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Supply Function Equilibria in a
Hydropower Market

by

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Abstract: The purpose of this working paper is to study the implications of market power in a hydropower based electricity market within the framework of the Supply Function Equilibria (SFE) model. We start by developing the standard SFE model with two symmetric players. We then develop a simple numerical example to illustrate the effects of production constraints both related to installed effect capacity and to energy capacity. We illustrate that binding constraints on energy production reduce the number of allowable supply functions.

1 Introduction

The purpose of this working paper is to study the implications of market power in a hydropower based electricity market within the framework of the Supply Function Equilibria (SFE) model. In addition to the usual problems related to models of this type we face the problem of modelling hydro-power production. Not taking this aspect into account would be wrong in a market dominated by this kind of production.

The idea of competition in supply functions origins from the debate on whether firms choose prices or quantities as strategic variables. The idea first outlined by Grossman (1981) was that firms may not be able to set a price or a given quantity for every possible state of the market in advance of trade taking place. Instead, firms may resort to specifying supply functions relating quantity to price. Grossman (1981) studied supply function equilibria in abscence of uncertainty. According to Klemperer and Meyer (1989), this approach led to a vast number of possible Nash equilibria in supply functions. In addition, without uncertainty, there is no reason to choose a more general supply function because firms can maximize profits either by fixing price or quantity.

Klemperer and Meyer (1989) introduced exogenous uncertainty into the supply function framework. They prove that under these conditions it is more profitable for firms to rely on supply functions rather than fixing price or quantity. With uncertainty, a supply function provides valuable flexibility to the firm. Furthermore, they also show that with uncertainty in demand, the number of possible Nash equilibria is dramatically reduced.

The supply function equilibria (SFE) concept developed by Klemperer and Meyer seems to fit quite closely to competition in several market where firms must commit to bids in advance, including electricity spot markets. Thus, not surprisingly, several papers[1] used this approach in order to analyze electricity market competition. These papers typically focus on competition in a thermal based electricity market. In such a market the focus is on production constraints at a particular time, not on constraints on energy produced over a time period.

This working paper is divided in three parts. In the first part we set up the Supply Function Equilibria (SFE) model based on the analysis by Baldick and Hogan (2001) and Green and Newbery (1992). On the basis of this model we develop a simple numerical example. In the second part

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[1] This includes Green and Newbery (1992), Green (1996), Newbery (1998), Baldick et al (2000) and Baldick and Hogan (2001).

of the paper we use this example to illustrate how competition in supply functions may be affected both by constraints in effect capacity (referred to as constraints on power produced) and constraints related to energy capacity.

2 The SFE model

In order to describe the SFE model we use a standard development closely related to the presentation made by Baldick and Hogan (2001). First we discuss demand, generation costs and capacities. In our discussion on generation costs and capacities we will also address the problem of defining costs in the presents of hydropower production. Then we discuss price, the assumptions on the form of the supply functions, profit and equilibrium conditions.

2.1 Demand

Baldick and Hogan (2001) use the following definition of the demand function Q:

$$\forall p \in R_+, \forall t \in [0,1], \qquad Q(p,t) = a(t) - bp. \tag{1}$$

Where,

- p is the price,

- t is the (normalized) time,

- $a : [0,1] \to R_+$ is the load-duration characteristic and

- $b \in R_+$ is the slope of the demand curve.

This definition implies that demand is additively separable in its two variables price and time. Furthermore, the time variable is normalized to be between 0 and 1. The time variable describes the share of clearing periods or hours below the peak demand period. The load-duration characteristic is assumed to be non-increasing so that $t = 0$ correspond to peak demand while $t = 1$ correspond to minimum demand. The load-duration curve gives the time (number of hours) that demand exceeds a given level, so at $t = 0$ we would only have the highest demand period left.

This approach resemble the approach made by Klemperer and Meyer (1989) where they let demand at a particular time be subject to an exogenous shock. Here, instead of a shock to demand at a particular time we look

at the variations in demand facing a supplier of electricity in the spot market. Producers are assumed to know the shape of the load duration curve. However, they face uncertainty with regard to the actual level of demand realized at a particular time.

2.2 Generation costs and capacities

A standard assumption is to let the total variable cost function be represented by a quadratic function of the form:

$$\forall i, \ C_i(q_i) = \frac{1}{2}c_i q_i^2 + v_i q_i. \tag{2}$$

The firms are labeled $i = 1..n$, with $n \geq 2$. q_i represents power produced by firm i and c_i and v_i are positive constants. $c_i \geq 0$ satisfy Klemperer and Meyer's condition that costs be non-decreasing. If we differentiate the cost function we get the firms marginal cost[2].

$$\forall i, \ C_i'(q_i) = c_i q_i + v_i. \tag{3}$$

Furthermore, we assume that all the firms are able to produce down to 0 output. Thus, the minimum capacity constraint is 0 for all firms. Baldick and Hogan (2001) note that firms may also face a maximum production capacity constraint, $\overline{q_i}$. The capacity constraint then becomes

$$\forall i, \ 0 \leq q_i \leq \overline{q_i}. \tag{4}$$

With hydropower production there is an additional constraint related to the amount of energy produced over time and the capacity constraint related to production of electricity at one point in time may not be as relevant as it is to thermal production. In the following we shall refer to the first type of constraint as the power constraint, while the second type of constraint is referred to as the energy constraint. We let $k = 1..n$ represent the number of load duration periods associated with the producer's planning horizon. Furthermore, we let $\overline{W}_i$ be the amount of energy available for hydropower firm i over all k load duration periods. This implies that the following constraint must hold.

$$\sum_k \int_{t=0}^1 q_{ikt} dt \leq \overline{W}_i. \tag{5}$$

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[2] The trick used by Klemperer and Meyer here was to let $C_i'(0) = 0$, so if $C_i'(0) = v_i > 0$ then the supply functions would be expressed in terms of $\widetilde{p} = p + v_i$. The marginal cost curve for symmetric producers where normalized to start at 0.

Using this constraint we implicitly assume there is a fixed amount of energy (water) available for production over all load duration periods.

2.3 Feasible and allowable supply functions

Following Green and Newbery (1992) and Baldick and Hogan (2001) we assume that each firm bid a supply function into the market. The supply function represents the amount of power the firm is willing to produce at a specified price per unit electricity.

Formally, a supply function S_i is a function that maps the level of prices into levels of output. In Klemperer and Meyer (1989) the functions are required to be defined for every price in the interval $[0, \infty]$. This implies that all firms will produce at prices in this interval. Furthermore, in order for a supply function to be feasible they only require that the function is contained in the quantity interval $[-\infty, +\infty]$. The lower boundary on quantities implying negative production has no meaning in real markets. Nor has it any implications for the result, so following Newbery (1998) we let this interval range from 0 to $+\infty$. The supply function in this case would be $S_i : [0, \infty] \rightarrow [0, \infty]$. Also, we shall require that the bid curves are monotonically increasing or as Baldick and Hogan (2001) puts it, that the supply functions must be non-decreasing.

If there are capacity constraints related to power production at a particular time, only supply functions mapping prices into the interval $[0, \overline{q_i}]$ would be feasible. Furthermore, following Baldick and Hogan (2001) (if the marginal costs are not normalized to 0) no firm would be willing to submit any bids at prices below operating costs corresponding to 0 output. A good candidate for the price minimum would be $\underline{p} = v_i$. In addition, with linear demand we would have a price $\overline{p}$ corresponding to $D(\widehat{p}, t) = 0$ at $t = 0$. With our definition of demand this price is $\overline{p} = N(0)/b$, called the "choke price". The supply function does not have to be defined for higher prices. Baldick and Hogan (2001) also discuss price caps and bid caps. This problem will be omitted here.

$\underline{p} = v_i$ may be a good candidate for the minimum price in a thermal system, but will a Hydro-producer be satisfied with this price? If the energy constraint is not effective then the producer might want to produce at this price, but if the energy constraint is binding then receiving v_i would not make the firm produce. The reason is that production has a positive alternative value in production at other times. This might be seen as lost income or cost at the time in question. If the number of load duration periods are large, then production within one particular load duration period have a

small effect on this alternative value. We assume we can neglect this effect and let λ denote the alternative value which is exogenous when looking at a particular load duration period. The relevant minimum price to a hydro-producer would have to cover operating cost corresponding to 0 output and the opportunity cost of producing in other periods, $\underline{p} = v_i + \lambda_i$.

Following these requirements, a feasible and allowable supply function for firm i is a function $S_i : [\underline{p}, \overline{p}] \to [0, \overline{q_i}]$.

2.4 Price, profit and equilibrium conditions

We need a market clearing condition. We assume that at each time $t \in [0, 1]$, the dispatcher chooses the lowest price $p(t)$ that clears the market. That is the price determined by

$$a(t) - bp = \sum_i S_i(p), \tag{6}$$

provided that such a price exist. This is a uniform price auction where all firms receive the marginal clearing price for their supply. We assume that the solution to our market clearing condition corresponds to prices within the range $[\underline{p}, \overline{p}]$ and do not consider prices outside this range.

The profit at time t for firm i is

$$\pi_{it} = S_i(p(t))p(t) - C_i(S_i(p(t))). \tag{7}$$

The profit over the whole load duration period is

$$\pi_i(S_i, S_{-i}) = \int_0^1 S_i(p(t))p(t) - C_i(S_i(p(t)) \, dt, \tag{8}$$

where $S_{-i} = S_{j \neq i}$.

Green and Newbery (1992) maximizes the contributed profit per unit time as defined in (7) and use the assumption[3] made by Klemperer and Meyer (1989) to justify that the resulting first order condition (13) would also maximize profit over the whole time horizon. To see how this is possible we follow the argumentation laid out by Baldick and Hogan (2001).

The first point is to note that for each firm i we consider that all the other j firms have commited to differentiable supply functions, S_j. This ensures a solution. Now, if we say that firm i is committed to supply the residual demand at any given price then we have that

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[3] The assumption is that firm i's residual demand at any price is the difference between demand and the quantity that other producers are willing to supply at that price.

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$$\forall t \in [0,1], q_{it} = Q(p(t),t) - \sum_{j \neq i} S_j(p(t)). \tag{9}$$

We can now rearrange equation (7) as follows:

$$\pi_{it} = p(t)[Q(p(t),t) - \sum_{j \neq i} S_j(p(t))] - C_i(Q(p(t),t) - \sum_{j \neq i} S_j(p(t))). \tag{10}$$

Since the supply functions S_j are assumed to be differentiable, we can derive the necessary conditions for maximizing the profit per unit time π_{it} at each time t over choices of $p(t)$.

$$\frac{\partial \pi_{it}}{\partial p(t)} = [Q(p(t),t) - \sum_{j \neq i} S_j(p(t))] + [p(t) - C_i'][\frac{\partial Q(p(t),t)}{\partial p(t)} - \sum_{j \neq i} \frac{\partial S_j(p(t))}{\partial p(t)}].$$
$$\tag{11}$$

Since we know that firm i will produce the difference between supply by other firms and total demand, we can find the price output pair optimizing firm i's production at time t by solving the first order condition with respect to q_{it}. Now, we assume that the relationship between price and quantity is monotonically non-decreasing over the time period. That is, a higher price means higher quantity and this quantity is unique. Then we can define a non-decreasing supply function for firm i that is infinitely differentiable.

$$S_i(p(t)) = q_{it}. \tag{12}$$

This function[4] also maximizes the integrated profit over the time horizon and can be calculated without reference to the load-duration characteristic.

$$S_i(p) = [p - C_i'][Q_p - \sum_{j \neq i} S_j'(p)], \tag{13}$$

where $Q_p = \frac{\partial Q(p(t),t)}{\partial p(t)}$ and $S_j'(p) = \frac{\partial S_j(p(t))}{\partial p(t)}$.

In the next subsection we develop a simple numerical example to be used as a benchmark throughout our analysis.

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[4]The solution to this first order differential equation is shown in Appendix A.

2.5 Numerical example: Two firms with marginal cost normalized to 0

Let us consider the solution to the SFE model in a situation with two symmetric firms, linear demand and constant marginal cost normalized to 0. This gives us the following coefficient and term related to the differential equation derived in Appendix A, equation (26):

$$u(p) = -\frac{1}{p}, \qquad w(p) = -b.$$

The supply function is

$$q(p) = e^{-\int \left(-\frac{1}{p}\right) dp} \left[A + \int -b e^{\int \left(-\frac{1}{p}\right) dp} dp \right]. \tag{14}$$

Rearranging (14) by $\int \frac{1}{x} dx = \ln x + c$ and $\int -f(x) dx = -\int f(x) dx$ we get:

$$q(p) = e^{\ln p + c} \left[A - b \int e^{-\ln p + c} dp \right].$$

Furthermore, by $e^{-\ln x} = \frac{1}{x}$, $\int \frac{1}{x} dx = \ln x + c$ and including the constants c in the term A the supply function becomes

$$q(p) = pA - pb \ln p. \tag{15}$$

According to Klemperer and Meyer (1989) any SFE where the firms do not know the uncertainty is intermediate between Cournot and Bertrand equilibrium levels. Thus, we know that the intersection between the supply curve and the demand curve would have to lie below the Cournot level and above the Bertrand level. However, in order to define the SFE we need a boundary solution. Here, we follow Green and Newbery (1992) and use the Cournot equilibrium as the boundary solution in order to define the SFE. The resulting supply schedule is called the Cournot supply schedule.

With linear demand, $Q = a(t) - bp$, we then write the first order Nash-Cournot solution for two symmetric players $i = 1, 2$ where the marginal cost have been normalized to 0. For maximum demand this solution will form an upper boundary on the supply function.

$$q_i = \frac{a}{3} \text{ and } p = \frac{a}{3b}. \tag{16}$$

We use this result to calculate the value of A by inserting the solution into the supply function (16).

$$\left(\frac{a}{3}\right) = \left(\frac{a}{3b}\right)A - \left(\frac{a}{3b}\right)b\ln\left(\frac{a}{3b}\right),$$

rearranging,

$$A = b + b\ln\left(\frac{a}{3b}\right),$$

furthermore, by $\ln\left(\frac{a}{b}\right) = -\ln\left(\frac{b}{a}\right)$,

$$A = b[1 - \ln\left(\frac{3b}{a}\right)], \tag{17}$$

and replacing A in the supply function (16) we get

$$q(p) = pb[1 - \ln\left(\frac{3b}{a}\right)] - pb\ln p. \tag{18}$$

We let $a = 56$ at peak demand when $t = 0$, $b = 0.3$ and $Q = 2q$, where $q_1 = q_2$[5]. We can now calculate the (Nash-Cournot) upper boundary for the symmetric duopoly, $(Q, p) = (37, 62)$. We observe that the aggregated supply function should pass through the origo $(p = 0)$ and that $\frac{dq}{dp} = 0$ at the boundary solution. The solution is shown in figure 1 along with the downwards sloping linear demand curve corresponding to the hour with highest demand $(t = 0)$.

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[5] The numerical values chosen here compares to the values used in an example in Newbery (1998).
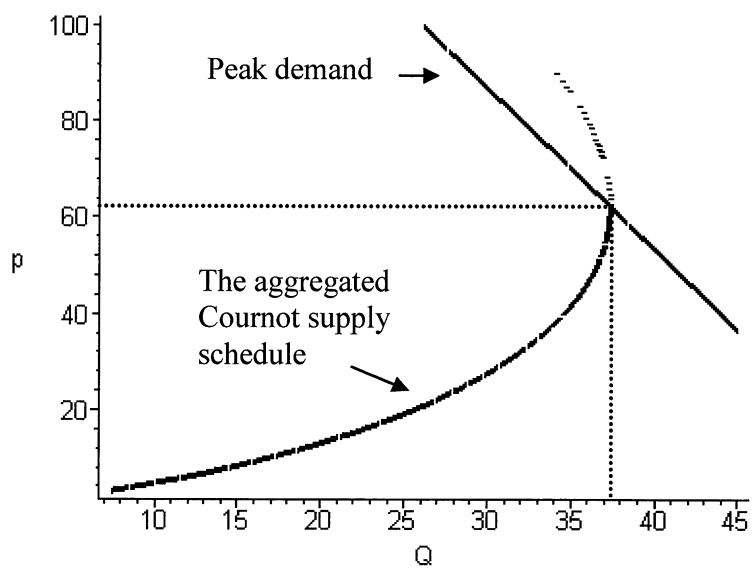
Figure 1: The figure shows the aggregated supply curve resulting from an SFE equilibrium with two symmetric firms, zero marginal cost and assuming a Cournot solution at the period of highest demand.

3 Constraints on power and energy produced

In this section we look at how constraints on power and energy produced, limit the number of possible SFE. The first issue is described in Newbery (1998) and in Baldick and Hogan (2001). The second issue mentioned above, is more relevant to a hydropower based electricity system. First, we briefly restate the model used by Newbery (1998). In subsection 3.2 we discuss constraints on energy produced.

3.1 Power constraints

We look at the symmetric duopoly case. Now, however we introduce the power constraint from equation (4) above. Neither firm can supply more than a specific capacity $q = \overline{q}$.

With linear demand $Q = a(t) - bp$ and constant marginal cost c, the maximization problem becomes:

$$\pi_i = (p - C(q_i))q_i + \mu(\overline{q} - q_i).$$

We then find the Nash-Cournot solution for two symmetric players assuming positive output values. For maximum demand this solution will form an upper boundary on the supply function.

$$\frac{a - b(c + \mu)}{3} = q_i \text{ and } \frac{a + 2b(c + \mu)}{3b} = p.$$

If we assume a positive shadow value on the constraint, then we have that $q_1 = q_2 = \overline{q}$ and the price $p = (a - 2\overline{q})/b$. We use this result to calculate the value of A.

$$\overline{q} = (\frac{a - 2\overline{q}}{b} - c)A - (\frac{a - 2\overline{q}}{b} - c)b\ln(\frac{a - 2\overline{q}}{b} - c),$$

rearranging,

$$A = b[\frac{\overline{q}}{a - 2\overline{q} - bc} + \ln(\frac{a - 2\overline{q} - bc}{b})]. \tag{19}$$

By replacing A from (20) in the supply function (16) we get that

$$q(p) = (p - c')b[\frac{\overline{q}}{a - 2\overline{q} - bc} + \ln(\frac{a - 2\overline{q} - bc}{b})] - (p - c')b\ln(p - c'). \tag{20}$$

If we let $a = 56$, $b = 0.3$, $c = 19$ and $\overline{q} = 20$, we can calculate the upper boundary $(Q, p) = (40, 53.3)$ corresponding to the intersection between the
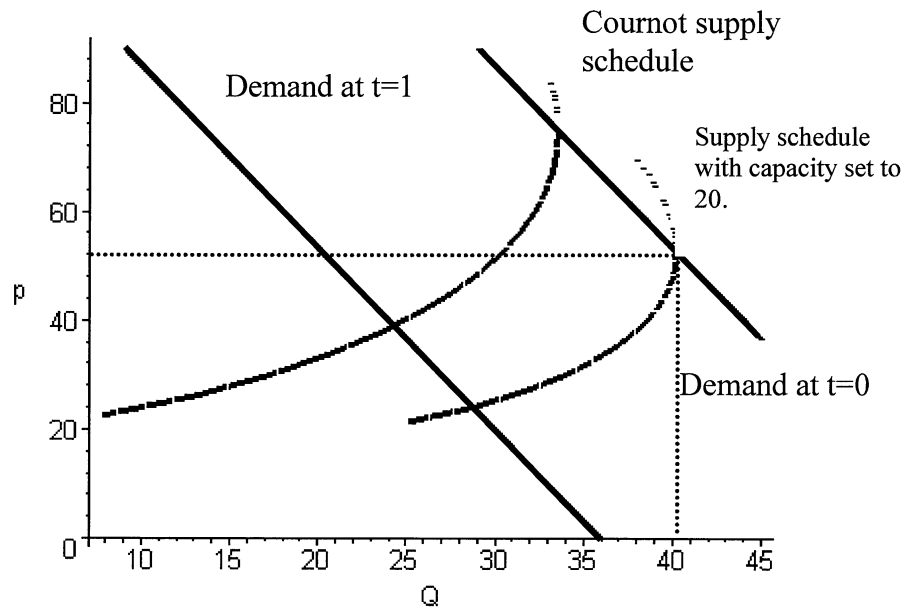
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Figure 2: Here we compare the Cournot supply schedule to the supply schedule where producers are facing a binding production constraint. Any supply schedule below the supply schedule at $\bar{q} = 20$ would violate the production constraint and are thus not feasible.

capacity limit and the demand curve. The solution is shown in figure 2 along with the downwards sloping linear demand curves for the lower and upper support and together with the supply function where the upper boundary is set to the Nash-Cournot solution.

The example shown in figure 2 is similar to the one shown in figure 4 in Newbery (1998). Newbery finds that the effect of a capacity constraint on power is to reduce the number of feasible supply function equilibria. The allowable supply functions have to lie between the Cournot supply schedule and the supply scheduele which intersect the capacity constraint on power in the hour of highest demand. Thus, more competitive bidding strategies are ruled out.

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3.2 Constraints on energy produced

In this subsection we shall look at the competition between two symmetric hydropower producers.

The problem for each of the two hydro-producing firms is almost identical to the symmetric two-firm problem described in subsection 2.1. The only difference is that marginal cost now is represented by the firms water value, λ. For now we shall assume this value to be fixed.

$$q(p) = (p - \lambda)A - (p - \lambda)b\ln(p - \lambda). \tag{21}$$

Let us use the same numerical values as in the example of the previous section, namely $a(t; t = 0) = a_0 = 56$, $b = 0.3$ and $\lambda = 19$ and calculate the (Nash-Cournot) upper boundary, $(Q, p)=(33.53, 74.8)$. The aggregated supply function for the hydro-producing firms would be identical to the one described in figure 2.

Using a simple example we can measure exactly how much energy is produced. We simplify to three periods where $a_0 = 56$, $a_{0.5} = 46$ and $a_1 = 36$ and each of the three periods have a duration of one hour. If the producers submit Cournot supply schedules we have already seen that the total production of energy in period 0 is 33.5 GWh. In the next period we find the price simply by setting the aggregated supply function equal to demand in this period:

$$(p - \lambda)b[1 - \ln(\frac{3b}{a_0 - b\lambda})] - (p - \lambda)b\ln(p - \lambda) = a_1 - bp \tag{22}$$

This gives us the values of price and total energy in period $t = 0.5$, (Q, p) = (30.31, 52.3). For the last period with the lowest demand, the quantity price pair would be $(Q, p)=(24.3, 39)$. In total, the amount of energy produced during the load duration period under this strategy is approximately 88 GWh. Now, we let each of the producers have k times 44 GWh available for production during all k load duration periods. Furthermore, we assume all load duration periods to be identical. Then if they choose to submit Cournot supply schedules, we see that the existing energy capacity and corresponding water value do not constrain their bids.

However, more competitive supply schedules would violate the energy capacity constraint. Assuming a fixed alternative value of water ($\lambda = 19$) and that the energy constraint is just binding in the case of two symmetric hydro power producers submitting Cournot supply schedules, then any pair of symmetric supply schedules below the Cournot schedules (more competitive) would violate the energy constraint and would thus not be feasible.

More competitive supply schedules would imply that the realized price corresponding to realized demand would lie below the price resulting from Cournot supply schedules. Since demand is downwards sloping this implies that quantity is higher for all realizations of demand. Thus, with more power produced throughout the load duration period, more energy is produced as well and the energy constraint is violated.

3.3 Introducing a competitive supply schedule

With a a fixed alternative value of water, we saw in the previous subsection that any supply schedule based on a more competitive solution than the Cournot equilibrium at the period of highest demand would violate the energy constraint. This does not mean that a more competitive strategy is ruled out, just that the producer's water is to low. With reference to the numerical example in the previous subsection, if a more competitive strategy is followed, then there would not be enough water for production in all k load duration periods. The combination of lower prices and higher supply compared to the Cournot supply schedule would leave water for production in only some of the k periods in question. Thus, when firms bid more competitive supply schedules, they would take this into account and place a higher alternative value on the use of water resources.

We use the same numerical example as in the previous subsection and compare the Cournot supply schedule to the most competitive schedule possible where price equals marginal cost. This is shown in figure 3.

As can be seen from figure 3, the water value or marginal cost in the case of a competitive supply schedule would have to be as high as 55.5 if the energy constraint is not to be violated. This implies that prices are higher and supply is lower at periods of low demand while prices are lower and supply is higher at periods of high demand compared to the Cournot supply schedule.

Recognising that all possible SFE are intermediate between the Cournot supply schedule and a supply schedule corresponding to Bertrand competition at the boundary solution, we see that an energy constraint will reduce the number of possible SFE.

4 Concluding remarks

The idea that firms compete in supply functions and the SFE model developed by Klemperer and Meyer (1989) fits quite closely to how firms compete in electricity markets. Firms face uncertainty with regard to demand at the
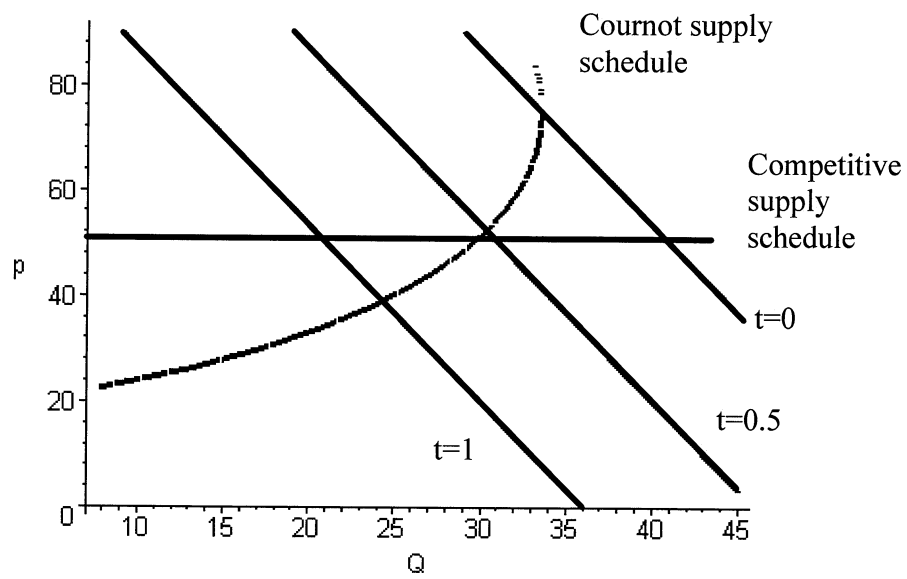
Figure 3: Here we compare the Cournot supply schedule to a competitive schedule. In both cases the total amount of energy produced is 88 GWh throughout the load duration period.

time of trade, but are nevertheless required to deliver binding supply schedules before trading actually takes place.

Thus, in spite of the technical difficulties associated with the SFE framework, several authors have tried to use this model as a basis for analyzing strategic behavior in electricity markets. In several papers R. J. Green and D. M. Newbery have used the SFE model in order to analyze electricity market competition in England and Wales. The success of their application of the SFE model relates to a large extent to the market structure of the England and Wales electricity market. Following deregulation, this market consisted for a significant period of time of only two rather symmetric firms. In this setting the SFE model is relatively easy to apply.

One central problem associated with the SFE model is that the model only predicts a range possible equilibria depending on the boundary solution. Following the argumentation of Klemperer and Meyer (1989) and Green and Newbery (1992) we define an upper boundary where Cournot quantities and prices are realized at the period of highest demand. The lower bound corresponds to Bertrand price and quantities at the same period. This leaves a rather wide range of possible SFE equilibria.

However, as shown by Newbery (1998) the number of possible SFE equilibria is reduced when we take into account that firms face production constraints. In addition it was shown that the existence of forward contracts could also contribute to a reduction in the number of possible SFE equilibria. Constraints on energy produced however has not been considered as this is a special feature of hydropower production.

In this working paper we have, by the use of a simple numerical example, illustrated that constraints on energy produced may contribute to a further reduction in the number of possible SFE equilibria. In particular, we looked at a case where the energy constraint was just binding when the SFE equilibrium was characterized by Cournot quantities and price at the upper boundary. Furthermore, we illustrated that a more competitive schedule would imply higher water values with more energy produced during periods of high demand and less during periods of low demand.

A   Solving the differential equation

Here we look at how to solve the first order conditions for a SFE when having two and n symmetric firms in the market. We start with the case of two symmetric firms. Firm i's (the same for firm j) profit is

$$\pi_i(p) = p[Q(p,t) - q_j(p)] - C\{Q(p,t) - q_j(p)\},$$

where $i \neq j$ and we assume that there are constant marginal costs in production (c). The residual demand facing producer i is

$$q_i = Q(p,t) - q_j(p).$$

The first order condition is

$$\frac{\partial \pi_i}{\partial p} = q_i + [p - c][\partial Q/\partial p - \frac{\partial q_j}{\partial p}].$$

Knowing that $q_i = q_j$ we only need to solve one equation,

$$\frac{dq}{dp} - \frac{q}{p - c} + Q_p = 0.$$

This is a differential equation. The equation may be written in the following form:

$$\frac{dq(p)}{dp} - u(p)q = w(p).$$

where $u(p)$ is the coefficient and $w(p)$ is the term of the differential equation. The equation may be described as a first order linear differential equation with variable coefficient,

$$u(p) = \frac{1}{p - c}$$

and constant term where we assume that $Q_{pt} = 0$,

$$w(p) = \frac{dQ}{dp}.$$

In order to solve this equation we need first to find out if the equation is exact, we form a new equation

$$dq + dp(uq - w) = 0.$$

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If we let $M = df/dq$ and $N = df/dp$ we can see that the equation is not exact. That is

$$\frac{\partial M}{\partial p} \neq \frac{\partial N}{\partial q}.$$

Thus, we need to multiply the equation with a factor that makes the equation exact. The factor is

$$e^{\int u\,dp},$$

resulting in

$$e^{\int u\,dp}dq + e^{\int u\,dp}(uq - w)dp = 0. \tag{23}$$

Now the equation is exact

$$\frac{\partial M}{\partial p} = ue^{\int u\,dp}, \quad \frac{\partial N}{\partial q} = ue^{\int u\,dp}.$$

In order to solve the equation (24) we integrate the first part and add on a second element representing the other part.

$$F(q,p) = \int e^{\int u\,dp}dq + \psi(p).$$

(by $\int k\,dx = k(x + d)$ where we omit the constant of integration d). We can rewrite this as

$$F(q,p) = qe^{\int u\,dp} + \psi(p). \tag{24}$$

In order to find the value of the second term, we differentiate the function above (25) with respect to the variable p

$$\frac{\partial F}{\partial p} = uqe^{\int u\,dp} + \psi'(p)$$

Since we know the value of $N = e^{\int u\,dp}(uq - w)$ and that this equals the partial differentiate of F with respect to p, we can write the equation as follows:

$$e^{\int u\,dp}(uq - w) = uqe^{\int u\,dp} + \psi'(p).$$

and we can find $\psi'(p)$

$$\psi'(p) = -we^{\int u\,dp}.$$

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Now the solution to $F(q,p) = d$ becomes

$$uqe^{\int u\,dp} - \int we^{\int u\,dp}dp = A.$$

And if we solve for q we get (here $d = A$)

$$q(p) = e^{-\int u\,dp}\left[A + \int we^{\int u\,dp}dp\right]. \tag{25}$$

This is the general solution to a first order differential equation of the first degree.

Next, we look at how to reach the general solution in the case of n symmetric firms. Each of the n firms face the same problem

$$\pi_i(p) = p[Q(p,t) - \sum_j q_j(p)] - C\{Q(p,t) - \sum_j q_j(p)\},$$

where $i \neq j$ and as before we assume that there are constant marginal costs in production (c). The residual demand facing producer i is:

$$q_i = Q(p,t) - \sum_j q_j(p)$$

Knowing that the firms are symmetric we have that $\sum_j \frac{\partial q_j}{\partial p} = (n-1)\frac{\partial q_j}{\partial p}$. The first order condition then becomes

$$\frac{\partial \pi_i}{\partial p} = q_i + [p - c][\partial q/\partial p - (n-1)\frac{\partial q_j}{\partial p}].$$

Rearranging for the symmetric case, we have that

$$\frac{\partial q}{\partial p} = (\frac{1}{n-1})(\frac{q}{p-c} + Q_p).$$

We see that this is a differential equation similar to the one discussed above for the symmetric two-firm case

$$\frac{dq(p)}{dp} - u(p)q = w(p),$$

where the coefficient $u(p)$ and the term $w(p)$ in this case is

$$u(p) = -\frac{1}{(p-c)(n-1)}, \text{ and } w(p) = D_p\frac{1}{n-1}.$$

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