

The Equity Premium Puzzle and Stochastic Population

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Abstract

This thesis aims to examine the link between the equity premium and demographic uncertainty. First I will present the theoretical background for the equity premium puzzle and overlapping generations models, before building an overlapping generations model; with two stochastic variables, population growth and technology. The model is a standard general equilibrium model, where agents maximize their objective functions, subject to some constraints. The stochastic variables are jointly log-normally distributed. Derivations are shown in detail to make it easy to read. Lastly I calibrate the model. The calibration shows that the stochastic population cannot account for the high equity premium. The results are similar to those of Mehra and Prescott (1985) and others, predicting that equity premium will be less than 1%.

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“Demographics are about everyone: who you are, and where you’ve been and where you are going. Demographics explain about two-thirds of everything: which products will be in demand, where job opportunities will occur, what school enrollments will be, when house values will rise or drop, what kinds of food people will buy and what kinds of cars they will drive. The further ahead in the future you are looking, the more relevant demographics will be to you.”

(David K. Foot –Boom, Bust and Echo:1996)

Contents

Introduction.....	6
1. Expected Utility Framework.....	8
Risk Aversion.....	9
Asset Pricing.....	10
Consumption-Based Capital Asset Pricing, the CCPM.....	11
The Equity Premium Puzzle.....	13
Overlapping Generations Models.....	17
Equity Premium and Population.....	19
2. Diamond Model with Log-normally Distributed Population Growth and Productivity Growth.....	20
Assumptions.....	20
The Model.....	23
Stochastic Population.....	23
The Stochastic Production Sector.....	24
Households.....	29
Equilibrium Capital Accumulation.....	32
Marginal Product of Capital.....	41
The Riskless Rate.....	44
Equity Premium.....	46
3. Calibration.....	48
The Case of Norway.....	49
The case of the US.....	54
Analysis.....	60
Conclusion.....	65
Appendix.....	66
Bibliography.....	75

Tables and Figures

Table 1: Summary statistics for the logarithms of the growth factors in technology and population, Norway 1973-2005.....	52
Table 2: Equity premium on different data set in the US.....	57
Table 3: Summary statistics for the logarithms of the growth factors in technology and population, US 1949-2005.....	58
Table 4: The distributions of the growth factors, Norway and the US.....	61
Figure 1: Life cycle and decisions.....	31
Figure 2: Phase diagram in the (k_{t+1}, k_t) -space.....	36
Figure 3: Transition to steady state.....	41
Figure 4: Employed 1000 persons 1972-2005, Norway.....	49
Figure 5: Growth factor of employed persons, 1973-2005.....	49
Figure 6: Annual stock market return and risk free rate in Norway 1960-2005.....	51
Figure 7: Growth factor of technology in Norway 1973-2005.....	52
Figure 8: Employed 1000 persons 1948-2005.....	54
Figure 9: Growth factor of employed persons, 1948-2005.....	55
Figure 10: Real annual return on S & P 500, 1889-2000.....	56
Figure 11: Real annual return on relatively riskless asset 1889-2000.....	56
Figure 12: Capital and labor share in the US 1929-1999.....	58
Figure 13: Growth factor of technology in the US 1949-2005.....	58
Figure 14: US equity premium as a function of rho.....	61
Figure 15: US equity premium as a function of rho and the covariance.....	62
Figure 16: Norwegian rate of risky return as a function of alpha and rho.....	62

Introduction

The aim of this thesis is to analyze the consequences of a stochastic population growth on the equity premium in an overlapping generations (OLG) model. This to see if this stochastic variable can account for part of the Equity Premium Puzzle. The population in an economy changes due to births, deaths and migration. The risks of demographic change, here interpreted as fertility risk, resulting in a baby boom or a baby bust, may be significant and it is not insurable in the market. Thus fertility risk requires a premium to be born. As documented by Davis and Li (2003), the patterns of the elderly dependency ratio are largely a consequence of changes in fertility, although longevity are also important.

The model incorporates a stochastic growth production sector. Economic growth is exogenous. Asset returns are determined by time preference, the marginal utility of wealth and attitudes toward risk. In the case of a small open economy, the asset returns are determined independently of the rate of growth¹, but in a closed or in a large open economy, they may be linked.

The idea is motivated by the assertion that the entry of the baby boom generation, those born roughly in the two decades following World War II, into its peak saving years was a key explanatory factor in the rise of stock market values in the 1990s. Examples are Passell (1996) and Moon et. al. (1998).² Individuals aged 40 to 60 years old are the prime savers in the economy in the US. That prices of stocks and other real assets are bid up are accompanied by the prediction that when the Baby Boomers reach retirement, they start consuming their savings (selling their assets) which result in declining asset prices and increasing expected returns.

¹ A small open economy takes the rate of return as exogeneous, given from abroad. According to Poterba (1998) shifts in the demand for financial assets in a small open economy, resulting from a demographic change, changes the amount of capital owned by the country's inhabitants, but not the capital-labor ratio or the rate of return.

² Analogously Mankiw and Weil (1989) argues that the increase in homebuying population –people in their late twenties and thirties- explains part of the increased real house prices at the late 1970s and early 1980s.

Population (forecasts) are widely used in various planning situations, such as schooling, health care and pension systems. In the very short run, the uncertainty expressed by stochastic forecasts is limited. On a five-year planning horizon you may safely use a deterministic forecast. In the long run however, planners interested in the age structure of the population 30 or more into the future, should take uncertainty into account.

1. Expected utility framework

The usefulness or satisfaction from an outcome x is in economics typically modeled through a utility function. A utility function $u(x)$ assigns a numerical value to each outcome in X , the set of possible outcomes, ranking the elements in accordance with the individual's preferences. The purpose of an ordinal³ utility function is to rank the outcomes from least to most preferred. For the preference relation \succ , to be rational⁴, preferences must be complete, i.e. all outcomes are ranked and transitive, i.e. if A is preferred to B , and B to C , then A must be preferred to C .

Mas-Colell et. al. distinguish between utility functions $U(.)$ defined on lotteries, referred to as von-Neumann-Morgenstern (v.N-M) expected utility functions, and utility functions $u(.)$ defined on sure amount of money, named Bernoulli utility functions. To apply this framework to the study of preferences over risky alternatives (v.N-M utility functions), in addition to the assumptions for rationality, the preference relation has to satisfy the continuity and independence axioms. Continuity means that small changes in probabilities do not change the nature of ordering between the lotteries. Independency refers to independence between lotteries, that is if two lotteries are combined equally with a third one, then the ordering of the two mixed lotteries should be independent of the particular third one used. If fulfilled, then the expected utility theorem says that the decision maker's preferences are representable by a utility function with the expected utility form. The expected utility is the mathematical expectation over the Bernoulli utilities of the realizations

$$U(X) = \sum_i u(x_i) p_i$$

where p_i is the probability of outcome x_i . The v.N-M expected utility theorem is crucial to a vast literature in economics, but it is not without difficulties. There are several paradoxes and challenges to the expected utility framework⁵.

³ Ordinal meaning that what is important is the ranking of the outcomes, in contrast to the cardinal utility which gives the absolute satisfaction of how much an outcome is preferred to another.

⁴ Rationality is a normative concept, stating how to make decisions

⁵ To mention some: The Allais paradox violates the independence axiom because typical preferences here appear to cycle. The Ellsberg paradox also violates the independence axiom, incorporating subjective probabilities to the model. The assumption of completeness may fail if it is hard to evaluate the alternatives.

Risk aversion

An individual who is risk averse prefers a certain given income to a risky income with the same expected value. He is risk neutral if he's indifferent between the two. In the context of expected utility theory, risk aversion is equivalent to the concavity of $u(\cdot)$. The risk neutral expected utility is linear $u(x) = x$.

Expected utility is typically defined over consumption or indirectly over final wealth. The desirability of more is captured by a positive marginal utility $u'(\cdot) > 0$. Risk aversion by concavity $u''(\cdot) < 0$. Strict concavity means that marginal utility of money is decreasing. At any level of wealth, the gain of an additional unit is less than the loss of the last unit obtained. The degree of risk aversion is measured by the Arrow-Pratt coefficient of absolute risk aversion, defined as

$$r_A(x) = -\frac{u''(x)}{u'(x)}$$

and by the coefficient of relative risk aversion, given by $r_R(x) = xr_A(x)$. The Arrow-Pratt coefficient measures the rate at which the probability premium, the excess in winning probability over fair odds to be indifferent between a certain outcome and a fifty-fifty gamble with the same expected value $(x+e, x-e)$, increases with the small risk e . The relative risk aversion shows how risk aversion varies with wealth. I will later make use of the constant relative risk aversion (CRRA) class of utility functions, in which relative risk aversion is independent of wealth, i.e. $r_R(x)$ is constant⁶.

Most people are risk averse most of the time, they buy insurance of different kinds and they seek occupations with relatively stable wages.⁷ In economic literature, utility is often represented by a standard concave utility function such as the CRRA-class mentioned.

⁶ The CRRA utility function is given by $U(c, \theta) = \frac{c^{1-\theta}}{1-\theta}$ and then $r_A(c) = -\frac{c(-\theta)c^{-\theta-1}}{c^{-\theta}} = \theta$ is constant.

⁷ Kahneman and Tversky (1979) empirically found that people are risk averse over gains, but risk loving over losses. They constructed Prospect Theory on basis of their empirical findings.

With the standard expected utility representation, risk and time preferences are closely linked. Time preferences are defined over the marginal utility of consumption over two points in time, which in the case of CRRA gives $\frac{\partial u(c_1)}{\partial u(c_0)} = \frac{1}{1 + \rho} \left(\frac{c_1}{c_0} \right)^\theta$ where ρ is the discount factor of future consumption (to be discussed in more detail later). Risk preferences are defined over the marginal utility of consumption over two states s and z , for the CRRA case $\frac{\partial u(c_s)}{\partial u(c_z)} = \left(\frac{c_s}{c_z} \right)^\theta$. Thus for the CRRA class the coefficient of relative risk aversion and the elasticity of intertemporal substitution are reciprocal to each other (Selden, 1978), and if the utility function is logarithmic time and risk preferences coincide.⁸

Asset pricing

The asset pricing models of financial economics describe the prices and expected rates of return of financial assets which are claims traded in financial markets. Examples of financial assets are stocks, bonds and options. Pricing models are typically based on either of two fundamental assumptions, arbitrage or equilibrium models. The no arbitrage principle states that market forces set prices to eliminate arbitrage opportunities. An arbitrage opportunity exists if assets can be combined in a portfolio with zero cost, no chance of a loss and a positive probability of gain (a free lunch).

In a financial market equilibrium, the investor's desired investment is derived from an optimization problem. The first order conditions for the investor require that he on the margin is indifferent to small changes in asset holdings. The market-clearing condition states that the aggregate of investor's wanted allocations must be equal to the aggregate "market portfolio" of securities in supply. A general equilibrium requires that prices and quantities are decided simultaneously, a partial equilibrium takes some prices or quantities as given ("given" usually in the sense of viewed as already a competitive result).

⁸ Epstein and Zin (1989) has found an expected utility representation separating the two aspects, the elasticity of intertemporal substitution ψ , and the relative risk aversion θ .

The purpose of asset pricing models is to value uncertain future cash flows at some point in time. Price equals its expected discounted payoff. The valuing have to account for the delay and the risk of the cash flow. Following Cochrane (2005) a basic pricing equation can be represented

$$p_t = E_t [m_{t+1} x_{t+1}]$$

where the price today p_t is the product of the stochastic cash flow x_{t+1} and the stochastic discount factor (SDF) m_{t+1} conditional on the information held by the consumer at time t .

There are several methods developed for valuation of financial assets⁹. By the so-called fundamental theorem of asset pricing¹⁰, they are equivalent in the sense that one method is applicable if and only if some of the others are. The SDF is convenient in consumption-based models because it is also the intertemporal marginal rate of substitution.

Consumption-based capital asset pricing, the CCAPM¹¹

To see why the SDF is appropriate in this context, consider an economy in which there is one productive unit producing the period dividend y_t in period t . There is one equity share with price p_t which is the claim to the stochastic process $\{y_t\}$, competitively traded. The intertemporal choice of a typical investor at time t is to equate the loss in utility associated with buying an additional unit of equity. To obtain one additional unit of equity, p_t units of consumption must be sacrificed, giving a loss in utility of $p_t U'(c_t)$. Selling this additional equity next period will yield

⁹ Examples are the time state preference model (TSP), risk free discounting of the certainty equivalent, the stochastic discount factor (SDF) and the single risk adjusted discount rate (RADR).

¹⁰ According to the fundamental theorem of asset pricing the following are equivalent:
-the absence of arbitrage

-the existence of a positive linear pricing rule (state prices)

-the existence of an optimal portfolio for some agent who prefers more to less

¹¹ The present model is a general equilibrium model in contrast to the older partial equilibrium capital asset pricing model (as in Sharpe(1964)). In the CAPM model assets whose returns are positively correlated with the world market portfolio (taken as already a competitive result) must offer a higher expected return.

$p_{t+1} + y_{t+1}$ to be consumed. The incremented consumption next period has the expected utility $\frac{1}{1+\rho} E_t \{ (p_{t+1} + y_{t+1}) U'(c_{t+1}) \}$

Hence the fundamental relation that prices assets, called an intertemporal Euler equation¹², is:

$$p_t U'(c_t) = \frac{1}{1+\rho} E_t \{ (p_{t+1} + y_{t+1}) U'(c_{t+1}) \}$$

Or if expressed by means of return

$$1 = \frac{1}{1+\rho} E_t \left\{ \frac{u'(c_{t+1})}{u'(c_t)} (1 + \tilde{r}_{t+1}) \right\}$$

Where the return is $(1 + \tilde{r}_{t+1}) = \frac{p_{t+1} + y_{t+1}}{p_t}$

And it can be expressed by means of the stochastic discount factor (SDF), mentioned above

$$1 = E_t [m_{t,t+1} (1 + \tilde{r}_{t+1})] \text{ where } m_{t,t+1} = \frac{1}{1+\rho} \frac{u'(c_{t+1})}{u'(c_t)}$$

The SDF is also the rate of marginal substitution (MRS(t+1,t)), the rate at which the consumer is willing to trade consumption tomorrow¹³ for consumption today. This is the ratio of the marginal utility of getting a bit more income at date t+1, $\frac{1}{1+\rho} u'(c_{t+1})$

to the marginal utility of losing a bit at date t, $u'(c_t)$. If future consumption is very valuable to you, then your MRS will be higher; you weight the future benefits $(1 + \tilde{r}_{t+1})$ strongly.

The Euler equation thus links two endogenous variables, the consumption and the rate of return. For the riskless one period bond the analog expression is

$$1 = \frac{1}{1+\rho} E_t \left\{ \frac{u'(c_{t+1})}{u'(c_t)} \right\} (1 + r_{t+1}^f)$$

¹² From the Swiss mathematician Leonard Euler (1707-1783). The dynamic equation arose originally in the problem of finding the so-called brachistochrone, which is the least-time path in a vertical plane for an object pulled by gravity between two specified points.

¹³ Or any else time s in the future. Then

$$m_{t,s} = \left(\frac{1}{1+\rho} \right)^{s-t} \frac{u'(c_s)}{u'(c_t)}$$

Expected return on equity can be written¹⁴

$$E_t(1 + \tilde{r}_{t+1}) = 1 + r_{t+1}^f + \text{cov}_t \left\{ \frac{-u'(c_{t+1})(1 + \tilde{r}_{t+1})}{E_t[u'(c_{t+1})]} \right\}$$

Expected asset returns equal the riskfree rate plus a premium for bearing risk, which depends on the covariance of the asset returns with the marginal utility of consumption. Idiosyncratic risk is not priced, it is the covariance between payoffs and marginal utility that matters, not the variance of payoffs. Assets that covary positively with consumption, i.e. assets that pay off in states when the consumption is high and the marginal utility is low, command a high premium because they destabilize consumption by exaggerating the state of the economy. Conversely, if an asset has high return when consumption is low (that is when marginal utility of consumption is high), the covariance term is positive and the asset's expected return may be below the riskless rate of interest, i.e. a negative risk premium. This because the asset has a value as consumption hedge and therefore will command a price above its "risk-neutral" price.

The Equity premium puzzle

"The equity premium puzzle" is a phenomenon discovered and dubbed by Mehra and Prescott (1985)¹⁵. They found that the historical U.S. equity premium (the return earned by a risky security in excess of that earned by a relatively risk free allocation) was far too great to be rationalized in the standard neoclassical paradigm¹⁶ of financial economics. The question they investigated was whether the magnitude of the covariance (the CCPM pricing equation) between the marginal utility of consumption and the stochastic return of the equity market was large enough to fit the observed 6% equity premium in the US. Stocks are obviously riskier than bills, having a standard

¹⁴ The derivation is given in Appendix A.

¹⁵ The original statement is presented in Appendix B.

¹⁶ E. Roy Weintraub defines neoclassical economics to rest on three assumptions: i) people have rational preferences among outcomes that can be identified and associated with a value ii) individuals maximize utility and firms maximize profits and iii) people act independently on the basis of full and relevant information. The basic assumptions imply equilibria, which are the solutions of the maximization problems. The equity premium is the premium for bearing additional non-diversifiable risk.

deviation of the returns about 20% a year contrary to 4%¹⁷. But do bearing this additional risk require a premium that large?

Mehra and Prescott find that it does not. Their analysis employ a variation of Lucas (1978) pure exchange model, a partial equilibrium model treating the growth rate of the endowment/consumption as an exogenous variable following a Markov process and asset prices as endogenous. When calibrating their model, the U.S. per capita real consumption of non-durables and services, its mean, variance and serial correlation is defined by a two states symmetric transition probability matrix. The paper defines and establishes the existence of a Debreu (1954) competitive equilibrium with a price system having a dot product representation under certain conditions. Consumption is stationary in growth rate, unconditional prices and returns are stationary. The single representative household has utility of the CRRA-class. They use real return on the S&P 500 Stock Price Index as the stochastic return and short-term government T-bills as the real riskless return for comparison to the calibrated model. They calculated predicted risk premium, restricting the values of the coefficient of relative risk aversion α ¹⁸ and the subjective discount factor β ¹⁹ based on evidence from various studies. The model does not fit the historical data on equity premium. The observed riskfree return of 0.80 % and the equity premium of 6 % is inconsistent with the predictions of the model. Largest premium obtainable within the model is 0.35%. To fit the historical data they have to relax the restrictions on the coefficient of relative risk aversion, which results in an extremely high degree of risk aversion. Intuitively, if people are more risk averse then equilibrium features higher expected returns on equity to compensate them for bearing risk. The high value of α required to fit the historical data implies an unacceptable high risk-free rate, which is the risk free rate puzzle, Weil (1989). Weil argues that households would need to have a negative subjective time discount rate to reproduce the historically low risk-free rates.

¹⁷ Historical U.S data from Mehra (2003). Other countries with significant capital markets yield similar differences.

¹⁸ “Certainly less than 10” (p59, Mehra, The Equity Premium: Why Is It a Puzzle?)

¹⁹ $\beta = \frac{1}{1 + \rho}$, between zero and one to place greatest weight on the first period.

The Equity Premium Puzzle is a quantitative puzzle, it is the order of magnitude that theory cannot account for. Various models attempt to explain the historical equity premium by adjusting or adding assumptions.

Research modifying preferences is either modifying the time-and-state-separability of utility or incorporating habit formation. Epstein and Zin (1989) presents “generalized expected utility” which allows separating the coefficient of relative risk aversion and the elasticity of intertemporal substitution. But to calibrate the model they have to make specific assumptions about the consumption process to obtain first-order conditions in observables. The framework decrease the risk-free rate puzzle, but it does not solve the equity premium puzzle. Internal habit formation, an approach initiated by Constantinides (1990), capture the notion that an individual’s utility is affected not only by current consumption but also by his past consumption. External habit formation means that utility depends on how one is doing relative to others (average per capita consumption). Habit models have also decreased the risk-free rate puzzle, but have had limited success in addressing the equity premium puzzle.

A model modifying probability distributions to admit rare but disastrous events in means of consumption, due to Rietz (1988), imply that the real interest rate and the probability of the occurrence of such an event move inversely. The perceived probability of such an event must have changed over time, it must have been low before 1945, the use of the atom bomb and higher after. But real interest rates have not moved as predicted by Rietz’s disaster scenario.

Market completeness is implicitly incorporated into asset pricing models by the assumption of a representative household. In complete markets heterogeneous households equalize, state by state, their marginal rate of substitution. Relaxing the assumption of complete markets, agents faced with uninsurable income shocks in an infinite-horizon model, will dynamically self-insure. Agents stock up on bonds when times are good and sell them when times are bad. Thus the difference between the equity premium in incomplete and complete markets is small (Heaton and Lucas 1996,1997). The difficulty of explaining the equity premium as a premium of bearing risk maybe because it is not a premium but rather due to other factors.

Bansal and Coleman (1996) use a monetary model where assets other than money play the role of facilitating transactions. They argue that Treasury bills (and monetary-like assets) may include a transaction-service component in their return. On the margin, the transaction service return of money relative to interest bearing checking accounts should be the interest paid on these accounts. They estimate this to be 6% based on the rate offered on NOW accounts. So the equity premium could in part be a liquidity premium, a premium demanded for illiquidity and not only a risk premium. But this is challenged by the facts that the majority of T-bills are held by institutions not as compensatory balances for checking accounts, the returns on NOW have varied, not in accordance with this model and the long term government bonds do in case have a significant transaction service component, which they shouldn't have.

Constantinides, Donaldson and Mehra (2002) impose borrowing constraints on the young in a three-generations overlapping model (see next section for a general description of overlapping generations models). The economy consists of the borrowing-constrained young, the saving middle-aged and the dissaving old. The young are prohibited from borrowing because human capital alone does not collateralize major loans. As noted by the CCAPM, the attractiveness of an asset depends on the correlation between consumption and equity income. Then as the correlation of equity income with consumption changes over the life cycle of an individual, so does the attractiveness of equity as an asset. A young person has both uncertain future wage and equity income and the correlation between of equity income with consumption is not particularly high²⁰. Equity is at this stage therefore a desirable asset to hold. For the middle-aged investors equity income is highly correlated with consumption and therefore requires a higher rate of return²¹. If equity is a desirable asset for the marginal investor, then the observed equity premium will be low relative to an economy in which the marginal investor finds holding equity not that attractive. In the presence of borrowing constraints, equity is held and priced by the middle-aged and the equity premium is high. The equity premium decreases when the borrowing constraint is relaxed, but the mean bond return roughly doubles, i.e. the risk-free rate puzzle is not solved.

²⁰ Empirically documented by Davis and Willen (2000)

²¹ Compare to the CAPM framework where the return on the market is a proxy for consumption. High-beta stocks pay off when the market return is high/marginal utility is low. Their price is relatively low and their rate of return high.

Recently McGrattan and Prescott (2001,2003) argue that some of the assumptions made in the original statement, Mehra and Prescott (1985), should be revised. They claim that the T-Bill rate is not to be used as the riskfree rate since most households hold long-term debt in their portfolios rather than short-term government paper, that the costs of holding diversified equity portfolios have to be accounted for, that taxes on dividends should be deducted from equity returns and that equilibrium conditions did not hold during the WW2 and the Korean War as the government imposed various restrictions. Then making adjustments for costs and taxes to equity returns, abstracting from the regulated sub-period 1935-1960 and using long-term high-grade bonds (and municipal bonds) as the riskfree instrument, they find that the average excess real return is less than one percent, and they claim there is no equity premium puzzle.

Overlapping generations models

The most important aspect of the overlapping generations models, contrary to the representative-agent models, is that it allows for heterogeneity across any T age cohorts of consumers. An individual's life span is divided into these T stages, where each stage describes a general "stage of life". T can be uncertain and/or infinite. To undergo a life cycle is important in such areas as the analysis of social security, effects of taxes on retirement decisions, distributive effects of taxes and effects of life-cycle saving on capital accumulation in the economy. This is why the overlapping generations model is a very useful tool for applied policy analysis.

In the typical overlapping generations model (OLG), all persons are assumed identical and to live for two periods. At any time t, two generations are alive, the young and the old. Each individual of generation t, i.e. born at time t, allocates his resources between consumption in the two periods according to a utility function $U_t(c_{1,t}, c_{2,t+1})$ where $c_{1,t}$ is consumption when young and $c_{2,t+1}$ when old. Normalizing the consumption good to have a price equal to one, then the intertemporal price ratio of consumption between period t and t+1 equals the real interest rate $(1 + r_{t+1})$. Maximizing utility subject to budget, given by the individual's resources, will in general imply that he

prefers to save (referred to as the Samuelson case by Gale(1973)) or dissave (called the classical case).

In the 50s and 60s Modigliani (Modigliani and Brumberg(1954), Modigliani(1966)) used an OLG to show that identical savings behavior of all generations over their lifetime would result in a constant savings ratio. Then the level of savings was dependent on growth rates of population and technology.

Microeconomic analysis of OLG started with Samuelson (1958), who considered the determination of interest rates in a pure exchange economy. He considered a single-perishable-commodity economy, in which the transfer of resources over time only could be in the form of consumption loans between the young and the old. If a durable good, such as money, exists which has intrinsic value and retains its value, then it is possible to invest in this good and later sell it to the next generation. Given some initial value of savings, current savings and interest rates are determined by the condition of market equilibrium and population growth. Over time this may converge to some equilibrium.

Diamond (1965) extended Samuelson's model by introducing production. Individuals who prefer to save can lend to entrepreneurs. In the Diamond model there is no labor income when old. The rate of interest is here determined by equilibrium in the capital market and by the characteristics of the production function. In this economy, which has an infinitely long life, he showed that despite the absence of usual sources of inefficiency, the competitive solution can be inefficient.

In the Samuelson case, when the young want to transfer value to the next period, Samuelson (1958) noticed that the market is not able to realize the contracts needed to store the monetary equilibrium²². Only the generation t can pay generation $t-1$, but generation t is not the one to which the member of $t-1$ lent their money in $t-1$, because generation t was not yet born. In the monetary equilibrium each young generation must lend to the old generation and they must be refunded the next period by the

²² Excess demand of young consumers is the negative of the excess demand of the old in the economy. An autarkic equilibrium has zero excess demand for any generation at any time, being homogeneous within their generation, consumers have no incentive to trade claims with consumers of their own generation. A monetary equilibrium has a non-zero excess demand.

subsequent young generation. Equilibrium can be realized through some i) storable commodity that retains its value, like fiat money. In my model I will use productive capital as the possible store of value. ii) A pay-as-you-go pension system can exactly realize the transfers, as young the consumers pay a premium and the proceeds are distributed among the old. The model gives the optimal amounts of premiums and benefits.

Incorporating a life-cycle feature to asset pricing means incorporating that the attractiveness of equity as an asset changes over the life cycle because the correlation of equity income with consumption changes.

Equity premium and population

The impact of demographic factors on asset prices is usually modeled using an OLG framework. These kind of models all share the feature that demographic shocks affect asset returns even in economies where rational agents anticipate the population growth. Typically they assume that people sell their financial assets in order to consume when retired. In such a framework, an ageing population generally implies a decrease of asset prices (both equity and bonds) and an increase of required expected excess returns.

Ang and Maddaloni (2003) find by pooling international data, that on average faster growth in the fraction of retired persons significantly decreases risk premiums. This demographic predictability of risk premiums is strongest in countries with well-developed social security systems and lesser-developed financial markets.

Donaldson and Maddaloni (2002) extend the OLG model of Constantinides, Donaldson and Mehra (2002) to include an exogenous and fixed population growth rate n . Supply of two financial assets, equity and risk-free bonds, grows at the same rate as the population. Calibrated simulations of the model shows that the risk premium is a decreasing function of n , but the effect is generally small. Relaxing the link between the supply of financial assets and population growth potentially produces even larger effects.

2. Diamond model with log normally distributed population growth and productivity growth

Assumptions

The rational expectations hypothesis is the stochastic version of perfect foresight. It states that the distributions of all future variables are known, given the available information, and thus they can be correctly predicted in distribution. Introducing uncertainty arises several complications. The formation of expectations of future prices has to be specified and that affect the properties of the model. A simple specification is not satisfactory because it accepts that agent will permanently make false predictions in a systematic way. The specification model should contain a model on learning to capture that agents will learn from their errors.

Even if there is no causal relationship between the state and the economy, the agents may believe that there is and there may exist a sunspots equilibrium where expectations are realized. This kind of equilibrium is not taken into account here.

Assuming that capital and output is the same commodity, the numeraire, one can consume one's capital. There are no market frictions like taxes or transactions costs. Imposing binding borrowing constraints on the young, such as Constantinides et. al. (2002), is relevant when the young would like to smooth consumption by borrowing, but are prevented by doing so because their human capital can't apply as guarantee.

The model contains no bequest. Offspring of individuals currently alive live together, and people are indifferent about their children's welfare. If introducing altruistically motivated transfers (Barro, 1974), then current generations are connected to future by this altruistic chain, and the equilibrium will yield the same as if there were one single infinite-horizon decision maker. Such a planner gives weight to all individuals (dynasties), including those not yet born. Thus a population-utility function puts more weight on the future, or equivalently has a lower discount rate ρ , more close to zero.

Population changes only by birth and death, i.e. this is a “closed economy” without migration. Every individual enters the economy as adult and lives for two periods. A more realistic model need an uncertain life-span, that is capturing a longevity risk in addition to the fertility risk. If taken this variable as stochastic as well, given that these individuals save for themselves, there would be a chance that they would die with some wealth left unconsumed. This kind of model need to take into account into who such means succeed. In such models buying an insurance against the risk of old age will leave the individuals better off (Yaari, 1965). The reason why I use fertility risk is that patterns of the elderly dependency ratio²³ are largely a consequence of changes in fertility.

At the beginning there is a generation -1, who only live for one period, called “the initial old”. The initial old generation has an exogenous capital stock $k_0 > 0$ to start the economy, equally owned by the generation.

Labor income changes over the life cycle. To capture the hump shape of earnings over life span I should have used more periods. Two periods do capture the assumption that people sell their financial assets in order to consume when retired. This is as mentioned the typical assumption which capture that when a larger proportions of agents retire, they dissave to fund their consumption, pushing asset prices down and increasing expected returns. However under other assumptions the opposite may be true. Storesletten, Telmer and Yaron (2001) extend the Constantinides and Duffie (1996) model with idiosyncratic labor risk to include a retirement state with no income shock. Here retirees face no labor market risk, and thus are less averse to bear aggregate risk and hold substantial amounts of equities. Such an economy with an increasing share of old people would see decreasing risk premiums.

Preferences are restricted to the CRRA class. But risk aversion itself may depend on demographic variables. Bakshi and Chen (1994) find empirical evidence that an investor’s relative risk aversion increases with age. Poterba (2001) finds that this relationship is not monotonic, thus simple summary measures, such as the average age may not be appropriate.

²³ Defined as the percentage of population over 65 years old as a ratio of the economically active population aged 15-64.

Production is Cobb-Douglas with two inputs, capital and labor. Other factors of production, such as land and human capital do not contribute to output here. The function is based on restrictive assumptions of perfect competition in factor and product markets. Research has indicated that for countries as a whole the assumption of constant return to scale is not unrealistic²⁴. For particular industries however there may in some cases be increasing returns to scale, and in others decreasing returns. Unitary elasticity of substitution is unrealistic. Labor and capital are correlated and the estimates are bound to be biased.

Capital fully depreciates in production²⁵. If depreciation was lower, it would have been necessary to specify how capital would be passed on from the old to the young. Since each period is about 30 years 100% depreciation is empirically plausible.

The country considered here is a closed economy. In a “small open economy” the world interest rate would determine returns. A change in demand for financial assets resulting from a demographic change would affect the amount of capital owned by the residents of the country, but not the capital per capita used in production. The rate of return would not depend on demographic changes or growth within the country. Closing the economy permits the real interest rate to be endogenized. To which degree world capital markets are really integrated is another question. There is substantial “home bias” in ownership. French and Poterba (1991) shows that more than ninety percent of the equity assets of the investors in the United States and Japan are held in their domestic equity markets. International interactions would complicate the analysis (see Baxter and King (2001) for an analysis). Abstracting from this is reasonable in the view that aging (interpreted as a fertility shock) is a world-wide phenomenon that cannot be avoided by going abroad.

²⁴ www.rrojasdatabank.org/brit08.htm

²⁵ This is also for convenience. It allows for the derivation of an explicit solution later.

The model

An economic model consists of different types of entities that take decisions subject to constraints. First I need to specify what the agents of the model are, which decisions they take, what constraints they have and what information they possess when making their decisions. My model has two types of agents, households and firms. Households have preferences over commodities and endowments of these. They maximize their preferences subject to budget. Firms maximize profits, subject to their plans being technologically feasible. The source of uncertainty²⁶ in this model is the two stochastic variables, the population growth and the technological growth, which distribution is known to all agents. The decisions make up a resulting equilibrium, which tells about the economy's dynamics, i.e. how the different decisions interact. The uncertainty accounts for the non-diversifiable market risk²⁷.

Stochastic population

The population growth factor gives the factor of increase (or decrease) in the number of persons in the population during a certain period of time. Assuming the population growth factor²⁸ G_t in any period t , is independently and identically log normally distributed such that

$$L_{t+1} = L_t G_{t+1}$$

Where L_{t+1} is the population at $t+1$.

With $E(\ln G) = g$ and $Var(\ln G) = \sigma_g^2$

The expected population growth factor at any time t is by²⁹

²⁶ In standard nonstochastic models there is no equity premium, in equilibrium all assets yields a common rate of return.

²⁷ There is no idiosyncratic risk in the economy as individuals within a cohort can share risks perfectly.

²⁸ When G_t is a growth factor, then the rate of growth is $G_t - 1$

²⁹ If a random variable is log-normally distributed, i.e. $\log(X) \sim N(\mu, \sigma)$ then

$$\log(EX) = E \log(X) + \frac{1}{2} Var(\log(X)) = \mu + \frac{\sigma^2}{2}$$

The log is a concave function. The mean of the log of the random variable X is smaller than the log of the mean. And

$$a \log(X) \sim N(a\mu, a^2\sigma^2)$$

$$\ln(EG) = E(\ln G) + \frac{1}{2}Var(\ln G), \quad EG = e^{g + \frac{1}{2}\sigma_g^2}$$

And it has a variance of

$$Var(G) = E(G^2) - (EG)^2 = e^{2g+2\sigma_g^2} - e^{2g+\sigma_g^2} = e^{2g+\sigma_g^2} (e^{\sigma_g^2} - 1)$$

The economy starts out with a population of L_0 and G_1 is the first shock to the economy. The shock is revealed at the very beginning of the period, thus L_{t+1} is the level of population through $t+1$, i.e. the labor force in $t+1$.

Expected population at time t may be written as

$$(1) \quad E_0 L_t = L_0 e^{t(g + \frac{1}{2}\sigma_g^2)}$$

The population changes through changes in fertility, which subsequently induce changes in the age distribution.

The stochastic production sector

The representative firm produces a single, perishable commodity maximizing profit π

$$\text{Max } \pi_t = p_t Y_t - R_t K_t - w_t N_t$$

which is its revenue, price times the output, less the payments to the factors of production, where K_t and N_t are the use of capital and labor, respectively in period t . Here capital is for simplicity assumed to fully depreciate in production, that is the rate of depreciation of physical capital is set equal to one. At the beginning of period t production takes place with the labor of generation t , the just revealed $N_t = N_{t-1} G_t$, and capital saved by the now old generation $t-1$. At the end of period t the firm pays its factors of production, a gross rent R_t to the capital and wage w_t to each employee. The gross return R_t indicates use from time $t-1$ to t .

The firm chooses how much to use of inputs subject to the technology, which is of the Cobb-Douglas type:

$$Y_t = A_t F(K_t, N_t) = A_t K_t^\alpha N_t^{1-\alpha} = A_t K_t^\alpha (N_{t-1} G_t)^{1-\alpha}$$

The production function describes the available technology, i.e. how commodities (inputs) can be transformed into output. When the amount of output obtained from given quantities of capital and labor rises over time, there is technological progress, A^{30} . A is a positive constant representing the productivity level. This may enter in the model as $Y=F(K,AL)$ labor-augmenting (Harrod-neutral), $Y=F(AK,L)$ capital-augmenting or $Y=AF(K,L)$, referred to as Hicks-neutral. Technological progress in the latter form is referred to as neutral in the sense that it does not directly affect the marginal rate of substitution between capital and labor. For, as is the case here, the Cobb-Douglas production function, they are all equivalent since $K^\alpha (EL)^{1-\alpha} = AK^\alpha L^{1-\alpha}$ if $A = E^{1-\alpha}$. I will use this latter kind of progress to estimate A directly from the data later.

Random technological growth evolves according to

$$A_{t+1} = V_{t+1}A_t$$

Assuming that the growth factor³¹ of technological progress is independently and identically log normally distributed, as the population growth factor. It has mean and variance given by

$$E(\ln V) = \nu \text{ and } Var(\ln V) = \sigma_\nu^2.$$

After V_{t+1} becomes known output is divided into payments to the factors of production.

Production does not exhibit constant returns³² to scale, that is is homogeneous of degree one in K and A . The production function is homogeneous of degree one (which is equivalent to an assumption of $\alpha + (1 - \alpha) = 1$) in K and N . This competitive firm is price-taker on both output and input markets. Labor supply is exogenous, i.e. $N_t = L_t$. The fact that factor inputs are multiplicative reflects the

³⁰ Neo-classical models based on capital accumulation need exogenous technological change to explain/incorporate growth. Here growth is not a result, it is an assumption. It is assumed for simplicity, I do not address the question of from what source the growth comes. Models of endogenous growth offer typically three fundamental sources of growth: human capital accumulation due to education investments, technological progress due to R&D investments and/or technological progress due to learning-by-doing externalities.

³¹ When V_t is a growth factor, then the rate of growth is $V_t - 1$

³² Constant returns to scale means that scaling all inputs up or down by some amount t scales output exactly the same way by the same t .

notion that one factor can be substituted for another. The following conditions apply to the production function:

i) $F(0,0) = 0$ It is not possible to produce something from nothing. ii) $\frac{\partial Y}{\partial K} > 0$ and

$\frac{\partial Y}{\partial N} > 0$. $\frac{\partial^2 Y}{\partial K^2} < 0$ and $\frac{\partial^2 Y}{\partial N^2} < 0$. Production is strictly increasing in both capital and

labor, and it is subject to diminishing marginal productivity, i.e. the additional output resulting from the use of an additional unit of input is decreasing. iii) The Inada

conditions, following Inada (1963), $\lim_{K \rightarrow 0} Y_K = \lim_{L \rightarrow 0} Y_L = \infty$ and

$\lim_{K \rightarrow \infty} Y_K = \lim_{L \rightarrow \infty} Y_L = 0$ ensure that the solution is interior, making sure that the

nonnegativity constraints are irrelevant. They demand that both inputs are required for production because $AF(0,N)=AF(K,0)=0$.

The firm solves

$$(2) \quad \text{Max } \pi_t = p_t Y_t - R_t K_t - w_t N_t$$

$$(3) \quad \text{Subject to } Y_t = A_t K_t^\alpha N_t^{1-\alpha}$$

Deriving first order conditions for the profit maximization

$$(4) \quad \frac{\partial \pi_t}{\partial K_t} = p_t \alpha A_t K_t^{\alpha-1} N_t^{1-\alpha} - R_t = 0$$

And thus $R_t = p_t \alpha A_t K_t^{\alpha-1} N_t^{1-\alpha}$ which means that capital is employed up to the point where the marginal revenue product, the product of the output price and the marginal product of the input, equals the cost R_t .

$$(5) \quad \frac{\partial \pi_t}{\partial N_t} = p_t (1-\alpha) A_t K_t^\alpha N_t^{-\alpha}$$

And in the same way $w_t = p_t (1-\alpha) A_t K_t^\alpha N_t^{-\alpha}$, the cost of hiring labor, the wage w_t must be equal to the rate at which revenue increases per additional labor employed.

Taken all variables in real terms, i.e. the price p_t of output normalized to 1, I want to define productivity adjusted worker and output per productivity adjusted worker

First I need to convert A_t into something that is constant returns to scale with

K_t , i.e. something to the power of $(1-\alpha)$. Note that

$$Y_t = K_t^\alpha \left(A_t^{\frac{1}{1-\alpha}} N_t \right)^{1-\alpha}$$

The expression in the parenthesis is the productivity adjusted labor force, now technology augments the productivity of labor. Defining productivity adjusted worker and output per productivity adjusted worker as

$$k_t \equiv \frac{K_t}{A_t^{\frac{1}{1-\alpha}} N_t} \quad \text{and} \quad y_t \equiv \frac{Y_t}{A_t^{\frac{1}{1-\alpha}} N_t}$$

Then the intensive form production function is

$$\frac{Y_t}{A_t^{\frac{1}{1-\alpha}} N_t} = F\left(\frac{K_t}{A_t^{\frac{1}{1-\alpha}} N_t}, \frac{A_t^{\frac{1}{1-\alpha}} N_t}{A_t^{\frac{1}{1-\alpha}} N_t}\right) = F\left(\frac{K_t}{A_t^{\frac{1}{1-\alpha}} N_t}, 1\right) \equiv f(k_t)$$

Explicit given as

(6)

$$y_t = \frac{Y_t}{A_t^{\frac{1}{1-\alpha}} N_t} = \frac{K_t^\alpha (A_t^{\frac{1}{1-\alpha}} N_t)^{1-\alpha}}{A_t^{\frac{1}{1-\alpha}} N_t} = \left(\frac{K_t}{A_t^{\frac{1}{1-\alpha}} N_t} \right)^\alpha \left(\frac{A_t^{\frac{1}{1-\alpha}} N_t}{A_t^{\frac{1}{1-\alpha}} N_t} \right)^{1-\alpha} = \left(\frac{K_t}{A_t^{\frac{1}{1-\alpha}} N_t} \right)^\alpha = k_t^\alpha$$

The stochastic real rate of return equals the marginal product of capital

$$1 + r_t = R_t = f'(k_t), \text{ and from } Y_t = A_t^{\frac{1}{1-\alpha}} N_t f\left(\frac{K_t}{A_t^{\frac{1}{1-\alpha}} N_t}\right) \text{ then}$$

$$\text{MPK} = \frac{\partial Y_t}{\partial K_t} = A_t^{\frac{1}{1-\alpha}} N_t f'(k_t) \frac{1}{A_t^{\frac{1}{1-\alpha}} N_t} = f'(k_t) = \alpha k_t^{\alpha-1}, \text{ the stochastic rate of return is}$$

$$(7) \quad 1 + r_t = R_t = \alpha \left(\frac{K_t}{A_t^{\frac{1}{1-\alpha}} N_t} \right)^{\alpha-1}$$

On a balanced growth path³³ the marginal product of capital will have a time-invariant expectancy, because of the steady state level of the capital per productivity adjusted worker. Although the level of aggregate variables such as capital stock and output increases, the resulting equilibrium return process is stationary.

³³ Balanced growth is growth consistent with the Kaldor facts. Definitions and assumptions will be given under the section of equilibrium.

The real wage equals the marginal product of labor

$$\text{MPN} = \frac{\partial Y_t}{\partial N_t} = A_t^{\frac{1}{1-\alpha}} f(k_t) + A_t^{\frac{1}{1-\alpha}} N_t f'(k_t) \left(- \frac{K_t}{A_t^{\frac{1}{1-\alpha}} N_t^2} \right), \text{ that is}$$

$$(8) \quad w_t = A_t^{\frac{1}{1-\alpha}} f(k_t) - A_t^{\frac{1}{1-\alpha}} f'(k_t) k_t = (1-\alpha) A_t^{\frac{1}{1-\alpha}} \left(\frac{K_t}{A_t^{\frac{1}{1-\alpha}} N_t} \right)^\alpha$$

In the steady state path the wage will grow at the rate $V^{\frac{1}{1-\alpha}}$. The growth rate is a trend-stationary process, it is stationary about its time trend, as well as weakly dependent. Factor prices (7) and (8) are determined by the capital-productivity adjusted labor ratio and the technology shock.

The aggregate economy will grow at the rate

$$\begin{aligned} \frac{Y_{t+1}}{Y_t} &= \frac{A_{t+1} K_{t+1}^\alpha (N_t G_{t+1})^{1-\alpha}}{A_t K_t^\alpha (N_t)^{1-\alpha}} \\ &= V_{t+1} \left(\frac{K_{t+1}}{K_t} \right)^\alpha G_{t+1}^{1-\alpha} \end{aligned}$$

Aggregate capital will grow at the same rate as the wage $V^{\frac{1}{1-\alpha}}$, because with the logarithmic utility people save a constant fraction of their wage³⁴ and in this model savings, investment and capital are equal, times the growth in population G . Thus

$$\frac{Y_{t+1}}{Y_t} = V_{t+1} \left(V_{t+1}^{\frac{1}{1-\alpha}} G_{t+1} \right)^\alpha G_{t+1}^{1-\alpha} = V_{t+1}^{\frac{1}{1-\alpha}} G_{t+1}$$

Which shows that the economy grows at the labor-augmented rate $V^{\frac{1}{1-\alpha}}$ times the labor growth G .

The national income

$$Y = wN + RK$$

³⁴ Returning to this in the section of households.

Both Y and K grows at the rate $V^{\frac{1}{1-\alpha}}$. The rate of return grows at rate zero. N grows of course at G and then the wage must grow at rate $V^{\frac{1}{1-\alpha}}$, otherwise labor share of output would vanish or become arbitrarily large.

This production function is chosen because it is compatible with the stylized facts³⁵ characterizing modern economies in the long run, as noted in an exercise by Ríos-Rull (2005): increasing (i) capital per capita, (ii) income per capita and (iii) real wage, and (iv) a considerably constant real interest rate. For the moment ignoring growth, payments to the production factors, capital K and labor N, at time t are given by

$$\begin{aligned} w_t N_t + R_t K_t &= p_t \frac{\partial Y_t}{\partial N_t} N_t + p_t \frac{\partial Y_t}{\partial K_t} K_t = p_t \left[(1-\theta) \left(\frac{K_t}{N_{t-1} G_t} \right)^\theta N_{t-1} G_t + \theta \left(\frac{K_t}{N_{t-1} G_t} \right)^{\theta-1} K_t \right] \\ &= p_t (1-\theta + \theta) K_t^\theta (N_{t-1} G_t)^{1-\theta} = p_t Y_t \end{aligned}$$

where p_t is the price of output at time t. Now observing that $\frac{Y_t}{N_t} = \frac{K_t^\alpha N_t^{1-\alpha}}{N_t} = \left(\frac{K_t}{N_t} \right)^\alpha$,

that is (i) and (ii) are consistent with each other. And the real wage is

$$\frac{w_t}{p_t} = \frac{p_t (1-\alpha) \left(\frac{K_t}{N_t} \right)^\alpha}{p_t} = (1-\alpha) \left(\frac{K_t}{N_t} \right)^\alpha$$

which increase is consistent with (i). The

interest rate is given by $R_t = p_t \alpha (K_t / N_t)^{\alpha-1}$, compare (iv), which variations can be explained by movements of prices.

Households

Each individual lives two periods and supplies in elastically one unit of labor in the first period. Consumption in second period will be what is saved from wage earned in the first period plus capital earnings. This model is therefore considered a Samuelson case.

³⁵ Referred to as the Kaldor facts, empirical regularities encapsulating important features of modern economies.

Household's preferences over commodities are specified through the utility function. I will use utility functions from the class of constant relative risk aversion (CRRA), because their relative risk aversion do not depend on wealth, which is realistic according to Campbell and Viceira (2002)³⁶:

$$(9) \quad \text{Max } U_t = \frac{1}{1-\theta} c_{1,t}^{1-\theta} + E_t \left\{ \frac{1}{1+\rho} \frac{1}{1-\theta} c_{2,t+1}^{1-\theta} \right\}, \theta > 0, \rho > 0$$

E_t is the expectation operator conditional upon the information available at the time t . ρ is the discount rate at which the individuals value future relative to current consumption. The assumption that $\rho > 0$ ³⁷ assures that they place greatest weight on the first period. The larger the discount rate the larger weight the consumer places on the consumption in the near term over that in the future. θ is referred to as the coefficient of relative risk aversion as mentioned earlier. This utility is time separable, which means that the period utility at time t depends only on consumption in period t and not on consumption in other periods. This formulation rules out, among other things, habit persistence. Leisure does not enter the utility function as I abstain from issues of labor participation, thus treating the labor supply as exogenous. I will check the outcome of the economy for the \log ³⁸ utility, which is the limit of CRRA as θ approaches 1.

At the end of period t the young generation receives a wage w_t and decides how much to consume and how much to save. The consumer is uncertain about the next period returns on the asset. Maximizing utility

$$(10) \quad \text{Max } U_t = \log c_{1,t} + \frac{1}{1+\rho} E_t \log c_{2,t+1}, \rho > -1$$

Subject to budget

$$c_{1,t} + s_t = w_t$$

$$c_{2,t+1} = (1 + r_{t+1})s_t = R_{t+1}(w_t - c_{1,t})$$

³⁶ "The long-run behavior of the economy suggests that relative risk aversion cannot depend strongly on wealth. Per capita consumption and wealth has increased greatly over the two past centuries. Since financial risks are multiplicative, this means that the absolute scale of financial risks has also increased while the relative scale is unchanged. Interest rates and risk premium do not show any evidence of long-term trends in response to this long-term growth; this implies that investors are willing to pay almost the same relative costs to avoid given relative risks as they did when they were much poorer, which is possible only if relative risk aversion is almost independent of wealth" (p 24)

³⁷ This corresponds to an assumption of $\beta \in (0,1)$

³⁸ The notations \log and \ln are used interchangeable here, both meaning the natural logarithm

where s_t is the amount saved.

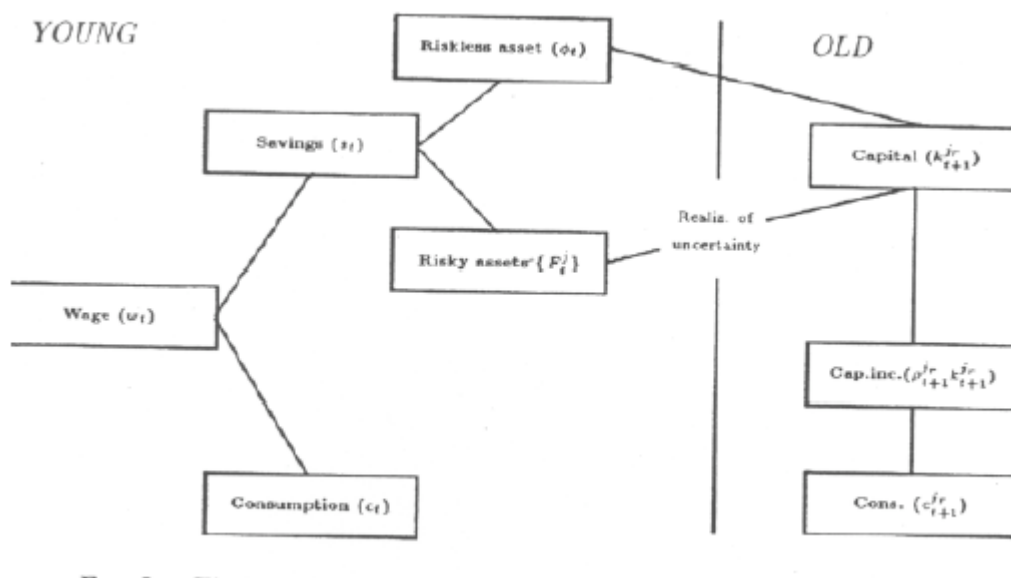


Figure 1: Life cycle and decisions, this is the figure at page 229, Acemoglu (2006)

Deriving first order condition for lifetime utility for the representative household

$$\frac{\partial U_t}{\partial c_{1,t}} = \frac{1}{c_{1,t}} + \frac{1}{1+\rho} E_t \left\{ \frac{1}{c_{2,t+1}} (-R_{t+1}) \right\} = 0$$

Which can be written as

$$(11) \quad \frac{1}{c_{1,t}} = \frac{1}{1+\rho} E_t \left\{ \frac{R_{t+1}}{c_{2,t+1}} \right\}$$

$$\text{And then } c_{1,t} = E_t \frac{(1+\rho)c_{2,t+1}}{R_{t+1}} = E_t \frac{(1+\rho)R_{t+1}(w_t - c_{1,t})}{R_{t+1}} = (1+\rho)(w_t - c_{1,t})$$

$$(2+\rho)c_{1,t} = (1+\rho)w_t$$

Which at optimum distributes $\frac{1+\rho}{2+\rho} w_t$ for consumption the first period and

$\frac{1}{2+\rho} w_t$ for saving. At utility maximum the consumer cannot gain from shifts of

consumption between periods, a unit reduction of first period consumption lowers

utility by $u'(c_{1,t})$ and raises second period utility by $\frac{R_{t+1}}{1+\rho} u'(c_{2,t+1})$ (11), these must

be equal as discussed earlier. The primary dynamic equation relates the agent's

consumption at time t to his consumption at $t+1$. This difference is important in means of distribution because labor's share of output goes to the young and capital's share goes to the old. The poorest generations are those which are large relative both to the preceding and the succeeding generations, because they will have both low wages and low returns on their savings, compare (7) and (8). Conversely, generations which are small relative to both preceding and succeeding generations enjoy high wages and high profits. Due to logarithmic utility, the optimal savings $s_t = \frac{1}{2+\rho} w_t = s_t(w_t)$ are independent of the interest rate³⁹.

Equilibrium capital accumulation

All markets clear. There is no unemployment in the labor market, labor is taken as exogenous in the production function:

$$(12) \quad N_t = L_t$$

Asset market

$$(13) \quad K_{t+1} = s_t N_t$$

Total savings of the currently young people makes up the capital stock for tomorrow since physical capital is the only asset in this economy.

Goods market

$$(14) \quad N_t c_{1,t} + N_{t-1} c_{2,t} + K_{t+1} = A_t F(K_t, N_t)$$

Total consumption plus gross investment equals output, i.e. what is available in the economy. This is taken into account in the budget constraint, there is no wasted resources as long as they give utility.

³⁹ With separable CRRA utility complete markets imply that individual consumption at each date, in each state of the world is a constant fraction of aggregate income. It does not imply that individual consumption is constant across time and states of the world, because it still varies with aggregate income and interest rate. A change in the interest rate has an ambiguous effect. A rise in the interest rate makes savings more attractive and people reduce consumption today. This is a substitution effect toward future consumption. A rise in interest rate also allows higher consumption in the future given the present value of resources. This income effect, i.e. expansion of feasible consumption set make people raise current consumption. At logarithmic utility the two effects cancel out and the saving rate is independent of r (and therefore changes in the capital-labor ratio of the economy). For proof# appendix?

From Walra's law⁴⁰ one of the market clearing conditions is redundant. Equilibrium in the labor market is straight forward, so dropping the goods market condition I exploit the asset market equilibrium condition to describe the equilibrium.

Dividing (13) by the labor force the same period, the capital per head available for any generation is determined by the total amount of savings by the previous generation.

$$k_{t+1} = \frac{K_{t+1}}{N_{t+1}} = \frac{s_t N_t}{N_{t+1}} = \frac{s_t}{G_{t+1}}$$

A demographic change, *ceteris paribus*, induces a change in capital per capita, which again by (7) and (8) induces changes in the rate of interest and the wage rate. If population growth at time t turns out to be higher (compared to some steady-state or expectancy), capital per worker falls, which increases r_t and decreases w_t .

The capital stock can be written

$$(15) \quad K_{t+1} = \frac{1}{2 + \rho} (1 - \alpha) A_t^{\frac{1}{1-\alpha}} \left(\frac{K_t}{A_t^{\frac{1}{1-\alpha}} N_t} \right)^\alpha N_t$$

which is a first-order nonlinear stochastic difference equation in capital. Dynamics enters the model from the fact that the present level depend upon the past. The capital stock and the stochastic outcome today determines labor income, which in turn determines saving and the capital stock next period. The difference equation is autonomous, i.e. t does not appear as an independent argument.

Stationarity

Wooldridge (2003) defines a stationary process to be a time series process where the marginal and all joint distributions are invariant over time. A stationary process,

⁴⁰ As each consumer satisfies his or her budget constraint, so the economy as a whole has to satisfy an aggregate budget constraint.

Or formally: define the aggregate excess demand function $Z(\mathbf{p})$ as the aggregate consumer demand function less the aggregate supply from consumers and the aggregate net supply of firms, and then Walra's law: If $Z(\mathbf{p})$ defined as described then $\mathbf{p}Z(\mathbf{p}) = 0$ for all \mathbf{p} .

$\{x_t : t = 1, 2, \dots\}$ is said to be weakly dependent if x_t and x_{t+h} are “almost independent” as h increases without bound.

For stationarity to hold I need that

$$(16) \quad \frac{1}{1 + \rho} * EV_t \leq 1$$

If this is not fulfilled then the future becomes relative more attractive to the consumer than the present and the consumer would save all his income.

For this discrete stochastic dynamic system there is a stationary expectancy of $\ln(k_t)$ ⁴¹, assumed to be weakly dependent. The unconditional expected level of capital per productivity-adjusted capita is a steady state if it is a solution to this equation such that $E \ln k_{t+1} = E \ln k_t$ ⁴². The steady state defines a stochastic balanced growth path. A balanced growth equilibrium (BGE) is a stationary equilibrium which allows perpetual growth in the steady state. In the BGE all endogenous accumulated variables grows at a stationary rate, not necessarily the same. This imply that along the path, the “great ratios” –K/Y, C/Y, I/Y- are stationary⁴³. The existence of BGE implies that the utility function must be additive separable and homogenous and the production function must be linearly homogenous⁴⁴ (Jones and Manuelli, 1990). Balanced growth requires factor shares to be constant, which can only be the case when total inputs grow at the same rate. The exception is the Cobb-Douglas function where the technological change can be represented as purely labor-augmented (shown to be $V^{\frac{1}{1-\alpha}}$ here).

⁴¹ The stationarity of this variable is proved in Appendix C.

⁴² Krugman, Delong and Baker (2005) define steady state in a deterministic Diamond model to be where capital per effective worker is constant. Edwards (2003) assumes the national economy to remain in a “steady state” where the ratio of capital to effective labor stays constant.

⁴³ Ji (2003) investigated the great ratios of Australia derived from the neoclassical stochastic growth model of Campbell (1994) and he finds that technology-capital, capital-output and consumption-output ratios are stationary. He concludes that this is in part support for the long-run implications of the one-sector neoclassical stochastic growth model.

⁴⁴ These requirements are satisfied by (3) and (10)

Remember that $k = \frac{K}{A^{1-\alpha} N} = \left(\frac{K}{N} \right)^{\frac{1}{1-\alpha}}$ so that capital per worker grows at the same rate

as $A^{\frac{1}{1-\alpha}}$, i.e. $V^{\frac{1}{1-\alpha}}$. The per-worker capital is obtained as the efficiency-unit value multiplied by the productivity level $A^{\frac{1}{1-\alpha}}$. Along the balanced growth path, the aggregate of output, capital, consumption and investment are all growing at the natural rate of the economy, $G_t V_t^{\frac{1}{1-\alpha}}$.

Take the equation for the capital stock (15) and divide both sides by the productivity-adjusted labor force $A_{t+1}^{\frac{1}{1-\alpha}} N_{t+1}$ in t+1:

$$\frac{K_{t+1}}{A_{t+1}^{\frac{1}{1-\alpha}} N_{t+1}} = \frac{1}{2 + \rho} \frac{(1 - \alpha) A_t^{\frac{1}{1-\alpha}} \left(\frac{K_t}{A_t^{\frac{1}{1-\alpha}} N_t} \right)^\alpha N_t}{A_{t+1}^{\frac{1}{1-\alpha}} G_{t+1} N_t}$$

Collecting the capital per adjusted capita terms for periods t and t+1

$$\frac{K_{t+1}}{A_{t+1}^{\frac{1}{1-\alpha}} N_{t+1}} = \frac{(1 - \alpha) A_t^{\frac{1}{1-\alpha}}}{(2 + \rho) (V_{t+1} A_t)^{\frac{1}{1-\alpha}}} \left(\frac{K_t}{A_t^{\frac{1}{1-\alpha}} N_t} \right)^\alpha$$

Written in the per capita adjusted form, the behavior of capital is

$$(17) \quad k_{t+1} = \frac{(1 - \alpha)}{(2 + \rho) V_{t+1}^{\frac{1}{1-\alpha}} G_{t+1}} k_t^\alpha$$

The global dynamics of this system can be analyzed qualitatively by phase diagram in the (k_{t+1}, k_t) -space.

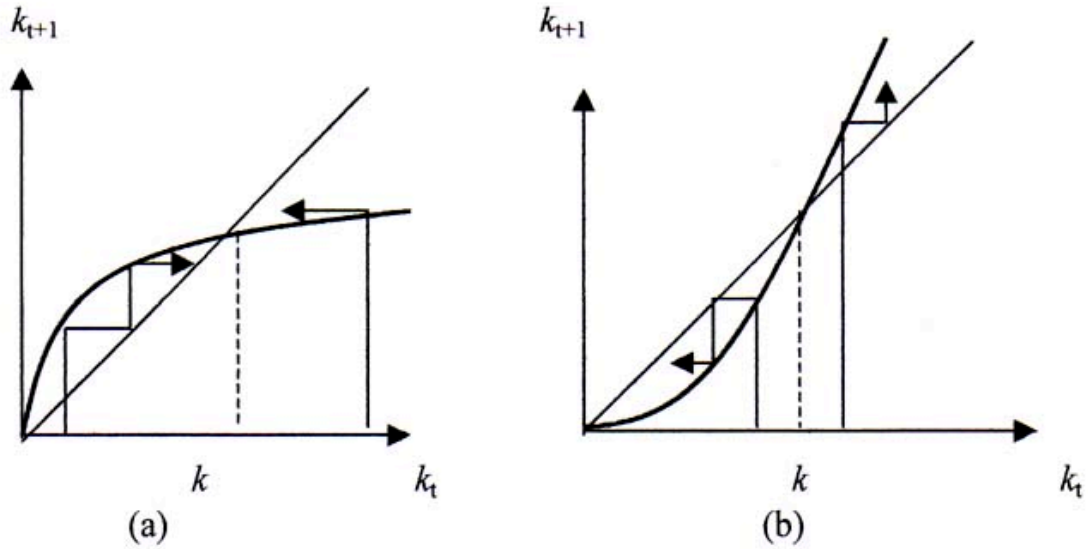


Figure 2: Phase diagram in the (k_{t+1}, k_t) -space.

Stability requires that the absolute value of the slope must be <1 (as in figure a). The slope in figure b) is >1 ⁴⁵. k_t is increasing if it lies above the 45° line (representing steady state where $k_{t+1} = k_t$) and decreasing when it lies below. The steady state in a) is locally asymptotically stable and in b) unstable, because capital moves away from its steady state.

To find the steady state, take natural logarithms to linearize the model

$$\ln k_{t+1} = \ln \left(\frac{1 - \alpha}{(2 + \rho)V_{t+1}^{\frac{1}{1-\alpha}} G_{t+1}} \right) + \alpha \ln k_t$$

And now taking expectations

$$E \ln k_{t+1} = E \ln \left(\frac{1 - \alpha}{(2 + \rho)V_{t+1}^{\frac{1}{1-\alpha}} G_{t+1}} \right) + \alpha E \ln k_t$$

Or

⁴⁵ If slope=1 then the steady state is a non-hyperbolic equilibrium, and it may be (locally) asymptotically stable or unstable. This is not the case here. Appendix C shows that the capital per productivity adjusted labor is stationary, i.e. shows that this slope is less than one as the model converges to a steady state.

$$E \ln k_{t+1} = \ln\left(\frac{1-\alpha}{2+\rho}\right) - \frac{1}{1-\alpha} E \ln V_{t+1} - E \ln G_{t+1} + \alpha E \ln k_t$$

Then using the steady state condition that $E \ln k_t = E \ln k_{t+1}$ the unconditional expectation of the log-capital level per adjusted worker on the balanced growth path is

$$E(\ln k_t) = \frac{1}{1-\alpha} \ln\left(\frac{1-\alpha}{2+\rho}\right) - \frac{1}{(1-\alpha)^2} E(\ln V_{t+1}) - \frac{1}{1-\alpha} E(\ln G_{t+1})$$

Which is

$$(18) \quad E(\ln k_t) = \frac{1}{1-\alpha} \ln\left(\frac{1-\alpha}{2+\rho}\right) - \frac{v}{(1-\alpha)^2} - \frac{g}{1-\alpha}$$

And it has a variance of

$$Var(\ln k_t) = \left(\frac{1}{1-\alpha}\right)^4 Var(\ln V_{t+1}) + \left(\frac{1}{1-\alpha}\right)^2 Var(\ln G_{t+1}) + 2\left(\frac{1}{1-\alpha}\right)^3 Cov(\ln V_{t+1}, \ln G_{t+1})$$

Which can be written as

$$(19) \quad Var(\ln k_t) = \left(\frac{1}{1-\alpha}\right)^4 \sigma_v^2 + \left(\frac{1}{1-\alpha}\right)^2 \sigma_g^2 + 2\left(\frac{1}{1-\alpha}\right)^3 \sigma_{\ln V, \ln G}$$

Where $\sigma_{\ln V, \ln G}$ is the covariance between the logarithm of population growth and the logarithm of the technological growth. The log-capital has a higher variance than the stochastic log-variables in the model. Having assumed that population growth is lognormal, all endogenous variables in the model multiplicative to the population growth are as well lognormal.⁴⁶ The unconditional expectation of k_t on the balanced growth path is then

$$\ln(Ek_t) = E(\ln k_t) + \frac{1}{2} Var(\ln k_t)$$

$$\ln(Ek_t) = \frac{1}{1-\alpha} \ln\left(\frac{1-\alpha}{2+\rho}\right) - \frac{v}{(1-\alpha)^2} - \frac{g}{1-\alpha} + \frac{1}{2}\left(\frac{1}{1-\alpha}\right)^4 \sigma_v^2 + \frac{1}{2}\left(\frac{1}{1-\alpha}\right)^2 \sigma_g^2 + \left(\frac{1}{1-\alpha}\right)^3 \sigma_{\ln V, \ln G}$$

And now taking exponentials yields

⁴⁶Distributions of gross returns and prices are bounded below because they cannot be negative. The lognormal distribution does account for this. Another advantage is that it takes account for compounding, because the random variable grows at every instant by a rate that is a normal random variable, i.e. reflecting continuous compounding. The disadvantage of log-returns is the property that the sum of lognormal returns is not itself lognormal. This is a problem because the log-normal-property of individual assets cannot be extended to a portfolio.

$$(20) \quad Ek_t = \left(\frac{1-\alpha}{2+\rho} \right)^{\frac{1}{1-\alpha}} e^{-\frac{v}{(1-\alpha)^2} - \frac{g}{1-\alpha} + \frac{1}{2} \left(\frac{1}{1-\alpha} \right)^4 \sigma_v^2 + \frac{1}{2} \left(\frac{1}{1-\alpha} \right)^2 \sigma_g^2 + \left(\frac{1}{1-\alpha} \right)^3 \sigma_{\ln v, \ln G}}$$

The steady state capital per productivity-adjusted capita⁴⁷ depend on the subjective time discount rate and characteristics of the production function. It is decreasing in g and v and increasing in σ_g^2 , σ_v^2 and the covariance between the two. The reason it is decreasing in the expected growth factors, is that a larger growth makes it harder to hold on to the steady state value and at the same time consumers wish to smooth their life time consumption by borrowing against future consumption. Growth in technology has bigger impact because it hits production directly, compare (3). Whilst population growth goes through a power of N . The capital per productivity-adjusted capita increases with the variances and covariance due to a demand for precautionary saving. Adding uncertainty, agents want to hedge against future unfavorable consumptions realizations by building buffer stocks, i.e. saving more and altering the capital level.

By evaluating its partial derivatives, ceteris paribus, I can infer to what extent permanent changes in variables and parameters affect the long run unconditional capital level:

$$\frac{\partial Ek_t}{\partial g} = -\frac{1}{1-\alpha} \left(\frac{1-\alpha}{2+\rho} \right)^{\frac{1}{1-\alpha}} e^{-\frac{v}{(1-\alpha)^2} - \frac{g}{1-\alpha} + \frac{1}{2} \left(\frac{1}{1-\alpha} \right)^4 \sigma_v^2 + \frac{1}{2} \left(\frac{1}{1-\alpha} \right)^2 \sigma_g^2 + \left(\frac{1}{1-\alpha} \right)^3 \sigma_{\ln v, \ln G}}$$

Thus an increasing g (which is the mean of the logarithm of the population growth factor and therefore also increased expected population growth, the log of the mean) decreases the capital per productivity-adjusted capita. Higher population decreases the wage, as noted earlier by (2), and thereby savings (capital per capita) which with logarithmic utility is a constant fraction of wage.

$$\frac{\partial Ek_t}{\partial v} = -\frac{1}{(1-\alpha)^2} \left(\frac{1-\alpha}{2+\rho} \right)^{\frac{1}{1-\alpha}} e^{-\frac{v}{(1-\alpha)^2} - \frac{g}{1-\alpha} + \frac{1}{2} \left(\frac{1}{1-\alpha} \right)^4 \sigma_v^2 + \frac{1}{2} \left(\frac{1}{1-\alpha} \right)^2 \sigma_g^2 + \left(\frac{1}{1-\alpha} \right)^3 \sigma_{\ln v, \ln G}}$$

An increased rate of growth of the log-technology growth decreases capital per effective labor. This because the effective labor (the denominator in the relationship) becomes more effective (bigger).

⁴⁷ $k_t = 0$ is not a solution to equation (17), because of the Inada conditions interest rates would be infinite and no solution to the consumer problem would exist.

$$\frac{\partial Ek_t}{\partial \sigma_g^2} = \frac{1}{2} \left(\frac{1}{1-\alpha} \right)^2 \left(\frac{1-\alpha}{2+\rho} \right)^{\frac{1}{1-\alpha}} e^{-\frac{v}{(1-\alpha)^2} \frac{g}{1-\alpha} + \frac{1}{2} \left(\frac{1}{1-\alpha} \right)^4 \sigma_v^2 + \frac{1}{2} \left(\frac{1}{1-\alpha} \right)^2 \sigma_g^2 + \left(\frac{1}{1-\alpha} \right)^3 \sigma_{\ln V, \ln G}}$$

An increase in the volatility of the lognormal distributed population growth implies a higher level of capital per capita. This because the consumer respond to increased income uncertainty by saving more. Such precautionary saving occurs if $u'''(c) > 0$ ⁴⁸, then the marginal utility $u'(c)$ is a convex function of c . Jensen's inequality⁴⁹ implies that a raise in uncertainty about period $t+1$ income, a more variable c_{t+1} with the same mean lowers $u(c_{t+1})$ and then raises $E_t \{u'(c_{t+1})\}$ and then to still satisfy the Euler condition, consumption the first period must fall, i.e. the consumer saves more⁵⁰.

$$\frac{\partial Ek_t}{\partial \sigma_v^2} = \frac{1}{2} \left(\frac{1}{1-\alpha} \right)^4 \left(\frac{1-\alpha}{2+\rho} \right)^{\frac{1}{1-\alpha}} e^{-\frac{v}{(1-\alpha)^2} \frac{g}{1-\alpha} + \frac{1}{2} \left(\frac{1}{1-\alpha} \right)^4 \sigma_v^2 + \frac{1}{2} \left(\frac{1}{1-\alpha} \right)^2 \sigma_g^2 + \left(\frac{1}{1-\alpha} \right)^3 \sigma_{\ln V, \ln G}}$$

Uncertainty in technology growth also induces a higher level of capital per effective capita by this precautionary savings motive.

$$\frac{\partial Ek_t}{\partial \sigma_{\ln V, \ln G}} = \left(\frac{1}{1-\alpha} \right)^3 \left(\frac{1-\alpha}{2+\rho} \right)^{\frac{1}{1-\alpha}} e^{-\frac{v}{(1-\alpha)^2} \frac{g}{1-\alpha} + \frac{1}{2} \left(\frac{1}{1-\alpha} \right)^4 \sigma_v^2 + \frac{1}{2} \left(\frac{1}{1-\alpha} \right)^2 \sigma_g^2 + \left(\frac{1}{1-\alpha} \right)^3 \sigma_{\ln V, \ln G}}$$

⁴⁸ With log utility $u'''(c) = \frac{2}{c^3} > 0$. Kimball (1990) showed that precautionary savings are

determined by the coefficient of relative prudence, defined as $\frac{-u'''(c)c}{u''(c)}$. The CRRA utility has a

relative prudence of $(1+\theta)$. Investors with a high θ subject to uncertain income save more to protect consumption against states of low income.

⁴⁹ If $u(c)$ is strictly concave, this implies Jensen's inequality

$$E[u(c)] \leq u(E[c])$$

⁵⁰ To see this consider the risk-free rate which also has to fulfil the Euler equation

$$c_{1,t}^{-1} = \frac{1}{1+\rho} R_{t+1}^f E_t c_{2,t+1}^{-1}$$

or $\frac{1}{R_{t+1}^f} = c_{1,t} \frac{1}{1+\rho} E_t c_{2,t+1}^{-1}$ Increasing the variance here in $c_{2,t+1}$, everything else being equal,

raises the marginal utility $E_t c_{2,t+1}^{-1}$ which has to offset by a decrease in $\frac{1}{R_{t+1}^f}$, i.e. a lowering of the

riskfree rate. Therefore, in this economy the interest rate is lower than in an otherwise identical economy with the same average second-period consumption but lower variance in it. With uncertainty in future income people save more, thereby bidding up prices of second period consumption relative to that of first period.

An increased covariance means that the risk of the two moves in the same direction, aggregate risk increases and again the precautionary savings motive implies a higher level of capital per effective capita.

$$\frac{\partial Ek_t}{\partial \rho} = -\frac{1}{(2+\rho)^2} \left(\frac{1-\alpha}{2+\rho} \right)^{\frac{\alpha}{1-\alpha}} e^{-\frac{v}{(1-\alpha)^2} - \frac{g}{1-\alpha} + \frac{1}{2} \left(\frac{1}{1-\alpha} \right)^4 \sigma_v^2 + \frac{1}{2} \left(\frac{1}{1-\alpha} \right)^2 \sigma_g^2 + \left(\frac{1}{1-\alpha} \right)^3 \sigma_{\ln V, \ln G}}$$

An increase in ρ , the intertemporal discount rate, reduces Ek_t . This because a higher ρ means placing higher weight on first period consumption, and individuals will then save less and thereby reducing capital, to obtain this higher first period consumption.

$$\frac{\partial Ek_t}{\partial \alpha} = \left[\underbrace{\frac{1}{(1-\alpha)^2} \ln \left(\frac{1-\alpha}{2+\rho} \right)}_{-because \ln(<1)} - \frac{1}{(1-\alpha)^2} - \frac{2v}{(1-\alpha)^3} - \frac{g}{(1-\alpha)^2} + \frac{2\sigma_v^2}{(1-\alpha)^5} + \frac{\sigma_g^2}{(1-\alpha)^3} + \frac{3\sigma_{\ln V, \ln G}}{(1-\alpha)^4} \right] e^{\frac{1}{1-\alpha} \ln \left(\frac{1-\alpha}{2+\rho} \right) - \frac{v}{(1-\alpha)^2} - \frac{g}{1-\alpha} + \frac{1}{2} \left(\frac{1}{1-\alpha} \right)^4 \sigma_v^2 + \frac{1}{2} \left(\frac{1}{1-\alpha} \right)^2 \sigma_g^2 + \left(\frac{1}{1-\alpha} \right)^3 \sigma_{\ln V, \ln G}}$$

If the capital's share α increases, then capital per capita will decrease if the expectancies of the growth rates are sufficiently higher than their variances, which is most likely the case⁵¹. This comes from the fact that increasing α gives a lower income, which again gives lower saving and capital. Though if variances and their covariance are high, capital per capita will increase, again due to the precautionary savings motive.

Transitory shocks to the economy will affect the economy temporarily and then it will gradually fall back to the steady state value. To see how suppose there is a one-time positive shock ε to population, out of steady state, so that

$$(21) \quad \dots N_t = (EG_t)N_{t-1}, N_{t+1} = (EG_{t+1})N_t + \varepsilon, N_{t+2} = (EG_{t+2})N_{t+1} \dots$$

Running the sequence of N's through the difference equation (17), and assuming that productivity shocks sticks to expectation gives the following transition path for the US economy experiencing the growth factor 1.7 in population instead of the expected 1.5980.

⁵¹ Which they are in the case of Norway and the US. Check Table 5 for an overview.

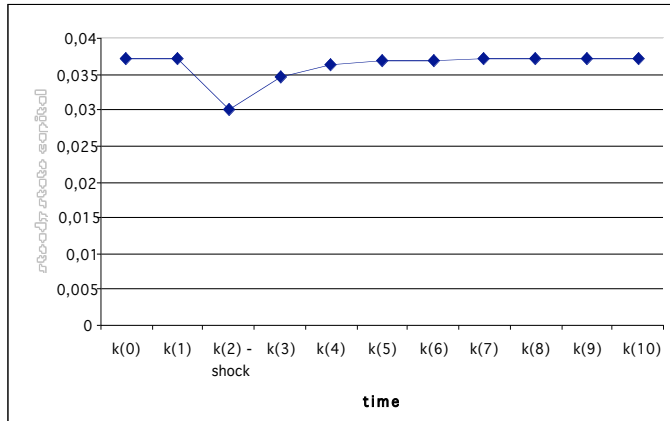


Figure 3: Transition to steady state per productivity adjusted capital

The capital will first fall as a consequence of more workers than expected, before it adjusts back to the steady state value⁵².

Marginal product of capital

The marginal product of capital shows the increase in the value of the firms output when one more unit of capital is employed. Take the marginal product of capital (7) and iterated a period ahead it is

$$(22) \quad R_{t+1} = \alpha k_{t+1}^{\alpha-1}$$

Which is a decreasing function of the efficiency-adjusted capital per capita

$$\frac{\partial R_{t+1}}{\partial k_{t+1}} = (\alpha - 1)\alpha k_{t+1}^{\alpha-2} \text{ because } \alpha < 1$$

This is the property of diminishing marginal returns. Adding more and more capital yields less and less additional output.

From the motion of capital (17)

$$\ln k_{t+1} = \ln \left(\frac{1 - \alpha}{2 + \rho} \right) - \frac{1}{1 - \alpha} \ln V_{t+1} - \ln G_{t+1} + \alpha \ln k_t$$

Conditional on information at time t expected capital in t+1 is

$$E_t(\ln k_{t+1}) = \ln \left(\frac{1 - \alpha}{2 + \rho} \right) - \frac{v}{1 - \alpha} - g + \alpha \ln k_t$$

⁵² Steady state value in the US is found to be 0.037065

And it has a variance

$$\text{Var}_t(\ln k_{t+1}) = \left(\frac{1}{1-\alpha} \right)^2 \sigma_v^2 + \sigma_g^2 + \frac{2}{1-\alpha} \sigma_{\ln v, \ln G}$$

Now log-linearizing equation (22) and taking expectations conditional on time t

$$E_t(\ln R_{t+1}) = \ln \alpha + (\alpha - 1)E_t(\ln k_{t+1})$$

And inserting the capital

$$E_t(\ln R_{t+1}) = \ln \alpha + (\alpha - 1) \left(\ln \frac{1-\alpha}{2+\rho} - \frac{v}{1-\alpha} - g + \alpha \ln k_t \right)$$

And its variance

$$\text{Var} \ln R_{t+1} = (\alpha - 1)^2 \text{Var} \ln k_{t+1} = \sigma_v^2 + (\alpha - 1)^2 \sigma_g^2 + 2(1-\alpha)\sigma_{\ln v, \ln G}$$

Now again exploiting the relation between the log of the mean and the mean of the log

$$\begin{aligned} \ln(E_t R_{t+1}) &= E_t \ln R_{t+1} + \frac{1}{2} \text{Var} \ln R_{t+1} \\ &= \ln \alpha + (\alpha - 1) \left[\ln \frac{1-\alpha}{2+\rho} - \frac{v}{1-\alpha} - g + \alpha \ln k_t \right] + \frac{1}{2} \left[\sigma_v^2 + (\alpha - 1)^2 \sigma_g^2 + 2(1-\alpha)\sigma_{\ln v, \ln G} \right] \\ (23) \quad E_t R_{t+1} &= \alpha \left(\frac{1-\alpha}{2+\rho} \right)^{\alpha-1} e^{v+(1-\alpha)g + \frac{1}{2}\sigma_v^2 + \frac{1}{2}(\alpha-1)^2\sigma_g^2 + (1-\alpha)\sigma_{\ln v, \ln G}} k_t^{\alpha^2 - \alpha} \end{aligned}$$

At time t expected rate of return depends on the capital per capita. If there is a high capital per capita, expected returns are lower because alpha is less than one and capital therefore has a negative and less than one exponent (that is is an decreasing function).

I find the unconditional mean by the log-linearized equation (22), taking the unconditional expectations

$$E(\ln R_{t+1}) = \ln \alpha + (\alpha - 1)E(\ln k_{t+1})$$

And using the expression for the unconditional capital from (17) to find:

$$E(\ln R_{t+1}) = \ln \alpha + (\alpha - 1) \left(\frac{1}{1-\alpha} \ln \left[\frac{1-\alpha}{2+\rho} \right] - \frac{v}{(1-\alpha)^2} - \frac{g}{1-\alpha} \right)$$

Summing up the notation, the expectation of the logarithm of the return to capital is

$$E(\ln R_{t+1}) = \ln \left(\frac{\alpha(2+\rho)}{1-\alpha} \right) + \frac{v}{1-\alpha} + g$$

And it has a variance of

$$Var(\ln R_{t+1}) = (\alpha - 1)^2 Var(\ln k_{t+1}) = \left(\frac{1}{1-\alpha}\right)^2 \sigma_v^2 + \sigma_g^2 + \frac{2}{1-\alpha} \sigma_{\ln V, \ln G}$$

The expected marginal product of capital is

$$\ln(ER_{t+1}) = E(\ln R_{t+1}) + \frac{1}{2} Var(\ln R_{t+1})$$

$$\ln(ER_{t+1}) = \ln\left(\frac{\alpha(2+\rho)}{1-\alpha}\right) + \frac{v}{1-\alpha} + g + \frac{1}{2}\left(\frac{1}{1-\alpha}\right)^2 \sigma_v^2 + \frac{1}{2}\sigma_g^2 + \frac{1}{1-\alpha}\sigma_{\ln V, \ln G}$$

Then

$$(24) \quad ER_{t+1} = \frac{\alpha(2+\rho)}{1-\alpha} e^{\frac{v}{1-\alpha} + g + \frac{1}{2}\left(\frac{1}{1-\alpha}\right)^2 \sigma_v^2 + \frac{1}{2}\sigma_g^2 + \frac{1}{1-\alpha}\sigma_{\ln V, \ln G}}$$

The steady state marginal product of capital is increasing both in the expectancies, in the variances and in the covariance of the lognormally distributed variables, the population growth and the technological growth. The higher α (=the more capital-intensive production) the higher impact has technological growth and the covariance between the two stochastic variables, and the higher the first fraction, the higher is the marginal product of capital. Changes in labor productivity growth has a greater effect on rates of return than do changes in labor force growth. The change of returns is equal to $e^{\frac{1}{1-\alpha}}$ times the change in labor productivity growth $e^{\Delta v}$ whereas a change in population growth $e^{\Delta g}$ is multiplied by e . The equation shows that a slower economic growth or a slower population growth comes with lower returns on capital.

The riskless rate

There is no riskless asset in this economy, but I can find out what its equilibrium rate of return would be. Introducing this riskless rate of return on the margin, $1 + r_t^f = R_t^f$, paying one unit of consumption next period, it must just like the risky investment satisfy the intertemporal Euler equation (11), replacing R_{t+1} by R_{t+1}^f

$$c_{1,t}^{-1} = \frac{1}{1 + \rho} R_{t+1}^f E_t c_{2,t+1}^{-1}$$

And replacing $c_{1,t}$ and $c_{2,t+1}$ with their values from the household maximization problem

$$R_{t+1}^f = \frac{(1 + \rho)c_{1,t}^{-1}}{E_t c_{2,t+1}^{-1}} = \frac{(1 + \rho) \frac{2 + \rho}{(1 + \rho)w_t}}{E_t \left[R_{t+1} \frac{1}{2 + \rho} w_t \right]^{-1}} = \frac{1}{E_t R_{t+1}^{-1}}$$

Using the expression for the stochastic rate of return (22)

$$R_{t+1}^f = \frac{1}{E_t (\alpha k_{t+1}^{\alpha-1})^1}$$

And now replacing capital in t+1 by its law of movement (17)

$$R_{t+1}^f = \alpha \frac{1}{E_t \left(\frac{1 - \alpha}{(2 + \rho)V_{t+1}^{1-\alpha} G_{t+1}} k_t^\alpha \right)^{1-\alpha}} = \alpha \left(\frac{2 + \rho}{1 - \alpha} \right)^{1-\alpha} k_t^{-\alpha + \alpha^2} \frac{1}{E_t (V_{t+1}^{-1} G_{t+1}^{\alpha-1})}$$

Because both growth factors are i.i.d., the conditional and unconditional expectations are the same. By the use of their moment generating functions⁵³, I can find the product of the moments of the log-normally distributed variables^{54,55} as

⁵³ The moment generating function defined as

$$m(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x) dx \text{ is useful for finding the different moments } E[X^t]$$

⁵⁴ The log-normal distribution has the moment generating function

$$E[X^t] = E[\exp(t \ln X)] = e^{t\mu + \frac{1}{2}t^2\sigma^2} \text{ where } \mu \text{ is the expected value of the log } X, \text{ and } \sigma^2 \text{ its variance.}$$

⁵⁵ When X and Z are jointly log-normally distributed

$$E(z^a x^t) = \exp \left[a\mu_z + t\mu_x + \frac{1}{2} (a^2\sigma_z^2 + t^2\sigma_x^2 + 2at\sigma_{z,x}) \right]$$

$$E_t(V_{t+1}^{-1}G_{t+1}^{\alpha-1}) = e^{-v+(1-\alpha)g+\frac{1}{2}\sigma_v^2+\frac{1}{2}(\alpha-1)^2\sigma_g^2+2(1-\alpha)\sigma_{\ln V, \ln G}}$$

And using this now

$$(25) \quad E_t R_{t+1}^f = \alpha \left(\frac{2+\rho}{1-\alpha} \right)^{1-\alpha} k_t^{-\alpha+\alpha^2} e^{v+(1-\alpha)g-\frac{1}{2}\sigma_v^2-\frac{1}{2}(\alpha-1)^2\sigma_g^2-2(1-\alpha)\sigma_{\ln V, \ln G}}$$

The riskless rate at time t+1 is therefore not constant. At time t it depends on the capital per capita in the same period. If there is a high capital per capita, expected riskless rate are lower because of the same reasoning as for the stochastic return.

To find the unconditional riskless rate, I first need that of the capital

$$ER_{t+1}^f = \alpha \left(\frac{2+\rho}{1-\alpha} \right)^{1-\alpha} e^{v+(1-\alpha)g-\frac{1}{2}\sigma_v^2-\frac{1}{2}(\alpha-1)^2\sigma_g^2-2(1-\alpha)\sigma_{\ln V, \ln G}} Ek_t^{\alpha^2-\alpha}$$

Where the unconditional moment of capital is

$$Ek_t^{-\alpha+\alpha^2} = e^{(-\alpha+\alpha^2)E \ln k_t + \frac{1}{2}(-\alpha+\alpha^2)^2 Var(\ln k_t)}$$

$$= e^{-\alpha(1-\alpha) \left[\frac{1}{1-\alpha} \ln \left(\frac{1-\alpha}{2+\rho} \right) - \frac{v}{(1-\alpha)^2} - \frac{g}{1-\alpha} \right] + \frac{1}{2}(-\alpha(1-\alpha))^2 \left[\left(\frac{1}{1-\alpha} \right)^4 \sigma_v^2 + \left(\frac{1}{1-\alpha} \right)^2 \sigma_g^2 + 2 \left(\frac{1}{1-\alpha} \right)^3 \sigma_{\ln V, \ln G} \right]}$$

Which can be written

$$Ek_t^{-\alpha+\alpha^2} = \left(\frac{1-\alpha}{2+\rho} \right)^{-\alpha} e^{\frac{\alpha v}{1-\alpha} + \alpha g + \frac{1}{2} \frac{\alpha^2}{(1-\alpha)^2} \sigma_v^2 + \frac{1}{2} \alpha^2 \sigma_g^2 + \frac{\alpha^2}{1-\alpha} \sigma_{\ln V, \ln G}}$$

And now inserting this into the riskfree expectation

$$ER_{t+1}^f = \alpha \left(\frac{2+\rho}{1-\alpha} \right)^{1-\alpha} e^{v+(1-\alpha)g-\frac{1}{2}\sigma_v^2-\frac{1}{2}(\alpha-1)^2\sigma_g^2-2(1-\alpha)\sigma_{\ln V, \ln G}} \left(\frac{1-\alpha}{2+\rho} \right)^{-\alpha} e^{\frac{\alpha v}{1-\alpha} + \alpha g + \frac{1}{2} \frac{\alpha^2}{(1-\alpha)^2} \sigma_v^2 + \frac{1}{2} \alpha^2 \sigma_g^2 + \frac{\alpha^2}{1-\alpha} \sigma_{\ln V, \ln G}}$$

To obtain the unconditional expectation

$$(26) \quad ER_{t+1}^f = \alpha \left(\frac{2+\rho}{1-\alpha} \right) e^{\frac{1}{1-\alpha}v+g+\frac{\alpha-1}{2}\sigma_v^2+(\alpha-\frac{1}{2})\sigma_g^2+\frac{-\alpha^2+4\alpha-2}{1-\alpha}\sigma_{\ln V, \ln G}}$$

With no uncertainty or growth the unique gross riskless rate would equal $\frac{\alpha(2+\rho)}{1-\alpha}$

which is positive. The incremental effect from the growth terms arises because consumption is likely to be higher in the future. Agents with concave utility would like to borrow against future consumption in order to smooth life time consumption. The higher the curvature of the utility function and the larger the expected growth, the greater is the desire to smooth consumption. In equilibrium this will lead to a higher

interest rate because aggregate consumers cannot simultaneously increase their current consumption.

The negative impact of the variances (α assumed to be <0.5) arises due to the demand for precautionary saving (as discussed in footnote 48-50). In a world of uncertainty agents will hedge against future unfavorable realizations by saving more. Thus at equilibrium the interest rate must fall to meet the enhanced demand for savings. The riskfree rate decreases⁵⁶ in the covariance between the two, which is also a consequence of precautionary savings. An increased covariance means that the aggregate risk increases, as the two risks move in the same direction, i.e. the gain of diversification decreases.

Equity premium

Given the unconditional gross marginal rate of capital and the riskless rate I can compute the unconditional expected equity premium as the difference between the two, thus

$$E \left[R_{t+1} - R_{t+1}^f \right] = ER_{t+1} - ER_{t+1}^f$$

$$= \frac{\alpha(2 + \rho)}{1 - \alpha} e^{\frac{v}{1-\alpha} + g + \frac{1}{2} \left(\frac{1}{1-\alpha} \right)^2 \sigma_v^2 + \frac{1}{2} \sigma_g^2 + \frac{1}{1-\alpha} \sigma_{\ln v, \ln G}} - \frac{\alpha(2 + \rho)}{1 - \alpha} e^{\frac{v}{1-\alpha} + g + \frac{\alpha-1}{(1-\alpha)^2} \sigma_v^2 + (\alpha-\frac{1}{2}) \sigma_g^2 + \frac{-\alpha^2 + 4\alpha - 2}{1-\alpha} \sigma_{\ln v, \ln G}}$$

Which can be written as

(27)

$$E \left[R_{t+1} - R_{t+1}^f \right] = \frac{\alpha(2 + \rho)}{1 - \alpha} e^{\frac{v}{1-\alpha} + g + \frac{1}{2} \left(\frac{1}{1-\alpha} \right)^2 \sigma_v^2 + \frac{1}{2} \sigma_g^2 + \frac{1}{1-\alpha} \sigma_{\ln v, \ln G}} \left(1 - e^{-\frac{1}{1-\alpha} \sigma_v^2 - (1-\alpha) \sigma_g^2 + \frac{-\alpha^2 + 4\alpha - 3}{1-\alpha} \sigma_{\ln v, \ln G}} \right)$$

The equity premium is increasing in v , g , the variances and the covariance⁵⁷. It increases with increased expectancies because it grows exponentially and thus the

⁵⁶ The expression $-\alpha^2 + 4\alpha - 2$ has roots $2 - \sqrt{2}$ and $2 + \sqrt{2}$. Because α is assumed less than $\frac{1}{2}$, the expression is negative

⁵⁷ Obviously increasing v and g increases the expression. This is also the case for the variances and covariance, proved by

impact on the risky return is higher than that on the riskfree. Increasing variances and covariance gives as discussed a higher return on capital and a lower riskfree rate, resulting in a higher equity premium. The impact of the logarithm of growth in technology is higher than that of the population. The equity premium is sensitive to technology which I will explore in the calibration.

$$\frac{\partial EP}{\partial \sigma_v^2} = \frac{1}{2} \left(\frac{1}{1-\alpha} \right)^2 \frac{\alpha(2+\rho)}{1-\alpha} e^{\frac{v}{1-\alpha} + g + \frac{1}{2} \left(\frac{1}{1-\alpha} \right)^2 \sigma_v^2 + \frac{1}{2} \sigma_g^2 + \frac{1}{1-\alpha} \sigma_{\ln V, \ln G}} \left[1 - e^{-\frac{1}{1-\alpha} \sigma_v^2 - (1-\alpha) \sigma_g^2 + \frac{-\alpha^2 + 4\alpha - 3}{1-\alpha} \sigma_{\ln V, \ln G}} \right]$$

$$+ \left(\frac{\alpha(2+\rho)}{1-\alpha} e^{\frac{v}{1-\alpha} + g + \frac{1}{2} \left(\frac{1}{1-\alpha} \right)^2 \sigma_v^2 + \frac{1}{2} \sigma_g^2 + \frac{1}{1-\alpha} \sigma_{\ln V, \ln G}} * \frac{1}{1-\alpha} e^{-\frac{1}{1-\alpha} \sigma_v^2 - (1-\alpha) \sigma_g^2 + \frac{-\alpha^2 + 4\alpha - 3}{1-\alpha} \sigma_{\ln V, \ln G}} \right)$$

which is positive

$$\frac{\partial EP}{\partial \sigma_g^2} = \frac{\alpha(2+\rho)}{2(1-\alpha)} e^{\frac{v}{1-\alpha} + g + \frac{1}{2} \left(\frac{1}{1-\alpha} \right)^2 \sigma_v^2 + \frac{1}{2} \sigma_g^2 + \frac{1}{1-\alpha} \sigma_{\ln V, \ln G}} \left[1 - e^{-\frac{1}{1-\alpha} \sigma_v^2 - (1-\alpha) \sigma_g^2 + \frac{-\alpha^2 + 4\alpha - 3}{1-\alpha} \sigma_{\ln V, \ln G}} \right]$$

$$+ \left(\frac{\alpha(2+\rho)}{1-\alpha} e^{\frac{v}{1-\alpha} + g + \frac{1}{2} \left(\frac{1}{1-\alpha} \right)^2 \sigma_v^2 + \frac{1}{2} \sigma_g^2 + \frac{1}{1-\alpha} \sigma_{\ln V, \ln G}} * (1-\alpha) e^{-\frac{1}{1-\alpha} \sigma_v^2 - (1-\alpha) \sigma_g^2 + \frac{-\alpha^2 + 4\alpha - 3}{1-\alpha} \sigma_{\ln V, \ln G}} \right)$$

is also positive and

$$\frac{\partial EP}{\partial \sigma_{\ln V, \ln G}} = \frac{\alpha(2+\rho)}{(1-\alpha)^2} e^{\frac{v}{1-\alpha} + g + \frac{1}{2} \left(\frac{1}{1-\alpha} \right)^2 \sigma_v^2 + \frac{1}{2} \sigma_g^2 + \frac{1}{1-\alpha} \sigma_{\ln V, \ln G}} \left[1 - e^{-\frac{1}{1-\alpha} \sigma_v^2 - (1-\alpha) \sigma_g^2 + \frac{-\alpha^2 + 4\alpha - 3}{1-\alpha} \sigma_{\ln V, \ln G}} \right]$$

$$+ \left(\frac{\alpha(2+\rho)}{1-\alpha} e^{\frac{v}{1-\alpha} + g + \frac{1}{2} \left(\frac{1}{1-\alpha} \right)^2 \sigma_v^2 + \frac{1}{2} \sigma_g^2 + \frac{1}{1-\alpha} \sigma_{\ln V, \ln G}} * \left(-\frac{-\alpha^2 + 4\alpha - 3}{1-\alpha} \right) e^{-\frac{1}{1-\alpha} \sigma_v^2 - (1-\alpha) \sigma_g^2 + \frac{-\alpha^2 + 4\alpha - 3}{1-\alpha} \sigma_{\ln V, \ln G}} \right)$$

is positive. The expression $-\alpha^2 + 4\alpha - 3$ is negative as long as α is less than one.

3. Calibration

Returns

Mehra and Prescott (1985) reported arithmetic averages⁵⁸ to summarize the historical information. I will do the same. This is the appropriate statistic to use when the objective is to obtain the mean value of the investment. If the objective had been the median, it should have been computed as the geometric sample average⁵⁹. When returns are serially correlated, then the arithmetic average can lead to misleading estimates “The best available evidence indicated that stock returns were uncorrelated over time” (p3, Mehra and Prescott, 2003)

Technological growth

The technological growth V_t is to be measured as Solow residual from the data. It is given⁶⁰ by

$$(28) \quad V_t - 1 = \frac{\frac{\partial A}{\partial t}}{A} = \frac{\frac{\partial Y}{\partial t}}{Y} - \alpha \frac{\frac{\partial K}{\partial t}}{K} - (1 - \alpha) \frac{\frac{\partial N}{\partial t}}{N}$$

Robert Solow defined rising productivity as rising output with constant capital and input. It is residual because it cannot be accounted for by capital accumulation or population growth. The Solow residual measures total factor productivity and it is here simply assumed exogenous.

⁵⁸ The arithmetic average of an n period investment is given by

$$r_A = \frac{1}{n} \sum_{t=1}^n r_t$$

⁵⁹ For an n-period investment the geometric average rate of return is given by

$$1 + r_G = \left[(1 + r_1)(1 + r_2) \dots (1 + r_n) \right]^{\frac{1}{n}}$$

⁶⁰ The derivation is given in Appendix D

The case of Norway

The time series used

Population L_t :

Is employed persons, employees and self-employed, aged 16-74⁶¹. I use the yearly series dating back to 1972⁶². The data is shown in the figure:



Figure 4: Employed 1000 persons in Norway, 1972-2005

And the corresponding annual growth in the labor force, G_t is illustrated in the next figure:

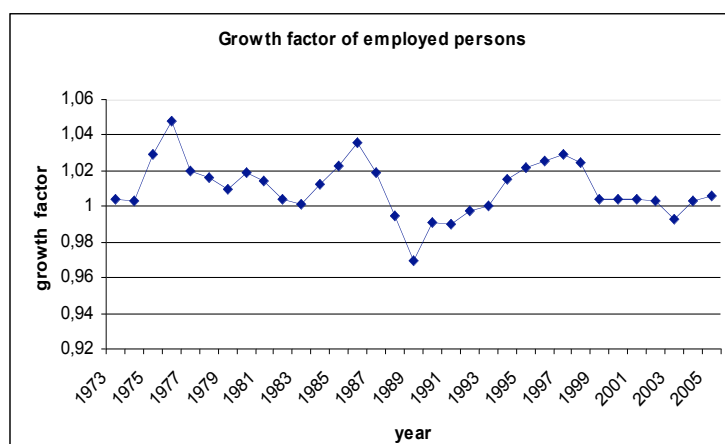


Figure 5: Growth factor of employed persons, 1973-2005

⁶¹Received from Statistics Norway (SSB), Table 05111, http://statbank.ssb.no/statistikkbanken/Default_FR.asp?PXSid=0&nvl=true&PLanguage=0&tilside=selecttable/hovedtabellHjem.asp&KortnavnWeb=aku

⁶²Earlier data is available for every decade at the Folketellingene, Statistics Norway; not considered here because the more frequent data gives a longer time series.

The proportion of employed persons to the total population over 15 years old, has decreased over time. In 1875 it was 61,7%, 56,1% in 1946 and 43,2% in 1990. There has been a movement from agricultural employment to production and service employment. Women entering the labor force accounted for about $\frac{3}{4}$ of the increased employment in the 70s and up to 1987. From 1987 to 1991 there was a decrease, mainly affecting men. People aged 67-74 are a declining group in the labor force, this may be due to the declining employment in farming. Especially men in the age 55-66 retires use different early retirement programs. The labor market participation of youth 16-24 has varied, it declined in 1988 when full-time work was replaced by full-time studies.

Total production Y_t ⁶³:

Real Gross Domestic Product measures the gross income generated from domestic production. The production approach⁶⁴ adds compensation of employees, operating surplus, consumption of fixed capital and taxes on production and subtracts subsidies.

Capital K_t ⁶⁵:

I use the series of total fixed assets. An asset is considered fixed if it is a product of a production process and used repeatedly or continuously over a horizon of minimum a year. It includes both material capital, such as buildings, machinery and hardware, and immaterial capital such as software and the search for minerals and oils. Not-produced capital is not included. The series is given in current prices and therefore adjusted by the yearly Consumer Price Index⁶⁶. The yearly index is an average of the monthly indexes. The CPI measures the changes in prices for household goods and services including charges and fees.

Stock market returns R_t ⁶⁷:

The return on equity consists of data from Norway Statistics and Oslo Stock Exchange for the years 1960-1966, the “NHH market index/Amadeus” over the years

⁶³ <http://www.ssb.no/histstat/aarbok/ht-0901-355.html>

⁶⁴ It can also be approached by expenditure or income.

⁶⁵ <http://www.ssb.no/histstat/tabeller/22-22-19.txt>

⁶⁶ <http://www.ssb.no/emner/08/02/10/kpi/tab-01.html>

⁶⁷ Received from prof. Thore Johnsen.

1967-1982 and the OSE index from 1983 to 2005. They are based on reinvestment of dividends throughout the year. The series are assumed to have similar properties.

Interest rates R_t^f ⁶⁸:

The 3 month Norwegian interbank interest rates (NIBOR) dates back to 1980. The old Nevi interest rates are added for the period 1966-1979 and an official discount rate is used for the years 1960-1965. The series are assumed to have similar properties.

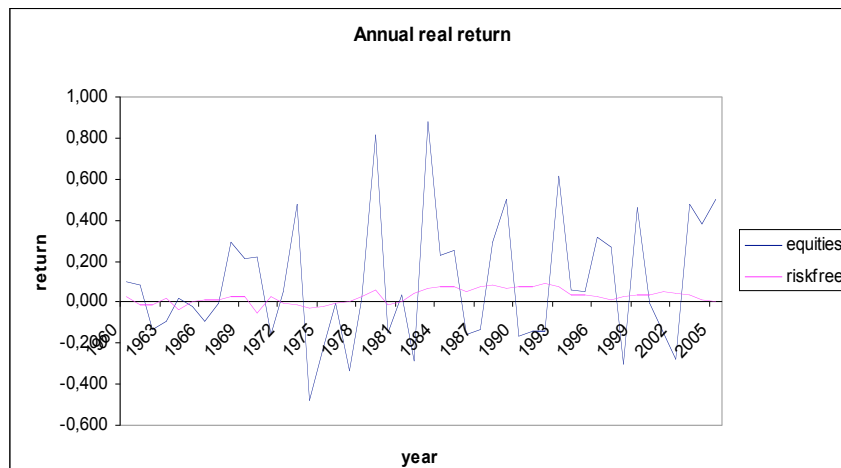


Figure 6: Annual stock market return and risk free rate in Norway 1960-2005

In the model outlined each period is 30 years. The real risky return in the stock market and the real risk free rate for such a period are

$$(28) \quad ER_{t+1} = (30 * 0.0900) + 1 = 3.700$$

$$(29) \quad ER_{t+1}^f = (30 * 0.0270) + 1 = 1.810$$

Which yields an equity premium of 1.89

I estimate α be 0.323 by simply saying that a simplified GDP consists of payments to labor and payments to capital. This is an average of the annual shares from 1970 to 2005⁶⁹. Mostly the other components that it ideally should be adjusted for concern the government, which is absent in this model. Moreover they make up a tiny fraction of the GDP.

⁶⁸ Also received from prof. Thore Johnsen at NHH.

⁶⁹ http://www.ssb.no/emner/09/01/nr/tab_1997-2005_04.html

Calibrating the model, I first calculate the technological growth⁷⁰ V_t , according to (28) using the fact that $\alpha = 0.323$. It is the growth in GDP less what is accounted for by capital and labor. It is shown as annual growth factor⁷¹ in the next figure:

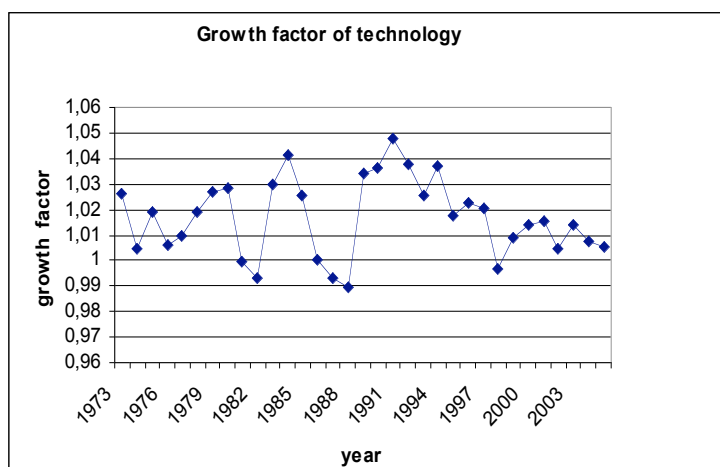


Figure 7: Growth factor of technology in Norway 1973-2005

Summary statistics

Having assumed that growth factors have log-normal distributions, taking logs on the data, summarizing statistics and multiplying them⁷² by 30^{73} to get the horizon of the model, the model has the following means, variances and covariance:

Summary statistics		
	Mean	Variance
$\ln G(t)$	0,298684	0,006997
$\ln V(t)$	0,505592	0,006621
Covariance		
Covar($\ln G(t)$, $\ln V(t)$)	-0,001914	

Table 1: Summary statistics for the logarithm of the growth factors in technology and population

⁷⁰ Some have argued that the measure of productivity growth should be adjusted for the quality improvements of output. This is not done here. When not done the productivity growth will be biased downwards.

⁷¹ 1 is added to the rate of growth

⁷² Except the covariance from which I extract the correlation coefficient of -0.27788

⁷³ Remember that population can be written

$$L_t = L_0 G_1 G_2 \dots G_t \text{ or in logs } \ln L_t = \ln L_0 + \ln G_1 + \ln G_2 + \dots + \ln G_t$$

where all G_i has the expectation $E(\ln G_i) = g$ and a variance of $Var(\ln G_i) = \sigma_g^2$

Then both the expectation $E_0(\ln L_t) = \ln L_0 + t * g$ and the variance $Var_0(\ln L_t) = t * \sigma_g^2$ increases linearly with time.

The mean of the logarithm of the technology growth is higher than that of the population growth. Their variances are about the same. The covariance is negative which means that the variables tend to move slightly in opposite directions.

Now calibrating the empirical data into equations (24) and (26), leaves a rho of 145.77.⁷⁴ This is an unrealistically high value. It corresponds to an annual rho of 0.1809. Imposing an $\alpha = 0.36$ and 0.4 result in a $\rho = 102.65$ and 74.75⁷⁵.

⁷⁴ Maple printouts in Appendix E

⁷⁵ Changing α changes the technological growth given from the Solow residual (28)

The case of the US

The US is a large open economy. Blanchard, Giavazzi and Sa's (2005) estimates are that U.S. financial assets are currently half of the world total. As a large open economy, I expect that it will better fit my model than the Norwegian small open economy.

The time series used

Population L_t ⁷⁶:

The series is employed persons from the age 16 and over. The series dates back to 1948 and the periodicity is quarterly. I estimate the annual observation as the average of the quarterly. I will just use the annual data for simplicity, because I need them together with output and capital to compute the Solow residual and the covariance. Data for capital are only available as annual data.



Figure 8: Employed 1000 persons 1948-2005, US

Or if illustrated by the growth factor G_t :

⁷⁶ Table created from <http://data.bls.gov/PDQ/outside.jsp?survey=ln>

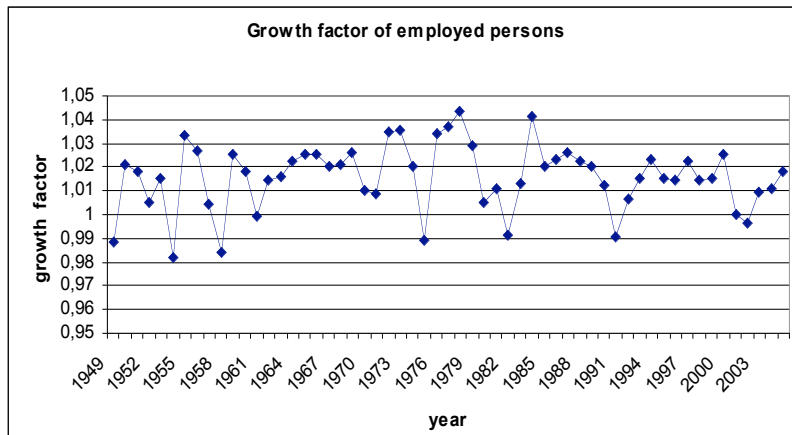


Figure 9: Growth factor of employed persons, 1948-2005

The US population experience some of the same tendencies as the Norwegian. There is a declining teen labor force participation due to an increased emphasis on school rather than work⁷⁷. The historical trend of an increasing labor force participation rate for women goes on with a declining rate for men. People retire earlier than before. The growth of employed persons is higher than in Norway and the mean age is younger.

Total production Y_t ⁷⁸:

The series of Gross Domestic Product dates back to 1929. It is adjusted for inflation by the Consumer Price Index⁷⁹.

Capital K_t ⁸⁰:

The series of fixed assets dates back to 1925. It consists of assets that provide capacity to produce output and income, such as equipment, software, and structures, including owner-occupied housing. It does not include human capital and land. It is adjusted for depreciation using BEA's assumed patterns.

Stock market returns R_t :

⁷⁷ <http://www.bls.gov/opub/ils/pdf/opbils49.pdf>

⁷⁸ Table 1.1.5 at <http://www.bea.gov/bea/dn/nipaweb/SelectTable.asp?Selected=N>

⁷⁹ <http://minneapolisfed.org/Research/data/us/calc/hist1913.cfm>

⁸⁰ Table 1.1 at <http://www.bea.gov/bea/dn/FA2004/SelectTable.asp>

Mehra and Prescott (2003)'s series is an updated version of the (1985). The data for the period 1802-1871 is based on Schwert (1990). Shiller (1989) is the source for the period 1871 to 1926. The yield on the Standard and Poor 500 Index (S & P). The index is based on reinvestment of dividends. From 1921 the data are obtained from NYSE database at the Center for Research in Security Prices (CRSP).

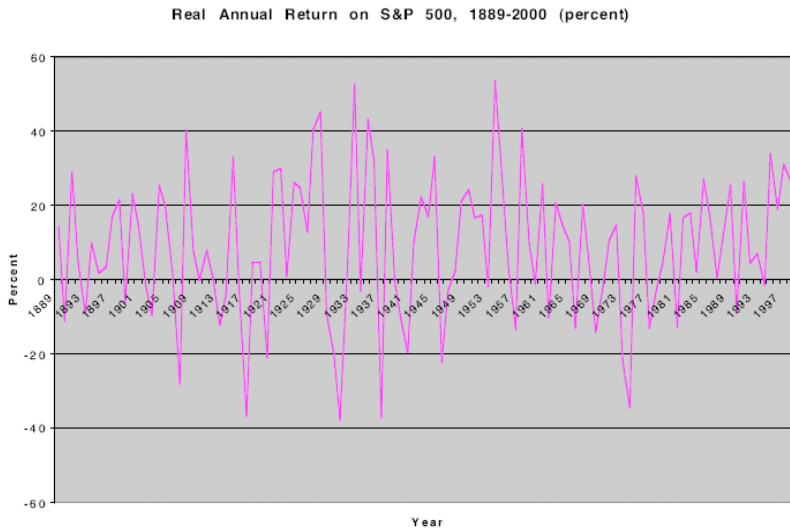


Figure 10: Real annual return on S&P 500, 1889-2000

Interest rates R_t^f ⁸¹

Siegel (1998) has constructed data for the period 1802-1871, later data is taken from Homer (1963) and Treasury bills is the estimate from its origination in 1931.

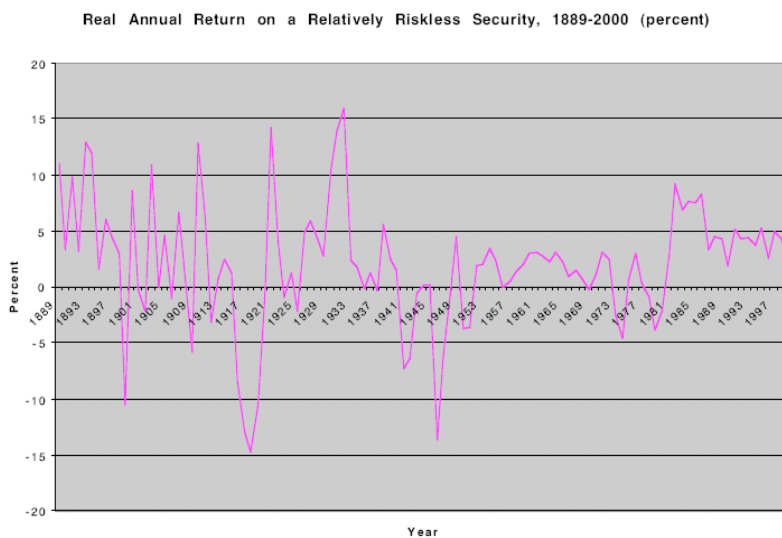


Figure 11: Real annual return on relatively riskless asset 1889-2000

⁸¹ The figures are taken from Mehra and Prescott (2003)

The equity premium has varied a lot over time and been negative from time to time. Variation in equity premium has been counter-cyclical⁸².

The equity premium vary based on which data set are used. This is illustrated in the table, which is Table 1 in Mehra and Prescott (2003b):

U.S. equity premium using different data sets			
Data set	% real return on a market index (mean)	% real return on a relatively riskless security (mean)	% equity premium (mean)
1802–1998 (Siegel)	7.0	2.9	4.1
1871–1999 (Shiller)	6.99	1.74	5.75
1889–2000 (Mehra–Prescott)	8.06	1.14	6.92
1926–2000 (Ibbotson)	8.8	0.4	8.4

Table 2: Equity premium on different data set in the US

I will use the Mehra/Prescott equity premium of 6.92 % in my model. For a 30 years period the real risky return in the stock market and the real risk free rate are

$$(30) \quad ER_{t+1} = (30 * 0.0806) + 1 = 3.418$$

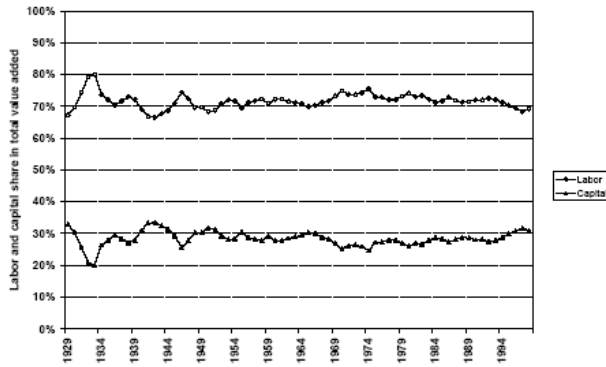
$$(31) \quad ER_{t+1}^f = (30 * 0.0114) + 1 = 1.342$$

Which yields an equity premium of 2.076, which is higher than the Norwegian.

In the US the capital share in national income is about $1/3$ ⁸³, while labor is about $2/3$, according to Acemoglu (2006).

⁸² As documented by Mehra and Prescott (2003)

⁸³ There are several estimates on alpha, ranging from 0.25 to 0.4. The estimates depends on how to measure capital.



Capital and Labor Share in the U.S. GDP.

Figure 12: Capital and labor share in the US 1929-1999. This is the figure at page 52, Acemoglu (2006). It shows that the shares are stable over time.

Computing the rate of growth of technology according to (28) by using the fact that $\alpha = 0.3333$, then the annual growth factor can be illustrated as:

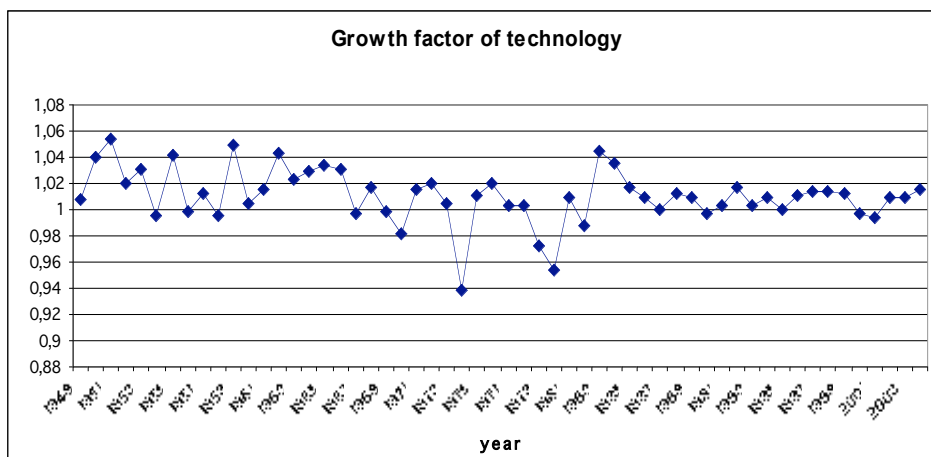


Figure 13: Growth factor of technology in the US, 1949-2005

The logarithms of the growth factors⁸⁴ have the following characteristics

Summary statistics		
	Mean	Variance
$\ln G(t)$	0,46586512	0,00581717
$\ln V(t)$	0,30639714	0,01278900
Covariance		
$\text{Cov}(\ln G(t), \ln V(t))$	0,00177169	

⁸⁴ The correlation coefficient between the two variables is 0.20541

Table 3: Summary statistics for the logarithms of the growth factors in technology and population, US 1949-2005

The expectancy of the logarithm of the population growth is higher than that of the technological growth, whereas the variance of the logarithm of the technological growth is higher than that of the population growth. There is a positive covariance between the two variables.

Calibrating the empirical values, the distributions of the growth factors and the historical equity premium into equation (27) requires a ρ of 56.88⁸⁵. It imply a discounting of 0.0173, corresponding to an annual ρ of 0.1449. This is a tough discounting, placing very high weight to first-period consumption.

If following the work of Kydland and Prescott (1982) then $\alpha = 0.36$ or Cooley and Prescott (1995) when $\alpha = 0.4$, give respectively values of rho of 50.28 and 40.48 in this model⁸⁶.

⁸⁵ Maple printouts in Appendix E

⁸⁶ Changing α also changes the technological growth through the Solow residual

Analysis

The historical average risk premium of 6.3 % in Norway and Mehra and Prescott (2003)'s equity premium of 6.92 % in the US is about the same, the US yields an excess equity premium of 0.62% over the Norwegian⁸⁷. The US time series account for the period 1889-2000, which is a longer series than the Norwegian. In the original statement Mehra and Prescott (1985) found an equity premium of 6.18%. The Norwegian risk-free rate of 2.7% is higher than that of the US of 1.14 %, and the return in the stock market is correspondingly higher, 9.00 % compared to 8.06 %. The higher return in the Norwegian stock market comes with a higher volatility, which can be seen comparing Figure 6 vs Figure 10. The risk-free rate in the US includes a higher variance than the Norwegian. The volatility of the Norwegian risk premium has been higher than the American, which can be seen comparing Figure 6 , Figure 10 and 11.

A rho of 0.01⁸⁸ or 0.3478 for a period of 30 years predicts on the Norwegian data a risky return of 3.2119 and a riskfree of 3.1818 which yields an equity premium of 0.0301 (0.100% annually). Corresponding values for the US are 3.0210 and 2.9382, and an equity premium of 0.0828 (or annually 0.276%). The predicted risk free interest rate is far higher than real world observations, just as found earlier by Mehra and Prescott (1985), Kocherlakota (1996) and others. The model predicts nearly three times higher equity premium in the US than in Norway. But it is not even close to the observed value of 2.076. Plotting the equity premium as a function of rho shows that we need an unrealistic high rho, calculated to be 56.88.

⁸⁷ Kvalvik and Medbøen (2002) found that the Norwegian equity premium was 0.35 % higher than the US. Their analysis considered a shorter time series 1967-2001, and it was compared to Mehra and Prescott (1985).

⁸⁸ By the use of another standard value from the literature, as Eisfeldt (2006), for the subjective discount rate $\beta = 0.96$, i.e. $\rho = 0.04167$ or on the 30 years horizon $\rho = 2.403$, into equations (1.19) and (1.21), I get an expected risky return of 6.0234 on the Norwegian data and 5.6654 on the American and an expected riskless return of 5.9671 in the case of Norway and 5.5101 in the US. This yields an equity premium of 0.0563 (0.188% annually) in Norway and 0.1552 (0.517% annually) in the US. In this case the model predicts returns to the equities far higher than observed.

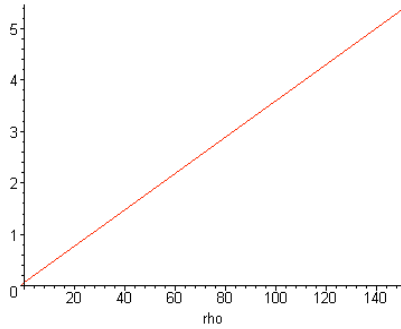


Figure 14: US equity premium as a function of rho

But the standard assumption of $\rho = 0.01$ does not fit the stationarity condition (16). It requires that $EV_t \leq 1 + \rho$. $EV_t = 1.6635$ in Norway demands a ρ of minimum 0.6635 (or annually 0.0137) and $EV_t = 1.3672$ in the US requires a ρ of minimum 0.3672 (0.0105 annually).

The differences in the predictions between the countries stems from different characteristics in the economies:

	Mean G(t)	Mean V(t)	Variance G(t)	Variance V(t)	Covariance(lnG(t),lnV(t))
Norway	0,29868	0,50559	0,00700	0,00662	-0,00191
The US	0,46587	0,30638	0,00582	0,01279	0,00177

Table 4: The distributions of the growth factors, Norway and the US

The US has a relatively higher growth in population than Norway. Norway has on the other side a higher growth in technology. There is more variability in both variables in Norway than in the US. This is what we would expect as Norway is a small economy and therefore more affected by any kind of shock. The higher means and variances account for a higher equity premium. The covariance is negative between the variable in the case of Norway, and positive in the US. A negative covariance decreases the equity premium according to (27). The alpha's are about the same, the slightly higher one in the US implies more weight on the variables involved (except g), altering the equity premium.

Different characteristics have different influence on the excess return according to the model. The figure shows excess return according to (26) in the US as a function of rho

and the covariance between the variables. A higher covariance and a higher rho both imply a higher equity premium.

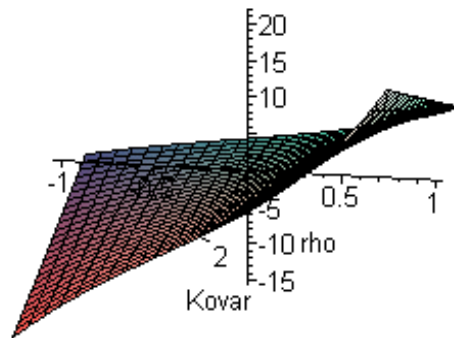


Figure 15: US equity premium as a function of rho and the covariance

The return to equity (24) moves with α and ρ as illustrated when using the Norwegian data for V and G

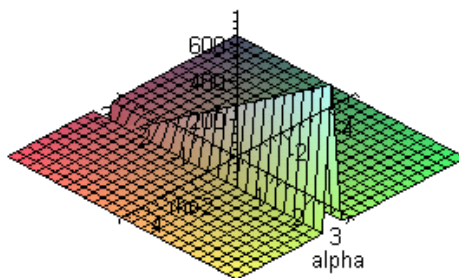


Figure 16: Norwegian rate of risky return as a function of alpha and rho

From the figure we can see as mentioned earlier that the rate of return is extremely sensitive to technology. Especially in the region $0.5 < \alpha < 1.5$. This is an unattractive feature of the Cobb-Douglas production function. Blanchard and Weil (2002) uses a necessary and sufficient condition (they consider a stationary economy) which is also

dependent on the technology mainly that $\alpha \geq (1 - \alpha)\beta$, but this cannot be fulfilled with realistic values of the parameters.

Omitting the log-growth in population, its variance and covariance with technology on the US (setting $g = 0, \sigma_g^2 = 0$ and $\sigma_{\ln G, \ln V} = 0$), gives a calibrated value of $\rho = 134.06$ which is significantly higher than 56.88 in the previous. Imposing $\rho = 0.3478$ implies a risky return of 1.8854, riskless rate of 1.8496 and equity premium of 0.0358 (0.119% annually). This means that growth and variation in population, and its interaction with technology may account for about the difference, 0.047, or 0.157% on an annual basis. A corresponding analysis on the Norwegian data leaves a calibrated value of $\rho = 189.49$. A $\rho = 0.3478$ gives a risky return of 2.3809 and a riskless rate of 2.3578. The equity premium of 0.0232 over 30 years corresponds to an annual equity premium of 0.023%.

The analysis shows that the stochastic population growth cannot account for the high observed equity premium. ρ must be unreasonable high to match the realized excess return to equities, and which entails even more unrealistically values of return. $\rho = 56.88$ in the US gives a risky return of 75.76 (or net annual return of 249%) and riskless of 73.68 (net annual return of 242%). Bullard and Feigenbaum (2004) claims that a ρ in the interval 0.027 to 0.029 produces real interest rates close to the U.S. data.

More risk aversion can be added to the model by use of the CRRA utility (9). This would be an appropriate augmentation because more risk aversion necessarily requires a higher equity premium. When people are more risk averse they demand a higher premium for bearing risk. According to Mehra and Prescott (1985) risk aversion should not exceed $\theta = 10$. Higher risk aversion will also alter precautionary savings, people subject to more uncertain income will save more in case of future states with low income.

The actual probability distributions of the variables are not tested. There may be another distribution fitting the data better. The data are neither tested for serial

correlation. If there is substantial serial correlation a more appropriate approximation would be as a Markov chain.

Conclusion

The calibrated values of a rho of 145.77 in Norway and 56.88 in the US indicate that the equity premium puzzle is not solved by the stochastic population. The values are far too high compared to standards of literature. That the Norwegian rho is more than twice as high as the American suggests that the equity premium puzzle is even bigger here.

But the calibration shows that stochastic population might explain a small small part of the equity premium. And the part explained is larger in the case of the US than in the case of Norway. This was as expected because the US is a large open economy, and rates of return are therefore more a consequence of internal events than in a small open economy, which takes the rate of return as exogenous from abroad.

The best available explanation of the equity premium in my opinion is McGrattan and Prescott (2003). They adjust equity returns by subtracting diversification costs and taxes and raises the riskfree rate by use of long-term debt, which is what most households hold instead of T-bills. The periods of the WWII and the Korean War are dismissed from the data set because markets were not functioning under governmental restrictions. The adjusted data gives an average excess real return that is less than one percent, and they claim that there is no longer a puzzle.

Appendix A

Given that

$$(1A) \quad 1 = \frac{1}{1+\rho} E_t \left\{ \frac{u'(c_{t+1})}{u'(c_t)} (1 + \tilde{r}_{t+1}) \right\}$$

And

$$(2A) \quad 1 = \frac{1}{1+\rho} E_t \left\{ \frac{u'(c_{t+1})}{u'(c_t)} \right\} (1 + r_{t+1}^f)$$

Then by use of the covariance decomposition⁸⁹ in equation (1A)

$$1 = \frac{1}{1+\rho} E_t \left[\frac{u'(c_{t+1})}{u'(c_t)} \right] E_t (1 + \tilde{r}_{t+1}) + \frac{1}{1+\rho} \text{cov}_t \left\{ \frac{u'(c_{t+1}), (1 + \tilde{r}_{t+1})}{u'(c_t)} \right\}$$

Then replacing 1 by its value from (2A) results in

$$E_t (1 + \tilde{r}_{t+1}) = 1 + r_{t+1}^f + \text{cov}_t \left\{ \frac{-u'(c_{t+1}), (1 + \tilde{r}_{t+1})}{E_t [u'(c_{t+1})]} \right\}$$

The equation in the text

Appendix B

The original analysis⁹⁰

Mehra and Prescott (1985) derive a variation of Lucas' (1978) pure exchange model. They assume that the growth rate of the endowment follows a Markov process. The Lucas model assumes that the endowment level follows a Markov process. The assumption of Mehra and Prescott enables them to include the non-stationarity of the consumption series in their model. The specification gives stationary and easily determined aggregate per capita consumption and asset prices.

The economy has a single representative household. It has preferences given by

⁸⁹ The covariance decomposition states that for two random variables, x and y:

$$E(xy) = E(x)E(y) + \text{cov}(x, y)$$

⁹⁰ This draws on Mehra and Prescott (1985), Haug (2003) and Appendix B, Mehra & Prescott (2003)

$$(1B) \quad E_0 \left\{ \sum_{t=0}^{\infty} \beta^t U(c_t) \right\}, 0 < \beta < 1$$

where c_t is per capita consumption, β is the subjective time discount factor, $E\{\cdot\}$ is the expectation operator conditional on information available at the time, and $U: R_+ \rightarrow R$ is the increasing concave utility function. The equilibrium return process is stationary when the utility function is restricted to be of the constant relative risk aversion (CRRA) class

$$(2B) \quad U(c, \alpha) = \frac{c^{1-\alpha}}{1-\alpha}, 0 < \alpha < \infty$$

They assume there is one productive unit which produces output y_t in period t , the period dividend. There is one equity share with price p_t^e that is competitively traded as a claim to the stochastic process $\{y_t\}$. The growth rate is a Markov⁹¹ process:

$$(3B) \quad y_{t+1} = x_{t+1} y_t$$

where $x_{t+1} = \frac{c_{t+1}}{c_t} \in \{\lambda_1, \dots, \lambda_n\}$ is the growth rate, having a transition probability matrix

$\Pr \{x_{t+1} = \lambda_i \mid x_t = \lambda_j\} = \phi_{ij}$. In matrix notation:

$$\Phi = \begin{bmatrix} \phi_{11} & \dots & \phi_{1n} \\ \dots & & \dots \\ \phi_{n1} & \dots & \phi_{nn} \end{bmatrix}$$

$\phi_{ij}, y_0 > 0$. ϕ_{ij} denotes the conditional probability of going from state j to state i . The Markov chain is ergodic⁹².

There is also risk free asset paying one unit of consumption the next period and having a price of p_t^f . The superscript tells the type of asset in consideration. Both

⁹¹ A Markov process for x_t is:

$$\Pr \{x_{t+1} = x_j \mid x_t = x_i, x_{t-1} = x_k, x_{t-2} = x_l, \dots\} = \Pr \{x_{t+1} = x_j \mid x_t = x_i\}$$

Current realization contains all information needed to make a forecast.

⁹² All states are recurrent and aperiodic. Recurrence means that it is possible to return to a given state, aperiodicity that it can be entered at any time. When assumed that all $\phi_{ij} > 0$ the process is irreducible, which says that from any state it is possible to reach all other states. Then the process converges to a limiting probability, an unconditional distribution for x_t .

assets are in zero-net supply⁹³. They are priced and traded ex-dividend at time t, in terms of the time t consumption good.

Pricing according to the CCPM, as described earlier⁹⁴, a security with a process $\{y_s\}$ yields

$$(4B) \quad p_t = E_t \left\{ \sum_{s=t+1}^{\infty} \beta^{s-t} \frac{u'(c_s)}{u'(c_t)} d_s \right\}$$

The dividend process for the equity share is $\{y_s\}$, the marginal utility is $u'(c) = c^{-\alpha}$ and we can write the price of the equity as

$$(5B) \quad p_t^e = p^e(x_t, y_t) = E_t \left\{ \sum_{s=t+1}^{\infty} \beta^{s-t} \frac{y_t^\alpha}{y_s^\alpha} y_s \right\}$$

Using the fact that $m_{t,s} = \left(\frac{1}{1+\rho} \right)^{s-t} \frac{u'(c_s)}{u'(c_t)}$ it can be written⁹⁵

$$(6B) \quad p_t^e = E_t \left[m_{t,t+1} (y_{t+1} + p_{t+1}^e) \right]$$

The equilibrium is time invariant functions of the state (x_t, y_t) . Now consider a given state where $y_t = c$ and $x_t = \lambda_t$ and denote it as (c, i) . The price of the equity in this state is⁹⁶

$$(7B) \quad p^e(c, i) = \beta \sum \phi_{ij} \lambda_j^{-\alpha} \left[p^e(\lambda_j c, j) + \lambda_j c \right]$$

Since $y_s = y_t x_{t+1} \dots x_s$ the price of the equity is homogeneous of degree one in y_t , or now c. This allows us to represent it as

⁹³ Any positive demand $z_t^+ > 0$ from one agent must be met by a negative demand from another $z_t^- = z_t^+$ so that at the aggregate it becomes $\sum z_t = 0$. But in this model there is a representative agent (or all agents are identical) and then there is no one on the other side of the market if you want to buy or sell. The equilibrium we are looking for is such that the representative consumer neither wishes to buy nor sell. The prices must be such that his asset demand is $z_t = z_{t-1} = 0$ for all t. Zero-net supply imply that $c_t = y_t$ at all dates. At equilibrium we are looking for the prices that make it optimal for the agent to consume his endowment.

⁹⁴ Here p_{t+1} (in the section of CCPM) is replaced by substitution by the perpetual series of dividends into infinity.

⁹⁵ Taking period t+1 outside and using the law of iterated expectations

$$p_t^e = E_t \left[m_{t,t+1} \left(y_{t+1} + E_{t+1} \left[\sum_{s=t+2}^{\infty} m_{t+1,s} y_s \right] \right) \right]$$

⁹⁶ $c_{t+1} = c[\lambda_1, \dots, \lambda_n]$ is conditional on current state

$$(8B) \quad p^e(c, i) = w_i c$$

where w_i is some constant. Substituting this relation into (7B) and dividing by c gives

$$(9B) \quad w_i = \beta \sum \phi_{ij} \lambda_j^{1-\alpha} (w_j + 1), \quad i=1, \dots, n$$

This can be written in matrix form

$$k = A(k + 1)$$

k is a column vector of k_i 's and 1 a column vector of ones. The matrix A , with elements $a_{ij} = \beta \phi_{ij} \lambda_j^{1-\alpha}$ is stable, i.e. $\lim A^m = 0$ as $m \rightarrow \infty$ ⁹⁷. This assumption assures that the equation system has a unique and positive solution.

The net period return when the current state is (c, i) and next period state is $(\lambda_j c, j)$ is⁹⁸

$$(10B) \quad r_{ij}^e = \frac{p^e(\lambda_j c, j) + \lambda_j c - p^e(c, i)}{p^e(c, i)} \\ = \frac{\lambda_j (w_j + 1)}{w_i} - 1$$

Given the current state i , the expected period return is

$$(11B) \quad E_i(r_i^e) = \sum_{j=1}^n \phi_{ij} r_{ij}^e$$

The expectation is conditional on the state being (c, i) , The conditional return on the riskfree asset, which pays 1 for sure, is

$$(12B) \quad r_i^f = \frac{1}{p^f(c, i)} - 1$$

The most suitable measure to summarize the historical information on returns is the arithmetic averages, as discussed in the section of calibration. To calculate the unconditional or average returns of the model, we need the unconditional probabilities π , the unconditional probability of being in state i , denoted π_i , which can be calculated by taking the following limit:

⁹⁷ Mehra (1988) shows that this is necessary and sufficient for the existence of a consumption of y_t every period in accordance with expected utility.

⁹⁸ By use of (8B)

$$(13B) \quad \pi = \begin{pmatrix} \pi_1 \\ \dots \\ \pi_n \end{pmatrix} = \begin{pmatrix} \pi_1 & \pi_1 & \dots & \pi_n \\ \dots & \dots & & \dots \\ \pi_n & \pi_n & \dots & \pi_n \end{pmatrix} = \lim_{k \rightarrow \infty} \Phi^k$$

Which is the unique solution to $\Phi\pi = \pi$. The unconditional probabilities gives the long-run probabilities of being in the different states. Expected returns are

$$(14B) \quad E(r^e) = \sum_{i=1}^n \pi_i R_i^e \quad \text{and} \quad E(r^f) = \sum_{i=1}^n \pi_i R_i^f$$

The long run being in each state times the return of the state gives the average return of the asset. The equity premium is the difference between the return on the risky asset and the return on the risk free one.

Mehra and Prescott assumes a two-states Markov process where

$$x_t = [\lambda_1, \lambda_2] = [1 + \mu + \delta, 1 + \mu - \delta]$$

The transition probability matrix is symmetric

$$\Phi = \begin{bmatrix} \phi & (1-\phi) \\ (1-\phi) & \phi \end{bmatrix}, \quad \phi \in (0,1)$$

This specification makes it possible to vary the growth rate of output by changing μ , the variation in consumption by δ and the serial correlation of the growth rate by changing ϕ . The correlation coefficient is given by $\rho(x_t, x_{t+1}) = 2\phi - 1$

To fit the US consumption data from the period 1889-1978, with sample values 0.018, 0.036 and -0.14, the Markov chain was defined to be $\mu = 0.018$, $\delta = 0.036$ and $\phi = 0.43$. Mehra and Prescott placed restrictions on the not measurable subjective parameters, $\beta \in (0,1)$ and $\alpha \in (0,10)$ based on a number of studies. Calibrating the consumption data and keeping the restriction on β and α , results in average real risk-free rates between zero and four percent. The largest premium obtainable within the model is 0.35 %, contrary to the observed 6%.

Appendix C

$$k_{t+1} = \frac{(1-\alpha)}{(2+\rho)V_{t+1}^{\frac{1}{1-\alpha}}G_{t+1}}k_t^\alpha$$

Taking logarithms at both sides

$$\ln k_{t+1} = \ln \left(\frac{1-\alpha}{(2+\rho)V_{t+1}^{\frac{1}{1-\alpha}}G_{t+1}} \right) + \alpha \ln k_t$$

Or explicitly

$$\ln k_{t+1} = \ln \frac{1-\alpha}{(2+\rho)V_{t+1}^{\frac{1}{1-\alpha}}G_{t+1}} + \alpha \left[\ln \frac{1-\alpha}{(2+\rho)V_t^{\frac{1}{1-\alpha}}G_t} + \alpha \left(\ln \frac{1-\alpha}{(2+\rho)V_{t-1}^{\frac{1}{1-\alpha}}G_{t-1}} + \dots + \alpha \left[\ln \frac{1-\alpha}{(2+\rho)V_1^{\frac{1}{1-\alpha}}G_1} \right] \right) \right]$$

Given that the economy starts out with a capital per efficiency adjusted worker

$$k_0 \text{ and } G_1 \text{ and } V_1 \text{ are the first shocks, i.e. } k_1 = \frac{1-\alpha}{(2+\rho)V_1^{\frac{1}{1-\alpha}}G_1}k_0^\alpha$$

More compactly the expression can be written as

$$\ln k_{t+1} = \left(\sum_{k=1}^{K=t+1} \alpha^{t+1-k} \ln \frac{1-\alpha}{(2+\rho)V_k^{\frac{1}{1-\alpha}}G_k} \right) + \alpha^{t+1} \ln k_0$$

Or

$$\ln k_{t+1} = \left(\sum_{k=1}^{K=t+1} \alpha^{t+1-k} \left[\ln(1-\alpha) - \ln(2+\rho) - \frac{1}{1-\alpha} \ln V_k - \ln G_k \right] \right) + \alpha^{t+1} \ln k_0$$

Then taking expectations,

$$E \ln k_{t+1} = \left(\sum_{k=1}^{K=t+1} \alpha^{t+1-k} \left[\ln(1-\alpha) - \ln(2+\rho) - \frac{v}{1-\alpha} - g \right] \right) + \alpha^{t+1} \ln k_0$$

Then letting $t \rightarrow \infty$, this is an infinite geometric series⁹⁹. And

$$E \ln k_{t+1} = \frac{\ln \frac{1-\alpha}{2+\rho} - \frac{v}{1-\alpha} - g}{1-\alpha} = \frac{1}{1-\alpha} \ln \left(\frac{1-\alpha}{2+\rho} \right) - \frac{v}{(1-\alpha)^2} - \frac{g}{1-\alpha}$$

The effect of the initial level k_0 disappears over time (α is by assumption strictly less than 1), i.e. the distribution is independent of time when the system has run for a long

⁹⁹ The geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ converges to $\frac{a}{1-r}$ if $|r| < 1$.

time. This also shows that as $t \rightarrow \infty$, the value of $\ln k_{t+1}$ is independent of k_0 , i.e. the series is weakly dependent¹⁰⁰ and it converges to a steady state.

Appendix D

Let production be written

$$Y(t) = A(t)[K(t)]^{\beta}[N(t)]^{1-\alpha}$$

The change in output can be represented as relations in labor-to-output, capital-to-output and productivity-to-output by differentiating

$$\frac{\partial Y}{\partial t} = \frac{\partial Y}{\partial K} \frac{\partial K}{\partial t} + \frac{\partial Y}{\partial N} \frac{\partial N}{\partial t} + \frac{\partial Y}{\partial A} \frac{\partial A}{\partial t} \quad (\text{aa})$$

The partial derivatives are

$$\frac{\partial Y}{\partial K} = \alpha A(t)[K(t)]^{\beta-1}[N(t)]^{1-\alpha} = \frac{\alpha Y(t)}{K(t)}$$

$$\frac{\partial Y}{\partial N} = (1-\alpha)A(t)[K(t)]^{\beta}[N(t)]^{-\alpha} = \frac{(1-\alpha)Y(t)}{N(t)}$$

$$\frac{\partial Y}{\partial A} = [K(t)]^{\beta}[N(t)]^{1-\alpha} = \frac{Y(t)}{A(t)}$$

Now inserting these values into (aa) yields

$$\frac{\partial Y}{\partial t} = \frac{\alpha Y(t)}{K(t)} \frac{\partial K}{\partial t} + \frac{(1-\alpha)Y(t)}{N(t)} \frac{\partial N}{\partial t} + \frac{Y(t)}{A(t)} \frac{\partial A}{\partial t}$$

Dividing both sides by $Y(t)$ shows that growth in technology is given by

$$\frac{\frac{\partial A}{\partial t}}{A} = \frac{\frac{\partial Y}{\partial t}}{Y} - \alpha \frac{\frac{\partial K}{\partial t}}{K} - (1-\alpha) \frac{\frac{\partial N}{\partial t}}{N}$$

Appendix E

Maple output for Norwegian data, alpha=.323 and rho=0.3478

```
> g:=0.29868419;
```

```
g := .29868419
```

```
> v:=0.505591736;
```

¹⁰⁰ From the definition a stationary process, $\{x_t : t = 1, 2, \dots\}$ is weakly dependent if x_t and x_{t+h} are

“almost independent” as h increases without bound, which was shown here.


```

v := .505591736
> Varg:=0.00699747;
Varg := .00699747
> Varv:=0.006621239;
Varv := .006621239
> Kovar:=-0.001914279;
Kovar := -.001914279
> Er:= (rho) -> (alpha*(2+rho)/(1-alpha))*exp((v/(1-alpha))+g+(0.5*(1/(1-
alpha)^2)*Varv)+(0.5*Varg)+((1/(1-alpha))*Kovar));
Er := ρ → 
$$\frac{\alpha(2+\rho)e^{\left(\frac{v}{1-\alpha}+g+\frac{.5\text{Varv}}{(1-\alpha)^2}+.5\text{Varg}+\frac{\text{Kovar}}{1-\alpha}\right)}}{1-\alpha}$$

> alpha:=.323;
α := .323
> Erf:= (rho) -> (alpha*(2+rho)/(1-alpha))*exp((v/(1-alpha))+g+((alpha-
0.5)*Varv/(1-alpha)^2)+((alpha-0.5)*Varg)+((-alpha^2+(4*alpha)-2)*Kovar/(1-
alpha)));
Erf := ρ → 
$$\frac{\alpha(2+\rho)e^{\left(\frac{v}{1-\alpha}+g+\frac{(\alpha-.5)\text{Varv}}{(1-\alpha)^2}+(\alpha-.5)\text{Varg}+\frac{(-\alpha^2+4\alpha-2)\text{Kovar}}{1-\alpha}\right)}}{1-\alpha}$$

>
> 'E(r^f)'=Erf(.3478);
E(rf) = 3.181832643
> 'E(r^m)'=Er(.3478);
E(rm) = 3.211860485
> %-%;
E(rm) - E(rf) = .030027842
> fsolve(Er(rho)-Erf(rho)=1.89);
145.7742500

```

Maple outprint for the US, alpha = .3333 and rho= 2.403

```

> g:=0.46586512;
g := .46586512
> v:=.30639714;
v := .30639714
> Varg:=.00581717;
Varg := .00581717
> Varv:=.012789;
Varv := .012789
> Kovar:=.00177169;
Kovar := .00177169
> Er:= (rho) -> (alpha*(2+rho)/(1-alpha))*exp((v/(1-alpha))+g+(0.5*(1/(1-
alpha)^2)*Varv)+(0.5*Varg)+((1/(1-alpha))*Kovar));

```

$$Er := \rho \rightarrow \frac{\alpha (2 + \rho) e^{\left(\frac{v}{1-\alpha} + g + \frac{.5 \text{Var}v}{(1-\alpha)^2} + .5 \text{Varg} + \frac{\text{Kovar}}{1-\alpha} \right)}}{1-\alpha}$$

> alpha:=.3333;

$$\alpha := .3333$$

> Erf:= (rho) -> (alpha*(2+rho)/(1-alpha))*exp((v/(1-alpha))+g+(alpha-0.5)*Varv/(1-alpha)^2)+(alpha-0.5)*Varg+((-alpha^2+(4*alpha)-2)*Kovar/(1-alpha));

$$Erf := \rho \rightarrow \frac{\alpha (2 + \rho) e^{\left(\frac{v}{1-\alpha} + g + \frac{(\alpha - .5) \text{Var}v}{(1-\alpha)^2} + (\alpha - .5) \text{Varg} + \frac{(-\alpha^2 + 4\alpha - 2) \text{Kovar}}{1-\alpha} \right)}}{1-\alpha}$$

> 'E(r^f)'=Erf(.3478);

$$E(r^f) = 2.938167673$$

> 'E(r^m)'=Er(.3478);

$$E(r^m) = 3.020950630$$

> %%-%%;

$$E(r^m) - E(r^f) = .082782957$$

> rr:=fsolve(Er(rho)-Erf(rho)=2.076);

$$rr := 56.87724831$$

> Er(56.88);

$$75.76180810$$

> Erf(56.88);

$$73.68571111$$

> %%-%%;

$$2.07609699$$

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