ds + \int_\tau^{\tau \vee t} e_s\, ds$ is an admissible perturbation.

In particular, I check if $c \equiv \epsilon/2$ satisfies hypothesis b). Hypothesis c) will then be satisfied by considering a strictly positive budget feasible $\hat{e}$ which is ϵ-close to e.

By Assumption 3.1 $P\{|e| \gg 0\} = 1$. Further, since $\pi(e)$ is a smooth functional of $e \in L^2(P \times \lambda)$, and securities prices necessarily will be Itô processes, $G^{0\pi} \in L^2(P \times \lambda)$ by Hölder's Inequality. Define $M_{s,t} = M_{s,t}(r) = \exp\{\int_s^t r_u\, du\}$ and $\bar{M}_{s,t} = M_{s,t}(|r|)$, and let

$$V^n_t \triangleq S^n_t + \int_0^t M_{s,t}\, dD^n_s, \qquad t \in [0,T],$$

Let $\tau \leq T$ be an arbitrary stopping time. It must be established that the second moment of $V^n_\tau \bar{M}_{\tau,T}$ is finite $\forall n \in \{0, \ldots, N\}$. By Itô's Lemma

$$\begin{aligned}\left(V^n_t \bar{M}_{t,T}\right)^2 &= \left(S^n_0 \bar{M}_{0,T}\right)^2 \\ &\quad + 2\int_0^t \bar{M}^2_{s,T} V^n_s \left[V^n_s\, |r_s| + \mu^n_G(s) + \sigma^n_G(s) \cdot \sigma^n_G(s)\right] ds \\ &\qquad\qquad + 2\int_0^t \bar{M}^2_{s,T} V^n_s \sigma^n_G(s)\, dB_s\end{aligned}$$

The first term in this expression has finite expectation by hypothesis i). Consider now the terms in the Lebesgue integral in order of appearance. First,

$$\int_0^{\tau} \bar{M}_{s,T}^2 V_s^{n2} |r_s| \, ds \leq \bar{r} \int_0^{\tau} e^{2\bar{r}(T-s)} \left[S_s^n + \int_0^s e^{\bar{r}(s-u)} \, dD_u^n \right]^2 ds$$

Since D_t is a submartingale it has the Doob-Meyer Decomposition $D_t = Z_t + X_t$ for Z_t an increasing process and X_t a martingale (see e.g. Karatzas and Shreve 1991). Since D_t is also an Itô process the Lebesgue integral must be increasing as the Itô integral is a martingale. The former integral is increasing only if $\mu_D(t) \geq 0$ $\forall t \in [0,T]$. Since $M_{t,T}\mu_D(t) \geq 0$ it is clear that V_t^n is also a submartingale as the converse of the Doob-Meyer Decomposition Theorem trivially obtains. Being a convex functional of a submartingale, the integrand is also a submartingale. Using Fubini's Theorem

$$\begin{aligned} \mathrm{E}\left\{ \int_0^{\tau} \bar{M}_{s,T}^2 V_s^{n2} |r_s| \, ds \right\} &\leq \bar{r} \int_0^T \mathrm{E}\left\{ 1_{[0,\tau]}(t) \bar{M}_{t,T}^2 V_t^{n2} \right\} dt \\ &\leq \bar{r} \int_0^T \mathrm{E}\left\{ e^{2\bar{r}(T-t)} \left[S_t^n + \int_0^t e^{\bar{r}(t-s)} \, dD_s^n \right]^2 \right\} dt \qquad (38) \end{aligned}$$

since $\tau \leq T$ and T trivially is a stopping time (Karatzas and Shreve 1991, Problem 1.3.26).

Recall that $\pi \in L^2(P \times \lambda)$ since it is the Riesz representation of a linear bounded functional on $L^2(P \times \lambda)$. It follows from hypothesis iii) that $S^n \in L^2(P \times \lambda)$ by repeated use of Hölder's Inequality. Since S^n is a square integrable Itô process it follows from hypothesis iii), the Itô isometry, and Hölder's Inequality that $S_t^n \int_0^t e^{\bar{r}(t-s)} \, dD_s^n$ has finite expectation. $\left(\int_0^t e^{\bar{r}(t-s)} \, dD_s^n\right)^2$ has finite expectation by the Itô isometry and hypothesis iii). This establishes that (38) is finite.

The remaining terms in the Lebesgue integral are similarly proved to have finite expectation. By Minkowski's Inequality the Lebesgue integral has finite expectation. The Itô integral is a martingale under the standing assumptions, and has therefore zero expectation.

Finally, it is clear that $\int_{\tau}^{\tau \vee t} e_s \, ds \in L^2(P \times \lambda)$, establishing the last hypothesis of Proposition 4 by Duffie and Skiadas (1994). □

2. Comparative Statics

The effect of a change in α_y on r_t^d is strictly positive iff

$$\left(\tfrac{1}{\delta_y}\pi_t^y|_{\delta_z=0}\right)^2 < \left\{\left[v_{yy}(t,y_t)\frac{\partial}{\partial\alpha_y}y_t\right]\tfrac{1}{\delta_y}\pi_t^y|_{\delta_z=0}\right.$$
$$\left. + v_y(t,y_t)\mathrm{E}\left\{\int_t^T e^{-\alpha_y(s-t)}\left((s-t)v_y(s,y_s) - v_{yy}(s,y_s)\frac{\partial}{\partial\alpha_y}y_s\right)|\mathcal{F}_t\right\}\right\},$$

where

$$\frac{\partial}{\partial\alpha_y}y_t = -te^{-\alpha_y t} - \delta_y\int_0^t e^{-\alpha_y(t-s)}(t-s)e_s\,ds < 0,$$

and $\delta_y^{-1}\pi_t^y|_{\delta_z=0}$ is the denominator in the expression for r_t^d. Both sides of the inequality are strictly positive, but their relative magnitudes are indeterminate without additional restrictions on preferences and endowments.

The effect of a change in δ_y on r_t^d is strictly positive iff

$$\left[v_{yy}(t,y_t)\frac{\partial}{\partial\delta_y}y_t\right]\tfrac{1}{\delta_y}\pi_t^y|_{\delta_z=0} > v_y(t,y_t)\mathrm{E}\left\{\int_t^T e^{-\alpha_y(s-t)}v_{yy}(s,y_s)\frac{\partial}{\partial\delta_y}y_s\,ds\,|\mathcal{F}_t\right\},$$

where

$$\frac{\partial}{\partial\delta_y}y_t = \int_0^t e^{-\alpha_y(t-s)}e_s\,ds > 0.$$

Both sides of the inequality are therefore strictly negative. Again, their relative magnitude is indeterminate.

CHAPTER 4

Additive Nonseparable Utility

This chapter presents results on equilibrium asset prices and returns in a representative consumer pure exchange economy. The consumer's utility functional has the von Neumann-Morgenstern representation, and is allowed to depend on past purchases of the commodities. The economy nests a large class of existing models in the literature, and is sufficiently general to allow for further developments in the modelling of intertemporal expected utility as applied to asset pricing theory. The suggested economy offers a rich environment for studying how modelling choices affect equilibrium returns.

JEL CLASSIFICATION: D51, G12.

KEY WORDS: Additive nonseparable utility; Utility gradients; Equilibrium asset pricing; ICAPM; CCAPM; Pure exchange economy; Malliavin calculus.

1. Introduction

The present chapter studies restrictions on equilibrium prices and returns when preferences are restricted to those that allow the von Neumann-Morgenstern utility representation, and information surprises occur according to a Brownian filtration. The suggested economy nests all models in the literature that use additive nonseparable von Neumann-Morgenstern utility. This level of generality is attained while retaining concrete restrictions on equilibrium returns, generalizing Merton's (1973) Intertemporal Capital Asset Pricing Model (ICAPM).

Table 1 shows the interrelation between models in the asset pricing literature that use preferences that allow a von Neumann-Morgenstern utility representation. No effort has been made to produce a complete listing of references to all papers, or a fair listing of originators. Further, the generic model is presented in a continuous-time economy,[1] with absolutely continuous cumulative consumption choices.[2]

TABLE 1. Asset pricing models using additive nonseparable von Neumann-Morgenstern utility

All models of additive nonseparable von Neumann-Morgenstern utility in the extant literature are special cases of

$$v(t,c,x) = \frac{1}{\gamma}\left[\dot{y}(t)^A + y(t)^B - z(t,x)^C\right]^\gamma \left[z(t,x) + x(t)^E\right]^D. \tag{39}$$

$$dy(t) = [\delta_y c(t) - \alpha_y y(t)]\,dt \tag{40}$$

$$dz(t,x) = [\delta_{zc} x(t) + \delta_{zy} y(t) - \alpha_z z(t)]\,dt. \tag{41}$$

$\dot{x}(t)$ is notation for $dx(t)/dt$. • denotes coefficients that take values different from zero. Setting coefficients equal to zero does not affect the felicity by properly adjusting the initial conditions for (40) and (41).

	Felicity					"Services"		"Habits"		
Model	A	B	C	D	E	α_y	δ_y	α_z	δ_{zc}	δ_{zy}
Panel A: State independent felicity; $x \equiv c$.										
Additive habit formation[a]	•		•				•	•	•	
Multiplicative habit formation[b]	•			•			•	•	•	
Durabilities[c]		•				•	•			
Durabilities and habits I[d]		•		•		•	•	•	•	
Durabilities and habits II[e]		•	•			•	•	•		•
Panel B: State dependent felicity; $x \neq c$.										
Add. external habit formation[f]	•		•				•		•	
Mul. external habit formation[g]	•			•			•		•	
Services and durabilities[h]		•		•		•	•	•	•	
Multi-attribute goods[i]	•			•	•		•	•	•	

[a]Sundaresan (1989), Constantinides (1990), and chapter 2. Detemple and Zapatero (1991) also derive results for a general felicity $v(t, c_t, z_t^c)$.
[b]Abel (1990), and chapter 2.
[c]Hindy and Huang (1993). Detemple and Giannikos's (1996) economy nests these preferences, although their main analysis is conducted with $\dot{y}(t)$ as numeraire. Hence the effects of local substitution are not present in their main analysis.
[d]Heaton (1993), Hindy, Huang, and Zhu (1997b), and chapter 3.
[e]Ferson and Constantinides (1991), and Heaton (1993).
[f]Campbell and Cochrane (1999).
[g]Abel (1990).
[h]Dunn and Singleton (1986).
[i]Detemple and Giannikos (1996) derive results for a general felicity $v(t, c_t, z_{1t}^x, g[x_t, z_{2t}^x])$, and for the special case of Cobb-Douglas felicity. The latter gives the gist of their analysis. The results do not change much qualitatively by changing the structure of the felicity as they use $\dot{y}(t)$ as numeraire (see e.g. their equation (35)).

This analysis does not try to determine which particular modelling choice is ideal. Instead I deduce general asset pricing restrictions without any of the restrictions in table 1, and treat them as examples. An example of particular interest is that parametrized by (B, D), (α_y, δ_y), and (α_z, δ_{zy}) different from zero. Empirical

[1]Dunn and Singleton (1986), Abel (1990), Ferson and Constantinides (1991), and Campbell and Cochrane (1999) use discrete-time economies.
[2]Hindy and Huang (1993), and Hindy, Huang, and Zhu (1997b) study economies with singular cumulative consumption.

results suggest that these preferences have a good fit to observed stock returns (Heaton 1993). I emphasize, though, that it will only serve as an example of how to apply the general theory in the present chapter.

To generalize the models in table 1 it is necessary to work with a model of preferences of the form

$$U(c) = \mathrm{E}\left\{ \int_0^T v(t, f(t, c, x))\, dt \right\},$$

where c is a vector of commodities and x is a vector of state variables. The distinction between c and x is interesting as we want to determine the equilibrium spot prices for all the traded commodities in the economy. It is also of interest to let the consumer's felicity v depend on factors that are not traded, or simply outside the control of the consumer. Letting C^j denote the consumption space of a commodity, it is also clear that the admissible vector valued *aggregator*, f, should encompass mappings $f^j : C^j \to \mathbb{R}^n$, $n \in \{2, 3, \dots\}$. This will for instance allow treatment of habit formation.

The basic structure of the economy is presented in section 2, together with assumptions on its primitive elements. Equilibrium restrictions on prices are studied in section 3, while equilibrium restrictions on returns are studied in section 4. Section 5 investigates more closely the structure of equilibrium prices and returns. The last section concludes and offers some ideas for future research.

2. The Economy

Consider a representative consumer pure exchange economy as in Lucas (1978). It is identified by its assumptions on the flow of information, the available consumption bundles, the consumer's preferences, and the securities market.

2.1. Information. The consumer's beliefs and information during the timespan $[0, T]$, $T < \infty$, are represented by the complete, filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$. Ω is the set of possible states of nature, in which a generic element $\omega \in \Omega$ completely determines a particular state of nature.[3] $\mathcal{F}$ is the σ-algebra of observable events of Ω. The probability measure P on $\mathcal{F}$ represents the beliefs that the consumer holds about the likelihood of realizations of these observable

[3]Not to be confused with the state vector $x(t, \omega)$.

events. At each point in time the consumer has access to the cumulative information represented by the augmented filtration $\mathbb{F}$, generated by the standard Brownian motion $B : [0,T] \times \Omega \to \mathbb{R}^d$ on $(\Omega, \mathcal{F}, P)$. The latter serves as the canonical uncertainty in the economy.

2.2. Consumption and States. All the commodities in the economy are perishable in the sense that produced output cannot be stored for later consumption. Still, they are allowed to be durable in the sense that purchases of a commodity can affect the felicity of the consumer after the time of the purchase. More formally, the consumption space consists of elements in the positive cone of $L^2(P \times \lambda)^k$ that are adapted to $\mathbb{F}$.[4] A typical element is denoted $c \in C \triangleq L^2_+(P \times \lambda)^k$. Produced output in the economy is represented by the endowment process $e \in C_+ \triangleq \text{int}(C)$. The exogenous states affecting the consumer's felicity are represented by a vector $x \in X \triangleq L^2(P \times \lambda)^l$.

ASSUMPTION 4.1. *The endowment is an Itô process*

$$e(t) = e_0 + \int_0^t \mu_e(s)\, ds + \int_0^t \sigma_e(s)\, dB_s,$$

where $\mu_e : [0,T] \times \Omega \to \mathbb{R}^k$ *and* $\sigma_e : [0,T] \times \Omega \to \mathbb{R}^{k \times d}$ *are adapted to* $\mathbb{F}$. *Similarly, the vector of states is an Itô process*

$$x(t) = x_0 + \int_0^t \mu_x(s)\, ds + \int_0^t \sigma_x(s)\, dB_s,$$

with $\mu_x : [0,T] \times \Omega \to \mathbb{R}^l$ *and* $\sigma_x : [0,T] \times \Omega \to \mathbb{R}^{l \times d}$, *also adapted to* $\mathbb{F}$. *Additionally, considered as functionals on the Fréchet space* $C_0(\mathbb{R}_+)$, *the drift vectors* $\mu_e, \mu_x : C_0([0,T] \times \mathbb{R}^d) \to \mathbb{R}^k, \mathbb{R}^l$ *respectively, and the dispersion matrices* $\sigma_e, \sigma_x : C_0([0,T] \times \mathbb{R}^d) \to \mathbb{R}^{k \times d}, \mathbb{R}^{l \times d}$ *respectively, are continuously Fréchet differentiable.*

REMARK. Neither endowments nor states are restricted to be Markov processes. They can depend on the entire history of the economy.

The restriction that e is $P \times \lambda$-square integrable is necessary to ensure that the Malliavin derivative $\mathfrak{D}_t e(s)$ exists as a stochastic distribution (Aase, Øksendal, and Ubøe 1998). The possibility that $\mathfrak{D}_t e(s) \notin L^2(P \times \lambda)^k$ does not pose any problems as

[4] λ is the Lebesgue measure on $([0,T], \mathcal{B}([0,T]))$, where $\mathcal{B}([0,T])$ is the Borel measurable subsets of $[0,T]$. $L^2(P \times \lambda)^k = L^2(\Omega \times [0,T], \mathcal{F} \otimes \mathcal{B}([0,T]), P \times \lambda) \times \cdots \times L^2(\Omega \times [0,T], \mathcal{F} \otimes \mathcal{B}([0,T]), P \times \lambda)$ k times.

all results in the following analysis will at most depend on $\mathrm{E}\{\mathfrak{D}_t e(s)\,|\mathcal{F}_t\}$. This projection of $\mathfrak{D}_t e(s)$ on $\mathcal{F}_t$ smooths $\mathfrak{D}_t e(s)$ sufficiently to ensure that $\mathrm{E}\{\mathfrak{D}_t e(s)\,|\mathcal{F}_t\} \in L^2(P \times \lambda)^k$ (Aase, Øksendal, and Ubøe, Theorem 4.1).

An unintended side-effect from this restriction on e is that it is not necessary to appeal to localization arguments (Karatzas and Shreve 1991, p. 34) in results that depend on the martingale property of functionals of e. That is, for smooth f the Itô integral $\int f(e)\,dB$ is not only a local martingale but also a martingale.

2.3. Preferences. The preferences represented by the models listed in table 1 can be folded into a larger class of preferences.

ASSUMPTION 4.2. *The representative consumer's preferences allow the additive nonseparable von Neumann-Morgenstern representation*

$$U(c) = \mathrm{E}\left\{\int_0^T v(t, f(t,c,x))\,dt\right\}, \tag{42}$$

where $f : [0,T]\times C\times X \to \mathbb{R}_+^{m_1}\times\cdots\times\mathbb{R}_+^{m_k}\times\mathbb{R}^{m_{k+1}}\times\cdots\times\mathbb{R}^{m_{k+l}}$, *and* $v : [0,T]\times\mathbb{R}_+^K\times\mathbb{R}^L \to \mathbb{R}$, *where* $K = \sum_{i=1}^k m_i$ *and* $L = \sum_{i=k+1}^{k+l} m_i$, $v \in C^{1,3,2}([0,T]\times\mathbb{R}_+^K\times\mathbb{R}^L)$.[5] *The aggregator is further specialized to*

$$\begin{aligned} f(t,c,x) &= [f^1(t,c^1),\ldots,f^k(t,c^k),f^{k+1}(t,x^1),\ldots,f^{k+l}(t,x^l)] \\ &= [f^{1,1}(t,c^1),\ldots,f^{k+l,m_{k+l}}(t,x^l)]. \end{aligned}$$

where $c = (c^1,\ldots,c^k)$ *is a vector of purchases of publicly traded commodities, while* $x = (x^1,\ldots,x^l)$ *is a vector of state variables or nontraded goods. Let* v_f *denote a vector of partial derivatives,*

$$v_f \triangleq (v_{f^1},\ldots,v_{f^{k+l}}) = \left(\sum_{j=1}^{m_1}\frac{\partial v}{\partial f^{1,j}},\ldots,\sum_{j=1}^{m_{k+l}}\frac{\partial v}{\partial f^{k+l,j}}\right).$$

$v_{f^{i,j}} > 0$ *and* $v_{f^{i,j}f^{i,j}} < 0\ \forall\, i = 1,\ldots,k$, *and some* $j \in \{1,\ldots,m_i\}$. *Finally,* v *is strictly concave in* c, *and its Fréchet derivative with respect to the commodity vector*

[5] It will at times be convenient to start the indexing of x at $k+1$, without formally introducing a shift-operator. The meaning of the superscript will be clear from the context.

satisfies a uniform growth condition

$$\left|v_{f^{i,j}}(t,f(t,c,x))\right| \leq V^{i,j}(1+\sum_{i=1}^{k}\left\|c^i\right\|_{L^1([0,t])})^{F^{i,j}}, \quad t \in [0,T]$$

$\forall x(\omega,\cdot): L^2(\lambda) \to \mathbb{R}$, $i=1,\dots,k$, $j=1,\dots,m_i$, *and some* $V^{i,j}, F^{i,j} \in \mathbb{R}_+$.

REMARK. This growth condition places a joint condition on v and f. Separately placing a linear growth condition on v in $(f^{1,1},\dots,f^{k+l,m_{k+l}})$, and a polynomial condition on f in c implies the condition given above. It is an easy task to verify this alternative condition. Another feasible restriction is to assume that $f-g \in L^2(P\times\lambda)$ is homogeneous of degree $n \in \mathbb{N}$, where $g \in L^2(\lambda)$.

The assumption that v is three times continuously differentiable in the entire commodity vector is more restrictive than necessary. It is sufficient that this holds only for the numeraire commodity (hereafter simply referred to as the 'numeraire') when the felicity is defined on the rate of purchases of this commodity. It suffices that it is twice continuously differentiable for the remaining commodities, or for all commodities when the felicity does not depend on the rate of purchases of the numeraire.

ASSUMPTION 4.3. *$f^i(t,c^i)$ is continuously Fréchet differentiable, $i=1,\dots,k$. Further, the Riesz representation $\hat{\varphi}^{i,j}$ of the differential $\delta f^{i,j}(t,c^1;h^i)$ is independent of the history of c^i, so $\hat{\varphi}^{i,j}(t,s)=\hat{\varphi}^{i,j}(t,s,c(s))$ is adapted to $\mathbb{F}$, and satisfies the uniform linear growth condition*

$$\left|\hat{\varphi}^{i,j}(t,s,c)\right| \leq \phi^{i,j}(1+|c|), \quad 0 \leq s \leq t \leq T,$$

for $c \in \mathbb{R}_+$, and some $\phi^{i,j} \in \mathbb{R}_+$ for $i=1,\dots,k$ and $j=1,\dots,m_i$.

The adaptedness of $\delta f^{i,j}(t,c^i;h^i)$ implies that we can write $\delta f^{i,j}(t,c^i;\cdot)(\omega)$ as an integral on $[0,t] \subset [0,T]$. This signifies that the consumer does not derive felicity from *prospects* of future consumption—aka. "no fun in gambling". In other words, $v(t,f(t,c,x))$ is allowed to depend on past purchases of the commodities $\{c_s\}_{0\leq s\leq t} \in \mathcal{F}_t$ and previous realizations of states $\{x_s\}_{0\leq s\leq t} \in \mathcal{F}_t$, but not on knowledge about the possible future outcomes $\{c_s\}_{t<s\leq T}$ and $\{x_s\}_{t<s\leq T}$. The assumption is stricter than necessary. It need hold only for the numeraire. Still, it

eases the following exposition. $\hat{\varphi}(t,s)$ can be interpreted as the weight the consumer places on time s consumption relative to time t consumption at time t, made clear in an example below.

With the current level of generality one can conceivably experience negative state and spot prices in the economy. It is therefore necessary to require the primitives of the economy to satisfy the following condition.[6]

CONDITION 4.1 (Consistency). *The parameters of e, x, and v are such that*

$$\pi^i(t,e,x) \triangleq \sum_{j=1}^{m_i} \mathrm{E}\left\{ \int_t^T v_{f^{i,j}}(s, f(s,e,x))\hat{\varphi}^{i,j}(s,t)\, ds \,|\mathcal{F}_t \right\} > 0 \quad P\text{-a.s.}$$

$\forall\ t \in [0,T)$, $i = 1,\ldots,k$. In addition, if $\pi^i(T,\hat{e},\hat{x}) \neq 0$ for some $(\hat{e},\hat{x}) \in C \times X$ then $\pi^i(T,e,x) > 0$.

If $v_{f^{i,j}}(s, f(s,e,x))\hat{\varphi}^{i,j}(s,t) > 0$ for all i and j, $P \times \lambda$-a.e., then the condition is automatically satisfied. Hindy, Huang, and Zhu (1997b), for instance, study an economy with a dual-attribute commodity where this consistency condition is satisfied without any further restrictions on endowments or preferences.

EXAMPLE 4.1. The following specifications illustrate how the utility functional (42) incorporates the models in table 1. The first model with additive separable utility is standard in much of the literature. The second model with external habit formation has been extensively studied in discrete time, but no analysis has been done in continuous time. The last model, with living standards derived from the service flow from purchases of a commodity, has been studied neither in discrete nor in continuous time. They will thus serve as examples throughout the chapter.

Additive separable: This class is attained by letting $f(t,c,x) = (c,x)(t)$. Breeden (1986) studies this model in a Markov production economy. The Gâteaux differential $\delta f(t,c,x;h) = h(t)$ can be considered a bounded linear functional on $L^2(\lambda)^k$ by introducing the generalized Dirac function, $\delta(t) = 0\ \forall\ t \neq 0$ and $\int_{\mathbb{R}} \delta d\lambda = 1$. Thereby $h(t) = \int_0^T \delta(t-s)h(s)\,ds$. The interpretation of $\hat{\varphi}(t,s)$ is that current consumption is "infinitely important" to the consumer's current welfare. Neither historical nor prospects

[6] Chapman (1998) gives an example of an economy with negative state prices. He finds this for viable parameter values in an exchange economy with additive habit formation (also known as 'linear' habit formation).

of future consumption play a role in how current consumption affects the consumer's current welfare.

External habit formation: Preferences that exhibit external habit formation, or "catching up with the Joneses," (Abel 1990, Campbell and Cochrane 1999) result when

$$\begin{aligned} f(t,c,x) &= (f^1(t,c^1), f^2(t,x^1)) \\ &= \left(c^1(t), x^1(0)e^{-\alpha t} + \eta \int_0^t e^{-\alpha(t-s)} x^1(s)\, ds\right), \end{aligned}$$

where $\alpha, \eta \in \mathbb{R}_+$. $c^1 \neq x^1$ in the consumer's optimization problem, while $c^1 = x^1 = e^1$ in equilibrium.

Durabilities and habits II: To arrive at a felicity that retains the local substitution effects of Hindy and Huang (1992) and that incorporates habit formation, let it depend on the history of service flows from a durable commodity simply by letting

$$\begin{aligned} f^{1,j}(t,y) &= \int_0^t g^j(t,s) y(s)\, ds, \\ f(t,c,x) &= [f^{1,1}(t,c^1), f^{1,2}(t, f^{1,1})] \triangleq [f^{1,1}(t,c^1), f^{1,2}(t,c^1)]. \end{aligned}$$

2.4. Securities. There are $N+1$ securities available for trade, with price processes $S = (S^0, \ldots, S^N)$. Each security entitles its holder to a cumulative dividend process $D = (D^0, \ldots, D^N)$. The sum of cumulative dividends and current securities prices, $G = S + D$, is called the *gains process*. G is assumed to have the decomposition $dG(t) = \mu_G(t)\, dt + \sigma_G(t)\, dB(t)$. If $\mathbf{I}_S(t)$ is the diagonal matrix with $S^j(t)$ on entry (j,j) then this decomposition can be rewritten as

$$G(t) = G(0) + \int_0^t \mathbf{I}_S(s)\mu(s)\, ds + \int_0^t \mathbf{I}_S(s)\sigma(s)\, dB(s),$$

where $\mu(t) = \mathbf{I}_S^{-1}(t)\mu_G(t)$ and $\sigma(t) = \mathbf{I}_S^{-1}(t)\sigma_G(t)$. To simplify the derivation of restrictions on equilibrium returns assume also that G^0 is of bounded variation.[7]

[7] Alternatively assume there is some portfolio (defined in the next subsection) whose gains process is of bounded variation. This assumption leads to the same result, but complicates notation.

In addition to the $N+1$ financial securities the consumer can purchase commodities in k different spot markets. Let $p \triangleq (p^1, \ldots, p^k)$ denote the prices in these markets for each of the commodities.

2.5. Trade. A *trading strategy* is any adapted, square integrable process θ : $\Omega \times [0,T] \to \mathbb{R}^{N+1}$. Total gains from trade are given by $\int_0^t \theta^\top(s)\,dG(s)$, where $\top$ is the transpose operator. A trading strategy θ is said to *finance* c if

$$\theta^\top(t)S(t) = \theta^\top(0)S(0) + \int_0^t \theta^\top(s)\,dG(s) - \int_0^t p^\top(s)c(s)\,ds,$$
$$\theta^\top(T)S(T) = 0,$$

Since $c \in L^2_+(P \times \lambda)^k$ the last condition is the real budget constraint, forcing the consumer to pay any debt left at time T. Having defined a trading strategy it is meaningful to say what it means to deflate prices.

DEFINITION 4.1 (Deflation). *For any strictly positive Itô process Y let* deflation *of G by Y, denoted G^Y, mean*

$$G^Y(t) = S(t)Y(t) + \int_0^t Y(s)\,dD(s) + \int_0^t \sigma_D(s)\sigma_Y^\top(s)\,ds.$$

This definition of deflation is necessary to ensure *numeraire invariance* (Huang 1985)—that a trading strategy θ finances c when trading in G, e.g. prices are denominated in US \$, iff it finances c when trading in G^Y, e.g. prices are denominated in Japanese ¥.

DEFINITION 4.2 (Admissible Trading). *Let Y be a strictly positive Itô process. The set of admissible trading strategies, Θ, consists of those θ satisfying*

i) *If G^Y is a $(P, \mathcal{F}_t)$-martingale for some Y, then $\int_0^t \theta^\top(s)\,dG^Y(s)$ is a $(P, \mathcal{F}_t)$-martingale* $\forall t \in [0,T]$.

ii) $\exists\, c \in L^2(P \times \lambda)^k$ *such that θ finances $c - e$.*

3. Equilibrium Prices

The pair (θ, c), $\theta \in \Theta$ and $c \in C$, is *budget feasible* if θ finances $c - e$, and $\theta^\top(0)S(0) \le p^\top(0)[e(0) - c(0)]$. A budget feasible (θ^*, c^*) is *optimal* if for all other budget feasible (θ, c), $U(c^*) \ge U(c)$. $\{(\theta^*, c^*); (S, p)\}$ is a securities-spot market *equilibrium* for the economy if (θ^*, c^*) is optimal for the single consumer economy given the prices (S, p), and markets clear; $c^* = e$ and $\theta = \mathbf{0}$. Given a budget feasible portfolio-consumption strategy $(\hat{\theta}, \hat{c})$, let the set of *feasible directions* be given by

$$F(\hat{\theta}, \hat{c}) \triangleq \{(\theta, c) \in \Theta \times L^2(P \times \lambda)^k : \theta \text{ finances } c,$$
$$(\hat{\theta}, \hat{c}) + \epsilon(\theta, c) \in \Theta \times C \ \forall \ \epsilon \in (0, \gamma), \gamma > 0\}.$$

Let F_Θ be the projection of F on Θ, and F_C the projection on C.

Dividends and securities prices are linked to the consumer's preferences and endowments through the gradient of the utility functional. If the *Gâteaux differential* of the utility functional,

$$\delta U(c; h) \triangleq \lim_{\epsilon \to 0} \frac{U(c + \epsilon h) - U(c)}{\epsilon}, \qquad h \in F_C(c)$$

is a bounded linear functional in h then the *utility gradient* (See e.g. Luenberger 1969) is defined as the adapted stochastic k-vector process $\nabla U(t, c)$ such that

$$\delta U(c; h) = \mathrm{E}\left\{ \int_0^T \nabla U(t, c) \cdot h(t)\, dt \right\}.$$

The existence of $\nabla U(t, c)$ is ensured by the Riesz Representation Theorem. If commodity j is chosen as numeraire then $\nabla U^j(t, c^*)$ turns out to be the *state price deflator* in the economy. I.e., $\nabla U^j(t, c^*)$-deflated gains are $(P, \mathcal{F}_t)$-martingales.

THEOREM 4.1 (Duffie and Skiadas). *Consider commodity j as numeraire, and assume that Assumption 4.2 and 4.3, and Condition 4.1 hold. If, given any stopping time τ and security n $\exists$ $\{\theta_m\}_{m \in \mathbb{N}}$ such that*

i) $\pm\theta_m \in F_\Theta \ \forall \ m \in \mathbb{N}$.

ii) $\forall \ m \in \mathbb{N}$, $\theta_m^n(t) = 1_{[0,\tau]}(t)$ *and* $\theta_m^k \equiv 0 \ \forall \ k \neq 0, n$, *where* $1_{[0,t]}(\cdot)$ *is the indicator function.*

iii) $\lim_{m \to \infty} \mathrm{E}\left\{ \int_\tau^T \theta_m^\top(t)\, dG^{\nabla U^j(t, c^*)}(t) \right\} = 0$.

then (θ^*, c^*) is optimal iff $G^{\nabla U^j(t, c^*)}$ is a $(P, \mathcal{F}_t)$-martingale.

LEMMA 4.1. *The conditions in Theorem 4.1 hold.*

PROOF. The proof follows that given in the Appendix of chapter 3, with only minor changes in notation. □

Given an aggregator f, the choice of numeraire determines the qualitative structure of equilibrium spot and securities prices. The functional form of f^j translates directly into the functional form of the state price deflator $\pi(t, c^{j*})$. Spot prices $(p^1, \ldots, p^k)$, $p^j \equiv 1$, and securities prices S are in turn defined in terms of $\pi(t, c^{j*})$ when c^{j*} is equal to aggregate (optimal) consumption of commodity j. The next three subsections derive conditions on optimal consumption-portfolio choices, spot prices, and securities prices that are consistent with rational expectations.

3.1. The State Price Deflator. First, the utility gradient is computed. The state price deflator then drops out of this result by selecting a specific commodity as numeraire and by determining the optimal purchase of this commodity.

THEOREM 4.2 (The Utility Gradient). *If Assumption 4.1, 4.2, and 4.3 hold then*

$$\nabla U^i(t, c) = \mathrm{E}\left\{\sum_{j=1}^{m_i} \int_t^T v_{f^{i,j}}(s, f(s, c, x))\hat{\varphi}^{i,j}(s, t)\, ds \,|\mathcal{F}_t\right\} \tag{43}$$

$i = 1, \ldots, k$.

Expression (43) represents the marginal utility of consumption of commodity i. Denoting element i by $\pi(t, c^i)$ equation (43) can be written as

$$\nabla U(t, c) = \left(\pi(t, c^1), \ldots, \pi(t, c^k)\right) \tag{44}$$

PROOF. Due to the nice properties of the felicity one need only show that it is viable to interchange limit and integration in the Gâteaux differential to establish that it exists. The natural tool is therefore to use the Lebesgue Dominated Convergence Theorem.

Consider the sequence $\{\epsilon_n\}_{n\in\mathrm{N}}$, where $\epsilon_n \in (0, 1]$, and $\epsilon_n \to 0$ as $n \to \infty$. For each ω and t consider $v(\omega, t, f)$ as a functional on $L^2([0, t], \mathcal{B}([0, t]), \lambda)$. Since v has a Fréchet derivative v_c by Assumptions 4.2 and 4.3, the Mean Value Theorem for functionals implies that (Luenberger 1969, Proposition 7.3.2)

$$\begin{aligned}
&\frac{1}{\epsilon_n}\left|v(\omega, t, f(c + \epsilon_n h, x)) - v(\omega, t, f(t, c, x))\right| \\
&\quad \le \|h\|_{L^2([0,t])} \sup_{\alpha\in(0,1)} \|v_c(\omega, t, f(t, c + \alpha\epsilon_n h, x))\|
\end{aligned}$$

and using the Minkowski inequality

$$\begin{aligned}
&\leq \|h\|_{L^2([0,t])} \sup_{\alpha\in(0,1)} \sum_{i=1}^{k}\sum_{j=1}^{m_i} \left|v_{f^{i,j}}(\omega,t,f(t,c+\alpha\epsilon_n h,x))\right| \\
&\quad\times \left(\int_0^t \left|\hat{\varphi}^{i,j}(\omega,t,s,c^i(s)+\alpha\epsilon_n h^i(s))\right|^2 ds\right)^{\frac{1}{2}},
\end{aligned}$$

and by the growth conditions in Assumption 4.2 and 4.3, another application of Minkowski's inequality, and using the fact that $\alpha\epsilon_n \in (0,1)$

$$\begin{aligned}
&\leq \|h\|_{L^2([0,t])} \sum_{i=1}^{k}\sum_{j=1}^{m_i} V^{i,j}\phi^{i,j} \left(\int_0^t \left|1+\left|c^i(\omega,s)\right|+\left|h^i(\omega,s)\right|\right|^2 ds\right)^{\frac{1}{2}} \\
&\quad\times \left(1+k\max_i\left\{\left\|c^i(\omega)\right\|_{L^1([0,t])}+\left\|h^i(\omega)\right\|_{L^1([0,t])}\right\}\right)^{F^{i,j}},
\end{aligned}$$

and by another application of Minkowski's Inequality, and using Hölder's Inequality

$$\begin{aligned}
&\leq \|h\|_{L^2([0,t])} \sum_{i=1}^{k}\sum_{j=1}^{m_i} F^{i,j}\phi^{i,j}2\sqrt{2t}\left\|c^i(\omega)\right\|_{L^1([0,t])}\left\|h^i(\omega)\right\|_{L^1([0,t])} \\
&\quad\times \left\|c^1(\omega)\right\|^2_{L^2([0,t])}\left\|h^1(\omega)\right\|^2_{L^2([0,t])} \\
&\quad\times \left(1+k\max_i\left\{\left\|c^i(\omega)\right\|_{L^1([0,t])}+\left\|h^i(\omega)\right\|_{L^1([0,t])}\right\}\right)^{F^{i,j}} \\
&\triangleq V(\omega,t).
\end{aligned}$$

Since $\|\cdot\|_{L^p([0,t])} \leq \|\cdot\|_{L^p([0,T])}$ it follows from repeated use of Fubini's Theorem and Hölder's Inequality that $V \in L^1(P\times\lambda)$. Appealing to the Lebesgue Dominated Convergence Theorem it follows that[8]

$$\lim_{n\to\infty}\frac{1}{\epsilon_n}[U(c+\epsilon_n h)-U(c)] = \int_{\Omega\times[0,T]} v_c\cdot h\, d(P\times\lambda).$$

The result obtains by computing v_c, interchanging the order of integration, observing that $v_c(t,f)$ and h are adapted to $\mathbb{F}$ and using the Law of Total Probability. □

THEOREM 4.3 (The State Price Deflator). *Without loss of generality, consider commodity c^1 as the numeraire. If the conditions of Theorem 4.2 hold then the*

[8]Any element in the equivalence class of modifications of V will serve as a dominator.

state price deflator is given by

$$\pi(t;e^1) = \sum_{j=1}^{m_1} \mathrm{E}\left\{ \int_t^T v_{f^{1,j}}(s, f(s,e,x))\hat{\varphi}^{1,j}(s,t)\,ds \,|\mathcal{F}_t \right\}.$$

PROOF. We are done if we can show that (θ^*, c^*) exists, that $c^* = e$, and that the securities markets clear. Since L^2 is a Hilbert space C is reflexive, meaning that $C^{**} = C$. Since v and f are Fréchet differentiable U is weakly continuous and therefore weak* continuous. The budget set is bounded by e, and hence absorbed by the closed unit sphere. By Alaoglu's Theorem the budget set is therefore weak* compact (Luenberger 1969, Theorem 5.10.1). Being a real-valued functional U therefore attains its maximum on the budget set (Luenberger 1969, Theorem 5.10.2).

Condition 4.1 assures that the gradient is strictly positive at e, and thereby $c^* = e$. This allocation is trivially financed by $\theta^* = \mathbf{0}$, and markets clear. □

Consider henceforth commodity 1 being the numeraire as a maintained assumption.

3.2. Spot Prices. As expected, real spot prices are loosely speaking characterized by marginal rates of substitution.

PROPOSITION 4.1 (Real Spot Prices). *If Condition 4.1 holds then the price of commodity $i = 2, \ldots, k$ in units of commodity 1 is given by*

$$p^i(t) = \pi(t, e^i)/\pi(t, e^1).$$

The following proof shows that the conclusion of Proposition 4.1 is sufficient for the conclusion of Lemma 2 in Duffie and Skiadas (1994) to hold. Their lemma effectively states that there can be "no expected deflated gains from trade in equilibrium". The following proof includes the necessary adjustments to their economy to facilitate multiple commodities.

PROOF. Define utility in terms of trading strategies in Θ by letting $U(\theta) \triangleq U(c)$ whenever θ finances c. Let $\theta_h \in F_\Theta(\theta)$ be a trading strategy financing $h \in F_C(c)$. The Gâteaux differential and Lagrangian associated with $U(\theta)$ are (For details, see pp. 112-13 of Duffie and Skiadas 1994)

$$\delta U(\theta;\theta_h) = \mathrm{E}\left\{\int_0^T \nabla U(t,c)\cdot h(t)\,dt\right\} \triangleq \mathrm{E}\left\{H_T^{\pi}\right\}$$

and

$$\mathcal{L}(\theta_h,\psi) \triangleq U(\theta_h) - \psi\left\{\theta^{\top}(0)S(0) - p^{\top}(0)[e(0)-c(0)]\right\}.$$

The Slater condition is trivially satisfied as $e(0) > 0$ and $\mathbf{0} \in \Theta$. Since U is strictly concave the first order condition for maximizing $\mathcal{L}$ is necessary and sufficient. Thereby, for any $\theta_h \in \Theta$, the first order condition is simply

$$\delta U(\theta;\theta_h) = \mathrm{E}\left\{H_T^{\pi}\right\} \le \bar{\psi}\theta_h^{\top}(0)S(0), \quad \forall\, \theta_h \in F_{\Theta}(\theta), \qquad \bar{\psi} \ge 0.$$

Recall now that θ_h finances h iff it finances h^{π^1} (using commodity 1 as numeraire), i.e.,

$$\theta_h^{\top}(t)S^{\pi^1}(t) = \theta_h^{\top}(0)S^{\pi^1}(0) + \int_0^t \theta_h^{\top}(s)\,dG^{\pi^1}(s) - \int_0^t p^{\top}(s)h^{\pi^1}(s)\,ds,$$

where $h^{\pi^1}(t) = \pi(t,e^1)h(t)$. From the budget constraint $\theta_h^{\top}(T)S^{\pi^1}(T) = 0$. Hence, there are no expected deflated gains from trade only if

$$\begin{aligned}\mathrm{E}\left\{H_T^{\pi}\right\} &= \mathrm{E}\left\{\theta_h^{\top}(0)S^{\pi^1}(0) + \int_0^T \theta_h^{\top}(s)\,dG^{\pi^1}(s)\right\}\\ &= \mathrm{E}\left\{\int_0^T p^{\top}(s)\pi(s,e^1)h(s)\,ds\right\}.\end{aligned}$$

Recalling the definition of H_T^{π} it follows that

$$\mathrm{E}\left\{\int_0^T \nabla U(s,c)\cdot h(t)\,ds\right\} = \mathrm{E}\left\{\int_0^T p^{\top}(s)\pi(s,e^1)h(s)\,ds\right\}, \qquad \forall\, h \in F_C(c).$$

By Condition 4.1 this holds only if $p(s)\pi(s,e^1) = \nabla U(s,c)$. □

EXAMPLE 4.2. This example computes state price densities for the economies introduced in Example 4.1.

Additive separable: As $f(t,c,x) = (c,x)(t)$ we have that $m_i = 1$, $k \geq 1$ and $l \geq 1$. Further, $\hat{\varphi}^i(t,s) = \delta(t-s)$ so that

$$\pi(t,e^1) = \mathrm{E}\left\{\int_t^T v_{c^1}(s,e(s),x(s))\delta(t-s)\,ds\,|\mathcal{F}_t\right\} = v_{c^1}(t,e(t),x(t)),$$

which reduces to $v_{c^1}(t,e^1(t))$ in the special case of a single-commodity economy (v_{c^1} denotes the partial derivative of v with regard to its second argument).

External habit formation: In this case $m_1 = 1 = k = l$. From the additive separable case it is immediate that

$$\pi(t,e^1) = v_{c^1}\left(t,e^1(t),X^1(t)\right),$$

where $X^1(t) = \int_0^t g(t,s)x^1(s)\,ds$.

Durabilities and habits II: In this case $m_1 = 2$, $k = 1$, and $l = 0$. For simplicity, denote $f^{1,1}$ by y and $f^{1,2}$ by z. Obviously $\hat{\varphi}^{1,1}(t,s) = g^y(t,s)$. To compute $\hat{\varphi}^{1,2}$ recall that

$$\begin{aligned}\delta f^z(t,c^1;h^1) &= \int_0^T \varphi^{1,2}(t,s)h^1(s)\,ds \\ &= \int_0^T \left[1_{[0,s]}(t)\int_t^s g^z(s,u)g^y(u,t)\,du\right]h^1(t)\,dt,\end{aligned}$$

so $\hat{\varphi}^{1,2}(s,t) = \int_t^s g^z(s,u)g^y(u,t)\,du$, and

$$\begin{aligned}\pi(t,e^1) = {}&\mathrm{E}\left\{\int_t^T v_y(s,y(s),z(s))g^y(s,t)\,ds\,|\mathcal{F}_t\right\} \\ &+\mathrm{E}\left\{\int_t^T v_z(s,y(s),z(s))\int_t^s g^z(s,u)g^y(u,t)\,du\,ds\,|\mathcal{F}_t\right\}\end{aligned}$$

3.3. Securities Prices. It follows readily from Theorem 4.1 that securities prices must satisfy a relation in the spirit of Lucas (1978). An important adjustment is necessary to account for the possibility of unbounded variation in dividends.

COROLLARY 4.1 (Generalization of Lucas' Asset Pricing Formula). *If Condition 4.1 holds*

$$S^j(t) = \frac{1}{\pi(t,e^1)}\mathrm{E}\left\{\int_t^T \pi(s,e^1)\,dD^j(s) + \int_t^T \sigma_{\pi^1}(s)\cdot\sigma_{D^j}(s)\,ds\,|\mathcal{F}_t\right\}, \tag{45}$$

$j = 0, 1, \ldots, N$.

The last term is the expected future covariation between the security's dividend payouts and the marginal utility of the consumer. A security will thus command a higher price the higher its dividends are correlated with marginal utility. This occurs if the security pays high dividends when the endowment of the numeraire is low, *ceteris paribus.*

4. Equilibrium Returns

It is in principle an easy exercise to derive a consumption-based model for equilibrium returns, using the procedure of Duffie and Zame (1989), and Back (1991). Assume for now, as in Corollary 4.1, that $d\pi(t, e^1) = \mu_{\pi^1}(t)\, dt + \sigma_{\pi^1}(t)\, dB(t)$, and recall that G^0 is of bounded variation. Since π^1-deflated gains are $(P, \mathcal{F}_t)$-martingales it follows that the absolutely continuous part of G^{π^1} must be a constant. Thereby, the equilibrium locally riskless rate of return and risk premia satisfy:

$$r(t) \triangleq \frac{\mu_G^0(t)}{S^0(t)} = -\frac{\mu_{\pi^1}(t)}{\pi(t, e^1)}, \tag{46}$$

$$\mu(t) - r(t)\mathbf{1} = -\frac{1}{\pi(t, e^1)}\sigma(t)\sigma_{\pi^1}(t)^\top. \tag{47}$$

As before $\mu(t) \triangleq \mathbf{I}_S^{-1}(t)\mu_G(t)$ and $\sigma(t) \triangleq \mathbf{I}_S^{-1}(t)\sigma_G(t)$.

Even though only the utility gradient of the numeraire enters the equilibrium returns, the presence of the other commodities and state variables affect both (46) and (47) through $\pi(t, e^1)$'s dependence on e and x through $v_{f^1}(t, f(t, e, x))$.

To justify the assumption that $\pi(t, e^1)$ is an Itô process (or any $\pi(t, e^i)$ for that matter), the next lemma characterizes the coefficients of the process. Looking back at the structure of $\pi(t, e^1)$ we see that we can write the state price deflator as $\sum_{j=1}^{m_1} A^j(t)M^j(t)$ where A are absolutely continuous functions and the M's are $(P, \mathcal{F}_t)$-martingales, as long as $\hat{\varphi}^{1,j}(t, s) \neq \delta(t - s)$. In the latter case the conditional expectation collapses to $v_{f^{1,j}}(t, f(t, e, x))$. By assumption this is an adapted, smooth function of (t, e, x), and its decomposition can be derived using Itô's Lemma alone. The decomposition in the former case, when the summand is a nontrivial conditional expectation, can be derived by the Clark-Ocone Theorem (For a general discussion of Malliavin calculus see, e.g., Nualart 1995, Øksendal 1996). Hence, one must explicitly allow for the possibility that the felicity depends on the *rate* of purchases of the numeraire.

It is useful at this stage to make the simplifying assumption that $v(t, f) = \exp\{-\int_0^t \beta(u)\,du\}u(f)$, with $\beta \in L^2(P \times \lambda)$. This does not represent any real loss of generality as a simple change of notation will take care of the more general case.[9]

THEOREM 4.4 (Equilibrium Returns). *Assume the conditions in Theorem 4.2 hold. If* $\mathcal{A} \triangleq \{j : \hat{\varphi}^{1,j}(t,s) = \delta(t-s)\} = \emptyset$ *then the equilibrium locally riskless rate of return is*

$$r(t) = \frac{1}{\pi(t,e^1)} \sum_{j=1}^{m_1} \Bigg[v_{f^{1,j}}(t, f(t,e,x))\hat{\varphi}^{1,j}(t,t) - \mathrm{E}\left\{ \int_t^T v_{f^{1,j}}(s, f(s,e,x)) \frac{\partial}{\partial t}\hat{\varphi}^{1,j}(s,t)\,ds \,|\mathcal{F}_t \right\} \Bigg] \tag{48}$$

and the equilibrium risk premia are

$$\begin{aligned} \mu(t) - r(t)\mathbf{1} = -\frac{1}{\pi(t,e^1)}\sigma(t) \sum_{j=1}^{m_1} \mathrm{E}\Bigg\{ \int_t^T \Bigg[&-v_{f^{1,j}}(s, f(s,e,x)) \int_t^s \mathfrak{D}_t\beta(u)\,du \\ &+ \sum_{i=1}^{k+l} v_{f^{1,j} f^i}(s, f(s,e,x)) \cdot \mathfrak{D}_t f^i(s,e^i,x^i) \Bigg] \hat{\varphi}^{1,j}(s,t) \\ &+ v_{f^{1,j}}(s, f(s,e,x))\mathfrak{D}_t\hat{\varphi}^{1,j}(s,t)\,ds|\mathcal{F}_t \Bigg\}. \end{aligned} \tag{49}$$

$\mathfrak{D}_u$ *is the Malliavin derivative operator.*

The result is proved in two stages. It is first shown that the assumption that $\pi(t,e^1)$ is an Itô process is justified. Subsequently, since G^{π^1} is a $(P,\mathcal{F}_t)$-martingale and both G and $\pi(t,e^1)$ are Itô processes, equilibrium security prices must ensure that the drift of G^{π^1} is zero P-a.s. This second part of the proof is identical to that of Duffie and Zame (1989), but care is taken to properly take into account that the dividends generally have *unbounded variation.* The complete steps are shown in Back (1991), and are thus not included here. The first part of the proof is stated in the following lemma.

[9] Case in point, introduce the notation $v(t,f) = v(g(t;\beta), f)$ and assume $v_t(t,f) = v_g(g,f)g_t(t,\beta)$. This obviously includes the common specification with $v(t,f) = g(t;\beta)u(f)$ where $g(t;\beta) = \exp\{-\int_0^t \beta(u)\,du\}$.

LEMMA 4.2. *If $\mathcal{A} = \emptyset$ the state price deflator $\pi(t, e^1)$ is an Itô process with drift coefficient*

$$\mu_{\pi^1}(t) = \sum_{j=1}^{m_1}\Bigg[\mathrm{E}\left\{\int_t^T v_{f^{1,j}}(s, f(s,e,x))\frac{\partial}{\partial t}\hat{\varphi}^{1,j}(s,t)\,ds\,|\mathcal{F}_t\right\} - v_{f^{1,j}}(t, f(t,e,x))\hat{\varphi}^{1,j}(t,t)\Bigg]$$

and diffusion coefficient

$$\sigma_{\pi^1}(t) = \sum_{j=1}^{m_1}\mathrm{E}\Bigg\{\int_t^T\Bigg[-v_{f^{1,j}}(s, f(s,e,x))\int_t^s \mathfrak{D}_t\beta(u)\,du + \sum_{i=1}^{k+l} v_{f^{1,j}f^i}(s, f(s,e,x))\cdot \mathfrak{D}_t f^i(s, e^i, x^i)\Bigg]\hat{\varphi}^{1,j}(s,t) + v_{f^{1,j}}(s, f(s,e,x))\mathfrak{D}_t\hat{\varphi}^{1,j}(s,t)\,ds|\mathcal{F}_t\Bigg\}.$$

If $\mathcal{A} \neq \emptyset$ then the state price deflator has decomposition

$$d\pi^{\mathcal{A}}(t, e^1) = d\pi^{-\mathcal{A}}(t, e^1) + \mu_{\pi^1}^{\mathcal{A}}\,dt + \sigma_{\pi^1}^{\mathcal{A}}(t)\,dB(t),$$

where

$$\mu_{\pi^1}^{\mathcal{A}} = \sum_{n\in\mathcal{A}}\Bigg\{v_{f^{1,n}t}(t, f(t,e,x)) + \sum_{i=1}^{k+l} v_{f^{1,n}f^i}(t, f(t,e,x))\cdot\mu_{f^i}(t) + \frac{1}{2}\sum_{i,j=1}^{k+l}\sigma_{f^i}(t)v_{f^{1,n}f^if^j}(t, f(t,e,x))\sigma_{f^j}^{\top}(t)\Bigg\},$$

and

$$\sigma_{\pi^1}^{\mathcal{A}}(t) = \sum_{n\in\mathcal{A}}\sum_{i=1}^{k+l} v_{f^{1,n}f^i}(t, f(t,e,x))\cdot\sigma_{f^i}(t),$$

and $\pi^{-\mathcal{A}}(t, e^1)$ is equal to the state price in the first part of the Lemma, when summing over $j \in \{1, \ldots, m_1\}\setminus\mathcal{A}$.

PROOF. Case 1: $\mathcal{A} = \emptyset$. It suffices to consider only one term of the state price deflator. Let $X^{1,j}(T,t) \triangleq \int_0^T v_{f^{1,j}}(s, f(s,c,x))\hat{\varphi}^{1,j}(s,t)\,ds$. Rewriting the state price deflator as $\pi^j(t, c^1) = \mathrm{E}\{X^{1,j}(T,t)\,|\mathcal{F}_t\} - X^{1,j}(t,t)$ we see that it consists of a Lebesgue integral and an Itô integral, since by the Clark-Ocone Theorem (Ocone and Karatzas 1991), the Law of Total Probability, and the martingale property of the Itô integral,

$$\mathrm{E}\left\{X^{1,j}(T,t)\,|\mathcal{F}_t\right\} = \mathrm{E}\left\{X^{1,j}(T,t)\right\} + \int_0^t \mathrm{E}\left\{\mathfrak{D}_u X^{1,j}(T,t)\,|\mathcal{F}_u\right\}\,dB(u).$$

The part of the integrand inside the conditional expectation in this Itô integral is given by

$$\begin{aligned}\mathfrak{D}_u X^{1,j}(T,t) &= \int_0^T \left\{\mathfrak{D}_u v_{f^{1,j}}(s,f(s,c,x))\hat{\varphi}^{1,j}(s,t)\right.\\ &\quad \left.+ v_{f^{1,j}}(s,f(s,c,x))\mathfrak{D}_u\hat{\varphi}^{1,j}(s,t)\right\}1_{[0,s]}(u)\,ds\\ &= \int_u^T \left[-v_{f^{1,j}}(s,f(s,c,x))\int_0^s \mathfrak{D}_u\beta(v)1_{[0,v]}(u)\,dv\right.\\ &\quad \left.+\sum_{i=1}^{k+l} v_{f^{1,j}f^i}(s,f(s,c,x))\cdot\mathfrak{D}_u f^i(s,c^i,x^i)\right]\hat{\varphi}^{1,j}(s,t)\\ &\quad + v_{f^{1,j}}(s,f(s,c,x))\mathfrak{D}_u\hat{\varphi}^{1,j}(s,t)\,ds.\end{aligned}$$

By Itô's Lemma and the Lebesgue Dominated Convergence Theorem,[10]

$$\begin{aligned}d\pi(t,c^1) &= \left\{\sum_{j=1}^{m_1}\mathrm{E}\left\{\frac{\partial}{\partial t}X^{1,j}(T,t)\,|\mathcal{F}_t\right\} - \int_0^t v_{f^{1,j}}(s,f(s,c,x))\frac{\partial}{\partial t}\hat{\varphi}^{1,j}(s,t)\,ds\right.\\ &\quad \left.- v_{f^{1,j}}(t,f(t,c,x))\hat{\varphi}^{1,j}(t,t)\right\}dt + \left\{\sum_{j=1}^{m_1}\mathrm{E}\left\{\mathfrak{D}_t X^{1,j}(T,t)\,|\mathcal{F}_t\right\}\right\}dB(t).\end{aligned}$$

The first part of the result now follows from Theorem 4.3 by setting $c = e$, and by the unique decomposition property of continuous semimartingales (Karatzas and Shreve 1991, Problem 3.3.2).

Case 2: $\mathcal{A} \neq \emptyset$. By Assumptions 4.2 and 4.3 the marginal felicity and the aggregators are smooth functionals of the commodity and state vectors. Hence, the decomposition of this summand of the state price deflator is given by an application of Itô's Lemma. □

$\mathcal{A} \neq \emptyset$ whenever the consumer derives felicity from the rate of purchases of the numeraire. This is the most frequently used modelling choice in the extant literature, for instance in models of habit formation (Sundaresan 1989, Constantinides

[10]It is an easy exercise to construct an integrable positive stochastic process that dominates $v_{f^{1,j}}(s,f)\left|\hat{\varphi}^{1,j}(s,t+\epsilon_n) - \hat{\varphi}^{1,j}(s,t)\right|$.

1990, Detemple and Zapatero 1991, and chapter 2). For all practical purposes one can always consider $\mathcal{A} \neq \emptyset$ a singleton. The class of preferences represented by these utility functionals will not treat consumption at nearby dates as close substitutes (Hindy and Huang 1992, Proposition 9).

COROLLARY 4.2 (No Local Substitution). *Assume the conditions in Theorem 4.2 hold. Let $r^{-\mathcal{A}}$ and $\mu^{-\mathcal{A}}(t)$ denote the rates in expression* (48) *and* (49) *when summing over $j \in \{1, \dots, m_1\} \setminus \mathcal{A}$. Let*

$$v_{f^{1,n} f^i f^j} \triangleq \begin{bmatrix} v_{f^{1,n} f^{i,1} f^{j,1}}, & \cdots & v_{f^{1,n} f^{i,1} f^{j,m_j}} \\ \vdots & \ddots & \vdots \\ v_{f^{1,n} f^{i,m_i} f^{j,1}}, & \cdots & v_{f^{1,n} f^{i,m_i} f^{j,m_j}} \end{bmatrix}, \quad n \in \mathcal{A}$$

be the $m_i \times m_j$ Hessian matrix of "substitution effects" between commodity/state i and commodity/state j. Further, let $df^i(t,c,x) = \mu_{f^i}(t)\,dt + \sigma_{f^i}(t)\,dB(t)$ be the Itô m_i-tuple decomposition of aggregator i.[11] *If $\mathcal{A} \neq \emptyset$ then the locally riskless rate of return is*

$$\begin{aligned} r^{\mathcal{A}}(t) = r^{-\mathcal{A}}(t) - \frac{1}{\pi(t,e^1)} \sum_{n \in \mathcal{A}} \Bigg\{ & v_{f^{1,n} t}(t, f(t,e,x)) \\ & + \sum_{i=1}^{k+l} v_{f^{1,n} f^i}(t, f(t,e,x)) \cdot \mu_{f^i}(t) \\ & + \frac{1}{2} \sum_{i,j=1}^{k+l} \sigma_{f^i}(t) v_{f^{1,n} f^i f^j}(t, f(t,e,x)) \sigma_{f^j}^{\top}(t) \Bigg\}, \end{aligned}$$

and the risk premia are

$$\begin{aligned} \mu^{\mathcal{A}}(t) - r^{\mathcal{A}}(t)\mathbf{1} = {} & \mu^{-\mathcal{A}}(t) - r^{-\mathcal{A}}(t)\mathbf{1} \\ & - \frac{1}{\pi(t,e^1)} \sigma(t) \sum_{n \in \mathcal{A}} \sum_{i=1}^{k+l} v_{f^{1,n} f^i}(t, f(t,e,x)) \cdot \sigma_{f^i}(t). \end{aligned}$$

[11] I.e.,

$$\mu_{f^i}(t,c,x) = \begin{bmatrix} \mu_{f^{i,1}}(t) \\ \vdots \\ \mu_{f^{i,m_i}}(t), \end{bmatrix}, \quad \text{and} \quad \sigma_{f^i}(t,c,x) = \begin{bmatrix} \sigma_{f^{i,1}}(t) \\ \vdots \\ \sigma_{f^{i,m_i}}(t) \end{bmatrix}$$

is an $m_i \times d$ matrix.

$r^{-\mathcal{A}}$ and $r^{\mathcal{A}}$ are naturally defined relative to the same state price deflator, and similarly for the risk premia.

The following result shows the structure of equilibrium returns under the strongest possible assumptions that encompass those used in previous studies. It turns out that only the risk premia are affected by these simplifying assumptions.

COROLLARY 4.3. *If $f(t,e,x)$ is absolutely continuous and $\beta, \varphi \in L^2(\lambda)$ then $r(t) = r^{\mathcal{A}}(t)$, and*

$$\begin{aligned}\mu(t) - r(t)\mathbf{1} &= \mu^{-\mathcal{A}}(t) - r^{-\mathcal{A}}(t)\mathbf{1} \\ &= \frac{-1}{\pi(t,e^1)}\sigma(t)\sum_{j=1}^{m_1}\sum_{i=1}^{k+l}\mathrm{E}\left\{\int_t^T v_{f^{1,j}}(s,f)\hat{\varphi}^{1,j}(s,t)\right. \\ &\qquad\qquad \left.\cdot\, \mathfrak{D}_t f^i(s,e^i,x^i)\,ds\,|\mathcal{F}_t\right\}.\end{aligned}$$

Hence, there is an asymmetry in how preferences affect equilibrium returns. Absence of local substitution affects the qualitative structure of the locally riskless rate of return, but does not affect risk premia. On the other hand, stochastic parameters in the consumer's preferences affect the qualitative structure of risk premia, but do not affect the locally riskless rate of return.

EXAMPLE 4.3. Consider again the economies introduced through Example 4.1 and 4.2. Using the previous computations as input to Theorem 4.4 and Corollary 4.2 it is straightforward to compute the risk premia.

Additive separable: Allowing for non-Markovian endowments and states, Breeden's (1986) model implies that $\mathcal{A}^{\mathsf{C}} = \emptyset$. I.e. there is no local substitution, and $r^{-\mathcal{A}}(t) \equiv 0$. From Corollary 4.2 (suppressing obvious arguments)

$$\begin{aligned}r(t) = \frac{-1}{v_{c^1}(t,e(t),x(t))}&\Big\{v_{c^1 t} + v_{c^1 c^1}\mu_{e^1}(t) + \cdots + v_{c^1 x^l}\mu_{x^l}(t) \\ &+ \tfrac{1}{2}v_{c^1 c^1 c^1}\sigma_{e^1}(t)\sigma_{e^1}^{\top}(t) + v_{c^1 c^1 c^2}\sigma_{e^1}(t)\sigma_{e^2}^{\top}(t) + \ldots \\ &+ \tfrac{1}{2}v_{c^1 x^l x^l}\sigma_{x^l}(t)\sigma_{x^l}^{\top}(t)\Big\}.\end{aligned}$$

Similarly, $\mu^{-\mathcal{A}}(t) \equiv 0$, so

$$\mu(t) - r(t)\mathbf{1} = \frac{-1}{v_{c^1}(t,e(t),x(t))}\sigma(t)\left\{v_{c^1 c^1}\sigma_{e^1}(t) + \cdots + v_{c^1 x^l}\sigma_{x^l}(t)\right\}^{\top}.$$

Trivially, these restrictions on equilibrium returns collapse to those of Breeden (1979) in the single-commodity economy.

External habit formation: Clearly $\mathcal{A}^{\complement} = \emptyset$ so $r^{-\mathcal{A}} \equiv 0$. Because $\mu_{f^{1,1}}(t) = \mu_{e^1}(t)$ and $\mu_{f^{2,1}}(t) = \mu_{X^1}(t) = [\eta x^1(t) - \alpha X^1(t)]$ it follows from Corollary 4.2 that

$$r(t) = \frac{-1}{v_{c^1}(t, e^1(t), X^1(t))}\Big\{v_{c^1 t} + v_{c^1 c^1}\mu_{e^1}(t) + v_{c^1 X^1}[\eta x^1(t) - \alpha X^1(t)] + \tfrac{1}{2}v_{c^1 c^1 c^1}\sigma_{e^1}(t)\sigma_{e^1}^{\top}(t)\Big\},$$

where $X^1(t) \triangleq x^1(0)e^{-\alpha t} + \eta\int_0^t e^{-\alpha(t-s)}x^1(s)\,ds$, and α and η are constants. From example 4.1 it is immediate that $\hat{\varphi}^1(t,s) = \delta(t-s)$ and $\hat{\varphi}^2 = \eta e^{-\alpha(t-s)}$. Since also $\mu^{-\mathcal{A}} \equiv 0$ Corollary 4.2 applies. Because $\sigma_{f^{2,1}}(t) = \sigma_{X^1}(t) = 0$,

$$\mu(t) - r(t)\mathbf{1} = -\frac{v_{c^1 c^1}(t, e^1(t), X^1(t))}{v_{c^1}(t, e^1(t), X^1(t))}\sigma(t)\sigma_{e^1}^{\top}(t).$$

Durabilities and habits II: Recall from Example 4.2 that the aggregator f has representation $\hat{\varphi}^{1,1}(s,t) = g^y(s,t)$ and $\hat{\varphi}^{1,2}(s,t) = \int_t^s g^z(s,u)g^y(u,t)\,du$. Therefore $\mathcal{A} = \emptyset$ and Theorem 4.4 applies. Assuming $g^y, g^z \in L^2(\lambda)$,

$$r(t) = \frac{1}{\pi(t,e^1)}\Big[v_y(t, y(t), z(t))g^y(t,t) - \mathrm{E}\Big\{\int_t^T v_y(s, y(s), z(s))g_t^y(s,t) + v_z(s, y(s), z(s))\Big(\int_t^s g^z(s,u)g_t^y(u,t)\,du - g^z(s,t)g^y(t,t)\Big)\,ds\,|\mathcal{F}_t\Big\}\Big],$$

where the state price deflator is as in Example 4.2. Further,

$$\begin{aligned}\mu(t) - r(t)\mathbf{1} = \frac{-1}{\pi(t,e^1)}\sigma(t)\mathrm{E}\Big\{\int_t^T\Big[&-v_y(s,y,z)\int_t^s \mathfrak{D}_t\beta(u)\,du \\ &+ (v_{yy}(s,y,z), v_{yz}(s,y,z))\cdot(\mathfrak{D}_t y(s), \mathfrak{D}_t z(s))\Big]g^y(s,t) \\ &+ \int_t^s g^z(s,u)g^y(u,t)\,du\Big[-v_y(s,y,z)\int_t^s \mathfrak{D}_t\beta(u)\,du \\ &+ (v_{zz}(s,y,z), v_{zy}(s,y,z))\cdot(\mathfrak{D}_t y(s), \mathfrak{D}_t z(s))\Big]\Big\},\end{aligned}$$

where

$$\mathfrak{D}_t y(s) = \int_t^s g^y(s,u)\mathfrak{D}_t e^1(u)\,du$$

$$\mathfrak{D}_t z(s) = \int_t^s g^z(s,u) \int_t^u g^y(u,v) \mathfrak{D}_t e^1(v),$$

and

$$\mathfrak{D}_t e^1(u) = \int_t^u \mathfrak{D}_t \mu_{e^1}(v)\, dv + \int_t^u \mathfrak{D}_t \sigma_{e^1}(v)\, dB(v) + \sigma_{e^1}(t).$$

5. Economic Implications

While Theorem 4.4 and Corollary 4.2 completely describe equilibrium security returns in the present economy, they do not say much about the information needed by a consumer to make optimal portfolio decisions. It is therefore of interest to investigate if the returns are related to common risk factors, which can each be fully hedged by a mutual fund. This investigation will naturally lead to Merton's (1973) ICAPM, since it is immediate that the classical separation theorem does not hold in the general case with k traded commodities and l state variables. To arrive at any useful characterizations of mutual funds it is necessary to say what the structure of $\mathfrak{D}_t f^i$ is.

LEMMA 4.3. *If Assumption 4.1, 4.2, and 4.3 hold then*

$$\mathfrak{D}_t f^i(s,e,x) = \begin{cases} \delta f^i(s, e^i; \mathfrak{D}_t e^i) & 1 \le i \le k \\ \delta f^i(s, x^i; \mathfrak{D}_t x^i) & k < i \le k + l \end{cases}$$

PROOF. By assumption f^i, e^i, and x^i are continuously Fréchet differentiable. The result therefore follows from the chain rule for Fréchet derivatives (Luenberger 1969, Proposition 7.3.1). □

In principle, $2(k+l+1)$ mutual funds can pool funds into portfolios that are perfectly correlated with each risk factor. The consumer is then indifferent between choosing a linear combination of the $N+1$ assets, or the $2(k+l+1)$ mutual funds.[12]

PROPOSITION 4.2 (ICAPM). *Let* $1_k \triangleq 1_{\{1 \le i \le k\}}$ *and* $1_l \triangleq 1_{\{k < i \le k+l\}}$, *and define* $\mathfrak{C}_t e(s) = \mathfrak{D}_t e(s) - \sigma_e(t)$. *Under the conditions in Theorem 4.4 there exists*

[12]A true mutual fund theorem should ideally be derived in a multi-consumer economy. It is not clear in the present representative consumer economy what the admissible felicities of the individual consumers are, and how their optimal consumption and portfolio choices aggregate.

$2(k+l)+1$ risk factors such that

$$\begin{aligned}\mu(t) - r(t)\mathbf{1} = &\sum_{i=1}^{k+l} \Lambda_L^i(t)\,\mathrm{Cov}_t\left(\mathbf{I}_S^{-1}(t)\,dG(t), 1_k de^i(t) + 1_l dx^i(t)\right)\\ &+ \sum_{i=1}^{k+l} \mathrm{Cov}_t\left(\mathbf{I}_S^{-1}(t)\,dG(t), \Lambda_T^i(t)\,dB(t)\right)\\ &+ \mathrm{Cov}_t\left(\mathbf{I}_S^{-1}(t)\,dG(t), \Lambda_D(t)\,dB(t)\right),\end{aligned}$$

where

$$\begin{aligned}\Lambda_L^1(t) = &\frac{-1}{\pi(t,e^1)}\mathrm{E}\left\{\int_t^T \sum_{j=1}^{m_1} v_{f^{1,j}f^i}(s,f)\cdot \delta f^i(s,e,x;1)\hat{\varphi}^{1,j}(s,t)\right.\\ &\left. + v_{f^{1,j}}(s,f)\frac{\partial \hat{\varphi}^{1,j}(s,t,x)}{\partial x}(s,t,e^1(t))\,ds\,|\mathcal{F}_t\right\},\\ \Lambda_L^i(t) = &\frac{-1}{\pi(t,e^1)}\mathrm{E}\left\{\int_t^T \sum_{j=1}^{m_1} v_{f^{1,j}f^i}(s,f)\cdot \delta f^i(s,e,x;1)\hat{\varphi}^{1,j}(s,t)\,ds\,|\mathcal{F}_t\right\},\\ \Lambda_T^i(t) = &\frac{-1}{\pi(t,e^1)}\mathrm{E}\left\{\int_t^T \sum_{j=1}^{m_i} v_{f^{1,j}f^i}(s,f)\cdot \delta f^i(s,e,x;1_k\mathfrak{C}_t e^i + 1_l\mathfrak{C}_t x^i)\right.\\ &\left.\times\,\hat{\varphi}^{1,j}(s,t)\,ds\,|\mathcal{F}_t\right\},\end{aligned}$$

and finally

$$\Lambda_D(t) = \frac{1}{\pi(t,e^1)}\mathrm{E}\left\{\int_t^T \sum_{j=1}^{m_1} v_{f^{1,j}}(s,f)\hat{\varphi}^{1,j}(s,t)\int_t^s \mathfrak{D}_t\beta(u)\,du\,ds\,|\mathcal{F}_t\right\}.$$

One fund will invest in a locally riskless portfolio. $2(k+l)$ of the funds invest in portfolios perfectly correlated with

i) changes in the level of endowments and state variables, and

ii) changes in the exogenous production technology, $\mathfrak{C}_t e(u)$, and changes in the dynamics of the exogenous environment, $\mathfrak{C}_t x(u)$,

and one fund set up to hedge against

iii) changes in the structure of the subjective discount rate, $\mathfrak{D}_t\beta(u)$.

Of the studies cited in table 1 only Detemple and Zapatero (1991), and Detemple and Giannikos (1996) use preferences that call for a hedge fund to handle changes in the market's subjective discount rate.

PROOF. Consider $1 \leq i \leq k$. By Lemma 4.3 $\mathfrak{D}_t f^i(s,e^i) = \delta f^i(s,e^i;\mathfrak{D}_t e^i)$. From Assumption 4.3 δf is a linear functional, and $\delta f^i(s,e^i;\mathfrak{D}_t e^i) = \delta f^i(s,e^i;\mathfrak{C}_t e^i) + \delta f^i(s,e^i;\sigma_{e^i}(t))$. The case of $k < i \leq k+l$ is similar with x^i in place of e^i. Further, $\mathfrak{D}_t \hat{\varphi}^1(s,t)$ reduces to a problem of ordinary partial differentiation with respect to $B(t)$, noting that $\mathfrak{D}_t B(t) = \mathbf{1}$. It remains only to collect terms in each risk factor; $\sigma_{e^i}(t)$, $\mathfrak{C}_t e^i(u)$ (and similar for x^i), and $\mathfrak{D}_t \beta(u)$. □

Under mild regularity conditions the qualitative structure of the ICAPM is invariant to the absence of local substitution. The following result formalizes the notion that no new risk factors will be added to the ICAPM.

COROLLARY 4.4. *Assume without loss of generality that $\mathcal{A} = \{1\}$. If the conditions of Corollary 4.2 hold then the risk factors remain as in Proposition 4.2. Denoting the risk factors with local substitution by $\Lambda^{-\mathcal{A}}$ (using the convention of Corollary 4.2), $\Lambda_T^{\mathcal{A}i} = \Lambda_T^{-\mathcal{A}i}$, $i \in \{1,\ldots,k+l\}$, $\Lambda_D^{\mathcal{A}} = \Lambda_D^{-\mathcal{A}}$, and*

$$\Lambda_L^{\mathcal{A}i} = \Lambda_L^{-\mathcal{A}i} + \sum_{i=1}^{k+l} v_{f^{1,1} f^i}(t,f) \cdot \sigma_{f^i}.$$

PROOF. Let $\sigma_{f^{i,j}} = 1_k \sigma_{e^i}(t) + 1_l \sigma_{x^i}(t)$ if $\hat{\varphi}^{i,j}(t,s) = \delta(t-s)$, and $\sigma_{f^{i,j}} = 0$ otherwise. The characterization follows by combining Proposition 4.2 and Corollary 4.2. □

The trivially necessary and sufficient condition for invariance of risk factors is that $[\mu^{\mathcal{A}}(t) - r^{\mathcal{A}}(t)\mathbf{1}] - [\mu^{-\mathcal{A}}(t) - r^{-\mathcal{A}}(t)\mathbf{1}]$ is perfectly correlated with any of the risk factors $\Lambda^{-\mathcal{A}}$. The a priori most likely candidate is $\Lambda_L^{-\mathcal{A}}$, the factor related to uncertainty about the level of production and state variables. This condition is not easily verified, and is so general that it is not of any interest. It is therefore not formally recorded.

Duffie and Zame (1989), and Back (1991) show that the Consumption-based Capital Asset Pricing Model (CCAPM) of Breeden (1979) is valid for general Itô processes when consumers have additive separable von Neumann-Morgenstern utility functionals. Studying two special cases of time nonseparable utility Bergman

(1985) finds that one cannot expect Breeden's CCAPM to hold with time nonseparable utility,[13] while Duffie and Zame note that the CCAPM will not obtain whenever the felicity is state-dependent.

Detemple and Zapatero (1991) study additive nonseparable utility in the form of habit formation without local substitution. They show that the CCAPM is still valid in the weak sense that any security's equilibrium risk premium is proportional to the instantaneous covariance between the aggregate endowment and the return on the security.[14] The assumption imposed to ensure the CCAPM is that the endowment processes $de(t)/e(t)$ has deterministic drift coefficient and dispersion vector, and that β is also deterministic. The next result summarizes these results and observations for the present economy. It shows that the result of Detemple and Zapatero is ensured to hold only if the subjective discount rate is sufficiently smooth, there is only one commodity, and there are no exogenous state variables that are imperfectly correlated with the numeraire.

PROPOSITION 4.3 (CCAPM). *If the conditions in Corollary 4.2 or Theorem 4.4 hold,*

i) *endowments* $de^i(t) = a^i(t)e^i(t)\,dt + b^i(t)e^i(t)\,dB(t)$, *with* $a^i \in L^2(\lambda)$ *and* $b^i \in L^2(\lambda)^d$, $i = 1, \dots, k$,
ii) *states* $dx^i(t) = c^i(t)x^i(t)\,dt + d^i(t)x^i(t)\,dB(t)$, *with* $c^i \in L^2(\lambda)$ *and* $d^i \in L^2(\lambda)^d$, $i = 1, \dots, l$, *and*
iii) *the subjective discount rate* $\beta(\omega, u) = \beta(u, e(\omega, u), x(\omega, u))$ *is smooth;* $\beta \in C^{1,3,3}([0,T] \times \mathbb{R}^k_+ \times \mathbb{R}^l)$,

then

$$\mu(t) - r(t)\mathbf{1} = \sum_{i=1}^{k+l} \kappa^i(t)\,\mathrm{Cov}_t\left(\mathbf{I}_S^{-1}(t)\,dG(t), \left[1_k\,\frac{de^i(t)}{e^i(t)} + 1_l\,\frac{dx^i(t)}{x^i(t)}\right]\right)$$

[13]Note that while Bergman (1985) assumes a Markovian economy, there are no such restrictions on the present economy. All of the important parameters are allowed to depend on the entire history of the economy. Bergman's parametrization of the Uzawa felicity, with consumption-dependent subjective discount rate, obtains by setting $\beta(\omega, t) = \beta(t, e^1(\omega, t))$.

[14]It does not hold in the strong sense that the coefficient of proportionality is equal to the Arrow-Pratt coefficient of absolute risk aversion for the consumer's felicity.

for a suitable choice of $\{\kappa^i\}_{i=1}^{k+l}$. *If in addition* $k = 1$, *and* $l = 0$ *or* $\sigma_{x^i} = \sigma_{e^1}$ *all* i,

$$\mu(t) - r(t)\mathbf{1} = \kappa(t)\,\mathrm{Cov}_t\left(\mathbf{I}_S^{-1}(t)\,dG(t), \frac{de^1(t)}{e^1(t)}\right),$$

for a suitable choice of $\kappa(t)$.

PROOF. Let Λ refer back to the risk factors in Proposition 4.4. If these risk factors collapse to a single risk factor then so will the risk factors in Proposition 4.2. Consider first the special case $\beta \in L^2(\lambda)$, so that $\mathfrak{D}_t\beta(u) = 0$, and therefore $\Lambda_D(t) \equiv 0$. The solution to the SDE for e^i (and x^i) is easily found by Itô's Lemma,

$$e^i(t) = e^i(0)\exp\left\{\int_0^t a^i(s) - \tfrac{1}{2}b^i(s)\cdot b^i(s)\,ds + \int_0^t b^i(s)\,dB(s)\right\}.$$

Thus, by the chain rule $\mathfrak{D}_t e^i(s) = e^i(s)b^i(t)$. Since $\delta f^i(t, e^i; \cdot)$ is a linear functional $\delta f^i(s, e^i; \mathfrak{D}_t e^i) = b^i(t)\,\delta f^i(s, e^i; e^i)$, applying δf^i on each element of b^i. Obviously, x^i is handled in the same fashion. Thereby, $\Lambda_L^i + \Lambda_T^i$, $i = 2, \ldots, k$, reduce to proportionality constants to $de^i(t)/e^i(t)$, and similarly to $dx^i(t)/x^i(t)$ for $i = k + 1, \ldots, k + l$. Using the chain rule $\mathfrak{D}_t\hat{\varphi}^i(s, t, e^i(t)) = \frac{\partial\hat{\varphi}^i(s,t,x)}{\partial x}(s, t, e^i(t))e^i(t)b^i(t)$. Hence, $\Lambda_L^1(t) + \Lambda_T^1(t)$ also reduces to a proportionality constant, to $de^1(t)/e^1(t)$.

Now consider without loss of generality the case of $\beta(\omega, u) = \beta(u, e^1(\omega, u))$. Introduce the notation $\beta_t(t, e(t)) = \frac{\partial\beta(t,x)}{\partial t}(t, e(t))$, $\beta_x = \frac{\partial\beta(t,x)}{\partial x}(t, e(t))$, with the obvious extension to β_{xx} and β_{xxx}. Using Itô's Lemma and the chain rule for Malliavin differentiation

$$\begin{aligned}\mathfrak{D}_t\mu_\beta(s) = \Big\{&\beta_{tx}(s) + \beta_x(s)a^1(s) + \beta_{xx}(s)\left[a^1(s)e^1(s) + b^1(s)\cdot b^1(s)\right]\\ &+ \tfrac{1}{2}\beta_{xxx}(s)e^1(s)b^1(s)\cdot b^1(s)e^1(s)\Big\}e^1(s)b^1(t),\end{aligned}$$

and

$$\mathfrak{D}_t\sigma_\beta(s) = \left[\beta_x(s) + \beta_{xx}(s)e^1(s)\right]e^1(s)b^1(s)\mathbf{I}_{b^1}(t),$$

where $\mathbf{I}_{b^1}$ denotes the $d \times d$ matrix with $b^{1,n}$ on all rows of the n'th column. Note also that $\mathrm{E}\left\{\int \mathfrak{D}_t\beta(u)\,du\,|\mathcal{F}_t\right\}$ is well defined as $e^1 \in L^p(P \times \lambda)$ $\forall p \geq 1$. Thereby, $\Lambda_D(t)$ also becomes a proportionality constant relative to $b^1(t)$. This proves the first part of the proposition.

The second part of the proposition is trivial in that all risk factors become proportionality constants to $de^1(t)/e^1(t)$. □

6. Concluding Remarks

An obvious extension of the present analysis is to allow endowments and dividends to be special semimartingales as done by Back (1991). I will argue that the former is not necessarily very interesting and that the latter is easily accommodated within the present economy.

The question of which class of stochastic processes that represents a good approximation of aggregate endowment cannot be resolved empirically. Even so, if individual consumption is lumpy, aggregate consumption will typically be much smoother.[15] How smooth aggregate endowment is depends on how synchronized individual demands are. Considering the number of individuals in say the US economy it is unlikely that one makes significant errors by assuming it is a Itô process.[16] On the other hand, a smooth process can be approximated arbitrarily close by a jump model (See e.g. Back 1991, Aase 1993b). Hence, the issue is largely one of taste.

Stock prices are typically lumpy, and it is desirable to be able to derive equilibrium restrictions for individual securities prices. Luckily, the extension to lumpy gains processes is far easier to incorporate than allowing lumpy aggregate endowments. To see this, let the part of the decomposition of the state price deflator that is a martingale be continuous. The generalization of Lucas' asset pricing formula (45) still holds (Hindy and Huang 1992, Duffie and Skiadas 1994):

$$S(t) = \frac{1}{\pi(t,e^1)} \mathrm{E}\left\{ \int_t^T \left(\pi(s,e^1)\, dD(s) + d\langle \pi^1, D\rangle(s)\right) |\mathcal{F}_t \right\}, \tag{50}$$

where $\mathbb{F} = \{\mathcal{F}_t\}_{0 \le t \le T}$ is not necessarily generated by Brownian motion. Letting D^c denote the continuous martingale part of D the covariation process $\langle \pi^1, D\rangle$ reduces to $\langle \pi^1, D^c\rangle$. (50) thereby reduces to (45). If A denotes the absolutely continuous

[15] I am referring to the nature of *unexpected* changes in aggregate consumption. The bounded variation part of its decomposition is a Lebesgue integral, and can therefore have a countable number of discontinuities, none of which will take the consumer by surprise.

[16] Note that if the filtration $\mathbb{F}$ is generated by a Brownian motion then the state price deflator will be an Itô process if $\mathcal{A} = \emptyset$, regardless of the assumptions made on the aggregate endowment (Hindy and Huang 1992, Proposition 7).

part of the decomposition of D, the jumps $D^d = (D - A) - D^c$ represent diversifiable risk in the economy since securities prices are unaffected by D^d (Back 1991).

Securities prices are no longer assured to be Itô processes, but rather inherit the special semimartingale properties of G. Still, the restrictions on equilibrium returns in the present economy hold. This is easily seen from the analysis of Back, which utilizes only that $\pi(t, e^1)$-deflated *gains* are martingales. In particular, since by assumption the martingale part of $\pi(t, e^1)$ is continuous $\langle \pi^1, G \rangle = \langle \pi^1, G^c \rangle$. The last expression equals $\int \sigma_{G^c} \sigma_{\pi^1}^\top dt$ if G^c is of unbounded variation, and zero otherwise (Protter 1990, Theorem IV.42). The covariation process thereby reduces to the one in (47). Back (p. 386) notes that "continuity of consumption would be an exceptional circumstance when gains processes jump, since undiversified jumps would cause wealth to jump and therefore consumption to jump." This observation is still pertinent in the present economy when $k = 1$. In the multi-commodity economy, however, the numeraire is no longer tied directly to wealth.

Despite the robustness of the present economy to more general information structures, one must also allow aggregate consumption to jump to gain any additional economic insight from allowing jumps in securities prices. If aggregate consumption is allowed to be lumpy when the utility functional is additive nonseparable, then the main challenge is to characterize the integrand in the representation of a special martingale as a stochastic integral. The need for such characterizations will always be present in these economies, as the state price deflator will consist of a sum of conditional expectations. Although the version of the Clark-Ocone Theorem developed by Ocone and Karatzas (1991) applies only to Brownian filtrations, there are well known extensions to Poisson filtrations (Bichteler, Gravereaux, and Jacod 1987, who develop the Malliavin calculus), and combinations of Brownian and Poisson filtrations (Aase, Øksendal, and Ubøe 1998).

Regardless of extensions, the current framework should be sufficient to address if von Neumann-Morgenstern utility is flexible enough to supply an answer to the following research program: Given historical dividends and the relatively smooth historical US per capita consumption,

i) explain the first moments of securities returns,

ii) explain the volatility in equities and fixed-income markets,

iii) explain autocorrelation patterns in securities returns,

iv) imply a reasonable *implicit* Arrow-Pratt coefficient of relative risk aversion.

Issue i) is natural to solve by posing simultaneous restrictions on the equilibrium returns in Theorem 4.4 (or Corollary 4.2 if one belive in that story). Issue ii) can be investigated through the price functional in Corollary 4.1, together with the state price deflator in Theorem 4.3. With the aid of the Clark-Ocone Theorem it is feasible to study the decomposition of securities prices, given the stochastic properties of dividends and per capita consumption. Issue iii) is again probably best resolved by studying the implied time series properties of Theorem 4.4. The last issue is handled by making sure the indirect utility function for wealth has the desired property. In equilibrium it is equal to $U(e)$.

The main challenge is to make all these partial solutions compatible. A promising example is the external habit formation economy of Campbell and Cochrane (1999). It answers most of these issues, but seems to have problems being consistent with atemporal lotteries for wealth, issue iv).

For the ambitious, the asset pricing literature claims there are even more "anomalies" to resolve. Still, the above four seem to be the most important to address. At least, they are the ones having received the most attention.

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