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A new method for robustness in rolling horizon planning

by

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Abstract

In this paper we describe a new method to solve Linear Programming (LP) problems with uncertain parameters. We apply this to planning problems where a rolling planning horizon is used. The method is based on a decomposition scheme where we iteratively solve an upper level problem for the first time period where the parameters are assumed to be known. The lower level problem use the upper level solution and computes a worst case scenario for an anticipation period with uncertain parameters. Information in how the worst case scenario is affected by the upper level decisions is given back as a valid inequality. This process is repeated until the upper level solution satisfy the last generated valid inequality. We test the proposed method on a integrated production, transportation and inventory planning problem. We compare our approach with a deterministic approach with and without safety stocks. The result shows that the method works well and perform better than the deterministic approach with safety stock.

Keywords: Robust optimization, Uncertainty, Transportation, Distribution, Production planning

Contents

1 Introduction

There are many industrial applications where decisions are made on a rolling planning horizon. Typically this is used in tactical models where the planning horizon is one to several months and where the demand is uncertain. We will study a general problem for integrated production, transportation and inventory planning problem. The decisions to make are how much to produce, how much to store and how to transport products between production units to customers. In the tactical planning, the planning horizon, say one year, is divided into time periods, say 12 months. For each time period we need information on production, transportation and storage capacities and demand. In addition we need information of transportation and inventory costs. Once we have solved the planning problem, we only implement the decisions for the first period as the situation may change for the remaining periods. During this month we may use the tactical decisions and use other operational models, for example, routing and production planning models to decide the short term operational decisions. Towards the end of the period we again go back and solve the tactical model which now covers the next 12 months. Any information collected during the month is used to reformulate the model. In tactical planning there is some data that is more uncertain than others. Production data is typically known whereas demand information is very uncertain. At the same time, demand can be uncertain over the next 12 months but fixed through contracts for the next 1-2 months.

One common solution approach is to collect or estimate all data and then formulate and solve a deterministic model. The advantage with this approach is that the planning problem is often tractable i.e. it is easy to solve the underlying optimization model. The disadvantage is that the solution may be quite wrong if the estimated information, e.g. demand, changes a lot to the next time period. There are two main approaches to include uncertainty. The first is to formulate the problem into a stochastic optimization model. This models typically becomes very large and are difficult to solve. There is also a need to provide detailed information and parameters on stochastic distribution functions. The second approach is to use so-called robust optimization. Here the uncertainty of the parameters is modeled as lower and upper bounds. The classical approach to robust optimization is to search for a optimal solution which has the property that the solution will satisfy all possible outcomes, and is called a static robust solution.

Bertsimas and Thiele [2006] explores the current status of static robust optimization for linear programming problems. These approaches do however find a solution over all time periods. When the planning is dynamic on a rolling horizon it is reasonable to expect that better solutions can be found, as we dynamically can adjust the planning when more information is known. In the dynamic planning situation it is assumed that there are two sets of variables where one set must be determined before all parameters are determined. The other set of variables model future decisions that need not be determined until a later stage.

Ben-Tal et al. [2004] introduce a computationally tractable robust formulation for the special case when the future decision variables can be expressed with an affine function of the uncertainty set. Although they observe many situations where this assumption holds, it is not an easy task to verify if the problem on hand satisfies all the requirements. The method has no flexibility in elaborations with uncertainty sets since a minor adjustment could change the robust counterpart to an intractable formulation.

Bertsimas and Caramanis [2007] approach a more general problem where the uncertainty set may be a general polytope. In the solution approach they use a partitioning of the uncertainty set and find a static robust solution for each partition. In a later stage, without uncertainty, at least one of the static solutions fulfills the now realized parameters and the best static solution is suggested to be selected for implementation. The difficulty with this approach is to select a well-performing partitioning such that static robust solutions are reasonable, while at the same time keeping the number of partitions low for efficiency.

In this paper, we consider problems where we have an uncertainty with the following properties.

- The uncertainty is limited into lower and upper bounds.
- There is no assumption on the underlying distribution functions.
- The overall uncertainty is limited.

The first two properties is included in robust optimization. The third is however not addressed when dealing with the rolling planning horizon. A typical example occur for annual planning of raw material for heating plants. The demand is expressed in energy content, i.e./ MWh, in the fuel delivered (typically fuel wood or wood chips). The overall energy demand is based on the average consumption from the last years. However, during the winter one month may be much warmer than normal and hence less energy is used. This is solved with flexible contracts where lower and upper bound of the energy delivered in a month can vary between, say, +-10%. However, there is a need to fix the exact level at some specific time before the next month. The remaining months still have the flexibility expressed through the contracts. It is obvious that the worst case scenario is that all months would be very cold and the maximum in the contracts need to be delivered. However, on an annual basis the overall uncertainty is not large as cold and warm weather balances over a longer period. This aspect must be expressed in the model and hence we introduce a limit on the overall uncertainty. We also note that this introduces a coupling between the periods.

We propose a new solution approach where we want to make decisions in the first time period such that in reduces the affect of the worst case scenario. The methodology is based on decomposing the problem into two separate problems. In the upper level problem we make business decisions i.e. decisions that will be implemented in the business process (e.g. fixing contract levels). The lower level problem finds the anticipated worst case solution given a business solution. This information is given back to the business planning model as a valid inequality integrating potential business solutions with worst case scenarios. These two problems are iteratively solved until the business solution satisfies the constructed valid inequality from the anticipation model. The business model is an LP problem whereas the anticipation model is a bilinear problem. The latter is solved with a heuristic approach.

We test the approach on an integrated production, transportation and inventory model. This model is very general and there are many applications that fits within this modeling framework. The result shows that the approach provides solutions that outperforms a standard deterministic approach. The solution approach is also efficient as it only need to solve sequences of LP problems of the same size (or smaller) as the deterministic approach.

The outline of the paper is as follows. In Section 2 we describe the general problem. We also introduce an illustrative example. In Section 3 we describe the proposed solution method. We also show how it can be applied to the example. In Section 4 we discuss the model used in the numerical experiments. In Section 5 we describe the results obtained and in Section 6 we give some conclusions.

2 Problem formulation

In this paper, we consider the following general optimization model.

$$
\begin{aligned}
[P] \quad & \min \sum_{t=1}^{T} \mathbf{c}_t^T \mathbf{x}_t && (1)\\
& s.t. \quad \mathbf{E}\mathbf{x}_1 = \mathbf{e} && (2)\\
& \sum_{t=1}^{T} \mathbf{A}_t \mathbf{x}_t = \mathbf{b} + \Delta \mathbf{b} && (3)\\
& \mathbf{x}_t \geq 0, \ t = 1, \ldots, T && (4)
\end{aligned}
$$

We have decision variables, $\mathbf{x}_t$, associated with each time period t. In [P] we have some constraints, (2), that are specific for the first period and where only variables for the first period appears. This can be, for example, inventory constraints coupled with initial inventory. In constraint (3) we have a combination of all variables and over all time periods. The right hand side consists of two parts. First we have $\mathbf{b}$ which is the estimated, for example, demand and resource limits. Second we have $\Delta \mathbf{b}$ which is the possible variation of $\mathbf{b}$. We want to use a general description of $\Delta \mathbf{b}$ and will assume that the possible values can be described by a polyhedron B, i.e. $\Delta \mathbf{b} \in B$. We assume an equal probability in B.

The planning period is divided into two parts, see Figure 1. The first part is the business planning period. Decisions here are to be implemented. The second part is the anticipation planning period. The solution from this part is only used to determine an anticipated scenario. In our case we want this to be the worst case scenario given the business solution. Once the model is solved and the business decision implemented, we will reformulate the model for the second period given what actually happen and what data is updated. We then resolve the model. This process is repeated in a rolling horizon environment.

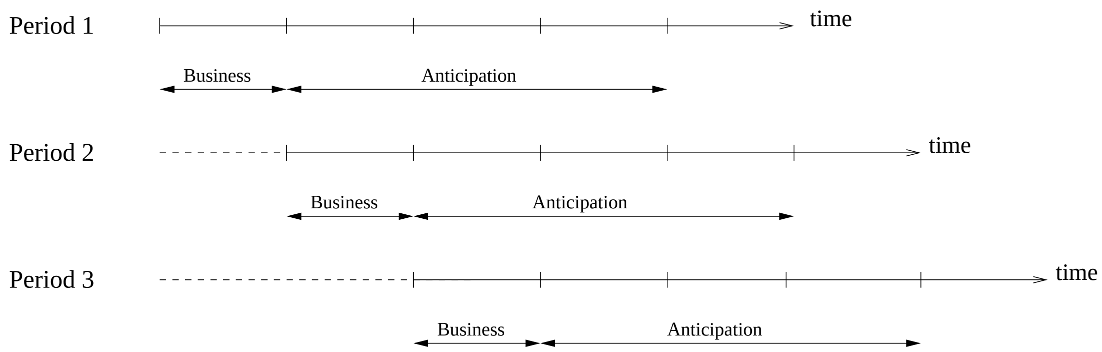

Figure 1: In a rolling horizon environment we first solve the problem with period 1 as business period. This solution is then fixed and new information is used to reformulate a new model (with one additional period in the end) where the second period is the business period. This process is repeated on a continuous basis.

The uncertainty is only coupled with the anticipation period as we assume that the information for the business period is known without uncertainty. For the anticipation period we assume that the uncertainty is bounded. In Figure 2 we illustrate how the uncertainty for demand can be modelled. Assume that we have an estimated and average demand d_t for time period t. For the anticipation period, i.e. $t \geq 2$, we have that the demand can be described as $d_t + \Delta d_t$ where $\Delta \underline{d}_t \leq \Delta d_t \leq \Delta \underline{d}_t$. In addition we also have a restriction that links the uncertainty for different time periods together. It can be, for example, that $d^{min} \leq \sum_{t=1}^{T} \Delta d_t \leq d^{max}$.

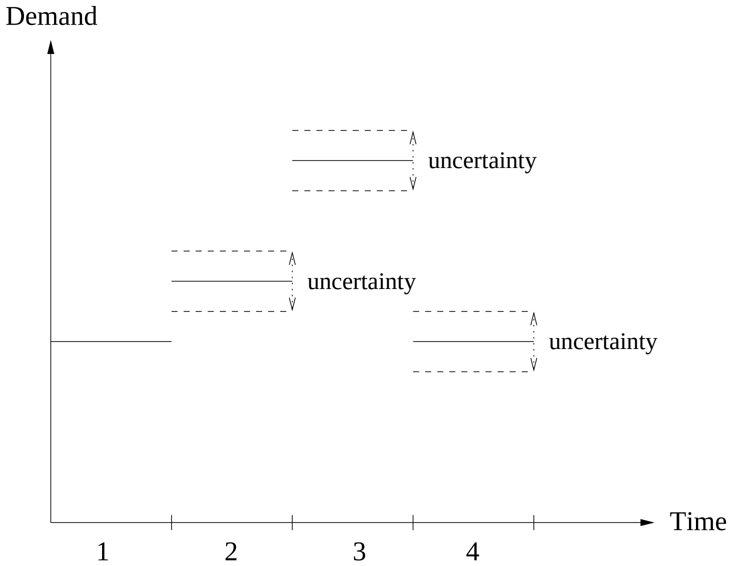

Figure 2: The demand is known (without uncertainty) in the first period (business period) and has an uncertainty with lower and upper bounds for the remaining periods (anticipation period).

To illustrate the problem we consider the following small illustrative example.

Example 1 *We have two supply points and two demand points, see Figure 3. There are two planning periods. The transportation costs between supply point i and demand point j are $(c_{11}, c_{12}, c_{21}, c_{22}) = (1, 5, 10, 6)$. The supply, s_{it} and demand, d_{jt}, during the two time periods are $(s_{1t}, s_{2t}, d_{1t}, d_{2t}) = (10, 5, 8, 2)$. However, in time period 2 there are uncertainties of the supply in $S1$ and demand in $D1$. The uncertainty in supply and demand are both in the interval $[-2, 2]$ i.e. $s_{12} \in [8, 12]$ and $d_{12} \in [6, 10]$. There is a possibility to have an inventory at the supply points and the inventory cost is 0. The aim is to minimize the total cost of transportation and inventory.*

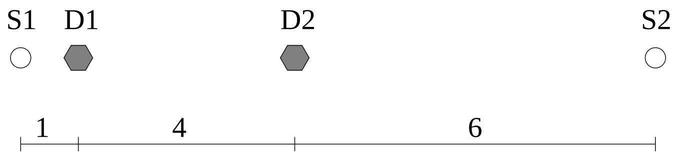

Figure 3: Spatial distribution of the example.

If we formulate the problem we can use x_{ijt} for the flow between supply i to demand j in period t and l_{it} for the inventory at supply i in period t. If we solve the deterministic problem with averaged supply and demand, i.e. assuming that the uncertainties are 0 ($s_{12} = 10$ and $d_{12} = 8$), we get the following solution. The total cost is 18+18=36.

Period	*Flow*	*Inventory*	*Objective*
1	$x_{111} = 8, x_{121} = 2$	-	*18*
2	$x_{112} = 8, x_{122} = 2$	-	*18*

Suppose we solve sequentially and fix the first period solution with cost 18. The worst case scenario in the second period is when $d_{12} = 10$ and $s_{12} = 8$. In this case, the second period solution is $x_{112} = 8, x_{212} = 2, x_{222}$ with a total cost of 40. The overall worst case cost would then be 18+40=58. However, if we in the first period decide to have in inventory at $S1$ of 2 units, then we would have a total cost of 20+22=42 which is much less than 58. This solution together with the worst case scenario is given below. A robust solution in this case is a solution which avoids the extreme worst case scenarios.

Period	*Flow*	*Inventory*	*Objective*
1	$x_{111} = 8, x_{121} = 2$	$l_{11} = 2$	*20*
2	$x_{112} = 8, x_{212} = 2, x_{222} = 2$	-	*22*

3 Solution methodology

We introduce a decomposition where the business and anticipation time periods are separated. For the anticipation period ($t = 2, \ldots, T$) where $\mathbf{x}_1$ is fixed we have

$$[F] \quad V(\mathbf{x}_1, \Delta\mathbf{b}) = \min \sum_{t=2}^{T} \mathbf{c}_t^T \mathbf{x}_t \tag{5}$$

$$s.t. \quad \sum_{t=2}^{T} \mathbf{A}_t \mathbf{x}_t = \mathbf{b} + \Delta\mathbf{b} - \mathbf{A}_1\mathbf{x}_1 \tag{6}$$

$$\mathbf{x}_t \geq 0, \;\; t = 2, \ldots, T \tag{7}$$

We can now rewrite problem [P] as

$$[P2] \quad \min \; \mathbf{c}_1^T \mathbf{x}_1 + V(\mathbf{x}_1, \Delta\mathbf{b}) \tag{8}$$

$$s.t. \quad \mathbf{E}\mathbf{x}_1 = \mathbf{e} \tag{9}$$

$$\mathbf{x}_1 \geq 0 \tag{10}$$

With the uncertainty $\Delta\mathbf{b} \in \mathbf{B}$ we can reformulate the robust counterpart of [P2] as

$$[PR] \quad z^* = \min \; \mathbf{c}_1^T \mathbf{x}_1 + w \tag{11}$$

$$s.t. \quad \mathbf{E}\mathbf{x}_1 = \mathbf{e} \tag{12}$$

$$w \geq \max_{\Delta\mathbf{b} \in \mathbf{B}} V(\mathbf{x}_1, \Delta\mathbf{b}) \tag{13}$$

$$\mathbf{x}_1 \geq 0 \tag{14}$$

Let $\mathbf{u}$ be the dual variables associated with constraint (6). We assume that there exist a feasible solution to [F]. We can then use LP-duality to reformulate [F] to

$$[F2] \quad V(\mathbf{x}_1, \Delta\mathbf{b}) = \max \; (\mathbf{b} + \Delta\mathbf{b} - \mathbf{A}_1\mathbf{x}_1)^T \mathbf{u} \tag{15}$$

$$s.t. \quad \mathbf{A}_t^T \mathbf{u} \leq \mathbf{c}_t, \;\; t = 2, \ldots, T \tag{16}$$

We can rewrite the maximization problem in constraint (13) as a bilinear problem

$$
\begin{aligned}
[BPP] \quad v^* = \max\, (\mathbf{b} + \Delta\mathbf{b} - \mathbf{A}_1\mathbf{x}_1)^T\mathbf{u} & & & \quad (17)\\
s.t. \quad \mathbf{A}_t^T\mathbf{u} &\leq \mathbf{c}_t, \;\; t = 2, \ldots, T & & \quad (18)\\
\Delta\mathbf{b} &\in \mathbf{B} & & \quad (19)\\
\mathbf{u} &\geq 0 & & \quad (20)
\end{aligned}
$$

Suppose that we for $\bar{\mathbf{x}}_1$, have an optimal solution $\Delta\mathbf{b}^*$ and $\mathbf{u}^*$ to [BPP]. Then we can write

$$
\begin{aligned}
V(\bar{\mathbf{x}}_1, \Delta\mathbf{b}^*) &= (\mathbf{b} + \Delta\mathbf{b}^* - \mathbf{A}_1\bar{\mathbf{x}}_1)^T\bar{\mathbf{u}}^* & (21)\\
(\mathbf{b} + \Delta\mathbf{b}^*)\mathbf{u}^* &= V(\bar{\mathbf{x}}, \Delta\mathbf{b}^*) + \mathbf{u}^{*T}\mathbf{A}_1\bar{\mathbf{x}}_1 & (22)
\end{aligned}
$$

For a given $\bar{\mathbf{x}}_1 \in Y = \{\mathbf{x} \geq 0 \,|\, \mathbf{E}\mathbf{x} = \mathbf{e}\}$ we have the following valid inequality in $[PR]$

$$
V(\bar{\mathbf{x}}_1, \Delta\mathbf{b}^*) + \mathbf{u}^{*T}\mathbf{A}_1\bar{\mathbf{x}}_1 - \mathbf{u}^*\mathbf{A}_1\mathbf{x}_1 \quad \leq \quad w \qquad (23)
$$

The value of w provides a bound of the worst case scenario in the anticipation period given a solution $\bar{\mathbf{x}}_1$. By finding new solutions in the business period we can create new bounds of the worst case scenario in the anticipation period. The idea is to establish a solution $\bar{\mathbf{x}}_1$ where we no more can improve the worst case scenario. The idea of the algorithm is to generate valid inequalities until no more can be found. To establish new solutions in the business period we solve

$$
\begin{aligned}
[PR_P] \quad h_P = \min\, \mathbf{c}_1^T\mathbf{x}_1 + w & & & \quad (24)\\
s.t. \quad \mathbf{E}\mathbf{x}_1 &= \mathbf{e} & & \quad (25)\\
w &\geq V(\mathbf{x}_1^p, \Delta\mathbf{b}^p) + \mathbf{u}^{pT}\mathbf{A}_1\mathbf{x}_1^p - \mathbf{u}^{pT}\mathbf{A}_1\mathbf{x}_1, \;\; p = 1, \ldots, P & & \quad (26)\\
\mathbf{x}_1 &\geq 0 & & \quad (27)
\end{aligned}
$$

Here, $(\mathbf{x}_1^p, \Delta\mathbf{b}^p, \mathbf{u}^p), p = 1, \ldots, P$ is a set of solutions to $[BPP]$. The algorithm can be summarized in the following algorithmic description.

Algorithm 1 Robust

Require: $\mathbf{B}, \mathbf{E}, \mathbf{A}, \mathbf{b}, \mathbf{e}$
  $P \leftarrow 0$
  $p \leftarrow 1$ //Iteration counter
  loop
    Solve $[PR_P]$ (gives $\mathbf{x}_1^p, w_p$)
    Solve $[BPP]$ (gives $\Delta\mathbf{b}^p$ and $\mathbf{u}^p$)
    if $w_p > V(\mathbf{x}_1^p, \Delta\mathbf{b}^p)$ **then**
      Add valid inequality $w \geq V(\mathbf{x}_1^p, \Delta\mathbf{b}^p) + \mathbf{u}^{pT}\mathbf{A}_1\mathbf{x}_1^p - \mathbf{u}^{pT}\mathbf{A}_1\mathbf{x}_1$ and update $P \leftarrow p$
    else
      Business solution is $\mathbf{x}_1^p$ and the worst case objective for the anticipation period is w_p.
    end if
    $p \leftarrow p + 1$ //New iteration
  end loop

Problem $[BPP]$ is a bilinear problem and is very difficult to solve. We will apply an iterative heuristic approach to solve [BPP] where we alternatively fix $\Delta\bar{\mathbf{b}}$ and $\bar{\mathbf{u}}$ and solve the resulting LP-problems. We assume that for $\bar{\mathbf{x}}_1$, we satisfy necessary optimality conditions for [BPP] (LP optimality for $\Delta\bar{\mathbf{b}}$ and $\bar{\mathbf{u}}$). We have

$$V(\bar{\mathbf{x}}_1, \Delta\bar{\mathbf{b}}) = (\mathbf{b} + \Delta\bar{\mathbf{b}} - \mathbf{A}_1\bar{\mathbf{x}}_1)^T\bar{\mathbf{u}} \tag{28}$$
$$(\mathbf{b} + \Delta\bar{\mathbf{b}})\bar{\mathbf{u}} = V(\bar{\mathbf{x}}, \Delta\bar{\mathbf{b}}) + \bar{\mathbf{u}}^T\mathbf{A}_1\bar{\mathbf{x}}_1 \tag{29}$$

This solution will provide a valid inequality in $[PR]$

$$V(\bar{\mathbf{x}}_1, \Delta\bar{\mathbf{b}}) + \bar{\mathbf{u}}^T\mathbf{A}_1\bar{\mathbf{x}}_1 - \bar{\mathbf{u}}\mathbf{A}_1\mathbf{x}_1 \leq w \tag{30}$$

Algorithm 2 BPP-Heuristic

Require: $\mathbf{B}, \mathbf{E}, \mathbf{A}, \mathbf{b}, \mathbf{e}$
  $k \leftarrow 1$ //Iteration counter
  Choose a point $\Delta\bar{\mathbf{b}} \in \mathbf{B}$
  loop
    Solve $[F]$ (gives $V(\bar{\mathbf{x}}_1, \Delta\bar{\mathbf{b}})$ and $\bar{\mathbf{u}}$)
    Solve $[BPP]$ with $\bar{\mathbf{u}}$ from $[F]$ fixed (gives v^* and $\Delta\bar{\mathbf{b}}$)
    if $v^* \leq V(\bar{x}_1, \Delta\bar{b})$ **then**
      Local optimal solution found, stop. Solution is $(\Delta\bar{\mathbf{b}}, \bar{\mathbf{u}})$
    end if
    $k \leftarrow k + 1$ //New iteration
  end loop

The method does not guarantee an optimal solution to [BPP]. We may find different solution depending on the first selection of $\bar{\mathbf{b}}$.

Illustrative example revisited

Below we use Example 1 but apply the proposed approach.

Example 2 *When we solve problem [PR_P] (with $P = 0$ and $w \geq 0$) we get the same solution as for the deterministic solution i.e. with objective 18 for the first time period. When we solve [BPP] we get the valid inequality $w \geq 40 - 9l_{11}$ where l_{11} is the inventory at supply point 1 in period 1. This relates to the worst case scenario we studied earlier. If $l_{11} = 0$ we get the worst case solution 40 in time period 2. When we solve [PR_P] we get a solution with $c_1^T x_1 = 42$ and $w = 0$. If we use this solution and solve [BPP] we get the valid inequality $w \geq 20$. Then we get $l_{11} = 20/9$ and $w = 20$. When we generate another valid inequality we get $w \geq 24 - l_{11}$. This last solution from [PR_P] satisfy this solution and we are done. The full robust solution is given below.*

Period	*Flow*	*Inventory*	*Objective*
1	$x_{111} = 8, x_{221} = 2$	$l_{11} = 2, l_{21} = 3$	*20*
2	$x_{112} = 10, x_{222} = 2$	-	*22*
Tot			*42*

This solution of with value 42 is the robust solution from Example 1. In this case we found out an inventory level. Typically, there would be some safety stock that is based on some experience to handle the uncertainty. In our case, this level comes as a part of the solution.

4 Integrated model

The solution approach can be applied to any planning problem that can be formulated as a LP problem. We will study an integrated production, transportation and inventory planning problem. The decision variables used are

y_{it}	production at industry i in time period t.
x_{ijt}	transportation between supply point i and customer j in time period t.
l_{it}^{ind}	inventory at industry i at the end of time period t.
l_{jt}^{dem}	inventory at industry i at the end of time period t.

In order to formulate the model we also need some parameters. These are

c_{ijt}	unit transportation cost between supply point i and customer j in time period t.
c_{it}^{ind}	unit inventory cost at industry i at the end of time period t.
c_{jt}^{dem}	unit inventory cost at industry i at the end of time period t.
PC_t	Production capacity in time period t.
TC_t	Transportation capacity in time period t.
s_i	Overall production capacity over the entire planning period.

We can formulate the planning problem as

$$[A] \quad \min \sum_{t=1}^{T} \sum_{i \in I} \sum_{j \in J} c_{ijt} x_{ijt} + \sum_{t=1}^{T} \sum_{i \in I} c_{it}^{ind} l_{it}^{ind} + \sum_{t=1}^{T} \sum_{j \in J} c_{jt}^{dem} l_{jt}^{dem} \tag{31}$$

$$s.t. \quad l_{it}^{ind} - l_{i,t-1}^{ind} + \sum_{j \in J} x_{ijt} - y_{it} = 0, \ i \in I, \ t = 1, \ldots, T \tag{32}$$

$$-l_{jt}^{dem} + l_{j,t-1}^{dem} + \sum_{i \in I} x_{ijt} = d_{jt} + \Delta d_{jt}, \ j \in J, \ t = 1, \ldots, T \tag{33}$$

$$\sum_{i \in I} y_{it} \leq PC_t, \ t = 1, \ldots, T \tag{34}$$

$$\sum_{i \in I} \sum_{j \in J} x_{ijt} \leq TC_t, \ t = 1, \ldots, T \tag{35}$$

$$\sum_{t=1}^{T} y_{it} \leq s_i, \ i \in I \tag{36}$$

$$P_{it}^{max} \geq y_{it} \geq 0, \ i \in I, \ t = 1, \ldots, T \tag{37}$$

$$x_{ijt} \geq 0, \ i \in I, j \in J, \ t = 1, \ldots, T \tag{38}$$

$$l_{it}^{ind} \geq 0, \ i \in I, \ t = 1, \ldots, T \tag{39}$$

$$l_{jt}^{dem} \geq 0, \ j \in J, \ t = 1, \ldots, T \tag{40}$$

We can introduce other costs, e.g., production costs but we want to keep the model simple. Based on $[A]$ we can formulate the robust formulation. We have

$$[AF] \quad V(l_1^{ind}, l_1^{dem}, x_1, y_1, \Delta d) =$$

$$\min \sum_{t=2}^{T} \sum_{i \in I} \sum_{j \in J} c_{ijt} x_{ijt} + \sum_{t=2}^{T} \sum_{i \in I} c_{it}^{ind} l_{it}^{ind} + \sum_{t=2}^{T} \sum_{j \in J} c_{jt}^{dem} l_{jt}^{dem} \tag{41}$$

$$s.t. \quad l_{it}^{ind} - l_{i,t-1}^{ind} + \sum_{j \in J} x_{ijt} - y_{it} = 0, \ i \in I, \ t = 2, \ldots, T \ | \alpha_{it} \tag{42}$$

$$-l_{jt}^{dem} + l_{j,t-1}^{dem} + \sum_{i \in I} x_{ijt} = d_{jt} + \Delta d_{jt}, \ j \in J, \ t = 2, \ldots, T \ | \beta_{jt} \tag{43}$$

$$\sum_{i \in I} y_{it} \leq PC_t, \ t = 2, \ldots, T \tag{44}$$

$$\sum_{i \in I} \sum_{j \in J} x_{ijt} \leq TC_t, \ t = 2, \ldots, T \tag{45}$$

$$\sum_{t=1}^{T} y_{it} \leq s_i, \ i \in I \ | \gamma_i \tag{46}$$

$$P_{it}^{max} \geq y_{it} \geq 0, \ i \in I, \ t = 2, \ldots, T \tag{47}$$

$$x_{ijt} \geq 0, \ i \in I, j \in J, \ t = 2, \ldots, T \tag{48}$$

$$l_{it}^{ind} \geq 0, \ i \in I, \ t = 2, \ldots, T \tag{49}$$

$$l_{jt}^{dem} \geq 0, \ j \in J, \ t = 2, \ldots, T \tag{50}$$

We can formulate the following robust model where the constraints (52)-(59) define Y.

$$
\begin{aligned}
[PR] \quad \min \sum_{i\in I}\sum_{j\in J} c_{ij1}x_{ij1} + \sum_{i\in I} c_{1i}^{ind} l_{1i}^{ind} \sum_{j\in J} c_{1j}^{dem} l_{1j}^{dem} + w && (51)\\
s.t. \quad l_{i1}^{ind} - l_{i,0}^{ind} + \sum_{j\in J} x_{ij1} &= y_{i1}, \;\; i \in I & (52)\\
-l_{j1}^{dem} + l_{j,0}^{dem} + \sum_{i\in I} x_{ij1} &= d_{j1}, \;\; j \in J & (53)\\
\sum_{i\in I} y_{i1} &\le PC_1 & (54)\\
\sum_{i\in I}\sum_{j\in J} x_{ij1} &\le TC_1 & (55)\\
P_{i1}^{max} \ge y_{i1} &\ge 0, \;\; i \in I & (56)\\
x_{ij1} &\ge 0, \;\; i \in I, j \in J & (57)\\
l_{i1}^{ind} &\ge 0, \;\; i \in I & (58)\\
l_{j1}^{dem} &\ge 0, \;\; j \in J & (59)\\
\max_{\Delta b \in B} V(l_1^{ind}, l_1^{dem}, x_1, y_1, \Delta b) &\le w & (60)
\end{aligned}
$$

The contribution to $[BPP]$ from the business period is

$$
\begin{aligned}
W(l_1^{ind}, l_1^{dem}, x_1, y_1) &= \\
\sum_{i\in I}(l_{i1}^{ind} - \sum_{j\in J} x_{ij1})\alpha_{i2} + \sum_{j\in J}(l_{j1}^{dem} - \sum_{i\in I} x_{ij1})\beta_{j2} - \sum_{i\in I} y_{i1}\gamma_i & & (61)
\end{aligned}
$$

Then we can rewrite P_p as

$$
\begin{aligned}
[P_p] \quad \min \sum_{i\in I}\sum_{j\in J} c_{ij1}x_{ij1} + w && (62)\\
s.t. \quad (l_1^{ind}, l_1^{dem}, x_1, y_1) &\in Y & (63)\\
V(l_{1p}^{ind}, l_{1p}^{dem}, x_{1p}, y_{1p}, \Delta b_p) & & \\
-W(l_{2p}^{ind}, l_{2p}^{dem}, x_{2p}, y_{2p}) & & \\
+W(l_2^{ind}, l_2^{dem}, x_2, y_2) &\le w, \;\; p = 1, \ldots, P & (64)
\end{aligned}
$$

5 Computational results

The solution method is implemented in the modeling language AMPL using the CPLEX solver. All tests have been done on a standard PC with 2.67 GHz and 1 GB of internal memory. The experiments are done with an example with 20 production units and 7 customers. The geographical distribution is given in Figure 4. The planning period is 12 periods. In order to simulate a rolling horizon with a planning period for 12 periods we have generated data for 24 periods.

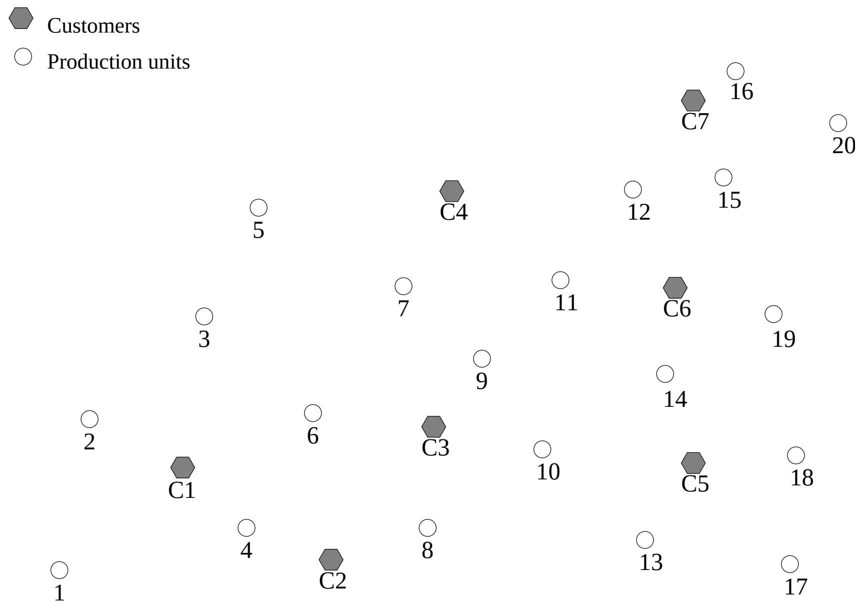

Figure 4: Spatial distribution of customers and production units.

In order to evaluate the behaviour of the proposed methodology we have randomly generated 100 scenarios. We have tested two set of uncertainties. The first is to use an uncertainty that increases with time. Here we have used an increment of 2% per period for one set of scenarios and 4% for another. The second set is to have an uncertainty of +-%10 in each anticipation period.

We have used six approaches to find a solution for the first set of uncertainties. These are:

Ideal In this situation the data is fully known through an oracle and represent the optimal solution.

Robust-100% In this approach we use the proposed solution method.

Robust-50% In this approach we assume that the uncertainty is 50% of the actual.

Deterministic In this approach the deterministic version with average demands are solved.

Minimal safety In case some part of the demand is unsatisfied in the deterministic approach we add a cost corresponding a safety stock level which is the minimal required to satisfy all demand. If we choose the maximum over all simulations we get a Max-min safety.

Full safety In case some part of the demand is unsatisfied in the deterministic approach we add a cost corresponding a safety stock level which is set such that we have as much safety stock as there is uncertainty in the demand. If the demand uncertainty is 2%, then the safety stock is 1% of the demand. (The demand uncertainty is +-1% and the worst case is for this problem to have over demand.)

The results are summarized in Table 5. Column "Uncertainty" gives either increments of 2% (0-22%) or 4% (0-44%) of uncertainty per period (where the first period is determined). Column "Method" give which of the six approaches used. Column "Costs" give the cost divided into transportation and inventory costs. Column "Compared to ideal" gives the total objective as compared to the ideal solution. The column "Worst case" provides the worst case scenario given the first period. This is the situation in case all decisions are implemented and the worst case occur over all time periods. Column "Plan" gives the result once the rolling horizon has been solved the full 12 periods. Column "Unsatisfied demand" gives the percentage of scenarios with unsatisfied demand (column "Scenarios") and the total quantity of the unsatisfied quantity (column "Quantity").

		Costs		Compared to ideal (%)		Unsatisfied demand	
Uncertainty	Method	Transport	Storage	Worst case	Plans	Scenarios (%)	Quantity
Ideal	Ideal	18398.4	3109.9				
0-22%	Robust	18916.8	4446.3	113.1	108.6		
0-22%	Robust-50%	18842.6	3941.7		105.9	2	3.6
0-22%	Deterministic	18377.7	3101.5		99.9	98	1227.1
0-22%	Minimal safety	18377.7	5518.5		111.1		
0-22%	Max-min safety	18377.7	10598.5		134.7		
0-22%	Full (1%) safety	18377.7	9757.0		130.8		
0-44%	Robust	18926.9	5493.3	129.9	113.5		
0-44%	Robust-50%	18859.6	4430.8		108.3	1	4.39
0-44%	Deterministic	18350.3	3090.2		99.7	97	2356.4
0-44%	Minimal safety	18350.3	7421.8		119.8		
0-44%	Max-min safety	18350.3	18036.3		169.2		
0-44%	Full (2%) safety	18350.3	16401.2		161.6		
0-44%	Robust-4 months	18720.6	4647.1		108.6	0	0
0-44%	Deterministic-4 months	18398.6	2929.3		99.2	100	3293.2
20%	Robust (unlimited)	19086.8	11042.4	140.6	140.1		
20%	Robust (20% annual)	19017.0	11067.5	140.1	139.9		
20%	Robust (20% season)	19019.8	10935.7	139.3	139.3		
20%	Robust (2% annual)	18794.2	9928.3	133.5	133.5		
20%	Deterministic	18249.1	3589.2		101.5	82	3684.6
20%	Minimal safety	18249.1	10603.7		134.1		
20%	Max-min safety	18249.1	31825.8		232.8		
20%	10% safety	18249.1	70144.2		411.0		

Table 1: Results from the comparison over 100 scenarios with the three different uncertainty levels.

From the results we can make the following comments.

- The deterministic approach provides a low cost but essentially always give some unsatisfied demand.
- The robust approach provides a average solution 8.6% higher than the ideal solution. However, this solution provides a feasible solution (i.e. no unsatisfied demand) in all cases.
- The robust solution is better than the deterministic approach with a minimal safety stock, 8.6% against 11.1%.
- The robust solution is much better than the deterministic approach with a 1% safety stock, 8.6% against 30.8%.
- By using a rolling horizon we can lower the first estimate of a worst case scenario from 13.1% to 8.6% for the case of 0-22% uncertainty. For the 0-44% case we reduce from 29.9% to 13.5%.

- The deterministic solution with 1% and 2% safety stock is an estimation of what could be a feasible solution. We have not accounted for refilling the safety stock and the increased transportation costs. To guarantee a feasible solution we would need a larger safety level, at least at some demand points and during some critical time periods. This is emphasized with the Max-min safety which is higher in both cases.
- The deterministic solution with 10% safety stock has almost a double cost as compared with the Max-min safety. But also when we count a 10% safety stock for only the difficult 6 months we land on more than 232 % from ideal, which is significantly higher than the robust worst case.
- We tend to obtain solutions close to the worst-case estimate when the cases have large uncertainties in the near future (as in the case of 20% per month). One reason for this is the fact that all business decisions have to prevent a sudden increase in demand in the next period which implies a need for a higher inventories.
- We can interpret the robust solution as an optimal (i.e. minimal) safety stock. This solution is feasible and not directly comparable with the reported minimal safety for the deterministic solution which is an optimistic approximation.

In Figures 5 and 6 we provide the bounds for all 100 scenarios for the scenarios of 1% and 2% uncertainty per period, respectively. With increased uncertainty we can see that the worst case estimate increases.

Note that the worst case values are estimates, they depend on the local optimum found in the heuristic algorithm for BPP. The worst case cost is estimated in the first business period. It is therefore possible for this first worst case estimate to be lower that the cost for the robust solution for all periods.

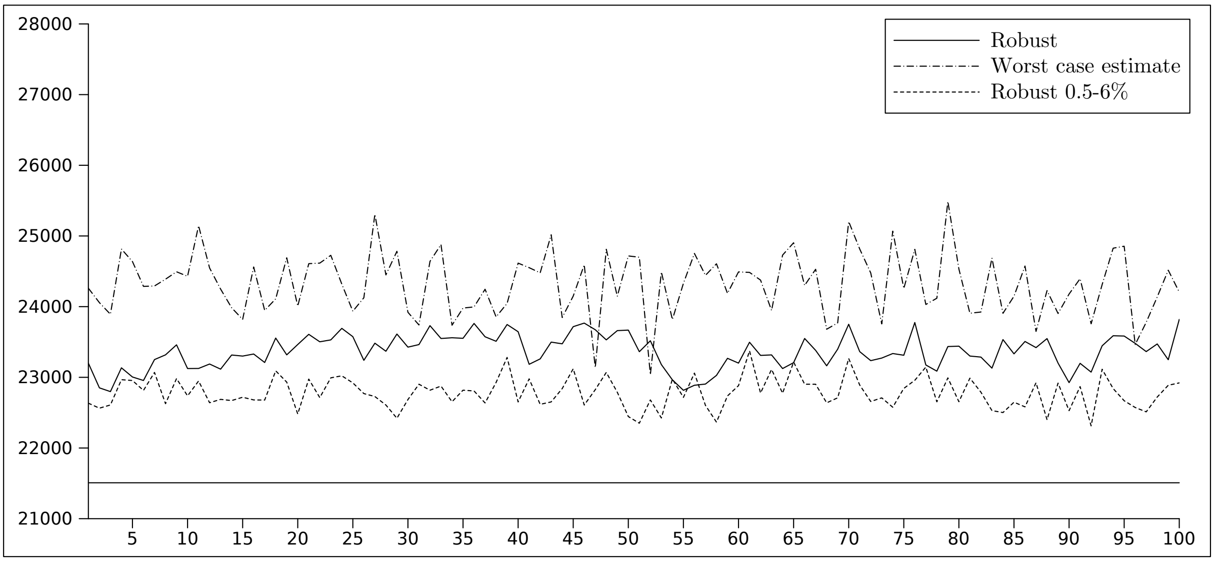

Figure 5: The bounds for robust, worst case estimate and robust with 50% estimated uncertainty with a 1% uncertainty.

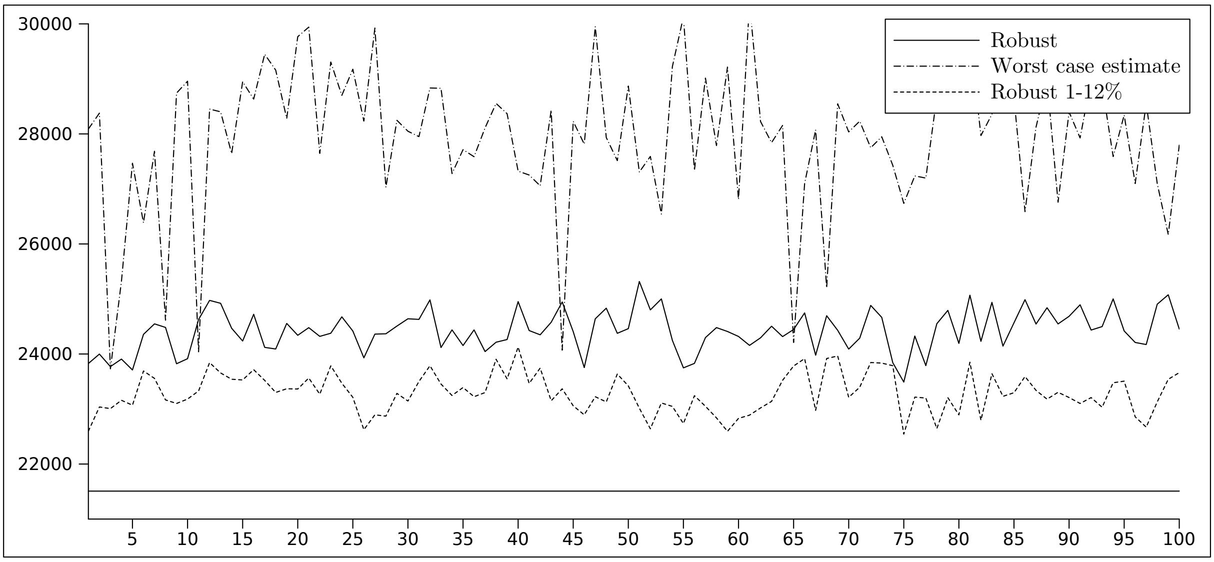

Figure 6: The bounds for robust, worst case estimate and robust with 50% estimated uncertainty with a 2% uncertainty.

Discussion on applications related to refinery planning

An obvious application is when the demand for different products or indirectly components are uncertain. By applying the proposed approach on a LP production planning model we would get answers on which safety stock levels of components that are optimal. The same is true also for uncertainty on availability of crude oils.

In the tests we have used an uncertainty in the demand for each time period. The linkage is expressed over the total uncertainty over the entire planning period. It is also possible to use the uncertainty for linked demand profiles. One example is when a tanker arriving to a refinery is loaded with a blend of components and the uncertainty is the arrival time. Then the actual volume (tanker capacity or order volume) is known but the demand profile may move between time periods. In such a case the rolling planning horizon is much shorter, for example, a few days with hourly time periods. The solution would provide a solution where the blending of components is optimized while taking account the product demand (combination of components) over several days. With such an approach it is possible to avoid shortages of components and/or less efficient blending solutions.

A very common problem is when the prices of petroleum products are uncertain. We can also introduce the uncertainty in the objective function. We need to reformulate the models but essentially the same approach can be used to solve this problem. If we assume that the problem is an LP problem we can simply formulate the dual problem and apply the proposed approach directly.

6 Conclusions

We have proposed a new solution method to solve LP problems with uncertain parameters in a rolling horizon planning approach. The new method is based on a decomposition principle where we iteratively solve a problem for the first period and then computes a worst case solution given the first period solution. The worst case scenario is found by solving an anticipation model. The information feed back to the first period problem is valid inequality that determines the impact of the worst case scenario by the business solution. The anticipation model is formulated as a bilinear problem. This is generally hard to solve and we make use of a heuristic approach where we iteratively solve an LP problem.

The proposed model only need to solve problems which has the same or smaller size than the deterministic approach. This is a very tractable property. The proposed method is more general than other approaches. It has no assumption on underlying assumptions on distribution functions and the uncertainty set can be described as a general polyhedron.

Numerical results show that the proposed solution method outperform the deterministic approach despite a minimal safety stock level is included to take care of any unsatisfied demand. The robust method can be used to find the optimal safety stock levels dynamically throughout the planning period. It also establish where inventory should be located.

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